

EC.1. Omitted proofs

EC.1.1. Proof of Theorem 1

To prove this, we prove that $Z_{\text{DR}}^* \geq Z_{\text{RR}}^*$. For any $R \in \mathcal{R}$, and any distribution F supported on \mathcal{P} , we have

$$\int_{\mathcal{P}} R(\mathbf{p}) dF(\mathbf{p}) \leq R \left(\int_{\mathcal{P}} \mathbf{p} dF(\mathbf{p}) \right), \quad (\text{EC.1})$$

which follows by Jensen's inequality and the concavity of R . This implies that for any $F \in \mathcal{F}$,

$$\inf_{R \in \mathcal{R}} \int_{\mathcal{P}} R(\mathbf{p}) dF(\mathbf{p}) \leq \inf_{R \in \mathcal{R}} R \left(\int_{\mathcal{P}} \mathbf{p} dF(\mathbf{p}) \right). \quad (\text{EC.2})$$

Therefore, we have that

$$\begin{aligned} Z_{\text{RR}}^* &= \max_{F \in \mathcal{F}} \left\{ \inf_{R \in \mathcal{R}} \int_{\mathcal{P}} R(\mathbf{p}) dF(\mathbf{p}) \right\} \\ &\leq \max_{F \in \mathcal{F}} \left\{ \inf_{R \in \mathcal{R}} R \left(\int_{\mathcal{P}} \mathbf{p} dF(\mathbf{p}) \right) \right\} \\ &\leq \max_{\mathbf{p} \in \mathcal{P}} \left\{ \inf_{R \in \mathcal{R}} R(\mathbf{p}) \right\} \\ &= Z_{\text{DR}}^*, \end{aligned}$$

where the second inequality follows because \mathcal{P} is assumed to be convex, and thus for any $F \in \mathcal{F}$, $\int_{\mathcal{P}} \mathbf{p} dF(\mathbf{p})$ is contained in \mathcal{P} . \square

EC.1.2. Proof of Theorem 2

We will establish this result by drawing a connection with game theory. Consider a two-player zero-sum game, where player 1 and 2 have strategy spaces \mathcal{P} and \mathcal{U} , respectively, and payoff functions g_1 and g_2 given by

$$\begin{aligned} g_1(\mathbf{p}, \mathbf{u}) &= R(\mathbf{p}, \mathbf{u}), \\ g_2(\mathbf{p}, \mathbf{u}) &= -R(\mathbf{p}, \mathbf{u}). \end{aligned}$$

Player 1 can choose the action $\mathbf{p} \in \mathcal{P}$ (a pure strategy) or some distribution F over \mathcal{P} (a mixed strategy), and player 2 chooses \mathbf{u} or some distribution Q over \mathcal{U} . Observe that by assumption, g_1 and g_2 are both continuous. Additionally, g_1 is quasiconcave in \mathbf{p} by assumption, and for g_2 , recall that the negative of a quasiconvex function is quasiconcave, so g_2 is also quasiconcave because $R(\mathbf{p}, \mathbf{u})$ is quasiconvex in \mathbf{u} . We now invoke the Debreu-Glicksberg-Fan theorem (see Theorem 1.2 in Fudenberg and Tirole 1991), which asserts that there exists a pure strategy Nash equilibrium for this game. In particular, there exists a $(\mathbf{p}^*, \mathbf{u}^*) \in \mathcal{P} \times \mathcal{U}$ such that

$$\begin{aligned} g_1(\mathbf{p}^*, \mathbf{u}^*) &\geq g_1(\mathbf{p}, \mathbf{u}^*), \quad \forall \mathbf{p} \in \mathcal{P}, \\ g_2(\mathbf{p}^*, \mathbf{u}^*) &\geq g_2(\mathbf{p}^*, \mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{U}. \end{aligned}$$

Note that this is equivalent to $(\mathbf{p}^*, \mathbf{u}^*)$ being a saddle point of $R(\mathbf{p}, \mathbf{u})$:

$$R(\mathbf{p}, \mathbf{u}^*) \leq R(\mathbf{p}^*, \mathbf{u}^*) \leq R(\mathbf{p}^*, \mathbf{u}), \quad \forall \mathbf{p} \in \mathcal{P}, \mathbf{u} \in \mathcal{U}. \quad (\text{EC.3})$$

This implies that

$$\begin{aligned} R(\mathbf{p}^*, \mathbf{u}^*) &\leq \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}^*, \mathbf{u}) \\ &\leq \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}), \end{aligned}$$

$$\begin{aligned} R(\mathbf{p}^*, \mathbf{u}^*) &\geq \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}^*) \\ &\geq \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}), \end{aligned}$$

which together imply that

$$\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) \leq \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}). \quad (\text{EC.4})$$

Observe now that

$$\begin{aligned} Z_{\text{RR}}^* &= \max_{F \in \mathcal{F}} \min_{\mathbf{u} \in \mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) \\ &\leq \min_{\mathbf{u} \in \mathcal{U}} \max_{F \in \mathcal{F}} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) \\ &= \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) \\ &\leq \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \\ &= Z_{\text{DR}}^*, \end{aligned}$$

and hence $Z_{\text{RR}}^* = Z_{\text{DR}}^*$. \square

EC.1.3. Proof of Theorem 3

To establish this result, observe that

$$\begin{aligned} &\inf_{Q \in \mathcal{Q}} \max_{\mathbf{p} \in \mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \\ &= \inf_{Q \in \mathcal{Q}} \max_{\pi \in \Delta_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \\ &= \max_{\pi \in \Delta_{\mathcal{P}}} \inf_{Q \in \mathcal{Q}} \sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \\ &= \max_{\pi \in \Delta_{\mathcal{P}}} \inf_{Q \in \mathcal{Q}} \int_{\mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \\ &= \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot R(\mathbf{p}, \mathbf{u}) \\ &= Z_{\text{RR}}^*. \end{aligned}$$

In the above, the steps are justified as follows. The first step follows because maximizing a function of \mathbf{p} over the finite set \mathcal{P} is the same as maximizing the expected value of that same function with respect to all probability mass functions $\boldsymbol{\pi}$ supported on \mathcal{P} .

The second step follows by Sion's minimax theorem, because the quantity $\sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u})$ is linear and therefore quasiconcave in $\boldsymbol{\pi}$ for a fixed Q , and is linear and therefore quasiconvex in Q for a fixed $\boldsymbol{\pi}$; additionally, the set $\Delta_{\mathcal{P}} = \{\boldsymbol{\pi} \in \mathbb{R}^{|\mathcal{P}|} \mid \mathbf{1}^\top \boldsymbol{\pi} = 1, \boldsymbol{\pi} \geq \mathbf{0}\}$ is a compact convex set, and \mathcal{Q} is a convex set. Additionally, note that $\sum_{\mathbf{p} \in \mathcal{P}} \pi(\mathbf{p}) \cdot \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u})$ is clearly continuous in $\boldsymbol{\pi}$. It is also continuous in Q , because each term $\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u})$ is continuous in Q when \mathcal{Q} is endowed with the topology of weak convergence, and there are finitely many such terms.

The third step follows by the linearity of integration. The fourth step follows by the fact that \mathcal{Q} contains the Dirac delta distribution that places unit mass on \mathbf{u} , for every $\mathbf{u} \in \mathcal{U}$. The final step just follows from the definition of Z_{RR}^* .

With this result in hand, observe that the existence of a $Q \in \mathcal{Q}$ such that for all $\mathbf{p} \in \mathcal{P}$, $\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \leq Z_{\text{DR}}^*$ is equivalent to the existence of $Q \in \mathcal{Q}$ such that

$$\max_{\mathbf{p} \in \mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \leq Z_{\text{DR}}^*,$$

which is equivalent to

$$\inf_{Q \in \mathcal{Q}} \max_{\mathbf{p} \in \mathcal{P}} \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \leq Z_{\text{DR}}^*.$$

Since the left-hand side of this inequality is equal to Z_{RR}^* , the existence of the distribution $Q \in \mathcal{Q}$ as in the theorem statement is equivalent to $Z_{\text{RR}}^* \leq Z_{\text{DR}}^*$; since it is always the case that $Z_{\text{RR}}^* \geq Z_{\text{DR}}^*$, this is equivalent to the problem being randomization-proof. \square

EC.1.4. Proof of Corollary 1

Observe that since $\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \leq \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$ always holds, equation (21) is equivalent to

$$\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \geq \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}),$$

or equivalently,

$$Z_{\text{DR}}^* \geq \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}).$$

Observe that the condition $\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) \leq Z_{\text{DR}}^*$ is exactly equivalent to the condition that there exists a $\mathbf{u} \in \mathcal{U}$ such that for all $\mathbf{p} \in \mathcal{P}$, $R(\mathbf{p}, \mathbf{u}) \leq Z_{\text{DR}}^*$.

To connect this to Theorem 3, consider $Q = \delta_{\mathbf{u}}$, where $\delta_{\mathbf{u}}$ is the Dirac delta distribution centered at \mathbf{u} . For any $\mathbf{p} \in \mathcal{P}$, $R(\mathbf{p}, \mathbf{u}) = \int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}') dQ(\mathbf{u}')$. Thus, for this choice of Q , it is the case that for all $\mathbf{p} \in \mathcal{P}$, $\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}') dQ(\mathbf{u}') \leq Z_{\text{DR}}^*$. By Theorem 3, this is equivalent to randomization-proofness. Thus, it follows that the strong duality condition (21) implies that the RPO problem is randomization-proof. \square

EC.1.5. Proof of Corollary 2

Proof of a): Observe that by the definition of \mathbf{u}^* , we have that $R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*) \leq R(\mathbf{p}_{\text{DR}}^*, \mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}$. By the hypothesis that \mathbf{p}_{DR}^* solves the nominal problem under \mathbf{u}^* , that is, $\mathbf{p}_{\text{DR}}^* \in \arg \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}^*)$, we also have that $R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*) \geq R(\mathbf{p}, \mathbf{u}^*)$ for all $\mathbf{p} \in \mathcal{P}$. This automatically implies that $(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*)$ is a saddle point of R , which implies that strong duality holds, i.e.,

$$\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) = \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.5})$$

and thus that the problem is randomization-proof.

Proof of b): By using Theorem 3, we know that there exists a distribution Q such that for every $\mathbf{p} \in \mathcal{P}$,

$$\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) \leq Z_{\text{DR}}^*. \quad (\text{EC.6})$$

Recall that $Z_{\text{DR}}^* = \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}) = R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*)$. Since the minimizer of this problem is unique, we can examine the above condition when $\mathbf{p} = \mathbf{p}_{\text{DR}}^*$:

$$\int_{\mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}) dQ(\mathbf{u}) \leq \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}). \quad (\text{EC.7})$$

As a result, we must have that $Q = \delta_{\mathbf{u}^*}$, as any other Q would result in $\int_{\mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}) dQ(\mathbf{u})$ being strictly more than $R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*)$, due to the uniqueness of \mathbf{u}^* .

With this insight in hand, observe that for any $\mathbf{p} \in \mathcal{P}$, we have

$$\int_{\mathcal{U}} R(\mathbf{p}, \mathbf{u}) dQ(\mathbf{u}) = R(\mathbf{p}, \mathbf{u}^*).$$

Using the property of Q in (EC.7) at any arbitrary \mathbf{p} and the fact that $Z_{\text{DR}}^* = R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*)$, we have

$$R(\mathbf{p}, \mathbf{u}^*) \leq R(\mathbf{p}_{\text{DR}}^*, \mathbf{u}^*) \quad (\text{EC.8})$$

for all \mathbf{p} , which completes the proof. \square

EC.1.6. Equivalence of strong duality and randomization-proofness when \mathcal{P} is finite

In this section, we develop the following result, which provides a condition under which strong duality and randomization-proofness are equivalent.

PROPOSITION EC.1. *Suppose that \mathcal{P} is finite, $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} and \mathcal{U} is a convex set. Then the robust price optimization problem is randomization-proof if and only if strong duality holds.*

Proof: Observe that

$$\begin{aligned} Z_{\text{RR}}^* &= \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \\ &= \min_{\mathbf{u} \in \mathcal{U}} \max_{\pi \in \Delta_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \\ &= \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}), \end{aligned}$$

where the second equality follows by Sion's minimax theorem, which follows because (1) $\sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u})$ is linear (and hence continuous and quasiconcave) in π , and convex (and hence continuous and quasiconvex) in \mathbf{u} , (2) $\Delta_{\mathcal{P}}$ is a compact convex set, and (3) \mathcal{U} is a convex set; and the third equality follows because the inner maximum is attained at a deterministic distribution that places all mass on a single \mathbf{p} . Observe that this last min-max expression is an upper bound on $Z_{\text{DR}}^* = \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u})$ by weak duality, and thus that the problem is exactly randomization-proof if and only if strong duality holds. \square

EC.1.7. Proof of Proposition 1

Let $(\boldsymbol{\mu}^*, \mathbf{p}^*)$ be an optimal solution of problem (36), which is guaranteed to exist because the objective function is continuous and the feasible set $\Delta_{[I]} \times \mathcal{P}$ is compact. Observe that the objective function in (36) can be re-arranged as

$$\begin{aligned} & \max_{\boldsymbol{\mu} \in \Delta_{[I]}, \mathbf{p} \in \mathcal{P}} \left\{ \sum_{i=1}^I \mu_i (\alpha_i + \log p_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\ &= \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \max_{\mathbf{p} \in \mathcal{P}} \left\{ \sum_{i=1}^I \mu_i \cdot \alpha_i + \sum_{i=1}^I \left[\mu_i \cdot (1 - \beta_i) + \sum_{j \neq i} \mu_j \gamma_{j,i} \right] \cdot \log p_i - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\ &= \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left[\sum_{i=1}^I \mu_i \cdot \alpha_i + \sum_{i=1}^I \max_{p_i \in \mathcal{P}_i} \left\{ \left[\mu_i \cdot (1 - \beta_i) + \sum_{j \neq i} \mu_j \gamma_{j,i} \right] \cdot \log p_i \right\} - \sum_{i=1}^I \mu_i \log \mu_i \right] \end{aligned}$$

where the first step follows by algebra, and the second by the separability of the objective in p_1, \dots, p_I and Assumption 1 (since the price set is a Cartesian product and the objective is separable, each product's price can be optimized independently). Consider now holding $\boldsymbol{\mu}$ fixed, and optimizing over \mathbf{p} . With $\boldsymbol{\mu}^*$ fixed, the optimal value of p_i for the above objective depends on the sign of $(1 - \beta_i + \sum_{j \neq i} \gamma_{j,i})$. If this coefficient is positive, then since $\log p_i$ is increasing in p_i , it is optimal to set $p'_i = \max \mathcal{P}_i$. If this coefficient is negative, then it is optimal to set $p'_i = \min \mathcal{P}_i$. It thus follows that if $(\boldsymbol{\mu}^*, \mathbf{p}^*)$ is an optimal solution, then $(\boldsymbol{\mu}^*, \mathbf{p}')$ is also an optimal solution, where $\mathbf{p}' \in \prod_{i=1}^I \{\min \mathcal{P}_i, \max \mathcal{P}_i\}$. \square

EC.1.8. Proof of Theorem 4

Let t_{floor} be any real number that satisfies

$$t_{\text{floor}} < \min_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}). \quad (\text{EC.9})$$

Note that the right-hand side is well-defined: $R(\mathbf{p}, \mathbf{u})$ is convex and therefore continuous in \mathbf{u} , and \mathcal{U} is a compact set, and hence $\min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u})$ is finite by the extreme value theorem. Since \mathcal{P} is finite, the quantity $\min_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u})$ is therefore finite.

Observe that the RRPO problem $\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u})$ is equivalent to

$$\text{RR}(\mathcal{P}): \quad \underset{t, \mathbf{u}}{\text{minimize}} \quad t \quad (\text{EC.10a})$$

$$\text{subject to} \quad t \geq R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{p} \in \mathcal{P}, \quad (\text{EC.10b})$$

$$t \geq t_{\text{floor}}, \quad (\text{EC.10c})$$

$$\mathbf{u} \in \mathcal{U}, \quad (\text{EC.10d})$$

Note that this formulation is almost the same as formulation (22) in Section 5.1, with the exception of the constraint $t \geq t_{\text{floor}}$; we shall discuss the reason for including this constraint momentarily, after problem (EC.11) is presented. Note that since t_{floor} by construction is always strictly less than $R(\mathbf{p}, \mathbf{u})$ for any $\mathbf{p} \in \mathcal{P}$ and any $\mathbf{u} \in \mathcal{U}$, this constraint is vacuous and cannot become active/binding (i.e., hold at equality).

Consider now the following generalization of the above problem, where we enforce the constraint $t \geq R(\mathbf{p}, \mathbf{u})$ at a specific set of price vectors, $\bar{\mathcal{P}} \subseteq \mathcal{P}$:

$$\text{RR}(\bar{\mathcal{P}}): \quad \underset{t, \mathbf{u}}{\text{minimize}} \quad t \quad (\text{EC.11a})$$

$$\text{subject to} \quad t \geq R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{p} \in \bar{\mathcal{P}}, \quad (\text{EC.11b})$$

$$t \geq t_{\text{floor}}, \quad (\text{EC.11c})$$

$$\mathbf{u} \in \mathcal{U}, \quad (\text{EC.11d})$$

Let us denote the optimal value by $Z_{\text{RR}}^*(\bar{\mathcal{P}})$, and the problem itself with this set of constraints by $\text{RR}(\bar{\mathcal{P}})$. Similarly to the previous formulation, which is $\text{RR}(\mathcal{P})$ and has objective value $Z_{\text{RR}}^*(\mathcal{P})$, the constraint $t \geq t_{\text{floor}}$ in $\text{RR}(\bar{\mathcal{P}})$ cannot be active so long as $\bar{\mathcal{P}}$ is nonempty. Here, we note that the reason for having the constraint $t \geq t_{\text{floor}}$ is to avoid a pathological case where the problem becomes unbounded if $\bar{\mathcal{P}}$ is empty; the inclusion of this constraint facilitates our analysis of the constructive procedure (Algorithm 2) that we will discuss shortly (see, in particular, the proof of Lemma EC.2).

With these definitions, to now appreciate our high-level proof strategy, observe that if we could succeed in showing that $Z_{\text{RR}}^*(\mathcal{P}) = Z_{\text{RR}}^*(\mathcal{P}')$ for a small set of prices $\mathcal{P}' \subseteq \mathcal{P}$ that satisfies $|\mathcal{P}'| \leq$

$M + 1$, then a similar application of Sion's minimax theorem used to derive the convex program in Section 5.1 would allow us to show the following:

$$\max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.12})$$

$$= Z_{\text{RR}}^*(\mathcal{P}) \quad (\text{EC.13})$$

$$= Z_{\text{RR}}^*(\mathcal{P}') \quad (\text{EC.14})$$

$$= \min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}'} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.15})$$

$$= \min_{\mathbf{u} \in \mathcal{U}} \max_{\pi \in \Delta_{\mathcal{P}'}} \sum_{\mathbf{p} \in \mathcal{P}'} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.16})$$

$$= \max_{\pi \in \Delta_{\mathcal{P}'}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}'} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.17})$$

which would imply that there exists a price distribution $\tilde{\pi}$ over \mathcal{P}' that achieves the same objective as the optimal solution of the RRPO problem when one considers all price vectors. In what follows, we will constructively establish that such a set \mathcal{P}' exists.

Before proceeding, we first recall the concept of a support constraint from the paper of Calafiore and Campi (2005).

DEFINITION EC.1. Consider the convex optimization problem

$$\text{CONV} : \underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \quad \mathbf{c}^\top \mathbf{x} \quad (\text{EC.18a})$$

$$\text{subject to} \quad \mathbf{x} \in \mathcal{X}_i, \quad \forall i \in [m], \quad (\text{EC.18b})$$

where m and n are positive integers, and each \mathcal{X}_i is a closed convex set. Consider the problem CONV_k , defined as

$$\text{CONV}_k : \underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \quad \mathbf{c}^\top \mathbf{x} \quad (\text{EC.19a})$$

$$\text{subject to} \quad \mathbf{x} \in \mathcal{X}_i, \quad \forall i \in [m] \setminus \{k\}, \quad (\text{EC.19b})$$

which is the problem obtained from removing the k th constraint $\mathbf{x} \in \mathcal{X}_k$ from CONV. Let \mathbf{x}^* and \mathbf{x}_k^* be optimal solutions of CONV and CONV_k , respectively. We say that constraint k is a *support constraint*, if $\mathbf{c}^\top \mathbf{x}^* > \mathbf{c}^\top \mathbf{x}_k^*$, i.e., if removing constraint k from CONV causes a change to the optimal objective value of the problem.

We now recall a critical result about support constraints from Calafiore and Campi (2005).

THEOREM EC.1 (**Theorem 2 of Calafiore and Campi (2005)**). *The number of support constraints for problem CONV is at most n .*

In our case, note that each inequality $t \geq R(\mathbf{p}, \mathbf{u})$ in $\text{RR}(\bar{\mathcal{P}})$ can be equivalently regarded as the constraint $(t, \mathbf{u}) \in \{(t', \mathbf{u}') \in \mathbb{R}^{M+1} \mid t' \geq R(\mathbf{p}, \mathbf{u}')\}$, and the set in this latter definition is clearly a closed, convex set.

With a slight abuse of terminology, we shall say that \mathbf{p} is a *support constraint* for $\text{RR}(\bar{\mathcal{P}})$ if the constraint $(t, \mathbf{u}) \in \{(t', \mathbf{u}') \in \mathbb{R}^{M+1} \mid t' \geq R(\mathbf{p}, \mathbf{u}')\}$ is a support constraint. Note that if \mathbf{p} is a support constraint, then the convex inequality $t \geq R(\mathbf{p}, \mathbf{u})$ must hold at equality for any optimal solution of $\text{RR}(\bar{\mathcal{P}})$. (This is true, because if there were an optimal solution (t, \mathbf{u}) at which the constraint for \mathbf{p} was non-binding, then one could take the improved optimal solution (t', \mathbf{u}') of $\text{RR}(\bar{\mathcal{P}} \setminus \{\mathbf{p}\})$ and take a convex combination with (t, \mathbf{u}) to obtain a better solution to $\text{RR}(\bar{\mathcal{P}})$.)

