Online Supplement for "Last-iterate Convergence in No-regret Learning: Games with Reference Effects Under Logit Demand"

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The supplemental materials are structured as follows. In Appendix A, we present a detailed discussion on the partial information structure, specifically focusing on the feasibility of accessing the first-order oracle. In Appendix B, we elaborate on the reasons why the alternative methods mentioned in Section 5 do not apply to our problem, and some supporting proofs are deferred to Appendix M. In Appendix C, we provide the proofs for all propositions. Then, Appendices D, E, and F correspond to the proofs for Theorems 1, 2, and 3, respectively, all of which pertain to the `OPGA` algorithm in the loss-neutral scenario. For the loss-averse scenario, Appendices G and H are dedicated to the proofs for Theorems 4 and 5, respectively. Next, Appendices I and J show the convergence of our algorithms with firm-differentiated step-sizes and inexact gradients, corresponding to Theorems 6 and 7, respectively. In Appendix K, we provide numerical experiments to illustrate the performance of `OPGA` in gain-seeking scenarios. In Appendix L, we present all supporting lemmas used in proofs. Finally, in Appendix N, we provide a summary for all constants used in the paper.

Appendix A Discussion on Partial Information Structure

In Section 3.3, we have introduced the partial information setting considered in this paper, where each firm i can access a first-order oracle but does not necessarily know the information about its competitors. The oracle for firm i outputs the derivative of its log-transformed revenue function, i.e., $\partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)/\partial p_i = 1/p_i + (b_i + c_i)\,[d_i(\mathbf{p},\mathbf{r}) - 1]$ in the loss-neutral scenario. Below, we provide further discussions and discuss the feasibility of obtaining the first-order information.

Since each firm i naturally knows its realized revenue $\Pi_i(\mathbf{p}^t,\mathbf{r}^t)$ after period t, firm i can directly deduce its market share through $d_i(\mathbf{p}^t,\mathbf{r}^t) = \Pi_i(\mathbf{p}^t,\mathbf{r}^t)/p_i^t$. Therefore, if firm i precisely knows $b_i + c_i$ as prior knowledge, it can compute the derivative $\partial \log\big(\Pi_i(\mathbf{p}^t,\mathbf{r}^t)\big)/\partial p_i$ solely based on the market feedback, making our partial information setting equivalent to the *bandit feedback*. Even when the sensitivity parameters are not known prior, it is feasible for firm i to estimate $b_i + c_i$ through temporary collaboration with other firms. Below, we describe the estimation scheme.

First, it is legitimate to assume that each firm has insights into its reference price, inferred from the historical pricing data. Then, let firms engage in the following two-period cooperation:

Period 1: Let every firm set the price equal to its current reference price, i.e., $p_k = r_k,\ \forall k \in N$.

Period 2: Let firm i slightly perturb its price to $p_i = r_i + \delta$ with small δ, whereas other firms retain the previous prices, i.e., $p_j = r_j,\ \forall j \in N \backslash \{i\}$.

We remark that during the cooperation, firms do not need to disclose their prices to the competitors if they prefer to preserve confidentiality. Below, we provide a detailed explanation of why this two-step approach is sufficient for parameter estimation. First, the left-hand and right-hand derivatives of the market share with respect to price change can be computed as follows

$$\lim_{\delta \to 0} \frac{d_i\big((p_i+\delta, \mathbf{p}_{-i}), \mathbf{r}\big) - d_i(\mathbf{p}, \mathbf{r})}{\delta} = \begin{cases} -(b_i + c_i^-) \cdot d_i(\mathbf{p}, \mathbf{r})\big(1 - d_i(\mathbf{p}, \mathbf{r})\big) & \text{if } \delta \to 0_+, \\ -(b_i + c_i^+) \cdot d_i(\mathbf{p}, \mathbf{r})\big(1 - d_i(\mathbf{p}, \mathbf{r})\big) & \text{if } \delta \to 0_-. \end{cases} \tag{A.1}$$

