Online Appendix: Matchmaking Strategies for Maximizing Player Engagement in Video Games

Appendix A: Omitted Discussions in Section 5

In this appendix, we discuss how we can generalize our baseline model to fit more realistic and generalized settings, serving as a setup to the case study.

A.1. Player Behavior

We extend the baseline model in three directions: Having 1) multiple skill levels, 2) general winrate between players, and 3) general churn behaviors.

First, instead of having only two skill levels in the baseline model, here we consider there are K ordered skill levels, where level 1 is the lowest and level K is the highest.

Second, we extend the outcome of a match as a Bernoulli random variable depending on the skill levels of the two players. Let p_{kj} be the winrate of a level k player versus a level j player, implying that $p_{kj} = 1 - p_{jk}$. Players of the same skill level are equally likely to win, i.e., $p_{kk} = 0.5$. A player with a higher skill level than their opponent has a strictly larger than 0.5 probability of winning, i.e., $p_{kj} > 0.5$ if $k > j$. We still assume that a player's skill level is fixed over their lifespan in the matching system. In Huang et al. (2019), players' skill level is an evolving metric that monotonically increases as the players play more. In our setting, skill levels reflect *relative* competence and are more stable because others are also getting more familiar with the game.

Third, we generalize players' churning behaviors. In particular, we assume that a player's engagement state is determined by the win-loss record of the last m matches and transitions according to a Markov chain when the player plays a new match. We use q to denote the 'churn state', i.e., a player quitting the game permanently. Let $\mathcal{G}$ be the set of all possible states of a player, which has cardinality at most $2^m + 1$ (history of wins/losses and the churn state). A player is then fully characterized by their skill level k and engagement state $g \in \mathcal{G}$, which we refer to as the demographic of players. Let $P^k_{win}, P^k_{lose} \in [0,1]^{|\mathcal{G}|\times|\mathcal{G}|}$ be the transition matrix of a level k player's engagement state, given that they win/lose the next match. Hence, if they are matched with a level j player, their aggregate transition matrix is given by $M_{kj} = p_{kj}P^k_{win} + (1 - p_{kj})P^k_{lose}$. For ease of notation, we also define $\bar{\mathcal{G}}$ to be the set of all the active states except the churn state q and $\bar{M}_{kj}$ be the reduced aggregate transition matrix without the churn state. We define an active player as one who has not churned and is thus in one of the states in $\bar{\mathcal{G}}$. We use a simple example to illustrate the notations introduced in this section.

Example EC.1. Suppose that players have two skill levels, either high or low (denoted by level 2 and 1, respectively). They quit with a probability 0.2 if they experience two consecutive losses and with a probability 0.5 if they experience three consecutive losses. This implies that $m = 2$, as only two previous matches plus the current match outcome affect the transition state. We further assume that a high-skilled player wins against a low-skill player with probability $p_{21} = 0.8$. Hence, for each skill level, there are 4 engagement states in $\mathcal{G}$: the player may experience 0, 1 or 2 consecutive losses or reach state q. We use 20, 21, 22, $2q$ and 10, 11, 12, $1q$ to denote the states of high- and low-skilled players, respectively. For $k = 1, 2$, the transition matrix P^k_{win} and P^k_{lose} is given by

$$P^k_{win} = \begin{array}{cc} & \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \\ kq \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad P^k_{lose} = \begin{array}{cc} & \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \\ kq \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The aggregate transition matrix is given by

$$M_{kk} = \begin{array}{cc} & \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \\ kq \end{array} & \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.4 & 0.1 \\ 0.5 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad M_{21} = \begin{array}{cc} & \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \\ kq \end{array} & \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.8 & 0 & 0.16 & 0.04 \\ 0.8 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad M_{12} = \begin{array}{cc} & \begin{array}{cccc} k0 & k1 & k2 & kq \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \\ kq \end{array} & \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.2 & 0 & 0.64 & 0.16 \\ 0.2 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

and the reduced transition matrix $\bar{M}_{kj}$ is defined as

$$\bar{M}_{kk} = \begin{array}{cc} & \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \end{array} & \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.4 \\ 0.5 & 0 & 0.25 \end{pmatrix} \end{array}, \quad \bar{M}_{21} = \begin{array}{cc} & \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \end{array} & \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.8 & 0 & 0.16 \\ 0.8 & 0 & 0.1 \end{pmatrix} \end{array}, \quad \bar{M}_{12} = \begin{array}{cc} & \begin{array}{ccc} k0 & k1 & k2 \end{array} \\ \begin{array}{c} k0 \\ k1 \\ k2 \end{array} & \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.2 & 0 & 0.64 \\ 0.2 & 0 & 0.4 \end{pmatrix} \end{array}.$$

□

A.2. Firm's Dynamic Optimization Problem

Now we describe the firm/matchmaker's problem in the generalized comprehensive model.

Let s^t_{kg} be the number of players at time t in the demographic with skill level k and engagement state g. The population of level k active players in period t is given by the vector $\mathbf{s}^t_k \in \mathbb{R}^{|\bar{\mathcal{G}}|}$, and we use $\mathbf{s}^t = [\mathbf{s}^t_1, \ldots, \mathbf{s}^t_K]$ to denote the system state. Let $f^t_{kg,jg'} \geq 0$ be the amount of level k players in state g that are matched to a level j opponent in state g' in time t. A feasible match given $\mathbf{s}^t$ is a set of matching flows $f^t_{kg,jg'}$ that satisfies:

$$\begin{aligned} \sum_{j=1}^{K} \sum_{g' \in \bar{\mathcal{G}}} f^t_{kg,jg'} &= s^t_{kg},\ k = 1, \ldots, K, \forall g \in \bar{\mathcal{G}}, \\ \sum_{j=1}^{K} \sum_{g' \in \bar{\mathcal{G}}} f^t_{jg',kg} &= s^t_{kg},\ k = 1, \ldots, K, \forall g \in \bar{\mathcal{G}}, \\ f^t_{kg,jg'} &= f^t_{jg',kg},\ j = 1, \ldots, K, k = 1, \ldots, K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}} \\ f^t_{kg,jg'} &\geq 0,\ j = 1, \ldots, K, k = 1, \ldots, K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}} \end{aligned} \tag{FB}$$

Namely, (FB) are *flow balance* constraints that make sure every active player is matched. The first equation ensures that every level k player in state g is matched with some opponents, and the second equation ensures that the total amount of matches against level k players in state g equals the number of such players. The third equation makes sure that for every pair of demographics, a match results in an equal effect on supply and demand.

Next, we depict the evolution of the system. Using $\mathbf{f}^t_{kj} = \{f^t_{kg,jg'}\} \in \mathbb{R}^{\bar{\mathcal{G}} \times \bar{\mathcal{G}}}_{\geq 0}$ to denote the flow matrix between level k and j, the *evolution of demographics* is given by

$$\mathbf{s}^{t+1}_k = \sum_{j=1,\ldots,K} \left(\mathbf{f}^t_{kj}\mathbf{1}\right)^\top \bar{M}_{kj},\ k = 1, \ldots, K, \tag{ED}$$

where $\mathbf{1}$ is a $|\bar{\mathcal{G}}| \times 1$ unit vector. Note that in (ED), $\mathbf{f}_{kj}^t \mathbf{1}$ is the vector describing how many level k players are matched to level j players for all states in $\bar{\mathcal{G}}$, and recall that $\bar{M}_{kj}$ is the state transition matrix for level k players matched to level j players. The engagement at period t is given by $\sum_{k=1}^{K} \sum_{g\in\bar{\mathcal{G}}} s_{kg}^t$, the total amount of active players.

The firm's objective is to maximize engagement, which we measure by the cumulative amount of active players across all periods, $\sum_{t=1}^{\infty} \gamma^{t-1} \sum_{k=1}^{K} \sum_{g\in\bar{\mathcal{G}}} s_{kg}^t$, where $\gamma \in (0,1]$ is the discount factor. The engagement maximization problem can be formulated as a Markov decision process, where the states are $\mathbf{s}^t$, the amount of active players in each demographic. Let V^π be the value function of a feasible policy π. The value-to-go function is given by

$$V^\pi(\mathbf{s}^t) = \sum_{k=1}^{K} \sum_{g\in\bar{\mathcal{G}}} s_{kg}^{t+1} + \gamma V^\pi(\mathbf{s}^{t+1}) \qquad \text{(EC.1)}$$
$$\text{subject to (FB), (ED).}$$

Our goal is to find the optimal policy with a value function $V^*(\cdot)$, such that $V^*(\mathbf{s}^t) \geq V^\pi(\mathbf{s}^t)$ for any feasible policy π. Note that the system dynamics are again all linear, just like the baseline model, so when the initial size of each demographic is given, maximizing the engagement is equivalent to solving an infinite linear program, which we describe in details in Appendix A.3.

Also in Appendix A.3, we discuss another four more possible extensions of our model: 1) adding a draw outcome 2) only a fraction of active players join the matchmaking pool 3) the game may last multiple (and possibly random) periods; 4) new players arrive depending on the entire history of active players. All of the above extensions can be incorporated into the LP formulation.

We conclude this section by outlining a few technical results. First, with the comprehensive model, we can still derive the value function under SBMM where players are only matched to others in the same skill level. Second, we can guarantee that the total engagement under any policy is finite. Finally, we can show that strong duality and complementary slackness hold for the infinite linear program associating to (EC.1) (formally presented in Appendix A.4 (EC.2)). We leave the detailed technical discussions to Appendix A.4.

A.3. Linear Program Formulation of the Comprehensive model and Possible Extensions

First, we provide the complete linear program of the engagement maximization problem with the comprehensive model:

$$\begin{aligned}
V^*(\mathbf{s}^0) = \max & \sum_{t=1}^{\infty} \gamma^{t-1} \sum_{k} \sum_{g\in\bar{\mathcal{G}}} s_{kg}^t \qquad \text{(EC.2)}\\
\text{s.t. } & \sum_{j=1}^{K} \sum_{g'\in\bar{\mathcal{G}}} f_{kg,jg'}^t = s_{kg}^t, \forall k, \forall g \in \bar{\mathcal{G}}, t = 0,1,\dots\\
& \sum_{j=1}^{K} \sum_{g'\in\bar{\mathcal{G}}} f_{jg',kg}^t = s_{kg}^t, \forall k, \forall g \in \bar{\mathcal{G}}, t = 0,1,\dots\\
& f_{kg,jg'}^t = f_{jg',kg}^t, j = 1,\dots,K, k = 1,\dots,K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}}, t = 0,1,\dots\\
& f_{kg,jg'}^t \geq 0, j = 1,\dots,K, k = 1,\dots,K, \forall g \in \bar{\mathcal{G}}, g' \in \bar{\mathcal{G}}, t = 0,1,\dots\\
& \mathbf{s}_k^{t+1} = \sum_{j=1,\dots,K} \left(\mathbf{f}_{kj}^t \mathbf{1}\right)^\top \bar{M}_{kj}, \forall k, t = 0,1,\dots.
\end{aligned}$$

Next, we point out that our framework (EC.2) is flexible enough to allow for the following various practical extensions while still resulting in a nice LP formulation.

1. A draw/tie outcome can be easily added, since our model only depends on the aggregate transition matrix M_{kk}. Let $P^k_{tie} \in [0,1]^{|\mathcal{G}|\times|\mathcal{G}|}$ be the transition matrix of a level k player's engagement state, given that they experienced a tie. Let $p^{win}_{kj}, p^{lose}_{kj}$ and p^{tie}_{kj} be the probability of win, lose and tie for a level k player who faces a level j player. Then the aggregate transition matrix is given by $M_{kj} = p^{win}_{kj} P^k_{win} + p^{lose}_{kj} P^k_{lose} + p^{tie}_{kj} P^k_{tie}$. Then the computation follows Eq. (EC.2).

2. If in each period, only *part* fraction of the idle players want to play, then we can simply multiply *part* on the right-hand-side of (FB) and add $(1-part)\mathbf{s}^t_k$ on the right-hand-side of (ED). For Eq. (FB), it now becomes

$$\begin{aligned}
&\sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f^t_{kg,jg'} = part\cdot s^t_{kg},\ k=1,\dots,K, \forall g\in\bar{\mathcal{G}},\\
&\sum_{j=1}^{K}\sum_{g'\in\bar{\mathcal{G}}} f^t_{jg',kg} = part\cdot s^t_{kg},\ k=1,\dots,K, \forall g\in\bar{\mathcal{G}},\\
&\qquad f^t_{kg,jg'} = f^t_{jg',kg},\ j=1,\dots,K, k=1,\dots,K, \forall g\in\bar{\mathcal{G}}, g'\in\bar{\mathcal{G}}\\
&\qquad f^t_{kg,jg'} \ge 0,\ j=1,\dots,K, k=1,\dots,K, \forall g\in\bar{\mathcal{G}}, g'\in\bar{\mathcal{G}}
\end{aligned} \tag{EC.3}$$

For Eq. (ED), it now becomes

$$\mathbf{s}^{t+1}_k = (1-part)\mathbf{s}^t_k + \sum_{j=1,\dots,K}\left(\mathbf{f}^t_{kj}\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.4}$$

3. If the match duration is not one period, we can modify (ED) so that the match flow returns to the demographics after a positive and random delay. Suppose a match may last at most D periods. Let w_d be the probability that a match lasts d periods, and $\sum_{d=1}^{D} w_d = 1$. Then Eq. (ED) can be modified as

$$\mathbf{s}^{t+1}_k = \sum_{d=1}^{D} w_d \sum_{j=1,\dots,K}\left(\mathbf{f}^{t+1-d}_{kj}\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.5}$$

4. New players whose amounts are linear functions of past history can be introduced easily by modifying (ED). Let $New^t_k(\{\mathbf{f}^\tau_{kj}\}^{t-1}_{\tau=1})$ be a linear function of $\mathbf{f}^\tau_{kj}$, $\tau=1,\dots,t-1$. Then we can modify Eq. (ED) as

$$\mathbf{s}^{t+1}_k = New^t_k(\{\mathbf{f}^\tau_{kj}\}^{t-1}_{\tau=1}) + \sum_{j=1,\dots,K}\left(\mathbf{f}^t_{kj}\mathbf{1}\right)^\top \bar{M}_{kj}\ \ k=1,\dots,K. \tag{EC.6}$$

A.4. SBMM and Technical Results

To make our problem well-defined, we make the following mild assumption: there exists a positive integer m such that the probability of churning after $m+1$ consecutive losses is strictly positive. This assumption clearly holds in reality. Similar to our baseline model, in the generalized comprehensive model, we use SBMM to refer to any policy that lets players in the same skill level match with each other (they are all equivalent with respect to engagement). Once again, SBMM is always feasible by letting players in the same state match each other, i.e., $f^t_{kg,kg} = s^t_{kg}$. Let $V^{\text{SBMM}}(\cdot)$ be the value function of SBMM. We first characterize the value of SBMM in Lemma EC.1, which is analogous to Lemma 1. Next, we show the finiteness of the value function under any policy.

LEMMA EC.1 **(Value of Skill-based Policy)**. *For any* $\gamma \in (0,1]$ *and any initial state of demographics* $\mathbf{s}^0$*, the value function of SBMM is*

$$V^{SBMM}(\mathbf{s}^0) = \sum_{k=1}^{K} \mathbf{v}_k^\top \mathbf{s}_k^0,$$

where $\mathbf{v}_k$ *is given by*

$$\mathbf{v}_k = \gamma^{-1}\left(\left(I - \gamma \bar{M}_{kk}\right)^{-1} - I\right)\mathbf{1}.$$

Under SBMM, a level k player starts from an active state g and transitions to the next engagement state according to the Markov chain M_{kk}. Note that each unit of players generates one unit of engagement in each period. Thus, the total engagement a unit of player can generate, which we refer to as their shadow price, is the average number of periods they stay active and is described by $\mathbf{v}_k$. The eigenvalue of $\bar{M}_{kk}$ is less than 1 and the total engagement is finite even when $\gamma = 1$.

LEMMA EC.2 **(Finiteness of the Value Function)**. *For any policy* π *and initial state of demographics* $\mathbf{s}^0$*,* $V^\pi(\mathbf{s}^0)$ *is finite.*

Lemma EC.2 is somewhat counter-intuitive: because a myopic policy gains non-negative advantages in every period compared to SBMM, it is intuitive that the power of optimal policy should grow exponentially with time (also conjectured by Chen et al. (2017b)). However, Lemma EC.2 shows that the benefit of the optimal policy is finite.

Finally, because (EC.2) is an infinite LP, strong duality is not always guaranteed, although we prove it does hold in our setting.

LEMMA EC.3 **(Strong Duality)**. *Strong duality and complementary slackness hold for* (EC.2) *for any initial demographics* $\mathbf{s}^0$*.*

Note that the stylized model presented in Section 3 is a special case of the comprehensive model in this section, so we also have strong duality and complementary slackness for (P). Proofs of all technical results in this Section will be provided in Appendix B.2.

