Mass Vaccination Scheduling: Trading off Infections, Throughput, and Overtime

Shanshan Luo, shanshan.luo@sauder.ubc.ca
Steven M. Shechter, steven.shechter@sauder.ubc.ca

Electronic Companion

This electronic companion contains all the proofs of our theoretical results and other supplementary material.

EC.1. Estimates of model parameters

Our estimates of transmission rate, α, and initially infectious probability, p_0, mainly rely on the model from Barnett and Fleming (2022). We elaborate on the estimation process in the following sections.

EC.1.1. Estimates of α

We adopt the infection risk model of Barnett and Fleming (2022) (see their Equation (10)), which estimates the per-minute probability of an infectious airline passenger infecting an uninfected passenger as a function of the distance, masking, and seat obstruction between them. We drop the seat interference factor for our setting, resulting in the following equation:

$$P(\text{infection within one minute}) = \pi_0 e^{-wd}(1 - p_{\text{masks}}), \tag{EC.1}$$

where π_0 represents the probability of infection at a distance of 0 without masks, w is an exponential decay factor per meter of distance, d is the distance between them, and p_{masks} is the probability that viral transmission is blocked when both individuals are masked. Table EC.1 reports the estimated distributions and mean for each parameter provided in Barnett and Fleming (2022). To estimate

Table EC.1 Estimated distribution of each model parameter

Parameter	Estimated Distribution	Mean
π_0(probability of infection at a distance of 0 without masks)	Beta(1,520)	1/521
w(exponential decay factor per meter of distance)	Lognormal(-0.703,0.318)	0.521
p_{masks}(mask blocking effects)	1-Normal(0.3,0.075)	0.7

the infection rate α, we first derive the distribution of α by setting $1 - e^{-\alpha}$ equal to the right-hand side of Equation (EC.1). Although we do not know the closed expression for the density function of α, we can still estimate its mean value via simulation, with 100,000 replications. Table EC.2 shows the α estimates for different physical distances, with all wearing masks. We use these values in Sections 3 and 5. For example, if the physical distance is 2 meters, then $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{0.0002/\text{min}, 0.0001/\text{min}, 0.00004/\text{min}, 0.00002/\text{min}\}$

Table EC.2 Estimates of Transmission rate α of COVID at early stage at different physical distances

Physical distance (m)	1	2	3	4	6	8	9
Infection probability (t=1 minute)	0.0003	0.0002	0.00013	0.0001	0.00004	0.00002	0.00001
$\alpha\,(\text{min}^{-1})$	0.0003	0.0002	0.00013	0.0001	0.00004	0.00002	0.00001
$1/\alpha\,(\text{min})$	2915.08	4722.03	7491.85	11670.66	27005.22	59189.23	86091.00

EC.1.2. Estimates of p_0

To estimate the initially infectious probability p_0, we use Equation (3) from Barnett and Fleming (2022), which they use to estimate the percentage of potential air passengers infected with COVID-19. Since we care about the health status of the entire population, we modify their equation by removing the "healthy passenger" factor (denoted as β in their paper). Then, the resulting formula is:

$$p_0 = \left(\frac{C_7}{POP}\right)\rho,$$

where C_7 stands for the number of confirmed cases of COVID over the past seven days, POP represents the population, and ρ is a multiplier that adjusts for unreported COVID-19 cases due to mild symptoms and inadequate testing.

To estimate C_7 for a particular area, we examine the historical data of newly reported daily cases (using 7-day averages). We calculate C_7 by taking the highest daily number observed during the peak of COVID-19 and multiplying it by 7, or using the highest weekly number directly. We notice that for many areas, the peak of COVID-19 happened around January 2022. Barnett and Fleming (2022) estimate the distribution of the multiplier, which is time-varying (see their Figure 1). Based on their estimation, we approximate the value of ρ for January as between 2 and 3. Table EC.3 shows a varied range of estimates of p_0 for a few major cities.

Table EC.3 The estimated values of p_0 in different areas

Area	C_7	POP	p_0
Miami-Dade County	110,439	2,800,000	0.08 ∼ 0.12
New York City	275,330	8,800,000	0.06 ∼ 0.09
Harris County	86,450	4,800,000	0.03 ∼ 0.04

Note. We access historical dataset from The New York Times (2024) [1].

EC.2. Proof of Lemma 1

Proof of Lemma 1. Since the infection processes are independent, the probability that the uninfected individual remains uninfected is $\Pi_{i=1}^{n} e^{-\alpha_i s_i}$, and thus the infection probability is $1 - e^{-\sum_{i=1}^{n}\alpha_i s_i}$.

[1] The New York Times (2024). Track coronavirus cases in places important to you. Accessed July 25, 2025, `https://www.nytimes.com/interactive/2023/us/covid-personalized-tracker.html`.

EC.3. Derivation of $p_{i^-}(j)$ for $k=1$

To summarize the construction of $p_{i^-}(j)$ from Kaandorp and Koole (2007), assume $\boldsymbol{x}$ is a schedule for a single-server system with service time distribution given by Exp(μ). Define $p_{i-}(j)$ and $p_{i+}(j)$ as

$$p_{i^-}(j) = \mathrm{P}(j \text{ people in the system just before the arrival of batch } x_i),$$
$$p_{i^+}(j) = \mathrm{P}(j \text{ people in the system just after the arrival of batch } x_i).$$

These are developed recursively as follows. Since the system is initially empty, $p_{1^-}(0) \equiv 1$. Letting j represent the number of people in the system just after the batch of x_i people arrives, then $p_{i^+}(j)$ is just the same as $p_{i^-}(j - x_i)$ for all i. What remains is to construct $p_{i^-}(j)$ for $i > 1$. Let $n \geq j$ denote the number of people in the system just after the arrival of batch x_{i-1}. Then, the number of potential service completions in the time slot of duration d follows a Poisson process with rate μ. Therefore, if $j > 0$, the queue reduces from n to j with a probability denoted as a_{n-j}, where $a_{n-j} \equiv e^{-\mu d}(\mu d)^{n-j}/(n-j)!$. If $j = 0$, the queue reduces from n to 0 with a probability denoted as b_n, where $b_n \equiv 1 - \sum_{j=0}^{n-1} a_j$. Unconditioning on n leads to the following final expressions for $p_{i^-}(j)$ for $i > 1$:

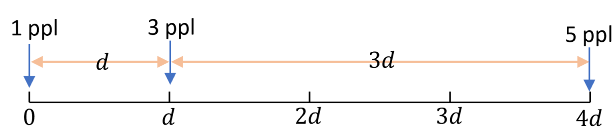
$$p_{i^-}(0) = \sum_{n=0}^{\|\boldsymbol{x}\|_1} p_{(i-1)^+}(n) b_n, \tag{EC.2}$$

$$p_{i^-}(j) = \sum_{n=j}^{\|\boldsymbol{x}\|_1} p_{(i-1)^+}(n) a_{n-j}, \quad j > 0. \tag{EC.3}$$

EC.4. A closed-form expression of of $C(\boldsymbol{x})$ for $\bar{z} = 1, k = 1$

Before stating the exact expression for $C(\boldsymbol{x})$, we introduce notation for convenience. Let $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_T)$, where δ_i denotes that after x_i people arrive at time slot i, the next *non-empty* batch will arrive δ_i time units later. For example, if there is a schedule $\boldsymbol{x} = (1, 3, 0, 0, 5)$ (see Figure EC.1), then $\boldsymbol{\delta} = (d, 3d, 2d, d, \infty)$, with $\delta_i = \infty$ meaning that no batch will arrive after time slot i. The total

Figure EC.1 Illustration for the definition of δ. "ppl" = "people".



number of new infections under schedule $\boldsymbol{x}$, $C(\boldsymbol{x})$, is calculated as

$$C(\boldsymbol{x}) = \sum_{i=1}^{T} \sum_{j=0}^{\|\boldsymbol{x}\|_1} p_{i^-}(j) R_{x_i}^{\delta_i}(j). \tag{EC.4}$$

where $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{T} x_i$ is the total number of arrivals (i.e., the day's throughput), and $p_{i^-}(j)$ denotes the probability that when x_i people arrive at time slot i, they find j people in the system (including the person in service). The expression $R_{x_i}^{\delta_i}(j)$ denotes the expected number of new infections acquired across those x_i people who find j people remaining in the system when the next batch of people is scheduled to arrive δ_i time units later. The calculation of $p_{i^-}(j)$ is described in EC.3. Next, we focus on the derivation of $R_{x_i}^{\delta_i}(j)$.

EC.4.1. Derivation of $R_{x_i}^{\delta_i}(j)$

When x_i people arrive together at time slot i, we assign each a unique position index (between 1 and x_i). Let $R_{x_i,n}^{\delta_i}(j)$ denote the probability that the individual with index n is initially uninfected and becomes infected while waiting. Then,

$$R_{x_i}^{\delta_i}(j) = \sum_{n=1}^{x_i} R_{x_i,n}^{\delta_i}(j).$$

The following proposition provides a high-level expression for $R_{x_i,n}^{\delta_i}(j)$.

Proposition EC.1. *Consider a batch of x_i people arriving at a single-server system with j people already in it, with the next batch scheduled to arrive δ_i time units later. The probability that individual n of the batch ($1 \le n \le x_i$) acquires an infection while waiting is given by*

$$R_{x_i,n}^{\delta_i}(j) = \begin{cases} 0 & \text{for } n+j=1 \\ p_0(1-p_0)E[1-e^{-\alpha W_n}] & \text{for } n+j=2, n<x_i \\ p_0(1-p_0)^2\left(E[1-e^{-\alpha W_{n-1}}]+E[1-e^{-\alpha W_n}]\right) \\ +p_0^2(1-p_0)E[1-e^{-\alpha(W_{n-1}+W_n)}] & \text{for } 2<n+j, n<x_i \\ p_0(1-p_0)^2\left(E[1-e^{-\alpha W_{n-1}}]+E[1-e^{-\alpha[W_n-\delta_i]_+}]\right) \\ +p_0^2(1-p_0)E[1-e^{-\alpha(W_{n-1}+[W_n-\delta_i]_+)}] & \text{for } 1<n+j, n=x_i \end{cases} \tag{EC.5}$$

where W_n represents the waiting time of individual n, W_0 represents the remaining waiting time of the last individual from the previous batch, and $[W_n-\delta_i]_+ = \max\{W_n-\delta_i, 0\}$.

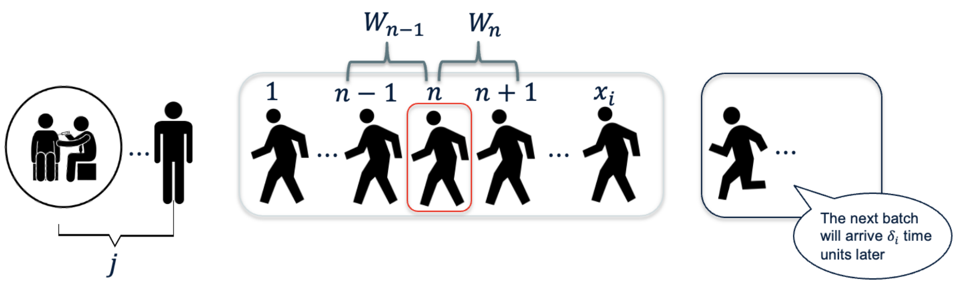
Proof of Proposition EC.1 We explain Proposition EC.1 through the following four cases:

- **Case 1**: If $n+j=1$, then individual n receives immediate service without the risk of infection while waiting.
- **Case 2**: If $n+j=2$, $n<x_i$, then individual n is at the head of the queue, with at least one person behind them. Only the individual indexed $n+1$ behind them can spread the virus to individual n. The time they spend together in the queue (i.e., their overlap time) is W_n since they all arrive simultaneously. Therefore, we have

$$R_{x_i,n}^{\delta_i}(j) = p_0(1-p_0)\mathrm{E}[1-e^{-\alpha W_n}] \tag{EC.6}$$

where the expression $(1-p_0)$ represents the probability that individual n is initially uninfected, and the expression p_0 represents the probability that individual $n+1$ is initially infectious.

Figure EC.2 Illustration for Case 3



- **Case 3**: If $2 < n+j$, $n < x_i$, as illustrated in Figure EC.2, then individual n waits between two other people in the queue. Both individual $n-1$ and individual $n+1$ can spread the virus to individual n. As discussed in Case 2, the overlap time between individual $n-1$ and individual n is W_{n-1}, and the overlap time between individual n and individual $n+1$ is W_n. Next, we divide this case into three subcases based on the initial health status of individual $n-1$, n, and $n+1$.

$$R^{\delta_i}_{x_i,n}(j) = (1-p_0)\Bigg(\underbrace{p_0(1-p_0)\mathrm{E}[1-e^{-\alpha W_{n-1}}]}_{\text{only individual } n-1 \text{ is infectious}} + \underbrace{p_0(1-p_0)\mathrm{E}[1-e^{-\alpha W_n}]}_{\text{only individual } n+1 \text{ is infectious}} + \underbrace{p_0^2\mathrm{E}[1-e^{-\alpha(W_n+W_{n-1})}]}_{\text{both individual } n-1 \text{ and } n+1 \text{ are infectious}}\Bigg) \tag{EC.7}$$

The expression $(1-p_0)$ outside of the parentheses represents the probability that individual n is uninfected, as they cannot become infected if they are initially infectious. In addition, the first two terms inside the parentheses are directly derived from the virus transmission model, while the last term is derived from Lemma 1.

- **Case 4**: If $1 < n+j$, $n = x_i$, then individual n is at the end of batch i. This is handled differently than Case 2 and 3, as individual $n+1$ who waits behind them would be from the next batch, which arrives after δ_i time units. Therefore, the overlap time between them is $[W_n - \delta_i]_+$. □

To calculate the expectations in Equation (EC.5), we need to determine the distributions of W_n for $n = 0, \cdots, x_i$. If $n+j=1$, then $W_n = 0$, as individual n receives immediate service without waiting. Otherwise, under the assumptions of the system, W_n follows an Erlang$(n+j-1, \mu)$ distribution.

Next, we unpack the expected values of Equation (EC.5) and present exact expressions for $R^{\delta_i}_{x_i,n}(j)$ in Proposition EC.2.

PROPOSITION EC.2. *Consider a batch of x_i people arriving to a single-server system with j people already in it, with the next batch scheduled to arrive δ_i time units later. The probability that individual n of the batch ($1 \le n \le x_i$) acquires an infection while waiting is given by*

$$R^{\delta_i}_{x_i,n}(j) = \begin{cases} 0 & \text{for } n+j=1, \\ p_0(1-p_0)(1-\gamma) & \text{for } 2=n+j, n<x_i, \\ p_0(1-p_0)^2\left(2-\gamma^{n+j-2}-\gamma^{n+j-1}\right) \\ +p_0^2(1-p_0)(1-\gamma\eta^{n+j-2}) & \text{for } 2<n+j, n<x_i, \\ p_0(1-p_0)^2\left(1-\gamma^{n+j-2}+\sum_{l=0}^{n+j-2}\frac{(\mu\delta_i)^l e^{-\mu\delta_i}(1-\gamma^{n+j-1-l})}{l!}\right) \\ +p_0^2(1-p_0)\left(1-\gamma^{n+j-2}+e^{-\mu\delta_i}\gamma\theta^{n+j-3}-\mu\theta^{n+j-2}A\right) & \text{for } 1<n+j, n=x_i. \end{cases} \quad \text{(EC.8)}$$

where $A=\sum_{l=0}^{n+j-3}\sum_{r=0}^{l}\frac{\alpha^l {\delta_i}^r e^{-(\alpha+\mu)\delta_i}}{r!}\left((\alpha+\mu)^{r-l-1}-(2\alpha+\mu)^{r-l-1}\right)$, $\gamma=\mu/(\alpha+\mu)$, $\eta=\mu/(2\alpha+\mu)$ *and* $\theta=\mu/\alpha$.