Due to the structural property of the MNL model, we observe that the derivatives in Eq. (A.1) only involve firm i's own information. In our partial information setting, the market share $d_i(\mathbf{p}, \mathbf{r})$ is accessible to firm i, as it can be derived from the realized sales volumes. Thus, firm i can approximate its sensitivity parameters using its market shares from periods 1 and 2, i.e., $d_i\big(\mathbf{r}, \mathbf{r}\big)$ and $d_i\big((r_i + \delta, \mathbf{r}_{-i}), \mathbf{r}\big)$, such that

$$\frac{d_i(\mathbf{r}, \mathbf{r}) - d_i\big((r_i+\delta, \mathbf{r}_{-i}), \mathbf{r}\big)}{\delta \cdot d_i(\mathbf{r}, \mathbf{r})\big(1 - d_i(\mathbf{r}, \mathbf{r})\big)} \approx \begin{cases} b_i + c_i^- & \text{if } \delta > 0, \\ b_i + c_i^+ & \text{if } \delta < 0. \end{cases} \tag{A.2}$$

Since the first-order oracle used in Algorithms 1 and 2 only relies on the sensitivity parameters through $b_i + c_i^-$ and $b_i + c_i^+$, knowing these two sums through Eq. (A.2) is sufficient for the firms.

In this two-period cooperation, firms are not required to disclose their proprietary information to competitors, thereby ensuring confidentiality and competitive advantage are upheld. Moreover, it would be considered reasonable for all firms to set their prices equal to the current reference prices, as these prices mirror consumers' price expectations, indicating a practical pricing strategy. Furthermore, even when the reference price is unknown, firms can still employ this procedure, albeit with extended collaboration periods necessary to gather enough data for the reference price. However, when all firms need to estimate their sensitivity parameters, it is also worth mentioning that a total number of $\mathcal{O}(n)$ periods are required for executing the above cooperation mechanism. If the number of firms involved is large, these additional periods are not negligible in the final complexity bound.

Appendix B Discussion on Alternative Methods

As briefly mentioned in Section 3, our problem can also be translated into a standard $2n$-player online game or a dynamical system. In this appendix, we will discuss these alternative methods and explain why existing tools from the literature on online games and nonlinear dynamical systems cannot be applied.

B.1 Standard $2n$-player Online Game Formulation

The oligopoly competition in our study involves a varying underlying state, i.e., reference price, which depends on firms' price decisions and changes every period. To convert this problem to a standard game without the varying state, we can view the reference price $\mathbf{r} = (r_i)_{i\in N}$ as the decision variables of n additional virtual players with carefully designed objective functions. In each period, these virtual players update its decision variable, i.e., the reference price, using gradient ascent with fixed step-sizes. Specifically, for each virtual player $i \in N$, its objective function $R_i(p_i, r_i)$ and step-sizes $\{\eta_r^t\}_{t\geq 0}$ are defined as follows

$$
\begin{aligned}
R_i(p_i, r_i) &= -\frac{1}{2}r_i^2 + r_i p_i, \quad \forall i \in N;\\
\eta_r^t \equiv \eta_r &:= 1-\alpha, \quad \forall t \geq 0,\ \forall i \in N.
\end{aligned}
\tag{B.1}
$$

To summarize, in this standard $2n$-player game, each real firm $i \in N$ has its log-revenue $\log\big(\Pi_i(\mathbf{p}, \mathbf{r})\big)$ as the objective function and updates its variable p_i via projected gradient ascent with step-size $\{\eta^t\}_{t\geq 0}$; each virtual firm $i \in N$ has the objective function $R_i(p_i, r_i)$ and updates its variable r_i using the standard gradient ascent with fixed step-size η_r. Below, we detail the update rule for this $2n$-player game in Algorithm 3, which essentially generates the same sequence $\{(\mathbf{p}^t, \mathbf{r}^t)\}_{t\geq 0}$ as Algorithm 1.

