Proofs of Statements and Supplementary Numerical Examples

EC.1. Proofs

EC.1.1. Proof of Proposition 1

Proof of Proposition 1. The proof is by induction. In **Step 1** we start with the customer at the first LP position ($m=1$), whose strategy does not depend on that of other LP customers. We formulate her join/balk/abandon problem as a dynamic program and show that the strategy has the claimed threshold structure. In **Step 2** we show that, assuming the first $m \geq 1$ customers follow the claimed threshold strategy, the customer at LP position $m+1$ also follows a threshold strategy and the thresholds $\bar{n}(m)$ and $\bar{n}(m+1)$ satisfy properties (i) and (ii) in Part 2 of the proposition. Finally, in **Step 3**, we characterize the thresholds K and L.

Step 1. Consider a tagged customer at LP position $m=1$, that is, there are no LP customers in front of her. Any waiting cost incurred so far is sunk, so given the customer is still in system, she maximizes her expected future utility. Since there can be a maximum of $\bar{n}_h$ HP customers in system, the position of the tagged customer is $(1,n)$ with $n \in \{1,...,\bar{n}_h+1\}$. For system positions $n \leq \bar{n}_h$ the next event is either a service completion, with probability $\bar{\mu} \equiv \mu/(\mu+\lambda_h)$, or an HP arrival with probability $\bar{\lambda} \equiv \lambda_h/(\mu+\lambda_h)$. For position $(1,\bar{n}_h+1)$ since the maximum number of HP customers is reached, any HP arrival balks and does not change the position of the tagged customer. Note that, due to the memoryless property of the arrival and service processes, no new information becomes available between arrivals and service completions, so the customer would only abandon at these discrete epochs. At each decision epoch, the customer can either abandon and receive an immediate reward of 0 or stay until the next event. We consider the uniformized system with transition rate $\mu+\lambda_h$. Let $v(n,m)$ denote the value function of the customer at position (n,m), i.e., her maximum future expected utility until she exits the system with $v(0,0)=R_l$. Then, writing $g^+ \equiv \max(0,g)$, the value function for the customer at LP position $m=1$ satisfies,

$$v(1,1) = [-\bar{c} + \bar{\lambda} v(1,2) + \bar{\mu} v(0,0)]^+, \tag{EC.1}$$

$$v(1,n) = [-\bar{c} + \bar{\lambda} v(1,n+1) + \bar{\mu} v(1,n-1)]^+, \quad n \in \{2,...,\bar{n}_h\}. \tag{EC.2}$$

$$v(1,\bar{n}_h+1) = [-\bar{c} + \bar{\lambda} v(1,\bar{n}_h+1) + \bar{\mu} v(1,\bar{n}_h)]^+, \tag{EC.3}$$

Applying Lemma EC.2 (Appendix B) with $i=1$ and $T=\bar{n}_h$ to (EC.1)–(EC.3), and noting that $v(0,0)=R_l > c_l/\mu$ by assumption, we have that $v(1,n)$ strictly decreases in $n \leq \bar{n}(1)$, where (EC.84) in Part (iii) of Lemma EC.2 implies

$$\bar{n}(1) = \begin{cases} 1, & \text{if } v(1,1) < c_l/\mu, \\ \max\{n \in \{2,...,\bar{n}_h+1\}; v(1,n-1) \geq c_l/\mu\}, & \text{otherwise.} \end{cases} \tag{EC.4}$$

That is, a customer in LP position $m = 1$ joins/stays if and only if her system position $n \le \bar{n}(1)$. Therefore, the solution of (EC.1)–(EC.3) satisfies

$$v(1,1) = -\bar{c} + \bar{\lambda} v(1,2) + \bar{\mu} v(0,0), \tag{EC.5}$$

$$v(1,n) = -\bar{c} + \bar{\lambda} v(1,n+1) + \bar{\mu} v(1,n-1), \quad n \in \{2,...,\bar{n}(1)-1\}, \tag{EC.6}$$

$$v(1,\bar{n}(1)) = \begin{cases} -\bar{c} + \bar{\lambda} v(1,\bar{n}(1)) + \bar{\mu} v(1,\bar{n}(1)-1), & \bar{n}(1) = \bar{n}_h + 1, \\ -\bar{c} + \bar{\mu} v(1,\bar{n}(1)-1), & 1 < \bar{n}(1) < \bar{n}_h + 1, \end{cases} \tag{EC.7}$$

and $v(1,n) = 0$ for $n > \bar{n}(1)$, where

$$\bar{n}(1) > 1 \Leftrightarrow v(1,\bar{n}(1)-1) \ge c_l/\mu, \text{ and } \bar{n}(1) < \bar{n}_h + 1 \Rightarrow c_l/\mu > v(1,\bar{n}(1)). \tag{EC.8}$$

Step 2. We turn to the induction step. Assume that customers are willing to join in the first $m \ge 1$ LP positions and follow a join/balk/abandon strategy with thresholds $\{\bar{n}(1),\dots,\bar{n}(m)\}$ that satisfy properties (i) and (ii) in Part 2 of the proposition, that is:

$$\bar{n}(j) = \bar{n}_h + j \Rightarrow \bar{n}(i) = \bar{n}_h + i, \qquad \text{and } \bar{n}(i) < \bar{n}_h + i \Rightarrow \bar{n}(j) = \bar{n}(i), \qquad \text{for } 1 \le i < j \le m.$$

Then $\bar{n}(j) \le \bar{n}(j-1) + 1$ for $1 < j \le m$, which implies that the next customer to abandon (if any) is the customer at LP position m, she stays if and only if her system position $n \le \bar{n}(m)$, and her value function must satisfy:

$$v(m,m) = -\bar{c} + \bar{\lambda} v(m,m+1) + \bar{\mu} v(m-1,m-1), \tag{EC.9}$$

$$v(m,n) = -\bar{c} + \bar{\lambda} v(m,n+1) + \bar{\mu} v(m,n-1), \quad n \in \{m+1,...,\bar{n}(m)-1\}, \tag{EC.10}$$

$$v(m,\bar{n}(m)) = \begin{cases} -\bar{c} + \bar{\lambda} v(m,\bar{n}(m)) + \bar{\mu} v(m,\bar{n}(m)-1), & \bar{n}(m) = \bar{n}_h + m, \\ -\bar{c} + \bar{\mu} v(m,\bar{n}(m)-1), & m < \bar{n}(m) < \bar{n}_h + m, \end{cases} \tag{EC.11}$$

and $v(m,n) = 0$ for $n > \bar{n}(m)$, where

$$\bar{n}(m) > m \Leftrightarrow v(m,\bar{n}(m)-1) \ge c_l/\mu, \text{ and } \bar{n}(m) < \bar{n}_h + m \Rightarrow c_l/\mu > v(m,\bar{n}(m)). \tag{EC.12}$$

Note that for $m = 1$, (EC.9)–(EC.12) agree with (EC.5)–(EC.8).

Next consider LP position $m+1$. Given the strategy of the customers in the first m LP positions, a customer at LP position $m+1$ has at most $\bar{n}(m)$ customers (of either class) in front of her. Hence, her maximum system position is $\bar{n}(m)+1$. We consider the following two cases separately: $\bar{n}(m) = m$, and $\bar{n}(m) > m$.

Case 1 ($\bar{n}(m) = m$): We show that customers prefer balking to joining LP position $m+1$, and therefore $m = L$. Since $v(m+1,n)$ is decreasing in n, it suffices to show that LP customers prefer balking to joining position $(m+1, m+1)$, i.e., with no HP customers in the system. To this end, note that since $m+1=\bar{n}(m)+1$ by assumption, a customer who considers joining position $(m+1,m+1)$ solves

$$v(m+1,\bar{n}(m)+1)=[-\bar{c}+\bar{\lambda}v(m,\bar{n}(m)+1)+\bar{\mu}v(m,m)]^{+}=[-\bar{c}+\bar{\mu}v(m,m)]^{+}, \tag{EC.13}$$

where the first equality follows because upon arrival of an HP customer, the customer at LP position m abandons as she moves to position $(m,\bar{n}(m)+1)$, which in turn moves the customer at LP position $m+1$ to position $(m,\bar{n}(m)+1)$. The second equality in (EC.13) follows because $v(m,\bar{n}(m)+1)=0$ by the induction assumption. Further, note that $-\bar{c}+\bar{\mu}v(m,m)<0$ on the RHS of the second equality in (EC.13), which implies that the maximum expected utility from joining position $(m+1,m+1)$ is strictly negative. That $-\bar{c}+\bar{\mu}v(m,m)<0$ follows because $\bar{n}(m)=m$ implies $c_l/\mu>v(m,m)$ by the second relationship in (EC.12), and $c_l/\mu=\bar{c}/\bar{\mu}$.

Case 2 ($\bar{n}(m)>m$): We show that the customer at LP position $m+1$ follows a threshold join/balk/abandon strategy with threshold $\bar{n}(m+1)\geq m+1$. Furthermore, the thresholds $\bar{n}(m)$ and $\bar{n}(m+1)$ satisfy: (i) if $\bar{n}(m+1)=\bar{n}_h+m+1$, then $\bar{n}(m)=\bar{n}_h+m$; and (ii) if $\bar{n}(m)<\bar{n}_h+m$ then $\bar{n}(m+1)=\bar{n}(m)$.

First, for the threshold structure, note that a customer in LP position $m+1$ solves:

$$v(m+1,m+1)=[-\bar{c}+\bar{\lambda}v(m+1,m+2)+\bar{\mu}v(m,m)]^{+}, \tag{EC.14}$$

$$v(m+1,n)=[-\bar{c}+\bar{\lambda}v(m+1,n+1)+\bar{\mu}v(m+1,n-1)]^{+},\quad n\in\{m+2,...,\bar{n}(m)\}, \tag{EC.15}$$

$$v(m+1,\bar{n}(m)+1)=\begin{cases}[-\bar{c}+\bar{\lambda}v(m+1,\bar{n}(m)+1)+\bar{\mu}v(m+1,\bar{n}(m))]^{+}, & \bar{n}(m)=\bar{n}_h+m,\\ [-\bar{c}+\bar{\mu}v(m+1,\bar{n}(m))]^{+}, & \bar{n}(m)<\bar{n}_h+m.\end{cases} \tag{EC.16}$$

Note that (EC.16) captures two cases for position $(m+1,\bar{n}(m)+1)$ depending on the strategy of the customer at LP position m, who is then in position $(m,\bar{n}(m))$. If $\bar{n}(m)=\bar{n}_h+m$, then any new HP arrival would balk and hence does not affect the position of the customer in LP position $m+1$. However, for $\bar{n}(m)<\bar{n}_h+m$ we have, similar to (EC.13), that

$$v(m+1,\bar{n}(m)+1)=[-\bar{c}+\bar{\lambda}v(m,\bar{n}(m)+1)+\bar{\mu}v(m+1,\bar{n}(m))]^{+}=[-\bar{c}+\bar{\mu}v(m+1,\bar{n}(m))]^{+},$$

because an HP arrival triggers the abandonment of the customer at LP position m.

Applying Lemma EC.2 with $i=m+1$ and $T=\bar{n}(m)$ to (EC.14)-(EC.16), we have by Part (i) that $v(m+1,n)$ strictly decreases in $n\leq\bar{n}(m+1)$. Furthermore, Part (iii) of Lemma EC.2 applies because $v(m,m)\geq c_l/\mu$ (this follows since $\bar{n}(m)>m$ and by (EC.12) we have $v(m,m)\geq v(m,\bar{n}(m)-1)\geq c_l/\mu$). That is, a customer in LP position $m+1$ joins/stays in the system if and only if her system position $n\leq\bar{n}(m+1)$, where

$$\bar{n}(m+1)=\begin{cases}m+1, & \text{if } v(m+1,m+1)<c_l/\mu,\\ \max\{n\in\{m+2,...,\bar{n}(m)+1\};v(m+1,n-1)\geq c_l/\mu\}, & \text{otherwise.}\end{cases} \tag{EC.17}$$

Second, we prove the relationships (i) and (ii) between the thresholds $\bar{n}(m)$ and $\bar{n}(m+1)$.

(i) $\bar{n}(m+1) = \bar{n}_h + m + 1 \Rightarrow \bar{n}(m) = \bar{n}_h + m$: First note that by (EC.17) the maximum system position of a customer in LP position $m+1$ satisfies $\bar{n}(m+1) \leq \bar{n}(m)+1$; hence, $\bar{n}(m) \geq \bar{n}(m+1)-1 = \bar{n}_h + m$. But we also have $\bar{n}(m) \leq \bar{n}_h + m$ since the system has at most $\bar{n}_h$ HP customers, so $\bar{n}(m) = \bar{n}_h + m$.