Proof of Proposition EC.2 Before introducing the proof, we remind that if $X\sim Erlang(k,\mu)$, then by Laplace–Stieltjes transform,

$$\mathrm{E}[e^{-tX}]=\left(\frac{\mu}{\mu+t}\right)^k, \quad \text{for } t>0, \quad \text{(EC.9)}$$

which we will frequently use later. Let W_n, $n=1,\cdots,x_i$, denote individual n's waiting time, and let W_0 be the remaining waiting time of the last individual from the previous batch, starting from the arrival of the current batch. We detail the expressions from the Case 2-4 in Proposition EC.1 as follows:

- **Case 2**: If $n+j=2, n<x_i$, since $W_n\sim \mathrm{Exp}(\mu)$, we have

$$R^{\delta_i}_{x_i,n}(j)=p_0(1-p_0)\mathrm{E}[1-e^{-\alpha W_n}]=p_0(1-p_0)\left(1-\frac{\mu}{\alpha+\mu}\right),$$

  where the last equality is derived by Equation (EC.9). Replace $\mu/(\alpha+\mu)$ with γ, we obtain the corresponding closed form in Equation (EC.8).

- **Case 3**: If $2<n+j, n<x_i$, from Equation (EC.5), we have

$$R^{\delta_i}_{x_i,n}(j)=p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_{n-1}}]+p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_n}] \\ +p_0^2(1-p_0)\mathrm{E}[1-e^{-\alpha(W_n+W_{n-1})}]. \quad \text{(EC.10)}$$

  We next calculate the terms in Equation (EC.10) one by one:

(1) Since $W_{n-1} \sim \text{Erlang}(n+j-2, \mu)$, by Equation (EC.9), we have

$$p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_{n-1}}] = p_0(1-p_0)^2\left(1-\left(\frac{\mu}{\alpha+\mu}\right)^{n+j-2}\right). \tag{EC.11}$$

(2) Similarly, with $W_n \sim \text{Erlang}(n+j-1, \mu)$, by Equation (EC.9), we have

$$p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_n}] = p_0(1-p_0)^2\left(1-\left(\frac{\mu}{\alpha+\mu}\right)^{n+j-1}\right). \tag{EC.12}$$

(3) Since individual $n-1$ still waits in the queue, it follows that $W_n = W_{n-1} + S$, where S represents their service time. S is distributed as $\text{Exp}(\mu)$ and independent of W_{n-1}, while W_n is not independent of W_{n-1}. Then, we have

$$\begin{aligned} p_0^2(1-p_0)\mathrm{E}[1-e^{-\alpha(W_{n-1}+W_n)}] &= p_0^2(1-p_0)\left(1-\mathrm{E}[e^{-\alpha(2W_{n-1}+S)}]\right) \\ &= p_0^2(1-p_0)(1-\mathrm{E}[e^{-2\alpha W_{n-1}}]\mathrm{E}[e^{-\alpha S}]) \\ &= p_0^2(1-p_0)\left(1-\frac{\mu^{n+j-1}}{(2\alpha+\mu)^{n+j-2}(\alpha+\mu)}\right), \end{aligned} \tag{EC.13}$$

where the second equality follows from the independence between W_{n-1} and S, and the third equality is derived by Equation (EC.9).

For notational convenience, let $\gamma = \mu/(\alpha+\mu)$ and $\eta = \mu/(2\alpha+\mu)$. Plugging Equation (EC.11), (EC.12), (EC.13) into Equation (EC.10), we obtain the closed expression as follows:

$$R^{\delta_i}_{x_i,n}(j) = p_0(1-p_0)^2(2-\gamma^{n+j-2}-\gamma^{n+j-1}) + p_0^2(1-p_0)(1-\gamma\eta^{n+j-2}). \tag{EC.14}$$

- **Case 4**: If $1 < n+j, n = x_i$, again, from Equation (EC.5), we have

$$\begin{aligned} R^{\delta_i}_{n,n}(j) = p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_{n-1}}] + p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha[W_n-\delta_i]_+}] \\ + p_0^2(1-p_0)\mathrm{E}[1-e^{-\alpha(W_{n-1}+[W_n-\delta_i]_+)}]. \end{aligned} \tag{EC.15}$$

Similarly as Case 3, we calculate the terms in Equation (EC.15) one by one:

(1) From Case 2 and Case 3 (1), $p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha W_{n-1}}] = p_0(1-p_0)^2(1-\gamma^{n+j-2})$.

(2) Since $W_n \sim \text{Erlang}(n+j-1, \mu)$, we have

$$\begin{aligned} p_0(1-p_0)^2\mathrm{E}[1-e^{-\alpha[W_n-\delta_i]_+}] &= p_0(1-p_0)^2\int_0^\infty\left(1-e^{-\alpha[w_n-\delta_i]_+}\right)\underbrace{\frac{\mu^{n+j-1}w_n^{n+j-2}e^{-\mu w_n}}{(n+j-2)!}}_{\text{PDF of } W_n}dw_n \\ &= p_0(1-p_0)^2\left(\sum_{l=0}^{n+j-2}\frac{(\mu\delta_i)^l e^{-\mu\delta_i}\left(1-\left(\frac{\mu}{\alpha+\mu}\right)^{n+j-1-l}\right)}{l!}\right). \end{aligned} \tag{EC.16}$$

Equation (EC.16) follows from integration by parts.

(3) If $n+j=2$, then $W_{n-1}=0$, $W_n \sim \text{Exp}(\mu)$, and we have

$$p_0^2(1-p_0)\mathrm{E}[1-e^{-\alpha(W_{n-1}+[W_n-\delta_i]_+)}] = p_0^2(1-p_0)e^{-\mu\delta_i}(1-\gamma).$$

If $n+j>2$, then $W_{n-1}\sim \text{Erlang}(n+j-2,\mu)$. To calculate $\text{E}[1-e^{-\alpha(W_{n-1}+[W_n-\delta_i]_+)}]$, we need to know the joint PDF of W_{n-1} and W_n. Again, as we discuss above Equation (EC.13), $W_n=W_{n-1}+S$, where $S\sim\text{Exp}(\mu)$ and independent of W_{n-1}. Then, the conditional PDF of W_n on $W_{n-1}=w_{n-1}$ is $f(w_n|w_{n-1})=\mu e^{-\mu(w_n-w_{n-1})}$. Therefore, we have

$$f(w_n,w_{n-1})=f(w_n|w_{n-1})f(w_{n-1})=\frac{\mu^{n+j-1}(w_{n-1})^{n+j-3}e^{-\mu w_n}}{(n+j-3)!},\quad 0\le w_{n-1}\le w_n, \tag{EC.17}$$

and

$$\begin{aligned}
&p_0^2(1-p_0)\text{E}[1-e^{-\alpha(W_{n-1}+[W_n-\delta_i]_+)}]\\
&=p_0^2(1-p_0)\left(1-\int_0^\infty\int_0^{w_n}e^{-\alpha(w_{n-1}+[w_n-\delta_i]_+)}\underbrace{\frac{\mu^{n+j-1}(w_{n-1})^{n+j-3}e^{-\mu w_n}}{(n+j-3)!}}_{\substack{\text{joint of PDF } W_{n-1} \text{ and } W_n\\ \text{in Equation (EC.17)}}}dw_{n-1}dw_n\right)\\
&=p_0^2(1-p_0)\left(1-\frac{\mu^{n+j-1}}{\alpha^{n+j-2}}\int_0^\infty e^{-\alpha[w_n-\delta_i]_+-\mu w_n}\left[1-\sum_{l=0}^{n+j-3}\frac{e^{-\alpha w_n}(\alpha w_n)^l}{l!}\right]dw_n\right) &\text{(EC.18)}\\
&=p_0^2(1-p_0)\left(1-\left(\frac{\mu}{\alpha+\mu}\right)^{n+j-2}+\frac{e^{-\mu\delta_i}\mu^{n+j-2}}{(\alpha+\mu)\alpha^{n+j-3}}-\frac{\mu^{n+j-1}}{\alpha^{n+j-2}}A\right), &\text{(EC.19)}
\end{aligned}$$

where $A=\sum_{l=0}^{n+j-3}\sum_{r=0}^{l}\frac{\alpha^l{\delta_i}^r e^{-(\alpha+\mu)\delta_i}}{r!}((\alpha+\mu)^{r-l-1}-(2\alpha+\mu)^{r-l-1})$. Equation (EC.18) and (EC.19) are obtained from integration by parts.

Let $\theta=\mu/\alpha$, and replace $\mu/(\alpha+\mu)$ with γ and $\mu/(2\alpha+\mu)$ with η. Thus, we have

$$\begin{aligned}
R_{n,n}^{\delta_i}&=p_0(1-p_0)^2\left(1-\gamma^{n+j-2}+\sum_{l=0}^{n+j-2}\frac{(\mu\delta_i)^l e^{-\mu\delta_i}(1-\gamma^{n+j-1-l})}{l!}\right)\\
&+p_0^2(1-p_0)\left(1-\gamma^{n+j-2}+e^{-\mu\delta_i}\gamma\theta^{n+j-3}-\mu\theta^{n+j-2}A\right). \qquad\square
\end{aligned}\tag{EC.20}$$

EC.5. Relationship between $C(\boldsymbol{x})$ and $O(\boldsymbol{x})$

In this section, we show that $C(\boldsymbol{x})\approx 2\alpha p_0(1-p_0)O(\boldsymbol{x})$ for small values of $\{\alpha_z\}_{z=1}^{\bar{z}}$. Therefore, given the approximate positive linear relationship between $C(\boldsymbol{x})$ and $O(\boldsymbol{x})$, minimizing the latter serves as a good proxy for minimizing the former.

For any given schedule $\boldsymbol{x}$, for small values of $\{\alpha_z\}_{z=1}^{\bar{z}}$, we have (by the Taylor series expansion):

$$\text{E}[1-e^{-(\sum_{z=1}^{\bar{z}}I_i^{i-z}\cdot\alpha_z O_i^{i-z}(\boldsymbol{x})+\sum_{z=1}^{\bar{z}}I_i^{i+z}\cdot\alpha_z O_i^{i+z}(\boldsymbol{x}))}]\approx\sum_{z=1}^{\bar{z}}I_i^{i-z}\cdot\alpha_z O_i^{i-z}(\boldsymbol{x})+\sum_{z=1}^{\bar{z}}I_i^{i+z}\cdot\alpha_z O_i^{i+z}(\boldsymbol{x}).$$

Similarly, we have

$$C(\boldsymbol{x})=\sum_{i=1}^{\|\boldsymbol{x}\|_1}\sum_{\boldsymbol{I}}\text{P}(\boldsymbol{I})\cdot(1-p_0)\cdot\text{E}[1-e^{-(\sum_{z=1}^{\bar{z}}I_i^{i-z}\cdot\alpha_z O_i^{i-z}(\boldsymbol{x})+\sum_{z=1}^{\bar{z}}I_i^{i+z}\cdot\alpha_z O_i^{i+z}(\boldsymbol{x}))}]$$

$$\approx \sum_{i=1}^{\|\boldsymbol{x}\|_1} \sum_{\boldsymbol{I}} \mathrm{P}(\boldsymbol{I}) \cdot (1-p_0) \cdot \left(\sum_{z=1}^{\bar{z}} I_i^{i-z} \cdot \alpha_z O_i^{i-z}(\boldsymbol{x}) + \sum_{z=1}^{\bar{z}} I_i^{i+z} \cdot \alpha_z O_i^{i+z}(\boldsymbol{x}) \right)$$

$$= (1-p_0) \sum_{i=1}^{\|\boldsymbol{x}\|_1} \left(\sum_{z=1}^{\bar{z}} \sum_{\boldsymbol{I} \in \{\boldsymbol{I}: I_i^{i-z}=1\}} \mathrm{P}(\boldsymbol{I}) \alpha_z \mathrm{E}[O_i^{i-z}(\boldsymbol{x})] + \sum_{z=1}^{\bar{z}} \sum_{\boldsymbol{I} \in \{\boldsymbol{I}: I_i^{i+z}=1\}} \mathrm{P}(\boldsymbol{I}) \alpha_z \mathrm{E}[O_i^{i+z}(\boldsymbol{x})] \right)$$

$$= (1-p_0) \sum_{i=1}^{\|\boldsymbol{x}\|_1} \left(\sum_{z=1}^{\bar{z}} p_0 \cdot \alpha_z \mathrm{E}[O_i^{i-z}(\boldsymbol{x})] + \sum_{z=1}^{\bar{z}} p_0 \cdot \alpha_z \mathrm{E}[O_i^{i+z}(\boldsymbol{x})] \right) \tag{EC.21}$$

$$= p_0(1-p_0) \sum_{z=1}^{\bar{z}} \alpha_z \sum_{i=1}^{\|\boldsymbol{x}\|_1} (\mathrm{E}[O_i^{i-z}(\boldsymbol{x})] + \mathrm{E}[O_i^{i+z}(\boldsymbol{x})])$$

$$= 2p_0(1-p_0) \sum_{z=1}^{\bar{z}} \alpha_z \sum_{i=1}^{\|\boldsymbol{x}\|_1} \mathrm{E}[O_i^{i-z}(\boldsymbol{x})] \tag{EC.22}$$

$$= 2p_0(1-p_0) O(\boldsymbol{x}). \tag{EC.23}$$

We explain Equation (EC.21)-(EC.23) as follows. For individual i, $\{\boldsymbol{I} : I_i^{i-z} = 1\}$ is a collection of all the cases when individual $i-z$ exists and is initially infectious. Since each individual's initial health status is independent and $\mathrm{P}(\boldsymbol{I})$ can be calculated exactly using the binomial distribution, $\sum_{\boldsymbol{I} \in \{\boldsymbol{I}: I_i^{i-z}=1\}} \mathrm{P}(\boldsymbol{I}) = p_0$. Similarly, $\sum_{\boldsymbol{I} \in \{\boldsymbol{I}: I_i^{i+z}=1\}} \mathrm{P}(\boldsymbol{I}) = p_0$. Thus, Equation (EC.21) holds. Equation (EC.22) follows from the fact that $O_i^{i-z}(\boldsymbol{x}) = 0$ for $z \geq i$ and $O_i^{i+z}(\boldsymbol{x}) = 0$ for $i+z > \|\boldsymbol{x}\|_1$. Equation (EC.23) is from the definition of $O(\boldsymbol{x})$.

EC.6. Proof of Proposition 1

Before we give the proof of Proposition 1, we first transform (OPT-$OVERLAP2$) with the decision variable $\boldsymbol{t}$ and the objective function $\tilde{O}(\boldsymbol{t})$. We then prove a structural property of $\tilde{O}(\boldsymbol{t})$ in the following Lemma EC.1 and use it later to Proposition 1.