Algorithm 3: Standard $2n$-player Game with No Varying State

Input: Initial reference price $\mathbf{r}^0$, initial price $\mathbf{p}^0$, and step-sizes $\{\eta^t\}_{t\geq 0}$ and $\eta_r = 1-\alpha$.

for $t = 0, 1, 2, \dots$ **do**

  for $i \in N$ **do**

    Update posted price: $p_i^{t+1} = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t \cdot \dfrac{\partial \log\big(\Pi_i(\mathbf{p}^t, \mathbf{r}^t)\big)}{\partial p_i}\right) = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t D_i^t\right)$.

    Update reference price: $r_i^{t+1} = r_i^t + \eta_r \cdot \dfrac{\partial R_i(p_i^t, r_i^t)}{\partial r_i} = \alpha r_i^t + (1-\alpha) p_i^t$.

  end

end

It can be easily seen that the pure strategy Nash equilibrium of this $2n$-player static game is equivalent to the SNE (see Definition 2) of the original n-player dynamic game. However, even after converting to the static game, no general convergence results are readily applicable in this problem, primarily for the following two reasons.

B.1.1 Lack of Variational Stability for $2n$-player Game. The first obstacle comes from the lack of critical properties in our $2n$-player game, such as monotonicity (Lin et al. 2020) or variational stability (Mertikopoulos and Zhou 2019), where the latter is a strictly weaker version of the former. Without loss of generality, we demonstrate the absence of the variational stability using a duopoly competition (i.e., $n=2$), which can be transformed to a standard 4-player game, with the price and reference price updating via Algorithm 3. To show this standard 4-player game is not variationally stable, it suffices to demonstrate that the second-order test for variational stability, outlined in Mertikopoulos and Zhou (2019, Table 1), is not consistently satisfied. This test essentially requires the negative definiteness of a 4×4 symmetric matrix $H^{\mathcal{G}}$, which will be formally defined below. First, let $\mathbf{x}=(\mathbf{p},\mathbf{r})$ be the 4-dimensional decision variable for all players, and let $f_k(\mathbf{x})$ be the objective functions where $k\in\{1,2,v_1,v_2\}$. Specifically, indices $1,2$ represent two real firms and index v_i represents the virtual firm that corresponds to the reference price of product $i\in\{1,2\}$. By construction, we have that

$$f_k(\mathbf{x})=\begin{cases}\log\left(\Pi_i(\mathbf{p},\mathbf{r})\right) & \text{if } k=i,\\ R_i(p_i,r_i) & \text{if } k=v_i,\end{cases}\qquad \text{where } i\in\{1,2\}. \tag{B.2}$$

Next, we define the 4×4 matrix $H^{\mathcal{G}}$, where we let $1,2,v_1,v_2$ correspond to matrix indices $1,2,3,4$, respectively. Then, the (m,l)-th entry of $H^{\mathcal{G}}$ are defined as follows

$$\left(H^{\mathcal{G}}\right)_{ml}:=\frac{\partial^2 f_m(\mathbf{x}^\star)}{\partial x_m\partial x_l}+\frac{\partial^2 f_l(\mathbf{x}^\star)}{\partial x_m\partial x_l},\quad \forall 1\le m,l\le 4, \tag{B.3}$$

where we denote $\mathbf{x}^\star=(\mathbf{p}^\star,\mathbf{p}^\star)$. Using direct computation and the optimality condition in Eq. (C.16), it holds that

$$H^{\mathcal{G}}=\begin{bmatrix} -2(\tilde{b}_1)^2(1-d_1^\star) & 2\tilde{b}_1\tilde{b}_2 d_1^\star d_2^\star & 1+\tilde{b}_1 c_1 d_1^\star(1-d_1^\star) & -\tilde{b}_1 c_2 d_1^\star d_2^\star \\ 2\tilde{b}_1\tilde{b}_2 d_1^\star d_2^\star & -2(\tilde{b}_2)^2(1-d_2^\star) & -\tilde{b}_2 c_1 d_1^\star d_2^\star & 1+\tilde{b}_2 c_2 d_2^\star(1-d_2^\star) \\ 1+\tilde{b}_1 c_1 d_1^\star(1-d_1^\star) & -\tilde{b}_2 c_1 d_1^\star d_2^\star & -2 & 0 \\ -\tilde{b}_1 c_2 d_1^\star d_2^\star & 1+\tilde{b}_2 c_2 d_2^\star(1-d_2^\star) & 0 & -2 \end{bmatrix}, \tag{B.4}$$