A.5. General Model Estimation and Winrate model

We present a general estimation procedure that can accommodate more general player behaviors. To be more specific, we illustrate how to estimate P^k_{win} and P^k_{lose} through maximum log-likelihood estimation without assuming players' churn behavior is based on aversion to losing streaks. To simplify our problem and reduce the number of parameters, we assume that the players' churn behavior is independent of skill level and drop the superscript k. In the state transition matrix, the row that represents state g in P_{win} (P_{lose}) only has two positive entries: the probability of churning after winning (losing) the next match and the probability of moving to a certain non-churn state. Our goal is then to estimate ρ^{win}_g (ρ^{lose}_g), the churn probability of a player in state $g \in \mathcal{G}$ after winning (losing) the next match (Note that $\rho_q^{win} = \rho_q^{lose} = 1$ since we assume that a churned player will stay churned.) Let g_t^i be the state player i reaches before playing t-th match, $\mathbb{W}_t^i$ be the indicator of whether player i wins the t-th match, and $\mathbb{I}_t^i$ be the indicator of whether the player i churns right after reaching g_t^i. For a player who played T^i total matches, we record his/her states and churn decisions starting from the 5th match because they only have a valid state right before the 5th match if $m = 4$. If the state sequence is given by $(g_5^i, g_6^i, \dots, g_{T^i}^i)$, the outcome sequence is $(\mathbb{W}_5^i, \mathbb{W}_6^i, \dots, \mathbb{W}_{T^i}^i)$, and churn decision sequence is $(\mathbb{I}_5^i, \mathbb{I}_6^i, \dots, \mathbb{I}_{T^i}^i)$, then the log-likelihood (LLH) function is given by

$$\sum_{i=1}^{N} \sum_{t=5}^{T^i} \log \left(\rho_{g_t^i}^{win} \mathbb{W}_t^i \mathbb{I}_t + (1 - \rho_{g_t^i}^{win}) \mathbb{W}_t^i (1 - \mathbb{I}_t) + \rho_{g_t^i}^{lose} (1 - \mathbb{W}_t^i) \mathbb{I}_t + (1 - \rho_{g_t^i}^{lose})(1 - \mathbb{W}_t^i)(1 - \mathbb{I}_t) \right).$$

Because the state transition is exogenous, the estimation is computationally simple and usually has a closed form. We show the closed form of the parameters for a winrate model with $m+2$ parameters below. *General Estimation Procedure and Winrate Model:* The winrate model assumes that the players' churn probability depends on the number of wins over the last m matches plus the next match (total m+1 matches). Hence, the model can be described by $m+2$ parameters, the churn probabilities when the player has 0 to $m+1$ wins. Note that although we only need to estimate $m+2$ parameters, we still need $2^m + 1$ states. This is because, for the LP formulation, the customer transition matrix needs the full win-loss record. Hence, a state g is a length-m binary sequence denoting the win-loss record, and we use $|g|$ to denote the wins in g. The solution for the MLE is

$$\rho_g^{win} = \frac{\text{Number of churn decision when players won } |g+1| \text{ matches over last } m+1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over last } m+1 \text{ matches}},$$

$$\rho_g^{lose} = \frac{\text{Number of churn decision when players won } |g| \text{ matches over the } m+1 \text{ matches}}{\text{Number of times that players won } \tilde{g} \text{ matches over the } m+1 \text{ matches}}.$$

The next table summarizes the out-of-sample negative LLH for some winrate models and their performance (relative power) compared to SBMM.

Table EC.1 Out-of-sample LLH and Relative Power Compared to SBMM for Winrate Models

	LLH	Optimal Policy	Random Policy
$m=1$	139639.4	4.33%	-2.12%
$m=2$	139496.3	5.92%	-4.18%
$m=3$	139309.0	6.63%	-6.17%

A.6. Robustness Check of Alternative Initial Populations

In Section 5.3, we provide performances of different policies under an initial state calibrated from Lichess. To ensure that our insights are robust, we consider three alternative initial states and report the performance in this section. We consider the following representative states:

A. Uniform: The initial state of all user types are the same, i.e., $s_{k,g}^0 = 1$ for all k, g.

B. Mostly low-skilled: the low-skilled players are exponentially more than high-skilled players. Specifically, we set $s_{k,g}^0 = 2^{(13-k)/2}$ for all k, g.

C. Mostly high-skilled: the high-skilled players are exponentially more than low-skilled players. Specifically, we set $s^0_{k,g} = 2^{(k-1)/2}$ for all k, g.

Table EC.2-EC.4 reports the results under these three initial states. The insights are exactly the same: The LF myopic performs better than HF myopic in all situations, highlighting the need for careful tiebreaking. The dual-based policy has almost zero gap with the optimal policy, suggesting that our constructed shadow prices capture the trade-off in the optimal policy.

Table EC.2 **Performance Comparison under Uniform Initial State**

	Random	Optimal	LF Myopic	HF Myopic	Dual Policy
$m = 1$	-2.70%	5.41%	5.28%	4.70%	5.41%
$m = 2$	-4.97%	6.91%	6.62%	6.39%	6.86%
$m = 3$	-6.67%	6.91%	6.62%	6.39%	6.86%
$m = 4$	-7.77%	6.62%	6.36%	6.14%	6.57%

Table EC.3 **Performance Comparison under Mostly Low-skilled Initial State**

	Random	Optimal	LF Myopic	HF Myopic	Dual Policy
$m = 1$	-1.78%	3.50%	3.46%	2.96%	3.49%
$m = 2$	-3.07%	4.33%	4.16%	3.82%	4.30%
$m = 3$	-3.89%	4.33%	4.14%	3.86%	4.29%
$m = 4$	-4.31%	4.19%	4.02%	3.75%	4.16%

Table EC.4 **Performance Comparison under Mostly High-skilled Initial State**

	Random	Optimal	LF Myopic	HF Myopic	Dual Policy
m=1	-1.45%	3.24%	3.13%	2.54%	3.24%
m=2	-2.80%	5.64%	5.34%	4.94%	5.64%
m=3	-3.88%	6.17%	5.89%	5.73%	6.15%
m=4	-4.62%	5.89%	5.67%	5.56%	5.89%

Appendix B: Omitted Proofs

B.1. Omitted Proofs from Section 4

Before proving results in Section 4, we first present a simplified formulation to the matchmaker's problem in (P). Recall that $\mathcal{P} := \{2w, 2\ell, 1w, 1\ell\}$. To begin, in any period $t = 0, 1, 2, \ldots$, matching flows $f^t_{2w,2\ell}$, $f^t_{1w,1\ell}$, $f^t_{2\ell,2w}$, and $f^t_{1\ell,1w}$ can be set to zero without loss of generality. Taking flows $f^t_{2w,2\ell} = f^t_{2\ell,2w} = a \in (0, \min\{s^t_{2w}, s^t_{2\ell}\}]$ as an example, it can be represented by $f^t_{2w,2w} = a$ and $f^t_{2\ell,2\ell} = a$, since they induce the same evolution to players' demographics $\{s^{t+1}_{2w}, s^{t+1}_{2\ell}\}$ in the next period. We also use the fact that $f^t_{i,j} =$

$f^t_{j,i}, \forall i,j \in \mathcal{P}$ to reduce the problem to 8 flow variables for each period, which is half of the original described in (P). Thus, we can rewrite the matchmaker's problem in (P) as

$$
\begin{aligned}
\max_{\{\mathbf{f}^t\}_{t=0}^{\infty}} \; & \sum_{t=1}^{\infty} \left(s^t_{2w} + s^t_{2\ell} + s^t_{1w} + s^t_{1\ell} \right) \\
\text{s.t. } & s^0_{2w} = f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell}, \\
& s^0_{2\ell} = f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell}, \\
& s^0_{1w} = f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w}, \\
& s^0_{1\ell} = f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell}, \\
& s^t_{2w} = f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell}, \; t = 1,2,..., && \text{(FB)} \\
& s^t_{2\ell} = f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell}, \; t = 1,2,..., \\
& s^t_{1w} = f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w}, \; t = 1,2,..., \\
& s^t_{1\ell} = f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell}, \; t = 1,2,..., \\
& s^t_{2w} = \frac{1}{2}\left(f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell}\right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell}, \; t = 1,2,..., && \text{(ED)} \\
& s^t_{2\ell} = \frac{1}{2} f^{t-1}_{2w,2w}, \; t = 1,2,..., \\
& s^t_{1w} = \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right), \; t = 1,2,..., \\
& s^t_{1\ell} = \frac{1}{2} f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w}, \; t = 1,2,..., \\
& f^t_{i,j} \geq 0, \; \forall i,j,t \in \mathcal{P}.
\end{aligned}
$$

By merging the flow balance and evolution of demographic constraints, we can further remove s^t_i for $t > 1$, with only 8 decision variables per period:

$$
\max_{\{\mathbf{f}^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(f^t_{2w,2w} + \frac{1}{2} f^t_{2\ell,2\ell} + f^t_{1w,1w} + \frac{1}{2} f^t_{1\ell,1\ell} + 2 f^t_{2w,1w} + f^t_{2\ell,1\ell} + f^t_{2w,1\ell} + 2 f^t_{2\ell,1w} \right) \tag{P'}
$$

s.t.

$$
\begin{aligned}
& s^0_{2w} = f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell}, \\
& s^0_{2\ell} = f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell}, \\
& s^0_{1w} = f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w}, \\
& s^0_{1\ell} = f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell}, \\
& f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell} = \frac{1}{2}\left(f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell}\right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell}, \; t = 1,2,..., \\
& f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{2w,2w}, \; t = 1,2,..., \\
& f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} = \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right), \; t = 1,2,..., \\
& f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w}, \; t = 1,2,..., \\
& f^t_{i,j} \geq 0, \; \forall i,j,t \in \mathcal{P}.
\end{aligned}
$$

Proof of Lemma 1. Under SBMM, for each player, their state transition follows a Markov chain. After one round of match, a winning player garners one unit of engagement, and becomes either a winning or losing player with the probability 1/2. A losing player may become a winning player with probability 1/2, or transfer to an absorbing state *quit* otherwise. The average engagement is the time before they transit to state *quit*. Let v_{2w}, $v_{2\ell}$, v_{1w}, $v_{1\ell}$ be the expected staying time of each type. For winning players, we have

$$v_{iw} = 1 + \frac{1}{2}v_{iw} + \frac{1}{2}v_{il},\ i = 1, 2.$$

For losing players, we have

$$v_{il} = \frac{1}{2} + \frac{1}{2}v_{iw},\ i = 1, 2.$$

Solving the above two equations, we have $v_{iw} = 5$ and $v_{il} = 3$, for $i = 1, 2$. The total engagement give state $\mathbf{s}^t$ is then $5(s^t_{2w} + s^t_{1w}) + 3(s^t_{2\ell} + s^t_{1\ell})$. ☐

Since our problem does not induce steady-states due to finiteness, we thus consider a generalization of steady-state referred to as *decaying steady-state*. A policy admits a decaying steady state $\mathbf{s}^t$, if there exists some $c \in (0, 1]$ such that under the given policy, we have $\mathbf{s}^{t+1} = c\mathbf{s}^t$. Unfortunately, Lemma EC.4 shows that no matching policy, besides SBMM, results in its demographics reaching a decaying steady state.

Lemma EC.4 **(No Steady State Exists)**. *Consider a fixed time period t.*

(i) SBMM can induce a non-zero decaying steady-state, but only for $c = (1 + \sqrt{5})/4$ and $\mathbf{s}^t$ a positive multiple of the vector $((1 + \sqrt{5})/2, 1, (1 + \sqrt{5})/2, 1)$.

(ii) For any matching policy that involves matching flows between high-skilled and low-skilled players, there is no non-zero decaying steady-state for any $c \in (0, 1]$.

Proof of Lemma EC.4. We work with the alternative formulation in (P') which has 8 flow variables each period.

(i) Let $f^t = (f^t_{2w,2w}, f^t_{2\ell,2\ell}, f^t_{1w,1w}, f^t_{1\ell,1\ell}, f^t_{2w,1w}, f^t_{2w,1\ell}, f^t_{2\ell,1w}, f^t_{2\ell,1\ell})$ be the flow vector at time t. With ($\mathrm{FB_S}$) conditions, the state $\mathbf{s}^t$ can then be expressed as $B\mathbf{f}^t$, where

$$B = \begin{array}{c} \\ s^t_{2w} \\ s^t_{2\ell} \\ s^t_{1w} \\ s^t_{1\ell} \end{array} \!\!\begin{array}{c} \begin{array}{cccccccc} f^t_{2w,2w} & f^t_{2\ell,2\ell} & f^t_{1w,1w} & f^t_{1\ell,1\ell} & f^t_{2w,1w} & f^t_{2w,1\ell} & f^t_{2\ell,1w} & f^t_{2\ell,1\ell} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Similarly, using the ($\mathrm{ED_S}$) conditions, the state $\mathbf{s}^{t+1}$ can be expressed as $A\mathbf{f}^t$, where

$$A = \begin{array}{c} \\ s^t_{2w} \\ s^t_{2\ell} \\ s^t_{1w} \\ s^t_{1\ell} \end{array} \!\!\begin{array}{c} \begin{array}{cccccccc} f^t_{2w,2w} & f^t_{2\ell,2\ell} & f^t_{1w,1w} & f^t_{1\ell,1\ell} & f^t_{2w,1w} & f^t_{2w,1\ell} & f^t_{2\ell,1w} & f^t_{2\ell,1\ell} \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

If there exists a decaying steady state for some $c \in [0, 1]$, then there exists vector $\mathbf{f}^t \geq 0$ such that

$$A\mathbf{f}^t = cB\mathbf{f}^t \iff (A - cB)\mathbf{f}^t = 0.$$

Now we provide the null space of $A - cB$. When $c \neq (1 + \sqrt{5})/4$, the null space is given by

$$\begin{pmatrix} 0.5c/g(c) & (-c^2 + 0.5c + 0.5)/g(c) & -0.5c/g(c) & (-c^2 + 0.5c)/g(c) & 0 & 0 & 0 & 1 \\ 0.5c/g(c) & (-c^2 + 0.5c + 0.5)/g(c) & (-c^2 + 0.5)/g(c) & (0.5c - 0.5)/g(c) & 0 & 0 & 1 & 0 \\ (-c^2 + c)/g(c) & (-0.5c + 0.5)/g(c) & -0.5c/g(c) & (-c^2 + 0.5c)/g(c) & 0 & 1 & 0 & 0 \\ (-c^2 + c)/g(c) & (-0.5c + 0.5)/g(c) & (-c^2 + 0.5)/g(c) & (0.5c - 0.5)/g(c) & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{EC.7}$$

where $g(c) = c^2 - 0.5c - 0.25$. Note that $g(c) = 0$ if $c = (1+\sqrt{5})/4$. In that case, the null space of $A - cB$ is given by

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{2}{-3+\sqrt{5}} & 0 & 0 & 1 \\ \frac{1+\sqrt{5}}{2} & 0 & \frac{3+\sqrt{5}}{3-\sqrt{5}} & 0 & \frac{-5+\sqrt{5}}{3-\sqrt{5}} & 1 & 1 & 0 \\ 0 & 0 & \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If there exists a linear combination of rows in the null space resulting in all non-negative elements and at least one non-zero element, then we have found a valid flow vector $\mathbf{f}^t$ and thus a valid demographic $\mathbf{s}^t$ that decays steadily at a rate of c.

First, consider $c \in (\frac{1+\sqrt{5}}{4}, 1]$, which implies $g(c) > 0$. Then observe that elements in the third column of Eq. (EC.7), representing flow $f^t_{1w,1w}$ are all negative. Hence, for any linear combination of rows in Eq. (EC.7), as long as the flow $f_{1w,1w}$ is positive, at least one of the elements representing flows $f^t_{2w,1w}, f^t_{2w,1\ell}, f^t_{2\ell,1w}, f^t_{2\ell,1\ell}$ is negative. Thus, no steady state exists when $c \in (\frac{1+\sqrt{5}}{4}, 1]$.

Next, consider $c \in (0, \frac{1+\sqrt{5}}{4})$, which implies $g(c) < 0$. Note that elements in the second column of Eq. (EC.7), representing the flow $f^t_{2\ell,2\ell}$ are all positive. Hence, for any linear combination of rows in Eq. (EC.7), as long as the flow $f_{2\ell,2\ell}$ is positive, then at least one of the elements representing flows $f^t_{2w,1w}, f^t_{2w,1\ell}, f^t_{2\ell,1w}, f^t_{2\ell,1\ell}$ is negative. Thus, no steady state exists when $c \in (0, \frac{1+\sqrt{5}}{4})$.