(ii) $\bar{n}(m) < \bar{n}_h + m \Rightarrow \bar{n}(m+1) = \bar{n}(m)$: This property follows from the definition of $\bar{n}(m+1)$ in (EC.17) and the following two facts. First, by (EC.12) we have $v(m, \bar{n}(m)-1) \geq c_l/\mu > v(m, \bar{n}(m))$), where the first inequality holds since $\bar{n}(m) > m$ and the second holds since $\bar{n}(m) < \bar{n}_h + m$. Second, we have by Part (ii) of Lemma EC.3 that $\bar{n}(m) < \bar{n}_h + m \Rightarrow v(m+1, n) = v(m, n)$ for $n \in \{m+1, \ldots, \bar{n}(m)+1\}$.

Now consider the following two cases; $\bar{n}(m) = m+1$ and $\bar{n}(m) > m+1$:

If $\bar{n}(m) = m+1$, then $v(m+1, \bar{n}(m)) = v(m, \bar{n}(m)) < c_l/\mu$, where the equality holds by Part (ii) of Lemma EC.3, and the inequality by (EC.12); therefore, $v(m+1, m+1) < c_l/\mu$ and (EC.17) implies $\bar{n}(m+1) = m+1$.

If $\bar{n}(m) > m+1$, then $v(m+1, \bar{n}(m)-1) = v(m, \bar{n}(m)-1) \geq c_l/\mu > v(m+1, \bar{n}(m)) = v(m, \bar{n}(m))$), where the equalities hold by Part (ii) of Lemma EC.3 and the inequalities by (EC.12), so (EC.17) implies $\bar{n}(m+1) = \bar{n}(m)$.

Step 3. Finally, we establish the claimed properties of the thresholds K and L. We have $K \equiv 0$ if $\bar{n}(1) < \bar{n}_h + 1$ and $K \equiv \max\{m \geq 1; \bar{n}(m) = \bar{n}_h + m\}$ otherwise. For the definition of L, first note that since $\bar{n}(K) > K$, the proof of Step 2 for Case 2 ($\bar{n}(m) > m$) applied to $m = K$ establishes $\bar{n}(K+1) \geq K+1$; then let $L \equiv \min\{m \geq K+1; \bar{n}(m) = m\}$.

Further, the fact that $L \leq \bar{n}_h + K$ follows because $\bar{n}(K+1) < \bar{n}_h + K + 1$ by definition of K, and since $K+1 \leq L$ by definition of L, the proof of Step 2 establishes that $\bar{n}(K+1) = \bar{n}(L) = L$.

It remains to show that L, and hence K and the function $\bar{n}(m)$, are finite. For any LP position $m \in \{1, 2, ..., L\}$ that a customer joins, i.e., with $\bar{n}(m) \geq m$, we have from (EC.85) in Part (iii) of Lemma EC.2 that

$$v(m, m) \leq v(m-1, m-1) - c_l/\mu. \tag{EC.18}$$

Expanding the expression for $v(m, m)$, we have for $m \in \{1, 2, ..., L\}$ that

$$v(m, m) = v(0, 0) + \sum_{i=1}^{m} v(i, i) - v(i-1, i-1) \leq R_l - mc_l/\mu, \tag{EC.19}$$

where the inequality holds since $v(0, 0) = R_l$ and by (EC.18). It follows from (EC.19) that $L < R_l\mu/c_l$. $\square$

EC.1.2. Proof of Proposition 2

Proof of Proposition 2. Define a generalized version of the stopping time in (8) as

$$\tau^{s^*} = \inf\{t \geq 0; X(t) = (m', n') \text{ or } X_2(t) > L\},$$

and let

$$T_{(m,n),(m',n')} \equiv \mathbb{E}[\tau^{s^*} | X(0) = (m,n)],$$
$$\upsilon_{(m,n),(m',n')} \equiv \mathbb{P}[X(\tau^{s^*}) = (m', n') | X(0) = (m,n)].$$

In words, $T_{(m,n),(m',n')}$ is the expected time it takes for a customer at position (m,n) to either abandon or reach position (m', n'), and $\upsilon_{(m,n),(m',n')}$ is the probability that starting from (m,n) the customer reaches (m', n') before abandoning.

We begin with characterizing the expected utility of the non-abandoning positions denoted by $u_s(m,n)$. Clearly, $Q(m,n) = 1$ for $m \leq K$ as the customers do not abandon from these positions. To obtain $W(m,n)$ we write it as the expected time it takes for a LP customer starting from (m,n) to reach the next LP position with no HP customers in the system $(m-1, m-1)$, plus the expected time it takes to reach $(0,0)$ starting from $(m-1, m-1)$, that is

$$W(m,n) = T_{(m,n),(m-1,m-1)} + T_{(m-1,m-1),(0,0)}. \tag{EC.20}$$

Note that starting from (m,n) with $m \leq K$ and before the next LP position $(m-1, m-1)$ is reached, the LP position of the customer remains constant while the shifted system position $n - (m-1)$ evolves on $\{0, 1, \cdots, \bar{n}_h + 1\}$ and according to the same transition rates as the birth-death process $\bar{N}(t)$. Therefore, we can express $T_{(m,n),(m-1,m-1)}$ in terms of the expected first passage time $\bar{w}$ given in (14) as

$$T_{(m,n),(m-1.m-1)} = \bar{w}(n - m + 1, \bar{n}_h + 1). \tag{EC.21}$$

Similarly, we can write,

$$\begin{aligned} T_{(m-1,m-1),(0,0)} &= T_{(m-1,m-1),(m-2,m-2)} + \ldots + T_{(1,1),(0,0)} \\ &= (m-1)\bar{w}(1, \bar{n}_h + 1), \end{aligned} \tag{EC.22}$$

which together with (EC.21) and substituting in (EC.20) establishes the claim.

Next, we turn to the expected utility of abandoning positions, i.e., $u_a(m,n;K,L)$. The probability that the customer eventually receives service $Q(m,n)$ is equal to that of reaching the first non-abandoning state (K,K) before abandoning the system, i.e., $\upsilon_{(0,0),(K,K)}$. To obtain $\nu_{(m,n),(K,K)}$ we note that the probability of reaching (K,K) before abandoning is the same as the probability that the system position reaches K before $L+1$. The latter can be obtained noting that the shifted system position $n-K$ evolves on $\{0,1,\ldots,\bar{n}+1\}$ and according to the same law as the birth-death process $N(t)$. It follows that,

$$Q(m,n)=q(n-K,L+1-K). \tag{EC.23}$$

To obtain the expected sojourn time $W(m,n)$ we write it as

$$W(m,n,\bar{n})=T_{(m,n),(K,K)}+\upsilon_{(m,n),(K,K)}T_{(K,K),(0,0)}. \tag{EC.24}$$

The first part of (EC.24) is the expected time it takes for a customer starting from (m,n) to either abandon or reach the first non-abandoning position (K,K). The second part is the expected *additional* time the customer spends in system before departure. This expected additional time is equal to zero if the customer abandons before reaching (K,K), and $T_{(K,K),(0,0)}$ if the customer reaches (K,K) first, which happens with probability $\upsilon_{(m,n),(K,K)}$. The second part is readily available noting that,

$$\upsilon_{(m,n),(K,K)}T_{(K,K),(0,0)}=Q(m,n)K\bar{w}(1,\bar{n}_h+1).$$

To obtain the first part we note that the time it takes for the customer in position (m,n) to reach (K,K) is identical to the time it takes for the system position n to reach K. Again considering the shifted position $n-K$ we have

$$T_{(m,n),(K,K)}=w(n-K,\bar{n}+1-K),$$

which after substituting in (EC.24) gives the result.

Next, we turn to the characterization of the threshold K. We show that the equilibrium threshold K must satisfy (18), i.e.,

$$K=\begin{cases}0, & \text{if } u_s(1,\bar{n}_h+1)<0,\\ \max\{m\geq 1; u_s(m,\bar{n}_h+m)\geq 0\}, & \text{otherwise.}\end{cases} \tag{EC.25}$$

First, note that,

$$u_s(m,\bar{n}_h+m)=R_l-c_l[\bar{w}(\bar{n}_h+1,\bar{n}_h+1)+(m-1)\bar{w}(1,\bar{n}_h+1)], \tag{EC.26}$$

and hence K as defined in (EC.25) is unique and well-defined. Now consider $u_s(m,n)$ in (16):

$$u_s(m,n)=R_l-c_l[\bar{w}(n-m+1,\bar{n}_h+1)+(m-1)\bar{w}(1,\bar{n}_h+1)]. \tag{EC.27}$$

Since $n-m\leq\bar{n}_h$ and by Lemma EC.5 part (i) $\bar{w}(i,J)$ is increasing in i, by Lemma EC.5 part (ii) we have $u_s(m,n)\geq 0$ for all $m\leq K$ and $n\leq\bar{n}_h+m$. Further, by definition $u_s(m,\bar{n}_h+m)<0$ for any $m\geq K+1$. Hence, by joining and not abandoning, customers receive a non-negative expected utility in LP positions $m \le K$ and a negative expected utility for $m > K$. Therefore, the equilibrium threshold K must satisfy (EC.25).

Finally, we turn to the characterization of the equilibrium threshold L. First, consider the case with $K > 0$. Then, by definition of threshold K we have $u_s(K, \bar{n}_h + K) \ge 0$. We next claim that this implies $u_a(K+1, \bar{n}_h + K; K, \bar{n}_h + K) \ge 0$. That is, if the customer prefers to join/stay in the non-abandoning position $(K, \bar{n}_h + K)$, she would also prefer to join/stay in position $(K+1, \bar{n}_h + K)$ and abandon if her system position reaches $\bar{n}_h + K + 1$. It follows that $L \ge \bar{n}_h + K$. To prove the claim, first note that $u_s(K, \bar{n}_h + K) \ge 0$ implies,

$$R_l - c_l \left[\bar{w}(\bar{n}_h + 1, \bar{n}_h + 1) + (K-1)\bar{w}(1, \bar{n}_h + 1)\right] \ge 0. \tag{EC.28}$$

It follows that,

$$\begin{aligned}
u_a(K+1, \bar{n}_h + K; K, \bar{n}_h + K) &= R_l q(\bar{n}_h, \bar{n}_h + 1) - c_l \left(w(\bar{n}_h, \bar{n}_h + 1) + q(\bar{n}_h, \bar{n}_h + 1) K \bar{w}(1, \bar{n}_h + 1)\right) && \text{(EC.29)}\\
&\ge c_l \left[q(\bar{n}_h, \bar{n}_h + 1)\left(\bar{w}(\bar{n}_h + 1, \bar{n}_h + 1) - \bar{w}(1, \bar{n}_h + 1)\right) - w(\bar{n}_h, \bar{n}_h + 1)\right] && \text{(EC.30)}\\
&= 0. && \text{(EC.31)}
\end{aligned}$$

To verify (EC.31), note that using the definitions of q, w, and $\bar{w}$ from (12)–(14),

$$\begin{aligned}
q(\bar{n}_h, \bar{n}_h + 1)\left(\bar{w}(\bar{n}_h + 1, \bar{n}_h + 1) - \bar{w}(1, \bar{n}_h + 1)\right) &= \frac{1-\rho_h}{1-\rho_h^{\bar{n}_h+1}} \left(\frac{(\mu-\lambda_h)\bar{n}_h + \lambda_h(\rho_h^{\bar{n}_h+1} - 1)}{(\mu-\lambda_h)^2}\right) && \text{(EC.32)}\\
&= \frac{(1-\rho_h)\bar{n}_h - \rho_h(1-\rho_h^{\bar{n}_h})}{(\mu-\lambda_h)(1-\rho_h^{\bar{n}_h+1})}, && \text{(EC.33)}
\end{aligned}$$

and,

$$\begin{aligned}
w(\bar{n}_h, \bar{n}_h + 1) &= \frac{\bar{n}_h(1-\rho_h^{\bar{n}_h+1}) - (\bar{n}_h + 1)(\rho_h - \rho_h^{\bar{n}_h+1})}{(\mu-\lambda_h)(1-\rho_h^{\bar{n}_h+1})} && \text{(EC.34)}\\
&= \frac{(1-\rho_h)\bar{n}_h - \rho_h(1-\rho_h^{\bar{n}_h})}{(\mu-\lambda_h)(1-\rho_h^{\bar{n}_h+1})}, && \text{(EC.35)}
\end{aligned}$$

which establishes (EC.31).