EC.6.1. Another mathematical formulation of (OPT-$OVERLAP2$)

Given any schedule $\boldsymbol{x} = (x_1, \cdots, x_T)$ with interval length d, we can construct a vector $\boldsymbol{t} = (t_1, \cdots, t_{\|\boldsymbol{x}\|_1})$, where t_i, $i = 1, \cdots, \|\boldsymbol{x}\|_1$, represents the arrival time of individual i. For example, consider the schedule $\boldsymbol{x} = (2,2,2)$, and suppose $d = 3$. Then, the arrival-time-based schedule $\boldsymbol{t} = (0,0,3,3,6,6)$. Let $\tilde{O}(\boldsymbol{t})$ denote the weighted sum of expected overlap time across arrivals under schedule $\boldsymbol{t}$. Since $\boldsymbol{x}$ and $\boldsymbol{t}$ have a one-to-one correspondence, $\tilde{O}(\boldsymbol{t}) = O(\boldsymbol{x})$.

We derive the expression for $\tilde{O}(\boldsymbol{t})$ as follows. Given any schedule $\boldsymbol{t}$, let $\tilde{W}_i(\boldsymbol{t})$ be a random variable representing the waiting time of individual i. Let $\tilde{O}_i^{i-z}(\boldsymbol{t})$ denote the overlap time between individual $i-z$ and individual i. Then,

$$\tilde{O}_i^{i-z}(\boldsymbol{t}) = [\tilde{W}_{i-z}(\boldsymbol{t}) - (t_i - t_{i-z})]_+, \quad i = z+1, \cdots, \|\boldsymbol{x}\|_1. \tag{EC.24}$$

Furthermore, we obtain $\tilde{O}(\boldsymbol{t})$ as $\tilde{O}(\boldsymbol{t}) = \sum_{z=1}^{\bar{z}} \sum_{i=\bar{z}+1}^{\|\boldsymbol{x}\|_1} \alpha_z \mathrm{E}[\tilde{O}_i^{i-z}(\boldsymbol{t})]$.

Next, we transform (OPT-$OVERLAP2$) as:

$$\begin{aligned} &\text{minimize} \quad \tilde{O}(\boldsymbol{t}), \\ &\text{subject to } t_1 \le t_2 \le \cdots t_m \le D, \\ &\qquad\qquad t_i/d \in \mathbb{Z}_{\ge 0}. \end{aligned}$$

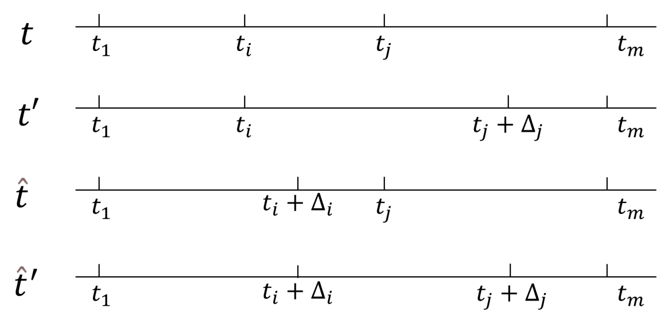
The following Lemma EC.1 and its proof is based on Lemma 1 of Vanden Bosch and Dietz (2001), who consider a scheduling problem with an objective of minimizing expected total waiting cost. We adapt their lemma for our objective of minimizing total expected overlap time. Whereas Vanden Bosch and Dietz (2001) use their Lemma 1 to show their objective function is *submodular*, we will use Lemma EC.1 to prove our objective function is multimodular.

EC.6.2. Property of $\tilde{O}(\boldsymbol{t})$

LEMMA EC.1. *For any feasible schedule* $\boldsymbol{t}$*, let* $\hat{\boldsymbol{t}}$ *be the same schedule as* $\boldsymbol{t}$*, except for individual* i*, who arrives at time* $t_i + \Delta_i \le t_{i+1}$ *instead of* t_i *(*$\Delta_i \ge 0$*). Similarly, let* $\boldsymbol{t}'$ *(*$\hat{\boldsymbol{t}}'$*) be the same as schedule* $\boldsymbol{t}$ *(*$\hat{\boldsymbol{t}}$*), except for individual* j *who arrives at* $t_j + \Delta_j \le t_{j+1}$ *instead of* t_j *(*$\Delta_j \ge 0$*). Then, for these four schedules (visualized in Figure EC.3), we have:*

$$\tilde{O}(\boldsymbol{t}) - \tilde{O}(\boldsymbol{t}') \le \tilde{O}(\hat{\boldsymbol{t}}) - \tilde{O}(\hat{\boldsymbol{t}}').$$

Figure EC.3 Illustration for Lemma EC.1



Proof of Lemma EC.1 Consider any arbitrary sample realization of service times $(S_1, \cdots, S_m)$ for individuals $1, \cdots, m$ (where the index is the order of arrival). For this realization, let $\tilde{W}_n(\boldsymbol{\tau})$ be the waiting time of individual n, and let $\tilde{O}_n^{n-z}(\boldsymbol{\tau})$ be the overlap time between individual $n-z$ and individual n for schedule $\boldsymbol{\tau} \in \{\boldsymbol{t}, \boldsymbol{t}', \hat{\boldsymbol{t}}, \hat{\boldsymbol{t}}'\}$. Additionally, to make $\tilde{O}_n^{n-z}(\boldsymbol{\tau})$ well defined on $n = 1, \cdots, m$ and $z = 0, \cdots, m-1$, for $z \ge n$, let $\tilde{O}_n^{n-z}(\boldsymbol{\tau}) = 0$. We will show that: for any $z \in \{0, \cdots, m-1\}$

$$\begin{aligned} &\tilde{O}_n^{n-z}(\boldsymbol{t}) - \tilde{O}_n^{n-z}(\boldsymbol{t}') = \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}) - \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}') = 0, \quad n = 2, \cdots, j-1, \\ &\tilde{O}_n^{n-z}(\boldsymbol{t}) - \tilde{O}_n^{n-z}(\boldsymbol{t}') \le \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}) - \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}'), \quad n = j, \cdots, m. \end{aligned} \tag{EC.25}$$

Then, since this holds for any sample path of service times, (EC.25) holds in expectation for n and z, and therefore holds for the weighted sum over n and z of the expected differences between the left and right-hand slides of (EC.25). In other words, showing (EC.25) holds for an arbitrary realization of service times suffices to show that $\sum_{z=1}^{\bar{z}}\sum_{n=z+1}^{m}\alpha_z(\mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t})]-\mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t}')])\le \sum_{z=1}^{\bar{z}}\sum_{n=z+1}^{m}\alpha_z(\mathrm{E}[\tilde{O}_n^{n-z}(\hat{\boldsymbol{t}})]-\mathrm{E}[\tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}')])$.

We will prove (EC.25) by induction.

- **Base case:** When $z=0$, (EC.25) is validated by the proof of Lemma 1 in Vanden Bosch and Dietz (2001), since $\tilde{O}_n^n(\boldsymbol{\tau})$ is defined as $\tilde{W}_n(\boldsymbol{\tau})$ for $\boldsymbol{\tau}\in\{\boldsymbol{t},\boldsymbol{t}',\hat{\boldsymbol{t}},\hat{\boldsymbol{t}}'\}$ and $n=1,\cdots,m$.
- **Inductive step:** Assume (EC.25) holds for $z=k$ (here we use k to denote a natural number instead of the number of servers). That is

$$\begin{aligned}&\tilde{O}_n^{n-k}(\boldsymbol{t})-\tilde{O}_n^{n-k}(\boldsymbol{t}')=\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}})-\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}}')=0,\ n=2,\cdots,j-1,\\&\tilde{O}_n^{n-k}(\boldsymbol{t})-\tilde{O}_n^{n-k}(\boldsymbol{t}')\le\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}})-\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}}'),\qquad n=j,\cdots,m.\end{aligned}\tag{EC.26}$$

If we can show (EC.25) holds for $z=k+1$ as well, the proof will be complete.

Before proceeding with our proof, we first derive the relationship between $\tilde{O}_n^{n-(k+1)}(\boldsymbol{\tau})$ and $\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{\tau})$, that is

$$\tilde{O}_n^{n-(k+1)}(\boldsymbol{\tau})=[\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{\tau})-(\tau_n-\tau_{n-1})]_+.\tag{EC.27}$$

We explain how to derive Equation (EC.27) as follows:

1) If $n>k+1$, then

$$\begin{aligned}\tilde{O}_n^{n-(k+1)}(\boldsymbol{\tau})&=[\tilde{W}_{n-(k+1)}(\boldsymbol{\tau})-(\tau_n-\tau_{n-(k+1)})]_+\\&=[\tilde{W}_{n-(k+1)}(\boldsymbol{\tau})-(\tau_{n-1}-\tau_{(n-1)-k})-(\tau_n-\tau_{n-1})]_+\\&=[[\tilde{W}_{(n-1)-k}(\boldsymbol{\tau})-(\tau_{n-1}-\tau_{(n-1)-k})]_+-(\tau_n-\tau_{n-1})]_+\\&=[\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{\tau})-(\tau_n-\tau_{n-1})]_+,\end{aligned}$$

where the third equality follows from $\tau_n-\tau_{n-1}\ge 0$.

2) If $2\le n\le k+1$, since individual $n-(k+1)$ does not exist, then $\tilde{O}_n^{n-(k+1)}(\boldsymbol{\tau})=\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{\tau})=0$, and thus Equation (EC.27) still holds.

We proceed to prove Equation (EC.25). Consider individuals indexed i and j, and without loss of generality, assume $j>i+1$ (we will discuss the case when $j=i+1$ later). We consider the following five cases:

(i) Consider $n\le j-1$. Since the only difference between $\boldsymbol{t}$ and $\boldsymbol{t}'$ (and between schedule $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$) is the arrival time of individual j, it follows that

$$t_n=t'_n\quad\text{and}\quad\hat{t}_n=\hat{t}'_n,\quad n=1,\cdots,j-1.$$

From (EC.26), we have

$$\tilde{O}_n^{n-k}(\boldsymbol{t})=\tilde{O}_n^{n-k}(\boldsymbol{t}')\quad\text{and}\quad\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}})=\tilde{O}_n^{n-k}(\hat{\boldsymbol{t}}'),\quad n=1,\cdots,j-2.$$

Then, by Equation (EC.27), we have

$$\tilde{O}_n^{n-(k+1)}(\boldsymbol{t})-\tilde{O}_n^{n-(k+1)}(\boldsymbol{t}')=\tilde{O}_n^{n-(k+1)}(\hat{\boldsymbol{t}})-\tilde{O}_n^{n-(k+1)}(\hat{\boldsymbol{t}}'),\quad n=2,\cdots,j-1.$$

(ii) Consider $n=j$. As in part (i), the only difference between schedule $\boldsymbol{t}$ and $\boldsymbol{t}'$ (and between schedule $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$) is the time of individual j's arrival, and so again from (EC.26) we have $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})=\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t}')$ and $\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})=\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}}')$. We use these equalities in the following:

$$\begin{aligned}\tilde{O}_{j}^{j-(k+1)}(\boldsymbol{t})-\tilde{O}_{j}^{j-(k+1)}(\boldsymbol{t}') &=[\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})-(t_j-t_{j-1})]_+-[\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})-(t_j+\Delta t_j-t_{j-1})]_+,\\ \tilde{O}_{j}^{j-(k+1)}(\hat{\boldsymbol{t}})-\tilde{O}_{j}^{j-(k+1)}(\hat{\boldsymbol{t}}') &=[\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})-(t_j-t_{j-1})]_+-[\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})-(t_j+\Delta t_j-t_{j-1})]_+.\end{aligned} \tag{EC.28}$$

Next, we compare $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})$ and $\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})$. Notice that the only difference between schedule $\boldsymbol{t}$ and $\hat{\boldsymbol{t}}$ is that individual i arrives later in schedule $\hat{\boldsymbol{t}}$ (i.e., $t_i+\Delta_i=\hat{t}_i$ and $t_n=\hat{t}_n$ for $n\neq i$). Therefore, it must follow that $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})\leq\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})$ for the following reasons:

(ii-a) If $j-1-k<i$, then the arrival time of individual i does not affect the waiting time of individual $(j-1-k)$, i.e., $\tilde{W}_{j-1-k}(\boldsymbol{t})=\tilde{W}_{j-1-k}(\hat{\boldsymbol{t}})$. Thus, since $t_{j-1-k}=\hat{t}_{j-1-k}$ and $t_{j-1}=\hat{t}_{j-1}$, by Equation (EC.24), we have

$$\begin{aligned}\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t}) &=[\tilde{W}_{j-1-k}(\boldsymbol{t})-(t_{j-1}-t_{j-1-k})]_+\\ &=[\tilde{W}_{j-1-k}(\hat{\boldsymbol{t}})-(\hat{t}_{j-1}-\hat{t}_{j-1-k})]_+\\ &=\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}}).\end{aligned}$$

(ii-b) If $j-1-k=i$, then the waiting time of individual i is less under schedule $\hat{\boldsymbol{t}}$, $\tilde{W}_{j-1-k}(\hat{\boldsymbol{t}})=[\tilde{W}_{j-1-k}(\boldsymbol{t})-\Delta_i]_+$. Thus, since $t_{j-1-k}+\Delta_i=\hat{t}_{j-1-k}$ and $t_{j-1}=\hat{t}_{j-1}$, by Equation (EC.24), we have

$$\begin{aligned}\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t}) &=[\tilde{W}_{j-1-k}(\boldsymbol{t})-(t_{j-1}-t_{j-1-k})]_+\\ &=[(\tilde{W}_{j-1-k}(\boldsymbol{t})-\Delta_i)-(t_{j-1}-t_{j-1-k}-\Delta_i)]_+\\ &=[[\tilde{W}_{j-1-k}(\boldsymbol{t})-\Delta_i]_+-(t_{j-1}-t_{j-1-k}-\Delta_i)]_+\\ &=[\tilde{W}_{j-1-k}(\hat{\boldsymbol{t}})-(\hat{t}_{j-1}-\hat{t}_{j-1-k})]_+\\ &=\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}}),\end{aligned}$$

where the third equation is from the fact that $t_{j-1}-t_{j-1-k}-\Delta_i\geq 0$.

(ii-c) If $j-1-k>i$, then the later arrival of individual i leads to a longer waiting time for individual $j-1-k$, i.e., $\tilde{W}_{j-1-k}(\boldsymbol{t})\leq\tilde{W}_{j-1-k}(\hat{\boldsymbol{t}})$. Since $t_{j-1-k}=\hat{t}_{j-1-k}$ and $t_{j-1}=\hat{t}_{j-1}$, similarly as case (ii-a) and (ii-b) above, by Equation (EC.24), we have $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})\leq\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}})$.

Furthermore, if we view the right-hand side of Equation (EC.28) as a function of $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t})$, then it is non-decreasing. Therefore, it follows that

$$\tilde{O}_{j}^{j-(k+1)}(\boldsymbol{t})-\tilde{O}_{j}^{j-(k+1)}(\boldsymbol{t}')\leq\tilde{O}_{j}^{j-(k+1)}(\hat{\boldsymbol{t}})-\tilde{O}_{j}^{j-(k+1)}(\hat{\boldsymbol{t}}').$$

(iii) Consider $n\in[j+1,j+k]$, which implies $k\geq 1$. Since $\tilde{W}_{n-(k+1)}(\boldsymbol{t})=\tilde{W}_{n-(k+1)}(\boldsymbol{t}')$, $t_n=t'_n$ and $t_{n-(k+1)}=t'_{n-(k+1)}$, by Equation (EC.24), it follows $\tilde{O}_{n}^{n-(k+1)}(\boldsymbol{t})=\tilde{O}_{n}^{n-(k+1)}(\boldsymbol{t}')$. Similarly, for $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$, we have $\tilde{O}_{n}^{n-(k+1)}(\hat{\boldsymbol{t}})=\tilde{O}_{n}^{n-(k+1)}(\hat{\boldsymbol{t}}')$. Thus,

$$\tilde{O}_{n}^{n-(k+1)}(\boldsymbol{t})-\tilde{O}_{n}^{n-(k+1)}(\boldsymbol{t}')=\tilde{O}_{n}^{n-(k+1)}(\hat{\boldsymbol{t}})-\tilde{O}_{n}^{n-(k+1)}(\hat{\boldsymbol{t}}')=0,\quad n=j+1,\cdots,j+k.$$

(iv) Consider $n = j+k+1$. We compare schedule $\boldsymbol{t}$ and $\boldsymbol{t}'$ in two subcases.