Finally, consider $c = (1+\sqrt{5})/4$. In this case, we can easily find a non-negative flow vector $\mathbf{f}^t$ by summing up the third and fourth row, which gives $\big((1+\sqrt{5})/2, 1, (1+\sqrt{5})/2, 1, 0, 0, 0, 0\big)$, representing SBMM, and any positive multiple of $\mathbf{s} = ((1+\sqrt{5})/2, 1, (1+\sqrt{5})/2, 1)$ can induce such flows. To see that no other policy can reach a steady state when $c = (1+\sqrt{5})/4$, note that we have to find a non-zero non-negative flow vector $\mathbf{f}^t$ with the use of the first two rows. For the first row, the sign of the fifth entry is the opposite of the eighth entry; for the second row, the sign of the fifth entry is opposite of the sixth and seventh entries. Hence, there is no way to construct a feasible flow vector $\mathbf{f}^t$ with non-negative elements. Thus, no other policy besides SBMM can induce a decaying steady state when $c = (1+\sqrt{5})/4$. $\square$

Proof of Lemma 2. For each flow that involves both high-skilled and low-skilled players, we can compare the outcome of one unit of such flow with SBMM flow that uses the same amount of players. A unit of $f_{2\ell,1w}$ uses the same amount of players as one unit of $f_{2\ell,2\ell}$ and $f_{1w,1w}$, but it leads to zero loss in the next period while SBMM loses a half unit of 2ℓ players. One unit of $f_{2\ell,1\ell}$ or $f_{2w,1w}$ leads to the same losses in the next period compared to the SBMM flow of one unit of $f_{2\ell,2\ell}, f_{1\ell,1\ell}, f_{2w,2w}$, and $f_{1w,1w}$. Finally, one unit of $f_{2w,1\ell}$ leads to a one unit loss of 1ℓ players in the next period, while the SBMM flow of one unit of $f_{1\ell,1\ell}$ and $f_{2w,2w}$ only leads to a half unit loss of 1ℓ players. Hence, to maximize the population in the next period, we should always maximize $f_{2\ell,1w}$ and set $f_{2w,1\ell} = 0$. For the rest of the players, they can be matched arbitrarily as the outcome in the next period is the same as SBMM. $\square$

B.1.1. Optimal Matching Policy The optimal matching flows in Theorem 1 are summarized in detail in the next table:

The optimal policy always maximizes matching flows between demographics 2ℓ and $1w$. The rest of the players are matched via SBMM in most scenarios (third row in Table EC.5), except when there are fewer

Table EC.5 **Optimal Matchmaking Policy,** $K_1 := \frac{18}{5}s^t_{2w} + \frac{9}{5}s^t_{2\ell} + \frac{3}{5}s^t_{1w}, K_2 := \frac{18}{5}s^t_{2w} + \frac{23}{5}s^t_{2\ell} - \frac{11}{5}s^t_{1w}$

States of demographics	Optimal matching policy
$s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1w}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $s^t_{1\ell} > K_2$	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$, $f^t_{2\ell,1\ell} = f^t_{1\ell,2\ell} = s^t_{2\ell} - s^t_{1w}$, and $f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}$
$s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1w}$, $s^t_{2\ell} \geq s^t_{1w}$, $s^t_{2w} < s^t_{1\ell}$, and $K_1 < s^t_{1\ell} \leq K_2$	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$, $f^t_{2\ell,2\ell} = \frac{9}{7}s^t_{2w} + \frac{23}{14}s^t_{2\ell} - \frac{11}{14}s^t_{1w} - \frac{5}{14}s^t_{1\ell}$, $f^t_{2\ell,1\ell} = f^t_{1\ell,2\ell} = \frac{5}{14}s^t_{1\ell} - \frac{9}{7}s^t_{2w} - \frac{9}{14}s^t_{2\ell} - \frac{3}{14}s^t_{1w}$, and $f^t_{1\ell,1\ell} = s^t_{1\ell} - f^t_{2\ell,1\ell}$
Otherwise	$f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = s^t_{2\ell} - \min\{s^t_{2\ell}, s^t_{1w}\}$, $f^t_{1w,1w} = s^t_{1w} - \min\{s^t_{2\ell}, s^t_{1w}\}$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = \min\{s^t_{2\ell}, s^t_{1w}\}$

high-skilled players compared to low-skilled players and too many 1ℓ players (first and second rows in Table EC.5). In this case, the matchmaker also matches players in demographic 2ℓ with players in 1ℓ.

B.1.2. Proof of Theorem 1 and Theorem 2 Before proving Theorem 1, we write out the dual problem of (P') explicitly. Denote λ^t_i for $i \in \mathcal{P}$ as the dual variables (shadow price) for each demographic in period $t = 0, 1, 2, \ldots$. Then we can write the dual problem of (P') as

$$\min_{\{\lambda^t\}} \sum_{i\in\mathcal{P}} s^0_i \lambda^0_i \tag{D'}$$

s.t.

$$\begin{aligned}
&1 \leq \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\ t = 0, 1, 2, \ldots,\\
&\frac{1}{2} \leq \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w},\ t = 0, 1, 2, \ldots,\\
&2 \leq \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell},\ t = 0, 1, 2, \ldots,\\
&\frac{1}{2} \leq \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},\ t = 0, 1, 2, \ldots,\\
&2 \leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell},\ t = 0, 1, 2, \ldots,\\
&1 \leq \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w},\ t = 0, 1, 2, \ldots,\\
&1 \leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w},\ t = 0, 1, 2, \ldots,\\
&1 \leq \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}\lambda^{t+1}_{1\ell},\ t = 0, 1, 2, \ldots
\end{aligned}$$

Proof of Theorem 1. We solve the problem in (P') by considering its dual problem (D'). In each period, there are four constraints in the primal problem (P'). Thus, we assign dual variable λ^t_i, where $i \in \{1w, 1\ell, 2w, 2\ell\}$, to each constraint representing the evolution of a players' demographics group in the primal problem. We will fully characterize the transition of primal and dual sequences, and show optimality by checking primal/dual feasibility and complementary slackness.

We break down the rest of this proof into 7 steps, corresponding to 7 scenarios of players' demographics. In each step, we analyze a scenario corresponding to a state of players' demographics. The states described by the 7 scenarios are mutually exclusive and collectively exhaustive. That is, in any period t, we have

- Scenario 1: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, and $s^t_{2w} \geq s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1w}$, $f^t_{1w,1w} = 0$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$;

- Scenario 2: $s^t_{2w}+s^t_{2\ell}\ge s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}<s^t_{1w}$, and $s^t_{2w}\ge s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=0$, $f^t_{1w,1w}=s^t_{1w}-s^t_{2\ell}$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{2\ell}$;
- Scenario 3: $s^t_{2w}+s^t_{2\ell}\ge s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, and $s^t_{2w}<s^t_{1\ell}$; The optimal matching flows are the same as those in Scenario 1;
- Scenario 4: $s^t_{2w}+s^t_{2\ell}<s^t_{1w}+s^t_{1\ell}$ and $s^t_{2\ell}<s^t_{1w}$; The optimal matching flows are the same as those in Scenario 2;
- Scenario 5: $s^t_{2w}+s^t_{2\ell}<s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, $s^t_{2w}<s^t_{1\ell}$, and $s^t_{1\ell}\le K_1=\frac{18}{5}s^t_{2w}+\frac{9}{5}s^t_{2\ell}+\frac{3}{5}s^t_{1w}$; The optimal matching flows are the same as those in Scenario 1;
- Scenario 6: $s^t_{2w}+s^t_{2\ell}<s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, $s^t_{2w}<s^t_{1\ell}$, and $s^t_{1\ell}>K_2=\frac{18}{5}s^t_{2w}+\frac{23}{5}s^t_{2\ell}-\frac{11}{5}s^t_{1w}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$, $f^t_{2\ell,1\ell}=f^t_{1\ell,2\ell}=s^t_{2\ell}-s^t_{1w}$, and $f^t_{1\ell,1\ell}=s^t_{1\ell}-f^t_{2\ell,1\ell}$;
- Scenario 7: $s^t_{2w}+s^t_{2\ell}<s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, $s^t_{2w}<s^t_{1\ell}$, and $K_1<s^t_{1\ell}\le K_2$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$, $f^t_{2\ell,2\ell}=\frac{9}{7}s^t_{2w}+\frac{23}{14}s^t_{2\ell}-\frac{11}{14}s^t_{1w}-\frac{5}{14}s^t_{1\ell}$, $f^t_{2\ell,1\ell}=f^t_{1\ell,2\ell}=\frac{5}{14}s^t_{1\ell}-\frac{9}{7}s^t_{2w}-\frac{9}{14}s^t_{2\ell}-\frac{3}{14}s^t_{1w}$, and $f^t_{1\ell,1\ell}=s^t_{1\ell}-f^t_{2\ell,1\ell}$.

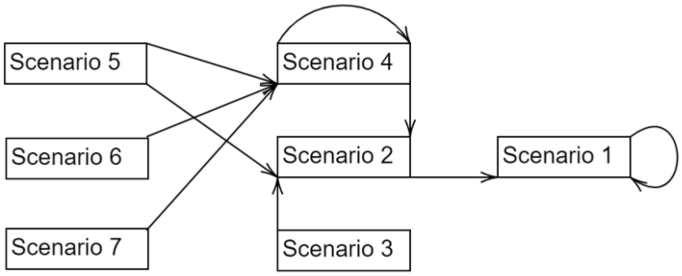

Figure EC.1 Evolution of Players' Demographics

Note that Scenarios 1 to 5 correspond to the third row of Table EC.5. In such cases, the optimal policy simply maximizes the matching between 2ℓ and $1w$ and does skill-based matching for the remaining players. Scenario 6 to 7 corresponds to the first two rows in Table EC.5. The reason we classify the aforementioned scenarios in this way is because of how the scenarios evolve over time, as shown in Figure EC.1. For example, at any period t, if we are in Scenario 1, then in the next period $t+1$, one can verify that we always stay in the same state of players' demographics described by Scenario 1 under the proposed matching policy. The dual variables for any state in Scenario 1 are constant over time. Similarly, once we reach Scenario 2 at time t, we always transfer to Scenario 1 at $t+1$.

For the rest of this proof, we show that the state evolves as Fig. EC.1, and our proposed matching policy is optimal in each scenario and its subsequent scenarios as the state demographics evolve. To be more specific, our proof goes through each scenario and shows the corresponding optimal matching flows in Table EC.5 are optimal. Note that by construction, the proposed matching policy in Table EC.5 is primal feasible, i.e., satisfying all constraints in (FB_S) and (ED_S). We establish optimality by constructing dual variables for each scenario and show that both complementary slackness and dual feasibility conditions hold (the primal feasibility can be easily verified for our proposed policy). When possible, we write the corresponding dual variables with closed-form expressions. Our proof starts with Scenario 1 as it is at the end of all transitions (does not transition to other scenarios), then works backward to parental scenarios. For each parental scenario, we readily construct the dual variables for all subsequent scenarios, representing all the following time periods. For example, when the demographic state is in Scenario 2, we show it will transition to Scenario 1 in one period under the proposed matching policy. Then we only need to construct the dual variables for the current period and use the dual variables constructed in Scenario 1 for all subsequent periods, which we already have, to establish optimality.

Scenario 1: $s^t_{2w}+s^t_{2\ell}\geq s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\geq s^t_{1w}$, and $s^t_{2w}\geq s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=s^t_{2\ell}-s^t_{1w}$, $f^t_{1w,1w}=0$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$.

We first consider Scenario 1, and prove that it is the "end" of all scenarios in Figure EC.1. In other words, we shall show that once the state of players' demographics falls in Scenario 1, it will remain in Scenario 1. To see this, for some $\mathbf{s}^t=\{s^t_{1w},s^t_{1\ell},s^t_{2w},s^t_{2\ell},\}$ in Scenario 1, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w}=\frac{1}{2}s^t_{1\ell}$, $s^{t+1}_{1\ell}=s^t_{1w}$, $s^{t+1}_{2w}=\frac{1}{2}(s^t_{2w}+s^t_{2\ell}+s^t_{1w})$, and $s^{t+1}_{2\ell}=\frac{1}{2}s^t_{2w}$. Then $s^{t+1}_{2\ell}\geq s^{t+1}_{1w}$ because $s^t_{2w}\geq s^t_{1\ell}$, and $s^{t+1}_{2w}\geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w}-s^{t+1}_{1\ell}=\frac{1}{2}(s^t_{2w}+s^t_{2\ell}-s^t_{1w})$ and $s^t_{2\ell}-s^t_{1w}\geq 0$. Hence, $\mathbf{s}^{t+1}$ still belongs to Scenario 1.

Next, we show that the proposed policy in Theorem 1 is optimal for all subsequent periods once players' demographics satisfy Scenario 1. The optimal solution in Theorem 1 suggests that in Scenario 1, only 4 variables are non-zero while all other flows are zero in each period. Therefore, we have

$$\begin{aligned}1&=\lambda^t_{2w}-\frac{1}{2}\lambda^{t+1}_{2w}-\frac{1}{2}\lambda^{t+1}_{2\ell},\\ \frac{1}{2}&=\lambda^t_{2\ell}-\frac{1}{2}\lambda^{t+1}_{2w},\\ 2&=\lambda^t_{2\ell}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ \frac{1}{2}&=\lambda^t_{1\ell}-\frac{1}{2}\lambda^{t+1}_{1w},\end{aligned}\tag{CS$_1$}$$

as the complementary conditions corresponding to primal non-zero variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$, respectively, and have

$$\begin{aligned}2&\leq\lambda^t_{2w}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ 1&\leq\lambda^t_{2w}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\\ 1&\leq\lambda^t_{2\ell}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\\ 1&\leq\lambda^t_{1w}-\frac{1}{2}\lambda^{t+1}_{1w}-\frac{1}{2}\lambda^{t+1}_{1\ell},\end{aligned}\tag{DF$_1$}$$

as the dual feasibility conditions corresponding to variables, $f^t_{2w,1w}$, $f^t_{2w,1\ell}$, $f^t_{2\ell,1\ell}$, and $f^t_{1w,1w}$, that are zero in the primal problem, respectively.

The following dual solutions:

$$\lambda^t_{2w}=5,\ \lambda^t_{2\ell}=3,\ \lambda^t_{1w}=9,\text{ and }\lambda^t_{1\ell}=5,\quad\forall t,\tag{EC.8}$$

satisfies complementary slackness in (CS$_1$) and feasibility conditions in (DF$_1$). Thus, the proposed policy in Theorem 1 is optimal once the players' demographics fall in scenario 1.

Scenario 2: $s^t_{2w}+s^t_{2\ell}\geq s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}<s^t_{1w}$, and $s^t_{2w}\geq s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=0$, $f^t_{1w,1w}=s^t_{1w}-s^t_{2\ell}$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{2\ell}$;

In the second step, we consider Scenario 2, which will transit to Scenario 1 after matching under the proposed policy as we have stated in Figure EC.1. To see that, for some $\mathbf{s}^t=\{s^t_{1w},s^t_{1\ell},s^t_{2w},s^t_{2\ell}\}$ in Scenario 2, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w}=\frac{1}{2}(s^t_{1w}+s^t_{1\ell}-s^t_{2\ell})$, $s^{t+1}_{1\ell}=\frac{1}{2}(s^t_{2\ell}+s^t_{1w})$, $s^{t+1}_{2w}=\frac{1}{2}s^t_{2w}+s^t_{2\ell}$, and $s^{t+1}_{2\ell}=\frac{1}{2}s^t_{2w},$. Then $s^{t+1}_{2\ell}\geq s^{t+1}_{1w}$ because $s^{t+1}_{2\ell}-s^{t+1}_{1w}=\frac{1}{2}(s^t_{2w}+s^t_{2\ell}-s^t_{1w}-s^t_{1\ell})\geq 0$. Also, $s^{t+1}_{2w}\geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w}-s^{t+1}_{1\ell}=\frac{1}{2}(s^t_{2w}+s^t_{2\ell}-s^t_{1w})\geq s^t_{1\ell}\geq 0$. Hence, $\mathbf{s}^{t+1}$ belongs to Scenario 1.

Therefore, we only need to show that in any period t such that players' demographics satisfy Scenario 2, we can find solutions to dual variables, induced by the proposed policy in Theorem 1, which satisfy the complementary slackness conditions and dual feasibility conditions. Note that in period t under Scenario 2, the non-zero primal variables are $f^t_{2w,2w}$, $f^t_{2\ell,1w}$, $f^t_{1w,1w}$ and $f^t_{1\ell,1\ell}$. Therefore, by taking out the condition for $f^t_{2\ell,1\ell}$ and replacing it with the one for $f^t_{1w,1w}$, the complementary slackness conditions in (CS$_1$) change to

$$\begin{aligned}1&=\lambda^t_{2w}-\frac{1}{2}\lambda^{t+1}_{2w}-\frac{1}{2}\lambda^{t+1}_{2\ell},\\ 2&=\lambda^t_{2\ell}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ 1&=\lambda^t_{1w}-\frac{1}{2}\lambda^{t+1}_{1w}-\frac{1}{2}\lambda^{t+1}_{1\ell},\\ \frac{1}{2}&=\lambda^t_{1\ell}-\frac{1}{2}\lambda^{t+1}_{1w},\end{aligned}\tag{CS$_2$}$$

in period t. Similarly, by taking out the condition for $f^t_{1w,1w}$ and replacing with the one for $f^t_{2\ell,1\ell}$, the dual feasibility conditions in (DF$_1$) turn into

$$\begin{aligned}2&\leq\lambda^t_{2w}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ 1&\leq\lambda^t_{2w}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\\ \frac{1}{2}&\leq\lambda^t_{2\ell}-\frac{1}{2}\lambda^{t+1}_{2w},\\ 1&\leq\lambda^t_{2\ell}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\end{aligned}\tag{DF$_2$}$$

in period t. Note that the complementary slackness conditions and the dual feasibility conditions switch back to those in (CS$_1$) and (DF$_1$), starting period $t+1$, as player's demographics transit into Scenario 1. Therefore, we have

$$\lambda^s_{2w}=5,\ \lambda^s_{2\ell}=3,\ \lambda^s_{1w}=9,\text{ and }\lambda^s_{1\ell}=5,\quad\forall s=t+1,...,T-1,$$

from (EC.8) and only need to find λ^t_i, where $i\in\{2w,2\ell,1w,1\ell\}$, satisfy conditions in (CS$_2$) and (DF$_2$). Indeed, such solutions exist and one can verify that

$$\lambda^t_{2w}=5,\ \lambda^t_{2\ell}=4,\ \lambda^t_{1w}=8,\text{ and }\lambda^t_{1\ell}=5,\tag{EC.9}$$

are the desired solution. Therefore, the proof for Scenario 2 is completed.

Scenario 3: $s^t_{2w}+s^t_{2\ell}\geq s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\geq s^t_{1w}$, and $s^t_{2w}<s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=s^t_{2\ell}-s^t_{1w}$, $f^t_{1w,1w}=0$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$;

In the third step, we consider Scenario 3, which shall transit to Scenario 2 after matching is done under the proposed policy. To see this, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 3, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{2w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w}$, $s^{t+1}_{1w} = \frac{1}{2}s^t_{1\ell}$, and $s^{t+1}_{1\ell} = s^t_{1w}$. Then $s^{t+1}_{2\ell} < s^{t+1}_{1w}$ because $s^t_{2w} < s^t_{1\ell}$, and $s^{t+1}_{2w} \geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w})$ and $s^t_{2\ell} - s^t_{1w} \geq 0$. Finally, $s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell} = \frac{1}{2}(2s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) \geq 0$. Hence, $\mathbf{s}^{t+1}$ belongs to Scenario 2.