The above implies the following:

$$u_a(K+1, \bar{n}_h + K; K, \bar{n}_h + K) = q(\bar{n}_h, \bar{n}_h + 1) u_s(K, \bar{n}_h + K). \tag{EC.36}$$

To see this, note that we have by definition,

$$\begin{aligned}
u_a(K+1, \bar{n}_h + K; K, \bar{n}_h + K) &= R_l q(\bar{n}_h, \bar{n}_h + 1) - c_l \left(w(\bar{n}_h, \bar{n}_h + 1) + c_l q(\bar{n}_h, \bar{n}_h + 1) K \bar{w}(1, \bar{n}_h + 1)\right)\\
&= q(\bar{n}_h, \bar{n}_h + 1)(R_l - c_l K \bar{w}(1, \bar{n}_h + 1)) - c_l w(\bar{n}_h, \bar{n}_h + 1).
\end{aligned}$$

Substituting $w(\bar{n}_h, \bar{n}_h+1) = q(\bar{n}_h, \bar{n}_h+1)\left(\bar{w}(\bar{n}_h+1, \bar{n}_h+1) - \bar{w}(1, \bar{n}_h+1)\right)$ yields:

$$u_a(K+1, \bar{n}_h+K; K, \bar{n}_h+K) = q(\bar{n}_h, \bar{n}_h+1)[R_l - c_l K\bar{w}(1, \bar{n}_h+1) - c_l\left(\bar{w}(\bar{n}_h+1, \bar{n}_h+1) - \bar{w}(1, \bar{n}_h+1)\right)].$$

The term in brackets equals $u_s(K, \bar{n}_h+K)$:

$$\begin{aligned} u_s(K, \bar{n}_h+K) &= R_l - c_l\left[\bar{w}(\bar{n}_h+1, \bar{n}_h+1) + (K-1)\bar{w}(1, \bar{n}_h+1)\right] \\ &= R_l - c_l K\bar{w}(1, \bar{n}_h+1) - c_l[\bar{w}(\bar{n}_h+1, \bar{n}_h+1) - \bar{w}(1, \bar{n}_h+1)]. \end{aligned}$$

Next, from Proposition 1 we know that $L = \bar{n}(K+1) = \ldots = \bar{n}(L)$ and by definition $\bar{n}(K+1) \leq \bar{n}_h + K$. Hence, we also have $L \leq \bar{n}_h + K$ which together with $L \geq \bar{n}_h + K$ establishes the claim. We proceed by characterizing the threshold L for the case with $K = 0$, i.e., where all positions are abandoning positions. We claim that the equilibrium threshold must satisfy,

$$L = \max\left\{n \geq 1; u_a(n, n; 0, n) \geq 0\right\}. \tag{EC.37}$$

From (17) we have,

$$u_a(n, n; 0, n) = R_l q(n, n+1) - c_l\left(w(n, n+1)\right).$$

In Lemma EC.5 Part (iii) we show that $u_a(n, n; 0, n)$ is decreasing in n when positive and once it becomes negative, it remains negative. Hence the threshold is well-defined. In addition, in Lemma EC.5 Part (iv) we show that $u(m, n, L)$ is decreasing in n for $n \leq L$ and independent of m. Therefore, $u(m, n, L) \geq 0$ for all positions with $m \geq 1$ and $m \leq n \leq L$. It follows that, when the threshold L is determined according to (EC.37), the expected utility of stay/join is non-negative for all positions with $n \leq L$ and maximized, i.e., $u_a(m, n; 0, L) > u_a(m, n; 0, L')$ for $n \leq L' < L$. Finally, since for any threshold larger than L there would be at least one position with negative utility, the unique equilibrium threshold must satisfy (EC.37). The proof is complete. □

EC.1.3. Proof of Proposition 3

Proof of Proposition 3. Recall that $\lambda \equiv \lambda_l + \lambda_h$, $\rho = \lambda/\mu$, and $\rho_h = \lambda_h/\mu$. Also, customers abandon from all positions, i.e., we have $\bar{n}_h > L$. Since there is a finite threshold on the maximum number of customers in the system, the continuous-time Markov chain that keeps track of the total number of customers in system is positive recurrent and the steady-state probabilities, denoted by $\{\pi_x\}$, satisfy the balance equations,

$$\lambda\pi_x = \mu\pi_{x+1}, \quad x \in \{0, 1, ..., m_b - 1\}, \tag{EC.38}$$

$$\lambda_h\pi_x = \mu\pi_{x+1}, \quad i \in \{m_b, ..., L-1\}. \tag{EC.39}$$

To see this, note that for $0 \leq x \leq m_b - 1$ high- and LP customers arrive at total rate λ and services are completed at rate μ. For $m_b \leq x \leq L-1$, only HP customers join the system and hence upward transitions occur at rate λ_h. From the balance equations, it follows that for $0 \le x \le L$ the steady-state probabilities satisfy,

$$\pi_x = \begin{cases} \pi_0 \rho^x, & x \in \{0, 1, ..., m_b\}, \\ \pi_0 \rho_h^{x-m_b} \rho^{m_b}, & x \in \{m_b + 1, ..., L\}. \end{cases} \tag{EC.40}$$

Next, consider the states $L + 1 \le x \le \bar{n}_h$. Then all the customers in the system are HP. Due to the preemptive priority, the number of HP customers in the system is independent of that of LP customers. Therefore, the steady-state probabilities are the same as that of an independent single-class $M/M/1$ queue with arrival rate λ_h and maximum number of customers in system $\bar{n}_h$. Therefore, we have

$$\pi_x = \frac{(1-\rho_h)\rho_h^x}{1-\rho_h^{\bar{n}_h+1}}, \quad L+1 \le x \le \bar{n}_h, \tag{EC.41}$$

for $\rho_h \ne 1$. The expression for $\rho_h = 1$ is obtained by taking the limit the limit $\rho_h \to 1$ and applying L'Hopital's rule. Finally, to obtain π_0, we use $\sum_{x=0}^{\bar{n}_h} \pi_x = 1$ which implies after simplification that,

$$\pi_0 = \left(1 - \frac{\rho_h^{\bar{n}_a+1} - \rho_h^{\bar{n}_h+1}}{1-\rho_h^{\bar{n}_h+1}}\right) / \left(\frac{1-\rho^{\bar{n}_b+1}}{1-\rho} + \rho^{\bar{n}_b} \frac{\rho_h - \rho_h^{\bar{n}_a - \bar{n}_b + 1}}{1-\rho_h}\right), \tag{EC.42}$$

when $\rho_h \ne 1$. For $\rho_h = 1$, taking the limit $\rho_h \to 1$ in the above and using L'Hopital's rule results in the claimed expression. In the limiting case where $\bar{n}_h = \infty$, i.e., HP customers do not balk, assuming that $\lambda_h < \mu$, the above simplifies to,

$$\pi_0 = \left(1 - \rho_h^{\bar{n}_a+1}\right) / \left(\frac{1-\rho^{\bar{n}_b+1}}{1-\rho} + \rho^{\bar{n}_b} \frac{\rho_h - \rho_h^{\bar{n}_a - \bar{n}_b + 1}}{1-\rho_h}\right). \tag{EC.43}$$

EC.1.4. Proofs for the Results in Section 6.1–6.2

Proof of Corollary 2. We show that the definitions of the joining and staying sets in (32)-(33) are equivalent to the characterizations of these sets in (37)-(38). This holds because the equilibrium expected utility function, $u(m, n; K, L; P_s)$, given in (15)-(17), satisfies the following three properties:

(i) *The function $u(m, n; K, L; P_s)$ is non-increasing in $m \le n$*: This follows because $v(m, n) \ge v(m+1, n)$ by Lemma EC.3.(i), where $v(m, n)$ is the equilibrium expected utility for $P_s = 0$, i.e., $v(m, n) = u(m, n; K, L; 0)$. The equilibrium expected utility function for non-zero service fee $P_s < R_s - c_l/\mu$ and reward R_l is equivalent to its counterpart for service fee $P_s = 0$ and *net* reward $R_l - P_s$. Therefore, writing $v(m, n; P_s)$ for the equilibrium expected utility for arbitrary $P_s < R_s - c_l/\mu$, Lemma EC.3.(i) implies that $v(m, n; P_s) \ge v(m+1, n; P_s)$, where $v(m, n; P_s) = u(m, n; K, L; P_s)$.

(ii) *The function $u(m, n; K, L; P_s)$ is decreasing in $n \ge m$*: This follows because (i) by (15) we have $u(m, n; K, L; P_s) = u_s(m, n; P_s)$ for $m \le K$ and $u(m, n; K, L; P_s) = u_a(m, n; K, L; P_s)$ for $m > K$; (ii) the function $u_s(m, n; P_s)$ decreases in $n \ge m$ by Lemma EC.5.(i) and (16); and (iii) the function $u_a(m, n; K, L; P_s)$ decreases in $n \ge m$ by Lemma EC.5.(iv) and (17).

(iii) For $P_e = 0$, we have by Proposition 2

$$u(m,n;K,L;P_S) \geq 0 \quad \Leftrightarrow \quad m \leq L \text{ and } m \leq n \leq \bar{n}(m) = min(m+\bar{n}_h, L). \tag{EC.44}$$

Now consider the *joining set* defined in (32): $S_J = \{(m,n) : 1 \leq m \leq n, u(m,n;K,L;P_S) \geq P_e\}$. To establish equivalence with (37), we show that

$$u(m,n;K,L;P_S) \geq P_e \quad \Leftrightarrow \quad (m,n) \in S_J(m_b, \mathbf{n}_b) = \{(m,n) : 1 \leq m \leq m_b, m \leq n \leq n_b(m)\},$$

where m_b and $n_b(m)$ are defined in (34)-(35). This equivalence follows from two sets of facts. First, properties (i)-(iii) imply that

$$u(m,n;K,L;P_S) \geq P_e \Leftrightarrow m \leq m_b \text{ and } m \leq n \leq n_b(m), \tag{EC.45}$$

where $m_b \leq L$ and $n_b(m) \leq \bar{n}(m)$ are uniquely defined by:

$$m_b = \max\{m \in \{1,2,..,L\} : u(m,m;K,L;P_s) \geq P_e\}, \tag{EC.46}$$

$$n_b(m) = \max\{n \in \{m, m+1, .., \bar{n}(m)\} : u(m,n;K,L;P_s) \geq P_e\}. \tag{EC.47}$$

Second, the definitions in (EC.46)-(EC.47) are equivalent to those in (34)-(35), because by (15) we have $u(m,n;K,L;P_s) = u_s(m,n;P_s)$ for $m \leq K$; and $u(m,n;K,L;P_s) = u_a(m,n;K,L;P_s)$ for $m > K$. Furthermore, when (34) yields $m_b > K$, note that $n_b(m) = m_b$ for $m \in \{K+1, ..., m_b\}$ as shown in (35); this follows because the definition of u_a in (16) implies that $u_a(m, m_b; K, L; P_s) = u_a(m_b, m_b; K, L; P_s)$ for $m \in \{K+1, ..., m_b\}$.

Next consider the *staying set* defined in (33): $S_S = \{(m,n) : 1 \leq m \leq n, u(m,m;K,L;P_S) \geq P_e \wedge u(m,n;K,L;P_S) \geq 0\}$. The equivalence with (38) follows from two facts. First, the proof for the joining set establishes that $u(m,m;K,L;P_S) \geq P_e \Leftrightarrow m \leq m_b$ where m_b is defined in (34). Second, for $m \leq m_b$, we have $u(m,n;K,L;P_S) \geq 0 \Leftrightarrow m \leq \bar{n}(m)$ by Proposition 2 (property (iii) above). □

Proof of Corollary 3. The results are immediate from Proposition 2, from the expressions for K, L in (30)-(31) and from the characterizations in Corollary 2 of the thresholds $\mathbf{n}_b$ and $\bar{\mathbf{n}}$, and the equilibrium joining set $S_J(m_b, \mathbf{n}_b)$ and staying set $S_S(m_b, \bar{\mathbf{n}})$. □

Proof of Proposition 4. To begin, we specialize (30)-(31), (34)-(35), and (40)-(41) in problem (42) for $\bar{n}_h = 1$. The expressions for the thresholds $K(P_s)$ and $L(P_s)$ in (30)-(31) specialize as follows:

$$K = \begin{cases} 1 + \left\lfloor \frac{R_l - P_s - c_l \bar{w}(2,2)}{c_l \bar{w}(1,2)} \right\rfloor, & R_l - c_l \bar{w}(2,2) \geq P_s, \\ 0, & R_l - c_l \bar{w}(2,2) < P_s, \end{cases} \tag{EC.48}$$

$$L = K + 1. \tag{EC.49}$$

The expressions for the balking thresholds $\mathbf{n}_b$ in (34)-(35) specialize as follows:

Fix $P_s < R_l - c_l/\mu$ and $P_e \le u(1,1;K,L;P_s)$, where K and L satisfy (EC.48)-(EC.49). Then

$$m_b = \begin{cases} \max\{1 \le m \le K : u_s(m,m;P_s) \ge P_e\}, & u_a(K+1,K+1;K,K+1;P_s) < P_e, \\ K+1, & u_a(K+1,K+1;K,K+1;P_s) \ge P_e, \end{cases} \tag{EC.50}$$

$$n_b(m) = \begin{cases} m+1, & 1 \le m < m_b, \\ \max\{n \in \{m,m+1\} : u_s(m,n;P_s) \ge P_e\}, & m = m_b \le K, \\ m_b, & m = m_b = K+1. \end{cases} \tag{EC.51}$$

The expression for the joining probability in (40) specializes as follows:

$$\begin{aligned} q_J(m_b,\mathbf{n}_b,L) &= \sum_{m=0}^{m_b-1} \sum_{n=m}^{n_b(m+1)-1} \pi_{m,n}(m_b,L) \\ &= \begin{cases} \sum_{m=0}^{m_b-2}\sum_{n=m}^{m+1} \pi_{m,n}(m_b,L) + \pi_{m_b-1,m_b-1}(m_b,L), & m_b \le K, \quad n_b(m_b) = m_b, \\ \sum_{m=0}^{m_b-1}\sum_{n=m}^{m+1} \pi_{m,n}(m_b,L), & m_b \le K, \quad n_b(m_b) = m_b+1, \\ \sum_{m=0}^{K-1}\sum_{n=m}^{m+1} \pi_{m,n}(m_b,L) + \pi_{K,K}(m_b,L), & m_b = K+1 = L = n_b(m_b). \end{cases} \end{aligned} \tag{EC.52}$$

LP customers balk if they find the system in state $(m_b, n_b(m_b))$ upon arrival. If $n_b(m_b) = m_b$, the LP customers' largest state is $(m_b - 1, m_b - 1)$, corresponding to the first and last cases of (EC.52). If $n_b(m_b) = m_b + 1$, the LP customers' largest joining state is $(m_b - 1, m_b)$, the second case of (EC.52).