Additionally, given an optimal solution (t^*, \mathbf{u}^*) , we shall say that \mathbf{p} is an *active non-support constraint* for $\text{RR}(\bar{\mathcal{P}})$ if the constraint $t \geq R(\mathbf{p}, \mathbf{u})$ is active at (t^*, \mathbf{u}^*) , i.e., if $t^* = R(\mathbf{p}, \mathbf{u}^*)$, and \mathbf{p} is not a support constraint of $\text{RR}(\bar{\mathcal{P}})$.

Lastly, given an optimal solution (t^*, \mathbf{u}^*) , we shall say that \mathbf{p} is a *non-active constraint* for $\text{RR}(\bar{\mathcal{P}})$ if the constraint $t \geq R(\mathbf{p}, \mathbf{u})$ is not active at (t^*, \mathbf{u}^*) , i.e., if $t^* > R(\mathbf{p}, \mathbf{u}^*)$.

With these definitions, let us consider the following procedure, which we denote as Algorithm 2. In words, this algorithm starts with all of the price vectors in \mathcal{P} , solves the randomized robust pricing problem with this set, and then decomposes the price vector set into sets corresponding to support constraints, active non-support constraints and non-active constraints. The non-active constraints are also discarded, and if there are any active non-support constraints, one price vector from that set of active non-support constraints is also discarded. The optimization problem is solved again with the remaining smaller set of constraints, and the remaining price vectors are then decomposed again. This procedure repeats until one has a set of price vectors for which all active constraints are support constraints, or equivalently, the set of active non-support constraints \mathcal{N} is empty.

We now establish a few key properties of this procedure.

LEMMA EC.1. *Algorithm 2 terminates in a finite number of iterations.*

Proof: Observe that after line 7 of Algorithm 2, $|\bar{\mathcal{P}}| = |\mathcal{N}| - 1 + |\mathcal{S}|$, whereas immediately before line 7, $|\bar{\mathcal{P}}| = |\mathcal{N}| + |\mathcal{S}| + |\mathcal{O}|$, which is strictly higher than $|\mathcal{N}| - 1 + |\mathcal{S}|$. Since the size of $\bar{\mathcal{P}}$ is decreasing by at least 1 in each iteration, the initial set \mathcal{P} is finite, and \mathcal{N} is always updated to be a subset of $\bar{\mathcal{P}}$, it must be that \mathcal{N} eventually becomes empty. \square

LEMMA EC.2. *The set $\bar{\mathcal{P}}$ is non-empty throughout the execution of Algorithm 2.*

Algorithm 2 Constructive procedure for reducing the price vector set \mathcal{P} .

- 1: Initialize $\bar{\mathcal{P}} = \mathcal{P}$.
 - 2: Solve $\text{RR}(\bar{\mathcal{P}})$ to obtain an optimal solution (t^*, \mathbf{u}^*) .
 - 3: Set $\mathcal{S} \subseteq \bar{\mathcal{P}}$ to be the set of support constraints for $\text{RR}(\bar{\mathcal{P}})$.
 - 4: Set $\mathcal{N} \subseteq \bar{\mathcal{P}}$ to be the set of active non-support constraints for $\text{RR}(\bar{\mathcal{P}})$ and (t^*, \mathbf{u}^*) .
 - 5: Set $\mathcal{O} \subseteq \bar{\mathcal{P}}$ to be the set of non-active constraints for $\text{RR}(\bar{\mathcal{P}})$ and (t^*, \mathbf{u}^*) .
 - 6: **while** $|\mathcal{N}| > 0$ **do**
 - 7: Select any $\mathbf{p} \in \mathcal{N}$, and set $\bar{\mathcal{P}} \leftarrow \mathcal{N} \setminus \{\mathbf{p}\} \cup \mathcal{S}$.
 - 8: Solve $\text{RR}(\bar{\mathcal{P}})$ to obtain an optimal solution (t^*, \mathbf{u}^*) .
 - 9: Set $\mathcal{S} \subseteq \bar{\mathcal{P}}$ to be the set of support constraints for $\text{RR}(\bar{\mathcal{P}})$.
 - 10: Set $\mathcal{N} \subseteq \bar{\mathcal{P}}$ to be the set of active non-support constraints for $\text{RR}(\bar{\mathcal{P}})$ and (t^*, \mathbf{u}^*) .
 - 11: Set $\mathcal{O} \subseteq \bar{\mathcal{P}}$ to be the set of non-active constraints for $\text{RR}(\bar{\mathcal{P}})$ and (t^*, \mathbf{u}^*) .
 - 12: **end while**
 - 13: **return** $\mathcal{S}^* \equiv \mathcal{S}$.
-

Proof: Suppose for the sake of contradiction that $\bar{\mathcal{P}}$ becomes empty. This would mean that in some iteration of the while loop, at line 7, one would have that $\mathcal{N} \setminus \{\mathbf{p}\} \cup \mathcal{S}$ is empty, which would imply that $\mathcal{N} \setminus \{\mathbf{p}\} = \emptyset$ and $\mathcal{S} = \emptyset$. Since \mathbf{p} is chosen from \mathcal{N} , this would imply that $\mathcal{N} = \{\mathbf{p}\}$. This means that prior to line 7, $\text{RR}(\bar{\mathcal{P}})$ has only one active constraint, which is at \mathbf{p} , and that constraint is not a support constraint. This, however, leads to a contradiction, because by removing this constraint, one would be free to set $t = t_{\text{floor}}$, and recalling that t_{floor} is strictly smaller than $R(\mathbf{p}, \mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}$, this would mean we are able to improve the objective value. (In other words, \mathbf{p} has to be a support constraint.) \square

LEMMA EC.3. *Let \mathcal{S}^* be the output of Algorithm 2. Then \mathcal{S}^* is not empty, and $|\mathcal{S}^*| \leq M + 1$.*

Proof: Observe that Algorithm 2 terminates by Lemma EC.1, and that by Lemma EC.2, $\bar{\mathcal{P}}$ never becomes empty during Algorithm 2. Consider now what happens in lines 9 and 10 of Algorithm 2 immediately prior to termination. Since $\bar{\mathcal{P}}$ is not empty, then $\text{RR}(\bar{\mathcal{P}})$ must have at least one active constraint from among the constraints in $\bar{\mathcal{P}}$ (i.e., $\mathcal{S} \cup \mathcal{N}$ is not empty); since the algorithm terminates and the loop exits, \mathcal{N} must be empty. This implies that \mathcal{S}^* is not empty.

To see that $|\mathcal{S}^*| \leq M + 1$, observe that \mathcal{S}^* is the set of support constraints of $\text{RR}(\bar{\mathcal{P}})$ for some $\bar{\mathcal{P}} \subseteq \mathcal{P}$, which is a $(M + 1)$ -dimensional convex optimization problem; by Theorem 2 of Calafiore and Campi (2005), $|\mathcal{S}^*|$ can be at most $M + 1$. \square

LEMMA EC.4. *Let \mathcal{S}^* be the output of Algorithm 2. Then $\text{RR}(\mathcal{S}^*) = \text{RR}(\mathcal{P})$.*

Proof: We shall establish this result by induction. Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$ be any non-empty set of price vectors, and let \mathcal{S} , \mathcal{N} and \mathcal{O} be defined as in lines 9, 10, and 11 respectively. Let (t^*, \mathbf{u}^*) be any optimal solution of $\text{RR}(\bar{\mathcal{P}})$. We claim that

$$Z_{\text{RR}}^*(\bar{\mathcal{P}}) \stackrel{(a)}{=} Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N}) \stackrel{(b)}{=} Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\}), \quad (\text{EC.20})$$

where \mathbf{p}' is any price vector in \mathcal{N} . We now establish each of the two equalities, (a) and (b).

Proof of (a). Observe that we trivially have $Z_{\text{RR}}^*(\bar{\mathcal{P}}) \geq Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N})$, because $\mathcal{S} \cup \mathcal{N}$ is a subset of $\bar{\mathcal{P}}$, and so the objective value of the minimization problem with $\mathcal{S} \cup \mathcal{N}$ is either equal or lower than with $\bar{\mathcal{P}}$.

To see why $Z_{\text{RR}}^*(\bar{\mathcal{P}}) > Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N})$ cannot hold, suppose for the sake of contradiction that this was the case, and let (t', \mathbf{u}') be any optimal solution of $\text{RR}(\mathcal{S} \cup \mathcal{N})$. Observe that by taking any convex combination of (t^*, \mathbf{u}^*) and (t', \mathbf{u}') , we would obtain a new solution $(\tilde{t}, \tilde{\mathbf{u}})$ that satisfies the constraint $t \geq R(\mathbf{p}, \mathbf{u})$ for all $\mathbf{p} \in \mathcal{S} \cup \mathcal{N}$, because the function $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} and both (t^*, \mathbf{u}^*) and (t', \mathbf{u}') satisfy those constraints. Note also that while (t', \mathbf{u}') might not necessarily satisfy $t \geq R(\mathbf{p}, \mathbf{u})$ for every $\mathbf{p} \in \mathcal{O} = \bar{\mathcal{P}} \setminus (\mathcal{S} \cup \mathcal{N})$, by a suitable choice of the weight on (t^*, \mathbf{u}^*) , the convex combination $(\tilde{t}, \tilde{\mathbf{u}})$ can be made to satisfy these because by assumption, $t^* > R(\mathbf{p}, \mathbf{u}^*)$ for all $\mathbf{p} \in \mathcal{O}$. (Because of the slack in these constraints for (t^*, \mathbf{u}^*) , and because there are finitely many of these constraints, we can choose the weight in the convex combination so that none of these constraints are violated in the new solution $(\tilde{t}, \tilde{\mathbf{u}})$.) Lastly, since the objective function is linear (in fact, it is exactly t), any convex combination of t^* and t' with a positive weight on t' would ensure that \tilde{t} is strictly smaller than t^* . Since we can construct a new solution $(\tilde{t}, \tilde{\mathbf{u}})$ that is feasible for $\text{RR}(\bar{\mathcal{P}})$ but has an objective value smaller than t^* , we reach a contradiction, because (t^*, \mathbf{u}^*) was assumed to be optimal for $\text{RR}(\bar{\mathcal{P}})$.

Proof of (b). We now establish the second equality. We shall do this by showing that $Z_{\text{RR}}^*(\bar{\mathcal{P}}) = Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\})$; by then using (a), we will have that $Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N}) = Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\})$.

As in the case of (a), we again trivially have that $Z_{\text{RR}}^*(\bar{\mathcal{P}}) \geq Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\})$, because $\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\} \subseteq \bar{\mathcal{P}}$.

To see why $Z_{\text{RR}}^*(\bar{\mathcal{P}}) > Z_{\text{RR}}^*(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\})$ cannot hold, let us suppose that it does, and let (t', \mathbf{u}') be any optimal solution of $\text{RR}(\mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\})$. We thus have $t^* > t'$. Now, consider a new solution $(\tilde{t}, \tilde{\mathbf{u}})$, which is obtained by taking a convex combination of (t^*, \mathbf{u}^*) and (t', \mathbf{u}') . Observe that any convex combination of (t^*, \mathbf{u}^*) and (t', \mathbf{u}') satisfies $t \geq R(\mathbf{p}, \mathbf{u})$ for all $\mathbf{p} \in \mathcal{S} \cup \mathcal{N} \setminus \{\mathbf{p}'\}$, because both

(t^*, \mathbf{u}^*) and (t', \mathbf{u}') have to satisfy these constraints and $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} . Observe also that because the constraints in \mathcal{O} are not binding at (t^*, \mathbf{u}^*) and \mathcal{O} is finite, we can choose the positive weight on (t', \mathbf{u}') in the convex combination so that $(\tilde{t}, \tilde{\mathbf{u}})$ satisfies $t \geq R(\mathbf{p}, \mathbf{u})$ for all $\mathbf{p} \in \mathcal{O}$ (this is similar to the argument in the case of equality (a)). Lastly, by choosing a suitable positive weight on (t', \mathbf{u}') , the new solution $(\tilde{t}, \tilde{\mathbf{u}})$ can also be made to have an objective value strictly lower than t^* . Thus, by choosing a suitable weight on (t', \mathbf{u}') , we are able to construct the new solution $(\tilde{t}, \tilde{\mathbf{u}})$ which is a feasible solution of $\text{RR}(\bar{\mathcal{P}} \setminus \{\mathbf{p}'\})$, and has objective value $\tilde{t} < t^* = Z_{\text{RR}}^*(\bar{\mathcal{P}})$. This implies that $Z_{\text{RR}}^*(\bar{\mathcal{P}} \setminus \{\mathbf{p}'\}) < Z_{\text{RR}}^*(\bar{\mathcal{P}})$, which immediately leads to a contradiction, because it implies that \mathbf{p}' is a support constraint of $\text{RR}(\bar{\mathcal{P}})$. (By dropping the \mathbf{p}' constraint from $\text{RR}(\bar{\mathcal{P}})$, we get $Z_{\text{RR}}^*(\bar{\mathcal{P}} \setminus \{\mathbf{p}'\})$ which we have shown to be lower than $Z_{\text{RR}}^*(\bar{\mathcal{P}})$.) This contradicts the assumption that \mathbf{p}' is from \mathcal{N} and not a support constraint.

Having established equation (EC.20), suppose that the loop in Algorithm 2 performs k iterations, and let $\bar{\mathcal{P}}_\ell$ be the set $\bar{\mathcal{P}}$ after line 7 is performed in the ℓ th iteration of the loop. Recall that \mathcal{S}^* is the output of Algorithm 2 in the hypothesis of the lemma. Then by applying equation (EC.20) inductively, we have that

$$Z_{\text{RR}}^*(\mathcal{P}) = Z_{\text{RR}}^*(\bar{\mathcal{P}}_1) = Z_{\text{RR}}^*(\bar{\mathcal{P}}_2) = Z_{\text{RR}}^*(\bar{\mathcal{P}}_3) = \cdots = Z_{\text{RR}}^*(\bar{\mathcal{P}}_k) = Z_{\text{RR}}^*(\mathcal{S}^*), \quad (\text{EC.21})$$

where each equality except for the last follows by using both (a) and (b) in (EC.20), while the last equality $Z_{\text{RR}}^*(\bar{\mathcal{P}}_k) = Z_{\text{RR}}^*(\mathcal{S}^*)$ follows by using (a) in (EC.20) together with the fact that $\mathcal{N} = \emptyset$ (this is the case because it is the final iteration and the loop is exited). \square

We now complete the proof of Theorem 4. To establish the result, we apply Algorithm 2. The resulting \mathcal{S}^* that we obtain is nonempty and has at most $M + 1$ elements (Lemma EC.3); moreover, we have that $Z_{\text{RR}}^*(\mathcal{S}^*) = Z_{\text{RR}}^*(\mathcal{P})$ (Lemma (EC.4)). By our earlier argument (equations (EC.12) - (EC.17)), this implies that

$$\max_{\pi \in \Delta_{\mathcal{S}^*}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{S}^*} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) = \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.22})$$

which establishes the theorem. \square

EC.2. Accompanying examples for Section 4

In this section, we provide a number of examples that accompany our discussion of our theoretical results in Section 4. Section EC.2.1 provides examples to illustrate that certain conditions in Theorem 1 cannot be relaxed, and an example to show that strong duality does not hold in the setting

of this theorem. Section EC.2.2 documents a number of examples of applications of Theorem 1. Section EC.2.3 documents a number of examples of applications of Theorem 2. Section EC.2.4 documents an example to show that uniqueness is necessary for part (b) of Corollary 2, and also to show that randomization-proofness can hold without strong duality holding when \mathcal{P} is finite. Lastly, Section EC.2.5 documents an example that illustrates that Z_{DR}^* , Z_{RR}^* and the dual bound $\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$ can in general all be different.

EC.2.1. Accompanying examples for Theorem 1

As noted in Section 4.1, the condition that all functions in \mathcal{R} be concave cannot be relaxed, which we illustrate using the following example.

EXAMPLE EC.1. Consider a single-product RPO problem, and suppose that the revenue function uncertainty set $\mathcal{R} = \{R_1, R_2\}$, where $R_1(\cdot)$ and $R_2(\cdot)$ are defined as

$$\begin{aligned} R_1(p) &= p(10 - 2p), \\ R_2(p) &= p \cdot 10p^{-2}. \end{aligned}$$

Note that $R_1(p)$ is the revenue function corresponding to the linear demand function $d_1(p) = 10 - 2p$, while $R_2(p)$ is the revenue function corresponding to the log-log demand function $d_2(p) = \exp(\log(10) - 2\log(p)) = 10p^{-2}$. Note also that $R_1(\cdot)$ is concave, while $R_2(\cdot)$ is convex. Suppose additionally that $\mathcal{P} = [1, 4]$.

We first calculate the optimal value of the DRPO problem. Observe that in the interval $[1, 4]$, the only root of the equation $10 - 2p = 10p^{-2}$ is $p \approx p' = 1.137805\dots$. For $p < p'$, $d_2(p) > d_1(p)$, and for $p > p'$, $d_1(p) > d_2(p)$. Therefore, the optimal value of the of the DRPO problem can be calculated as

$$\begin{aligned} & \max_{p \in [1, 4]} \min_{R \in \mathcal{R}} R(p) \\ &= \max\left\{ \max_{p \in [1, p']} \min_{R \in \mathcal{R}} R(p), \max_{p \in [p', 4]} \min_{R \in \mathcal{R}} R(p) \right\} \\ &= \max\left\{ \max_{p \in [1, p']} p \cdot (10 - 2p), \max_{p \in [p', 4]} p \cdot 10p^{-2} \right\} \\ &= 10p' - 2p'^2 \\ &= 8.78885 \end{aligned}$$

In the above, the first step follows because the best value of the worst-case revenue over $[1, 4]$ is equivalent to taking the higher of the best worst-case revenue over either $[1, p']$ or $[p', 4]$. The second step follows because for every $p \in [1, p']$, $d_1(p) < d_2(p)$, and so $R_1(p) = p \cdot d_1(p) < p \cdot d_2(p) = R_2(p)$; similarly, for every $p \in [p', 4]$, $d_1(p) > d_2(p)$, and so $R_2(p) < R_1(p)$. The third step follows by carrying out the maximization of each of the two functions from the prior step over its corresponding interval.

Now, let us lower bound the optimal value of the RRPO problem. Consider a distribution F that randomizes over prices in the following way:

$$p = \begin{cases} 1 & \text{with probability } 17/21, \\ 2.5 & \text{with probability } 4/21. \end{cases} \quad (\text{EC.23})$$

The worst-case revenue for this distribution is

$$\begin{aligned} & \min_{R \in \mathcal{R}} \int_1^4 R(p) dF(p) \\ &= \min \left\{ \frac{17}{21} \cdot R_1(1) + \frac{4}{21} \cdot R_1(2.5), \frac{17}{21} \cdot R_2(1) + \frac{4}{21} \cdot R_2(2.5) \right\} \\ &= \min \left\{ \frac{17}{21} \cdot 8 + \frac{4}{21} \cdot 12.5, \frac{17}{21} \cdot 10 + \frac{4}{21} \cdot 4 \right\} \\ &= \min \left\{ \frac{62}{7}, \frac{62}{7} \right\} \\ &= \frac{62}{7} \\ &= 8.857143. \end{aligned}$$

This implies that $Z_{\text{RR}}^* \geq 8.857143$, whereas $Z_{\text{DR}}^* = 8.78885$, and thus $Z_{\text{RR}}^* > Z_{\text{DR}}^*$. \square

The condition that \mathcal{P} be a convex set also cannot be relaxed in general, which we illustrate using the next example.

EXAMPLE EC.2. Consider again a single-product RPO problem. Suppose that $\mathcal{R} = \{R_1, R_2\}$, where $R_1(p) = p(10 - p)$, $R_2(p) = p(4 - 0.2p)$; R_1 and R_2 correspond to linear demand functions $d_1(p) = 10 - p$, $d_2(p) = 4 - 0.2p$. Suppose that $\mathcal{P} = \{p_1, p_2\}$, where $p_1 = 5$, $p_2 = 10$. From this data, observe that:

$$\begin{aligned} R_1(p_1) &= 5(10 - 5) = 25, \\ R_1(p_2) &= 10(10 - 10) = 0, \\ R_2(p_1) &= 5(4 - 0.2(5)) = 15, \\ R_2(p_2) &= 10(4 - 0.2(10)) = 20. \end{aligned}$$

We first calculate the optimal value of the DRPO problem:

$$\begin{aligned} Z_{\text{DR}}^* &= \max_{p \in \{p_1, p_2\}} \min\{R_1(p), R_2(p)\} \\ &= \max\{\min\{25, 15\}, \min\{0, 20\}\} \\ &= \max\{15, 0\} \\ &= 15. \end{aligned}$$

For the RRPO problem, the optimal value is given by the following LP:

$$\underset{\eta, \pi}{\text{maximize}} \quad \eta \tag{EC.24a}$$

$$\text{subject to} \quad \eta \leq \pi_{p_1} \cdot p_1 \cdot (10 - p_1) + \pi_{p_2} \cdot p_2 \cdot (10 - p_2) \tag{EC.24b}$$

$$\eta \leq \pi_{p_1} \cdot p_1 \cdot (4 - 0.2p_1) + \pi_{p_2} \cdot p_2 \cdot (4 - 0.2p_2) \tag{EC.24c}$$

$$\pi_{p_1} + \pi_{p_2} = 1 \tag{EC.24d}$$

$$\pi_{p_1}, \pi_{p_2} \geq 0. \tag{EC.24e}$$

The optimal distribution over $\mathcal{P} = \{p_1, p_2\}$ is given by $\pi_{p_1} = 2/3$, $\pi_{p_2} = 1/3$, which leads to $Z_{\text{RR}}^* = 50/3 = 16.6667$. Since this is higher than Z_{DR}^* , we conclude that this particular instance is randomization-receptive. \square

Lastly, as noted in our discussion after Theorem 1, this result is not being driven by strong duality. The next example illustrates that a RPO problem can be randomization-proof without strong duality holding. (Note that this example is the same as our previous example, Example EC.2, with the price set $\{5, 10\}$ replaced by its convex hull, $[5, 10]$.)