where we use the shorthand notations $\tilde{b}_i:=b_i+c_i$ and $d_i^\star=d_i(\mathbf{p}^\star,\mathbf{p}^\star)$. Consider the principal minor of $H^{\mathcal{G}}$ formed by removing the first row and the first column of $H^{\mathcal{G}}$. Its determinant can be computed as follows

$$\begin{aligned}&\mathtt{det}\left(\begin{bmatrix} -2(\tilde{b}_2)^2(1-d_2^\star) & -\tilde{b}_2 c_1 d_1^\star d_2^\star & 1+\tilde{b}_2 c_2 d_2^\star(1-d_2^\star) \\ -\tilde{b}_2 c_1 d_1^\star d_2^\star & -2 & 0 \\ 1+\tilde{b}_2 c_2 d_2^\star(1-d_2^\star) & 0 & -2 \end{bmatrix}\right)\\ &=-8(\tilde{b}_2)^2(1-d_2^\star)+2\left[1+\tilde{b}_2 c_2 d_2^\star(1-d_2^\star)\right]^2+2\left(\tilde{b}_2 c_1 d_1^\star d_2^\star\right)^2\\ &>2-8(\tilde{b}_2)^2.\end{aligned} \tag{B.5}$$

If $H^{\mathcal{G}}$ is negative definite, then the above determinant should be negative. Yet, from Eq. (B.5), it is evident that determinant must be positive when $\tilde{b}_2 = b_2 + c_2 \leq 1/2$. Consequently, this implies that the second-order test for variational stability does not always hold, which indicates that our $2n$-player game is not variationally stable.

B.1.2 Inflexible Step-sizes for Virtual Players. The other obstacle stems from the asynchronous updates for the real firms (price players) and the virtual firms (reference price players). While the real firms have the flexibility in adopting time-varying step-sizes, the virtual firms must stick to the constant step-size of $(1-\alpha)$. As a result, this inflexibility perplexes the analysis, as the typical convergence results of online games require the step-sizes of multiple players to have the same pattern (all diminishing or sufficiently small constant step-sizes) (see, e.g., Nagurney and Zhang (1995), Scutari et al. (2010), Bravo et al. (2018), Mertikopoulos and Zhou (2019)).

We are aware that Golrezaei et al. (2020) also have the same challenge. Below, we would like to elaborate on how Golrezaei et al. (2020) handle the issue of heterogeneous step-sizes and clarify why this approach is not applicable to the oligopoly competition studied in this paper. The central lemma in their analysis is Golrezaei et al. (2020, Lemma 9.1), which essentially shows that

$$\sum_{i=1,2} (p_i^\star - p_i) \cdot \frac{\partial \pi_i(\mathbf{p}, r)}{\partial p_i}\bigg|_{r=\theta_1 p_1 + \theta_2 p_2} > 0, \quad \forall \mathbf{p} \in \mathcal{P}^n, \tag{B.6}$$

where $\pi_i(\mathbf{p}, r)$ denotes their revenue function for firm i and $\mathbf{p}^\star$ denotes their SNE. Note that under their reference price update model, the condition $r = \theta_1 p_1 + \theta_2 p_2$ indicates that the reference price already converges to the price. The inequality in Eq. (B.6) basically demonstrates a similar property as variational stability for their duopoly competition, except for requiring $r = \theta_1 p_1 + \theta_2 p_2$. When the real firms adopt decreasing step-sizes, the reference price would gradually converge towards the price. Then, together with Eq. (B.6), Golrezaei et al. (2020) manage to derive the global convergence of their algorithm. It is worth mentioning that both two parts of the proof for Golrezaei et al. (2020, Theorem 5.1) rely on Eq. (B.6).