The expression for the service probability in (41) specializes as follows:

$$\begin{aligned} q_S(m_b,\mathbf{n}_b,K,L) &= \sum_{m=0}^{m_b-1} \sum_{n=m}^{n_b(m+1)-1} \pi_{m,n}(m_b,L)(\mathbf{1}_{\{m<K\}} + \mathbf{1}_{\{m\ge K\}} q(n+1-K;L+1-K)) \\ &= \begin{cases} q_J(m_b,\mathbf{n}_b,L), & m_b \le K, \\ \sum_{m=0}^{K-1}\sum_{n=m}^{m+1} \pi_{m,n}(m_b,L) + \pi_{K,K}(m_b,L)q(1,2), & m_b = K+1 = L, \end{cases} \end{aligned} \tag{EC.53}$$

where the conditional service probability $q(1,2)$ satisfies (12). LP customers who join when $K-1$ or fewer LP customers are in the system get served for sure; LP customers who find the system in state (K,K) join into position $(L,L) = (K+1,K+1)$ and get served with probability $q(1,2) < 1$.

Therefore, for $\bar{n}_h = 1$ the revenue-maximization problem (42) is as follows:

$$\max_{\{P_e \ge 0, P_s \ge 0\}} \lambda_l P_e q_J(m_b,\mathbf{n}_b,L) + \lambda_l P_s q_S(m_b,\mathbf{n}_b,K,L) \quad s.t. \quad \text{(EC.48)} - \text{(EC.53)}. \tag{EC.54}$$

To prove the result, we need to show that problem (EC.54) is equivalent to the optimization problem (43)-(47), which we restate here for convenience:

$$\max_{\{K \le m_b \le K+1\}} \lambda_l P_e q_J(m_b,\mathbf{n}_b,L) + \lambda_l P_s q_S(m_b,\mathbf{n}_b,K,L) \tag{EC.55}$$

s.t.

$$L = K+1, \tag{EC.56}$$

$$n_b(m) = \begin{cases} m+1, & m < m_b, \\ m_b, & m = m_b. \end{cases} \tag{EC.57}$$

$$P_s = 0, \quad P_e = R_l - c_l m_b \bar{w}(1,2), \qquad \text{if } m_b = n_b(m_b) = K, \tag{EC.58}$$

$$P_e = 0, \quad P_s = R_l - c_l(m_b - 1)\bar{w}(1,2) - c_l/\mu, \quad \text{if } m_b = n_b(m_b) = K+1. \tag{EC.59}$$

This equivalence follows from two steps. We prove below that problem (EC.54) is equivalent to the following optimization problem:

$$\max_{\{K \le m_b \le n_b(m_b) \le K+1\}} \lambda_l P_e q_J(m_b, \mathbf{n}_b, L) + \lambda_l P_s q_S(m_b, \mathbf{n}_b, K, L) \tag{EC.60}$$

s.t.

$$L = K+1, \tag{EC.61}$$

$$n_b(m) = m+1, \quad m < m_b \tag{EC.62}$$

$$P_s = 0, \quad P_e = R_l - c_l m_b \bar{w}(1,2), \qquad \text{if } m_b = n_b(m_b) = K, \tag{EC.63}$$

$$P_e + P_s = R_l - c_l K \bar{w}(1,2) - c_l/\mu, \qquad \text{if } m_b = K, \quad n_b(m_b) = K+1, \tag{EC.64}$$

$$P_e = 0, \quad P_s = R_l - c_l(m_b - 1)\bar{w}(1,2) - c_l/\mu, \quad \text{if } m_b = n_b(m_b) = K+1. \tag{EC.65}$$

Second, we note that this problem and problem (EC.55)-(EC.59) have the same solution, for the following reasons: The only difference between the two problems is the case given in (EC.64), where the LP balking threshold $m_b = K$ and the system balking threshold $n_b(m_b) = K+1$. However, note that the revenue attained in case (EC.64) is dominated by the revenue attained in case (EC.65): This follows because (i) both cases charge the same total price (equal to $R_l - c_l K \bar{w}(1,2) - c_l/\mu$) and (ii) case (EC.65) attains a higher service probability since it yields a higher LP balking threshold ($m_b = K+1$) compared to case (EC.64) (where $m_b = K$) and both cases yield the same system balking threshold ($n_b(m_b) = K+1$).

Proof that (EC.54) is equivalent to problem (EC.60)-(EC.65):

To prove this equivalence we show that, for any solution of (EC.54), there exists a solution of problem (EC.60)-(EC.65) that weakly increases revenue and is feasible for problem (EC.54).
Fix arbitrary feasible prices $(\hat{P}_e, \hat{P}_s)$. By (EC.48)-(EC.51) these prices uniquely determine the thresholds $K, L = K+1, m_b$ and $\mathbf{n}_b$. In turn, by (EC.52)-(EC.53) the thresholds K, m_b and $n_b(m_b)$ uniquely

determine the joining and service probabilities q_J and q_S, respectively. Let $(\hat{K}, \hat{m}_b, \hat{\mathbf{n}}_b)$ denote the thresholds, and $\hat{q}_J$ and $\hat{q}_S$ denote the probabilities induced by $(\hat{P}_e, \hat{P}_s)$.

To prove the equivalence we show the following: Let P_e^* and P_s^* denote the prices that satisfy (EC.63)-(EC.65) with $m_b = \hat{m}_b$, $n_b(m_b) = \hat{n}_b(\hat{m}_b)$; and $K = min(\hat{m}_b, \hat{K})$, that is, set $K = \hat{K}$ if $\hat{m}_b = \hat{K} + 1$ and $K = \hat{m}_b$ if $\hat{m}_b \leq \hat{K}$. We show that the prices (P_e^*, P_s^*) are revenue-optimal under the probabilities $\hat{q}_J$ and $\hat{q}_S$ induced by the fees $(\hat{P}_e, \hat{P}_s)$.

Specifically, let $\Pi(P_e, P_s) = \lambda_l P_e \hat{q}_J + \lambda_l P_s \hat{q}_S$ denote the revenue (EC.54) as a function of prices (P_e, P_s), under the fixed probabilities $\hat{q}_J$ and $\hat{q}_S$. Formally, we show:

$$\begin{aligned}
&\Pi(P_e^*, P_s^*) \geq \Pi(P_e, P_s) \quad \forall (P_e, P_s) \text{ s.t.} \\
&\qquad\qquad q_J(m_b(P_e, P_s), \mathbf{n}_b(P_e, P_s)) = \hat{q}_J, \text{ and } q_S(m_b(P_e, P_s), \mathbf{n}_b(P_e, P_s)) = \hat{q}_S.
\end{aligned}$$

By (EC.63)-(EC.65), the thresholds $(\hat{K}, \hat{m}_b, \hat{\mathbf{n}}_b)$ satisfy one of three cases: Case (i) in (EC.63): $\hat{m}_b = \hat{n}_b(\hat{m}_b) \leq \hat{K}$; case (ii) in (EC.64): $\hat{m}_b \leq \hat{K}$ and $\hat{n}_b(\hat{m}_b) = \hat{m}_b + 1$, and case (iii) in (EC.65): $\hat{m}_b = \hat{n}_b(\hat{m}_b) = \hat{K} + 1$. We first consider cases (i) and (ii) jointly and then prove the claim for case (iii).

1. Cases (i) in (EC.63) and (ii) in (EC.64): $\hat{m}_b \leq \hat{K}$. This case implies that $\hat{K} > 0$, and no LP customer who joins later abandons (they only join into LP positions $m \leq \hat{K}$). Therefore, the joining and service probabilities induced by the prices $(\hat{P}_e, \hat{P}_s)$ are equal, see also (EC.52)-(EC.53). Therefore, the revenue function $\Pi(P_e, P_s)$ satisfies:

$$\Pi(P_e, P_s) = \lambda_l (P_e + P_s) \hat{q}_J = \lambda_l (P_e + P_s) \hat{q}_S.$$

That is, the revenue only depends on the total fee. Furthermore, (EC.52)-(EC.53) imply that for $\hat{m}_b \leq \hat{K}$ the joining and service probabilities are independent of $\hat{K} \geq \hat{m}_b$; hence, the revenue is also independent of $\hat{K} \geq \hat{m}_b$. Without loss of generality, we therefore set $K = \hat{m}_b$. Next, we show the optimal fees satisfy (EC.63)-(EC.64) with $m_b = K = \hat{m}_b$ and $n_b(m_b) = \hat{n}_b(\hat{m}_b)$.
Holding $m_b = K = \hat{m}_b$ and $n_b(m_b) = \hat{n}_b(\hat{m}_b)$ constant, the optimal fees P_e^* and P_s^* solve:

$$\begin{aligned}
P_e^* + P_s^* &= \max(P_e + P_s) \\
&\text{s.t.} \\
u_s(\hat{m}_b, \hat{m}_b + 1; P_s) &\geq 0, && \text{(EC.66)} \\
u_s(\hat{m}_b, n_b(\hat{m}_b); P_s) &\geq P_e > \max(u_s(\hat{m}_b + 1, \hat{m}_b + 1; P_s), u_a(\hat{m}_b + 1; \hat{m}_b, \hat{m}_b + 1; P_s)). && \text{(EC.67)}
\end{aligned}$$

The constraints follow from (EC.48)-(EC.51). Constraint (EC.66) ensures that customers do not abandon in LP position $\hat{m}_b$. Constraint (EC.67) ensures by the first inequality that customers join into (up to) position $(\hat{m}_b, \hat{n}_b(\hat{m}_b))$, and by the second inequality that they do not join into LP position $\hat{m}_b + 1$.

First note that the constraint (EC.67) defines a non-empty feasibility interval for P_e; that is the upper bound on P_e exceeds the lower bound on P_e. This holds because we have

$$u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); P_s) \geq u_s(\hat{m}_b, \hat{m}_b + 1; P_s) > \max(u_s(\hat{m}_b + 1, \hat{m}_b + 1; P_s), u_a(\hat{m}_b + 1, \hat{m}_b + 1; \hat{m}_b, \hat{m}_b + 1; P_s).$$

The first inequality holds because $\hat{n}_b(\hat{m}_b) \leq \hat{m}_b + 1$ by (EC.51) and by Lemma EC.5.(ii); the second inequality holds because $u_s(\hat{m}_b, \hat{m}_b + 1; P_s) > u_s(\hat{m}_b + 1, \hat{m}_b + 1; P_s)$, and because (EC.36) implies

$$u_a(\hat{m}_b + 1, \hat{m}_b + 1; \hat{m}_b, \hat{m}_b + 1; P_s) = q(1,2) u_s(\hat{m}_b, \hat{m}_b + 1; P_s),$$

where $q(1,2) < 1$. Specifically, the above equality follows by setting $K = \hat{m}_b$ and $\bar{n}_h = 1$ in (EC.36), which:

$$u_a(K+1, K+\bar{n}_h; K, K+\bar{n}_h) = q(\bar{n}_h, \bar{n}_h + 1) u_s(K, K+\bar{n}_h).$$

Next, note that the first inequality in (EC.67) must be binding at optimality. That is, the optimal entrance fee must satisfy $P_e^*(P_s) = u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); P_s)$. Equivalently, we have $P_e^*(P_s) + P_s = u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0)$, because $u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); P_s) = u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0) - P_s$ by definition of u_s.