EXAMPLE EC.3. Consider a single-product RPO problem, where:

$$\mathcal{P} = [5, 10], \tag{EC.25}$$

$$R_1(p) = p(10 - p), \tag{EC.26}$$

$$R_2(p) = p(4 - 0.2p). \tag{EC.27}$$

Observe that \mathcal{P} is convex and both R_1 and R_2 are concave quadratic functions, and so by Theorem 1, the problem is randomization-proof. However, observe that

$$\begin{aligned} & \max_{p \in [5, 10]} \min\{R_1(p), R_2(p)\} \\ & = 18.75, \end{aligned}$$

whereas

$$\begin{aligned} & \min\left\{ \max_{p \in [5, 10]} R_1(p), \max_{p \in [5, 10]} R_2(p) \right\} \\ & = \min\{25, 20\} \\ & = 20, \end{aligned}$$

which exactly implies that $\max_{\mathbf{p} \in \mathcal{P}} \min_{R \in \mathcal{R}} R(\mathbf{p}) \neq \min_{R \in \mathcal{R}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p})$. \square

EC.2.2. Applications of Theorem 1

Theorem 1 allows us to assert that a number of RPO problems are randomization-proof under certain conditions, which we illustrate through the following four examples.

EXAMPLE EC.4. (Single-product pricing under linear demand). Suppose that $I = 1$, which corresponds to a single-product pricing problem. Let $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ denote the vector of linear demand model parameters, and let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β) . Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the set of revenue functions that arise from the uncertainty set \mathcal{U} . Note that each revenue function is of the form $R(p) = \alpha p - \beta p^2$. Therefore, the condition that each $R \in \mathcal{R}$ is concave implies that $R''(p) = -2\beta \leq 0$. Thus, if \mathcal{U} is such that $\inf\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} \geq 0$, then the robust price optimization problem is randomization-proof. \square

EXAMPLE EC.5. (Multi-product pricing under linear demand). In the more general multi-product pricing problem, let $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathbb{R}^I \times \mathbb{R}^I \times \mathbb{R}^{I(I-1)}$ denote the vector of linear demand model parameters, and let \mathcal{U} be an arbitrary uncertainty set of these model parameter vectors. Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the set of revenue functions that arise from the uncertainty set \mathcal{U} . Observe that each revenue function $R(\cdot, \mathbf{u})$ is of the form

$$\begin{aligned} R(\mathbf{p}) &= \sum_{i=1}^I p_i (\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) \\ &= \boldsymbol{\alpha}^\top \mathbf{p} - \mathbf{p}^\top \mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \mathbf{p}, \end{aligned}$$

where $\mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}}$ is the matrix

$$\mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}} = \begin{bmatrix} \beta_1 & -\gamma_{1,2} & -\gamma_{1,3} & \cdots & -\gamma_{1,I-1} & -\gamma_{1,I} \\ -\gamma_{2,1} & \beta_2 & -\gamma_{2,3} & \cdots & -\gamma_{2,I-1} & -\gamma_{2,I} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ -\gamma_{I,1} & -\gamma_{I,2} & -\gamma_{I,3} & \cdots & -\gamma_{I,I-1} & \beta_I \end{bmatrix}.$$

This implies that

$$\nabla^2 R(\mathbf{p}) = -(\mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}} + \mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}}^\top).$$

The function R is therefore concave if the matrix $\mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}} + \mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}}^\top$ is positive semidefinite. Therefore, if \mathcal{U} is such that $\inf\{\lambda_{\min}(\mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}} + \mathbf{M}_{\boldsymbol{\beta}, \boldsymbol{\gamma}}^\top) \mid (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \in \mathcal{U}\} \geq 0$, where $\lambda_{\min}(\mathbf{A})$ denotes the minimum eigenvalue of a symmetric matrix \mathbf{A} , then the problem is randomization-proof. \square

EXAMPLE EC.6. (Single-product pricing under semi-log demand). For the single-product pricing problem under semi-log demand, $d(p) = \exp(\alpha - \beta p)$ is the demand function given the parameter

vector $\mathbf{u} = (\alpha, \beta)$. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β) , and assume that β is bounded away from zero, that is, $\inf\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} \geq 0$. $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the revenue function uncertainty set. For a given $R \in \mathcal{R}$, its second derivative is $R''(p) = \beta(\beta p - 2)e^{\alpha - \beta p}$. Thus, for $R''(p)$ to be nonpositive, we need $\beta p - 2 \leq 0$ or equivalently $\beta p \leq 2$ (since β is assumed to be nonnegative) for all $p \in \mathcal{P}$ in order for $R(p)$ to be concave. Thus, if $\sup_{p \in \mathcal{P}} \sup_{(\alpha, \beta) \in \mathcal{U}} \{\beta p\} \leq 2$ and $\inf\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} \geq 0$, then the RPO problem is randomization-proof. \square

EXAMPLE EC.7. (Single-product pricing under log-log demand). For the single-product pricing problem under log-log demand, $d(p) = \exp(\alpha - \beta \log p) = e^\alpha \cdot p^{-\beta}$ is the demand function and $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ is the vector of uncertain demand model parameters. Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an uncertainty set of possible values of (α, β) , and assume that β is bounded away from zero from below, that is, $\inf\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} \geq 0$. Let $\mathcal{R} = \{R(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$ be the revenue function uncertainty set. For a given $R \in \mathcal{R}$, its second derivative is $R''(p) = e^\alpha \cdot (\beta - 1)(\beta) \cdot p^{-\beta-1}$. Thus, for $R''(p)$ to be nonpositive, we need $\beta - 1 \leq 0$, or equivalently $\beta \leq 1$. Thus, if $\sup_{(\alpha, \beta) \in \mathcal{U}} \beta \leq 1$ and $\inf_{(\alpha, \beta) \in \mathcal{U}} \beta \geq 0$, then the RPO problem is randomization-proof. \square

EC.2.3. Applications of Theorem 2

Theorem 2 allows us to establish that a single product RPO problem under semi-log and log-log demand will be randomization-proof, even when the revenue function is not necessarily concave (and one cannot rely on Theorem 1).

EXAMPLE EC.8. Consider a single-product price optimization problem where the demand follows a semi-log model. The uncertain parameter is therefore $\mathbf{u} = (\alpha, \beta)$.

Observe that $R(p, \mathbf{u}) = pe^{\alpha - \beta p}$ is convex in \mathbf{u} . Thus, it is also quasiconvex in \mathbf{u} for a fixed p . Note also that the function R is quasi-concave in p . To see this, observe that $\log R(p, \mathbf{u}) = \log p + \alpha - \beta p$, which is concave in p ; this means that R is log-concave in p . Since any log-concave function is quasiconcave, it follows that R is quasiconcave in p .

Thus, if $\mathcal{P} \subseteq \mathbb{R}$ and $\mathcal{U} \subseteq \mathbb{R}^2$ are compact and convex, then Theorem 2 asserts that the RPO problem is randomization-proof. \square

EXAMPLE EC.9. Consider a single-product price optimization problem where the demand follows a log-log model. The uncertain parameter is $\mathbf{u} = (\alpha, \beta)$, and $R(p, \mathbf{u}) = pe^{\alpha - \beta \log p}$. Assume that $\mathcal{P} \subseteq \mathbb{R}$ is a compact convex set, and that $\min\{p \mid p \in \mathcal{P}\} > 0$.

Observe that $R(p, \mathbf{u}) = p \cdot e^{\alpha - \beta \log p}$ is convex in \mathbf{u} , and therefore quasiconvex in \mathbf{u} for a fixed p . Additionally, with regard to quasiconcavity in p , observe that $\log R(p, \mathbf{u}) = \log p + \alpha - \beta \log p =$

$(1 - \beta) \log p + \alpha$, which means that R is log-concave in p whenever $1 - \beta > 0$ or equivalently $\beta < 1$. Therefore, R will also be quasiconcave whenever $\beta < 1$.

Thus, if $\mathcal{P} \subseteq \mathbb{R}$ and $\mathcal{U} \subseteq \mathbb{R}^2$ are compact and convex, and $\max\{\beta \mid (\alpha, \beta) \in \mathcal{U}\} < 1$, then Theorem 2 guarantees that the RPO problem is randomization-proof. \square

EC.2.4. Example of necessity of uniqueness assumption in part (b) of Corollary 2

The following example illustrates that the implication in part (b) of Corollary 2 may fail to hold if \mathbf{u}^* is not the unique solution of $\min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u})$. Additionally, this example also illustrates that even when \mathcal{P} is finite, the RPO problem can be randomization-proof in the absence of strong duality.

EXAMPLE EC.10. Consider a single product pricing instance, i.e., $I = 1$, which we define as follows. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ where $p_1 = 5$, $p_2 = 8$, $p_3 = 9$. Let the demand model d be a linear demand model, so that the uncertain parameter $\mathbf{u} = (\alpha, \beta)$ and $d(p, \mathbf{u}) = \alpha - \beta p$. Finally, let $\mathcal{U} = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}$, where $(\alpha_1, \beta_1) = (10, 1)$, $(\alpha_2, \beta_2) = (3, 0.1)$, $(\alpha_3, \beta_3) = (3.6, 0.2)$.

We first calculate $\min_{\mathbf{u} \in \mathcal{U}} R(p, \mathbf{u})$ for each $p \in \mathcal{P}$. We have:

- For $p_1 = 5$: $p_1(\alpha_2 - \beta_2 p_1) = 12.5 < p_1(\alpha_3 - \beta_3 p_1) = 13 < p_1(\alpha_1 - \beta_1 p_1) = 25$. Hence, $\min_{\mathbf{u} \in \mathcal{U}} R(p_1, \mathbf{u}) = \min\{12.5, 13, 25\} = 12.5$.
- For $p_2 = 8$: $p_2(\alpha_1 - \beta_1 p_2) = p_2(\alpha_3 - \beta_3 p_2) = 16 < p_2(\alpha_2 - \beta_2 p_2) = 17.6$. Hence, $\min_{\mathbf{u} \in \mathcal{U}} R(p_2, \mathbf{u}) = \min\{16, 16, 17.6\} = 16$, and note also that the minimizing \mathbf{u} is not unique (the minimum is attained at both (α_1, β_1) and (α_3, β_3)).
- For $p_3 = 9$: $p_3(\alpha_1 - \beta_1 p_3) = 9 < p_3(\alpha_3 - \beta_3 p_3) = 16.2 < p_3(\alpha_2 - \beta_2 p_3) = 18.9$. Hence, $\min_{\mathbf{u} \in \mathcal{U}} R(p_3, \mathbf{u}) = \min\{9, 16.2, 18.9\} = 9$.

From this, we can see that the optimal deterministic robust price is $p_{\text{DR}}^* = p_2 = 8$ and the optimal deterministic robust objective value is $Z_{\text{DR}}^* = 16$. At $p = 8$, we can see that $\arg \min_{\mathbf{u} \in \mathcal{U}} R(p_2, \mathbf{u}) = \{(\alpha_1, \beta_1), (\alpha_3, \beta_3)\}$.

Let us now consider the RRPO problem. When we write the problem $\max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{p \in \mathcal{P}} \pi_p R(\mathbf{p}, \mathbf{u})$ as a linear program, we get the following problem:

$$\begin{aligned} \underset{\pi, t}{\text{maximize}} \quad & t & (\text{EC.28a}) \end{aligned}$$

$$\text{subject to} \quad t \leq \pi_{p_1} \cdot p_1(\alpha_1 - \beta_1 p_1) + \pi_{p_2} \cdot p_2(\alpha_1 - \beta_1 p_2) + \pi_{p_3} \cdot p_3(\alpha_1 - \beta_1 p_3), \quad (\text{EC.28b})$$

$$t \leq \pi_{p_1} \cdot p_1(\alpha_2 - \beta_2 p_1) + \pi_{p_2} \cdot p_2(\alpha_2 - \beta_2 p_2) + \pi_{p_3} \cdot p_3(\alpha_2 - \beta_2 p_3), \quad (\text{EC.28c})$$

$$t \leq \pi_{p_1} \cdot p_1(\alpha_3 - \beta_3 p_1) + \pi_{p_2} \cdot p_2(\alpha_3 - \beta_3 p_2) + \pi_{p_3} \cdot p_3(\alpha_3 - \beta_3 p_3), \quad (\text{EC.28d})$$

$$\pi_{p_1} + \pi_{p_2} + \pi_{p_3} = 1, \quad (\text{EC.28e})$$

$$\pi_{p_1}, \pi_{p_2}, \pi_{p_3} \geq 0, \quad (\text{EC.28f})$$

or equivalently,

$$\begin{aligned} \underset{\pi, t}{\text{maximize}} \quad & t & (\text{EC.29a}) \end{aligned}$$

$$\text{subject to} \quad t \leq 25\pi_{p_1} + 16\pi_{p_2} + 9\pi_{p_3} \quad (\text{EC.29b})$$

$$t \leq 12.5\pi_{p_1} + 17.6\pi_{p_2} + 18.9\pi_{p_3}, \quad (\text{EC.29c})$$

$$t \leq 13\pi_{p_1} + 16\pi_{p_2} + 16.2\pi_{p_3}, \quad (\text{EC.29d})$$

$$\pi_{p_1} + \pi_{p_2} + \pi_{p_3} = 1, \quad (\text{EC.29e})$$

$$\pi_{p_1}, \pi_{p_2}, \pi_{p_3} \geq 0, \quad (\text{EC.29f})$$

for which the optimal objective value is $Z_{\text{RR}}^* = 16$, which is the same as Z_{DR}^* . Thus, we can see that if the problem is randomization-proof, but the uniqueness condition on $\min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}_{\text{DR}}^*, \mathbf{u})$ is not met, then \mathbf{p}_{DR}^* is not necessarily a maximizer of the nominal problem at any \mathbf{u}^* . (For $\mathbf{u}^* = (\alpha_1, \beta_1)$, the p that maximizes $R(\cdot, \mathbf{u}^*)$ is p_1 , but for $\mathbf{u}^* = (\alpha_3, \beta_3)$, the p that maximizes $R(\cdot, \mathbf{u}^*)$ is p_3 .)

Additionally, observe that in this instance, strong duality does not hold. If we compute $\min_{\mathbf{u} \in \mathcal{U}} \max_{p \in \mathcal{P}} R(p, \mathbf{u})$, we get:

$$\begin{aligned} & \min_{\mathbf{u} \in \mathcal{U}} \max_{p \in \mathcal{P}} R(p, \mathbf{u}) \\ &= \min\{\max_{p \in \mathcal{P}} R(p, \mathbf{u}_1), \max_{p \in \mathcal{P}} R(p, \mathbf{u}_2), \max_{p \in \mathcal{P}} R(p, \mathbf{u}_3)\} \\ &= \min\{\max\{25, 16, 9\}, \max\{12.5, 17.6, 18.9\}, \max\{13, 16, 16.2\}\} \\ &= \min\{25, 18.9, 16.2\} \\ &= 16.2, \end{aligned}$$

which is strictly higher than $Z_{\text{RR}}^* = 16$. \square

EC.2.5. Example to illustrate that Z_{DR}^* , Z_{RR}^* and dual objective can be distinct when \mathcal{P} is finite

In this section, we revisit Example EC.2 from Section EC.2.1, which can be viewed as an instance of a finite \mathcal{P} robust price optimization problem.

EXAMPLE EC.11. Consider a single-product RPO problem, and suppose that $\mathcal{P} = \{p_1, p_2\}$ where $p_1 = 5$, $p_2 = 10$. Consider $\mathbf{u} = (u_1)$, where $\mathcal{U} = \{0, 1\}$, and let $R(\mathbf{p}, \mathbf{u})$ be defined as

$$R(p, \mathbf{u}) = u_1 \cdot p(10 - p) + (1 - u_1) \cdot p(4 - 0.2p). \quad (\text{EC.30})$$

Note that this is the same instance as Example EC.2, with the only difference being that we have re-defined the uncertain parameter \mathbf{u} to be a binary variable that toggles between the two revenue

functions $p(10 - p)$ and $p(4 - 0.2p)$. For this instance, $Z_{\text{DR}}^* = 15$ and $Z_{\text{RR}}^* = 50/3 = 16.6667$, as before, and the dual objective $\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$ is

$$\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) = \min\{\max\{25, 0\}, \max\{15, 20\}\} \quad (\text{EC.31})$$

$$= \min\{25, 20\} \quad (\text{EC.32})$$

$$= 20, \quad (\text{EC.33})$$

which is strictly larger than Z_{RR}^* . \square

Note that in the previous example, the difference between $\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$ and Z_{RR}^* is driven by \mathcal{U} being a discrete set. If we replaced \mathcal{U} with its convex hull, then Z_{RR}^* and the dual objective would be identical, which is consistent with Proposition EC.1.6 (when \mathcal{U} is convex and $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} , then Z_{RR}^* and the dual objective are equal, and strong duality is equivalent to randomization-proofness).

EXAMPLE EC.12. Consider the same single-product RPO problem as in the preceding example, but let $\mathcal{U} = [0, 1]$. In this case, both Z_{RR}^* and Z_{DR}^* are unchanged because the objective of the inner minimization over $\mathbf{u} \in \mathcal{U}$ is unchanged when we replace $\mathcal{U} = \{0, 1\}$ with $\mathcal{U} = [0, 1]$. For the dual objective, we get

$$\min_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) = \min_{0 \leq u_1 \leq 1} \max\{u_1 \cdot 25 + (1 - u_1)15, (1 - u_1) \cdot 20\} \quad (\text{EC.34})$$

$$= \min_{0 \leq u_1 \leq 1} \{10u_1 + 15, 20 - 20u_1\} \quad (\text{EC.35})$$

$$= 50/3, \quad (\text{EC.36})$$

which is the same as Z_{RR}^* , as predicted by Proposition EC.1.6. \square

EC.3. Mixed-integer exponential cone formulations for Section 5

In this section, we describe how the two mixed-integer convex programs in Section 5, for the semi-log demand model (problem (29) in Section 5.3) and the log-log demand model (problem (37) in Section 5.4) can be reformulated as mixed-integer exponential cone programs.

We first recall that the standard exponential cone is defined as

$$\mathcal{K}_{\text{exp}} = \{(r, 0, t) \in \mathbb{R}^3 \mid r \geq 0, t \leq 0\} \cup \{(r, s, t) \in \mathbb{R}^3 \mid s > 0, r \geq s \exp(t/s)\}. \quad (\text{EC.37})$$

We also recall that the function $f(\mu) = \mu \log \mu$ is convex, for $\mu > 0$. To model this function using the exponential cone, let us introduce the epigraph variable q . We can then model the epigraph of this function as follows:

$$q \geq \mu \log \mu \quad (\text{EC.38})$$

$$\Leftrightarrow \frac{q}{\mu} \geq \log \mu \quad (\text{EC.39})$$

$$\Leftrightarrow e^{q/\mu} \geq \mu \quad (\text{EC.40})$$

$$\Leftrightarrow 1 \geq \mu e^{-q/\mu} \quad (\text{EC.41})$$

$$\Leftrightarrow (1, \mu, -q) \in \mathcal{K}_{\text{exp}}. \quad (\text{EC.42})$$

Therefore, the semi-log demand formulation (29) can be written using the auxiliary epigraph variables q_1, \dots, q_I as

$$\begin{aligned} \underset{\mu, \mathbf{q}, \mathbf{w}, \mathbf{x}}{\text{maximize}} \quad & \sum_{i=1}^I \mu_i \alpha_i + \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} - \sum_{i=1}^I \beta_i \cdot \sum_{t \in \mathcal{P}_i} t \cdot w_{i,i,t} + \sum_{i=1}^I \sum_{j \neq i} \gamma_{i,j} \cdot \left(\sum_{t \in \mathcal{P}_j} t \cdot w_{i,j,t} \right) - \sum_{i=1}^I q_i \\ & (\text{EC.43a}) \end{aligned}$$

$$\text{subject to} \quad (1, \mu_i, -q_i) \in \mathcal{K}_{\text{exp}}, \quad \forall i \in [I], \quad (\text{EC.43b})$$

$$\text{constraints (29b) – (29h)}. \quad (\text{EC.43c})$$

In a similar fashion, the log-log demand formulation (37) can be written as

$$\begin{aligned} \underset{\mu, \mathbf{q}, \mathbf{w}, \mathbf{x}}{\text{maximize}} \quad & \sum_{i=1}^I \mu_i \alpha_i + \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} - \sum_{i=1}^I \beta_i \cdot \sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} + \sum_{i=1}^I \sum_{j \neq i} \gamma_{i,j} \sum_{t \in \mathcal{P}_j} \log t \cdot w_{i,j,t} - \sum_{i=1}^I q_i \\ & (\text{EC.44a}) \end{aligned}$$

$$\text{subject to} \quad (1, \mu_i, -q_i) \in \mathcal{K}_{\text{exp}}, \quad \forall i \in [I], \quad (\text{EC.44b})$$

$$\text{constraints (29b) – (29h)}. \quad (\text{EC.44c})$$

EC.4. Deterministic robust price optimization for finite \mathcal{P} , convex \mathcal{U} under the semi-log and log-log demand models

In this section, we describe how to formulate the DRPO problem as a mixed-integer exponential cone program for the semi-log and log-log demand models. In both cases, we assume that \mathcal{U} is a convex uncertainty set, and that Assumption 1 on the structure of \mathcal{P} holds.