(iv-a) If $\tilde{W}_j(\boldsymbol{t}) \le \Delta_j$, then $\tilde{W}_j(\boldsymbol{t}') = 0$. Therefore, we have

$$\tilde{O}^j_{j+k+1}(\boldsymbol{t}) = [\tilde{W}_j(\boldsymbol{t}) - (t_{j+k+1} - t_j)]_+ = 0 \text{ (by our assumption, } \Delta_j \le (t_{j+k+1} - t_j)),$$
$$\tilde{O}^j_{j+k+1}(\boldsymbol{t}') = [\tilde{W}_j(\boldsymbol{t}') - (t_{j+k+1} - t_j - \Delta_j)]_+ = 0.$$

(iv-b) If $\tilde{W}_j(\boldsymbol{t}) > \Delta_j$, then $\tilde{W}_j(\boldsymbol{t}') = \tilde{W}_j(\boldsymbol{t}) - \Delta_j$ and

$$\begin{aligned}\tilde{O}^j_{j+k+1}(\boldsymbol{t}') &= [\tilde{W}_j(\boldsymbol{t}') - (t_{j+k+1} - t_j - \Delta_j)]_+ \\ &= [\tilde{W}_j(\boldsymbol{t}) - (t_{j+k+1} - t_j)]_+ \\ &= \tilde{O}^j_{j+k+1}(\boldsymbol{t}).\end{aligned}$$

Therefore, $\tilde{O}^j_{j+k+1}(\boldsymbol{t}') = \tilde{O}^j_{j+k+1}(\boldsymbol{t})$. We can make a similar comparison between schedule $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$. Hence,

$$\tilde{O}^j_{j+k+1}(\boldsymbol{t}) - \tilde{O}^j_{j+k+1}(\boldsymbol{t}') = \tilde{O}^j_{j+k+1}(\hat{\boldsymbol{t}}) - \tilde{O}^j_{j+k+1}(\hat{\boldsymbol{t}}') = 0.$$

(v) Consider $n \ge j+k+2$. We still divide it into two subcases:

(v-a) If $\tilde{W}_j(\hat{\boldsymbol{t}}) > \Delta_j$, then $\tilde{W}_j(\hat{\boldsymbol{t}}') = \tilde{W}_j(\hat{\boldsymbol{t}}) - \Delta_j$. Let S_j represent the service time of individual j. For $\tilde{W}_{j+1}(\hat{\boldsymbol{t}}')$ and $\tilde{W}_{j+1}(\hat{\boldsymbol{t}})$, we have

$$\begin{aligned}\tilde{W}_{j+1}(\hat{\boldsymbol{t}}') &= [\tilde{W}_j(\hat{\boldsymbol{t}}') + S_j - (\hat{t}'_{j+1} - \hat{t}'_j)]_+ \\ &= [\tilde{W}_j(\hat{\boldsymbol{t}}) - \Delta_j + S_j - (\hat{t}_{j+1} - \hat{t}_j - \Delta_j)]_+ \\ &= [\tilde{W}_j(\hat{\boldsymbol{t}}) + S_j - (\hat{t}_{j+1} - \hat{t}_j)]_+ \\ &= \tilde{W}_{j+1}(\hat{\boldsymbol{t}}).\end{aligned}$$

Since $\hat{t}_n = \hat{t}'_n$ for $n \ge j+1$, we have $\tilde{W}_n(\hat{\boldsymbol{t}}) = \tilde{W}_n(\hat{\boldsymbol{t}}')$ for $n \ge j+1$. Thus, by Equation (EC.24), we have $\tilde{O}^{n-(k+1)}_n(\hat{\boldsymbol{t}}) - \tilde{O}^{n-(k+1)}_n(\hat{\boldsymbol{t}}') = 0$ for $n \ge j+k+2$.

With $t_n = t'_n$ for $n \ge j+k+1$, we write out the explicit formulas of $\tilde{O}^{n-(k+1)}_n(\boldsymbol{t})$ and $\tilde{O}^{n-(k+1)}_n(\boldsymbol{t}')$ for $n \ge j+k+2$ as

$$\begin{aligned}\tilde{O}^{n-(k+1)}_n(\boldsymbol{t}) &= [\tilde{O}^{n-1-k}_{n-1}(\boldsymbol{t}) - (t_n - t_{n-1})]_+, \\ \tilde{O}^{n-(k+1)}_n(\boldsymbol{t}') &= [\tilde{O}^{n-1-k}_{n-1}(\boldsymbol{t}') - (t_n - t_{n-1})]_+.\end{aligned}$$

From the inductive assumption (EC.26), we have

$$\tilde{O}^{n-k}_n(\boldsymbol{t}) - \tilde{O}^{n-k}_n(\boldsymbol{t}') \le \tilde{O}^{n-k}_n(\hat{\boldsymbol{t}}) - \tilde{O}^{n-k}_n(\hat{\boldsymbol{t}}') = 0, \quad n = j+k+1, \cdots, m.$$

Therefore,

$$\tilde{O}^{n-(k+1)}_n(\boldsymbol{t}) - \tilde{O}^{n-(k+1)}_n(\boldsymbol{t}') \le 0 = \tilde{O}^{n-(k+1)}_n(\hat{\boldsymbol{t}}) - \tilde{O}^{n-(k+1)}_n(\hat{\boldsymbol{t}}'), \quad n = j+k+2, \cdots, m.$$

(v-b) If $\tilde{W}_j(\hat{\boldsymbol{t}}) \le \Delta_j$, then $\tilde{W}_j(\hat{\boldsymbol{t}}') = 0$. With $\tilde{W}_j(\boldsymbol{t}') \le \tilde{W}_j(\hat{\boldsymbol{t}}')$ (since the only difference between schedule $\boldsymbol{t}'$ and schedule $\hat{\boldsymbol{t}}'$ is that individual i arrives later in schedule $\hat{\boldsymbol{t}}'$), then $\tilde{W}_j(\hat{\boldsymbol{t}}') = \tilde{W}_j(\boldsymbol{t}') = 0$. Recall that $t_j = \hat{t}_j$, $t'_j = \hat{t}'_j = t_j + \Delta_j$, and $t_n = t'_n = \hat{t}_n = \hat{t}_n$ for $n \ge j+1$. Then, for $\tilde{W}_{j+1}(\boldsymbol{t}')$ and $\tilde{W}_{j+1}(\hat{\boldsymbol{t}}')$, we have

$$\begin{aligned}\tilde{W}_{j+1}(\boldsymbol{t}') &= [\tilde{W}_j(\boldsymbol{t}') + S_j - (t'_{j+1} - t'_j)] \\ &= [\tilde{W}_j(\hat{\boldsymbol{t}}') + S_j - (\hat{t}'_{j+1} - \hat{t}'_j)] \\ &= \tilde{W}_{j+1}(\hat{\boldsymbol{t}}').\end{aligned}$$

Therefore, we have that $\tilde{W}_n(\boldsymbol{t}') = \tilde{W}_n(\hat{\boldsymbol{t}}')$ for $n \ge j+1$. Since $t'_n = \hat{t}'_n$ for $n \ge j+1$, then from Equation (EC.24), $\tilde{O}_n^{n-k}(\boldsymbol{t}') = \tilde{O}_n^{n-k}(\hat{\boldsymbol{t}}')$ for $n \ge j+k+1$. Additionally, the base case implies $\tilde{W}_n(\boldsymbol{t}) \le \tilde{W}_n(\hat{\boldsymbol{t}})$, $n \ge j+1$. Since $t_n = \hat{t}_n$ for $n \ge j+1$, then from Inequality (EC.24), we have $\tilde{O}_n^{n-k}(\boldsymbol{t}) \le \tilde{O}_n^{n-k}(\hat{\boldsymbol{t}})$ for $n \ge j+k+1$. It follows that

$$\begin{aligned}\tilde{O}_n^{n-(k+1)}(\boldsymbol{t}) - \tilde{O}_n^{n-(k+1)}(\boldsymbol{t}') &= [\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{t}) - (t_n - t_{n-1})]_+ - [\tilde{O}_{n-1}^{n-1-k}(\boldsymbol{t}') - (t_n - t_{n-1})]_+ \\ &\le [\tilde{O}_{n-1}^{n-1-k}(\hat{\boldsymbol{t}}) - (t_n - t_{n-1})]_+ - [\tilde{O}_{n-1}^{n-1-k}(\hat{\boldsymbol{t}}') - (t_n - t_{n-1})]_+ \\ &= \tilde{O}_n^{n-(k+1)}(\hat{\boldsymbol{t}}) - \tilde{O}_n^{n-(k+1)}(\hat{\boldsymbol{t}}'), \quad n = j+k+2, \cdots, m.\end{aligned}$$

Finally, we consider the case when $j = i+1$. We only need to discuss the subcase when $n = j$, since the results of other subcases are the same as those when $j > i+1$.

(a) If $\tilde{O}_i^{j-1-k}(\boldsymbol{t}) \le \Delta_i$, recall that $\Delta_i \le t_j - t_i < t'_j - t'_i$. By assumption (EC.25) and Equation (EC.27), we have $\tilde{O}_j^{j-(k+1)}(\boldsymbol{t}) = \tilde{O}_j^{j-(k+1)}(\boldsymbol{t}') = \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}}) = \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}}') = 0$.

(b) If $\tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t}) > \Delta_i$, then $\tilde{O}_{j-1}^{j-1-k}(\hat{\boldsymbol{t}}) = \tilde{O}_{j-1}^{j-1-k}(\boldsymbol{t}) - \Delta_i$. By Equation (EC.25), recalling that $\hat{t}_j - \hat{t}_i = t_j - t_i - \Delta_i$, then $\tilde{O}_j^{j-(k+1)}(\boldsymbol{t}) = \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}})$. Similarly, by assumption (EC.27), we obtain $\tilde{O}_j^{j-(k+1)}(\boldsymbol{t}') = \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}}')$.

So $\tilde{O}_j^{j-(k+1)}(\boldsymbol{t}) - \tilde{O}_j^{j-(k+1)}(\boldsymbol{t}') \le \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}}) - \tilde{O}_j^{j-(k+1)}(\hat{\boldsymbol{t}}')$ still holds. □

EC.6.3. Proof of Proposition 1

Proof of Proposition 1 Define $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\cdots$, $\boldsymbol{v}_T$ in $\mathbb{Z}^{T-1}$ as in Equation (2). By Definition 1, to show $\widehat{O}$ is multimodular in $\mathcal{A}$, we need to verify for any $\boldsymbol{y} \in \mathcal{A}$ and $\boldsymbol{v}_i$, $\boldsymbol{v}_j$, $1 \le i < j \le T$, such that $\{\boldsymbol{y}+\boldsymbol{v}_i, \boldsymbol{y}+\boldsymbol{v}_j, \boldsymbol{y}+\boldsymbol{v}_i+\boldsymbol{v}_j\} \subset \mathcal{A}$, we have

$$\widehat{O}(\boldsymbol{y}+\boldsymbol{v}_i) + \widehat{O}(\boldsymbol{y}+\boldsymbol{v}_j) \ge \widehat{O}(\boldsymbol{y}) + \widehat{O}(\boldsymbol{y}+\boldsymbol{v}_i+\boldsymbol{v}_j). \tag{EC.29}$$

Our next steps are based on the approach used by Kaandorp and Koole (2007) to prove their Theorem A.4. Define vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_T$ in $\mathbb{Z}^T$ as

$$\begin{aligned}\boldsymbol{u}_1 &= (-1, 0, \cdots, 0, 1), \\ \boldsymbol{u}_2 &= (1, -1, 0, \cdots, 0), \\ &\vdots \\ \boldsymbol{u}_T &= (0, \cdots, 0, 1, -1).\end{aligned} \tag{EC.30}$$

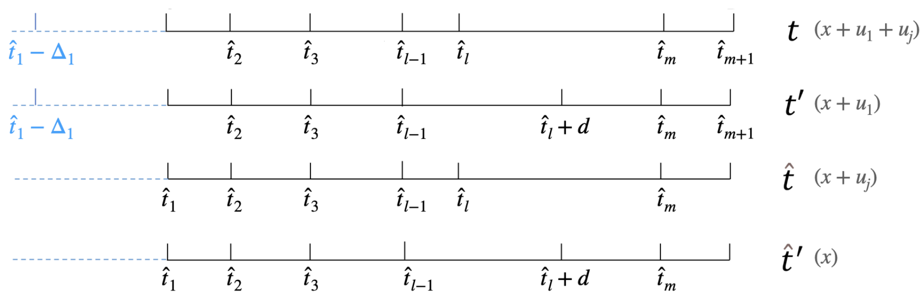
Given $\boldsymbol{y} \in \mathcal{A}$, construct $\boldsymbol{x} = (y_1, \cdots, y_{T-1}, m - \sum_{i=1}^{T-1} y_i)$. Hence, by the definition of the function $\widehat{O}$, it suffices to show that for any $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, $1 \le i < j \le T$, such that $\{\boldsymbol{x}+\boldsymbol{u}_i, \boldsymbol{x}+\boldsymbol{u}_j, \boldsymbol{x}+\boldsymbol{u}_i+\boldsymbol{u}_j\} \subset \mathbb{Z}^T_{\ge 0}$, we have

$$O(\boldsymbol{x}+\boldsymbol{u}_i) + O(\boldsymbol{x}+\boldsymbol{u}_j) \ge O(\boldsymbol{x}) + O(\boldsymbol{x}+\boldsymbol{u}_i+\boldsymbol{u}_j). \tag{EC.31}$$

Consider the following correspondences between four discrete batch-arrival schedules and their equivalent continuous-time arrival schedule versions: associate $(\boldsymbol{x}+\boldsymbol{u}_i+\boldsymbol{u}_j)$ with $\boldsymbol{t}$, $(\boldsymbol{x}+\boldsymbol{u}_i)$ with $\boldsymbol{t}'$, $(\boldsymbol{x}+\boldsymbol{u}_j)$ with $\hat{\boldsymbol{t}}$, and $\boldsymbol{x}$ with $\hat{\boldsymbol{t}}'$. We consider the following two cases:

(i) $2 \le i < j \le T$. Then, the relationships among $\boldsymbol{t}$, $\boldsymbol{t}'$, $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$ are similar to those in Figure EC.3. By Lemma EC.1, it follows that $O(\boldsymbol{x}+\boldsymbol{u}_i) - O(\boldsymbol{x}) \ge O(\boldsymbol{x}+\boldsymbol{u}_i+\boldsymbol{u}_j) - O(\boldsymbol{x}+\boldsymbol{u}_j)$, thus satisfying Inequality (EC.31).