Therefore, any optimal solution must satisfy the following conditions:

$$P_s^* \leq u_s(\hat{m}_b, \hat{m}_b + 1; 0), \tag{EC.68}$$

$$P_e^* + P_s^* = u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0), \tag{EC.69}$$

where (EC.68) is equivalent to (EC.66), and (EC.69) ensures that (EC.67) holds. Together, these conditions imply

$$P_e^* = u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0) - P_s^* \geq u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0) - u_s(\hat{m}_b, \hat{m}_b + 1; 0), \tag{EC.70}$$

where the equality follows from (EC.69) and the inequality follows from (EC.68).

Now consider the two cases for $\hat{n}_b(\hat{m}_b)$:

Case (i) in (EC.63): $\hat{n}_b(\hat{m}_b) = \hat{m}_b$, so that $u_s(\hat{m}_b, \hat{n}_b(\hat{m}_b); 0) - u_s(\hat{m}_b, \hat{m}_b + 1; 0) > 0$, where the inequality follows by Lemma EC.5.(ii). Therefore, (EC.70) implies that $P_e^* > 0$, and that any solution must satisfy

$$P_e^* = u_s(\hat{m}_b, \hat{m}_b; 0) - P_s^* > 0, \quad P_s^* \in [0, u_s(\hat{m}_b, \hat{m}_b + 1; 0)].$$

The *unique optimal single-fee solution* has $P_s^* = 0$ and $P_e^* = u_s(\hat{m}_b, \hat{m}_b; 0)$, as claimed in (EC.63), where $u_s(\hat{m}_b, \hat{m}_b; 0) = R_l - c_l \hat{m}_b \bar{w}(1,2)$ by definition of u_s.

Case (ii) in (EC.64): $\hat{n}_b(\hat{m}_b) = \hat{m}_b + 1$, so (EC.70) reduces to $P_e^* = u_s(\hat{m}_b, \hat{m}_b + 1; 0) - P_s^* \geq 0$, and any solution must satisfy

$$P_e^* + P_s^* = u_s(\hat{m}_b, \hat{m}_b + 1; 0), \quad P_s^* \in [0, u_s(\hat{m}_b, \hat{m}_b + 1; 0)],$$

as claimed in (EC.64), where $u_s(\hat{m}_b, \hat{m}_b + 1; 0) = R_l - c_l \hat{m}_b \bar{w}(1,2) - c_l/\mu$ by definition of u_s.

Therefore, there are *two optimal single-fee solutions*, $P_e^* = u_s(\hat{m}_b, \hat{m}_b + 1; 0)$ or $P_s^* = u_s(\hat{m}_b, \hat{m}_b + 1; 0)$.

2. Case (iii) in (EC.65): $\hat{m}_b = \hat{n}_b(\hat{m}_b) = \hat{K} + 1$. By (EC.48)-(EC.51) this case arises for any fees (P_e, P_s) that satisfy

$$u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; P_s) \geq P_e \geq 0, \tag{EC.71}$$

$$\begin{cases} u_s(1,2;P_s) < 0, & \hat{K} = 0, \\ u_s(\hat{K}, \hat{K}+1; P_s) \geq 0, & \hat{K} > 0. \end{cases} \tag{EC.72}$$

The constraint (EC.71) ensures that the entrance fee is (i) non-negative and (ii) consistent with the largest joining position $(\hat{K}+1, \hat{K}+1)$; the constraint (EC.72) ensures that P_s is consistent with $\hat{K}$, the largest non-abandoning LP position.

We show the optimal fees satisfy (EC.65) with $m_b = \hat{m}_b$, $n_b(m_b) = \hat{n}_b(\hat{m}_b)$ and $K = \hat{K}$: $P_e^* = 0$, and $u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; P_s^*) = 0$, i.e., $P_s^* = R_l - c_l \overline{w}(1,2)\hat{K} - c_l w(1,2)/q(1,2)$ by definition of u_a.

To this end, we first characterize the optimal entrance fee P_e for fixed service fee P_s and then show that the resulting revenue function strictly increases in P_s.

First, note that, holding constant P_s, we have from (EC.71)-(EC.72) that only (EC.71) depends on the entrance fee P_e, and the revenue $\Pi(P_e, P_s) = \lambda_l P_e \hat{q}_J + \lambda_l P_s \hat{q}_S$ strictly increases in P_e. Therefore, maximizing revenue requires $P_e^*(P_s) = u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; \hat{P}_s)$. Substituting $P_e^*(P_s) = u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; P_s)$ into the revenue function $\Pi(P_e, P_s)$, and noting $u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; P_s) = q(1,2)(R_l - P_s - c_l \overline{w}(1,2)\hat{K}) - c_l w(1,2)$, the revenue satisfies

$$\begin{aligned} \Pi(P_e^*(P_s), P_s) &= \lambda_l \hat{q}_J P_e^*(P_s) + \lambda_l \hat{q}_S P_s \\ &= \lambda_l \hat{q}_J [q(1,2)(R_l - c_l \overline{w}(1,2)\hat{K}) - c_l w(1,2)] + \lambda_l [\hat{q}_S - \hat{q}_J \cdot q(1,2)] P_s. \end{aligned}$$

Note that $\hat{q}_S - \hat{q}_J \cdot q(1,2) > 0$: This follows because from (EC.52)-(EC.53), we have for $m_b = K+1$ that $q_S(m_b, \mathbf{n}_b) - q(1,2) q_J(m_b, \mathbf{n}_b) > 0$. Therefore the revenue $\Pi(P_e^*(P_s), P_s)$ strictly increases in P_s so that maximizing revenue requires setting P_s to be the largest feasible service fee. Specifically, the optimal service fee P_s^* solves

$$P_s^* = \max P_s \quad \text{s.t. } \{P_e^*(P_s) = u_a(\hat{K}+1, \hat{K}+1; \hat{K}, \hat{K}+1; P_s) \geq 0, \quad \text{(EC.72)}\}. \tag{EC.73}$$

It is easy to verify that the optimal fee P_s^* satisfies $u_a(\hat{K}+1,\hat{K}+1;\hat{K},\hat{K}+1;P_s^*)=0$, which yields the expression for P_s^* in (EC.65). This holds because (i) the functions $u_a(\hat{K}+1,\hat{K}+1;\hat{K},\hat{K}+1;P_s)$, $u_s(1,2;P_s)$, and $u_s(\hat{K},\hat{K}+1;P_s)$ are decreasing in P_s; and (ii) if $\hat{K}>0$ then $u_a(\hat{K}+1,\hat{K}+1;\hat{K},\hat{K}+1;P_s)=q(1,2)u_s(\hat{K},\hat{K}+1;P_s)$ where $q(1,2)>0$. $\square$

Proof of Proposition 5. Consider the constraint (48). By Lemma EC.5(iii), the function $u_a(n,n;0,n;P_s)$, given in (49), is strictly decreasing in n so long as the function is positive. Therefore, the abandonment threshold L induced by the service fee P_s must satisfy

$$R_l - c_l\frac{w(L+1,L+2)}{q(L+1,L+2)} < P_s \leq R_l - c_l\frac{w(L,L+1)}{q(L,L+1)}. \tag{EC.74}$$

Similarly, from the constraint (50), given L the balking threshold m_b must satisfy

$$(R_l-P_s)q(m_b+1,L+1)-c_lw(m_b+1,L+1) < P_e \leq (R_l-P_s)q(m_b,L+1)-c_lw(m_b,L+1). \tag{EC.75}$$

Now note from (53) that for fixed thresholds m_b, L and service fee P_s, the revenue increases in P_e whereas the inequalities in (EC.74) are independent of P_e. Therefore, given any P_s and L that satisfy (EC.74), the revenue-maximizing entrance fee is the largest value of P_e that satisfies (EC.75):

$$P_e = (R_l-P_s)q(m_b,L+1)-c_lw(m_b,L+1). \tag{EC.76}$$

Substituting P_e from (EC.76) into the revenue function (53) we get

$$\begin{aligned}\Pi &= \lambda_l(P_eq_J(m_b,L)+\lambda_lP_sq_S(m_b,L)\\ &= \lambda_l\left[R_lq(m_b,L+1)-c_lw(m_b,L+1)\right]q_J(m_b,L)+\lambda_lP_s\left[q_S(m_b,L)-q_J(m_b,L)q(m_b,L+1)\right].\end{aligned}$$

Now observe that, for fixed thresholds m_b, L we have

$$q_S(m_b,L)=\sum_{i=0}^{m_b-1}\pi_iq(i+1,L+1)\geq\sum_{i=0}^{m_b-1}\pi_iq(m_b,L+1)=q_J(m_b,L)q(m_b,L+1),$$

where the inequality follows from the fact that $q(i,L+1)$ is decreasing in i. It follows that the revenue increases in P_s, and thus $P_s=R_l-c_lw(L,L+1)/q(L,L+1)$. $\square$

EC.1.5. Proof of Proposition 7

First, note that equations (60) and (61) imply that if it is optimal to admit a LP customer in state $(x_l-1,x-1)$ and hence transition to (x_l,x), it must also be optimal to keep a LP customer upon arrival of an HP customer in state $(x_l,x-1)$ and transition to (x_l,x). It follows that the set of positions into which the social planner admits and keeps (does not expel) LP customers, must be identical.

Next, observe that if it is optimal to reject a LP arrival at state (x_l,x) it is also optimal to expel a LP arrival upon arrival of an HP customer at state (x_l+1,x). That is, $\mathcal{T}_2 z(x_l+1,x)=\mathcal{T}_1(x_l,x)$ or $\mathcal{T}_2 z(x_l,x)=\mathcal{T}_1(x_l-1,x)$ for all $x_l>0$ and $x-x_l\geq 0$.

To prove the claimed structure, we show that for all $x_l\geq 0$ and $x\geq x_l$,

$$z(x_l+2,x+2)-z(x_l+1,x+1)\leq z(x_l+1,x+1)-z(x_l,x), \tag{EC.77}$$

and for all states with $x-x_l\geq 1$,

$$z(x_l+2,x+1)-z(x_l+1,x)\leq z(x_l+1,x+1)-z(x_l,x). \tag{EC.78}$$

Property (EC.77) implies that the marginal value of adding a LP customer is decreasing for a fixed number of HP customers in the system. Property (EC.78) implies that the marginal value of adding a LP customer is decreasing in the number of LP customers in the system.

Define $\bar{x}$ as follows:

$$\bar{x}=\min\left\{x>0; z(x+1,x+1)-z(x,x)<0\right\}, \tag{EC.79}$$

that is, the maximum number of LP customers that the social planner admits/keeps in the system when there are no HP customers in the system (i.e., along the diagonal; $x=x_l$). Note that $\bar{x}$ is well-defined since (even in the absence of HP arrivals) the expected utilities eventually become negative for large enough x. The above properties then imply that the welfare maximizing policy is characterized by a threshold for each $x_l\leq\bar{x}$, say $\bar{x}(x_l)$, such that it is optimal to accept a LP arrival at state (x_l,x), and keep a LP customer at state (x_l,x) upon arrival of an HP customer, if and only if $x<\bar{x}(x_l)$. Further, by property (EC.78) $\bar{x}(x_l)=x_l+\bar{n}_h$ for $x_l\in\{0,\ldots,\bar{x}-\bar{n}_h-1\}$ and non-increasing in x_l for $x_l\in\{\bar{x}-\bar{n}_h,\ldots,\bar{x}\}$.

We establish the above claims by induction on the steps of the value iteration algorithm applied to the value functions introduced above. In Lemma EC.1 we show that $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3$ preserve properties (EC.77) and (EC.78). We then initialize the value iteration algorithm for the value function z at zero for all states, where properties (EC.77)–(EC.78) trivially hold. Then assuming that the value function satisfies (EC.77)–(EC.78) in iteration n, Lemma 2 implies that the properties will continue to hold in iteration $n+1$. Therefore, by induction z satisfies properties (EC.77)–(EC.78).