EC.4.1. Semi-log model

For the semi-log demand model, we can write the DRPO problem as

$$\begin{aligned} & \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \\ & = \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}. \end{aligned} \quad (\text{EC.45})$$

To accomplish our reformulation, we will make use of the fact that the optimal solution set of the DRPO problem is unchanged upon log-transformation, that is,

$$\arg \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) = \arg \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log R(\mathbf{p}, \mathbf{u}).$$

Thus, instead of problem (EC.45), we can focus on the following problem:

$$\begin{aligned} & \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log \left(\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log \left(\sum_{i=1}^I e^{\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \end{aligned}$$

Here, we can again use the log-sum-exp biconjugate technique to reformulate the objective function in the following way:

$$\begin{aligned} & \log \left(\sum_{i=1}^I e^{\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right) \\ &= \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left\{ \sum_{i=1}^I \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\}. \end{aligned}$$

Thus, the overall problem becomes the following max-min-max problem:

$$\max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left\{ \sum_{i=1}^I \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\}.$$

Here, we observe that the objective function is linear in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, and is concave in $\boldsymbol{\mu}$; additionally, the feasible region of \mathbf{u} is assumed to be convex, and the feasible region of $\boldsymbol{\mu}$ is convex and compact (being just the $(|I| - 1)$ -dimensional unit simplex). Therefore, we can use Sion's minimax theorem to interchange the minimization over \mathbf{u} and the maximization over $\boldsymbol{\mu}$, which gives us

$$\begin{aligned} & \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left\{ \sum_{i=1}^I \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\ &= \max_{\mathbf{p} \in \mathcal{P}} \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \min_{\mathbf{u} \in \mathcal{U}} \left\{ \sum_{i=1}^I \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\ &= \max_{\mathbf{p} \in \mathcal{P}, \boldsymbol{\mu} \in \Delta_{[I]}} \min_{\mathbf{u} \in \mathcal{U}} \left\{ \sum_{i=1}^I \mu_i \cdot (\log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \end{aligned}$$

Under Assumption 1, this final problem can then be reformulated as robust mixed-integer exponential cone program, just as in Section 5.3. We introduce the same binary decision variable $x_{i,t}$ which is 1 if product i is offered at price $t \in \mathcal{P}_i$, and 0 otherwise, and we use $w_{i,j,t}$ to denote the linearization of $\mu_i \cdot x_{j,t}$ for $i, j \in [I]$, $t \in \mathcal{P}_j$. This gives rise to the following program:

$$\begin{aligned} \text{maximize}_{\boldsymbol{\mu}, \mathbf{w}, \mathbf{x}} \quad & \min_{\mathbf{u} \in \mathcal{U}} \left\{ \sum_{i=1}^I \mu_i \alpha_i + \sum_{i=1}^I \left(\sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} - \beta_i \cdot \sum_{t \in \mathcal{P}_i} t \cdot w_{i,i,t} + \sum_{j \neq i} \gamma_{i,j} \sum_{t \in \mathcal{P}_j} t \cdot w_{i,j,t} \right) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\ & \text{(EC.46a)} \end{aligned}$$

$$\text{subject to} \quad \sum_{t \in \mathcal{P}_j} w_{i,j,t} = \mu_i, \quad \forall i \in [I], j \in [I], \quad \text{(EC.46b)}$$

$$\sum_{i=1}^I w_{i,j,t} = x_{j,t}, \quad \forall j \in [I], t \in \mathcal{P}_j, \quad (\text{EC.46c})$$

$$\sum_{i=1}^I \mu_i = 1, \quad (\text{EC.46d})$$

$$\sum_{t \in \mathcal{P}_i} x_{i,t} = 1, \quad \forall i \in [I], \quad (\text{EC.46e})$$

$$w_{i,j,t} \geq 0, \quad \forall i \in [I], j \in [I], t \in \mathcal{P}_j, \quad (\text{EC.46f})$$

$$x_{i,t} \in \{0, 1\}, \quad \forall i \in [I], t \in \mathcal{P}_i, \quad (\text{EC.46g})$$

$$\mu_i \geq 0, \quad \forall i \in [I]. \quad (\text{EC.46h})$$

Note that the feasible region of this problem is identical to that of problem (29), which appeared in our discussion of the separation problem for the RRPO problem when \mathcal{U} is convex and \mathcal{P} is finite. The difference here is that the objective is now a robust objective; it is the worst-case value of the objective of problem (29), taken over the convex uncertainty set \mathcal{U} . Depending on the structure of \mathcal{U} , the overall problem can remain in the mixed-integer convex program problem class. For example, if \mathcal{U} is a polyhedron, then one can use LP duality to reformulate the robust problem exactly by introducing additional variables and constraints, as is normally done in robust optimization (Bertsimas and Sim 2004, Ben-Tal and Nemirovski 2000, Bertsimas et al. 2011). Similarly, if \mathcal{U} is a second-order cone representable set, then one can again use conic duality to reformulate the problem. Alternatively, one can also consider solving the problem using a cutting plane method/delayed constraint generation approach, whereby one reformulates the program in epigraph form and then solves the inner minimization over \mathbf{u} to identify new cuts to add (Bertsimas et al. 2016).

EC.4.2. Log-log model

For the log-log demand model, we can write the DRPO problem as

$$\begin{aligned} & \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} R(\mathbf{p}, \mathbf{u}) \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \\ &= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I e^{\alpha_i + (1 - \beta_i) \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \end{aligned}$$

Again, as with the semi-log model, solving the above problem is equivalent to solving the same problem with a log-transformed objective. Taking this log-transformed problem as our starting

point, replacing the log-sum-exp function with its biconjugate and applying Sion's minimax theorem gives us:

$$\begin{aligned}
& \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \log \left(\sum_{i=1}^I e^{\alpha_i + (1-\beta_i) \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \right) \\
&= \max_{\mathbf{p} \in \mathcal{P}} \min_{\mathbf{u} \in \mathcal{U}} \max_{\boldsymbol{\mu} \in \Delta_{[I]}} \left\{ \sum_{i=1}^I \left[\alpha_i \mu_i + (1-\beta_i) \mu_i \cdot \log p_i + \sum_{j \neq i} \gamma_{i,j} \mu_i \cdot \log p_j \right] - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\
&= \max_{\mathbf{p} \in \mathcal{P}, \boldsymbol{\mu} \in \Delta_{[I]}} \min_{\mathbf{u} \in \mathcal{U}} \left\{ \sum_{i=1}^I \left[\alpha_i \mu_i + (1-\beta_i) \mu_i \cdot \log p_i + \sum_{j \neq i} \gamma_{i,j} \mu_i \cdot \log p_j \right] - \sum_{i=1}^I \mu_i \log \mu_i \right\}.
\end{aligned}$$

Under Assumption 1, this last problem can be re-written as the following robust version of problem (37), with the decision variables defined identically:

$$\begin{aligned}
\text{maximize}_{\boldsymbol{\mu}, \mathbf{w}, \mathbf{x}} \quad & \min_{\mathbf{u} \in \mathcal{U}} \left\{ \sum_{i=1}^I \mu_i \alpha_i + \sum_{i=1}^I \left(\sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} - \beta_i \cdot \sum_{t \in \mathcal{P}_i} \log t \cdot w_{i,i,t} + \sum_{j \neq i} \gamma_{i,j} \sum_{t \in \mathcal{P}_j} \log t \cdot w_{i,j,t} \right) - \sum_{i=1}^I \mu_i \log \mu_i \right\} \\
& \text{(EC.47a)}
\end{aligned}$$

subject to constraints (29b) – (29h).

(EC.47b)

Again, this problem has exactly the same feasible region as the log-log separation problem (37) and the semi-log separation problem (29). Additionally, just as with the deterministic robust problem (EC.46) for the semi-log model, this problem can be further reformulated by exploiting the structure of \mathcal{U} , or otherwise one can design a cutting plane method that generates violated uncertain parameter vectors $\mathbf{u} \in \mathcal{U}$ on the fly.

EC.5. Incorporating constraints on price distributions

In this section, we present an extension of our framework to consider the possibility of constrained distributions over price vectors in the finite \mathcal{P} , convex \mathcal{U} case. In Section EC.5.1, we show how a particular type of constrained RRPO problem can be reformulated as a large-scale convex program. In Section EC.5.2, we present results of a small numerical experiment to show that incorporating constraints on the price distribution produced by the RRPO approach does not result in a large deterioration in performance compared to the DRPO approach.

EC.5.1. Randomized pricing with constraints

Suppose that we consider the following more general instance of the RRPO problem:

$$\text{RRPO-C: } \max_{\boldsymbol{\pi} \in \tilde{\Delta}_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \tag{EC.48}$$

where $\tilde{\Delta}_{\mathcal{P}}$ is the following subset of the probability simplex $\Delta_{\mathcal{P}}$:

$$\tilde{\Delta}_{\mathcal{P}} = \left\{ \boldsymbol{\pi} \in \Delta_{\mathcal{P}} \mid \sum_{\mathbf{p} \in \mathcal{P}} g_k(\mathbf{p}) \pi_{\mathbf{p}} = c_k, \forall k \in [K_1], \sum_{\mathbf{p} \in \mathcal{P}} h_k(\mathbf{p}) \pi_{\mathbf{p}} \leq d_k, \forall k \in [K_2], \right\}, \tag{EC.49}$$

where K_1 and K_2 are nonnegative integers. In words, $\tilde{\Delta}_{\mathcal{P}}$ is the set of all probability distributions that satisfy a set of *moment conditions* with respect to a set of moment functions $\{g_k(\cdot)\}_{k=1}^{K_1}$, $\{h_k(\cdot)\}_{k=1}^{K_2}$; in particular, the expected value of $g_k(\tilde{\mathbf{p}})$, where $\tilde{\mathbf{p}}$ is the random price vector that follows the distribution π , must be equal to c_k , for K_1 different moment functions. Similarly, the expected value of $h_k(\tilde{\mathbf{p}})$ must be less than or equal to d_k for K_2 different moment functions.

Let us now see how to reformulate this as a tractable problem. Let us assume that \mathcal{U} is convex and that $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} . In this case, Sion's minimax theorem again lets us assert

$$\max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.50})$$

$$= \min_{\mathbf{u} \in \mathcal{U}} \max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.51})$$

where we are also using the fact that $\tilde{\Delta}_{\mathcal{P}}$ is convex, as it is a polyhedron, and is compact, as it is a closed subset of the compact set $\Delta_{\mathcal{P}}$.

Now, let us introduce a K_1 -dimensional Lagrange multiplier vector $\boldsymbol{\lambda}$ and a K_2 -dimensional nonpositive Lagrange multiplier vector $\boldsymbol{\rho}$. Observe that the inner maximization over π can be re-written equivalently as

$$\max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.52})$$

$$= \max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \min_{\boldsymbol{\lambda} \in \mathbb{R}^{K_1}, \boldsymbol{\rho} \leq \mathbf{0}} \left\{ \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k \cdot \left(\sum_{\mathbf{p} \in \mathcal{P}} g_k(\mathbf{p}) \pi_{\mathbf{p}} - c_k \right) + \sum_{k=1}^{K_2} \rho_k \cdot \left(\sum_{\mathbf{p} \in \mathcal{P}} h_k(\mathbf{p}) \pi_{\mathbf{p}} - d_k \right) \right\} \quad (\text{EC.53})$$

$$= \max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \min_{\boldsymbol{\lambda} \in \mathbb{R}^{K_1}, \boldsymbol{\rho} \leq \mathbf{0}} \left\{ \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.54})$$

$$= \min_{\boldsymbol{\lambda} \in \mathbb{R}^{K_1}, \boldsymbol{\rho} \leq \mathbf{0}} \max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \left\{ \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.55})$$

$$= \min_{\boldsymbol{\lambda} \in \mathbb{R}^{K_1}, \boldsymbol{\rho} \leq \mathbf{0}} \left\{ \max_{\pi \in \tilde{\Delta}_{\mathcal{P}}} \left\{ \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) \right\} - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.56})$$

$$= \min_{\boldsymbol{\lambda} \in \mathbb{R}^{K_1}, \boldsymbol{\rho} \leq \mathbf{0}} \left\{ \max_{\mathbf{p} \in \mathcal{P}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.57})$$

where the interchange of maximization over $\pi \in \tilde{\Delta}_{\mathcal{P}}$ and minimization over $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ follows by linear programming strong duality, and the final equality follows because the optimal solution of

the inner maximization over $\pi \in \Delta_{\mathcal{P}}$ occurs when all of the probability mass is placed at a single $\mathbf{p} \in \mathcal{P}$. Combining (EC.57) with (EC.51), we obtain

$$\min_{\mathbf{u} \in \mathcal{U}} \max_{\pi \in \Delta_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.58})$$

$$= \min_{\mathbf{u} \in \mathcal{U}} \min_{\lambda \in \mathbb{R}^{K_1}, \rho \leq \mathbf{0}} \left\{ \max_{\mathbf{p} \in \mathcal{P}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.59})$$

$$= \min_{\mathbf{u} \in \mathcal{U}, \lambda \in \mathbb{R}^{K_1}, \rho \leq \mathbf{0}} \left\{ \max_{\mathbf{p} \in \mathcal{P}} \left(R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right) - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \right\} \quad (\text{EC.60})$$

This final problem can be re-written as the following convex optimization problem, where we introduce an epigraph variable t to model the innermost maximum over \mathbf{p} :

$$\text{minimize}_{\lambda, \rho, t, \mathbf{u}} \quad t - \sum_{k=1}^{K_1} \lambda_k c_k - \sum_{k=1}^{K_2} \rho_k d_k \quad (\text{EC.61a})$$

$$\text{subject to} \quad t \geq R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}), \quad \forall \mathbf{p} \in \mathcal{P}, \quad (\text{EC.61b})$$

$$\mathbf{u} \in \mathcal{U}, \quad (\text{EC.61c})$$

$$\lambda \in \mathbb{R}^{K_1}, \quad (\text{EC.61d})$$

$$\rho \leq \mathbf{0}, \quad (\text{EC.61e})$$

$$t \in \mathbb{R}. \quad (\text{EC.61f})$$

Like the unconstrained problem (22), this problem can again potentially be solved via constraint generation. The challenge in solving this problem now becomes how one solves the modified separation problem

$$\max_{\mathbf{p} \in \mathcal{P}} \left\{ R(\mathbf{p}, \mathbf{u}) + \sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p}) + \sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p}) \right\}. \quad (\text{EC.62})$$

This separation problem is more challenging than the separation problem in the unconstrained distribution setting, because of the presence of the $\sum_{k=1}^{K_1} \lambda_k g_k(\mathbf{p})$ and $\sum_{k=1}^{K_2} \rho_k h_k(\mathbf{p})$ terms in the objective function. In particular, because of these terms and the fact that the λ_k and ρ_k variables can be negative, one cannot straightforwardly apply the biconjugate reformulation technique for the case of log-log and semi-log demand model. However, one can solve the problem heuristically, using the same random improvement approach that we use in Section EC.8.1. We adopt this approach in our numerical experiment in Section EC.5.2.

With regard to the moment functions $g_k(\cdot)$ and $h_k(\cdot)$, one natural choice of moment functions correspond to the first and second moments of each product's random price. In particular, letting $\tilde{\mathbf{p}}$ be the random price vector, one may wish to enforce that the distribution π over \mathcal{P} is such that

$$\mathbb{E}_{\pi}[\tilde{p}_i] = \mu_i, \quad \forall i \in [I], \quad (\text{EC.63})$$

$$\mathbb{E}_{\pi}[\tilde{p}_i^2] \leq \mu_i^2 + \sigma_i^2, \quad \forall i \in [I], \quad (\text{EC.64})$$

where μ_i and σ_i are user-specified values for the desired mean and standard deviation of each product's random price. This corresponds to specifying $2I$ moment functions, where $g_i(\mathbf{p}) = p_i$ for $1 \leq i \leq I$, and $h_i(\mathbf{p}) = p_i^2$ for $1 \leq i \leq I$. The resulting instance of the RRPO problem is one where we seek a distribution π that maximizes the worst-case revenue, with the additional restriction that the mean of each product's random price be equal to a predefined value μ_i , and the standard deviation is at most σ_i . By appropriately setting μ_i and σ_i , one can obtain randomized pricing schemes with controlled behavior, e.g., where the average price of a product is fixed to a certain value and to ensure that the variability around that value is not too large. As another example, one could set μ_i and σ_i so that the resulting price distribution π matches historical price data for a collection of products, i.e., on average, π results in the same average prices and the same variability in those prices as in the empirical distribution of \mathbf{p} in a historical data set.

We close this section by noting that moment constraints have a rich history in the distributionally robust optimization (DRO) literature; see, for example, Zymler et al. (2013), Rujeerapaiboon et al. (2016) and the references therein. We remark here that our use of moment constraints differs from how it is typically used in DRO. In DRO, one seeks to solve a stochastic program where the distribution of a parameter is uncertain, and one constructs an ambiguity set of possible distributions that are consistent with known information and data through moment constraints (e.g., the first and second moments of the uncertain distribution should be equal to certain known values); thus, moment constraints serve to limit how conservative the decision maker is. This contrasts with our use of moment constraints, where the moment constraints are on the firm's decision (the distribution of the price vector $\tilde{\mathbf{p}}$), and their purpose is to facilitate the implementation of the randomized pricing strategy.

EC.5.2. Numerical experiment with orangeJuice data set

In this section, we demonstrate the value of constrained price distributions using the `orangeJuice` problem instances studied in Section 7.2. We consider the same set of robust price optimization problems as in Section 7.2, with the modification that \mathcal{P}_i includes $\hat{\mu}_i$, which is the target mean for product i ; the reason for this will be discussed shortly.

For each value of θ , we solve the constrained randomized problem, with the moment constraints (EC.63) and (EC.64). We specifically impose that

$$\mathbb{E}_{\pi}[\tilde{p}_i] = \hat{\mu}_i, \quad \forall i \in [I], \quad (\text{EC.65})$$

$$\mathbb{E}_{\pi}[\tilde{p}_i^2] \leq \hat{\mu}_i^2 + \alpha^2 \hat{\sigma}_i^2, \quad \forall i \in [I], \quad (\text{EC.66})$$

where $\hat{\mu}_i$ and $\hat{\sigma}_i$ are the mean and standard deviation, respectively, of the price of product i , calculated with respect to the optimal unconstrained randomized robust price distribution π' . In

other words, we seek a distribution π that has the same marginal means as π' , but has less variance in each product's price than π' . We test values of $\alpha \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$. Note that when $\alpha = 1$, the constrained RRPO problem is equivalent to the unconstrained RRPO problem. Additionally, we note that for any $\alpha \geq 0$, the set of distributions $\tilde{\Delta}_{\mathcal{P}}$ defined according to the moment constraints is non-empty, because \mathcal{P}_i includes $\hat{\mu}_i$ (as noted earlier).

Tables EC.1 and EC.2 display the results for the log-log and semi-log demand models. From these tables, we can see that in general, when the uncertainty set is small, constraining the distribution using the moment constraints described above results in weaker randomized pricing schemes than the unconstrained approach, and the resulting distributions do not improve on the deterministic robust price solution. For example, with the log-log demand model and $\theta = 0.1$, all of the constrained distributions with $\alpha \leq 0.9$ have a negative RI metric, indicating that their worst-case revenue is worse than the deterministic robust price solution. However, as the uncertainty set gets larger, there are values of $\alpha < 1$ for which the constrained RRPO solution yields a positive RI. For example, with the log-log demand model and $\theta = 0.5$, the constrained RRPO solution with α set to 0.8 or 0.9 outperforms the deterministic robust solution. As θ gets larger, it is possible to find distributions with less variability (i.e., with smaller α) that outperform the deterministic robust solution. For example, with the log-log demand model and $\theta = 1.0$, the constrained RRPO solution with $\alpha = 0.6$ continues to give a positive RI. These results suggest that by considering a constrained form of the RRPO problem, it is possible to trade-off some of the worst-case performance of the unconstrained RRPO solution to obtain price distributions that are more structured (in the examples shown above, ones with less variability than the unconstrained RRPO solution). Of course, the moment constraints that we have considered are only one way in which one can constrain the price distribution. An interesting direction for future research is to consider other types of constraints that could lead to structured price distributions that continue to provide a benefit over the deterministic robust scheme.

θ	Method m	Z_m^*	RI (%)
0.10	RRPO-C, $\alpha = 0.5$	237386.69	-58.05
0.10	RRPO-C, $\alpha = 0.6$	262704.91	-53.57
0.10	RRPO-C, $\alpha = 0.7$	285955.77	-49.47
0.10	RRPO-C, $\alpha = 0.8$	313211.80	-44.65
0.10	RRPO-C, $\alpha = 0.9$	415820.50	-26.52
0.10	RRPO	722647.22	27.71
0.50	RRPO-C, $\alpha = 0.5$	162117.27	-30.54
0.50	RRPO-C, $\alpha = 0.6$	190673.79	-18.30
0.50	RRPO-C, $\alpha = 0.7$	224890.46	-3.64
0.50	RRPO-C, $\alpha = 0.8$	251484.85	7.75
0.50	RRPO-C, $\alpha = 0.9$	286534.25	22.77
0.50	RRPO	342614.34	46.80
0.80	RRPO-C, $\alpha = 0.5$	131098.05	-19.21
0.80	RRPO-C, $\alpha = 0.6$	157189.92	-3.13
0.80	RRPO-C, $\alpha = 0.7$	175564.41	8.19
0.80	RRPO-C, $\alpha = 0.8$	200577.13	23.60
0.80	RRPO-C, $\alpha = 0.9$	228397.76	40.75
0.80	RRPO	260049.66	60.25
1.00	RRPO-C, $\alpha = 0.5$	111909.24	-12.72
1.00	RRPO-C, $\alpha = 0.6$	131534.71	2.58
1.00	RRPO-C, $\alpha = 0.7$	149969.36	16.96
1.00	RRPO-C, $\alpha = 0.8$	172717.05	34.70
1.00	RRPO-C, $\alpha = 0.9$	191272.60	49.17
1.00	RRPO	217580.86	69.69
1.50	RRPO-C, $\alpha = 0.5$	78581.62	3.54
1.50	RRPO-C, $\alpha = 0.6$	92204.17	21.48
1.50	RRPO-C, $\alpha = 0.7$	103533.52	36.41
1.50	RRPO-C, $\alpha = 0.8$	116139.87	53.02
1.50	RRPO-C, $\alpha = 0.9$	127482.81	67.97
1.50	RRPO	142307.66	87.50
2.00	RRPO-C, $\alpha = 0.5$	53377.06	8.23
2.00	RRPO-C, $\alpha = 0.6$	61205.36	24.10
2.00	RRPO-C, $\alpha = 0.7$	70425.84	42.80
2.00	RRPO-C, $\alpha = 0.8$	78122.02	58.40
2.00	RRPO-C, $\alpha = 0.9$	87114.25	76.63
2.00	RRPO	94847.37	92.31

Table EC.1 Results of comparison between RRPO, constrained RRPO and DRPO on orangeJuice instances with log-log demand model.