Figure EC.4 Schedule schema for Proposition 1 when $1 = i < j \le T$



(ii) $1 = i < j \le T$. We use Figure EC.4 to illustrate the interconnections among $\boldsymbol{t}$, $\boldsymbol{t}'$, $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$, and we construct it in the following steps:

1) (Schedule $\hat{\boldsymbol{t}}$) We assign indexes $1, 2, \cdots, m$ to individuals who arrive at the system following schedule $\hat{\boldsymbol{t}}$, according to their order of arrival. Let $\hat{t}_i$ denote individual i's arrival time.

2) (Schedule $\hat{\boldsymbol{t}}'$) Note that the difference between schedule $\hat{\boldsymbol{t}}$ $(\boldsymbol{x}+\boldsymbol{u}_j)$ and $\hat{\boldsymbol{t}}'$ $(\boldsymbol{x})$ is that one individual from $(j-1)$th batch arrives one-time slot (d time units) later in schedule $\boldsymbol{x}$. Suppose that particular individual is assigned with index l. Then, the individual l's arrival time is $\hat{t}_l + d$.

3) (Schedule $\boldsymbol{t}$ and $\boldsymbol{t}'$) Compare schedule $\boldsymbol{t}$ $(\boldsymbol{x}+\boldsymbol{u}_1+\boldsymbol{u}_j)$ and $\hat{\boldsymbol{t}}$ $(\boldsymbol{x}+\boldsymbol{u}_j)$. The addition of $\boldsymbol{u}_1$ corresponds to moving one individual from the first batch to the end, denoted as $\hat{t}_{m+1}$. To make Figure EC.4 analogous to Figure EC.3, assume an individual arriving earlier than $\hat{t}_1$ (e.g., at time $\hat{t}_1 - \Delta_1$, where $\Delta_1 > 0$). Similarly, we construct schedule $\boldsymbol{t}'$ based on schedule $\hat{\boldsymbol{t}}'$. Note that the total number of individuals in schedule $\boldsymbol{t}$ ($\boldsymbol{t}'$) is m.

Disregarding $\hat{t}_{m+1}$ in schedule $\boldsymbol{t}$ and $\boldsymbol{t}'$, Figure EC.4 is analogous to Figure EC.3. Specifically, $\hat{t}_1 - \Delta_1$ and $\hat{t}_1$ in Figure EC.4 correspond to t_i and $t_i + \Delta_i$ in Figure EC.3. Additionally, $\hat{t}_l$ and $\hat{t}_l + d$ in Figure EC.4 correspond to t_j and $t_j + \Delta_j$ in Figure EC.3.

Notably, after introducing the additional individual to schedule $\boldsymbol{t}$ ($\boldsymbol{t}'$), the total number of individuals in schedule $\boldsymbol{t}$ ($\boldsymbol{t}'$) is $m+1$. Similar to the proof of Lemma EC.1, consider any arbitrarily sample realization of service times $(S_1, \cdots, S_{m+1})$ for the individuals $1, \cdots, m+1$. For $\boldsymbol{\tau} \in \{\boldsymbol{t}, \boldsymbol{t}', \hat{\boldsymbol{t}}, \hat{\boldsymbol{t}}'\}$ and $z = 1, 2, \cdots, \bar{z}$, let $\tilde{W}_n(\boldsymbol{\tau})$ be the waiting time of individual n, and let $\tilde{O}_n^{n-z}(\boldsymbol{\tau})$ be the overlap time between individual $n-z$ and individual n under schedule $\boldsymbol{\tau}$ in this realization.

By the proof of Lemma EC.1, for any $\Delta_1 > 0$ and $z \in \{1, \cdots, \bar{z}\}$, we have

$$\tilde{O}_n^{n-z}(\boldsymbol{t}) - \tilde{O}_n^{n-z}(\boldsymbol{t}') \le \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}) - \tilde{O}_n^{n-z}(\hat{\boldsymbol{t}}'), \quad n = z+1, \cdots, m. \tag{EC.32}$$

Additionally, the late arrival of individual l would result in $\tilde{W}_n(\boldsymbol{t}) \le \tilde{W}_n(\boldsymbol{t}')$ for $n = l+1, \cdots, m$. Then, we compare $\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t})$ and $\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}')$ in the following three cases:

(a) If $m+1-z \le l-1$, since $\tilde{W}_{m+1-z}(\boldsymbol{t}) = \tilde{W}_{m+1-z}(\boldsymbol{t}')$ and $\hat{t}_n = \hat{t}'_n$ for $n \ne l$, then

$$\begin{aligned}\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}) - \tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}') &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [\tilde{W}_{m+1-z}(\boldsymbol{t}') - (\hat{t}'_{m+1} - \hat{t}'_{m+1-z})]_+ \\ &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [\tilde{W}_{m+1-z}(\boldsymbol{t}') - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ \\ &= 0.\end{aligned}$$

(b) If $m+1-z = l$, since $\tilde{W}_{m+1-z}(\boldsymbol{t}') = [\tilde{W}_{m+1-z}(\boldsymbol{t}) - d]_+$ and $\hat{t}'_{m+1-z} = \hat{t}_{m+1-z} + d$, then

$$\begin{aligned}\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}) - \tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}') &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [\tilde{W}_{m+1-z}(\boldsymbol{t}') - (\hat{t}'_{m+1} - \hat{t}'_{m+1-z})]_+ \\ &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [[\tilde{W}_{m+1-z}(\boldsymbol{t}) - d]_+ - (\hat{t}_{m+1} - \hat{t}_{m+1-z} - d)]_+ \\ &\le [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [(\tilde{W}_{m+1-z}(\boldsymbol{t}) - d) - (\hat{t}_{m+1} - \hat{t}_{m+1-z} - d)]_+ \\ &= 0.\end{aligned}$$

(c) If $m+1-z \ge l+1$, since $\tilde{W}_{m+1-z}(\boldsymbol{t}) \le \tilde{W}_{m+1-z}(\boldsymbol{t}')$ and $\hat{t}_n = \hat{t}'_n$ for $n \ne l$, then

$$\begin{aligned}\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}) - \tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}') &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [\tilde{W}_{m+1-z}(\boldsymbol{t}') - (\hat{t}'_{m+1} - \hat{t}'_{m+1-z})]_+ \\ &= [\tilde{W}_{m+1-z}(\boldsymbol{t}) - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ - [\tilde{W}_{m+1-z}(\boldsymbol{t}') - (\hat{t}_{m+1} - \hat{t}_{m+1-z})]_+ \\ &\le 0.\end{aligned}$$

Thus, we have $\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}) - \tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}') \le 0$ for $z = 1, \cdots, \bar{z}$ in any sample realization, and then

$$\begin{aligned}\tilde{O}(\boldsymbol{t}) - \tilde{O}(\boldsymbol{t}') &= \left(\sum_{z=1}^{\bar{z}} \sum_{n=z+1}^{m} \alpha_z (\mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t})] - \mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t}')])\right) + \sum_{z=1}^{\bar{z}} \alpha_z (\mathrm{E}[\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t})] - \mathrm{E}[\tilde{O}_{m+1}^{m+1-z}(\boldsymbol{t}')]) \\ &\le \left(\sum_{z=1}^{\bar{z}} \sum_{n=z+1}^{m} \alpha_z (\mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t})] - \mathrm{E}[\tilde{O}_n^{n-z}(\boldsymbol{t}')])\right) \\ &= \tilde{O}(\hat{\boldsymbol{t}}) - \tilde{O}(\hat{\boldsymbol{t}}').\end{aligned} \tag{EC.33}$$

Finally, as $\Delta_1 \to \infty$, we have

$$\begin{aligned}\lim_{\Delta_1 \to \infty} \tilde{O}(\boldsymbol{t}) &= \tilde{O}(\hat{t}_2, \cdots, \hat{t}_l, \cdots, \hat{t}_{m+1}) = O(\boldsymbol{x} + \boldsymbol{u}_1 + \boldsymbol{u}_j), \\ \lim_{\Delta_1 \to \infty} \tilde{O}(\boldsymbol{t}') &= \tilde{O}(\hat{t}_2, \cdots, \hat{t}_l + d, \cdots, \hat{t}_{m+1}) = O(\boldsymbol{x} + \boldsymbol{u}_1).\end{aligned}$$

Since $\tilde{O}(\hat{\boldsymbol{t}}) = O(\boldsymbol{x} + \boldsymbol{u}_j)$ and $\tilde{O}(\hat{\boldsymbol{t}}') = O(\boldsymbol{x})$, by Equation (EC.33), Equation (EC.31) is derived. □

EC.7. Proof of Proposition 2

Before presenting the proof of Proposition 2, we introduce a new notation, $o(x_i)$. Recall that $O(\boldsymbol{x})$ denotes the weighted sum of expected overlap time across all batches of arrivals given by schedule $\boldsymbol{x}$. Analogously, let $o(x_i)$ denote the weighted sum of expected overlap time across a single batch of x_i individuals arriving together to an empty system. Then, $o(x_i) = \sum_{z=1}^{\bar{z}} \alpha_z \sum_{j=1}^{x_i} [j - z - 1]_+ / \mu$. The following Lemma EC.2 shows its discrete convex property, and we will use it in the following (as well as in the proof of Proposition 3).

LEMMA EC.2.

(a) $o(x_i)$ *is non-decreasing in* x_i.

(b) for $x_i \geq 2$, $o(x_i+1) - o(x_i) > o(x_i) - o(x_i-1)$.

Proof of Lemma EC.2.

(a) It follows from the definition of $o(x_i)$.

(b) For $x_i \geq 2$,

$$\begin{aligned}(o(x_i+1) - o(x_i)) - (o(x_i) - o(x_i-1)) &= \sum_{z=1}^{\bar{z}} \alpha_z \left(\frac{[x_i+1-z-1]_+ - [x_i-z-1]_+}{\mu}\right) \\ &> 0.\end{aligned}$$

□

Proof of Proposition 2. Let $a \equiv m/T$. By assumption, a is an integer, and therefore the E-D schedule is given by $\bar{\boldsymbol{x}}(m,D) = (a,\cdots,a)$. Recall that $O(\boldsymbol{x})$ denotes the weighted sum of expected overlap time across all batches of arrivals given by schedule $\boldsymbol{x}$. Also, let $o(x_i)$ denote the weighted sum of overlap time across a single batch of x_i individuals arriving together to an empty system. Then, $o(x_i) = \sum_{z=1}^{\bar{z}} \alpha_z \sum_{j=1}^{x_i} [j-z-1]_+/\mu$. We use this in the following (as well as in the proof of Proposition 3).

(a) It suffices to show that for any feasible schedule $\boldsymbol{x}(m,D)$, there exists a D^* such that for all $D \geq D^*$, $O(\bar{\boldsymbol{x}}(m,D)) \leq O(\boldsymbol{x}(m,D))$. This would mean that $\bar{\boldsymbol{x}}(m,D)$ is an optimal schedule.

To do so, we first construct a function $f(D)$ such that $f(D) \leq O(\boldsymbol{x}(m,D)) - O(\bar{\boldsymbol{x}}(m,D))$. We will show that $f(D)$ is non-decreasing in D and that there exists a finite, positive D^* such that $f(D^*) \geq 0$, which indicates $O(\bar{\boldsymbol{x}}(m,D)) \leq O(\boldsymbol{x}(m,D))$ for $D > D^*$ as well.

We start with the construction of the function $f(D)$. Let $\bar{p}_{i^-}(j,D)$ denote the probability that there are j people in the system just before the arrival of ith batch under duration D and schedule $\bar{\boldsymbol{x}}(m,D)$. Define the function $f(D) \equiv \sum_{i=1}^{T} o(x_i) - \sum_{i=1}^{T}\sum_{j=0}^{m} \bar{p}_{i^-}(j,D) o(j+a)$. Then, $f(D) \leq O(\boldsymbol{x}(m,D)) - O(\bar{\boldsymbol{x}}(m,D))$ is satisfied, since

$$\begin{aligned} O(\boldsymbol{x}(m,D)) - O(\bar{\boldsymbol{x}}(m,D)) &\geq \sum_{i=1}^{T} o(x_i) - O(\bar{\boldsymbol{x}}(m,D)) \\ &= \sum_{i=1}^{T} o(x_i) - \sum_{i=1}^{T}\sum_{j=0}^{m} \bar{p}_{i^-}(j,D) \cdot \left(\sum_{z=1}^{\bar{z}}\sum_{n=1}^{a} \frac{\alpha_z [j+n-z-1]_+}{\mu}\right) \\ &\geq \sum_{i=1}^{T} o(x_i) - \sum_{i=1}^{T}\sum_{j=0}^{m} \bar{p}_{i^-}(j,D) o(j+a) \\ &= f(D). \end{aligned} \tag{EC.34}$$

In Equation (EC.34), the first inequality is from the fact that when x_i people arrive under schedule $\boldsymbol{x}(m,D)$, they may find people in the system from previous batches. The last inequality follows because $\sum_{z=1}^{\bar{z}}\sum_{n=1}^{a} \frac{\alpha_z[j+n-z-1]_+}{\mu} \leq \sum_{z=1}^{\bar{z}}\sum_{n=1}^{j+a} \frac{\alpha_z[n-z-1]_+}{\mu} = o(j+a)$.

Next, we want to show $f(D)$ in non-decreasing in D. Before proving it, we define additional random variables $\bar{P}_i^D$ to represent the number of people in the system just before ith batch's arrival under the E-D schedule and X_i^D to represent Poisson random variable with rate μd

(recall $d \equiv \frac{D}{T-1}$) for $i = 1, \cdots, T$. Note that $\mathrm{P}(\bar{P}_i^D = j) = \bar{p}_{i^-}(j, D)$, for $j = 0, 1, \cdots, m$. The recursive relationship between $\bar{P}_{i+1}^D$ and $\bar{P}_i^D$ is:

$$\bar{P}_1^D = 0,$$
$$\bar{P}_{i+1}^D = [\bar{P}_i^D + a - X_i^D]_+, \quad i = 1, \cdots, T-1.$$

Then, for $D_1 > D_2 > 0$, we have $X_i^{D_2} \preceq X_i^{D_1}$ and thus $\bar{P}_{i+1}^{D_1} \preceq \bar{P}_{i+1}^{D_2}$ inductively ($\preceq$ is the stochastically ordering) for $i = 1, \cdots, T-1$. It follows that

$$\begin{aligned} f(D_1) - f(D_2) &= \sum_{i=1}^{T} \sum_{j=0}^{m} \bar{p}_{i^-}(j, D_2) o(j+a) - \sum_{i=1}^{T} \sum_{j=0}^{m} \bar{p}_{i^-}(j, D_1) o(j+a) \\ &= \sum_{i=1}^{T} \mathrm{E}[o(\bar{P}_i^{D_2} + a)] - \sum_{i=1}^{T} \mathrm{E}[o(\bar{P}_i^{D_1} + a)] \\ &\geq 0 \quad (\text{since } \bar{P}_i^{D_2} \succeq \bar{P}_i^{D_1} \text{ and } o(j+a) \text{ is non-decreasing in } j). \end{aligned}$$

Finally, we would like to show the existence of $D^* \in (0, \infty)$ such that $f(D^*) > 0$. As $D \to \infty$, $\lim_{D\to\infty} \bar{p}_{i^-}(0, D) = 1$ and $\lim_{D\to\infty} \bar{p}_{i^-}(j, D) = 0$ for $j \geq 1$. It follows that, for any feasible schedule $\boldsymbol{x}(m, D) \neq \bar{\boldsymbol{x}}(m, D)$,

$$\begin{aligned} \lim_{D\to\infty} f(D) &= \sum_{i=1}^{T} o(x_i) - \sum_{i=1}^{T} \sum_{j=0}^{m} \lim_{D\to\infty} \bar{p}_{i^-}(j, D) o(j+a) \\ &= \sum_{i=1}^{T} o(x_i) - \sum_{i=1}^{T} o(a) \\ &> 0. \end{aligned}$$

The last inequality from above is from the convexity part of Lemma EC.2 and Jensen's inequality, since $a = \sum_{i=1}^{T} x_i / T$.