Lemma EC.1. *Denote by $\mathcal{Z}$ the set of functions satisfying properties* (EC.77)–(EC.78). *If $z \in \mathcal{Z}$, then (i) $\mathcal{T}_1 z \in \mathcal{Z}$; (ii) $\mathcal{T}_2 z \in \mathcal{Z}$; and (iii) $\mathcal{T}_3 z \in \mathcal{Z}$.*

Proof of Lemma EC.1. (*i*) First, consider property (EC.77). We need to show that for $x_l \geq 0$ and $x \geq x_l$,

$$\mathcal{T}_1 z(x_l+2, x+2) - \mathcal{T}_1 z(x_l+1, x+1) \leq \mathcal{T}_1 z(x_l+1, x+1) - \mathcal{T}_1 z(x_l, x),$$

or using the definition of $\mathcal{T}_1$,

$$\begin{aligned}\max\left[z(x_l+3, x+3), z(x_l+2, x+2)\right] - \max\left[z(x_l+2, x+2), z(x_l+1, x+1)\right] \leq \\ \max\left[z(x_l+2, x+2), z(x_l+1, x+1)\right] - \max\left[z(x_l+1, x+1), z(x_l, x)\right].\end{aligned}$$

Noting that $\max(a, b) = b + (a-b)^+$ we can rewrite the above as,

$$\begin{aligned}\left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^- + \left[z(x_l+3, x+3) - z(x_l+2, x+2)\right]^+ \leq \\ \left[z(x_l+1, x+1) - z(x_l, x)\right]^- + \left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^+.\end{aligned}$$

The inequality holds since by property (EC.77),

$$\left[z(x_l+3, x+3) - z(x_l+2, x+2)\right]^+ \leq \left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^+,$$

and

$$\left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^- \leq \left[z(x_l+1, x+1) - z(x_l, x)\right]^-.$$

Next consider property (EC.78). We need to show that for $x - x_l \geq 1$,

$$\mathcal{T}_1 z(x_l+2, x+1) - \mathcal{T}_1 z(x_l+1, x) \leq \mathcal{T}_1 z(x_l+1, x+1) - \mathcal{T}_1 z(x_l, x),$$

which using the definition of $\mathcal{T}_1$ is equivalent to,

$$\begin{aligned}\max\left[z(x_l+3, x+2), z(x_l+2, x+1)\right] - \max\left[z(x_l+2, x+1), z(x_l+1, x)\right] \leq \\ \max\left[z(x_l+2, x+2), z(x_l+1, x+1)\right] - \max\left[z(x_l+1, x+1), z(x_l, x)\right].\end{aligned}$$

Rewrite the above as,

$$\begin{aligned}\left[z(x_l+2, x+1) - z(x_l+1, x)\right]^- + \left[z(x_l+3, x+2) - z(x_l+2, x+1)\right]^+ \leq \\ \left[z(x_l+1, x+1) - z(x_l, x)\right]^- + \left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^+,\end{aligned}$$

which holds since by property (EC.78),

$$\left[z(x_l+3, x+2) - z(x_l+2, x+1)\right]^+ \leq \left[z(x_l+2, x+2) - z(x_l+1, x+1)\right]^+,$$

and

$$[z(x_l+2,x+1)-z(x_l+1,x)]^- \leq [z(x_l+1,x+1)-z(x_l,x)]^- .$$

(ii) Note that for all $x_l>0$ and $x-x_l<\bar{n}_h$, we have $\mathcal{T}_2 z(x_l,x)=\mathcal{T}_1(x_l-1,x)$ and for $x-x_l\geq\bar{n}_h$ we have $\mathcal{T}_2 z(x_l,x)=\mathcal{T}_1(x_l,x)$. We begin by property (EC.77). We need to show,

$$\mathcal{T}_2 z(x_l+2,x+2)-\mathcal{T}_2 z(x_l+1,x+1)\leq \mathcal{T}_2 z(x_l+1,x+1)-\mathcal{T}_2 z(x_l,x), \tag{EC.80}$$

for all $x_l\geq 0$ and $x\geq x_l$. For $x_l>0$ and $x-x_l<\bar{n}_h$ the inequality is equivalent to,

$$\mathcal{T}_1 z(x_l+1,x+2)-\mathcal{T}_1 z(x_l,x+1)\leq \mathcal{T}_1 z(x_l,x+1)-\mathcal{T}_1 z(x_l-1,x),$$

which holds since $\mathcal{T}_1 z$ satisfies property (EC.77). For $x-x_l\geq\bar{n}_h$ (EC.80) is equivalent to

$$z(x_l+2,x+2)-z(x_l+1,x+1)\leq z(x_l+1,x+1)-z(x_l,x),$$

which holds since z satisfies property (EC.77). It remains to consider the case with $x<\bar{n}_h$ and $x_l=0$. In this case (EC.80) is equivalent to

$$\mathcal{T}_1 z(1,x+2)-\mathcal{T}_1 z(0,x+1)\leq \mathcal{T}_1 z(0,x+1)-z(0,x+1),$$

which can be written as,

$$\max[z(2,x+3),z(1,x+2)]-\max[z(1,x+2),z(0,x+1)]\leq \max[z(1,x+2),z(0,x+1)]-z(0,x+1),$$
$$[z(1,x+2)-z(0,x+1)]^- + [z(2,x+3)-z(1,x+2)]^+ \leq (z(1,x+2)-z(0,x+1))^+,$$

which holds by property (EC.77) since if $z(1,x+2)-z(0,x+1)\geq 0$ then the above reduces to

$$[z(2,x+3)-z(1,x+2)]^+ \leq (z(1,x+2)-z(0,x+1))^+,$$

and if $z(1,x+2)-z(0,x+1)<0$, to

$$[z(1,x+2)-z(0,x+1)]^- \leq 0.$$

Next consider property (EC.78). We need to show that,

$$\mathcal{T}_2 z(x_l+2,x+1)-\mathcal{T}_2 z(x_l+1,x)\leq \mathcal{T}_2 z(x_l+1,x+1)-\mathcal{T}_2 z(x_l,x).$$

For $x_l>0$ and $x-x_l<\bar{n}_h$ the above is equivalent to,

$$\mathcal{T}_1 z(x_l+1,x+1)-\mathcal{T}_1 z(x_l,x)\leq \mathcal{T}_1 z(x_l,x+1)-\mathcal{T}_1 z(x_l-1,x),$$

which holds since $\mathcal{T}_1$ satisfies property (EC.78). For $x_l>0$ and $x-x_l=\bar{n}_h$, we need to show,

$$\mathcal{T}_1 z(x_l+1,x+1)-\mathcal{T}_1 z(x_l,x)\leq z(x_l+1,x+1)-z(x_l,x),$$

which is equivalent to,

$$[z(x_l+1,x+1)-z(x_l,x)]^- + [z(x_l+2,x+2)-z(x_l+1,x+1)]^+ \leq z(x_l+1,x+1)-z(x_l,x).$$

This holds since if $z(x_l+2,x+2)-z(x_l+1,x+1)\geq 0$ it reduces to

$$z(x_l+2,x+2)-z(x_l+1,x+1)\leq z(x_l+1,x+1)-z(x_l,x),$$

and if $z(x_l+2,x+2)-z(x_l+1,x+1)<0$, it reduces to

$$z(x_l+1,x+1)-z(x_l,x)\leq z(x_l+1,x+1)-z(x_l,x),$$

and both clearly hold. For $x-x_l>\bar{n}_h$, the above is satisfied since z satisfies property (EC.78) by assumption. For $x_l=0$ and $1\leq x<\bar{n}_h$,

$$\begin{aligned}\mathcal{T}_2z(2,x+1)-\mathcal{T}_2z(1,x)\leq&\mathcal{T}_2z(1,x+1)-\mathcal{T}_2z(0,x)\Rightarrow\\ \mathcal{T}_1z(1,x+1)-\mathcal{T}_1z(0,x)\leq&\ \mathcal{T}_1z(0,x+1)-z(0,x+1).\end{aligned}$$

which holds as shown above. For $x_l=0$ and $x=\bar{n}_h$,

$$\begin{aligned}\mathcal{T}_2z(2,x+1)-\mathcal{T}_2z(1,x)\leq&\ \mathcal{T}_2z(1,x+1)-\mathcal{T}_2z(0,x)\Rightarrow\\ \mathcal{T}_1z(1,x+1)-\mathcal{T}_1z(0,x)\leq& z(1,x+1)-z(0,x).\end{aligned}$$

This is equivalent to,

$$[z(1,x+1)-z(0,x)]^- + [z(2,x+2)-z(1,x+1)]^+ \leq z(1,x+1)-z(0,x),$$

which holds using a similar argument as above.

(iii) We first consider property (EC.77). We need to show that for $x_l\geq 0$ and $x\geq x_l$,

$$\mathcal{T}_3z(x_l+2,x+2)-\mathcal{T}_3z(x_l+1,x+1)\leq\mathcal{T}_3z(x_l+1,x+1)-\mathcal{T}_3z(x_l,x),$$

If $x_l\geq 0$ and $x-x_l>0$, this is equivalent to,

$$z(x_l+2,x+1)-z(x_l+1,x)\leq z(x_l+1,x)-z(x_l,x-1),$$

which holds by property (EC.78). If $x_l>0$ and $x-x_l=0$, we need to show that,

$$R_l+z(x_l+1,x+1)-R_l-z(x_l,x)\leq R_l+z(x_l,x)-R_l-z(x_l-1,x-1),$$

which holds by property (EC.77). If $x=x_l=0$, we need to show that,

$$R_l+z(1,1)-R_l-z(0,0)\leq R_l+z(0,0)-z(0,0),$$

which simplifies to

$$z(1,1)-z(0,0)\le R_l,$$

and holds since the marginal value of one additional LP customer is bounded by the service reward R_l.

Next we consider property (EC.78). We need to show that,

$$\mathcal{T}_3 z(x_l+2,x+1)-\mathcal{T}_3 z(x_l+1,x)\le \mathcal{T}_3 z(x_l+1,x+1)-\mathcal{T}_3 z(x_l,x).$$

For $x_l\ge 0$ and $x-x_l>1$ this is equivalent to,

$$z(x_l+2,x)-z(x_l+1,x-1)\le z(x_l+1,x)-z(x_l,x-1),$$

which holds by (EC.78). For $x_l\ge 0$ and $x-x_l=1$, we need to show that,

$$R_l+z(x_l+1,x)-R_l-z(x_l,x-1)\le z(x_l+1,x)-z(x_l,x-1),$$

which holds with equality. Since this property is only defined for $x-x_l>0$ the proof is complete. $\square$

EC.2. Auxiliary Lemmas

LEMMA EC.2. *Fix $i\ge 1$, $T\ge i$, and $v(i-1,i-1)\ge 0$. The value function*

$$v(i,i) = [-\bar{c}+\bar{\lambda}v(i,i+1)+\bar{\mu}v\,(i-1,i-1)]^+, \tag{EC.81}$$

$$v(i,n) = [-\bar{c}+\bar{\lambda}v(i,n+1)+\bar{\mu}v(i,n-1)]^+, \quad n\in\{i+1,...,T\}, \tag{EC.82}$$

$$v(i,T+1) = \begin{cases}[-\bar{c}+\bar{\lambda}v(i,T+1)+\bar{\mu}v(i,T)]^+, & T=\bar{n}_h+i-1,\\ [-\bar{c}+\bar{\mu}v(i,T)]^+, & T<\bar{n}_h+i-1,\end{cases} \tag{EC.83}$$

satisfies the following properties:

(i) $v(i,n+1)\le v(i,n)\le v(i-1,i-1)$ for $n\in\{i,\dots,T\}$.

(ii) If $v(i-1,i-1)<c_l/\mu$, customers prefer balking to joining in LP position i.

(iii) If $v(i-1,i-1)\ge c_l/\mu$ then a customer joins/stays in LP position i if and only if her system position $n\le\bar{n}(i)$, where

$$\bar{n}(i)=\begin{cases} i, & \textit{if } v(i,i)<c_l/\mu,\\ \max\{n\in\{i+1,...,T+1\}; v(i,n-1)\ge c_l/\mu\}, & \textit{otherwise},\end{cases} \tag{EC.84}$$

and the value function satisfies

$$v(i,i)\le v(i-1,i-1)-c_l/\mu \textit{ and } v(i,n)\le v(i,n-1)-c_l/\mu \textit{ for } i<n\le\bar{n}(i). \tag{EC.85}$$

Proof of Lemma EC.2. (i) By induction on the steps of the value iteration algorithm. Note that for any finite $T<\infty$ value iteration converges (see, e.g., Bertsekas 2011, Chapter 4). Let $v_0(i,n)=0$ for $n\in\{i,\ldots,T+1\}$. For $k\geq 0$ set

$$v_{k+1}(i,i) = [-\bar{c}+\bar{\lambda}v_k(i,i+1)+\bar{\mu}v_k\,(i-1,i-1)]^+, \tag{EC.86}$$

$$v_{k+1}(i,n) = [-\bar{c}+\bar{\lambda}v_k(i,n+1)+\bar{\mu}v_k(i,n-1)]^+, \quad n\in\{i+1,\ldots,T\}. \tag{EC.87}$$

$$v_{k+1}(i,T+1) = \begin{cases} [-\bar{c}+\bar{\lambda}v_k(i,T+1)+\bar{\mu}v_k(i,T)]^+, & T=\bar{n}_h+i-1, \\ [-\bar{c}+\bar{\mu}v_k(i,T)]^+, & T<\bar{n}_h+i-1. \end{cases} \tag{EC.88}$$

From (EC.81)–(EC.83) it follows that $v_k(i,n)\to v(i,n)$ as $k\to\infty$ by the convergence of the value iteration algorithm. The proof is complete if

$$v_k(i,n+1)\leq v_k(i,n)\leq v_k(i-1,i-1), \quad n\in\{i,\ldots,T\}, \quad \text{for } k\geq 0. \tag{EC.89}$$

This clearly holds for $k=0$. Next, if (EC.89) holds for some k then by using (EC.86)–(EC.89) and noting that $\bar{\lambda}+\bar{\mu}=1$, it is straightforward to verify that $v_{k+1}(i,n+1)\leq v_{k+1}(i,n)\leq v(i-1,i-1)$ for $n\in\{i,\ldots,T\}$.