θ	Method m	Z_m^*	RI (%)
0.10	RRPO-C, $\alpha = 0.5$	153375.29	-47.20
0.10	RRPO-C, $\alpha = 0.6$	147513.70	-49.22
0.10	RRPO-C, $\alpha = 0.7$	173292.78	-40.34
0.10	RRPO-C, $\alpha = 0.8$	171790.80	-40.86
0.10	RRPO-C, $\alpha = 0.9$	205353.74	-29.30
0.10	RRPO	342357.06	17.86
0.50	RRPO-C, $\alpha = 0.5$	117253.37	-20.64
0.50	RRPO-C, $\alpha = 0.6$	131225.11	-11.18
0.50	RRPO-C, $\alpha = 0.7$	133415.93	-9.70
0.50	RRPO-C, $\alpha = 0.8$	155148.87	5.01
0.50	RRPO-C, $\alpha = 0.9$	171173.27	15.85
0.50	RRPO	197517.06	33.68
0.80	RRPO-C, $\alpha = 0.5$	87201.93	-17.53
0.80	RRPO-C, $\alpha = 0.6$	98469.28	-6.87
0.80	RRPO-C, $\alpha = 0.7$	104328.93	-1.33
0.80	RRPO-C, $\alpha = 0.8$	111791.18	5.73
0.80	RRPO-C, $\alpha = 0.9$	122249.99	15.62
0.80	RRPO	149709.04	41.59
1.00	RRPO-C, $\alpha = 0.5$	74421.62	-14.44
1.00	RRPO-C, $\alpha = 0.6$	77826.59	-10.52
1.00	RRPO-C, $\alpha = 0.7$	90332.98	3.86
1.00	RRPO-C, $\alpha = 0.8$	103167.94	18.61
1.00	RRPO-C, $\alpha = 0.9$	107505.80	23.60
1.00	RRPO	125987.02	44.85
1.50	RRPO-C, $\alpha = 0.5$	53040.97	-6.08
1.50	RRPO-C, $\alpha = 0.6$	56740.90	0.47
1.50	RRPO-C, $\alpha = 0.7$	62651.62	10.94
1.50	RRPO-C, $\alpha = 0.8$	66842.72	18.36
1.50	RRPO-C, $\alpha = 0.9$	73892.42	30.84
1.50	RRPO	82880.96	46.76
2.00	RRPO-C, $\alpha = 0.5$	35416.94	-4.70
2.00	RRPO-C, $\alpha = 0.6$	38862.06	4.57
2.00	RRPO-C, $\alpha = 0.7$	41834.74	12.57
2.00	RRPO-C, $\alpha = 0.8$	45750.84	23.10
2.00	RRPO-C, $\alpha = 0.9$	50007.72	34.56
2.00	RRPO	54665.15	47.09

Table EC.2 Results of comparison between RRPO, constrained RRPO and DRPO on orangeJuice instances with semi-log demand model.

EC.6. Approximation of convex price sets by discrete price sets

In this section, we show that under certain conditions, the optimal randomized robust pricing scheme over a convex price set \mathcal{P} can be well-approximated by the optimal randomized robust pricing scheme over a finite subset of \mathcal{P} .

Let $Z_{\text{RR}}^*(\mathcal{P}')$ denote the optimal objective value of the RRPO problem with respect to the price set \mathcal{P}' , which could be a finite set or an uncountable set:

$$Z_{\text{RR}}^*(\mathcal{P}') = \sup_{F \in \mathcal{F}_{\mathcal{P}'}} \inf_{\mathbf{u} \in \mathcal{U}} \int R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}). \quad (\text{EC.67})$$

We additionally make the following assumption regarding the revenue function R .

ASSUMPTION EC.1. *There exists $L > 0$ such that for all $\mathbf{u} \in \mathcal{U}$, $R(\cdot, \mathbf{u})$ is Lipschitz continuous with constant L , i.e.,*

$$|R(\mathbf{p}, \mathbf{u}) - R(\mathbf{p}', \mathbf{u})| \leq L \|\mathbf{p} - \mathbf{p}'\|_{\infty}, \quad (\text{EC.68})$$

for all $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$.

Our main result can be stated as follows.

THEOREM EC.2. *Suppose that \mathcal{P} and \mathcal{U} are nonempty compact convex sets. Suppose that $R(\mathbf{p}, \mathbf{u})$ is convex in \mathbf{u} . Suppose also that Assumption EC.1 holds.*

Then, for any $\epsilon > 0$ there exists a finite subset $\hat{\mathcal{P}} \subseteq \mathcal{P}$, such that

$$Z_{\text{RR}}^*(\mathcal{P}) - Z_{\text{RR}}^*(\hat{\mathcal{P}}) \leq \epsilon. \quad (\text{EC.69})$$

To establish the result, we need to establish several auxiliary results. The first result states that if we are given a finite δ -net of \mathcal{P} , the nominal problem for a given \mathbf{u} with the exact set \mathcal{P} and the δ -net of \mathcal{P} will be close in their optimal objective values.

LEMMA EC.5. *Let $\mathcal{P} \subseteq \mathbb{R}^I$ be a nonempty set of price vectors, and let $\hat{\mathcal{P}} \subseteq \mathcal{P}$ be a finite set such that for every $\mathbf{p} \in \mathcal{P}$, there exists $\mathbf{p}' \in \hat{\mathcal{P}}$ such that $\|\mathbf{p} - \mathbf{p}'\|_{\infty} \leq \delta$. Suppose also that Assumption EC.1 holds. Then*

$$\sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \leq L\delta \quad (\text{EC.70})$$

for all $\mathbf{u} \in \mathcal{U}$.

Proof: Fix any $\epsilon > 0$ and any $\mathbf{u} \in \mathcal{U}$. Let \mathbf{p}^* be such that $R(\mathbf{p}^*, \mathbf{u}) \geq \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \epsilon$ or equivalently, $R(\mathbf{p}^*, \mathbf{u}) + \epsilon \geq \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$, which is guaranteed to exist because of the definition of the supremum.

Let $\hat{\mathbf{p}} \in \arg \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u})$. Note that $\hat{\mathbf{p}}$ is well-defined, because $\hat{\mathcal{P}}$ is a finite set.

Let $\tilde{\mathbf{p}} \in \hat{\mathcal{P}}$ be a price vector such that $\|\tilde{\mathbf{p}} - \mathbf{p}^*\|_\infty \leq \delta$, which is guaranteed to exist by the hypotheses of the lemma. We now have

$$\begin{aligned} & \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \\ &= R(\mathbf{p}^*, \mathbf{u}) + \epsilon - R(\hat{\mathbf{p}}, \mathbf{u}) \\ &= R(\mathbf{p}^*, \mathbf{u}) - R(\tilde{\mathbf{p}}, \mathbf{u}) + R(\tilde{\mathbf{p}}, \mathbf{u}) - R(\hat{\mathbf{p}}, \mathbf{u}) + \epsilon \\ &\leq |R(\mathbf{p}^*, \mathbf{u}) - R(\tilde{\mathbf{p}}, \mathbf{u})| + 0 + \epsilon \\ &\leq L\delta + \epsilon, \end{aligned}$$

where the first step follows by the definitions of \mathbf{p}^* and $\hat{\mathbf{p}}$; the second step follows by algebra; the third step follows by basic properties of absolute values, and also the fact that $\tilde{\mathbf{p}}$ is a feasible but not necessarily optimal solution to $\max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u})$, and hence $R(\tilde{\mathbf{p}}, \mathbf{u}) \leq R(\hat{\mathbf{p}}, \mathbf{u})$ (or equivalently, $R(\tilde{\mathbf{p}}, \mathbf{u}) - R(\hat{\mathbf{p}}, \mathbf{u}) \leq 0$); and the last step follows by Assumption EC.1.

Since $\epsilon > 0$ is arbitrary, it must therefore be that $\sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \leq L\delta$, as required. \square

The second result says that given a δ -net of \mathcal{P} , the objective value of a certain min-max problem, where the outer minimization is over \mathbf{u} and the inner maximization is over \mathcal{P} , will not change too much if the inner maximization is performed over the δ -net.

LEMMA EC.6. *Let $\mathcal{P} \subseteq \mathbb{R}^I$ be a nonempty set of price vectors, and let $\hat{\mathcal{P}} \subseteq \mathcal{P}$ be a finite set such that for every $\mathbf{p} \in \mathcal{P}$, there exists $\mathbf{p}' \in \hat{\mathcal{P}}$ such that $\|\mathbf{p} - \mathbf{p}'\|_\infty \leq \delta$. Suppose also that Assumption EC.1 holds. Then*

$$\inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \leq L\delta. \quad (\text{EC.71})$$

Proof: Fix any $\epsilon > 0$. Let $\hat{\mathbf{u}} \in \mathcal{U}$ be an uncertain parameter vector such that $\max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \hat{\mathbf{u}}) \leq \inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) + \epsilon$. Such a $\hat{\mathbf{u}}$ is guaranteed to exist by the definition of the infimum.

We now have:

$$\inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.72})$$

$$\leq \inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \hat{\mathbf{u}}) + \epsilon \quad (\text{EC.73})$$

$$\leq \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \hat{\mathbf{u}}) - \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \hat{\mathbf{u}}) + \epsilon \quad (\text{EC.74})$$

$$\leq L\delta + \epsilon, \quad (\text{EC.75})$$

where the first step follows by the definition of $\hat{\mathbf{u}}$ as an ϵ -approximate solution to the discretized problem $\inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u})$; the second step follows because $\hat{\mathbf{u}}$ is a feasible solution to the continuous (non-discretized) problem $\inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$; and the final step by Lemma EC.5. Since ϵ was arbitrary, it must be that $\inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \leq L\delta$ as required. \square

Finally, we prove Theorem EC.2.

Proof of Theorem EC.2: Let $\delta = \epsilon/L$. Since \mathcal{P} is compact, there exists a finite set $\hat{\mathcal{P}}$ which is a δ -net, that is, for every $\mathbf{p} \in \mathcal{P}$, there exists $\mathbf{p}' \in \hat{\mathcal{P}}$ such that

$$\|\mathbf{p} - \mathbf{p}'\|_{\infty} \leq \delta. \quad (\text{EC.76})$$

We now have

$$Z_{\text{RR}}^*(\mathcal{P}) - Z_{\text{RR}}^*(\hat{\mathcal{P}}) \quad (\text{EC.77})$$

$$= \sup_{F \in \mathcal{F}(\mathcal{P})} \inf_{\mathbf{u} \in \mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) - \sup_{F \in \mathcal{F}(\hat{\mathcal{P}})} \inf_{\mathbf{u} \in \mathcal{U}} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) \quad (\text{EC.78})$$

$$= \inf_{\mathbf{u} \in \mathcal{U}} \sup_{F \in \mathcal{F}(\mathcal{P})} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) - \inf_{\mathbf{u} \in \mathcal{U}} \sup_{F \in \mathcal{F}(\hat{\mathcal{P}})} \int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p}) \quad (\text{EC.79})$$

$$= \inf_{\mathbf{u} \in \mathcal{U}} \sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u}) - \inf_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.80})$$

$$\leq L\delta \quad (\text{EC.81})$$

$$= \epsilon, \quad (\text{EC.82})$$

where the first step follows by the definition of $Z_{\text{RR}}^*(\mathcal{P}')$, the second by applying Sion's minimax theorem, the third by the equivalence of $\sup_{F \in \mathcal{F}(\mathcal{P})} \int R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p})$ and $\sup_{\mathbf{p} \in \mathcal{P}} R(\mathbf{p}, \mathbf{u})$, the fourth by Lemma EC.6, and the final step by the definition of δ . Note that in the second step, Sion's minimax theorem applies as $\int_{\mathcal{P}} R(\mathbf{p}, \mathbf{u}) dF(\mathbf{p})$ is convex in \mathbf{u} (and hence both quasiconvex and continuous in \mathbf{u}), and linear in F (and hence both quasiconcave and continuous in the topology of weak convergence in F), and $\mathcal{F}(\mathcal{P})$ and \mathcal{U} are both convex, and \mathcal{U} is additionally compact. \square

We now pause to make a few comments about Theorem EC.2. First, the Lipschitz continuity assumption on the revenue function R is not particularly restrictive, and is satisfied by many

popular demand models. In particular, when \mathcal{P} and \mathcal{U} are compact sets and the gradient $\nabla_{\mathbf{p}}R(\mathbf{p}, \mathbf{u})$ with respect to \mathbf{p} is continuous in both \mathbf{p} and \mathbf{u} , then R will satisfy the requirement of the theorem for a Lipschitz constant L that is defined as

$$L = \max_{\mathbf{u} \in \mathcal{U}} \max_{\mathbf{p} \in \mathcal{P}} \|\nabla_{\mathbf{p}}R(\mathbf{p}, \mathbf{u})\|_1. \quad (\text{EC.83})$$

For the semi-log and log-log demand models, the revenue function R has a gradient $\nabla_{\mathbf{p}}R(\mathbf{p}, \mathbf{u})$ that is a continuous function of \mathbf{p} and \mathbf{u} .

Second, this theorem implies that when the revenue function is Lipschitz continuous with the same constant across every \mathbf{u} , discretization will perform well. In particular, for a price set of the form $\mathcal{P} = [p_{\min,i}, p_{\max,i}]^I$, a uniform discretization of the form $\hat{\mathcal{P}} = \{\delta \lceil p_{\min,i}/\delta \rceil, \delta(\lceil p_{\min,i}/\delta \rceil + 1), \delta(\lceil p_{\min,i}/\delta \rceil + 2), \dots, \delta \lfloor p_{\max,i}/\delta \rfloor\}^I$ for some $\delta > 0$ will yield the δ -net in the ℓ_∞ norm required in the proof of Theorem EC.2. By setting δ to be suitably small, we can thus ensure that the discretized RRPO problem returns a randomized pricing scheme that is arbitrarily close to the optimal randomized pricing scheme over the continuous price set \mathcal{P} . This result thus provides a theoretical justification for our focus on discrete price sets in Sections 5 and EC.7 of the paper.

EC.7. Solution method for finite \mathcal{P} , finite \mathcal{U}

The second solution approach we consider is for the case where both \mathcal{P} and \mathcal{U} are finite sets. In particular, we assume that the uncertainty set \mathcal{U} is a binary representable set. For fixed positive integers m and n , we let \mathcal{U} be defined as

$$\mathcal{U} = \{\mathbf{u} = \mathbf{F}\mathbf{z} \mid \mathbf{A}\mathbf{z} \leq \mathbf{b}, \mathbf{z} \in \{0, 1\}^n\}, \quad (\text{EC.84})$$

where \mathbf{b} is a m -dimensional real vector, \mathbf{A} is a m -by- n real matrix and \mathbf{F} is a M -by- n real matrix, where M is the dimension of the uncertain parameter vector \mathbf{u} . We note here that any finite set can be represented in this way: if $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_H\}$, then one can define \mathbf{z} as $\mathbf{z} = (z_1, \dots, z_H) \in \{0, 1\}^H$, use the linear inequality system $\mathbf{A}\mathbf{z} \leq \mathbf{b}$ to represent the constraints $\mathbf{1}^\top \mathbf{z} \leq 1$, $-\mathbf{1}^\top \mathbf{z} \leq -1$, and define $\mathbf{F} = [\mathbf{u}_1 \ \dots \ \mathbf{u}_H]$ (i.e., the matrix obtained from stacking the vectors $\mathbf{u}_1, \dots, \mathbf{u}_H$ as columns).

From a modeling standpoint, a discrete set \mathcal{U} can be used to model the firm's uncertainty in different ways. One could use this type of uncertainty set to model a finite set of scenarios in terms of the customer population's demand response to the firm's prices. Another possibility is that the firm faces a collection of different customer segments, where each one corresponds to a particular type of demand model, and at most one segment will respond to the firm's products; in this setting, the firm could use the uncertainty set to model its uncertainty as to which segment will respond. Given the factor structure $\mathbf{u} = \mathbf{F}\mathbf{z}$ within \mathcal{U} , one could also consider a setting where the firm estimates a richer demand model where the elasticity parameters (represented by \mathbf{u}) are dependent

on macroeconomic factors (represented by \mathbf{z}) that are uncertain and represented in a discrete way; for example, one could use binary variables z_1, z_2, z_3 to indicate three different unemployment scenarios, and z_4, z_5, z_6 to indicate three different interest rate scenarios. The firm would then obtain a randomized pricing strategy that accounts for uncertainty in these macroeconomic factors.

Recall that when \mathcal{P} is finite, then the RRPO problem is

$$Z_{\text{RR}}^* = \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}). \quad (\text{EC.85})$$

We can transform this problem into a dual problem where the outer problem is to optimize a distribution over uncertain parameter vectors, and the inner problem is to optimize over the price vector, as follows:

$$Z_{\text{RR}}^* = \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.86})$$

$$= \max_{\pi \in \Delta_{\mathcal{P}}} \min_{\lambda \in \Delta_{\mathcal{U}}} \sum_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \mathcal{U}} \pi_{\mathbf{p}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.87})$$

$$= \min_{\lambda \in \Delta_{\mathcal{U}}} \max_{\pi \in \Delta_{\mathcal{P}}} \sum_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \mathcal{U}} \pi_{\mathbf{p}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.88})$$

$$= \min_{\lambda \in \Delta_{\mathcal{U}}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \mathcal{U}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.89})$$

where the first equality follows because minimization of a function of \mathbf{u} over the finite set \mathcal{U} is the same as minimizing the expected value of that function over all probability mass functions supported on \mathcal{U} ; the second equality follows by linear programming duality; and the final equality follows because maximization of a function of \mathbf{p} over \mathcal{P} is the same as maximizing the expected value of that function over all probability mass functions supported on \mathcal{P} . We refer to problem (EC.85) as the *primal* problem and (EC.89) as the *dual* problem.

Consider now the *restricted primal problem*, where we replace \mathcal{P} with a subset $\hat{\mathcal{P}} \subseteq \mathcal{P}$ in problem (EC.85), and the *restricted dual problem*, where we replace \mathcal{U} with a subset $\hat{\mathcal{U}} \subseteq \mathcal{U}$ in problem (EC.89). Let us denote the objective values of these two problems with $Z_{P, \hat{\mathcal{P}}}$ and $Z_{D, \hat{\mathcal{U}}}$, respectively. These two problems are:

$$Z_{P, \hat{\mathcal{P}}} = \max_{\pi \in \Delta_{\hat{\mathcal{P}}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad (\text{EC.90})$$

$$Z_{D, \hat{\mathcal{U}}} = \min_{\lambda \in \Delta_{\hat{\mathcal{U}}}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}). \quad (\text{EC.91})$$

Observe that $Z_{P, \hat{\mathcal{P}}}$ and $Z_{D, \hat{\mathcal{U}}}$ bound Z_{RR}^* from below and above, that is,

$$Z_{P, \hat{\mathcal{P}}} \leq Z_{\text{RR}}^* \leq Z_{D, \hat{\mathcal{U}}}.$$

In the above, the justification for the first inequality is because maximizing over distributions supported on the smaller set of price vectors $\hat{\mathcal{P}}$ cannot result in a higher worst-case objective than solving the full primal problem with \mathcal{P} , which gives the value Z_{RR}^* . The second inequality similarly follows because minimizing over distributions supported on the smaller set of uncertainty realizations $\hat{\mathcal{U}}$ cannot result in a lower worst-case objective than solving the full dual problem with \mathcal{U} , which gives Z_{RR}^* .

The idea of double column generation is as follows. Let us pick some subset of price vectors $\hat{\mathcal{P}} \subseteq \mathcal{P}$ and some subset of uncertainty realizations $\hat{\mathcal{U}} \subseteq \mathcal{U}$. Observe that the restricted primal problem (EC.90) for $\hat{\mathcal{P}}$ can be written in epigraph form as

$$\underset{\pi, t}{\text{maximize}} \quad t \tag{EC.92a}$$

$$\text{subject to} \quad t \leq \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{U}, \tag{EC.92b}$$

$$\sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} = 1, \tag{EC.92c}$$

$$\pi_{\mathbf{p}} \geq 0, \quad \forall \mathbf{p} \in \hat{\mathcal{P}}. \tag{EC.92d}$$

This problem has a huge number of constraints (one for each $\mathbf{u} \in \mathcal{U}$). However, we can solve it using delayed constraint generation, starting from the set $\hat{\mathcal{U}}$. Upon solving it in this way, at termination we will have a subset \mathcal{U}' of uncertainty realizations from \mathcal{U} that were found during the constraint generation process. We update $\hat{\mathcal{U}}$ to be equal to \mathcal{U}' .