Thus, we only need to check we can find solutions to dual variables, which satisfy the complementary slackness conditions and dual feasibility conditions. According to the policy in Theorem 1, in period t under Scenario 3, primal variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$ are non-zero. Thus, in period t, the complementary slackness and dual feasibility conditions are

$$\begin{aligned} 1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell}, \\ \frac{1}{2} &= \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w}, \\ 2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\ \frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w}, \end{aligned} \tag{CS$_3$}$$

and

$$\begin{aligned} 2 &\leq \lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - \lambda^{t+1}_{1\ell}, \\ 1 &\leq \lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\ 1 &\leq \lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w}, \\ 1 &\leq \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}\lambda^{t+1}_{1\ell}, \end{aligned} \tag{DF$_3$}$$

respectively, where

$$\lambda^{t+1}_{2w} = 5,\ \lambda^{t+1}_{2\ell} = 4,\ \lambda^{t+1}_{1w} = 8, \text{ and } \lambda^{t+1}_{1\ell} = 5,$$

are from (EC.9), since players' demographic shall transit to Scenario 2 in the next period.

Finally, one can verify that

$$\lambda^t_{2w} = 5.5,\ \lambda^t_{2\ell} = 3,\ \lambda^t_{1w} = 9, \text{ and } \lambda^t_{1\ell} = 4.5, \tag{EC.10}$$

are the desired solutions we are looking for, satisfying (CS$_3$) and (DF$_3$). This completes the proof for Scenario 3.

Scenario 4: $s^t_{2w} + s^t_{2\ell} < s^t_{1w} + s^t_{1\ell}$ and $s^t_{2\ell} < s^t_{1w}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = 0$, $f^t_{1w,1w} = s^t_{1w} - s^t_{2\ell}$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{2\ell}$.

In the fourth step, we consider Scenario 4. For some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 4, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w} = \frac{1}{2}(s^t_{1w} + s^t_{1\ell} - s^t_{2\ell}), s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2w} = \frac{1}{2}s^t_{2w} + s^t_{2\ell}$, and $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w},$. Then $s^{t+1}_{2\ell} < s^{t+1}_{1w}$ because $s^{t+1}_{1w} - s^{t+1}_{2\ell} = \frac{1}{2}(s^t_{1w} + s^t_{1\ell} - s^t_{2w} - s^t_{2\ell}) > 0$. Hence, the state shall either transit to Scenario 2 or stay in Scenario 4 after a match. Suppose at time $t = s$, players' demographic is in Scenario 4. Denote $\tau := \min\{t \geq s \mid s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}\}$, representing the time period players' demographic transit to Scenario 2. First, we argue that $\tau < \infty$. Suppose otherwise, we have $\tau \to$

∞, which implies that the players' demographic stays in Scenario 4 forever. Note that as long as players' demographic belongs to Scenario 4, no high-skilled players shall depart from the matching system since there are always enough low-skilled players to be matched with. Thus, we have

$$\sum_{t=s}^{\infty}(s^t_{2w}+s^t_{2\ell})=\sum_{t=s}^{\infty}(s^s_{2w}+s^s_{2\ell})\to\infty,$$

which contradicts Lemma EC.2. Therefore, we have $\tau<\infty$.

Next, under Scenario 2, according to the optimal policy in Theorem 1, we have $\{f^t_{2w,2w}\}_{t=s}^{\tau-1}$, $\{f^t_{2\ell,1w}\}_{t=s}^{\tau-1}$, $\{f^t_{1w,1w}\}_{t=s}^{\tau-1}$, and $\{f^t_{1\ell,1\ell}\}_{t=s}^{\tau-1}$ as the sequences that contain non-zero primal variables, whereas each elements in sequences $\{f^t_{2w,1w}\}_{t=s}^{\tau-1}$, $\{f^t_{2w,1\ell}\}_{t=s}^{\tau-1}$, $\{f^t_{2\ell,2\ell}\}_{t=s}^{\tau-1}$, and $\{f^t_{2\ell,1\ell}\}_{t=s}^{\tau-1}$ are zero.

To proof the proposed policy is optimal in Scenario 4, we verify that there exists sequences $\{\lambda^t_{2w}\}_{t=s}^{\tau}$, $\{\lambda^t_{2\ell}\}_{t=s}^{\tau}$, $\{\lambda^t_{1w}\}_{t=s}^{\tau}$, and $\{\lambda^t_{1\ell}\}_{t=s}^{\tau}$, such that

$$\lambda^\tau_{2w}=5,\ \lambda^\tau_{2\ell}=4,\ \lambda^\tau_{1w}=8,\text{ and }\lambda^\tau_{1\ell}=5, \tag{EC.11}$$

and for all $s\le t\le\tau-1$, we have

$$\begin{aligned}1&=\lambda^t_{2w}-\frac12\lambda^{t+1}_{2w}-\frac12\lambda^{t+1}_{2\ell},\\ 2&=\lambda^t_{2\ell}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ 1&=\lambda^t_{1w}-\frac12\lambda^{t+1}_{1w}-\frac12\lambda^{t+1}_{1\ell},\\ \frac12&=\lambda^t_{1\ell}-\frac12\lambda^{t+1}_{1w},\end{aligned}\quad\Longleftrightarrow\quad\begin{aligned}\lambda^t_{2w}&=1+\frac12\lambda^{t+1}_{2w}+\frac12\lambda^{t+1}_{2\ell}\\ \lambda^t_{2\ell}&=1+\lambda^{t+1}_{2w}-\frac12\lambda^{t+1}_{1w}+\frac12\lambda^{t+1}_{1\ell}\\ \lambda^t_{1w}&=1+\frac12\lambda^{t+1}_{1w}+\frac12\lambda^{t+1}_{1\ell}\\ \lambda^t_{1\ell}&=\frac12+\frac12\lambda^{t+1}_{1w}\end{aligned} \tag{CS$_4$}$$

as complementary slackness conditions and

$$\begin{aligned}2&\le\lambda^t_{2w}+\lambda^t_{1w}-\lambda^{t+1}_{2w}-\lambda^{t+1}_{1\ell},\\ 1&\le\lambda^t_{2w}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\\ \frac12&\le\lambda^t_{2\ell}-\frac12\lambda^{t+1}_{2w},\\ 1&\le\lambda^t_{2\ell}+\lambda^t_{1\ell}-\lambda^{t+1}_{2w},\end{aligned} \tag{DF$_4$}$$

as dual feasibility conditions.

For convenience, we also rearrange (CS$_4$) forwardly:

$$\lambda^{t+1}_{2w}=\lambda^t_{2\ell}-\lambda^t_{1w}+2\lambda^t_{1\ell}-1, \tag{EC.12}$$

$$\lambda^{t+1}_{2\ell}=2\lambda^t_{2w}-\lambda^t_{2\ell}+\lambda^t_{1w}-2\lambda^t_{1\ell}-1, \tag{EC.13}$$

$$\lambda^{t+1}_{1w}=2\lambda^t_{1\ell}-1, \tag{EC.14}$$

$$\lambda^{t+1}_{1\ell}=2\lambda^t_{1w}-2\lambda^t_{1\ell}-1. \tag{EC.15}$$

Equation (EC.14) follows the third equation of (CS$_4$). Equation (EC.15) follows the last equation in (CS$_4$) and (EC.14). Equation (EC.12) follows the second equation in (CS$_4$), (EC.14), and (EC.15). Finally, (EC.13) follows the first equation in (CS$_4$) and (EC.13).

Taking (EC.12)-(EC.15) into (DF$_4$), we can rewrite the dual feasibility conditions as

$$0 \le \lambda_{2w}^t - \lambda_{2\ell}^t, \tag{EC.16}$$

$$0 \le \lambda_{2w}^t - \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{1\ell}^t, \tag{EC.17}$$

$$0 \le \lambda_{2\ell}^t + \lambda_{1w}^t - 2\lambda_{1\ell}^t, \tag{EC.18}$$

$$0 \le \lambda_{1w}^t - \lambda_{1\ell}^t, \tag{EC.19}$$

Note that (EC.17) is implied by summing up (EC.16) and (EC.19). For the rest of this step, we show that the updated dual feasibility conditions (EC.16), (EC.18),(EC.19) are satisfied for all $s \le t \le \tau$. We will prove a stronger result: (EC.16), (EC.18),(EC.19) together with the following (EC.20) are satisfied for all $s \le t \le \tau$.

$$0 \le -\lambda_{2w}^t + \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{1\ell}^t, \text{ and } -1 \le \lambda_{2w}^t - \lambda_{1w}^t + \lambda_{1\ell}^t. \tag{EC.20}$$

We prove this result by backward induction with the help of the following lemma.

Lemma EC.5. *Consider the dual sequences* $\{\lambda_{2w}^t\}_{t=s}^{\tau}$, $\{\lambda_{2\ell}^t\}_{t=s}^{\tau}$, $\{\lambda_{1w}^t\}_{t=s}^{\tau}$, *and* $\{\lambda_{1\ell}^t\}_{t=s}^{\tau}$ *that is defined by Eq.* (EC.11) *and Eq.* (CS$_4$). *Then we have:*

(1) $\lambda_{1w}^t \ge 5$, $\lambda_{1\ell}^t \ge 3$, *for* $t = s, ..., \tau$.

(2) $\lambda_{1w}^t \le \lambda_{1w}^{t+1}$, $\lambda_{1\ell}^t \le \lambda_{1\ell}^{t+1}$, *for* $t = s, \dots, \tau - 1$.

(3) $\lambda_{2w}^t \ge \lambda_{2w}^{t+1}$, $\lambda_{2\ell}^t \ge \lambda_{2\ell}^{t+1}$, *for* $t = s, \dots, \tau - 1$.

For the base step, we first verify that the conditions are satisfied for both τ and $\tau - 1$. The solution for τ is given by $\lambda_{2w}^\tau = 5$, $\lambda_{2\ell}^\tau = 4$, $\lambda_{1w}^\tau = 8$, and $\lambda_{1\ell}^\tau = 5$, and the solution for $\tau - 1$ is given by $\lambda_{2w}^{\tau-1} = 5.5$, $\lambda_{2\ell}^{\tau-1} = 4.5$, $\lambda_{1w}^{\tau-1} = 7.5$, and $\lambda_{1\ell}^{\tau-1} = 4.5$, and one can easily verify that all the conditions are satisfied. As for the induction step, suppose inequalities in (EC.16), (EC.18), (EC.19) and (EC.20) hold for all periods $t = k+1, ..., \tau$. Consider period $k \le \tau - 2$. For (EC.19), we have

$$\lambda_{1w}^k - \lambda_{1\ell}^k = (1 + \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) - (\frac{1}{2} + \frac{1}{2}\lambda_{1w}^{k+1}) = \frac{1}{2}(1 + \lambda_{1\ell}^{k+1}) \ge 0,$$

where the first equality uses (CS$_4$) to expand λ_{1w}^k and $\lambda_{1\ell}^k$, and the inequality follows Lemma EC.5(1).

Next, we check the inequalities in (EC.16) and (EC.18). Using (CS$_4$) to expand the two conditions, we have

$$\begin{aligned}
&0 \le \lambda_{2w}^k - \lambda_{2\ell}^k \\
\Longleftrightarrow &0 \le (1 + \frac{1}{2}\lambda_{2w}^{k+1} + \frac{1}{2}\lambda_{2\ell}^{k+1}) - (1 + \lambda_{2w}^{k+1} - \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) \\
\Longleftrightarrow &0 \le -\lambda_{2w}^{k+1} + \lambda_{2\ell}^{k+1} + \lambda_{1w}^{k+1} - \lambda_{1\ell}^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
&0 \le \lambda_{2\ell}^k + \lambda_{1w}^k - 2\lambda_{1\ell}^k \\
\Longleftrightarrow &0 \le (1 + \lambda_{2w}^{k+1} - \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) + (1 + \frac{1}{2}\lambda_{1w}^{k+1} + \frac{1}{2}\lambda_{1\ell}^{k+1}) - 2(\frac{1}{2} + \frac{1}{2}\lambda_{1w}^{k+1}) \\
\Longleftrightarrow &-1 \le \lambda_{2w}^{k+1} - \lambda_{1w}^{k+1} + \lambda_{1\ell}^{k+1}.
\end{aligned}$$

Note that these two inequalities are exactly (EC.20) in period $k+1$, which hold by our assumption.

Finally, we check (EC.20) for period k. Plugging in (CS_4) twice for period k and $k+1$, we can express $-\lambda_{2w}^k+\lambda_{2\ell}^k+\lambda_{1w}^k-\lambda_{1\ell}^k$ with variables from $k+2$:

$$\begin{aligned}
-\lambda_{2w}^k+\lambda_{2\ell}^k+\lambda_{1w}^k-\lambda_{1\ell}^k &= \frac{1}{2}(1+\lambda_{2w}^{k+1}-\lambda_{2\ell}^{k+1}-\lambda_{1w}^{k+1}+2\lambda_{1\ell}^{k+1})\\
&= \frac{1}{2}-\frac{1}{4}\lambda_{2w}^{k+2}+\frac{1}{4}\lambda_{2\ell}^{k+2}+\frac{1}{2}\lambda_{1w}^{k+2}-\frac{1}{2}\lambda_{1\ell}^{k+2}\\
&= \frac{1}{2}+\frac{1}{4}(-\lambda_{2w}^{k+2}+\lambda_{2\ell}^{k+2}+\lambda_{1w}^{k+2}-\lambda_{1\ell}^{k+2})+\frac{1}{4}(\lambda_{1w}^{k+2}-\lambda_{1\ell}^{k+2})\geq 0
\end{aligned}$$

where the the inequality follows (EC.20) and (EC.19) in period $k+2$. Similarly, we also have

$$\begin{aligned}
1+\lambda_{2w}^k-\lambda_{1w}^k+\lambda_{1\ell}^k &= \frac{1}{2}\left(\lambda_{2w}^{k+1}+\lambda_{2\ell}^{k+1}-\lambda_{1\ell}^{k+1}\right)+\frac{3}{2}\\
&\geq \frac{1}{2}\left(\lambda_{2w}^{\tau-1}+\lambda_{2\ell}^{\tau-1}-\lambda_{1\ell}^{\tau-1}\right)+\frac{3}{2}=4.25>0,
\end{aligned}$$

where the inequality uses the decreasing property of λ_{2w}^t, $\lambda_{2\ell}^t$ and the increasing property of $\lambda_{1\ell}^t$ from Lemma EC.5. Thus, all dual feasible conditions in (EC.16), (EC.18), (EC.19), and (EC.20) are satisfied, which imply that conditions in (DF_4) also hold for all $s\leq t\leq\tau$. This completes the proof for Scenario 4.

Scenario 5: $s_{2w}^t+s_{2\ell}^t<s_{1w}^t+s_{1\ell}^t$, $s_{2\ell}^t\geq s_{1w}^t$, $s_{2w}^t<s_{1\ell}^t$, and $s_{1\ell}^t\leq K_1=\frac{18}{5}s_{2w}^t+\frac{9}{5}s_{2\ell}^t+\frac{3}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t=s_{2w}^t$, $f_{2\ell,2\ell}^t=s_{2\ell}^t-s_{1w}^t$, $f_{1\ell,1\ell}^t=s_{1\ell}^t$, and $f_{2\ell,1w}^t=f_{1w,2\ell}^t=s_{1w}^t$;

According to the proposed matching flows above, the state of demographics either goes to Scenario 2 or Scenario 4 in period $t+1$. If it goes to Scenario 4, following the optimal solution in Scenario 4, players' demographic eventually evolves to Scenario 2 at period $\tau\leq t+4$. That is, we summarize players' demographics in period $k=t+1,...,\tau$ when $\tau=t+4$ in the next table. Note that in period $t+4$, we have

$$s_{2w}^{t+4}+s_{2\ell}^{t+4}-s_{1w}^{t+4}-s_{1\ell}^{t+4}=\frac{1}{16}\left(18s_{2w}^t+9s_{2\ell}^t+3s_{1w}^t-5s_{1\ell}^t\right)\geq 0,$$

where the inequality follows $s_{1\ell}^t\leq\dfrac{18}{5}s_{2w}^t+\dfrac{9}{5}s_{2\ell}^t+\dfrac{3}{5}s_{1w}^t$. Therefore, we have $\tau\leq t+4$.

Table EC.6 Players' Demographics for Scenario 5

k	s_{2w}^k	$s_{2\ell}^k$	s_{1w}^k	$s_{1\ell}^k$
$t+1$	$\frac{1}{2}\left(s_{2w}^t+s_{2\ell}^t+s_{1w}^t\right)$	$\frac{1}{2}s_{2w}^t$	$\frac{1}{2}s_{1\ell}^t$	s_{1w}^t
$t+2$	$\frac{1}{4}\left(3s_{2w}^t+s_{2\ell}^t+s_{1w}^t\right)$	$\frac{1}{4}\left(s_{2w}^t+s_{2\ell}^t+s_{1w}^t\right)$	$\frac{1}{4}\left(-s_{2w}^t+2s_{1w}^t+s_{1\ell}^t\right)$	$\frac{1}{4}\left(s_{1\ell}^t+s_{2w}^t\right)$
$t+3$	$\frac{1}{8}\left(5s_{2w}^t+3s_{2\ell}^t+3s_{1w}^t\right)$	$\frac{1}{8}(3s_{2w}^t+s_{2\ell}^t+s_{1w}^t)$	$\frac{1}{8}\left(-s_{2w}^t-s_{2\ell}^t+s_{1w}^t+2s_{1\ell}^t\right)$	$\frac{1}{8}\left(s_{2\ell}^t+3s_{1w}^t+s_{1\ell}^t\right)$
$t+4$	$\frac{1}{16}\left(11s_{2w}^t+5s_{2\ell}^t+5s_{1w}^t\right)$	$\frac{1}{16}\left(5s_{2w}^t+3s_{2\ell}^t+3s_{1w}^t\right)$	$\frac{1}{16}\left(-4s_{2w}^t-s_{2\ell}^t+3s_{1w}^t+3s_{1\ell}^t\right)$	$\frac{1}{8}(s_{2w}^t+s_{1w}^t+s_{1\ell}^t)$

In Table EC.7, we list the dual variables when $\tau=t+1,...,t+4$ respectively, and one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasible conditions.