For our oligopoly game with logit demand, we are able to prove a property analogous to Eq. (B.6) but only holds locally around $\mathbf{p}^\star$. The proof of Lemma EC.1 is deferred to Appendix M.1.

LEMMA EC.1. *In the loss-neutral scenario, define function $\mathcal{H}(\mathbf{p})$ as follows:*

$$\mathcal{H}(\mathbf{p}) := \sum_{i \in N} (p_i^\star - p_i) \cdot \frac{\partial \log\big(\Pi(\mathbf{p}, \mathbf{r})\big)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i \in N} \left[\frac{1}{p_i} + (b_i + c_i)\big(d_i(\mathbf{p}, \mathbf{p}) - 1\big)\right](p_i^\star - p_i), \tag{B.7}$$

where $\mathbf{p}^\star$ is the unique SNE. Then, there exist $\gamma > 0$ and a open set $U_\gamma \ni \mathbf{p}^\star$ such that

$$\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2, \quad \forall \mathbf{p} \in U_\gamma. \tag{B.8}$$

Leveraging Lemma EC.1, we proceed to establish local convergence in the subsequent proposition (see Appendix M.2 for its proof). It is important to note that Proposition EC.1 guarantees convergence only in the vicinity of $\mathbf{p}^\star$, since Lemma EC.1 is only applicable on a local scale.

Proposition EC.1 **(Local Convergence of OPGA)**. *In the loss-neutral scenario, let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$ hold. Then, there exists some neighborhood $\mathcal{B}$ of $\mathbf{p}^\star$ such that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters $\mathcal{B}$ with a sufficiently small step-size, the price path will stay in $\mathcal{B}$ during subsequent periods.*

Furthermore, suppose the step-sizes satisfy $\eta^t = \frac{C_\eta}{t}$ for all $t\geq 1$, where C_η is some general constant. Then, the local convergence rate of $\{(\mathbf{p}^t, \mathbf{r}^t)\}_{t\geq 0}$ after the path stays in $\mathcal{B}$ satisfies that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \mathcal{O}\left(\frac{1}{t}\right), \quad \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 \leq \mathcal{O}\left(\frac{1}{t}\right). \tag{B.9}$$

Since Lemma EC.1 does not always hold for general $\mathbf{p}\in\mathcal{P}^n$, we are unable to derive the global convergence via a similar two-step analysis employed in Golrezaei et al. (2020), which makes it necessary for us to devise new techniques. In our proofs for Theorems 1 and 2, we introduce a weighted ℓ^1-distance (defined in Eq. (11)) to measure the convergence, and our analysis mainly leverages a structural property of the SNE under the MNL model, as shown in Lemma EC.3.

Moreover, it should be highlighted that the analyses based on the variational stability itself (Mertikopoulos and Zhou 2019) or its variant (Golrezaei et al. 2020) achieve, at their best, an $\mathcal{O}(1/t)$ convergence rate in the noise-free setting. In comparison, by exploiting characteristics of the MNL model, we manage to derive a faster rate of $\mathcal{O}(1/t^2)$ in Theorem 2. This improvement further sets apart our convergence results in the loss-neutral scenario from those reported by Mertikopoulos and Zhou (2019) and Golrezaei et al. (2020).