Since $f(D)$ is a continuous and non-decreasing function, then we can find $D^* \in (0, \infty)$ such that $f(D^*) \in (0, \lim_{D\to\infty} f(D)]$ and thus $f(D^*) > 0$.

(b) It suffices to show that there exists a function $g(m)$ such that $\frac{O(\boldsymbol{x}(m,D)) - O(\boldsymbol{x}^{o*}(m,D))}{O(\boldsymbol{x}(m,D))} \leq g(m)$ and $\lim_{m\to\infty} g(m) = 0$. If so, then $0 \leq \lim_{m\to\infty} \dfrac{O(\boldsymbol{x}(m,D) - O(\boldsymbol{x}^{o*}(m,D)))}{O(\boldsymbol{x}(m,D))} \leq 0$.

First we want to show $O(\boldsymbol{x}^{o*}(m,D)) \geq o(m - \lceil \mu D \rceil)$, where $\lceil \cdot \rceil$ is the ceiling function, and we will use it to construct the function $g(m)$. Let x_T^* denote the Tth element of the vector $\boldsymbol{x}^{o*}(m,D)$. Let $p_{T^-}^*(j, m)$ denote the probability that j people are in the system just before the arrival of Tth batch under schedule $\boldsymbol{x}^{o*}(m,D)$. Then, we have

$$O(\boldsymbol{x}^{o*}(m,D)) \geq \sum_{j=0}^{m} p_{T^-}^*(j, D) o(j + x_T^*) \tag{EC.35}$$

$$\geq o\left(\left(\sum_{j=0}^{m} p_{T^-}^*(j, D) \cdot j\right) + x_T^*\right) \tag{EC.36}$$

$$\geq o(m - x_T^* - \lceil \mu D \rceil + x_T^*) \tag{EC.37}$$

$$= o(m - \lceil \mu D \rceil). \tag{EC.38}$$

Inequality (EC.35) follows from the fact that if there are j people still in the system just before the arrivals of x_T^* individuals, then due to the memoryless property of the service times' distributions, the weighted sum of overlap time is no less than $o(j + x_T^*)$. Inequality (EC.36) is from Lemma EC.2 and Jensen's inequality. To explain Inequality (EC.37), consider a schedule in which $m - x_T^*$ individuals arrive at the beginning; i.e., the schedule $\boldsymbol{y} = (m - x_T^*, 0, \cdots, 0, x_T^*)$. Then, the expected number of individuals remaining in the system just before the batch of x_T^* arrivals under schedule $\boldsymbol{y}$ is no more than that under schedule $\boldsymbol{x}^{o*}(m, D)$. Moreover, based on the assumption that the service times follow exponential distributions, the expected number just before final batch under schedule $\boldsymbol{y}$ is no less than $m - x_T^* - \lceil \mu D \rceil$. Also, recalling $o(x)$ is non-decreasing in x, we derive Inequality (EC.37).

Next, we define the function $g(m) \equiv \frac{o(m) - o(m - \lceil \mu D \rceil)}{o(m - \lceil \mu D \rceil)}$. Then, $\frac{O(\boldsymbol{x}(m,D)) - O(\boldsymbol{x}^{o*}(m,D))}{O(\boldsymbol{x}(m,D))} \leq g(m)$ is satisfied, since

$$\begin{aligned} \frac{O(\boldsymbol{x}(m,D)) - O(\boldsymbol{x}^{o*}(m,D))}{O(\boldsymbol{x}(m,D))} &\leq \frac{O(\boldsymbol{x}(m,D)) - O(\boldsymbol{x}^{o*}(m,D))}{O(\boldsymbol{x}^{o*}(m,D))} \\ &\leq \frac{o(m) - o(m - \lceil \mu D \rceil)}{o(m - \lceil \mu D \rceil)}. \end{aligned} \tag{EC.39}$$

Inequality (EC.39) holds since schedule $(m, 0, \cdots, 0)$ results in the largest weighted sum expected overlap time.

Finally, we show $\lim_{m \to \infty} g(m) = 0$:

$$\begin{aligned} \lim_{m \to \infty} g(m) &= -1 + \lim_{m \to \infty} \frac{o(m)}{o(m - \lceil \mu D \rceil)} \\ &= -1 + \lim_{m \to \infty} \frac{\sum_{z=1}^{\bar{z}} \alpha_z \sum_{n=1}^{m} \frac{[n-z-1]_+}{\mu}}{\sum_{z=1}^{\bar{z}} \alpha_z \sum_{n=1}^{m - \lceil \mu D \rceil} \frac{[n-z-1]_+}{\mu}} \\ &= -1 + \lim_{m \to \infty} \frac{\sum_{z=1}^{\bar{z}} \alpha_z (m-z)(m-z-1)/2}{\sum_{z=1}^{\bar{z}} \alpha_z (m - \lceil \mu D \rceil - z)(m - \lceil \mu D \rceil - z - 1)/2} \\ &= 0. \end{aligned} \tag{EC.40}$$

Since the numerator and denominator of expression to the right of the limit in Equation (EC.40) are polynomials of form m^2 plus some lower-degree terms, then the limit is 1, and thus the last equality is derived. □

EC.8. Proof of Proposition 3

Proof of Proposition 3. We prove it by contradiction. Suppose schedule $\boldsymbol{x}^* \equiv (\frac{d}{S}, \cdots, \frac{d}{S}, m - \frac{D}{S})$ is not optimal. Let $\boldsymbol{y} = (y_1, \cdots, y_T)$ be a schedule that optimizes *OPT-OVERLAP*. Then, $\boldsymbol{y} \neq \boldsymbol{x}^*$, and one of the following two cases must true:

(i) For all $i \le T-1$, $y_i \le \frac{d}{S}$. Moreover, there must exist at least one index $j \le T-1$ such that $y_j < \frac{d}{S}$ (otherwise, $\boldsymbol{y} = \boldsymbol{x}^*$, a contradiction);

(ii) There must exist at least one index $j \le T-1$ such that $y_j > \frac{d}{S}$;

For Case (i), since $y_i \le \frac{d}{S}$ for all $i \le T-1$, then batch y_i will finish their service within one slot and leave the system before the arrival of batch y_{i+1}. Then, the weighted sum of overlap under schedule $O(\boldsymbol{y}) = \sum_{i=1}^{T} o(y_i)$, where $o(y_i)$ is defined in the proof of Proposition 2. Since $y_j < \frac{d}{S}$, then $y_T > m - \frac{D}{S}$. Construct a new schedule $\boldsymbol{z} = (z_1, \cdots, z_T)$ such that

$$z_i = \begin{cases} \frac{d}{S} & \text{if } i = j, \\ y_T - \left(\frac{d}{S} - y_j\right) & \text{if } i = T, \\ y_i & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} O(\boldsymbol{z}) &= \textstyle\sum_{i=1}^{T} o(z_i) \quad (\text{since } z_i \le \frac{d}{S} \text{ for all } i \le T-1) \\ &= \textstyle\sum_{i=1}^{T} o(z_i) - (o(z_j) + o(z_T)) + (o(z_j) + o(z_T)) \\ &< \textstyle\sum_{i=1}^{T} o(z_i) - (o(z_j) + o(z_T)) + (o(y_j) + o(y_T)) \\ &= \textstyle\sum_{i=1}^{T} o(y_i) - (o(y_j) + o(y_T)) + (o(y_j) + o(y_T)) \\ &= O(\boldsymbol{y}). \end{aligned}$$

where the above inequality stems from the convexity of $o(x)$ function. Consequently, schedule $\boldsymbol{z}$ is better than $\boldsymbol{y}$, leading to a contradiction.

For Case (ii), since $y_j > \frac{d}{S}$, then there exists at least one individual from batch y_j remaining in the system when batch y_{j+1} arrives. If we create a new schedule $\boldsymbol{z}$ in which this individual arrives at the start of slot $j+1$ instead of slot j (i.e., the new schedule is identical as the old, except slot j has one less and slot $j+1$ has one extra individual compared to schedule $\boldsymbol{y}$), then it may reduce the expected overlap time between them and the individual ahead of them within a queue physical distance as $\bar{z}$ but will not influence the expected overlap time between other pairs. Hence, we have $O(\boldsymbol{z}) \le O(\boldsymbol{y})$. If $O(\boldsymbol{z}) < O(\boldsymbol{y})$, then $\boldsymbol{y}$ is not optimal (contradiction); if $O(\boldsymbol{z}) = O(\boldsymbol{y})$ and $\boldsymbol{z} = \boldsymbol{x}^*$, then $\boldsymbol{x}^*$ is also optimal (contradiction); if $O(\boldsymbol{z}) = O(\boldsymbol{y})$ and $\boldsymbol{z} \ne \boldsymbol{x}^*$, then let $\boldsymbol{y} \equiv \boldsymbol{z}$ and we still use the above Case (i) & (ii) to analysis schedule $\boldsymbol{y}$ iteratively until a contradiction occurs. □

EC.9. Proof of Proposition 4

Before presenting the proof of Proposition 4, we introduce a new notation, $o(x_i|j)$. Let $o(x_i|j)$ denote the weighted sum of expected overlap time across a single batch of x_i individuals arriving to a system with j people already in it. Then, $o(x_i|j) = \sum_{z=1}^{\bar{z}} \alpha_z \sum_{n=1}^{x_i} [j+n-z-1]_+/\mu$. The following Lemma EC.3 shows its connection to $o(x_i+j)$ and $o(j)$, and we will use it in the following.

Lemma EC.3.

(a) $o(x_i|j) = o(x_i+j) - o(j)$.

(b) for $2 \le x_1 < x_2$, $o(x_2|j) - o(x_1|j+x_2-x_1)$ *is non-decreasing in* j.

Proof of Lemma EC.3.

(a) It follows from the definition of $o(x_i|j)$:

$$\begin{aligned} o(x_i|j) &= \sum_{z=1}^{\bar z} \alpha_z \sum_{n=1}^{x_i} \frac{[j+n-z-1]_+}{\mu} \\ &= \sum_{z=1}^{\bar z} \alpha_z \sum_{n=j+1}^{j+x_i} \frac{[j+n-z-1]_+}{\mu} \\ &= \sum_{z=1}^{\bar z} \alpha_z \sum_{n=1}^{j+x_i} \frac{[j+n-z-1]_+}{\mu} - \sum_{z=1}^{\bar z} \alpha_z \sum_{n=1}^{j} \frac{[j+n-z-1]_+}{\mu} \\ &= o(j+x_j) - o(j). \end{aligned}$$

(b) For $2 \le x_1 < x_2$, we have

$$\begin{aligned} o(x_2|j) - o(x_1|j+x_2-x_1) &= (o(x_2+j) - o(j)) - (o(x_1+j+x_2-x_1) - o(j+x_2-x_1)) \\ &= o(j+x_2-x_1) - o(j). \end{aligned}$$

From Lemma EC.2(b), we have $o(j+x_2-x_1) - o(j)$ is non-decreasing in j. □

Proof of Proposition 4. Let $o(x_i|j) = \sum_{z=1}^{\bar z} \sum_{n=1}^{x_i} \alpha_z \frac{[j+n-z-1]_+}{\mu}$. In particular, when $j = 0$, $o(x_i|0) = o(x_i)$ ($o(x_i)$ is defined in Lemma EC.2). We have proved properties of $o(x_i)$ in Lemma EC.2 and properties of $o(x_i|j)$ in Lemma EC.3 and will use them later.

(a) For notational convenience, let $\boldsymbol{x} = (x_1, x_2)$ and $\hat{\boldsymbol{x}} = (\hat x_1, \hat x_2) = (x_2, x_1)$. Recalling the expression for $O(\boldsymbol{x})$ we derived as Equation (3), we have

$$\begin{aligned} O(\boldsymbol{x}) &= \sum_{i=1}^{2} \sum_{j=0}^{x_1+x_2} p_{i^-}(j) o(x_i|j), \\ O(\hat{\boldsymbol{x}}) &= \sum_{i=1}^{2} \sum_{j=0}^{x_1+x_2} \hat p_{i^-}(j) o(\hat x_i|j). \end{aligned}$$

where $p_{i^-}(j)$ ($\hat p_{i^-}(j)$) is the probability that when batch x_i ($\hat x_i$) people arrive, they find j people in the system.

Before proving $O(\boldsymbol{x}) - O(\hat{\boldsymbol{x}}) \le 0$, we first derive the relationships between $p_{i^-}(j)$ and $\hat p_{i^-}(j)$, which we will utilize in the subsequent proof.

$$p_{1^-}(0) = \hat p_{1^-}(0) = 1, \tag{EC.41}$$

$$p_{2^-}(j) = \hat p_{2^-}(n) = 0, \quad j > x_1, n > x_2, \tag{EC.42}$$

$$p_{2^-}(j) = \hat p_{2^-}(j + x_2 - x_1), \quad j = 1, 2, \cdots, x_1, \tag{EC.43}$$

$$p_{2^-}(0) = \sum_{j=0}^{x_2-x_1} \hat p_{2^-}(j). \tag{EC.44}$$

The difference between $O(\boldsymbol{x})$ and $O(\hat{\boldsymbol{x}})$ is calculated as:

$$\begin{aligned} O(\boldsymbol{x}) - O(\hat{\boldsymbol{x}}) &= \sum_{i=1}^{2} \sum_{j=0}^{x_1+x_2} p_{i^-}(j) o(x_i|j) - \sum_{i=1}^{2} \sum_{j=0}^{x_1+x_2} \hat p_{i^-}(j) o(\hat x_i|j) \\ &= \left(o(x_1) + \sum_{j=0}^{x_1+x_2} p_{2^-}(j) o(x_2|j) \right) - \left(o(x_2) + \sum_{j=0}^{x_1+x_2} \hat p_{2^-}(j) o(x_1|j) \right) \text{ (by Equation (EC.41))} \\ &= (o(x_1) - o(x_2)) + \left(\sum_{j=0}^{x_1} p_{2^-}(j) o(x_2|j) - \sum_{j=0}^{x_2} \hat p_{2^-}(j) o(x_1|j) \right) \text{ (by Equation (EC.42))} \end{aligned}$$

$$= (o(x_1) - o(x_2)) + \left(p_{2^-}(0)o(x_2) + \sum_{j=1}^{x_1} p_{2^-}(j)o(x_2|j)\right) - \left(\sum_{j=0}^{x_2-x_1} \hat{p}_{2^-}(j)o(x_1|j) + \sum_{j=x_2-x_1+1}^{x_2} \hat{p}_{2^-}(j)o(x_1|j)\right)$$

$$= (o(x_1) - o(x_2)) + \underbrace{\left(p_{2^-}(0)o(x_2) - \sum_{j=0}^{x_2-x_1} \hat{p}_{2^-}(j)o(x_1|j)\right)}_{\text{Define it as } A_1} + \underbrace{\left(\sum_{j=1}^{x_1} p_{2^-}(j)o(x_2|j) - \sum_{j=x_2-x_1+1}^{x_2} \hat{p}_{2^-}(j)o(x_1|j)\right)}_{\text{Define it as } A_2}.$$