With this (updated) subset $\hat{\mathcal{U}}$ in hand, we now solve the restricted dual problem (EC.91) for $\hat{\mathcal{U}}$. This problem can be written in epigraph form as

$$\underset{\lambda, \rho}{\text{minimize}} \quad \rho \tag{EC.93a}$$

$$\text{subject to} \quad \rho \geq \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{p} \in \mathcal{P}, \tag{EC.93b}$$

$$\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} = 1, \tag{EC.93c}$$

$$\lambda_{\mathbf{u}} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}. \tag{EC.93d}$$

This problem also has a huge number of constraints, but again we can solve it using delayed constraint generation, with the initial subset of price vectors set to $\hat{\mathcal{P}}$. At termination, we will have a new subset \mathcal{P}' of price vectors, which will contain the original set of price vectors in $\hat{\mathcal{P}}$. We then update $\hat{\mathcal{P}}$ to \mathcal{P}' , and go back to solving the restricted primal problem. The process then repeats: after solving the restricted primal, we will have a new (bigger) $\hat{\mathcal{U}}$; we then solve the restricted dual, after which we have a new (bigger) $\hat{\mathcal{P}}$; we then go back to the restricted primal, and so on.

After each iteration of solving the restricted primal and restricted dual, the set $\hat{\mathcal{P}}$ expands and the set $\hat{\mathcal{U}}$ expands. Thus, the bounds $Z_{P,\hat{\mathcal{P}}}$ and $Z_{D,\hat{\mathcal{U}}}$ get closer and closer to Z_{RR}^* . The algorithm can then be terminated either when $Z_{P,\hat{\mathcal{P}}} = Z_{D,\hat{\mathcal{U}}}$, which would imply that both restricted primal and restricted dual objective values exactly coincide with Z_{RR}^* ; or otherwise, one can terminate when $Z_{D,\hat{\mathcal{U}}} - Z_{P,\hat{\mathcal{P}}} < \epsilon$, where $\epsilon > 0$ is a user specified tolerance.

The overall algorithmic approach is formalized as Algorithm 3. This algorithm invokes two procedures, PRIMALCG (Algorithm 4) and DUALCG (Algorithm 5), which are delayed constraint generation algorithms for solving the restricted primal and dual problems respectively. We note that Algorithm 3 is an adaptation of the double column generation algorithm of Wang et al. (2024) for the randomized robust assortment optimization problem, which is itself adapted from the double column generation algorithm of Delage and Saif (2022) for solving mixed-integer distributionally robust optimization problems. The proof of correctness of this procedure follows similarly to Delage and Saif (2022), and is omitted for brevity. The novelty in our approach lies in how we handle the separation problems which are at the heart of PRIMALCG and DUALCG, which we discuss next. Sections EC.7.1, EC.7.2 and EC.7.3 provide the details for the linear, semi-log and log-log demand models. Section EC.7.4 discusses the complexity of randomization and provides a result that is analogous to Theorem 4 for the case of a finite \mathcal{P} and finite \mathcal{U} .

Algorithm 3 Double column generation method for solving the finite \mathcal{P} , finite \mathcal{U} RRPO problem.

- 1: Initialize $\hat{\mathcal{P}}$ to be a non-empty subset of \mathcal{P} , and $\hat{\mathcal{U}}$ to be a non-empty subset of \mathcal{U} .
 - 2: Set $\text{LB} \leftarrow -\infty$, $\text{UB} \leftarrow +\infty$
 - 3: **repeat**
 - 4: Run PRIMALCG($\hat{\mathcal{P}}, \hat{\mathcal{U}}$) to solve the restricted primal problem with $\hat{\mathcal{P}}$ and with $\hat{\mathcal{U}}$ as the initial uncertainty set. Let the objective value be $Z_{P,\hat{\mathcal{P}}}$ and the new uncertainty set be \mathcal{U}' .
 - 5: Set $\hat{\mathcal{U}} \leftarrow \mathcal{U}'$.
 - 6: Set $\text{LB} \leftarrow Z_{P,\hat{\mathcal{P}}}$.
 - 7: Run DUALCG($\hat{\mathcal{P}}, \hat{\mathcal{U}}$) to solve the restricted dual problem with $\hat{\mathcal{U}}$ and with $\hat{\mathcal{P}}$ as the initial price vector set. Let the objective value be $Z_{D,\hat{\mathcal{U}}}$ and the new price vector set be \mathcal{P}' .
 - 8: Set $\hat{\mathcal{P}} \leftarrow \mathcal{P}'$.
 - 9: Set $\text{UB} \leftarrow Z_{D,\hat{\mathcal{U}}}$.
 - 10: **until** $\text{UB} - \text{LB} < \epsilon$
-

Note that the doubly restricted primal and dual problems (EC.94) and (EC.96) solved in PRIMALCG and DUALCG are both linear programs, and can be thus be solved easily. The principal

Algorithm 4 PRIMALCG procedure.

-
- 1: Initialize $\mathcal{U}' \leftarrow \hat{\mathcal{U}}$
 - 2: **repeat**
 - 3: Solve the doubly restricted primal problem:

$$\underset{\boldsymbol{\pi}, t}{\text{maximize}} \quad t \tag{EC.94a}$$

$$\text{subject to} \quad t \leq \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{U}', \tag{EC.94b}$$

$$\sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} = 1, \tag{EC.94c}$$

$$\pi_{\mathbf{p}} \geq 0, \quad \forall \mathbf{p} \in \hat{\mathcal{P}}. \tag{EC.94d}$$

Let $(\boldsymbol{\pi}, t^*)$ be the optimal solution of the doubly restricted problem.

- 4: Solve the primal separation problem:

$$\min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot R(\mathbf{p}, \mathbf{u}). \tag{EC.95}$$

Let t' and \mathbf{u}^* be the optimal objective value and solution of this separation problem.

- 5: **if** $t^* > t'$ **then**
 - 6: Set $\mathcal{U}' \leftarrow \mathcal{U}' \cup \{\mathbf{u}^*\}$
 - 7: **end if**
 - 8: **until** $t^* \leq t'$
 - 9: Set $Z_{P, \hat{\mathcal{P}}} \leftarrow t^*$
 - 10: **return** $(Z_{P, \hat{\mathcal{P}}}, \mathcal{U}')$.
-

difficulty in these procedures comes from the primal and dual separation problems (EC.95) and (EC.97), which require optimizing over an uncertain parameter vector $\mathbf{u} \in \mathcal{U}$ and a price vector $\mathbf{p} \in \mathcal{P}$ respectively. In the following sections, we discuss how these two separation problems can be tackled for the linear, semi-log and log-log demand models. Note that in all three sections, we continue to make Assumption 1, which states that \mathcal{P} can be written as the Cartesian product of finite sets of prices for each product, i.e., $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_I$, where $\mathcal{P}_1, \dots, \mathcal{P}_I$ are finite sets.

EC.7.1. Primal and dual subproblems for linear demand model

For the linear demand model, the primal separation problem is

$$\min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^I p_i \cdot (\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) \right]. \tag{EC.98}$$

Algorithm 5 DUALCG procedure.1: Initialize $\mathcal{P}' \leftarrow \hat{\mathcal{P}}$ 2: **repeat**

3: Solve the doubly restricted dual problem:

$$\underset{\lambda, \rho}{\text{minimize}} \quad \rho \quad (\text{EC.96a})$$

$$\text{subject to} \quad \rho \geq \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{p} \in \mathcal{P}', \quad (\text{EC.96b})$$

$$\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} = 1, \quad (\text{EC.96c})$$

$$\lambda_{\mathbf{u}} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}. \quad (\text{EC.96d})$$

Let $(\boldsymbol{\lambda}, \rho^*)$ be the optimal solution of the doubly restricted problem.

4: Solve the dual separation problem:

$$\max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot R(\mathbf{p}, \mathbf{u}) \quad (\text{EC.97})$$

Let ρ' and \mathbf{p}^* be the optimal objective value and solution of this separation problem.5: **if** $\rho^* < \rho'$ **then**6: Set $\mathcal{P}' \leftarrow \mathcal{P}' \cup \{\mathbf{p}^*\}$ 7: **end if**8: **until** $\rho^* \geq \rho'$ 9: Set $Z_{D, \hat{\mathcal{U}}} \leftarrow \rho^*$ 10: **return** $(Z_{D, \hat{\mathcal{U}}}, \mathcal{P}')$.

Note that this objective function is linear in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Therefore, the whole problem can be expressed as

$$\underset{\mathbf{p} \in \hat{\mathcal{P}}}{\text{minimize}} \quad \sum \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^I p_i \cdot (\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) \right] \quad (\text{EC.99a})$$

$$\text{subject to} \quad \mathbf{u} = \mathbf{Fz}, \quad (\text{EC.99b})$$

$$\mathbf{Az} \leq \mathbf{b}, \quad (\text{EC.99c})$$

$$\mathbf{z} \in \{0, 1\}^n, \quad (\text{EC.99d})$$

which is a mixed-integer linear program.

The dual separation problem is

$$\max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I p_i (\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j) \right]. \quad (\text{EC.100})$$

By introducing the same binary variables as in the separation problem (24) (the linear demand separation problem for the convex \mathcal{U} setting), we obtain the following mixed-integer linear program:

$$\text{maximize}_{\mathbf{x}, \mathbf{y}} \quad \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I \sum_{t \in \mathcal{P}_i} \alpha_i \cdot t \cdot x_{i,t} - \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} t^2 \cdot \beta_i \cdot x_{i,t} + \sum_{i=1}^I \sum_{j \neq i} \sum_{t_1 \in \mathcal{P}_i} \sum_{t_2 \in \mathcal{P}_j} \gamma_{i,j} \cdot t_1 \cdot t_2 \cdot y_{i,j,t_1,t_2} \right] \quad (\text{EC.101a})$$

$$\text{subject to} \quad \sum_{t \in \mathcal{P}_i} x_{i,t} = 1, \quad \forall i \in [I], \quad (\text{EC.101b})$$

$$\sum_{t_2 \in \mathcal{P}_j} y_{i,j,t_1,t_2} = x_{i,t_1}, \quad \forall i, j \in [I], j \neq i, t_2 \in \mathcal{P}_j, \quad (\text{EC.101c})$$

$$\sum_{t_1 \in \mathcal{P}_i} y_{i,j,t_1,t_2} = x_{j,t_1}, \quad \forall i, j \in [I], j \neq i, t_1 \in \mathcal{P}_i, \quad (\text{EC.101d})$$

$$x_{i,t} \in \{0, 1\}, \quad \forall i \in [I], t \in \mathcal{P}_i, \quad (\text{EC.101e})$$

$$y_{i,j,t_1,t_2} \in \{0, 1\}, \quad \forall i, j \in [I], i \neq j, t_1 \in \mathcal{P}_i, t_2 \in \mathcal{P}_j. \quad (\text{EC.101f})$$

Importantly, note that the size of this problem does not scale with the number of uncertainty realizations inside $\hat{\mathcal{U}}$; the form of this problem is equivalent to problem (24) where \mathbf{u} is replaced with $\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \mathbf{u}$ (the ‘‘average’’ uncertain demand parameter). As we will see in the next couple of sections, the same will not be true for the semi-log and log-log demand models.

EC.7.2. Primal and dual subproblems for semi-log demand model

For the semi-log demand model, the primal separation problem is

$$\min_{\substack{\mathbf{u} \in \hat{\mathcal{U}} \\ \mathbf{p} \in \hat{\mathcal{P}}}} \sum \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right]. \quad (\text{EC.102})$$

Note that this objective function is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, because the weights $\pi_{\mathbf{p}}$ and p_i for a given $\mathbf{p} \in \hat{\mathcal{P}}$ and $i \in [I]$ are nonnegative, and because the function $e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j}$ is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, the whole problem can be expressed as

$$\text{minimize}_{\mathbf{u}, \mathbf{z}} \quad \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] \quad (\text{EC.103a})$$

$$\text{subject to} \quad \mathbf{u} = \mathbf{Fz}, \quad (\text{EC.103b})$$

$$\mathbf{Az} \leq \mathbf{b}, \quad (\text{EC.103c})$$

$$\mathbf{z} \in \{0, 1\}^n, \quad (\text{EC.103d})$$

which can be re-written as a mixed-integer exponential cone program.

The dual separation problem is

$$\max_{\substack{\mathbf{p} \in \hat{\mathcal{P}} \\ \mathbf{u} \in \hat{\mathcal{U}}}} \sum \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right]. \quad (\text{EC.104})$$

The objective function of this problem is in general not concave in \mathbf{p} . However, just as in Section 5.3, the related problem of optimizing the logarithm of this objective, which is

$$\max_{\mathbf{p} \in \mathcal{P}} \log \left[\sum_{\mathbf{u} \in \tilde{\mathcal{U}}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] \right] \quad (\text{EC.105})$$

$$= \max_{\mathbf{p} \in \mathcal{P}} \log \left[\sum_{\mathbf{u} \in \tilde{\mathcal{U}}} \sum_{i=1}^I e^{\log \lambda_{\mathbf{u}} + \log p_i + \alpha_i - \beta_i p_i + \sum_{j \neq i} \gamma_{i,j} p_j} \right] \quad (\text{EC.106})$$

can be reformulated as a mixed-integer exponential cone program using the same biconjugate-based technique in Section 5.3. In particular, when Assumption 1 holds, then problem (EC.106) is equivalent to

$$\begin{aligned}
\text{maximize}_{\mathbf{w}, \mathbf{x}, \boldsymbol{\mu}} \quad & \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \mu_{\mathbf{u},i} \cdot (\log \lambda_{\mathbf{u}} + \alpha_i) \\
& + \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} \log t w_{\mathbf{u},i,i,t} \\
& + \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} (-\beta_i) \cdot t \cdot w_{\mathbf{u},i,i,t} \\
& + \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \sum_{j \neq i} \gamma_{i,j} \cdot \sum_{t \in \mathcal{P}_j} t \cdot w_{\mathbf{u},i,j,t} \\
& - \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \mu_{\mathbf{u},i} \log \mu_{\mathbf{u},i} \tag{EC.107a}
\end{aligned}$$

$$\text{subject to} \quad \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \mu_{\mathbf{u},i} = 1, \tag{EC.107b}$$

$$\sum_{t \in \mathcal{P}_j} w_{\mathbf{u},i,j,t} = \mu_{\mathbf{u},i}, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i, j \in [I], \tag{EC.107c}$$

$$\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I w_{\mathbf{u},i,j,t} = x_{j,t}, \quad \forall j \in [I], t \in \mathcal{P}_j, \tag{EC.107d}$$

$$\sum_{t \in \mathcal{P}_j} x_{j,t} = 1, \quad \forall j \in [I], \tag{EC.107e}$$

$$w_{\mathbf{u},i,j,t} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i, j \in [I], t \in \mathcal{P}_j, \tag{EC.107f}$$

$$\mu_{\mathbf{u},i} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i \in [I], \tag{EC.107g}$$

$$x_{j,t} \in \{0, 1\}, \quad \forall j \in [I], t \in \mathcal{P}_j, \tag{EC.107h}$$

where $x_{j,t}$ is a binary decision variable that is 1 if product j is offered at price $t \in \mathcal{P}_j$, and 0 otherwise; $\mu_{\mathbf{u},i}$ is a nonnegative decision variable introduced as part of the biconjugate-based reformulation; and $w_{\mathbf{u},i,j,t}$ is a decision variable that represents the linearization of $\mu_{\mathbf{u},i} \cdot x_{j,t}$ for all $\mathbf{u} \in \hat{\mathcal{U}}, i, j \in [I]$, and $t \in \mathcal{P}_j$.

As with problem (29), this problem can be expressed as a mixed-integer exponential cone program. One notable difference between formulation (EC.107) and formulation (29) from earlier is that the number of decision variables and constraints is larger because the decision variable $\mu_{\mathbf{u},i}$ is introduced for every combination of an uncertainty realization in $\hat{\mathcal{U}}$ and each product i ; thus, $\boldsymbol{\mu}$ represents a probability mass function over the set $\hat{\mathcal{U}} \times [I]$.

EC.7.3. Primal and dual subproblems for log-log demand model

For the log-log demand model, the primal separation problem is

$$\min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j}. \quad (\text{EC.108})$$

Note that the objective function is convex in $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$; it is the nonnegative weighted combination of terms of the form $e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j}$, each of which are convex in $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Thus, the overall problem, which can be stated as

$$\underset{\mathbf{u}, \mathbf{z}}{\text{minimize}} \quad \sum_{\mathbf{p} \in \hat{\mathcal{P}}} \pi_{\mathbf{p}} \cdot \sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \quad (\text{EC.109a})$$

$$\text{subject to} \quad \mathbf{u} = \mathbf{Fz}, \quad (\text{EC.109b})$$

$$\mathbf{Az} \leq \mathbf{b}, \quad (\text{EC.109c})$$

$$\mathbf{z} \in \{0, 1\}^n, \quad (\text{EC.109d})$$

is a mixed-integer convex program, and can be expressed as a mixed-integer exponential cone program.

The dual separation problem is

$$\max_{\mathbf{p} \in \mathcal{P}} \sum_{\mathbf{u} \in \mathcal{U}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \right]. \quad (\text{EC.110})$$

The objective function of this problem is in general not concave in \mathbf{p} . However, following the same method as in Section 5.4, we can reformulate the related problem of maximizing the logarithm, which is

$$\max_{\mathbf{p} \in \mathcal{P}} \log \left(\sum_{\mathbf{u} \in \mathcal{U}} \lambda_{\mathbf{u}} \cdot \left[\sum_{i=1}^I p_i \cdot e^{\alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \right] \right) \quad (\text{EC.111})$$

$$= \max_{\mathbf{p} \in \mathcal{P}} \log \left(\sum_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I e^{\log \lambda_{\mathbf{u}} + \log p_i + \alpha_i - \beta_i \log p_i + \sum_{j \neq i} \gamma_{i,j} \log p_j} \right) \quad (\text{EC.112})$$

as a mixed-integer exponential cone program. Under Assumption 1, the resulting formulation is

$$\begin{aligned} \underset{\mathbf{x}, \mathbf{w}, \boldsymbol{\mu}}{\text{maximize}} \quad & \sum_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I \mu_{\mathbf{u},i} \cdot (\log \lambda_{\mathbf{u}} + \alpha_i) \\ & + \sum_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I \sum_{t \in \mathcal{P}_i} (1 - \beta_i) \log t \cdot w_{\mathbf{u},i,i,t} \\ & + \sum_{\mathbf{u} \in \mathcal{U}} \sum_{i=1}^I \sum_{j \neq i} \gamma_{i,j} \cdot \sum_{t \in \mathcal{P}_j} \log t \cdot w_{\mathbf{u},i,j,t} \end{aligned}$$

$$-\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \mu_{\mathbf{u},i} \log \mu_{\mathbf{u},i} \quad (\text{EC.113a})$$

$$\text{subject to } \sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I \mu_{\mathbf{u},i} = 1, \quad (\text{EC.113b})$$

$$\sum_{t \in \mathcal{P}_j} w_{\mathbf{u},i,j,t} = \mu_{\mathbf{u},i}, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i, j \in [I], \quad (\text{EC.113c})$$

$$\sum_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{i=1}^I w_{\mathbf{u},i,j,t} = x_{j,t}, \quad \forall j \in [I], t \in \mathcal{P}_j, \quad (\text{EC.113d})$$

$$\sum_{t \in \mathcal{P}_j} x_{j,t} = 1, \quad \forall j \in [I], \quad (\text{EC.113e})$$

$$w_{\mathbf{u},i,j,t} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i, j \in [I], t \in \mathcal{P}_j, \quad (\text{EC.113f})$$

$$\mu_{\mathbf{u},i} \geq 0, \quad \forall \mathbf{u} \in \hat{\mathcal{U}}, i \in [I], \quad (\text{EC.113g})$$

$$x_{j,t} \in \{0, 1\}, \quad \forall j \in [I], t \in \mathcal{P}_j, \quad (\text{EC.113h})$$

where the decision variables have the same meaning as those in formulation (EC.107).

EC.7.4. Complexity of randomization for finite \mathcal{P} , finite \mathcal{U}

In the case that \mathcal{P} and \mathcal{U} are both finite, it is possible to characterize the support size more directly than in the case where \mathcal{U} is convex, by using standard results in linear programming. In particular, we have the following result.