B.2 Nonlinear Dynamical System Formulation

The study of the limiting behavior of a competitive gradient-based learning algorithm is related to dynamical system theories (Mazumdar et al. 2020). In fact, the update of Algorithm 1 can be viewed as a nonlinear dynamical system. Assume a constant step-size is employed, i.e., $\eta^t\equiv\eta$, $\forall t\geq 0$. Then, Lines 5 and 7 in Algorithm 1 are equivalent to the dynamical system

$$(\mathbf{p}^{t+1}, \mathbf{r}^{t+1}) = \mathbf{f}(\mathbf{p}^t, \mathbf{r}^t), \quad \forall t\geq 0, \tag{B.10}$$

where $\mathbf{f}(\cdot)$ is a vector-valued function defined as

$$\mathbf{f}(\mathbf{p},\mathbf{r}) := \begin{pmatrix} \mathtt{Proj}_{\mathcal{P}}\Big(p_1 + \eta\Big(1/p_1 + (b_1+c_1)\cdot d_1(\mathbf{p},\mathbf{r}) - (b_1+c_1)\Big)\Big) \\ \cdots \\ \mathtt{Proj}_{\mathcal{P}}\Big(p_n + \eta\Big(1/p_n + (b_n+c_n)\cdot d_n(\mathbf{p},\mathbf{r}) - (b_n+c_n)\Big)\Big) \\ \alpha r_1 + (1-\alpha)p_1 \\ \cdots \\ \alpha r_n + (1-\alpha)p_n \end{pmatrix}. \tag{B.11}$$

Under the assumption that $\mathbf{p}^\star \in \mathcal{P}^n$, it is evident that $\mathbf{p}^\star$ is the unique fixed point of the system in Eq. (B.10). Generally, fixed points can be categorized into three classes:

- *Asymptotically stable* when all nearby solutions converge to it.
- *Stable* when all nearby solutions remain in close proximity.
- *Unstable* when almost all nearby solutions diverge away from the fixed point.

Hence, if we can demonstrate the asymptotic stability of $\mathbf{p}^\star$, we can at least prove the local convergence of the price and reference price.

Standard dynamical systems theory (Arrowsmith et al. 1990) states that $\mathbf{p}^\star$ is asymptotically stable if the spectral radius of the Jacobian matrix $\nabla\mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$ is strictly less than one. Yet, computing the spectral radius is not straightforward. The primary challenge stems from the fact that the entries of $\nabla\mathbf{f}(\mathbf{p}^\star, \mathbf{p}^\star)$ contain $\mathbf{p}^\star$ and $d_i(\mathbf{p}^\star, \mathbf{p}^\star)$, but there is no closed-form expression for $\mathbf{p}^\star$.

Apart from the above issue, it is worth noting that the function $\mathbf{f}(\cdot,\cdot)$ is not globally smooth due to the presence of the projection operator. Furthermore, the function $\mathbf{f}(\cdot,\cdot)$ also depends on the step-size η. When the firms adopt time-varying step-sizes, the dynamical system in Eq. (B.10) becomes non-stationary, i.e., $(\mathbf{p}^{t+1}, \mathbf{r}^{t+1}) = \mathbf{f}^t(\mathbf{p}^t, \mathbf{r}^t)$. Although the sequence of functions $\{\mathbf{f}^t(\cdot,\cdot)\}_{t\geq 0}$ shares the same fixed point, verifying the convergence (stability) of the system requires examining the spectral radius of $\nabla\mathbf{f}^t(\mathbf{p}^\star, \mathbf{p}^\star)$ for all $t \geq 0$.

Most significantly, even if asymptotic stability holds, it can only guarantee local convergence of Algorithm 1. Our goal, however, is to prove global convergence, such that both the price and reference price converge to the SNE for arbitrary initializations.

Appendix C Proofs of Propositions

C.1 Proof of Proposition 1

Proposition 1 **(Restated)**. *Let $\mathcal{S}$ be the set of SNE(s). Then, the following statements hold:*

- *If there exists any gain-seeking product, an SNE never exists, i.e., $\mathcal{S}$ is empty.*
- *Otherwise, with only loss-averse and loss-neutral products, an SNE always exists, and $\mathcal{S}$ can be expressed as*

$$\mathcal{S} = \left\{ \mathbf{p}^\star \;\middle|\; p_i^\star \in \left[\frac{1}{\left(b_i + c_i^-\right)\cdot\left(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\right)}, \frac{1}{\left(b_i + c_i^+\right)\cdot\left(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\right)} \right],\ \forall i \in N \right\}. \tag{C.1}$$

Proof of Proposition 1. We prove the two parts of the proposition separately.