(ii) Recall that LP customers join/stay in position (i,n) if and only if the expected utility of doing so is nonnegative. The expressions inside the brackets on the right-hand sides of (EC.81)–(EC.83) represent these expected utilities, for system positions $n\in\{i,\ldots,T+1\}$. From (EC.81) the expected utility of joining/staying in position (i,i) is nonnegative if and only if $v(i,i)=\bar{\lambda}v(i,i+1)+\bar{\mu}\,(v(i-1,i-1)-\bar{c}/\bar{\mu})\geq 0$. This holds if and only if $v\,(i-1,i-1)\geq\bar{c}/\bar{\mu}$ (where $\bar{c}/\bar{\mu}=c_l/\mu$), because $v(i,i)$ is a convex combination of $v(i,i+1)$ and $v\,(i-1,i-1)-\bar{c}/\bar{\mu}$, and we have $v(i,i+1)\leq v(i,i)$ from Part (i).

(iii) If $v\,(i-1,i-1)\geq\bar{c}/\bar{\mu}$, then the proof of Part (ii) implies that the expected utility of joining/staying in position (i,i) is nonnegative. Furthermore, the same line of argument, applied to (EC.82)–(EC.83) and $n\in\{i+1,\ldots,T+1\}$, establishes that the expected utility of joining/staying in position (i,n) is nonnegative if and only if $v(i,n-1)\geq c_l/\mu$. These conditions, together with Part (i), imply that a LP customer joins/stays in position (i,n) if and only if her system position $n\leq\bar{n}(i)$ with $\bar{n}(i)$ as defined in (EC.84).

We next prove (EC.85). For position (i,i) we have

$$v(i,i)=\bar{\lambda}v(i,i+1)+\bar{\mu}(v(i-1,i-1)-c_l/\mu)\leq v\,(i-1,i-1)-c_l/\mu, \tag{EC.90}$$

where the equality follows from (EC.81) and because by Part (ii) customers have nonnegative expected utility of joining/staying in system position i. The inequality in (EC.90) holds because $v(i,i)$ is a convex combination of $v(i,i+1)$ and $v(i-1,i-1)-c_l/\mu$, and $v(i,i+1)\leq v(i,i)$ by Part

(i).

Similarly, for positions (i,n) with $n \in \{i+1,\dots,\bar{n}(i)\}$ we have

$$v(i,n) = \bar{\lambda}v(i,n+1) + \bar{\mu}(v(i,n-1) - c_l/\mu) \leq v(i,n-1) - c_l/\mu. \tag{EC.91}$$

The equality follows from (EC.82)–(EC.83) because customers have nonnegative expected utility of joining/staying in system positions $n \in \{i,\dots,\bar{n}(i)\}$. The inequality holds because $v(i,n)$ is a convex combination of $v(i,n+1)$ and $v(i,n-1) - c_l/\mu$, and $v(i,n+1) \leq v(i,n)$ by Part (ii). □

LEMMA EC.3. *Consider the value functions $v(m,n)$ and $v(m+1,n)$ satisfying equations (EC.9)–(EC.11) and (EC.14)–(EC.16), respectively. We have*

(i) If $\bar{n}(m) = \bar{n}_h + m$, then $v(m+1,n) \leq v(m,n)$ for $n \in \{m+1,\dots,\bar{n}_h + m\}$.

(ii) If $\bar{n}(m) < \bar{n}_h + m$, then $v(m+1,n) = v(m,n)$ for $n \in \{m+1,\dots,\bar{n}(m)+1\}$.

Proof of Lemma EC.3. (i) The proof is by induction and convergence of the value iteration algorithm. Note that since $\bar{n}(m) = \bar{n}_h + m$, the value of staying at all system positions is nonnegative for the customer at LP position m. Therefore, using (EC.9)–(EC.11), $v(m,n)$ satisfies

$$\begin{aligned}
v(m,m) &= -\bar{c} + \bar{\lambda}v(m,m+1) + \bar{\mu}v(m-1,m-1), \\
v(m,n) &= -\bar{c} + \bar{\lambda}v(m,n+1) + \bar{\mu}v(m,n-1), \quad n \in \{m+1,...,\bar{n}_h + m - 1\}, \\
v(m,\bar{n}_h + m) &= -\bar{c} + \bar{\lambda}v(m,\bar{n}_h + m) + \bar{\mu}v(m,\bar{n}_h + m - 1).
\end{aligned}$$

To prove the claim we approximate the value function for the customer at LP position $m+1$, given in (EC.14)-(EC.16), using the recursion

$$\begin{aligned}
v_{k+1}(m+1,m+1) &= [-\bar{c} + \bar{\lambda}v_k(m+1,m+2) + \bar{\mu}v(m,m)]^+, \\
v_{k+1}(m+1,n) &= [-\bar{c} + \bar{\lambda}v_k(m+1,n+1) + \bar{\mu}v_k(m+1,n-1)]^+, \quad n \in \{m+2,...,\bar{n}_h + m\}, \\
v_{k+1}(m+1,\bar{n}_h + m + 1) &= [-\bar{c} + \bar{\lambda}v_k(m+1,\bar{n}_h + m + 1) + \bar{\mu}v_k(m+1,\bar{n}_h + m)]^+,
\end{aligned}$$

for $k \geq 0$ and with $v_0(m+1,n) = 0$ for $n \in \{m+1,...,\bar{n}_h + m + 1\}$. The claim is clearly satisfied for $k = 0$. Assuming

$$v_k(m+1,n) \leq v(m,n), \quad \text{for } n \in \{m+1,...,\bar{n}_h + m\}, \tag{EC.92}$$

we show that $v_{k+1}(m+1,n) \leq v(m,n)$ for $n \in \{m+1,...,\bar{n}_h + m\}$ and hence the result follows by induction. For $n = m+1$ the corresponding equations are

$$\begin{aligned}
v(m,m+1) &= -\bar{c} + \bar{\lambda}v(m,m+2) + \bar{\mu}v(m,m), \\
v_{k+1}(m+1,m+1) &= [-\bar{c} + \bar{\lambda}v_k(m+1,m+2) + \bar{\mu}v(m,m)]^+,
\end{aligned}$$

and for $n \in \{m+2, ..., \bar{n}_h + m - 1\}$, we have

$$v(m,n) = -\bar{c} + \bar{\lambda} v(m,n+1) + \bar{\mu} v(m,n-1),$$
$$v_{k+1}(m+1,n) = [-\bar{c} + \bar{\lambda} v_k(m+1,n+1) + \bar{\mu} v_k(m+1,n-1)]^+.$$

In each case, the claim directly follows by comparing the two equations and using (EC.92).
For $n = \bar{n}_h + m$, we have

$$v(m,\bar{n}_h+m) = -\bar{c} + \bar{\lambda} v(m,\bar{n}_h+m) + \bar{\mu} v(m,\bar{n}_h+m-1),$$
$$v_{k+1}(m+1,\bar{n}_h+m) = [-\bar{c} + \bar{\lambda} v_k(m+1,\bar{n}_h+m+1) + \bar{\mu} v_k(m+1,\bar{n}_h+m-1)]^+.$$

The claim follows by noting that $v_k(m+1,\bar{n}_h+m+1) \leq v_k(m+1,\bar{n}_h+m)$ (see the proof of Lemma (EC.2) part (i)) and using (EC.92).

(ii) For $\bar{n}(m) < \bar{n}_h + m$, the result follows because the solution of (EC.14)–(EC.16) is unique and the function $v(m,n)$ that satisfies (EC.9)–(EC.11) and $v(m,n) = 0$ for $n > \bar{n}(m)$ also satisfies (EC.14)–(EC.16) when setting $v(m+1,n) = v(m,n)$ for $n \in \{m+1, \ldots, \bar{n}(m)+1\}$. $\square$

Lemma EC.4. *Consider q and w given in (12) and (13), respectively.*
(i) $w(L,L+1)/q(L,L+1)$ is strictly increasing in L for $L \geq 1$.
(ii) $w(i,L+1)/q(i,L+1)$ is strictly increasing in i for $0 \leq i \leq L$.

Proof of Lemma EC.4. (i) First, consider the case with $\rho_h \neq 1$. Using (12) and (13) we have

$$\frac{w(L,L+1)}{q(L,L+1)} = \frac{L(1-\rho_h^{L+1}) - (L+1)(\rho_h - \rho_h^{L+1})}{\mu(1-\rho_h)^2}.$$
$$= \frac{L(1-\rho_h) - \rho_h(1-\rho_h^L)}{\mu(1-\rho_h)^2} = \frac{1}{\mu(1-\rho_h)} \sum_{i=1}^{L} (1-\rho_h^i),$$

which is clearly increasing in L. For $\rho_h = 1$, we get,

$$\frac{w(L,L+1)}{q(L,L+1)} = \frac{L(L+1)}{2\mu},$$

which is also increasing in L.

(ii) Assume $\rho_h < 1$. Using (12) and (13) we have after simplifying

$$\frac{w(i,L+1)}{q(i,L+1)} = \frac{i(1-\rho_h^{L+1}) - (L+1)(\rho_h^{L+1-i} - \rho_h^{L+1})}{\mu(1-\rho_h)(1-\rho_h^{L+1-i})}.$$

Adding and subtracting $L+1$ in the numerator and simplifying we get

$$\begin{aligned}
\frac{w(i,L+1)}{q(i,L+1)} &= \frac{i(1-\rho_h^{L+1}) - (L+1)(\rho_h^{L+1-i} - \rho_h^{L+1}) + (L+1) - (L+1)}{\mu(1-\rho_h)(1-\rho_h^{L+1-i})}.\\
&= \frac{i(1-\rho_h^{L+1}) + (L+1)(1-\rho_h^{L+1-i}) - (L+1)(1-\rho_h^{L+1})}{\mu(1-\rho_h)(1-\rho_h^{L+1-i})}\\
&= \frac{(L+1)(1-\rho_h^{L+1-i}) - (L+1-i)(1-\rho_h^{L+1})}{\mu(1-\rho_h)(1-\rho_h^{L+1-i})}\\
&= \frac{L+1}{\mu(1-\rho_h)} - \frac{(1-\rho_h^{L+1})(L+1-i)}{\mu(1-\rho_h)(1-\rho_h^{L+1-i})}.
\end{aligned}$$

Hence, noting that $\rho_h < 1$, it suffices to show that $(L+1-i)/(1-\rho_h^{L+1-i})$ is decreasing in i. To this end, let $\nu \equiv L+1-i$ and note that ν decreases as i increases. Thus, it remains to show that $\nu/(1-\rho_h^\nu)$ is increasing in ν. The first difference is

$$\frac{\nu+1}{1-\rho_h^{\nu+1}} - \frac{\nu}{1-\rho_h^\nu} = \frac{(1-\rho_h^\nu)-(1-\rho_h)\nu\rho_h^\nu}{(1-\rho_h^{\nu+1})(1-\rho_h^\nu)}.$$

The denominator is clearly positive. To show the numerator is also positive, we show that

$$\frac{1-\rho_h^\nu}{1-\rho_h} > \nu\rho_h^\nu. \tag{EC.93}$$

Consider $r(x) \equiv x^\nu$. By the mean value theorem, there exists $c_0 \in (\rho_h, 1)$ such that

$$\frac{1-\rho_h^\nu}{1-\rho_h} = \frac{r(1)-r(\rho_h)}{1-\rho_h} = r'(c_0).$$

But $r'(c_0) = \nu c_0^{\nu-1} > \nu c_0^\nu > \nu\rho_h^\nu$, which proves (EC.93).

The proof for $\rho_h > 1$ follows a similar argument.