Scenario 6: $s_{2w}^t+s_{2\ell}^t<s_{1w}^t+s_{1\ell}^t$, $s_{2\ell}^t\geq s_{1w}^t$, $s_{2w}^t<s_{1\ell}^t$, and $s_{1\ell}^t>K_2=\frac{18}{5}s_{2w}^t+\frac{23}{5}s_{2\ell}^t-\frac{11}{5}s_{1w}^t$; The optimal matching flows are: $f_{2w,2w}^t=s_{2w}^t$, $f_{2\ell,1w}^t=f_{1w,2\ell}^t=s_{1w}^t$, $f_{2\ell,1\ell}^t=f_{1\ell,2\ell}^t=s_{2\ell}^t-s_{1w}^t$, and $f_{1\ell,1\ell}^t=s_{1\ell}^t-f_{2\ell,1\ell}^t$.

Table EC.7 Dual Variables for Scenario 5

$\tau = t+1$	$k=t$	$k=t+1$			
λ^k_{2w}	5.5	5			
$\lambda^k_{2\ell}$	3	4			
λ^k_{1w}	9	8			
$\lambda^k_{1\ell}$	4.5	5			
$\tau = t+2$	$k=t$	$k=t+1$	$k=t+2$		
λ^k_{2w}	6	5.5	5		
$\lambda^k_{2\ell}$	3.25	4.5	4		
λ^k_{1w}	8.75	7.5	8		
$\lambda^k_{1\ell}$	4.25	4.5	5		
$\tau = t+3$	$k=t$	$k=t+1$	$k=t+2$	$k=t+3$	
λ^k_{2w}	6.5	6	5.5	5	
$\lambda^k_{2\ell}$	3.5	5	4.5	4	
λ^k_{1w}	8.75	7	7.5	8	
$\lambda^k_{1\ell}$	4	4.25	4.5	5	
$\tau = t+4$	$k=t$	$k=t+1$	$k=t+2$	$k=t+3$	$k=t+4$
λ^k_{2w}	7.0625	6.5	6	5.5	5
$\lambda^k_{2\ell}$	3.75	5.625	5	4.5	4
λ^k_{1w}	8.75	6.625	7	7.5	8
$\lambda^k_{1\ell}$	3.8125	4	4.25	4.5	5

Following the policy above, the state of demographics in the next period is given by $s^{t+1}_{1w} = \frac{1}{2}(s^t_{1\ell} - (s^t_{2\ell} - s^t_{1w}))$, $s^{t+1}_{1\ell} = s^t_{1w}$, $s^{t+1}_{2w} = \frac{1}{2}s^t_{2w} + s^t_{2\ell}$, $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w}$. This corresponds to Scenario 4 since

$$\begin{aligned} s^{t+1}_{2w} + s^{t+1}_{2\ell} - s^{t+1}_{1w} - s^{t+1}_{1\ell} &= s^t_{2w} + \frac{3}{2}s^t_{2\ell} - \frac{3}{2}s^t_{1w} - \frac{1}{2}s^t_{1\ell} \\ &< s^t_{2w} + \frac{3}{2}s^t_{2\ell} - \frac{3}{2}s^t_{1w} - \frac{1}{2}K_2 \\ &= -\frac{4}{5}s^t_{2w} - \frac{4}{5}s^t_{2\ell} - \frac{2}{5}s^t_{1w} < 0. \end{aligned}$$

Also, $s^{t+1}_{1w} - s^{t+1}_{2\ell} = \frac{1}{2}(s^t_{1w} + s^t_{1\ell} - s^t_{2w} - s^t_{2\ell}) < 0$.

Since $t+1$, we follow the proposed policy in Scenario 4 until we reach Scenario 2 at τ. One can verify that $\tau > t+5$, because following the proposed solution in Scenario 4, we have $s^{t+5}_{1w} = -\frac{1}{4}s^t_{2w} - \frac{1}{16}s^t_{2\ell} + \frac{3}{16}s^t_{1w} + \frac{3}{16}s^t_{1\ell}$, $s^{t+5}_{1\ell} = \frac{1}{8}s^t_{2w} + \frac{1}{8}s^t_{1w} + \frac{1}{8}s^t_{1\ell}$, $s^{t+5}_{2w} = \frac{11}{16}s^t_{2w} + \frac{5}{8}s^t_{2\ell}$, $s^{t+5}_{2\ell} = \frac{5}{16}s^t_{2w} + \frac{3}{8}s^t_{2\ell}$. To check if it still belongs to Scenario 4,

$$\begin{aligned} s^{t+5}_{2w} + s^{t+5}_{2\ell} - s^{t+5}_{1w} - s^{t+5}_{1\ell} &= \frac{9}{8}s^t_{2w} + \frac{23}{16}s^t_{2\ell} - \frac{11}{16}s^t_{1w} - \frac{5}{16}s^t_{1\ell} \\ &< \frac{9}{8}s^t_{2w} + \frac{23}{16}s^t_{2\ell} - \frac{11}{16}s^t_{1w} - \frac{5}{16}K_2 = 0. \end{aligned}$$

For periods $t+1$ and forward, we use the dual variables proposed in Scenario 4, 2, and 1. Hence, we only need to show that the proposed policy as well as the corresponding dual variables at period t satisfiy complementary slackness and dual feasibility in order to establish optimality. By complementary slackness, we have

$$\lambda^t_{2w} = 1 + \frac{1}{2}\lambda^{t+1}_{2w} + \frac{1}{2}\lambda^{t+1}_{2\ell} \tag{EC.21}$$

$$\lambda^t_{2\ell} = \frac{1}{2} + \lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{1w}, \tag{EC.22}$$

$$\lambda_{1w}^t = \frac{3}{2} + \lambda_{1\ell}^{t+1} + \frac{1}{2}\lambda_{1w}^{t+1}, \tag{EC.23}$$

$$\lambda_{1\ell}^t = \frac{1}{2} + \frac{1}{2}\lambda_{1w}^{t+1}. \tag{EC.24}$$

Then, we need to validate the dual feasibility condition corresponding to $f_{2\ell,2\ell}^t, f_{1w,1w}^t, f_{2w,1w}^t, f_{2w,1\ell}^t$, which are given by:

$$\lambda_{2\ell}^t - 0.5\lambda_{2w}^{t+1} \geq 0.5, \tag{EC.25}$$

$$\lambda_{1w}^t - 0.5\lambda_{1w}^{t+1} - 0.5\lambda_{1\ell}^{t+1} \geq 1, \tag{EC.26}$$

$$\lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - \lambda_{1\ell}^{t+1} \geq 2, \tag{EC.27}$$

$$\lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} \geq 1. \tag{EC.28}$$

Taking (EC.21)-(EC.24) into the above inequalities, it is equivalent to validate

$$\lambda_{2w}^{t+1} \geq \lambda_{1w}^{t+1}, \tag{EC.29}$$

$$1.5 + \lambda_{1\ell}^{t+1} \geq 1, \tag{EC.30}$$

$$0.5 + 0.5\lambda_{2\ell}^{t+1} + 0.5\lambda_{1w}^{t+1} \geq 0.5\lambda_{2w}^{t+1}, \tag{EC.31}$$

$$1.5 + 0.5\lambda_{2\ell}^{t+1} + 0.5\lambda_{1w}^{t+1} \geq 0.5\lambda_{2w}^{t+1}. \tag{EC.32}$$

Among them, (EC.30) is trivially true, because $\lambda_{1\ell}^{t+1}$ is in Scenario 4 and is greater than 5 by Lemma EC.5. (EC.31) and (EC.32) are directly from (EC.20). Thus, we only need to validate (EC.29). Note that in Scenario 6, we have $\tau \geq t+5$. At $k = \tau - 4 \geq t+1$, we have

$$\lambda_{2w}^{\tau-4} = 7.0625,\ \lambda_{2\ell}^{\tau-4} = 6.1875,\ \lambda_{1w}^{\tau-4} = 6.3125, \text{ and } \lambda_{1\ell}^{\tau-4} = 3.8125, \tag{EC.33}$$

by using the complementary slackness conditions recursively. By Lemma EC.5, we know that $\lambda_{2w}^{t+1} \geq \lambda_{2w}^{\tau-4} \geq \lambda_{1w}^{\tau-4} \geq \lambda_{1w}^{t+1}$, which completes the proof.

Scenario 7: $s_{2w}^t + s_{2\ell}^t < s_{1w}^t + s_{1\ell}^t$, $s_{2\ell}^t \geq s_{1w}^t$, $s_{2w}^t < s_{1\ell}^t$, and $K_1 < s_{1\ell}^t \leq K_2$; The optimal matching flows are: $f_{2w,2w}^t = s_{2w}^t$, $f_{2\ell,1w}^t = f_{1w,2\ell}^t = s_{1w}^t$, $f_{2\ell,2\ell}^t = \frac{9}{7}s_{2w}^t + \frac{23}{14}s_{2\ell}^t - \frac{11}{14}s_{1w}^t - \frac{5}{14}s_{1\ell}^t$, $f_{2\ell,1\ell}^t = f_{1\ell,2\ell}^t = \frac{5}{14}s_{1\ell}^t - \frac{9}{7}s_{2w}^t - \frac{9}{14}s_{2\ell}^t - \frac{3}{14}s_{1w}^t$, and $f_{1\ell,1\ell}^t = s_{1\ell}^t - f_{2\ell,1\ell}^t$.

In this scenario, the state of demographics will transit to Scenario 4 in the second period. Further, one can verify that the system reaches Scenario 2 in period $k = t+4$, with $s_{2w}^{t+4} + s_{2\ell}^{t+4} = s_{1w}^{t+4} + s_{1\ell}^{t+4}$. Further, in period $k = t+5$, the system goes to Scenario 1 with $s_{2\ell}^{t+5} = s_{1w}^{t+5}$. Hence, in period $k = t+5$, we reach a degenerate case with $f_{2\ell,2\ell}^{t+5} = 0$. From the view of the simplex method, under the solution of Scenario 5.1, the reduced cost of $f_{2\ell,1\ell}^t$ is positive, so we take it into the basic feasible solution, and move $f_{2\ell,2\ell}^{t+5}$ out of the basis. The positiveness of all the other flows remains. To see if the solution is optimal, we list out the dual variables for the first $t+6$ periods in Table EC.8, and for periods $k > t+6$, we always have

$$\lambda_{2w}^k = 5,\ \lambda_{2\ell}^k = 3,\ \lambda_{1w}^k = 9, \text{ and } \lambda_{1\ell}^k = 5.$$

Then one can easily verify that the proposed dual variables satisfy complementary slackness and dual feasibility. □

Table EC.8 Dual Variables for Scenario 7

k	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$
λ^k_{2w}	50/7	737/112	85/14	39/7	71/14	5	5
$\lambda^k_{2\ell}$	849/224	639/112	285/56	32/7	57/14	22/7	4
λ^k_{1w}	1963/224	737/112	389/56	52/7	111/14	62/7	8
$\lambda^k_{1\ell}$	849/224	445/112	59/14	125/28	69/14	5	5
Primal	Scenario 7	Scenario 4	Scenario 4	Scenario 4	Scenario 2	Scenario 1	Scenario 1

Proof of Lemma EC.5 (1) We prove the statements by backward induction from period τ.

First, we show that $\lambda^t_{1w} \geq 5$, $\lambda^t_{1\ell} \geq 3$ for all $t = s, ..., \tau$. This is true for τ according to (EC.11), which completes the base step.

The induction hypothesis is that $\lambda^k_{1w} \geq 5$, $\lambda^k_{1\ell} \geq 3$ for all $k = t, ..., \tau$. Then for period $t-1$, from Eq. (CS$_4$) we have

$$\lambda^{t-1}_{1w} = 1 + 0.5\lambda^t_{1w} + 0.5\lambda^t_{1\ell} \geq 1 + 0.5 \cdot 5 + 0.5 \cdot 3 = 5,$$

$$\lambda^{t-1}_{1\ell} = 0.5 + 0.5\lambda^t_{1w} \geq 0.5 + 0.5 \cdot 5 = 3,$$

and the induction step is completed.

(2) We show that $\lambda^t_{1w} \leq \lambda^{t+1}_{1w}$, $\lambda^t_{1\ell} \leq \lambda^{t+1}_{1\ell}$, for $t = s, \ldots, \tau - 1$. For the induction base step, we have $\lambda^{\tau}_{1w} = 7.5 \leq 8 = \lambda^{\tau}_{1w}$ and $\lambda^t_{1\ell} = 4.5 \leq 5 = \lambda^{\tau}_{1\ell}$. Our induction hypothesis it that for all $k = t, ..., \tau - 1$, we have $\lambda^k_{1w} \leq \lambda^{k+1}_{1w}$ and $\lambda^k_{1\ell} \leq \lambda^{k+1}_{1\ell}$. Now, consider period $t-1$. Take the difference between λ^{t-1}_{1w} and λ^t_{1w}, we have

$$\begin{aligned}
&\lambda^{t-1}_{1w} - \lambda^t_{1w} \\
=&1 + 0.5\lambda^t_{1w} + 0.5\lambda^t_{1\ell} - \lambda^t_{1w} \\
=&1 + 0.5\lambda^k_{1\ell} - 0.5\lambda^k_{1w} \\
=&0.75 - 0.25\lambda^{k+1}_{1\ell} \\
\leq&0.75 - 0.25 \cdot 3 = 0,
\end{aligned} \tag{EC.34}$$

where the first equality follows Eq. (CS$_4$), the third equality follows (EC.15), and the last inequality follows the fact that $\lambda^t_{1\ell} \geq 3$. Next, we show that $\lambda^{t-1}_{1\ell} \leq \lambda^t_{1\ell}$. Note that from the last equation in (CS$_4$), we have $\lambda^{t-1}_{1\ell} = 0.5 + 0.5\lambda^t_{1w}$. Since $\lambda^t_{1w} \leq \lambda^{t+1}_{1w}$, we have $\lambda^{t-1}_{1\ell} \leq 0.5 + 0.5\lambda^{t+1}_{1w} = \lambda^t_{1\ell}$.

(3) Finally, we show that $\lambda^t_{2w} \geq \lambda^{t+1}_{2w}$, $\lambda^t_{2\ell} \geq \lambda^{t+1}_{2\ell}$. For the base step, we have $\lambda^{\tau-1}_{2w} = 5.5 \geq 5 = \lambda^{\tau}_{2w}$ and $\lambda^{\tau-1}_{2\ell} = 4.5 \geq 4 = \lambda^{\tau}_{2\ell}$. The induction hypothesis is that for all $k = t, ..., \tau - 1$, we have $\lambda^k_{2w} \geq \lambda^{k+1}_{2w}$ and $\lambda^k_{2\ell} \geq \lambda^{k+1}_{2\ell}$. Now, consider period $t-1$. We have

$$\lambda^{t-1}_{2w} = 1 + 0.5\lambda^t_{2w} + 0.5\lambda^t_{2\ell} \geq 1 + 0.5\lambda^{t+1}_{2w} + 0.5\lambda^{t+1}_{2\ell} = \lambda^t_{2w},$$

where the equality on the two sides follows the first equation of (CS$_4$). Next, consider $\lambda^{t-1}_{2\ell}$. We have

$$\begin{aligned}
\lambda^{t-1}_{2\ell} &= 1 + \lambda^t_{2w} - 0.5\lambda^t_{1w} + 0.5\lambda^t_{1\ell} \\
&= 1 + \lambda^t_{2w} - 0.5(1 + 0.5\lambda^{t+1}_{1w} + 0.5\lambda^{t+1}_{1\ell}) + 0.5(0.5 + 0.5\lambda^{t+1}_{1w}) \\
&= 1 + \lambda^t_{2w} - 0.25 - 0.25\lambda^{t+1}_{1\ell} \\
&\geq 1 + \lambda^{t+1}_{2w} - 0.25 - 0.25\lambda^{t+2}_{1\ell} \\
&= \lambda^t_{2\ell},
\end{aligned}$$

where the first equality follows the second equation of (CS$_4$), the second equality follows the third and fourth equations of (CS$_4$), the first inequality follows the fact that $\lambda_{1\ell}^{t+1} \le \lambda_{1\ell}^{t+2}$ and $\lambda_{2w}^{t} \ge \lambda_{2w}^{t+1}$. Thus, the induction step is completed and this completes the proof. $\square$

Proof of Theorem 2. (a) In order to prove the first statement, we consider the linear program for the one-shot matching problem in (P$_1$) and use the same simplification tricks we used in Appendix B.1. Without loss of generality, we will set the engagement level of SBMM in the next period to be 1, which is equivalent to the constraint $1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}$. Thus, the following optimization problem selects the initial state of the demographics $\mathbf{s}$ to maximize the ratio of the optimal policy to SBMM for the one-period problem (we drop all the superscripts, representing time periods, since it is a one-shot problem):

$$\max_{\mathbf{f},\mathbf{s}} f_{2w,2w} + \frac{1}{2}f_{2\ell,2\ell} + f_{1w,1w} + \frac{1}{2}f_{1\ell,1\ell} + 2f_{2w,1w} + f_{2\ell,1\ell} + f_{2w,1\ell} + 2f_{2\ell,1w}$$

s.t.

$$\begin{aligned}
1 &= s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}\\
s_{2w} &= f_{2w,2w} + f_{2w,1w} + f_{2w,1\ell},\\
s_{2\ell} &= f_{2\ell,2\ell} + f_{2\ell,1w} + f_{2\ell,1\ell},\\
s_{1w} &= f_{1w,1w} + f_{2\ell,1w} + f_{2w,1w},\\
s_{1\ell} &= f_{1\ell,1\ell} + f_{2w,1\ell} + f_{2\ell,1\ell}.
\end{aligned}$$

which is the linear program for the one-shot matching problem in (P$_1$), maximizing over the initial demographics and matching flows, with an additional constraint. Without loss of generality, the additional constraint $1 = s_{2w} + \frac{1}{2}s_{2\ell} + s_{1w} + \frac{1}{2}s_{1\ell}$, normalizes the value of one-shot SBMM to 1.