Part 1. In this part where there is one or more gain-seeking product(s), i.e., $\exists i \in N$ such that $c_i^+ > c_i^-$, we show the non-existence of SNE by contradiction-based arguments. Suppose that there exists an SNE $\mathbf{p}^\star$ under gain-seeking reference effects. By Definition 2, $\mathbf{p}^\star$ must satisfy $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$, i.e., the price at SNE is equal to the corresponding reference price. This implies that the revenue function is non-smooth at an SNE as a result of gain-seeking reference effects. For a non-smooth point to be a Nash equilibrium, its left-hand derivative must be non-negative, and its right-hand derivative must be non-positive, implying the left-hand derivative is no greater than the right-hand derivative. Below, we take the left-hand and right-hand derivatives of $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i at its SNE, i.e., $(\mathbf{p},\mathbf{r})=(\mathbf{p}^\star,\mathbf{p}^\star)$

$$\lim_{\Delta p_i\to 0^-}\frac{\Pi_i\big((p_i^\star+\Delta p_i,\mathbf{p}_{-i}^\star),\mathbf{p}^\star\big)-\Pi_i\big(\mathbf{p}^\star,\mathbf{p}^\star\big)}{\Delta p_i}=d_i(\mathbf{p}^\star,\mathbf{p}^\star)\cdot\left[1-p_i^\star(b_i+c_i^+)\big(1-d_i(\mathbf{p}^\star,\mathbf{p}^\star)\big)\right], \tag{C.2a}$$

$$\lim_{\Delta p_i\to 0^+}\frac{\Pi_i\big((p_i^\star+\Delta p_i,\mathbf{p}_{-i}^\star),\mathbf{p}^\star\big)-\Pi_i\big(\mathbf{p}^\star,\mathbf{p}^\star\big)}{\Delta p_i}=d_i(\mathbf{p}^\star,\mathbf{p}^\star)\cdot\left[1-p_i^\star(b_i+c_i^-)\big(1-d_i(\mathbf{p}^\star,\mathbf{p}^\star)\big)\right], \tag{C.2b}$$

where the left-hand derivative Eq. (C.2a) has the effective reference price sensitivity c_i^+ because when p_i approaches $p_i^\star$ from left, it follows that $p_i\le r_i=p_i^\star$. For the similar reason, the right-hand derivative Eq. (C.2b) uses c_i^- as the effective reference price sensitivity. We notice that the left-hand derivative is smaller than the right-hand derivative since product i has the gain-seeking reference effect, i.e., $c_i^+>c_i^-$. This conflicts with the necessary condition for $\mathbf{p}^\star$ to be an NE. We conclude that no SNE exists in the gain-seeking scenario; hence, the price and reference price paths are cyclic in the long run for any given initial reference price.

Part 2. In this part where there are only loss-averse and loss-neutral products, our **first step** is to show that any SNE price must satisfy the characterization in Eq. (C.1). In the **second step**, we demonstrate that for any given pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ where $\tilde{c}_i\in[c_i^+,c_i^-]$, there exists a unique price vector $\mathbf{p}$ that satisfies

$$p_i=\frac{1}{(b_i+\tilde{c}_i)\cdot(1-d_i(\mathbf{p},\mathbf{p}))},\quad \forall i\in N, \tag{C.3}$$

and $\mathbf{p}$ is also an SNE. Together, these two steps prove the existence of SNE and show that $\mathcal{S}$ admits the expression in Eq. (C.1).