PROPOSITION EC.2. *Suppose that the uncertainty set \mathcal{U} is finite, and that $R(\mathbf{p}, \mathbf{u}) \geq 0$ for all $\mathbf{p} \in \mathcal{P}$ and $\mathbf{u} \in \mathcal{U}$. The RRPO problem $\max_{\boldsymbol{\pi} \in \Delta_{\mathcal{P}}} \min_{\mathbf{u} \in \mathcal{U}} \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u})$ has an optimal solution $\boldsymbol{\pi}^*$ with a support of size at most $|\mathcal{U}| + 1$.*

Proof: The RRPO problem can be equivalently written as

$$\underset{s, t, \boldsymbol{\pi}}{\text{maximize}} \quad t \quad (\text{EC.114a})$$

$$\text{subject to } t + s_{\mathbf{u}} = \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{U}, \quad (\text{EC.114b})$$

$$\sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} = 1, \quad (\text{EC.114c})$$

$$\pi_{\mathbf{p}} \geq 0, \quad \forall \mathbf{p} \in \mathcal{P}, \quad (\text{EC.114d})$$

$$s_{\mathbf{u}} \geq 0, \quad \forall \mathbf{u} \in \mathcal{U}, \quad (\text{EC.114e})$$

$$t \geq 0, \quad (\text{EC.114f})$$

where $s_{\mathbf{u}}$ can be regarded as a slack variable for the constraint $t - \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u}) \leq 0$ (or equivalently, $t \leq \sum_{\mathbf{p} \in \mathcal{P}} \pi_{\mathbf{p}} R(\mathbf{p}, \mathbf{u})$). Observe that this is a linear program in standard form, with

finitely many equality constraints and finitely many variables. Note that the feasible region is non-empty, because given any $\boldsymbol{\pi} \in \Delta_{\mathcal{P}}$, we can select t and $\mathbf{s} = (s_{\mathbf{u}})_{\mathbf{u} \in \mathcal{U}}$ to obtain a feasible solution to the LP above; note that additionally the problem is bounded, because $\{R(\mathbf{p}, \mathbf{u}) \mid \mathbf{p} \in \mathcal{P}, \mathbf{u} \in \mathcal{U}\}$ is a finite set. Hence, it follows that there exists an optimal solution. Since the problem is in standard form, there is at least one extreme point (Corollary 2.2 of Bertsimas and Tsitsiklis 1997). By Theorem 2.7 of Bertsimas and Tsitsiklis (1997), there must exist an extreme point solution that is optimal. Since extreme points correspond to basic feasible solutions, this optimal solution must be such that at most $|\mathcal{U}| + 1$ coordinates of $(\mathbf{s}, t, \boldsymbol{\pi})$ are non-zero. For the resulting distribution $\boldsymbol{\pi}$, at most $|\mathcal{U}| + 1$ prices therefore have non-zero probability. \square

EC.8. Additional numerical results

EC.8.1. Experiments with discrete \mathcal{U} and log-log and semi-log demand models

In this set of experiments, we consider linear, log-log and semi-log demand models, where uncertainty is modeled through a discrete uncertainty set. We specifically consider a discrete budget uncertainty set \mathcal{U} here:

$$\mathcal{U} = \{\mathbf{u} = \mathbf{u}_0 - (\mathbf{u}_0 - \bar{\mathbf{u}}) \circ \boldsymbol{\xi} - (\mathbf{u}_0 - \underline{\mathbf{u}}) \circ \boldsymbol{\eta} \mid \mathbf{e}^\top \boldsymbol{\xi} + \mathbf{e}^\top \boldsymbol{\eta} \leq \Gamma, \boldsymbol{\xi} + \boldsymbol{\eta} \leq \mathbf{1}, \boldsymbol{\xi}, \boldsymbol{\eta} \in \{0, 1\}^{I+I^2}\}, \quad (\text{EC.115})$$

where $\underline{\mathbf{u}}$ and $\bar{\mathbf{u}}$ are respectively the component-wise lower and upper bounds of \mathbf{u} , \mathbf{u}_0 is the nominal value of the uncertain parameter vector $\mathbf{u} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, \circ denotes the element-wise product of two vectors, Γ is the budget of uncertainty and $I + I + I(I - 1) = I + I^2$ is the total number of demand model parameters. Under the budget uncertainty set \mathcal{U} , up to Γ parameters can attain their lower bounds or upper bounds, whereas the remaining parameters can only attain their nominal values. We shall assume that the lower bound vector $\underline{\mathbf{u}}$ and upper bound vector $\bar{\mathbf{u}}$ are defined as $\underline{\mathbf{u}} = 0.7\mathbf{u}_0$ and $\bar{\mathbf{u}} = 1.3\mathbf{u}_0$, where \mathbf{u}_0 is the vector of nominal parameters.

For each of the three demand models (linear, semi-log and log-log), we vary the number of products I in $\{5, 10, 15\}$. For each value of I , we generate 24 random instances, where the values of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are independently randomly generated as follows:

1. *Linear demand.* Each $\alpha_i \sim \text{Uniform}(100, 200)$, $\beta_i \sim \text{Uniform}(5, 15)$, $\gamma_{i,j} \sim \text{Uniform}(-0.1, +0.1)$.
2. *Semi-log demand.* Each $\alpha_i \sim \text{Uniform}(8, 10)$, $\beta_i \sim \text{Uniform}(1.5, 2)$, $\gamma_{i,j} \sim \text{Uniform}(-0.5, +0.5)$.
3. *Log-log demand.* Each $\alpha_i \sim \text{Uniform}(10, 14)$, $\beta_i \sim \text{Uniform}(1.5, 2)$, $\gamma_{i,j} \sim \text{Uniform}(-0.8, +0.8)$.

We set the price set of each $i \in [I]$ as $\mathcal{P}_i = \{1, 2, 3, 4, 5\}$.

For each instance, we solve the nominal problem, the DRPO problem and the RRPO problem. For both RRPO and DRPO, we test a different collection of Γ values for the uncertainty set depending on the value of I .

To solve the RRPO problem for each instance, we execute the double column generation algorithm described in Section EC.7. For the linear demand model, we solve both primal and dual separation problems as mixed-integer programs in Gurobi.

For the log-log and semi-log demand models, we proceed differently. In our preliminary experimentation with the restricted dual problem, we observed that exactly solving the dual separation problem (EC.107) (for semi-log demand) or (EC.113) (for log-log demand) via Mosek takes quite a long time. Therefore, to reduce the computation time of RRPO with discrete \mathcal{U} , we instead use a random improvement heuristic to obtain the solution of dual separation problem. Specifically, we randomly select a price vector \mathbf{p}^0 as a starting point. We start with changing the price of product $i = 1$ and keeping the prices of all other products unchanged, to search for a price vector \mathbf{p}^1 that makes the objective value of the dual separation problem the largest. Then based on the current price vector \mathbf{p}^1 , we change the price of product $i = 2$ and keep the prices of all other products unchanged, to search for a better price vector \mathbf{p}^2 . We repeat this for all of the products, yielding the price vector \mathbf{p}^I . We repeat this procedure with 100 random starting points, and retain the best solution over these 100 repetitions. We note that in the heuristic form of the double column generation method, we continue to solve the primal separation problem (over $\mathbf{u} \in \mathcal{U}$) exactly using Mosek. The lower bound given by the restricted primal, $Z_{P,\hat{p}}$, therefore remains a valid lower bound on the true optimal RRPO objective, and will correspond to the worst-case expected revenue of a possibly suboptimal price distribution $\boldsymbol{\pi}$, at every iteration of the double column generation method. At termination, the objective value returned by the algorithm will therefore be a lower bound on the true optimal RRPO objective.

Although this approximate method cannot guarantee that the overall double column generation procedure converges to a provably optimal solution, our preliminary experimentation with small instances suggests that it obtains the exact solution of RRPO that one would obtain if the dual separation problem were solved to provable optimality. In Section EC.8.2 of the ecompanion, we undertake an in-depth comparison of this heuristic form of double column generation against the exact double column generation approach, where both separation problems are solved to provable optimality using Mosek. When one imposes a time limit on the exact approach, we find that the heuristic approach returns solutions of comparable quality (in cases where the exact approach terminates with the optimal solution), or better (in cases where the exact approach exhausts the time limit and does not return an optimal solution), and in general, the heuristic approach terminates in significantly less time than the exact approach.

With regard to the DRPO problem for each log-log and semi-log instance, we note that we do not have a solution algorithm or formulation to solve it exactly. Therefore, we again use the same random improvement heuristic to obtain an approximate solution of DRPO with these demand

models. We randomly pick a starting price vector, and change the price of one product at a time to improve the worst case objective value until we no longer get an improvement. We repeat this procedure 50 times and select the best resulting price vector from these 50 repetitions as the approximate solution of DRPO. We note that we use a smaller number of repetitions because each repetition involves solving worst-case problem over $\mathbf{u} \in \mathcal{U}$ repeatedly in order to evaluate the robust objective of each candidate price vector; this contributes to a large overall computation time for this approach. We denote this approximate solution by $\hat{\mathbf{p}}_{\text{DR}}$. With regard to the DRPO problem for linear demand, we observe that the objective function of DRPO is linear in the uncertain parameter vector \mathbf{u} , and that the description of the set polyhedron (EC.115) is integral (i.e., extreme points of this polyhedron naturally correspond to $\boldsymbol{\xi}, \boldsymbol{\eta} \in \{0, 1\}^{2I+I^2}$). Therefore, DRPO can be solved exactly by relaxing the requirement $\boldsymbol{\xi}, \boldsymbol{\eta} \in \{0, 1\}^{2I+I^2}$ in the uncertainty set (EC.115), and reformulating the worst-case objective using LP duality, leading to a mixed-integer linear program.

Lastly, for the nominal problem for each instance, we either formulate it as a mixed-integer linear program (in the linear demand case) or use the biconjugate technique to formulate it as a mixed-integer exponential cone program (in the log-log and semi-log demand cases).

We report the same metrics as in Section 7.1, with several modifications for the log-log and semi-log demand models. For those models, we use \hat{Z}_{DR} and $R(\hat{\mathbf{p}}_{\text{DR}}, \mathbf{u}_0)$ to denote the approximate objective value of the DRPO problem and revenue of the approximate DRPO solution $\hat{\mathbf{p}}_{\text{DR}}$ under the nominal parameter vector \mathbf{u}_0 . We also use \hat{Z}_{RR} to denote the approximate objective of the RRPO problem, and $\mathbb{E}[R(\hat{\mathbf{p}}_{\text{RR}}, \mathbf{u}_0)]$ denotes the expected revenue of the resulting randomized price vector $\hat{\mathbf{p}}_{\text{RR}}$ under the nominal demand model parameter vector \mathbf{u}_0 . Finally, the approximate relative improvement of RRPO over DRPO is defined as $\hat{\text{RI}} = (\hat{Z}_{\text{RR}} - \hat{Z}_{\text{DR}}) / \hat{Z}_{\text{DR}} \times 100\%$.

Table EC.3 shows the results for the linear demand model. Here, we interestingly find that the vast majority of instances are randomization-proof, i.e., the average RI is below 1%, if not exactly 0%. We note here that we tested other families of instances where $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and \mathcal{P}_i are generated differently, but in virtually every case we found that the relative improvement of randomized over deterministic robust pricing was very small. These results, together with those for the convex \mathcal{U} case, suggest that randomized pricing is of limited benefit compared to deterministic pricing for the uncertain linear demand model case.

Tables EC.4 and EC.5 show how the results vary for different values of discrete uncertainty budget Γ for semi-log and log-log demand.² We can see that, in most of the cases we test, the

² We note here that for the log-log model, we encountered one instance ($I = 10, \Gamma = 44$) where \hat{Z}_{DR} was higher than \hat{Z}_{RR} ; in general, when solved to perfect optimality, one should see Z_{RR}^* should be higher than Z_{DR}^* . We have verified that the reason for this anomaly was a numerical error in the solution of the worst-case subproblem in Mosek within the DRPO random improvement heuristic. This instance is omitted in our calculation of \hat{Z}_{DR} , $\hat{\text{RI}}$ and $\mathbb{E}[R(\hat{\mathbf{p}}_{\text{DR}}, \mathbf{u}_0)]$, and the affected entries are indicated by * in Table EC.5.

randomized robust pricing strategy provides a substantial benefit over the deterministic robust price solution. The percentage improvement given by randomization ranges from 0% to as much as 488.59% for semi-log instances, and from 0% to 175.18% for log-log instances. Similar to the cases with convex \mathcal{U} , both \hat{Z}_{RR} and \hat{Z}_{DR} decrease as the uncertainty set becomes larger. While the RI metric generally decreases as Γ increases, in some instances it can be increasing in Γ at small values of Γ (this is visible in the average results metrics for $I = 10$ with semi-log demand). When Γ is large enough, the RI metric often becomes very small or even zero. This makes sense when interpreted through Corollary 2. Specifically, when nature is able to make a large number of demand model parameters take their worst values, it is likely that at the \mathbf{u}^* at which the optimal objective of DRPO is attained is such that the price vector for the nominal problem with \mathbf{u}^* coincides with the optimal price vector for DRPO. Thus, by Corollary 2, the problem will be randomization-proof.

With regard to the computation time, the computation time for both RRPO and DRPO increases with I . Interestingly, the computation time required by RRPO does not necessarily increase as the discrete uncertainty budget Γ increases; in some cases, when Γ is large, the RRPO solution degenerates to the DRPO solution, allowing the double column generation algorithm to terminate quickly. By comparing t_{RR} and t_{DR} , we can see that RRPO in general takes less time than DRPO. The computation time of the RRPO problem in semi-log instances is no more than approximately two minutes on average ($I = 15, \Gamma = 36$), while in log-log instances, solving RRPO requires no more than 1.5 minutes on average ($I = 15, \Gamma = 18$). Lastly, for linear demand, the computation time for RRPO is extremely small, requiring no more than a few seconds on average.

EC.8.2. Comparison of heuristic and exact solutions of RRPO in the finite \mathcal{P} , finite \mathcal{U} case

In this section, we perform a detailed comparison of the performance of the heuristic form of RRPO, where the dual separation problem is solved approximately using the random improvement heuristic, against the exact form of RRPO, where both the primal and dual separation problem are solved exactly. We consider the log-log and semi-log demand models, and we test the same instances described in Section EC.8.1.

For the exact approach, we incorporated the following termination logic. When the relative gap between the lower bound from the restricted primal and the upper bound from the restricted dual is within 10^{-6} . In certain cases, due to numerical precision issues arising from computing the log-log and semi-log objective functions and solving the primal and dual separation problems within Mosek, we would alternatively terminate when the set of price vectors $\hat{\mathcal{P}}$ no longer changes (i.e., when the dual separation problem is unable to produce a price vector that is not contained in $\hat{\mathcal{P}}$). Finally, we also imposed a time limit of 3600 seconds for the double column generation algorithm,

I	Γ	t_{RR}	Z_{RR}^*	$\mathbb{E}[R(\mathbf{p}_{RR}^*, \mathbf{u}_0)]$	t_{DR}	Z_{DR}^*	RI(%)	$R(\mathbf{p}_{DR}^*, \mathbf{u}_0)$	t_N	Z_N^*	$Z_{N,WC}$
5	3	8.76	1724.38	2458.17	0.43	1723.46	0.06	2462.66	0.32	2473.30	1719.29
5	6	0.27	1316.39	2382.10	0.04	1313.68	0.23	2383.35	–	–	1261.84
5	9	0.27	1159.33	2294.44	0.03	1157.98	0.12	2293.84	–	–	1039.24
5	12	0.19	1126.85	2259.05	0.03	1126.85	0.00	2259.05	–	–	987.29
5	18	0.20	1124.87	2256.23	0.03	1124.87	0.00	2256.23	–	–	984.10
5	24	0.17	1123.78	2256.23	0.03	1123.78	0.00	2256.23	–	–	982.18
10	5	0.53	3717.70	4990.89	0.11	3714.08	0.10	4997.68	0.09	5009.03	3711.50
10	7	0.59	3315.59	4972.42	0.11	3312.91	0.08	4973.49	–	–	3297.07
10	9	0.70	2982.02	4941.26	0.11	2979.68	0.08	4944.40	–	–	2942.80
10	14	1.35	2589.70	4782.85	0.12	2583.93	0.23	4791.83	–	–	2437.09
10	19	0.84	2372.37	4662.16	0.11	2371.21	0.05	4657.17	–	–	2133.38
10	26	0.56	2342.57	4636.61	0.10	2342.57	0.00	4636.61	–	–	2086.03
10	33	0.57	2339.18	4633.87	0.11	2339.18	0.00	4636.61	–	–	2081.46
10	44	0.57	2334.90	4627.78	0.10	2334.90	0.00	4627.78	–	–	2075.15
15	6	0.92	5920.62	7495.77	0.22	5918.51	0.04	7506.26	0.17	7513.95	5917.60
15	12	1.79	4706.04	7435.56	0.25	4701.16	0.11	7442.32	–	–	4668.21
15	18	3.16	4051.33	7250.57	0.27	4045.35	0.15	7248.00	–	–	3889.25
15	24	4.50	3722.40	7071.15	0.29	3713.86	0.23	7068.49	–	–	3428.59
15	36	1.12	3500.73	6889.94	0.21	3500.73	0.00	6889.47	–	–	3112.83

Table EC.3 Results for linear instances with discrete \mathcal{U} .

I	Γ	t_{RR}	\hat{Z}_{RR}	$\mathbb{E}[R(\hat{\mathbf{p}}_{RR}, \mathbf{u}_0)]$	t_{DR}	\hat{Z}_{DR}	$\hat{R}I$ (%)	$R(\hat{\mathbf{p}}_{DR}, \mathbf{u}_0)$	t_N	Z_N^*	$Z_{N,WC}$
5	3	13.23	8.65×10^3	1.57×10^5	20.35	5.56×10^3	63.01	2.17×10^5	0.93	2.27×10^5	5.04×10^3
5	6	0.43	3.06×10^3	1.81×10^5	20.52	1.92×10^3	66.89	2.20×10^5	–	–	1.78×10^3
5	9	0.55	1.84×10^3	2.09×10^5	22.96	1.64×10^3	18.42	2.22×10^5	–	–	1.63×10^3
5	12	0.63	1.65×10^3	2.22×10^5	25.53	1.60×10^3	4.39	2.25×10^5	–	–	1.60×10^3
5	18	0.27	1.59×10^3	2.27×10^5	23.87	1.59×10^3	0.11	2.27×10^5	–	–	1.59×10^3
5	24	0.08	1.59×10^3	2.27×10^5	13.53	1.59×10^3	0.00	2.27×10^5	–	–	1.59×10^3
10	5	1.59	1.21×10^6	5.30×10^7	117.69	4.95×10^5	173.47	6.73×10^7	0.15	7.16×10^7	4.48×10^5
10	7	2.59	6.50×10^5	4.10×10^7	119.28	1.87×10^5	233.11	5.69×10^7	–	–	1.88×10^5
10	9	3.49	4.32×10^5	3.64×10^7	121.43	1.09×10^5	250.12	5.41×10^7	–	–	1.05×10^5
10	14	6.44	1.80×10^5	3.67×10^7	180.63	6.69×10^4	161.08	6.96×10^7	–	–	6.78×10^4
10	19	10.23	9.63×10^4	5.29×10^7	362.29	6.36×10^4	67.80	7.15×10^7	–	–	6.36×10^4
10	26	12.14	6.69×10^4	6.40×10^7	559.63	6.32×10^4	19.70	7.15×10^7	–	–	6.33×10^4
10	33	11.37	6.37×10^4	7.11×10^7	699.06	6.33×10^4	6.47	7.15×10^7	–	–	6.33×10^4
10	44	7.84	6.33×10^4	7.15×10^7	799.81	6.33×10^4	0.09	7.16×10^7	–	–	6.33×10^4
15	6	37.63	3.55×10^8	3.22×10^9	331.13	1.59×10^7	488.59	4.52×10^9	0.44	5.05×10^9	1.42×10^7
15	12	18.86	8.69×10^6	3.24×10^9	344.94	2.15×10^6	471.14	3.51×10^9	–	–	1.77×10^6
15	18	33.08	3.11×10^6	3.08×10^9	653.64	1.18×10^6	323.61	4.91×10^9	–	–	1.16×10^6
15	24	48.52	1.75×10^6	3.67×10^9	1525.41	1.12×10^6	165.75	5.03×10^9	–	–	1.11×10^6
15	36	101.38	1.21×10^6	4.82×10^9	3400.17	1.1×10^6	38.58	5.02×10^9	–	–	1.10×10^6

Table EC.4 Results for semi-log instances with discrete \mathcal{U} .

i.e., we exit from the loop in Algorithm 3 if the time limit exceeds 3600 seconds. We note here that the first two termination criteria were also implemented for the heuristic approach.

In Tables EC.6 and EC.7, we display the optimality gap G_E for the exact approach, the computation time t_E for the exact approach, the computation time t_H for the heuristic approach, and

I	Γ	t_{RR}	\hat{Z}_{RR}	$\mathbb{E}[R(\hat{\mathbf{p}}_{RR}, \mathbf{u}_0)]$	t_{DR}	\hat{Z}_{DR}	\hat{RI} (%)	$R(\hat{\mathbf{p}}_{DR}, \mathbf{u}_0)$	t_N	Z_N^*	$Z_{N,WC}$
5	3	13.33	3.26×10^5	1.86×10^6	31.23	2.80×10^5	22.03	1.89×10^6	0.96	4.31×10^6	1.49×10^5
5	6	0.20	6.37×10^4	4.03×10^6	20.39	6.15×10^4	3.16	4.29×10^6	–	–	6.1×10^4
5	9	0.27	4.91×10^4	4.19×10^6	18.39	4.87×10^4	0.65	4.24×10^6	–	–	4.77×10^4
5	12	0.35	4.71×10^4	4.20×10^6	16.82	4.70×10^4	0.20	4.18×10^6	–	–	4.57×10^4
5	18	0.17	4.66×10^4	4.18×10^6	11.67	4.66×10^4	0.01	4.18×10^6	–	–	4.52×10^4
5	24	0.14	4.66×10^4	4.18×10^6	8.73	4.66×10^4	0.05	4.16×10^6	–	–	4.52×10^4
10	5	4.64	2.33×10^6	3.89×10^7	161.57	1.38×10^6	83.54	6.45×10^7	0.31	7.66×10^7	1.23×10^6
10	7	5.39	1.31×10^6	5.22×10^7	165.21	8.59×10^5	63.34	7.12×10^7	–	–	8.39×10^5
10	9	5.30	8.74×10^5	5.15×10^7	149.63	6.23×10^5	38.99	7.28×10^7	–	–	6.12×10^5
10	14	5.10	5.07×10^5	5.67×10^7	138.67	3.82×10^5	29.31	7.30×10^7	–	–	3.80×10^5
10	19	3.52	3.74×10^5	6.35×10^7	136.70	3.31×10^5	12.90	7.51×10^7	–	–	3.33×10^5
10	26	3.56	3.27×10^5	7.38×10^7	138.77	3.18×10^5	4.28	7.59×10^7	–	–	3.21×10^5
10	33	4.79	3.18×10^5	7.60×10^7	137.07	3.15×10^5	2.34	7.59×10^7	–	–	3.17×10^5
10	44	3.19	3.15×10^5	7.62×10^7	131.47	$3.19 \times 10^{5*}$	1.19*	$7.53 \times 10^{7*}$	–	–	3.14×10^5
15	6	35.68	1.71×10^7	4.33×10^8	552.64	7.75×10^6	175.18	7.83×10^8	1.20	8.38×10^8	7.20×10^6
15	12	62.70	5.55×10^6	4.93×10^8	582.20	2.94×10^6	102.06	7.78×10^8	–	–	2.76×10^6
15	18	76.60	2.92×10^6	4.55×10^8	561.96	1.92×10^6	65.73	8.24×10^8	–	–	1.88×10^6
15	24	54.30	2.03×10^6	6.79×10^8	572.08	1.71×10^6	32.76	8.19×10^8	–	–	1.71×10^6
15	36	48.34	1.69×10^6	8.11×10^8	600.30	1.63×10^6	8.07	8.32×10^8	–	–	1.63×10^6

Table EC.5 Results for log-log instances with discrete \mathcal{U} .

the relative improvement $RI_{E \rightarrow H}$ of the heuristic approach over the exact approach, for the log-log and semi-log demand models respectively. The optimality gap G_E is defined as

$$G_E = 100\% \times \frac{Z_{RR,E,UB}^* - Z_{RR,E,LB}^*}{Z_{RR,E,UB}^*}, \quad (\text{EC.116})$$

where $Z_{RR,E,UB}^*$ and $Z_{RR,E,LB}^*$ are the final upper and lower bounds from the double column generation approach. The relative improvement $RI_{E \rightarrow H}$ is defined as

$$RI_{E \rightarrow H} = 100\% \times \frac{Z_{RR,H,LB}^* - Z_{RR,E,LB}^*}{Z_{RR,E,LB}^*}, \quad (\text{EC.117})$$

where $Z_{RR,H,LB}^*$ is the final lower bound from the heuristic approach. The relative improvement metric measures how much the price distribution from the heuristic double CG method improves on the price distribution from the exact double CG method. Note that this metric can be negative, in cases where the heuristic approach terminates with a suboptimal solution to the RRPO problem.