Here, we verify the statement for $\rho_h = 1$. In this case, we have,

$$\begin{aligned}\frac{w(i,L+1)}{q(i,L+1)} &= \frac{i(L+1-i)/(2\mu)}{1-i/(L+1)} \\ &= \frac{i(L+1)}{2\mu},\end{aligned}$$

which is increasing in i. The proof is complete. $\square$

Lemma EC.5. *The following statements hold:*

(i) The function $\bar{w}(i,J)$ as defined in (14) is strictly increasing in $i \le J$.

(ii) The function $u_s(m,n)$ as defined in (16) is strictly decreasing in $m \le n$.

(iii) Fix $K \ge 0$. Consider $u_a(m,n;K,L)$ as defined in (17) with $K+1 \le m \le n = L$, that is,

$$u_a(m,n;K,n) = q(n-K,n+1-K)(R_l - c_l\bar{w}(1,\bar{n}_h+1)K) - c_l w(n-K,n+1-K). \tag{EC.94}$$

The function $u_a(m,n;K,n)$ given in (EC.94) is strictly decreasing in n when positive. Further, if $u_a(m,n_0;K,n_0) < 0$ for some n_0, then $u_a(m,n_0;K,n_0) < 0$ for all $n \ge n_0$.

(iv) Consider $u_a(m,n;K,L)$ as defined in (17) with $K+1 \le m \le n \le L$, where the equilibrium thresholds K and L are given by (18) and (19), respectively, that is,

$$u_a(m,n;K,L) = q(n-K,L+1-K)(R_l - c_l\bar{w}(1,\bar{n}_h+1)K) - c_l(w(n-K,L+1-K). \tag{EC.95}$$

The function $u_a(m,n;K,L)$ given in (EC.95) is strictly decreasing in $n \le L$ with $u_a(m,n;K,L) \ge 0$ for $n \le L$.

Proof of Lemma EC.5. (i) First assume that $\rho_h \neq 1$. Then using (14) it suffices to show that the numerator of $\bar{w}(i,J)$, which we denote by $\omega(i) := i(\mu-\lambda_h) + \lambda_h(\rho_h^J - \rho_h^{J-i})$, is increasing in i. After simplifying we have,

$$\begin{aligned}\omega(i+1)-\omega(i) &= \mu + \lambda_h(\rho_h^{J-i} - \rho_h^{J-(i+1)} - 1)\\ &= \mu(1+\rho_h^{J-i+1} - \rho_h^{J-i} - \rho_h)\\ &= \mu(1-\rho_h)(1-\rho_h^{J-i}),\end{aligned}$$

which is positive for $\rho_h \neq 1$.

Next, assume that $\rho_h = 1$. In this case, we have from (14) that $\bar{w}(i,J) = i(1-i+2J)/2\mu$ which is clearly increasing in i for $i \le J$.

(ii) Using the definition of $u_s(m,n)$ in (16), we need to show for $n \ge m+1$ that

$$u_s(m+1,n) - u_s(m,n) = c_l\left(\bar{w}(n-m+1,\bar{n}_h+1) - \bar{w}(n-m,\bar{n}_h+1) - \bar{w}(1,\bar{n}_h+1)\right) < 0. \quad \text{(EC.96)}$$

First assume that $\rho_h \neq 1$. We have from (14) that

$$\bar{w}(n-m+1,\bar{n}_h+1) - \bar{w}(n-m,\bar{n}_h+1) - \bar{w}(1,\bar{n}_h+1) = \frac{\lambda_h(\rho_h^{\bar{n}_h+1-(n-m)} - \rho_h^{\bar{n}_h+1-(n-m+1)} + \rho_h^{\bar{n}_h} - \rho_h^{\bar{n}_h+1})}{(\mu-\lambda_h)^2} \quad \text{(EC.97)}$$

$$= \frac{\lambda_h \rho_h^{\bar{n}_h}(1-\rho_h)(1-\rho_h^{-(n-m)})}{(\mu-\lambda_h)^2}, \quad \text{(EC.98)}$$

which is clearly negative for $\rho_h \neq 1$.

Next, consider the case $\rho_h = 1$. Then (14) implies

$$\bar{w}(n-m+1,\bar{n}_h+1) - \bar{w}(n-m,\bar{n}_h+1) - \bar{w}(1,\bar{n}_h+1) = \frac{-(n-m)}{\mu} < 0. \quad \text{(EC.99)}$$

(iii) Letting $n' \equiv n-K$, we rewrite the function $u_a(m,n;K,n)$, given in (EC.94), as follows:

$$q(n',n'+1)\left(R_l - c_l\frac{w(n',n'+1)}{q(n',n'+1)} - c_l\bar{w}(1,\bar{n}_h+1)K\right).$$

We need to show that this function is strictly decreasing in n' when positive. This follows from two facts. First, $q(n',n'+1)$ is positive and strictly decreasing in n' by (12). Second, the term in parenthesis is strictly decreasing in n', because $w(n',n'+1)/q(n',n'+1)$ is strictly increasing in n' by Part (i) of Lemma EC.4.

(iv) For $K+1 \le m \le n \le L$ we have from (EC.95) that

$$u_a(m,n;K,L) = q(n-K,L-K+1)\left(R_l - c_l\frac{w(n-K,L-K+1)}{q(n-K,L-K+1)} - c_l\bar{w}(1,\bar{n}_h+1)K\right). \quad \text{(EC.100)}$$

Since $q(n-K, L-K+1)$ is positive and strictly decreasing in $n \leq L$ by inspection of (12), it suffices to show that the term in parenthesis in (EC.100) is positive and decreasing in $n \leq L$. Since $u_a(m, L; K, L) \geq 0$ by (19), this term is non-negative for $n = L$, i.e.,

$$R_l - c_l \frac{w(L-K, L-K+1)}{q(L-K, L-K+1)} - c_l \bar{w}(1, \bar{n}_h + 1)K \geq 0. \tag{EC.101}$$

Further, by Lemma EC.4 Part (ii) we have that the following fraction is strictly increasing in n:

$$\frac{w(n-K, L-K+1)}{q(n-K, L-K+1)}.$$

Therefore, $u_a(m, n; K, L)$ is strictly decreasing in $n \leq L$ with $u_a(m, n; K, L) \geq 0$ for $n \leq L$. $\square$

EC.3. Estimation Details for the Exogenous Model in Section 5.3

To estimate the exogenous patience distribution, we simulate the rational model for 150000 time units, removing observations from the first 5000 time units as warmup. We then use the observed patience times to estimate the exogenous distribution. Because we only observe the patience times for customers who eventually abandon, the patience times for served customers are censored. We hence use a Kaplan-Maier estimator using the same approach as in Brown et al. (2005). Figure EC.1 Provides the fitted Kaplan-Maier estimator from the rational model with $\lambda_h = \lambda_l = 0.5$, $\mu = 0.66$ and $c_l = 1, R_l = 20$ and $\bar{n}_h = 100$.

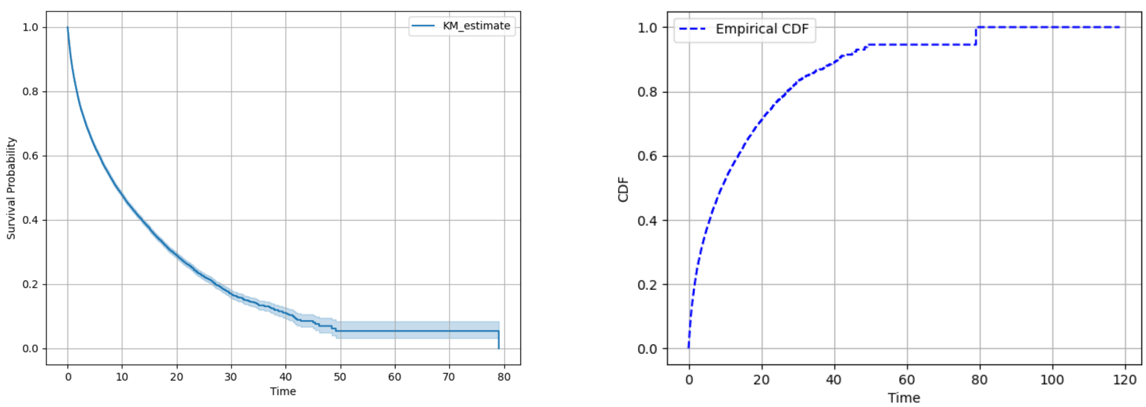

Figure EC.1 The estimated exogenous patience distribution using the Kaplan-Maier estimator.

EC.4. Numerical Examples in Support of Remark 3 in Section 6.3

As noted in Remark 3, we observe in numerical experiments that the socially optimal threshold $\bar{x}(x_l)$ (shown to be non-decreasing in Proposition 7) is in fact constant for LP positions x_l from which customers may be expelled. In this appendix we present two numerical examples for illustration.

We compute the value functions numerically using value iteration. To this end, we truncate the state-space as follows: we assume that there are no arrivals to the system if $x > M$ where M is a large threshold. We then increase the threshold M until the value functions of the truncated formulation converge up to one decimal place. Figure EC.2 presents the optimal value functions for different states of a system with $\lambda_l = 0.2, \lambda_h = 0.3, \mu = 1, R_l = 15, c_l = 1, \alpha = 0.01$, $n_h = 3$ (Top) and $n_h = 10$ (Bottom). In both cases, the welfare maximizing policy does not admit / expels a LP customer when the total number of customers in system reaches $x = 8$. When $\bar{n}_h = 3$, we have $\bar{x}(x_l) = x_l + \bar{n}_h$ for $x_l < 6$, and $\bar{x}(x_l) = 8$ for $6 \leq x_l \leq 8$. At $(x_l, x) = (8, 8)$, for instance, it is optimal to reject an arriving LP customer since $z(9, 9) < z(8, 8)$. Also, it is optimal to expel a LP customer upon arrival of an HP customer since $z(8, 9) < z(7, 8)$. When $\bar{n}_h = 10$, $\bar{x}(x_l) = 8$ for all $1 \leq x_l \leq 8$.

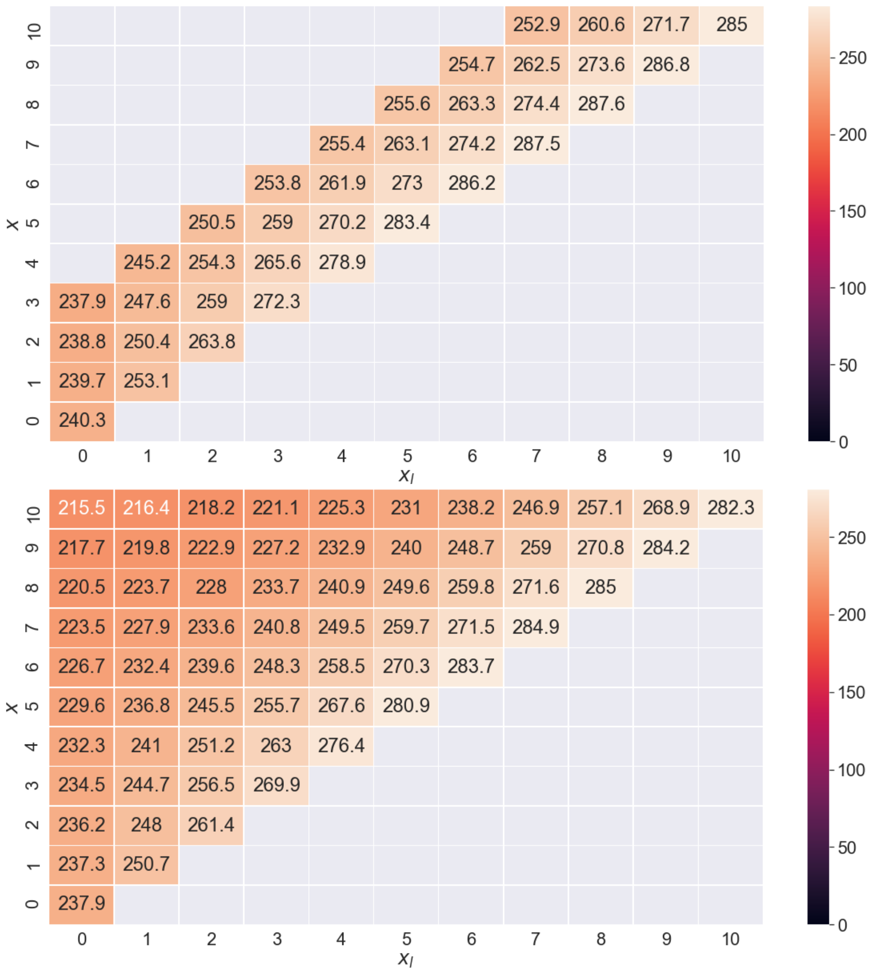

Figure EC.2 **Value function for a system with $\lambda_l = 0.2, \lambda_h = 0.3, \mu = 1, R_l = 15, c_l = 1, \alpha = 0.01$, $n_h = 3$ (top) and $n_h = 10$ (bottom).**