Next, we simplify A_1 and A_2:

$$\begin{aligned} A_1 &= p_{2^-}(0)o(x_2) - \sum_{j=0}^{x_2-x_1} \hat{p}_{2^-}(j)o(x_1|j) \\ &\le p_{2^-}(0)o(x_2) - \sum_{j=0}^{x_2-x_1} \hat{p}_{2^-}(j)o(x_1) \\ &= p_{2^-}(0)[o(x_2) - o(x_1)], \text{ (by Equation (EC.44))} \end{aligned}$$

$$\begin{aligned} A_2 &= \sum_{j=1}^{x_1} p_{2^-}(j)o(x_2|j) - \sum_{j=x_2-x_1+1}^{x_2} \hat{p}_{2^-}(j)o(x_1|j) \\ &= \sum_{j=1}^{x_1} p_{2^-}(j)o(x_2|j) - \sum_{j=1}^{x_1} p_{2^-}(j)o(x_1|j + x_2 - x_1) \text{ (by Equation (EC.43))} \\ &= \sum_{j=1}^{x_1} p_{2^-}(j)\left(o(x_2|j) - o(x_1|j + x_2 - x_1)\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} O(\boldsymbol{x}) - O(\hat{\boldsymbol{x}}) &= o(x_1) - o(x_2) + A_1 + A_2 \\ &\le o(x_1) - o(x_2) + p_{2^-}(0)[o(x_2) - o(x_1)] + \sum_{j=1}^{x_1} p_{2^-}(j)\left(o(x_2|j) - o(x_1|j + x_2 - x_1)\right) \\ &= \sum_{j=1}^{x_1} p_{2^-}(j)(o(x_1) - o(x_2) + o(x_2|j) - o(x_1|j + x_2 - x_1)) \\ &\le \sum_{j=1}^{x_1} p_{2^-}(j)(o(x_1) - o(x_2) + o(x_2|x_1) - o(x_1|x_2)) \text{ (by Lemma EC.3(b))} \\ &= 0 \text{ (by Lemma EC.3(a))} \end{aligned}$$

(b) Let $f(x,D)$ denote the value of $O((x,m-x)) - O((x+1,m-x-1))$ under duration D. Without loss of generality, we assume $\mu = 1$. Since $f(x,0) = 0$ and $\lim_{D\to\infty} f(x,D) > 0$ (explained by Equation (EC.45)), then to show the existence of the D_x (the positive root of $f(x,D)$ in D), it suffices to show $f(x,D)$ first decreases and then increases for $D \ge 0$.

$$\lim_{D\to\infty} f(x,D) = (o(x) + o(m-x)) - (o(x+1) + o(m-x-1)) > 0. \quad \text{(EC.45)}$$

where the inequality is from the strictly discrete convex property, which is verified in Lemma EC.2(b).

For notational convenience, let $\boldsymbol{x} = (x, m-x)$ and $\hat{\boldsymbol{x}} = (x+1, m-x-1)$. Similarly as in the proof of Proposition 4(i), we define probabilities $p_{i^-}(j)$ and $\hat{p}_{i^-}(j)$, and derive their relationships:

$$p_{2^-}(j) = \hat{p}_{2^-}(j+1) = \frac{e^{-D}D^{x-j}}{(x-j)!}, \quad j = 1, 2, \cdots, x, \quad \text{(EC.46)}$$

$$p_{2^-}(0) = \hat{p}_{2^-}(0) + \hat{p}_{2^-}(1) = 1 - \sum_{j=1}^{x} p_{2^-}(j). \tag{EC.47}$$

Then, $f(x,D)$ is calculated as:

$$\begin{aligned}
f(x,D) &= o(x) + \sum_{j=0}^{x} p_{2^-}(j)o(m-x|j) - o(x+1) - \sum_{j=0}^{x+1} \hat{p}_{2^-}(j)o(m-x-1|j) \\
&= o(x) - o(x+1) + \left(p_{2^-}(0)o(m-x) - \sum_{j=0}^{1} \hat{p}_{2^-}(j)o(m-x-1|j)\right) + \left(\sum_{j=1}^{x} p_{2^-}(j)o(m-x|j) - \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(m-x-1|j)\right) \\
&= -o(1|x) + \hat{p}_{2^-}(0)(o(m-x) - o(m-x-1)) + \left(\sum_{j=1}^{x} p_{2^-}(j)o(m-x|j) - \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(m-x-1|j)\right) \\
&\quad \text{(by Equation (EC.47) and Lemma EC.3(a), which implies } o(m-x) = o(m-x-1|1)) \\
&= -o(1|x) + \hat{p}_{2^-}(0)o(1|m-x-1) + \left(\sum_{j=1}^{x} p_{2^-}(j)o(m-x|j) - \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(m-x-1|j)\right) \\
&= -o(1|x) + \hat{p}_{2^-}(0)o(1|m-x-1) + \left(\sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(m-x|j-1) - \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(m-x-1|j)\right) \quad \text{(by Equation (EC.46))} \\
&= -o(1|x) + \hat{p}_{2^-}(0)o(1|m-x-1) + \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)(o(m-x|j-1) - o(m-x-1|j)) \\
&= -o(1|x) + \left(1 - \sum_{j=1}^{x+1} \hat{p}_{2^-}(j)\right)o(1|m-x-1) + \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)o(1|j-1) \\
&= o(1|m-x-1) - o(1|x) - \hat{p}_{2^-}(1)o(1|m-x-1) - \sum_{j=2}^{x+1} \hat{p}_{2^-}(j)(o(1|m-x-1) + o(1|j-1)) \\
&= o(1|m-x-1) - o(1|x) - \frac{e^{-D}D^x}{x!}o(1|m-x-1) - \sum_{j=2}^{x+1} \frac{e^{-D}D^{x+1-j}}{(x+1-j)!}(o(1|m-x-1) + o(1|j-1)). \quad \text{(by Equation (EC.46))}
\end{aligned}$$

Next, we calculate the first order derivative of $f(x,D)$ with respect to D, that is

$$\begin{aligned}
\frac{\partial f(x,D)}{\partial D} &= \frac{e^{-D}D^x - e^{-D}xD^{x-1}}{x!}o(1|m-x-1) + \sum_{j=2}^{x+1} \frac{e^{-D}D^{x+1-j} - (x+1-j)e^{-D}D^{x-j}}{(x+1-j)!}(o(1|m-x-1) + o(1|j-1)) \\
&= e^{-D}\left(\frac{o(1|m-x-1)}{x!}D^x - \sum_{n=0}^{x-2} \frac{o(1|x-n) - o(1|x-n-1)}{n!}D^n\right).
\end{aligned} \tag{EC.48}$$

If we view the term in Equation (EC.48), $\frac{o(1|m-x-1)}{x!}D^x - \sum_{n=0}^{x-2} \frac{o(1|x-n)-o(1|x-n-1)}{n!}D^n$, as a polynomial of D, defined as $g(D)$, and rewrite it in descending order of the variable exponents, we have

$$\begin{aligned}
g(D) &= \frac{o(1|m-x-1)}{x!}D^x - \sum_{n=0}^{x-2} \frac{o(1|x-n) - o(1|x-n-1)}{n!}D^n \\
&= \frac{o(1|m-x-1)}{x!}D^x - \frac{o(1|2) - o(1|1)}{(x-2)!}D^{x-2} - \frac{o(1|3) - o(1|2)}{(x-3)!}D^{x-3} - \cdots - (o(1|x-1) - o(1|x-2))D + (o(1|x-1) - o(1|x)).
\end{aligned}$$

Since there is one sign change between the first and second terms, by Descartes' rule of signs, $g(D)$ has exactly one positive root. Additionally, since $g(0) = o(1|x-1) - o(1|x) < 0$ and $\lim_{D\to\infty} g(D) > 0$, then $g(D)$ is a single-crossing function from negative to positive within $D \in [0, +\infty)$. Then, $\partial f(x,D)/\partial D$ is a single-crossing function from negative to positive as well. Therefore, $f(x,D)$ first decreases and then increases with respect to D, which indicates the existence of the single root D_x.

For all $D \le D_x$, we have

$$f(x,D) = O((x, m-x)) - O((x+1, m-x-1)) \le 0.$$

Since $O(\boldsymbol{x})$ is multimodular in $\boldsymbol{x}$, we have

$$O((x+1,m-x-1))-O((x+2,m-x-2))\leq O((x,m-x))-O((x+1,m-x-1))\leq 0,$$

which implies $f(x+1,D)\leq f(x,D)\leq 0$ for $D\leq D_x$. Hence, $D_{x+1}\geq D_x$.

Furthermore, for any $D\in[D_{x-1},D_x)$, we have $f(n,D)\geq 0$ for $2\leq n\leq x$ and $f(n,D)\leq 0$ for $x+1\leq n\leq \lfloor m/2-1\rfloor$. This implies that $O((n,m-x)\geq O((n+1,m-x-1))$ for $2\leq n\leq x-1$ and $O((n,m-n))\leq O((n+1,m-n-1))$ for $x\leq n\leq\lfloor m/2-1\rfloor$, indicating that schedule $(x,m-x)$ minimizes the $O(\boldsymbol{x})$ function. □

EC.10. Proof of Proposition 5

Proof of Proposition 5 We skip the proof of (b) and (c), since (b) has been proved in Kaandorp and Koole (2007), and it is easy to verify (c).

(a) Define vectors $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_{T+1}\in\mathbb{Z}^T$ as in Equation (2) and $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{T+1}\in\mathbb{Z}^{T+1}$ as Equation (EC.30). Recall Equation (EC.31) from the proof of Proposition 1, we have proved that given any vector $\boldsymbol{y}\in\mathbb{Z}^{T+1}_{\geq 0}$, if $\{\boldsymbol{y}+\boldsymbol{u}_i,\boldsymbol{y}+\boldsymbol{u}_j,\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_j\}\subset\mathbb{Z}^{T+1}_{\geq 0}$, then

$$O(\boldsymbol{y}+\boldsymbol{u}_i)+O(\boldsymbol{y}+\boldsymbol{u}_j)\geq O(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_j)+O(\boldsymbol{y}). \tag{EC.49}$$

Notably, we have

$$\|\boldsymbol{y}\|_1=\|\boldsymbol{y}+\boldsymbol{u}_i\|_1=\|\boldsymbol{y}+\boldsymbol{u}_j\|_1=\|\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_j\|_1. \tag{EC.50}$$

Additionally, for any schedule $\boldsymbol{\tau}\in\{\boldsymbol{y},\boldsymbol{y}+\boldsymbol{u}_i,\boldsymbol{y}+\boldsymbol{u}_j,\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_j\}$, assign indexes $n=1,\cdots,\|\boldsymbol{\tau}\|_1$ to each individual based on their order of arrival under the corresponding continuous-time arrival schedule. Let $W_n(\boldsymbol{\tau})$ denote individual n's waiting time, and let $O_n^{n-z}(\boldsymbol{\tau})$ denote the overlap time between individual $n-z$ and n. Then, we have $O(\boldsymbol{\tau})=\sum_{z=1}^{\bar{z}}\sum_{n=z+1}^{\|\boldsymbol{\tau}\|_1}\alpha_z\mathrm{E}[O_n^{n-z}(\boldsymbol{\tau})]$. We will use these notations and Equation (EC.49) & (EC.50) repeatedly later on.

To prove (a), we apply Definition 1 to the following cases:

(a-1) If $2\leq i<j\leq T$, define $\boldsymbol{z}\equiv(\boldsymbol{x},0)$. Then, $\boldsymbol{z}+\boldsymbol{u}_i=(\boldsymbol{x}+\boldsymbol{v}_i,0)$, $\boldsymbol{z}+\boldsymbol{u}_j=(\boldsymbol{x}+\boldsymbol{v}_i,0)$, and $\boldsymbol{z}+\boldsymbol{u}_i+\boldsymbol{u}_j=(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j,0)$. It follows that

$$\begin{aligned}O(\boldsymbol{x}+\boldsymbol{v}_i)+O(\boldsymbol{x}+\boldsymbol{v}_j)&=O(\boldsymbol{z}+\boldsymbol{u}_i)+O(\boldsymbol{z}+\boldsymbol{u}_j)\\&\geq O(\boldsymbol{z}+\boldsymbol{u}_i+\boldsymbol{u}_j)+O(\boldsymbol{z})\quad\text{(by Equation (EC.49))}\\&=O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j)+O(\boldsymbol{x}).\end{aligned}$$

(a-2) If $2\leq i<j=(T+1)$, define $\boldsymbol{y}\equiv(\boldsymbol{x},1)$ (we also use it in other cases). Note that compared to schedule $\boldsymbol{x}$, schedule $\boldsymbol{y}$ has one more arrival at $(T+1)$th time period. From Case (i) in the proof of Proposition 1 and Lemma EC.1, for $z=1,\cdots,\bar{z}$, we have

$$\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i)+O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})]\geq\mathrm{E}[O_n^{n-z}(\boldsymbol{y})+O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})],\quad n=z+1,\cdots,\|\boldsymbol{y}\|_1-1. \tag{EC.51}$$

Additionally, the difference between schedule $\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1}$ and schedule $\boldsymbol{y}+\boldsymbol{u}_{T+1}$ is that the former one shifts one individual at ith time period to an earlier time period. This individual cannot be individual $\|\boldsymbol{y}\|_1$. Our next goal is to show $\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \geq \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]$ for $z=1,\cdots,\bar{z}$. Let n denote the index of the shifted individual. We consider the following subcases:

(a-2.1) If $n<\|\boldsymbol{y}\|_1-z$, then $W_{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1}) \succeq W_{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})$. But the arrival times of individual $\|\boldsymbol{y}\|_1-z$ (individual $\|\boldsymbol{y}\|_1$) are same under these two schedules. Therefore, by Equation (EC.24), we have

$$\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \geq \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]. \tag{EC.52}$$

(a-2.2) If $n=\|\boldsymbol{y}\|_1-z$, then $W_{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})=[W_{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})-d]_+$, the arrival times of individual $\|\boldsymbol{y}\|_1$ are same, but the arrival time of individual $\|\boldsymbol{y}\|_1-z$ under schedule $\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1}$ is shifted d time units earlier. Again, by Equation (EC.24), we have

$$\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] = \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]. \tag{EC.53}$$

(a-2.3) If $\|\boldsymbol{y}\|_1-z<n<\|\boldsymbol{y}\|_1$ (since the shifted individual cannot be individual $\|\boldsymbol{y}\|_1$, then $n<\|\boldsymbol{y}\|_1$), the waiting times of individual $\|\boldsymbol{y}\|_1-z$ and the arrival times of individual $\|\boldsymbol{y}\|_1-z$ (individual $\|\boldsymbol{y}\|_1$) are same under these two schedules, which implies

$$\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] = \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]. \tag{EC.54}$$

Since $\boldsymbol{y}=(\boldsymbol{x},1)$, $\boldsymbol{y}+\boldsymbol{u}_i=(\boldsymbol{x}+\boldsymbol{v}_i,1)$, $\boldsymbol{y}+\boldsymbol{u}_{T+1}=(\boldsymbol{x}+\boldsymbol{v}_{T+1},0)$ and $\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1}=(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1},0)$, then by Equation (EC.50), we have

$$\begin{aligned}
O(\boldsymbol{x}+\boldsymbol{v}_i) &= \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}+\boldsymbol{u}_i\|_1-1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i)] = \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i)],\\
O(\boldsymbol{x}+\boldsymbol{v}_{T+1}) &= \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}+\boldsymbol{u}_{T+1}\|_1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] = \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}\|_1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})],\\
O(\boldsymbol{x}) &= \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y})],\\
O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1}) &= \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1}\|_1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})] = \sum_{z=1}^{\bar{z}}\alpha_z \sum_{n=z+1}^{\|\boldsymbol{y}\|_1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})].
\end{aligned}$$