We verify that the optimal solution to the above optimization problem is $s_{2\ell} = s_{1w} = 2/3$, $f_{2\ell,1w} = 2/3$, and all other matching flows are 0. The objective value is 4/3, which is the desired ratio. Denote λ_0 as the dual variable corresponding to the constraint normalizing the engagement for SBMM to be 1. We verify the proposed solution using complementary slackness conditions:

$$\begin{aligned}
0 &= \frac{1}{2}\lambda_0 - \lambda_{2\ell},\\
0 &= \lambda_0 - \lambda_{1w},\\
2 &= \lambda_{2\ell} + \lambda_{1w},
\end{aligned}$$

where the conditions correspond to primal non-zero variables $s_{2\ell}$, s_{1w}, and $f_{2\ell,1w}$, respectively. There is a unique solution of dual variables solving the complementary slackness conditions: $\lambda_0 = 4/3$, $\lambda_{2\ell} = 2/3$, and $\lambda_{1w} = 4/3$. To complete the proof, we need to check dual feasibility conditions:

$$\begin{aligned}
&0 \le \lambda_0 - \lambda_{2w},\ 0 \le \frac{1}{2}\lambda_0 - \lambda_{1\ell},\ 1 \le \lambda_{2w},\ \frac{1}{2} \le \lambda_{2\ell},\ 1 \le \lambda_{1w},\ \frac{1}{2} \le \lambda_{1\ell},\\
&2 \le \lambda_{2w} + \lambda_{1w},\ 1 \le \lambda_{2\ell} + \lambda_{1\ell},\ 1 \le \lambda_{2w} + \lambda_{1\ell},\ 1 \le \lambda_{2\ell} + \lambda_{1w},
\end{aligned}$$

representing zero state variables (s_{2w}, $s_{1\ell}$) and zero matching flows ($f_{2w,2w}$, $f_{2\ell,2\ell}$, $f_{1w,1w}$, $f_{1\ell,1\ell}$, $f_{2w,1w}$, $f_{2\ell,1\ell}$, $f_{2w,1\ell}$), respectively.

(b) Next, we turn our attention to the infinite horizon problem in (P'). Using a similar idea, we can solve an optimization problem to find the maximum ratio between the optimal matching policy and SBMM. Using Lemma 1, the value function of the baseline model under SBMM is

$$V^{\mathrm{SBMM}}(\mathbf{s}^0) = 5(s^t_{2w} + s^t_{1w}) + 3(s^t_{2\ell} + s^t_{1\ell}), \quad t = 1, 2, \ldots, \tag{EC.35}$$

which we normalize to 1 without loss of generality. Thus, the following optimization problem selects the initial state of the demographics $\mathbf{s^0}$ to maximize the ratio of the optimal policy to SBMM for the infinite-horizon problem (we set $t = 0$ without loss of generality):

$$\max_{\{\mathbf{f}^t\},\{\mathbf{s}^0\}} \sum_{t=0}^{\infty} \left(f^t_{2w,2w} + \frac{1}{2} f^t_{2\ell,2\ell} + f^t_{1w,1w} + \frac{1}{2} f^t_{1\ell,1\ell} + 2 f^t_{2w,1w} + f^t_{2\ell,1\ell} + f^t_{2w,1\ell} + 2 f^t_{2\ell,1w} \right) \tag{EC.36}$$

$$\tag{EC.37}$$

s.t.

$$1 = 5(s^0_{2w} + s^0_{1w}) + 3(s^0_{2\ell} + s^0_{1\ell}), \tag{EC.38}$$

$$\begin{aligned}
0 &= f^0_{2w,2w} + f^0_{2w,1w} + f^0_{2w,1\ell} - s^0_{2w}, \\
0 &= f^0_{2\ell,2\ell} + f^0_{2\ell,1w} + f^0_{2\ell,1\ell} - s^0_{2\ell}, \\
0 &= f^0_{1w,1w} + f^0_{2\ell,1w} + f^0_{2w,1w} - s^0_{1w}, \\
0 &= f^0_{1\ell,1\ell} + f^0_{2w,1\ell} + f^0_{2\ell,1\ell} - s^0_{1\ell},
\end{aligned}$$

and for all $t = 1, 2, \ldots$,

$$\begin{aligned}
& f^t_{2w,2w} + f^t_{2w,1w} + f^t_{2w,1\ell} = \frac{1}{2}\left(f^{t-1}_{2w,2w} + f^{t-1}_{2\ell,2\ell}\right) + f^{t-1}_{2w,1w} + f^{t-1}_{2w,1\ell} + f^{t-1}_{2\ell,1w} + f^{t-1}_{2\ell,1\ell}, \\
& f^t_{2\ell,2\ell} + f^t_{2\ell,1w} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{2w,2w}, \\
& f^t_{1w,1w} + f^t_{2\ell,1w} + f^t_{2w,1w} = \frac{1}{2}\left(f^{t-1}_{1w,1w} + f^{t-1}_{1\ell,1\ell}\right), \\
& f^t_{1\ell,1\ell} + f^t_{2w,1\ell} + f^t_{2\ell,1\ell} = \frac{1}{2} f^{t-1}_{1w,1w} + f^{t-1}_{2w,1w} + f^{t-1}_{2\ell,1w}, \\
& f^t_{i,j} \geq 0, \ \forall i, j \in \{2w, 2\ell, 1w, 1\ell\}.
\end{aligned}$$

Again, we use dual complementary slackness and feasibility conditions verifying the initial state $\mathbf{s}^0 = \{0, 1/8, 1/8, 0\}$, along with the optimal matching flows in Theorem 1, is the optimal solution to the above maximization problem. The objective value is $3/2$, which is the desired ratio.

With slight abuse of notation, denote the dual variable to the new constraint (EC.38) as λ_0. We show that under the optimal initial state where $s_{2\ell} = s_{1w} = 1/8$, we have $\lambda_0 = 3/2$, representing the maximum ratio. The proposed initial state is in Scenario 1. Based on the transition in Fig. EC.1, we shall always stay in Scenario 1. We list the states in periods 0, 1, and 2 in Table EC.9.

Note that in period 0, we reach a degenerate period with only one positive flow $f^0_{2l,1w}$, and in period 1 we reach a degenerate case with only two positive flows $f^1_{2w,2w}$ and $f^1_{1\ell,1\ell}$. Thus, for these two periods, we can use only part of the equations in (CS_1). To be specific, given our proposed primal solution, the complementary slackness equations are given by

$$
\begin{aligned}
0 &= 3\lambda_0 - \lambda^0_{2\ell},\\
0 &= 5\lambda_0 - \lambda^0_{1w},\\
2 &= \lambda^0_{2\ell} + \lambda^0_{1w} - \lambda^1_{2w} - \lambda^1_{1\ell},\\
1 &= \lambda^1_{2w} - \frac{1}{2}\lambda^2_{2w} - \frac{1}{2}\lambda^2_{2\ell},\\
\frac{1}{2} &= \lambda^1_{1\ell} - \frac{1}{2}\lambda^2_{1w},\\
&\quad (\mathrm{CS}_1),\ t=2,\ldots
\end{aligned}
$$

and the dual feasibility constraints we need to check is

$$
\begin{aligned}
&0 \le 5\lambda_0 - \lambda^0_{2w}, \quad 0 \le 3\lambda_0 - \lambda^0_{1w}, \quad 1 \le \lambda^0_{2w} - \frac{1}{2}\lambda^1_{2w} - \frac{1}{2}\lambda^1_{2\ell},\\
&\frac{1}{2} \le \lambda^0_{2\ell} - \frac{1}{2}\lambda^1_{2w}, \quad \frac{1}{2} \le \lambda^0_{1\ell} - \frac{1}{2}\lambda^1_{1w}, \quad \frac{1}{2} \le \lambda^1_{1\ell} - \frac{1}{2}\lambda^2_{1w}\\
&2 \le \lambda^1_{2\ell} + \lambda^1_{1w} - \lambda^2_{2w} - \lambda^2_{1\ell}, \quad (\mathrm{DF}_1) \text{ for } t=0,\ldots
\end{aligned}
$$

We then list out the dual variables inperiods 0, 1, and 2 in Table EC.9. For $t > 2$, we use $\lambda^0_{2w} = 5$ $\lambda^0_{2\ell} = 3$, $\lambda_{1w} = 9$, $\lambda_{1\ell} = 5$. Together with $\lambda_0 = 3/2$, one can verify that the proposed dual solution satisfies complementary slackness and dual feasibility constraints.

Table EC.9 Primal States and Dual Variables in Period 0, 1, and 2.

	s^t_{2w}	$s^t_{2\ell}$	s^t_{1w}	$s^t_{1\ell}$	λ^t_{2w}	$\lambda^t_{2\ell}$	λ^t_{1w}	$\lambda^t_{1\ell}$
$t=0$	0	1/8	1/8	0	11/2	9/2	15/2	9/2
$t=1$	1/8	0	0	1/8	5	4	8	5
$t=2$	1/16	1/16	1/16	0	5	3	9	5

□

Proof of Corollary 1. Denote the winrate between high- and low-skilled players as $\alpha \in [0.5, 1]$. In this part, we show that for $\alpha < 1$, the maximum power of optimal matching is non-increasing when α decreases. Note that under this setting, one unit of cross-level matching that induces high-skilled players to win with probability α can be decomposed into the following three matching flows: 1) $2\alpha - 1$ unit of perfect cross-level matching flow, where high-skilled players always win against low-skilled players; 2) $2(1-\alpha)$ unit of high-skilled players matching with each other; and 3) low-skilled players matching with each other. Take the flow between 2ℓ and $1w$ as an example. One can easily verify that it can be decomposed into $(2\alpha - 1)f_{2\ell,1w}$, $2(1-\alpha)f_{2\ell,2\ell}$, and $2(1-\alpha)f_{1w,1w}$, which induce the same player demographics in the next period.
As a result, to have a unit of cross-level flow in this setting, we also need $\dfrac{2\alpha-1}{2(1-\alpha)}$ unit of SBMM among both high- and low-skilled level players. Therefore, when solving we need to include the following additional constraints when solving the linear program in (EC.37) in the proof of part (b):

$$
f^t_{i,j} \le \frac{2\alpha-1}{2(1-\alpha)} f^t_{i,i}, \text{ and } f^t_{i,j} \le \frac{2\alpha-1}{2(1-\alpha)} f^t_{i,i}, \quad \forall i \in \{2w, 2\ell\},\, j \in \{1w, 1\ell\},\, t = 0,1,2,\ldots \tag{EC.39}
$$

Clearly, $\frac{2\alpha-1}{2(1-\alpha)}$ is a increasing function in α. Therefore, as α decreases, the feasible region also shrinks, leading to non-increasing optimal objective values. Lastly, note that when $\alpha = 1/2$, the problem reduces to pure SBMM and this completes the proof. □

B.1.3. Proof of Theorem 3 Before showing the results of the interplay between PTW strategies and matchmaking, we update the flow balancing and demographic evolution constraints, then formally state the matchmaker's problem when facing PTW.

In any period t, the flow balance constraints are now

$$\begin{aligned}
\sum_j f^t_{i,j} &= s^t_i,\ \forall i \in \bar{\mathcal{P}},\\
\sum_i f^t_{i,j} &= s^t_j,\ \forall j \in \bar{\mathcal{P}},\\
f^t_{i,j} &= f^t_{j,i},\ \forall i \neq j,\ i,j \in \bar{\mathcal{P}},\\
f^t_{i,j} &\geq 0,\ \forall i \neq j,\ i,j \in \bar{\mathcal{P}},
\end{aligned} \tag{FB$_{\text{ptw}}$}$$

where $\bar{\mathcal{P}} := \{\bar{2}w, \bar{2}\ell, \underline{2}w, \underline{2}\ell, \underline{1}w, \underline{1}\ell\}$, and evolution of demographics are now

$$\begin{aligned}
s^{t+1}_{\bar{2}w} &= \frac{1}{2}(f^t_{\bar{2}w,\bar{2}w} + f^t_{\bar{2}w,\bar{2}\ell} + f^t_{\bar{2}\ell,\bar{2}w} + f^t_{\bar{2}\ell,\bar{2}\ell} + f^t_{\bar{2}w,\underline{2}w} + f^t_{\bar{2}w,\underline{2}\ell} + f^t_{\bar{2}\ell,\underline{2}w} + f^t_{\bar{2}\ell,\underline{2}\ell}) + f^t_{\bar{2}w,\underline{1}w} + f^t_{\bar{2}w,\underline{1}\ell} + f^t_{\bar{2}\ell,\bar{1}w} + f^t_{\bar{2}\ell,\underline{1}\ell},\\
s^{t+1}_{\bar{2}\ell} &= \frac{1}{2}(f^t_{\bar{2}w,\bar{2}w} + f^t_{\bar{2}w,\bar{2}\ell} + f^t_{\bar{2}w,\underline{2}w} + f^t_{\bar{2}w,\underline{2}\ell}) + \beta(\frac{1}{2}f^t_{\underline{1}w,\underline{1}w} + \frac{1}{2}f^t_{\underline{1}w,\underline{1}\ell} + f^t_{\underline{1}w,\underline{2}w} + f^t_{\underline{1}w,\underline{2}\ell} + f^t_{\underline{1}w,\bar{2}w} + f^t_{\underline{1}w,\bar{2}\ell}),\\
s^{t+1}_{\underline{2}w} &= \frac{1}{2}(f^t_{\underline{2}w,\underline{2}w} + f^t_{\underline{2}w,\underline{2}\ell} + f^t_{\underline{2}\ell,\underline{2}w} + f^t_{\underline{2}\ell,\underline{2}\ell} + f^t_{\underline{2}w,\bar{2}w} + f^t_{\underline{2}w,\bar{2}\ell} + f^t_{\underline{2}\ell,\bar{2}w} + f^t_{\underline{2}\ell,\bar{2}\ell}) + f^t_{\underline{2}w,\underline{1}w} + f^t_{\underline{2}w,\underline{1}\ell} + f^t_{\underline{2}\ell,\underline{1}w} + f^t_{\underline{2}\ell,\underline{1}\ell},\\
s^{t+1}_{\underline{2}\ell} &= \frac{1}{2}(f^t_{\underline{2}w,\underline{2}w} + f^t_{\underline{2}w,\underline{2}\ell} + f^t_{\underline{2}w,\bar{2}w} + f^t_{\underline{2}w,\bar{2}\ell}),\\
s^{t+1}_{\underline{1}w} &= \frac{1}{2}(f^t_{\underline{1}w,\underline{1}w} + f^t_{\underline{1}w,\underline{1}\ell} + f^t_{\underline{1}\ell,\underline{1}w} + f^t_{\underline{1}\ell,\underline{1}\ell}),\\
s^{t+1}_{\underline{1}\ell} &= (1-\beta)(\frac{1}{2}f^t_{\underline{1}w,\underline{1}w} + \frac{1}{2}f^t_{\underline{1}w,\underline{1}\ell} + f^t_{\underline{1}w,\underline{2}w} + f^t_{\underline{1}w,\underline{2}\ell} + f^t_{\underline{1}w,\bar{2}w} + f^t_{\underline{1}w,\bar{2}\ell}).
\end{aligned} \tag{ED$_{\text{ptw}}$}$$

From (FB$_{\text{ptw}}$) and (ED$_{\text{ptw}}$) conditions, we can see that players in the new demographics with skill level $\bar{2}$ behave the same as those with skill level $\underline{2}\ell$. However, the matchmaker's objective is largely different from the one in (P). The matchmaker also needs to consider the revenue generated by paid subscriptions. Given β, r, and the initial demographics $\mathbf{s}^0 = \{s^0_{2w}, s^0_{2\ell}, s^0_{1w}, s^0_{1\ell}\}$, the matchmaker's problem is

$$\begin{aligned}
V^*(\beta, r, \mathbf{s}^0) := \max_{\{f^t_{i,j}\}^\infty_{t=1}} & \sum_{t=1}^{\infty}\sum_i s^t_i + \sum_{t=1}^{\infty} r(s^t_{\bar{2}w} + s^t_{\bar{2}\ell}) = \max_{\{f^t_{i,j}\}^\infty_{t=1}} ENG(\beta, r, \mathbf{s}^0) + REV(\beta, r, \mathbf{s}^0) \\
\text{s.t. } & s^0_{\bar{2}w} = 0,\\
& s^0_{\bar{2}\ell} = 0,\\
& s^0_{\underline{2}w} = s^0_{2w},\\
& s^0_{\underline{2}l} = s^0_{2\ell},\\
& s^0_{\underline{1}w} = s^0_{1w},\\
& s^0_{\underline{1}\ell} = s^0_{1\ell},\\
& (\text{FB}_{\text{ptw}}) \text{ and } (\text{ED}_{\text{ptw}})\ \forall t = 0,1,2,\ldots, \text{ and } i,j \in \bar{\mathcal{P}},
\end{aligned} \tag{P$_{\text{ptw}}$}$$

where the objective function is to maximize the sum of player engagement $ENG(\beta, r, \mathbf{s}^0)$ and revenue from paid subscription $REV(\beta, r, \mathbf{s}^0)$.

Proof of Theorem 3. We prove the three parts separately.

(a) We show that unsubscribed high-skilled players in $\underline{2}w$ ($\underline{2}\ell$) would only be matched with any unsubscribed low-skilled players after all the subscribed players in $\bar{2}w$ ($\bar{2}\ell$) have matched with unsubscribed low-skilled players.