Before continuing, we note that in some instances, we found the gap G_E to be either slightly larger than the tolerance of 10^{-6} , or otherwise to be a very small negative number; both of these phenomena are due to the aforementioned numerical issues we encountered. We also note that for larger instances, the computation time sometimes significantly exceeds 3600 seconds. As noted before, the outer loop of the double CG algorithm is terminated when the run time exceeds 3600 seconds, and in some cases, the final iteration of the outer loop before this criterion is triggered can require a huge amount of time (due to solving both the restricted primal and restricted dual problems, each of which can involve multiple solves of the primal and dual separation problems).

For the log-log results shown in Table EC.6, we see that in general, the exact double CG algorithm terminates with a provably optimal solution, as the average optimality gap is very close to zero across all values of I and Γ . With regard to the heuristic double CG method, we can see the average RI metric is in general zero or no smaller than -0.36% . This implies that the heuristic double CG method returns solutions that are in general either the same or only very slightly suboptimal compared to the exact double CG solution. We additionally see that the average computation time of the heuristic double CG method is much smaller than that of the exact double CG method. For example, for $I = 15$, the average computation time of the exact double CG method can range anywhere from around 15 minutes to about 45 minutes, whereas for the heuristic double CG method it is no more than 1.5 minutes. These results indicate that for the log-log model, the heuristic method can obtain solutions of essentially the same quality as the exact method but in substantially less time.

For the semi-log results shown in Table EC.7, we see similar behavior as we do for the log-log models for $I \in \{5, 10\}$. For $I = 15$, we find that the exact double CG method in general does not terminate with a provably optimal solution (i.e., a gap of 0%) under the soft 3600 second time limit. The average optimality gap for the exact method ranges from about 14% to 40% , and the average computation time can be as high as 90 minutes. In contrast, the average computation time for the heuristic double CG method is no more than two minutes. In addition, for $\Gamma \geq 12$, the heuristic solution has an objective value for which the average improvement over the (suboptimal) exact double CG solution ranges from approximately 6% to 20% . This highlights that in settings where the exact double CG method requires a large amount of computation time and must be terminated prematurely, the heuristic method obtains higher quality solutions than the exact method.

EC.8.3. Effect of parameter bounds on relative improvement in discrete \mathcal{U} experiments

In this section, we explore the effect of the width of the budget uncertainty set in our experiments in Section EC.8.1. Recall that in those experiments, we assumed a budget uncertainty set, where $\bar{\mathbf{u}}$ and $\underline{\mathbf{u}}$ denote the vectors of upper and lower bounds, respectively, for the uncertain parameter vector \mathbf{u} , and these parameter vectors were defined as $\bar{\mathbf{u}} = 1.3\mathbf{u}_0$, $\underline{\mathbf{u}} = 0.7\mathbf{u}_0$. By setting the bounds in this way, we assume that each uncertain parameter can vary by $+30\%$ or -30% relative to its nominal value.

We now consider setting these bound vectors as

$$\bar{\mathbf{u}} = (1 + \delta)\mathbf{u}_0, \tag{EC.118}$$

$$\underline{\mathbf{u}} = (1 - \delta)\mathbf{u}_0, \tag{EC.119}$$

I	Γ	G_E	t_E	t_H	$RI_{E \rightarrow H}$
5	3	0.00	16.16	13.33	0.00
5	6	0.00	1.58	0.20	0.00
5	9	-0.34	2.21	0.27	-0.35
5	12	-0.06	2.45	0.35	-0.07
5	18	-0.03	1.88	0.17	-0.05
5	24	0.00	1.49	0.14	0.00
10	5	0.00	63.35	4.64	-0.12
10	7	0.01	77.43	5.39	0.00
10	9	0.01	80.18	5.30	0.00
10	14	0.01	83.99	5.10	-0.01
10	19	0.01	72.05	3.52	-0.03
10	26	0.01	74.75	3.56	0.00
10	33	0.01	70.30	4.79	0.00
10	44	0.01	68.23	3.19	0.00
15	6	0.01	952.93	35.68	-0.25
15	12	0.02	2356.89	62.70	-0.30
15	18	0.23	2739.09	76.60	-0.36
15	24	0.04	2033.25	54.30	-0.24
15	36	0.04	1351.31	48.34	-0.02

Table EC.6 Comparison of exact double CG and heuristic double CG methods for log-log demand model instances from Section EC.8.1.

I	Γ	G_E	t_E	t_H	$RI_{E \rightarrow H}$
5	3	0.00	18.30	13.23	0.00
5	6	0.00	7.57	0.43	0.00
5	9	0.00	8.52	0.55	0.00
5	12	0.00	7.64	0.63	0.00
5	18	0.00	4.17	0.27	0.00
5	24	0.00	1.91	0.08	0.00
10	5	0.00	151.81	1.59	-0.01
10	7	0.01	344.41	2.59	0.00
10	9	0.02	585.85	3.49	0.00
10	14	0.04	1279.63	6.44	-0.29
10	19	0.07	1845.95	10.23	0.02
10	26	0.04	1526.40	12.14	0.00
10	33	0.51	1084.73	11.37	0.07
10	44	0.02	686.13	7.84	0.00
15	6	0.19	1930.95	37.63	0.14
15	12	14.02	4163.44	18.86	5.82
15	18	38.81	5119.94	33.08	19.47
15	24	39.67	4808.48	48.52	19.86
15	36	37.17	4508.40	101.38	15.97

Table EC.7 Comparison of exact double CG and heuristic double CG methods for semi-log demand model instances from Section EC.8.1.

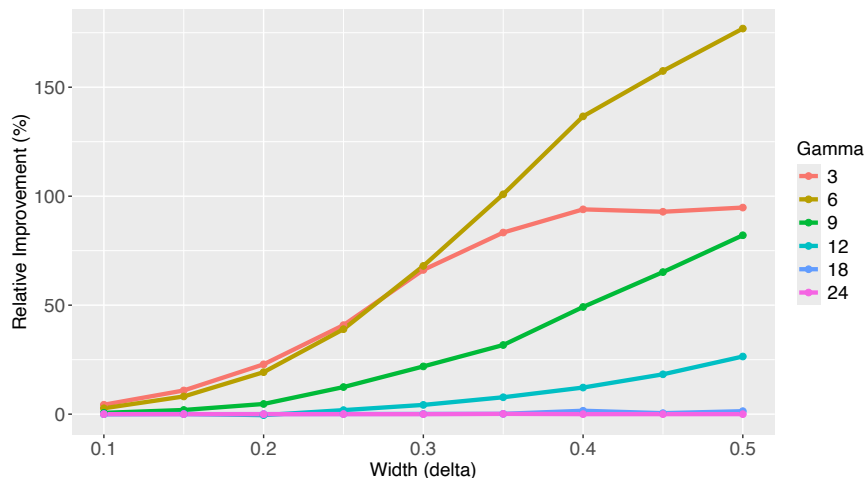


Figure EC.1 Plot of RI versus δ for semi-log demand models.

where $\delta \in (0, 1)$ is the *width parameter*.

We consider the same log-log and semi-log demand model instances from Section EC.8.1, and allow for δ to vary in the set $\{0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5\}$. For simplicity, we restrict our focus to instances with $I = 5$, and allow the uncertainty set budget Γ to vary in the same way as before, i.e., $\Gamma \in \{3, 6, 9, 12, 18, 24\}$. For each instance, we solve the RRPO problem and DRPO problem using the same methods used in Section EC.8.1 (i.e., the heuristic double column generation approach and the random improvement heuristic, respectively). We calculate the average value of the RI metric across all 24 instances for each Γ and δ .

For the semi-log demand model, Figure EC.1 plots the RI metric against δ for different values of Γ , while Figure EC.2 plots the RI metric against Γ for different values of δ . In general, we can see that the relative improvement of the randomized robust price distribution over the deterministic robust price vector increases as the width parameter δ increases, suggesting that randomization becomes more beneficial when there is more uncertainty in each individual parameter. Interestingly, as observed in Section EC.8.1, the RI metric is initially increasing with Γ but decreases after a certain point, and this peak seems to be the same across all values of δ . Lastly, another interesting insight is that as Γ increases, the rate at which the RI increases with δ is lower, and for sufficiently high Γ values, the RI seems to stay very close to zero across all δ values.

For the log-log demand model, Figure EC.3 plots the RI metric against δ for different values of Γ , and Figure EC.4 plots the RI metric against Γ for different values of δ . Like in the semi-log case, the benefit of randomization over the deterministic robust solution does appear to increase as δ increases. The rate at which the RI increases becomes smaller when Γ is higher, and for the highest values of Γ ($\Gamma = 18$ or $\Gamma = 24$), changing δ (at least within the 0.1 to 0.5 range) does not seem to change the RI.

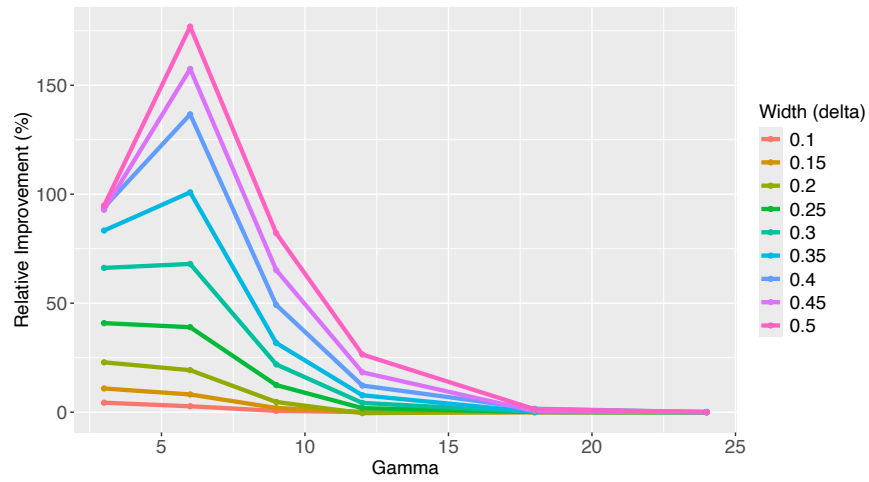


Figure EC.2 Plot of RI versus Γ for semi-log demand models.

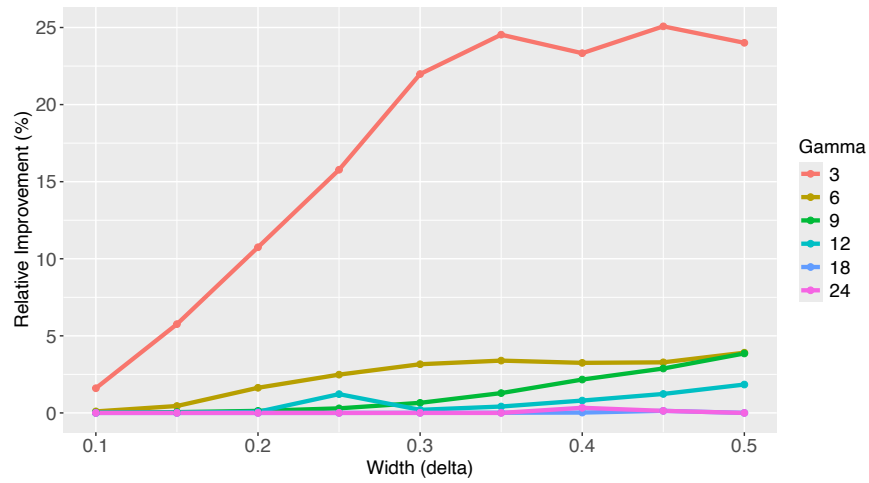


Figure EC.3 Plot of RI versus δ for log-log demand models.

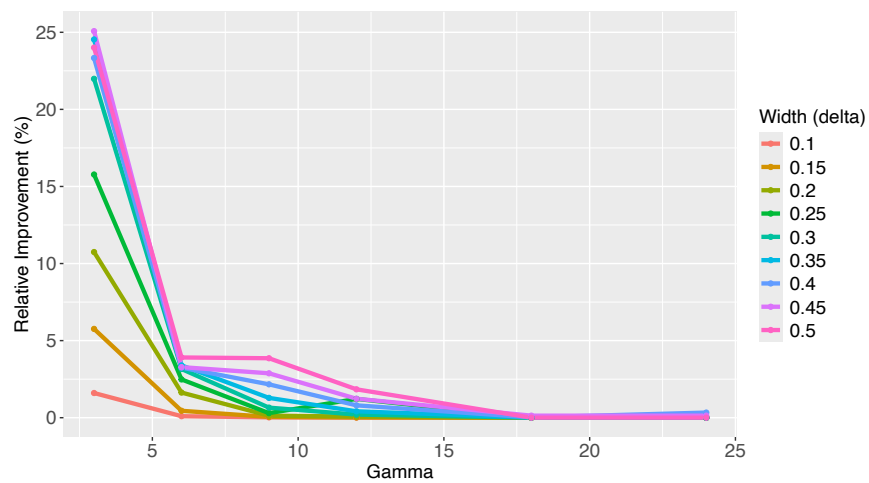


Figure EC.4 Plot of RI versus Γ for log-log demand models.

EC.8.4. Estimation results for orangeJuice data set

Tables EC.8 and EC.9 display the point estimates of α , β and γ for the semi-log and log-log demand models for the orangeJuice data set.

Parameters	Product										
	1	2	3	4	5	6	7	8	9	10	11
$\alpha_{\text{semi-log}}$	9.873	9.829	8.598	9.504	9.024	9.828	8.582	7.901	7.152	11.161	10.896
$\beta_{\text{semi-log}}$	1.0222	0.4581	1.2735	1.7888	1.3354	0.6507	1.6491	1.3945	2.0809	1.6290	0.0383
$\alpha_{\text{log-log}}$	10.140	10.956	8.266	8.421	9.045	10.613	7.832	7.127	6.563	11.326	11.198
$\beta_{\text{log-log}}$	2.7195	2.0410	3.3037	3.8855	2.9357	2.6101	3.6063	2.8209	3.9717	2.7942	0.1542

Table EC.8 Estimation results for α and β .

$\gamma_{\text{semi-log}}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$
$i=1$	–	0.0571	0.0813	0.0966	0.0193	-0.0232	0.1305	0.1904	0.1490	0.0582	0.0815
$i=2$	0.1384	–	0.0041	0.0009	0.0204	0.0153	0.0090	0.1040	-0.0023	0.0491	0.0394
$i=3$	0.3386	0.0916	–	0.1943	0.0702	-0.0062	0.0051	0.0950	-0.0310	0.0690	0.0950
$i=4$	0.4313	0.0976	-0.1112	–	0.4089	0.3518	0.2085	-0.0777	-0.0352	0.0383	-0.2290
$i=5$	0.1916	0.0490	0.3026	0.2966	–	-0.1538	0.1547	-0.0314	0.1034	0.3338	0.0370
$i=6$	0.0211	0.0493	-0.0194	-0.0018	0.0888	–	0.0340	0.0472	-0.0167	0.0297	0.1119
$i=7$	0.2007	0.0388	0.0706	0.0672	0.3233	0.0837	–	0.0377	0.2216	-0.0504	0.1405
$i=8$	0.0117	0.0119	0.0932	0.0757	0.1023	-0.0160	0.1345	–	0.1372	0.2143	0.2699
$i=9$	0.0955	0.0373	-0.0211	0.3651	0.4176	0.0358	0.2127	0.1462	–	0.2337	0.1627
$i=10$	0.0412	-0.3941	0.0764	0.4867	0.4810	0.0109	-0.0814	-0.1047	0.0878	–	0.0274
$i=11$	-0.0893	-0.1587	-0.1358	-0.0252	-0.0690	0.0079	-0.0574	-0.1117	-0.1271	0.0809	–
$\gamma_{\text{log-log}}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$
$i=1$	–	0.2196	0.1631	0.2129	0.0646	-0.0577	0.2576	0.3338	0.2494	0.0621	0.2939
$i=2$	0.3474	–	0.0403	0.0004	0.0338	0.0492	0.0193	0.1879	-0.0042	0.0739	0.1257
$i=3$	0.8673	0.5123	–	0.4400	0.1482	-0.0338	0.0527	0.1480	-0.1001	0.1267	0.3683
$i=4$	1.1581	0.3822	-0.2283	–	0.8367	1.2659	0.4495	-0.1569	-0.0115	0.1003	-0.7321
$i=5$	0.4624	0.2241	0.8344	0.6406	–	-0.6800	0.3223	0.0646	0.1426	0.5815	0.1782
$i=6$	0.0462	0.2424	-0.0343	-0.0173	0.2086	–	0.0975	0.1187	-0.0364	0.0561	0.4159
$i=7$	0.4644	0.2531	0.0971	0.1204	0.6997	0.2946	–	0.1728	0.4912	-0.0564	0.4497
$i=8$	0.0652	0.1430	0.1980	0.1587	0.2705	-0.0988	0.3198	–	0.3034	0.2992	0.9436
$i=9$	0.2971	-0.1190	-0.0216	0.7986	0.8825	0.3045	0.5869	0.1706	0.0	0.3171	0.4223
$i=10$	0.1406	-1.7987	0.1061	1.0453	1.0852	0.0811	-0.1213	-0.1760	0.0454	–	-0.0371
$i=11$	-0.2246	-0.7519	-0.3177	-0.0483	-0.1635	-0.0570	-0.1100	-0.1780	-0.2646	0.1341	–

Table EC.9 Estimation results for γ for orangeJuice data set.**EC.8.5. Performance results for orangeJuice data set**

Tables EC.10 and EC.11 below compare the performance of the nominal, deterministic robust and randomized robust pricing solutions under a discrete budget uncertainty set for the orangeJuice data set.

Γ	t_{RR}	Z_{RR}^*	$\mathbb{E}[R(\mathbf{p}_{RR}^*, \mathbf{u}_0)]$	t_{DR}	\hat{Z}_{DR}	RI(%)	$R(\hat{\mathbf{p}}_{DR}, \mathbf{u}_0)$	t_N	Z_N^*	$Z_{N,WC}$
5	14.64	162753.97	290939.28	225.36	102626.41	58.59	260321.17	0.81	590547.01	85304.36
10	6.29	70401.48	404458.22	208.42	47969.46	46.76	350396.25	–	–	38815.61
15	3.93	39567.50	349936.30	209.14	32757.43	20.79	334211.84	–	–	22798.44
20	16.70	31438.76	328664.19	197.37	25348.77	24.02	299970.20	–	–	15940.16

Table EC.10 Results for orangeJuice pricing problem with semi-log demand and discrete \mathcal{U} .

Γ	t_{RR}	Z_{RR}^*	$\mathbb{E}[R(\mathbf{p}_{RR}^*, \mathbf{u}_0)]$	t_{DR}	\hat{Z}_{DR}	RI(%)	$R(\hat{\mathbf{p}}_{DR}, \mathbf{u}_0)$	t_N	Z_N^*	$Z_{N,WC}$
5	12.85	272399.89	605265.00	306.63	174478.12	56.12	811254.69	0.87	1110000.00	117186.58
10	10.41	135761.31	750084.92	260.96	77297.42	75.63	896972.12	–	–	50458.15
15	15.59	72930.45	761785.89	193.13	44914.20	62.38	896972.12	–	–	27389.15
20	8.56	45153.74	770505.32	190.76	27675.07	63.16	409330.70	–	–	17502.32

Table EC.11 Results for orangeJuice pricing problem with log-log demand and discrete \mathcal{U} .

References

- A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88(3):411–424, 2000.
- D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.
- D. Bertsimas and J. N. Tsitsiklis. *Introduction to linear optimization*. Dynamic Ideas LLC, 1997.
- D. Bertsimas, D. B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501, 2011.
- D. Bertsimas, I. Dunning, and M. Lubin. Reformulation versus cutting-planes for robust optimization: A computational study. *Computational Management Science*, 13:195–217, 2016.
- G. Calafiore and M. C. Campi. Uncertain convex programs: randomized solutions and confidence levels. *Mathematical Programming*, 102:25–46, 2005.
- E. Delage and A. Saif. The value of randomized solutions in mixed-integer distributionally robust optimization problems. *INFORMS Journal on Computing*, 34(1):333–353, 2022.
- D. Fudenberg and J. Tirole. *Game theory*. MIT Press, 1991.
- N. Rujerapaiboon, D. Kuhn, and W. Wiesemann. Robust growth-optimal portfolios. *Management Science*, 62(7):2090–2109, 2016.
- Z. Wang, H. Peura, and W. Wiesemann. Randomized assortment optimization. *Operations Research*, 72(5):2042–2060, 2024.
- S. Zymler, D. Kuhn, and B. Rustem. Distributionally robust joint chance constraints with second-order moment information. *Mathematical Programming*, 137:167–198, 2013.