Therefore, by Equation (EC.51)- (EC.54), it follows that

$$\begin{aligned}
O(\boldsymbol{x}+\boldsymbol{v}_i)+O(\boldsymbol{x}+\boldsymbol{v}_{T+1}) &= \sum_{z=1}^{\bar{z}}\alpha_z\left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i)+O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})]\right)+\sum_{z=1}^{\bar{z}}\alpha_z\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})]\\
&\geq \sum_{z=1}^{\bar{z}}\alpha_z\left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1}\mathrm{E}[O_n^{n-z}(\boldsymbol{y})+O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]\right)+\sum_{z=1}^{\bar{z}}\alpha_z\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_i+\boldsymbol{u}_{T+1})]\\
&= O(\boldsymbol{x})+O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1}).
\end{aligned}$$

(a-3) If $1=i<j\leq(T+1)$, as shown in Case (ii) of the proof of Proposition 1, for $z=1,\cdots,\bar{z}$, we have

$$\begin{aligned}
&\mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_j)-O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1)] \leq \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_j)-O_n^{n-z}(\boldsymbol{y})], \quad n=z+1,\cdots,\|\boldsymbol{y}\|_1\\
\Longrightarrow\ &\mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_j)+O_n^{n-z}(\boldsymbol{y})] \leq \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_j)+O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1)], \quad n=z+1,\cdots,\|\boldsymbol{y}\|_1.
\end{aligned} \tag{EC.55}$$

(a-3.1) If $j \le T$, then $\boldsymbol{y}+\boldsymbol{u}_1=(\boldsymbol{x}+\boldsymbol{v}_1,2)$, $\boldsymbol{y}+\boldsymbol{u}_j=(\boldsymbol{x}+\boldsymbol{v}_j,1)$, $\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_j=(\boldsymbol{x}+\boldsymbol{v}_1+\boldsymbol{v}_j,2)$. Again, by Equation (EC.50), we have

$$\begin{aligned}
O(\boldsymbol{x}+\boldsymbol{v}_1)+O(\boldsymbol{x}+\boldsymbol{v}_j) &= \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-2} \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_1)] + \sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_j)] \right) \\
&= \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1) + O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_j)] \right) \quad (\text{since } O_z^0(\boldsymbol{y}+\boldsymbol{u}_1)\equiv 0) \\
&\ge \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_j) + O_n^{n-z}(\boldsymbol{y})] \right) \quad (\text{by Equation (EC.55)}) \\
&= O(\boldsymbol{x}+\boldsymbol{v}_1+\boldsymbol{v}_j)+O(\boldsymbol{x}). \quad (\text{since } O_z^0(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_j)\equiv 0)
\end{aligned}$$

(a-3.2) If $j=T+1$, then $\boldsymbol{y}+\boldsymbol{u}_1=(\boldsymbol{x}+\boldsymbol{v}_1,2)$, $\boldsymbol{y}+\boldsymbol{u}_{T+1}=(\boldsymbol{x}+\boldsymbol{v}_{T+1},0)$, $\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1}=(\boldsymbol{x}+\boldsymbol{v}_1+\boldsymbol{v}_{T+1},1)$, and thus

$$\begin{aligned}
O(\boldsymbol{x}+\boldsymbol{v}_1)+O(\boldsymbol{x}+\boldsymbol{v}_{T+1}) &= \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-2} \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_1)] + \sum_{n=z+1}^{\|\boldsymbol{y}\|_1} \mathrm{E}[O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \right) \\
&= \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1) + O_n^{n-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \right) + \sum_{z=1}^{\bar{z}} \alpha_z \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \\
&\quad (\text{since } O_z^0(\boldsymbol{y}+\boldsymbol{u}_1)\equiv 0) \\
&\ge \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1}) + O_n^{n-z}(\boldsymbol{y})] \right) + \sum_{z=1}^{\bar{z}} \alpha_z \mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] \\
&\quad (\text{by Equation (EC.55)}) \\
&= \sum_{z=1}^{\bar{z}} \alpha_z \left(\sum_{n=z+1}^{\|\boldsymbol{y}\|_1-1} \mathrm{E}[O_{n-1}^{n-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1}) + O_n^{n-z}(\boldsymbol{y})] \right) + \sum_{z=1}^{\bar{z}} \alpha_z \mathrm{E}[O_{\|\boldsymbol{y}\|_1-1}^{\|\boldsymbol{y}\|_1-1-z}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1})] \qquad \text{(EC.56)} \\
&= O(\boldsymbol{x})+O(\boldsymbol{x}+\boldsymbol{v}_1+\boldsymbol{v}_{T+1}). \quad (\text{since } O_z^0(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1})\equiv 0)
\end{aligned}$$

To explain Equation (EC.56), we compare schedules $\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1}$ and $\boldsymbol{y}+\boldsymbol{u}_{T+1}$. Adding $\boldsymbol{u}_1$ means shifting individual 1 to the end. Consequently, the individual $\|\boldsymbol{y}\|_1-z$ ($\|\boldsymbol{y}\|_1$) under schedule $\boldsymbol{y}+\boldsymbol{u}_{T+1}$ becomes individual $\|\boldsymbol{y}\|_1-z-1$ ($\|\boldsymbol{y}\|_1-1$) under schedule $\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1}$. This leads to the relationship $W_{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1}) = W_{\|\boldsymbol{y}\|_1-z-1}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1})$, which implies $\mathrm{E}[O_{\|\boldsymbol{y}\|_1}^{\|\boldsymbol{y}\|_1-z}(\boldsymbol{y}+\boldsymbol{u}_{T+1})] = \mathrm{E}[O_{\|\boldsymbol{y}\|_1-1}^{\|\boldsymbol{y}\|_1-z-1}(\boldsymbol{y}+\boldsymbol{u}_1+\boldsymbol{u}_{T+1})]$, resulting in the equivalence in weighted sum of overlap times as shown in Equation (EC.56).

□

EC.11. Model accounting for no-show behavior

In this section, we first derive the recursive formulas of $O(\boldsymbol{x},\theta)$, $L(\boldsymbol{x},\theta)$, and then state the proof of Proposition 6.

$O(\boldsymbol{x},\theta)$ has a recursive formula as

$$O(\boldsymbol{x},\theta) = \sum_{i=1}^{T}\sum_{j=0}^{m}\sum_{k=1}^{x_i}\sum_{z=1}^{\bar{z}}\sum_{n=1}^{k} p_{i^-}(j,\theta)\cdot\left(\alpha_z \frac{[j+n-z-1]_+}{k\mu}\right)\cdot\binom{x_i}{k}\theta^{x_i-k}(1-\theta)^k, \qquad \text{(EC.57)}$$

where $p_{i^-}(j,\theta)$ correspond to $p_{i^-}(j)$, as defined in Equations (EC.2) and (EC.3), in the setting considering no-shows. They can be generated recursively as follows:

$$p_{1^-}(0,\theta) = p_{1^-}(0) \equiv 1,$$

$$p_{i^+}(j,\theta) = \sum_{k=0}^{x_i} \binom{x_i}{k} \theta^{x_i-k}(1-\theta)^k \cdot p_{i^-}(j-k,\theta),\ j \geq 0,$$
$$p_{i^-}(0,\theta) = \sum_{n=0}^{\|\boldsymbol{x}\|_1} p_{(i-1)^+}(n,\theta) b_n,$$
$$p_{i^-}(j,\theta) = \sum_{n=j}^{\|\boldsymbol{x}\|_1} p_{(i-1)^+}(n,\theta) a_{n-j},\ j > 0.$$

Similarly, the recursive formulas of $L(\boldsymbol{x},\theta)$ is $L(\boldsymbol{x},\theta) = \sum_{j=0}^{m} p_{(T+1)^-}(j,\theta) \cdot \frac{j}{k\mu}$.

EC.11.1. Proof of Proposition 6

Proof of Proposition 5 Define vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{T+1} \in \mathbb{Z}^T$ as in Equation (2) in our paper.

(a) By the definition of multimodularity, it suffices to show that for any $1 \leq i < j \leq T+1$,

$$O(\boldsymbol{x}+\boldsymbol{v}_i,\theta) + O(\boldsymbol{x}+\boldsymbol{v}_j,\theta) \geq O(\boldsymbol{x},\theta) + O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j,\theta).$$

Give each individual a unique index k, $1 \leq k \leq \|\boldsymbol{x}\|_1$, according to the order of their arrival under schedule $\boldsymbol{x}$. Consider any arbitrary sample realization of their attendance $\boldsymbol{A} = (A_1, \cdots, A_{\|\boldsymbol{x}\|_1})$, where $A_k = 0$ (or $A_k = 1$) signifies individual k as a no-show (or show-up). Let $\boldsymbol{x}|\boldsymbol{A}$ represent the actual arrivals in realization $\boldsymbol{A}$; the ith element is $\sum_{i=1}^{x_1+\cdots+x_i} \boldsymbol{A}_i - \sum_{i=1}^{x_1+\cdots+x_{i-1}} \boldsymbol{A}_i$. Assume that if $\boldsymbol{x}+\boldsymbol{v}_i$ is feasible, then adding $\boldsymbol{v}_i$ to $\boldsymbol{x}$ corresponds to shifting individual with index l ($l = x_1 + \cdots + x_{i-1} + 1$) to arrive one time period earlier. Specifically, $\boldsymbol{x}+\boldsymbol{v}_1$ means the removal of individual 1, while $\boldsymbol{x}+\boldsymbol{v}_{T+1}$ introduces an additional individual $\|\boldsymbol{x}\|_1 + 1$ in the last time period. For $i \leq T$, if individual l no-shows in realization $\boldsymbol{A}$, $O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) = O(\boldsymbol{x}|\boldsymbol{A})$; otherwise, $O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) = O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i)$. For $i = T+1$, $O((\boldsymbol{x}+\boldsymbol{v}_{T+1})|\boldsymbol{A}) = \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1})$. Following the rule of conditional probability,

$$O(\boldsymbol{x},\theta) = \sum_{\boldsymbol{A}} \mathrm{P}(\boldsymbol{A}) O(\boldsymbol{x}|\boldsymbol{A}) = \sum_{\boldsymbol{A}} (1-\theta)^{\|\boldsymbol{A}\|_1} \theta^{\|\boldsymbol{x}\|_1 - \|\boldsymbol{A}\|_1} O(\boldsymbol{x}|\boldsymbol{A}). \quad \text{(EC.58)}$$

Since the no-show probability is the same for each individual, then

$$O(\boldsymbol{x}+\boldsymbol{v}_i,\theta) = \sum_{\boldsymbol{A}} P(\boldsymbol{A}) O(\boldsymbol{x}|\boldsymbol{A}), \quad \text{for } i \leq T,$$

and

$$O(\boldsymbol{x}+\boldsymbol{v}_{T+1},\theta) = \sum_{\boldsymbol{A}} P(\boldsymbol{A})(\theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1})).$$

Next, for $1 \leq i < j \leq T$, suppose adding vector $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ leads to the shifts in the arrivals of individuals k_i and k_j. For any realization $\boldsymbol{A}$, one of the following three cases will occur:

(i) If both individual k_i and k_j present, then

$$\begin{aligned} O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + O((\boldsymbol{x}+\boldsymbol{v}_j)|\boldsymbol{A}) &= O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_j) \\ &\geq O(\boldsymbol{x}|\boldsymbol{A}) + O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i+\boldsymbol{v}_j) \\ &= O(\boldsymbol{x}|\boldsymbol{A}) + O((\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j)|\boldsymbol{A}). \end{aligned}$$

(ii) If neither individual k_i or k_j presents, then

$$\begin{aligned} O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + O((\boldsymbol{x}+\boldsymbol{v}_j)|\boldsymbol{A}) &= O(\boldsymbol{x}|\boldsymbol{A}) + O(\boldsymbol{x}|\boldsymbol{A}) \\ &= O(\boldsymbol{x}|\boldsymbol{A}) + O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j|\boldsymbol{A}). \end{aligned}$$

(iii) If one of individual k_i presents and individual k_j presents, then

$$O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + O((\boldsymbol{x}+\boldsymbol{v}_j)|\boldsymbol{A}) = O((\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j)|\boldsymbol{A}) + O(\boldsymbol{x}|\boldsymbol{A}).$$

Therefore,

$$\begin{aligned} O(\boldsymbol{x}+\boldsymbol{v}_i,\theta) + O(\boldsymbol{x}+\boldsymbol{v}_j,\theta) &= \sum_{\boldsymbol{A}} \mathrm{P}(\boldsymbol{A}) O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + \sum_{\boldsymbol{A}} \mathrm{P}(\boldsymbol{A}) O((\boldsymbol{x}+\boldsymbol{v}_j)|\boldsymbol{A}) \\ &\geq \sum_{\boldsymbol{A}} \mathrm{P}(\boldsymbol{A}) \left(O(\boldsymbol{x}|\boldsymbol{A}) + O((\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j)|\boldsymbol{A}) \right) \\ &\geq O(\boldsymbol{x},\theta) + O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_j,\theta). \end{aligned}$$

For $1 \leq i < j = T+1$ and any realization $\boldsymbol{A}$, one of the following two cases will occur:

(i) If individual k_i presents, then

$$\begin{aligned} & O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1}) \\ =\ & O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1}) \\ =\ & \theta O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1}) \\ \geq\ & \theta O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta)(O(\boldsymbol{x}|\boldsymbol{A}) + O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1})) \\ =\ & O(\boldsymbol{x}|\boldsymbol{A}) + \theta O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1}) \\ =\ & O(\boldsymbol{x}|\boldsymbol{A}) + \theta O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + (1-\theta) O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}+\boldsymbol{v}_{T+1}), \end{aligned}$$

where the inequality is from the multimodularity of $O(\cdot)$ function.

(ii) If individual k_i does not show up, then

$$\begin{aligned} & O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1}) \\ =\ & O(\boldsymbol{x}|\boldsymbol{A}) + \theta O(\boldsymbol{x}|\boldsymbol{A}) + (1-\theta) O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1}) \\ =\ & O(\boldsymbol{x}|\boldsymbol{A}) + \theta O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}) + (1-\theta) O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}+\boldsymbol{v}_{T+1}). \end{aligned}$$

Therefore,

$$
\begin{aligned}
& O(\boldsymbol{x}+\boldsymbol{v}_i,\theta)+O(\boldsymbol{x}+\boldsymbol{v}_{T+1},\theta) \\
=& \sum_{\boldsymbol{A}} P(\boldsymbol{A})O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A})+\sum_{\boldsymbol{A}} P(\boldsymbol{A})(\theta O(\boldsymbol{x}|\boldsymbol{A})+(1-\theta)O(\boldsymbol{x}|\boldsymbol{A}+\boldsymbol{v}_{T+1})) \\
\geq & \sum_{\boldsymbol{A}} P(\boldsymbol{A})(O(\boldsymbol{x}|\boldsymbol{A})+\theta O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A})+(1-\theta)O((\boldsymbol{x}+\boldsymbol{v}_i)|\boldsymbol{A}+\boldsymbol{v}_{T+1})) \\
=& O(\boldsymbol{x},\theta)+O(\boldsymbol{x}+\boldsymbol{v}_i+\boldsymbol{v}_{T+1},\theta).
\end{aligned}
$$

(b) The proof of showing the multimodularity of $L(\boldsymbol{x},\theta)$ is similar to that of $O(\boldsymbol{x},\theta)$. □