We prove the statement above by contradiction. Suppose on the optimal trajectory, the flow between high-skilled non-paying players in $\underline{2}i$ and low-skilled unsubscribed players $\underline{1}j$ are positive for some $i,j=w,\ell$, while there exists subscribed players $\bar{2}i$ who are matched by skill levels. Then by matching $\bar{2}i$ with $\underline{1}j$, we can collect strictly more rewards in the current period, and a player in $\bar{2}i$ would replace a player $\underline{2}i$ in all the subsequent periods. Hence, the solution cannot be optimal.

(b) We show that if $s^0_{2w}+s^0_{2\ell}\ge s^0_{1w}+s^0_{1\ell}$, then $ENG(\beta,r,\mathbf{s}^0)<V^*(0,0,\mathbf{s}^0)$. Furthermore, there exists a threshold $\bar{r}$ such that $V^*(\beta,r,\mathbf{s}^0)\ge V^*(0,0,\mathbf{s}^0)$ if and only if $r\ge\bar{r}$.

We first show that $ENG(\beta,r,\mathbf{s}^0)\le V^*(0,0,\mathbf{s}^0)$. The engagement $ENG(\beta,r,\mathbf{s}^0)$ is at most $V^*(\beta,0,\mathbf{s}^0)$, which is the optimal engagement with the same demographic but without revenue. We now show that $V^*(\beta,0,\mathbf{s}^0)\le V^*(0,0,\mathbf{s}^0)$, and the equality only holds when we do not have low players at all.

To proceed, we will first show that when $\beta>0$, $r=0$, the optimal matching policy (for engagement maximization) is exactly the same as the baseline model. Recall that when $s^0_{2w}+s^0_{2\ell}\ge s^0_{1w}+s^0_{1\ell}$, the state space can be decomposed into three scenarios. We shall show that these three scenarios (as well as the scenario transition) are the same as the baseline model, and provide the shadow prices. Then, with the help of shadow prices, we shall show that for any initial states such that $s^0_{2w}+s^0_{2\ell}\ge s^0_{1w}+s^0_{1\ell}$, the optimal engagement decreases with β.

Step 1: Our first step is to show that the optimal policy remains the same, and provide closed-form shadow prices for Scenario 1-3. When $r=0$, we can merge level $\bar{2}$ and $\underline{2}$. Compared with (P'), a fraction of 1ℓ players will transit to state $2l$. The resulting LP formulation is different from (P') only on two constraints. That is, we replace

$$f^t_{2\ell,2\ell}+f^t_{2\ell,1w}+f^t_{2\ell,1\ell}=\frac{1}{2}f^{t-1}_{2w,2w},$$

$$f^t_{1\ell,1\ell}+f^t_{2w,1\ell}+f^t_{2\ell,1\ell}=\frac{1}{2}f^{t-1}_{1w,1w}+f^{t-1}_{2w,1w}+f^{t-1}_{2\ell,1w},$$

with

$$f^t_{2\ell,2\ell}+f^t_{2\ell,1w}+f^t_{2\ell,1\ell}=\frac{1}{2}f^{t-1}_{2w,2w}+\beta(\frac{1}{2}f^{t-1}_{1w,1w}+f^{t-1}_{2w,1w}+f^{t-1}_{2l,1w}),$$

$$f^t_{1\ell,1\ell}+f^t_{2w,1\ell}+f^t_{2\ell,1\ell}=(1-\beta)\frac{1}{2}f^{t-1}_{1w,1w}+f^{t-1}_{2w,1w}+f^{t-1}_{2\ell,1w}.$$

Next, we shall show that for Scenario 1-3 in Figure EC.1, the transition is still valid, and the proposed matching policy still satisfies dual feasibility and complementary slackness.

Scenario 1: $s^t_{2w}+s^t_{2\ell}\ge s^t_{1w}+s^t_{1\ell}$, $s^t_{2\ell}\ge s^t_{1w}$, and $s^t_{2w}\ge s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w}=s^t_{2w}$, $f^t_{2\ell,2\ell}=s^t_{2\ell}-s^t_{1w}$, $f^t_{1w,1w}=0$, $f^t_{1\ell,1\ell}=s^t_{1\ell}$, and $f^t_{2\ell,1w}=f^t_{1w,2\ell}=s^t_{1w}$.

We first show that once the state of players' demographics falls in Scenario 1, it will remain in Scenario 1. To see this, for some $\mathbf{s}^t=\{s^t_{1w},s^t_{1\ell},s^t_{2w},s^t_{2\ell},\}$ in Scenario 1, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w}=\frac{1}{2}s^t_{1\ell}$, $s^{t+1}_{1\ell}=(1-\beta)s^t_{1w}$, $s^{t+1}_{2w}=\frac{1}{2}(s^t_{2w}+s^t_{2\ell}+s^t_{1w})$, and $s^{t+1}_{2\ell}=\frac{1}{2}s^t_{2w}+\beta s^t_{1w}$. Then $s^{t+1}_{2\ell}\ge s^{t+1}_{1w}$

because $s^t_{2w} \ge s^t_{1\ell}$, and $s^{t+1}_{2w} \ge s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w}) + \beta s^t_{1w}$ and $s^t_{2\ell} - s^t_{1w} \ge 0$. Hence, $\mathbf{s}^{t+1}$ still belongs to Scenario 1.

Next, we show that the proposed policy in Theorem 1 is optimal for all subsequent periods once players' demographics satisfy Scenario 1. The optimal solution in Theorem 1 suggests that in Scenario 1, only 4 variables are non-zero while all other flows are zero in each period. Therefore, we have

$$\begin{aligned}
1 &= \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\\
\frac{1}{2} &= \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w},\\
2 &= \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - (1-\beta)\lambda^{t+1}_{1\ell} - \beta\lambda_{2\ell},\\
\frac{1}{2} &= \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},
\end{aligned}$$

as the complementary conditions corresponding to primal non-zero variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$, respectively. The following dual solutions:

$$\lambda^t_{2w} = 5,\ \lambda^t_{2\ell} = 3,\ \lambda^t_{1w} = \frac{9+5\beta}{1+\beta}, \text{ and } \lambda^t_{1\ell} = \frac{5+3\beta}{1+\beta}, \quad \forall t, \tag{EC.40}$$

satisfies complementary slackness. The dual feasibility conditions corresponding to variables, $f^t_{2w,1w}$, $f^t_{2w,1\ell}$, $f^t_{2\ell,1\ell}$, and $f^t_{1w,1w}$, that are zero in the primal problem, are given by

$$\begin{aligned}
\lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - (1-\beta)\lambda^{t+1}_{1\ell} - \beta\lambda_{2\ell} \ge 2, &\iff 2 \ge 0\\
\lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w} \ge 1, &\iff \frac{4+2\beta}{1+\beta} \ge 0\\
\lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w} \ge 1, &\iff \frac{2}{1+\beta} \ge 0\\
\lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}(1-\beta)\lambda^{t+1}_{1\ell} - \frac{1}{2}\beta\lambda^{t+1}_{2\ell} \ge 1, &\iff 1 \ge 0,
\end{aligned}$$

where the right-hand-side is by plugging in (EC.40), and is obviously true for $\beta > 0$. Thus, the proposed policy in Theorem 1 is optimal once the players' demographics fall in scenario 1.

Scenario 2: $s^t_{2w} + s^t_{2\ell} \ge s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} < s^t_{1w}$, and $s^t_{2w} \ge s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = 0$, $f^t_{1w,1w} = s^t_{1w} - s^t_{2\ell}$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{2\ell}$;

In the second step, we consider Scenario 2, which will transit to Scenario 1 after matching under the proposed policy as we have stated in Figure EC.1. To see that, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 2, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{1w} = \frac{1}{2}(s^t_{1w} + s^t_{1\ell} - s^t_{2\ell})$, $s^{t+1}_{1\ell} = \frac{1}{2}(1-\beta)(s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2w} = \frac{1}{2}s^t_{2w} + s^t_{2\ell}$, and $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w} + \frac{1}{2}\beta(s^t_{2\ell} + s^t_{1w})$. Then $s^{t+1}_{2\ell} \ge s^{t+1}_{1w}$ because $s^{t+1}_{2\ell} - s^{t+1}_{1w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) + \frac{1}{2}\beta(s^t_{2\ell} + s^t_{1w}) \ge 0$. Also, $s^{t+1}_{2w} \ge s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w}) + \frac{1}{2}\beta(s^t_{2\ell} + s^t_{1w}) \ge s^t_{1\ell} \ge 0$. Hence, $\mathbf{s}^{t+1}$ belongs to Scenario 1.

Therefore, we only need to show that in any period t such that players' demographics satisfy Scenario 2, we can find solutions to dual variables, induced by the proposed policy in Theorem 1, which satisfy the complementary slackness conditions and dual feasibility conditions. Note that in period t under Scenario 2, the non-zero primal variables are $f^t_{2w,2w}$, $f^t_{2\ell,1w}$, $f^t_{1w,1w}$ and $f^t_{1\ell,1\ell}$. The complementary slackness condition is given by

$$\begin{aligned}
&1 = \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\\
&2 = \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - (1-\beta)\lambda^{t+1}_{1\ell} - \beta\lambda^{t+1}_{2\ell},\\
&1 = \lambda^t_{1w} - \frac{1}{2}\lambda^{t+1}_{1w} - \frac{1}{2}(1-\beta)\lambda^{t+1}_{1\ell} - \frac{1}{2}\beta\lambda^{t+1}_{2\ell},\\
&\frac{1}{2} = \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w},
\end{aligned}$$

in period t. By plugging in λ^{t+1}_i from (EC.40), the shadow prices that satisfy the complementary slackness in Scenario 2 are given by

$$\lambda^t_{2w} = 5,\ \lambda^t_{2\ell} = 4,\ \lambda^t_{1w} = \frac{8+4\beta}{1+\beta},\ \text{and}\ \lambda^t_{1\ell} = \frac{5+3\beta}{1+\beta}. \tag{EC.41}$$

The dual feasibility conditions are given by

$$\begin{aligned}
\lambda^t_{2w} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - (1-\beta)\lambda^{t+1}_{1\ell} - \beta\lambda^{t+1}_{2\ell} \geq 2, &\iff 1 \geq 0,\\
\lambda^t_{2w} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w} \geq 1, &\iff \frac{4+2\beta}{1+\beta} \geq 0,\\
\lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w} \geq \frac{1}{2}, &\iff 1 \geq 0,\\
\lambda^t_{2\ell} + \lambda^t_{1\ell} - \lambda^{t+1}_{2w} \geq 1, &\iff \frac{3+\beta}{1+\beta} \geq 0,
\end{aligned}$$

where the right-hand-side is by plugging in (EC.40) and (EC.41). Therefore, the proof for Scenario 2 is complete.

Scenario 3: $s^t_{2w} + s^t_{2\ell} \geq s^t_{1w} + s^t_{1\ell}$, $s^t_{2\ell} \geq s^t_{1w}$, and $s^t_{2w} < s^t_{1\ell}$; The optimal matching flows are: $f^t_{2w,2w} = s^t_{2w}$, $f^t_{2\ell,2\ell} = s^t_{2\ell} - s^t_{1w}$, $f^t_{1w,1w} = 0$, $f^t_{1\ell,1\ell} = s^t_{1\ell}$, and $f^t_{2\ell,1w} = f^t_{1w,2\ell} = s^t_{1w}$;

In the third step, we consider Scenario 3. Different from the baseline model, Scenario 3 shall transit to either Scenario 2 or Scenario 1 after matching is done under the proposed policy. To see this, for some $\mathbf{s}^t = \{s^t_{1w}, s^t_{1\ell}, s^t_{2w}, s^t_{2\ell}\}$ in Scenario 3, following the proposed policy, the state at $t+1$ is given by $s^{t+1}_{2w} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} + s^t_{1w})$, $s^{t+1}_{2\ell} = \frac{1}{2}s^t_{2w} + \beta s^t_{1w}$, $s^{t+1}_{1w} = \frac{1}{2}s^t_{1\ell}$, and $s^{t+1}_{1\ell} = (1-\beta)s^t_{1w}$. Then $s^{t+1}_{2w} \geq s^{t+1}_{1\ell}$ because $s^{t+1}_{2w} - s^{t+1}_{1\ell} = \frac{1}{2}(s^t_{2w} + s^t_{2\ell} - s^t_{1w}) + \beta s^t_{1w}$ and $s^t_{2\ell} - s^t_{1w} \geq 0$. Also, $s^{t+1}_{2w} + s^{t+1}_{2\ell} - s^{t+1}_{1w} - s^{t+1}_{1\ell} = \frac{1}{2}(2s^t_{2w} + s^t_{2\ell} - s^t_{1w} - s^t_{1\ell}) + 2\beta s^t_{1w} \geq 0$. However, the relation between $s^{t+1}_{2\ell}$ and s^{t+1}_{1w} rely on the value of β. Hence, $\mathbf{s}^{t+1}$ belongs to either Scenario 2 or Scenario 1.

We first consider the case when $\mathbf{s}^{t+1}$ belongs to Scenario 2. According to the proposed policy, in period t under Scenario 3, primal variables $f^t_{2w,2w}$, $f^t_{2\ell,2\ell}$, $f^t_{2\ell,1w}$, and $f^t_{1\ell,1\ell}$ are non-zero. Thus, in period t, the complementary slackness conditions are given by

$$\begin{aligned}
&1 = \lambda^t_{2w} - \frac{1}{2}\lambda^{t+1}_{2w} - \frac{1}{2}\lambda^{t+1}_{2\ell},\\
&\frac{1}{2} = \lambda^t_{2\ell} - \frac{1}{2}\lambda^{t+1}_{2w},\\
&2 = \lambda^t_{2\ell} + \lambda^t_{1w} - \lambda^{t+1}_{2w} - (1-\beta)\lambda^{t+1}_{1\ell} - \beta\lambda^{t+1}_{2\ell},\\
&\frac{1}{2} = \lambda^t_{1\ell} - \frac{1}{2}\lambda^{t+1}_{1w}.
\end{aligned}$$

By plugging in (EC.41) for period $t+1$, the corresponding shadow prices are

$$\lambda_{2w}^t = 5.5,\ \lambda_{2\ell}^t = 3,\ \lambda_{1w}^t = \frac{(3+\beta)^2}{1+\beta},\ \text{and}\ \lambda_{1\ell}^t = \frac{9+5\beta}{2(1+\beta)}. \tag{EC.42}$$

The dual feasibility conditions are

$$\begin{aligned}
\lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1-\beta)\lambda_{1\ell}^{t+1} - \beta\lambda_{2\ell}^{t+1} \ge 2, &\iff 2.5 \ge 0,\\
\lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} \ge 1, &\iff \frac{4+2\beta}{1+\beta} \ge 0,\\
\lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} \ge 1, &\iff \frac{3-\beta}{2+2\beta} \ge 0,\\
\lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1} \ge 1, &\iff \frac{3+3\beta+2\beta^2}{2+2\beta} \ge 0,
\end{aligned}$$

respectively, where the right-hand-side is by plugging in (EC.41) and (EC.42). It is straightforward to see that the proposed shadow prices are dual feasible for $\beta \in [0,1]$.

Next, consider the case $\mathbf{s}^{t+1}$ belongs to Scenario 1. In period t, the complementary slackness conditions are given by

$$\begin{aligned}
1 &= \lambda_{2w}^t - \frac{1}{2}\lambda_{2w}^{t+1} - \frac{1}{2}\lambda_{2\ell}^{t+1},\\
\frac{1}{2} &= \lambda_{2\ell}^t - \frac{1}{2}\lambda_{2w}^{t+1},\\
2 &= \lambda_{2\ell}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1-\beta)\lambda_{1\ell}^{t+1} - \beta\lambda_{2\ell}^{t+1},\\
\frac{1}{2} &= \lambda_{1\ell}^t - \frac{1}{2}\lambda_{1w}^{t+1}.
\end{aligned}$$

By plugging in (EC.40) for period $t+1$, the corresponding shadow prices are

$$\lambda_{2w}^t = 5,\ \lambda_{2\ell}^t = 3,\ \lambda_{1w}^t = \frac{9+5\beta}{1+\beta},\ \text{and}\ \lambda_{1\ell}^t = \frac{5+3\beta}{1+\beta}. \tag{EC.43}$$

The dual feasibility conditions are

$$\begin{aligned}
\lambda_{2w}^t + \lambda_{1w}^t - \lambda_{2w}^{t+1} - (1-\beta)\lambda_{1\ell}^{t+1} - \beta\lambda_{2\ell}^{t+1} \ge 2, &\iff 2 \ge 0,\\
\lambda_{2w}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} \ge 1, &\iff \frac{4+2\beta}{1+\beta} \ge 0,\\
\lambda_{2\ell}^t + \lambda_{1\ell}^t - \lambda_{2w}^{t+1} \ge 1, &\iff \frac{2}{1+\beta} \ge 0,\\
\lambda_{1w}^t - \frac{1}{2}\lambda_{1w}^{t+1} - \frac{1}{2}\lambda_{1\ell}^{t+1} \ge 1, &\iff \frac{1}{1+\beta} \ge 0,
\end{aligned}$$

respectively, where the right-hand-side is by plugging in (EC.40) and (EC.43). It is straightforward to see that the proposed shadow prices are dual feasible for $\beta \in [0,1]$.

Step 2: With Step 1, we can compute the maximum engagement for any β, as long as the initial level 2 players are more than level 1 players. We now show that for $\beta > 0$, the engagement is always more than the case when $\beta = 0$.

If $\mathbf{s}^0$ belongs to Scenario 1, then the engagement without PTW is $5s_{2w}^0 + 3s_{2\ell}^0 + 9s_{1w}^0 + 5s_{1\ell}^0$. The engagement with PTW is $5s_{2w}^0 + 3s_{2\ell}^0 + \frac{9+5\beta}{1+\beta}s_{1w}^0 + \frac{5+3\beta}{1+\beta}s_{1\ell}^0$. The difference is $\frac{4\beta}{1+\beta}s_{1w}^0 + \frac{2\beta}{1+\beta}s_{1\ell}^0$, which is always positive for $\beta > 0$. Thus, the engagement must decrease.

If $\mathbf{s}^0$ belongs to Scenario 2, then the engagement without PTW is $5s_{2w}^0 + 4s_{2\ell}^0 + 8s_{1w}^0 + 5s_{1\ell}^0$. The engagement with PTW is $5s_{2w}^0 + 4s_{2\ell}^0 + \frac{8+4\beta}{1+\beta}s_{1w}^0 + \frac{5+3\beta}{1+\beta}s_{1\ell}^0$. The difference is $\frac{4\beta}{1+\beta}s_{1w}^0 + \frac{2\beta}{1+\beta}s_{1\ell}^0$, which is always positive for $\beta > 0$. Thus, the engagement must decrease.

If $\mathbf{s}^0$ belongs to Scenario 3, then the engagement without PTW is $5.5s^0_{2w}+3s^0_{2\ell}+9s^0_{1w}+4.5s^0_{1\ell}$. With PTW, if it transits to Scenario 2 in the next period, then the engagement is $5.5s^0_{2w}+3s^0_{2\ell}+\frac{(3+\beta)^2}{1+\beta}s^0_{1w}+\frac{9+5\beta}{2(1+\beta)}s^0_{1\ell}$. The difference is $\frac{6\beta-\beta^2}{1+\beta}s^0_{1w}+\frac{2\beta}{1+\beta}s^0_{1\ell}$, which is always positive for $\beta\in(0,1)$. Thus, the engagement must decrease. If it transits to Scenario 1 in the next period, then the engagement is $5s^0_{2w}+3s^0_{2\ell}+\frac{9+5\beta}{1+\beta}s^0_{1w}+\frac{5+3\beta}{1+\beta}s^0_{1\ell}$. The difference is $\frac{1}{2}s^0_{2w}+\frac{4\beta}{1+\beta}s^0_{1w}+\frac{1.5\beta-0.5}{1+\beta}s^0_{1\ell}$. To see that this value is positive, note that if $\mathbf{s}^1$ belongs to Scenario 1, then we have $s^1_{2\ell}\ge s^1_{1w}\iff \frac{1}{2}s^0_{2w}+\beta s^0_{1w}\ge\frac{1}{2}s^0_{1\ell}\iff 2\beta s^0_{1w}\ge s^0_{1\ell}-s^0_{2w}$. Thus,

$$\begin{aligned}&\frac{1}{2}s^0_{2w}+\frac{4\beta}{1+\beta}s^0_{1w}+\frac{1.5\beta-0.5}{1+\beta}s^0_{1\ell}\\ \ge&\frac{1}{2}s^0_{2w}+\left(\frac{2}{1+\beta}s^0_{1\ell}-\frac{2}{1+\beta}s^0_{2w}\right)+\frac{1.5\beta-0.5}{1+\beta}s^0_{1\ell},\\ =&1.5s^0_{1\ell}-\frac{1.5-0.5\beta}{1+\beta}s^0_{2w},\end{aligned}$$

where the inequality is by plugging in $2\beta s^0_{1w}\ge s^0_{1\ell}-s^0_{2w}$. Because $\frac{1.5-0.5\beta}{1+\beta}<1.5$ and $s^0_{2w}<s^0_{1\ell}$, which value is positive. Thus, the engagement always decreases in the presence of PTW.

(c) Fix $s^0_{2w}/s^0_{2\ell}$ and $s^0_{1w}/s^0_{1\ell}$ and vary $(s^0_{2w}+s^0_{2\ell})/(s^0_{1w}+s^0_{1\ell})$. Consider $\mathbf{s}^0$ such that $(s^0_{2w}+s^0_{2\ell})/(s^0_{1w}+s^0_{1\ell})=0$, i.e., there are only low players. Then the optimal matching is simply SBMM. When $r=0$, in the presence of PTW, some of the low player now becomes high player, which enables cross-level matchmaking, and we must have $ENG(\beta,0,\mathbf{s}^0)=V^*(\beta,0,\mathbf{s}^0)>V^*(0,0,\mathbf{s}^0)$. That said, when there is no revenue, the engagement is higher thanks to the change in demographic distribution. Because $ENG^*(\beta,r,\mathbf{s}^0)$ is a continuous function of r, for r that is sufficiently close to 0, the engagement is still higher than SBMM. □

B.1.4. Proof of Proposition 1 and Theorem 4

Proof of Proposition 1. Under SBMM, players are only matched with others within the same skill level. Thus, the only losses of the system are from losing players with high and low skills, respectively, and the two skill levels do not affect each other. As a result, in order to maximize engagement by adjusting the bot ratio, we need to minimize the churning rate of losing players. In other words, we need to maximize the winrate of losing players.

The winrate of losing players in either skill level can be written as the following:

$$\alpha(1-p(\alpha))+\frac{1-\alpha}{2},$$

where the first term represents that with probability α, a losing player shall be matched with a bot, and she has probability $1-p(\alpha)$ to stay after being matched with a bot; the second term represents that with probability $1-\alpha$, a losing player is still matched with another human player with the same skill level, and thus, she has 50% chance to win and stay in the system for subsequent matchings. This completes the proof of the desired result. □

Proof of Theorem 4. To better facilitate this proof, we first present an equivalent primal formulation in the presence of bots.

Note that when matched with a bot, a losing player still has a probability $(1-p(\alpha))$ to leave the system. As a result, we can split the flow that involves bots into two: one to a "perfect bot" that guarantees the return of the human player in the next period; one to a "skill-based bot" that perfectly mimics a human player with the same skill level. In other words, an imperfect bot (that induces churning) can be viewed as a combination of a perfect bot (that does not induce churning) and SBMM.

Mathematically, for one unit of flow of losing players there is $(1-\alpha)$ fraction shall participate in regular matching without bots. Among the α fraction that is matched with bots, there is $2p(\alpha)$ fraction matched to perfect bots, and $1-2p(\alpha)$ fraction matched based on SBMM. One can easily verify that we have $2p(\alpha)+1-2p(\alpha)=1$, representing the decomposition induces the same amount of total flow, and $1-p(\alpha)=\frac{1}{2}2p(\alpha)+(1-2p(\alpha))$, representing the decomposition gives the same non-churning players.

Using the flow decomposition above, we introduce $f^t_{i,PB}$, where $i\in\{2\ell,1\ell\}$ and PB abbreviated for *perfect bots*, to denote the flow of losing players who are matched to perfect bots. Then the engagement maximization problem is

$$\max_{\alpha} P_{bot}(\alpha), \tag{EC.44}$$

where

$$P_{bot}(\alpha):=\max_{\mathbf{f}^t}\sum_{t=0}^{\infty}\left(f^t_{2w,2w}+\frac{1}{2}f^t_{2\ell,2\ell}+f^t_{1w,1w}+\frac{1}{2}f^t_{1\ell,1\ell}+2f^t_{2w,1w}+f^t_{2\ell,1\ell}+f^t_{2w,1\ell}+2f^t_{2\ell,1w}+f^t_{2\ell,PB}+f^t_{1\ell,PB}\right)$$

such that

$$\begin{aligned}
s^0_{2w}&=f^0_{2w,2w}+f^0_{2w,1w}+f^0_{2w,1\ell},\\
s^0_{2\ell}&=f^0_{2\ell,2\ell}+f^0_{2\ell,1w}+f^0_{2\ell,1\ell}+f^0_{2\ell,PB},\\
s^0_{1w}&=f^0_{1w,1w}+f^0_{2\ell,1w}+f^0_{2w,1w},\\
s^0_{1\ell}&=f^0_{1\ell,1\ell}+f^0_{2w,1\ell}+f^0_{2\ell,1\ell}+f^0_{1\ell,PB},
\end{aligned}$$

and for all $t=1,2,\ldots$

$$\begin{aligned}
s^t_{2w}=f^t_{2w,2w}+f^t_{2w,1w}+f^t_{2w,1\ell},&=\tfrac{1}{2}f^{t-1}_{2w,2w}+\tfrac{1}{2}f^{t-1}_{2\ell,2\ell}+f^{t-1}_{2\ell,PB}+f^{t-1}_{2w,1w}+f^{t-1}_{2w,1\ell}+f^{t-1}_{2\ell,1w}+f^{t-1}_{2\ell,1\ell},\\
s^t_{2\ell}=f^t_{2\ell,2\ell}+f^t_{2\ell,1w}+f^t_{2\ell,1\ell}+f^t_{2\ell,PB}&=\tfrac{1}{2}f^{t-1}_{2w,2w},\\
s^t_{1w}=f^t_{1w,1w}+f^t_{2\ell,1w}+f^t_{2w,1w},&=f^{t-1}_{1\ell,PB}+\tfrac{1}{2}\left(f^{t-1}_{1w,1w}+f^{t-1}_{1\ell,1\ell}\right)\\
s^t_{1\ell}=f^t_{1\ell,1\ell}+f^t_{2w,1\ell}+f^t_{2\ell,1\ell}+f^t_{1\ell,PB}&=\tfrac{1}{2}f^{t-1}_{1w,1w}+f^{t-1}_{2w,1w}+f^{t-1}_{2\ell,1w},
\end{aligned}$$

and for all $t=0,1,2,\ldots$

$$\begin{aligned}
f^t_{2\ell,PB}&\le\alpha(1-2p(\alpha))s^t_{2\ell},\\
f^t_{2\ell,1\ell}+f^t_{2\ell,1w}&\le(1-\alpha)s^t_{2\ell},\\
f^t_{1\ell,PB}&\le\alpha(1-2p(\alpha))s^t_{1\ell},\\
f^t_{2w,1\ell}+f^t_{2\ell,1\ell}&\le(1-\alpha)s^t_{1\ell}.
\end{aligned}\tag{EC.45}$$

where constraints in (EC.45) represent the flow decomposition to perfect bots (fraction $\alpha(1-2p(\alpha))$) and regular matching (fraction $1-\alpha$, and the rests are to SBMM).

Immediately, we observe that α does not impact the objective in $P_{bot}(\alpha)$, but the constraints, in particular, constraints in (EC.45).

Next, we show that any $\alpha > \alpha^{SBMM}$ cannot be optimal. When $\alpha(1-p(\alpha))+\frac{1}{2}(1-\alpha)$ is concave, it renders a unique maximizer α^{SBMM}, which solves

$$\frac{1}{2} - p(\alpha) - \alpha p'(\alpha) = 0. \tag{EC.46}$$

Consider any $\alpha > \alpha^{SBMM}$, we must have

$$\frac{d[\alpha(1-2p(\alpha))]}{d\alpha}\Big|_{\alpha=\alpha'} = 1 - 2p(\alpha) - 2\alpha p'(\alpha) < 0,$$

according to concavity. Therefore, function $\alpha(1-2p(\alpha))$ decreases in α for all $\alpha > \alpha^{SBMM}$. Furthermore, $1-\alpha$ also decreases in α. Thus, the feasible region described by (EC.45) strictly decreases when increasing α for any $\alpha > \alpha^{SBMM}$. As a result, for any $\alpha < \alpha'$ where $\alpha' > \alpha^{SBMM}$, we have $P_{bot}(\alpha) \geq P_{bot}(\alpha')$. This completes the proof. □

B.2. Omitted Proofs from Appendix A

Proof of Lemma EC.1. Denote $\mathbf{v}_k = \{v_{kg}\}_{g\in\bar{\mathcal{G}}}$ as the vector of value (engagement) that 1 unit of players in level k can create. Note that v_{kg} is the average active time starting from state g in the absorbing Markov chain M_{kk} before absorption. By Theorem 3.2.4 in Kemeny et al. (1960), we have

$$\begin{aligned}\mathbf{v}_k &= (\bar{M}_{kk} + \gamma\bar{M}_{kk}^2 + \gamma^2\bar{M}_{kk}^3 + \cdots)\mathbf{1}\\ &= \left(\frac{1}{\gamma}\left(I + \gamma\bar{M}_{kk} + \gamma^2\bar{M}_{kk}^2 + \cdots\right) - \frac{1}{\gamma}I\right)\mathbf{1}\\ &= \gamma^{-1}\left(\left(I-\gamma\bar{M}_{kk}\right)^{-1} - I\right)\mathbf{1}.\end{aligned}$$

Note that by Theorem 3.2.1 in Kemeny et al. (1960), $\left(I-\gamma\bar{M}_{kk}\right)^{-1}$ always exists. Summing up players at all levels, we have

$$V^{\text{SBMM}}(\mathbf{s}^0) = \sum_{k=1}^{K} \mathbf{v}_k^\top \mathbf{s}_k^0.$$

□

Proof of Lemma EC.2. We prove this lemma by induction when $\gamma = 1$. For the base case, consider the engagement of level 1 players. Because their winrate is at most 0.5, a positive proportion of level 1 players would experience $m+1$ consecutive losses and quit, for every $m+1$ rounds. Let $\epsilon > 0$ be the probability that a player quits after $m+1$ consecutive losses. After $m+1$ rounds of matches, a level 1 player starting from any state has at least $2^{-(m+1)}\epsilon$ probability to quit. Hence, a player's engagement is bounded by

$$(m+1)(1-2^{-(m+1)}\epsilon) + (m+1)(1-2^{-(m+1)}\epsilon)^2 + (m+1)(1-2^{-(m+1)}\epsilon)^3 + \cdots = \frac{(m+1)(1-2^{-(m+1)}\epsilon)}{2^{-(m+1)}\epsilon}. \tag{EC.47}$$

The induction hypothesis is that the engagement is finite for players of level 1 to level k. We then show that the engagement is also finite for players of level $k+1$. From the induction hypothesis, the engagement from matches with players from levels 1 to k must be finite. Thus, we only need to consider engagement generated from matches with players of level $k+1$ to K, and show it is also finite. Note that level $k+1$ players' winrate is at most 0.5 since they are matched with players with at least the same skill level. Thus, following the exact same argument of the base case with players of level 1, we can show that a player's engagement is bounded by the same expression in (EC.47), which is finite.

Finally, when $\gamma < 1$, the claim is obvious since the amount of active players is non-increasing over time. □

Proof of Lemma EC.3. Our proof relies on Theorem 2.3 in Ghate (2015), which requires showing that our LP formulation (EC.2) satisfies the following five hypotheses. Let $X \subseteq \mathcal{R}^{\mathbb{N}}$ be a linear subspace. For an infinite primal LP

$$V(P) = \sup \sum_{j=1}^{\infty} c_j x_j \tag{EC.48}$$

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i,\ i = 1, 2, \cdots \tag{EC.49}$$

$$x_j \geq 0,\ j = 1, 2, \cdots \tag{EC.50}$$

$$x \in X, \tag{EC.51}$$

we assume for any $x \in X$

H1. $\sum_{j=1}^{\infty} c_j x_j < \infty$.

H2. $\sum_{j=1}^{\infty} a_{ij} x_j < \infty,\ j = 1, 2, \cdots$.

Further, let $Y \subseteq \mathcal{R}^{\mathbb{N}}$ be the subset of all $y \in R^{\mathbb{N}}$ such that

H3. $\sum_{i=1}^{\infty} b_i y_i < \infty$.

H4. For every $x \in X$, $\sum_{j=1}^{\infty} |a_{ij} x_j y_i|$ converges to some limit $L_i(x, y_i)$ for $i = 1, 2, \cdots$, and

H5. The above limits $L_i(x, y_i)$ have the property that $\sum_{i=1}^{\infty} L_i(x, y_i) < \infty$.

Then consider the dual problem

$$V(D) = \inf \sum_{i=1}^{\infty} b_i y_i \tag{EC.52}$$

$$\sum_{i=1}^{\infty} a_{ij} y_i \geq c_j,\ j = 1, 2, \cdots \tag{EC.53}$$

$$y \in Y. \tag{EC.54}$$

By Theorem 2.3 in Ghate (2015), suppose $x \in X$ and $y \in Y$ are feasible to the primal and dual problems and are complementary ($x_j(c_j - \sum_{i=1}^{\infty} a_{ij} y_i) = 0$ for all j). Then x and y are optimal solutions to the primal and dual problems, and $V(P) = V(D)$.

For our problem in (EC.2), let X to be the l_1 sequence space. Because of Lemma EC.2, letting X be the l_1 sequence space is without of generality. We check hypotheses H1 to H5, respectively. For dual variables, we only consider y from l_∞ space. As we will show, any y from l_∞ space satisfy H3 to H5, and we only use such y in the following proofs.

Hypothesis H1 is satisfied because X is the l_1 space. Hypothesis H2 and H4 are satisfied since we have finitely many primal variables in each constraint of problem EC.2 and X is the l_1 space. Hypothesis H3 is satisfied since only the constraints associate with the initial period ($t = 0$) leads to nonzero values (thus finite), and y is in l_∞ space. Finally, for hypothesis H5, let A_i be the set of nonzero columns for row i. Note that we have

$$\sum_{i=1}^{\infty} L_i(x, y_i) = \sum_{i=1}^{\infty} \sum_{j \in A_i} |a_{ij} x_j y_i| \leq \sum_{i=1}^{\infty} \sup_{i \in \mathbb{N}} |y_i| \sum_{j \in A_i} x_j = \sup_{i \in \mathbb{N}} |y_i| \sum_{i=1}^{\infty} \sum_{j \in A_i} x_j \leq 2|\bar{\mathcal{G}}| \sup_{i \in \mathbb{N}} |y_i| \sum_{j \in \mathbb{N}} |x_j| < \infty,$$

where the first inequality follows the fact $a_{ij} \in [0,1]$ and Y is l_∞ space, the second inequality follows the fact that each primal variable x_j appears in two periods, so it shows up in at most $2|\bar{\mathcal{G}}|$ constraints in problem (EC.2), and the last inequality follows from the fact that X is the l_1 space. □