We start with the first step. According to Definition 2, it holds that $\mathbf{p}^E(\mathbf{p}^\star)=\mathbf{p}^\star$ for any SNE $\mathbf{p}^\star\in\mathcal{S}$, i.e., the price output by the equilibrium pricing policy is the same as the input reference price. As the SNE is a special case of NE, each firm's revenue needs to satisfy the first-order condition. By expanding the derivative and incorporating the sub-gradient at non-smooth points, we find that equilibrium $\mathbf{p}^E(\mathbf{r})=(p_1^E(\mathbf{r}),\ldots,p_n^E(\mathbf{r}))$ for the given reference price $\mathbf{r}$ admits that

$$\frac{\partial\Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i}\bigg|_{(\mathbf{p}^E(\mathbf{r}),\mathbf{r})}=0\quad\Leftrightarrow\quad p_i^E(\mathbf{r})\cdot\big(1-d_i(\mathbf{p}^E(\mathbf{r}),\mathbf{r})\big)-\frac{1}{b_i+c_i\big(p_i^E(\mathbf{r}),r_i\big)}=0,\quad\forall i\in N, \tag{C.4}$$

where $c_i(p_i,r_i):=\mathbb{1}\{p_i<r_i\}\cdot c_i^++\mathbb{1}\{p_i>r_i\}\cdot c_i^-+\mathbb{1}\{p_i=r_i\}\cdot\tilde{c}_i$ represents the effective reference price sensitivity for product i at (p_i,r_i), and $\tilde{c}_i$ take the unique value between c_i^+ and c_i^- that makes the equality in Eq. (C.4) holds. For any SNE $\mathbf{p}^\star \in \mathcal{S}$, since $\mathbf{p}^E(\mathbf{p}^\star) = \mathbf{p}^\star$, we evaluate Eq. (C.4) at $(\mathbf{p}^E(\mathbf{r}), \mathbf{r}) = (\mathbf{p}^\star, \mathbf{p}^\star)$ to obtain that

$$p_i^\star = \frac{1}{\big(b_i + c_i(p_i^\star, p_i^\star)\big) \cdot \big(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star)\big)}, \quad \forall i \in N. \tag{C.5}$$

Since the $c_i(p_i^\star, p_i^\star) \in [c_i^+, c_i^-]$, this proves that $\mathcal{S}$ must be a subset of the set characterized by the right-hand side of Eq. (C.1). This completes the proof for the first step.

Now, we proceed to the second step and begin by showing that given any pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ where $\tilde{c}_i \in [c_i^+, c_i^-]$, Eq. (C.3) produces a unique price vector $\mathbf{p}$. By definition of the market share, we have that

$$d_i(\mathbf{p}, \mathbf{p}) = \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\in N} \exp(a_k - b_k p_k)} = d_0(\mathbf{p}, \mathbf{p}) \cdot \exp\left(a_i - \frac{b_i}{(b_i + \tilde{c}_i)\big(1 - d_i(\mathbf{p}, \mathbf{p})\big)}\right), \tag{C.6}$$

where the last equality follows from substituting p_i with the right-hand side of Eq. (C.3), and $d_0(\mathbf{p}, \mathbf{p}) := \frac{1}{1 + \sum_{k\in N} \exp(a_k - b_k p_k)}$, which is the no-purchase probability. Rearranging Eq. (C.6), we move all terms containing $d_i(\mathbf{p}, \mathbf{p})$ to the left-hand side to obtain that

$$d_i(\mathbf{p}, \mathbf{p}) \cdot \exp\left(\frac{b_i}{b_i + \tilde{c}_i} \cdot \frac{1}{1 - d_i(\mathbf{p}, \mathbf{p})}\right) = d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i). \tag{C.7}$$

Define function $V_{\tilde{c}_i}(x) : (0, \infty) \to (0, 1)$ as the unique real solution v to the following equation:

$$v \cdot \exp\left(\frac{b_i}{b_i + \tilde{c}_i} \cdot \frac{1}{1 - v}\right) = x. \tag{C.8}$$

Then, from Eq. (C.7), we can express $d_i(\mathbf{p}, \mathbf{p})$ in terms of $V_{\tilde{c}_i}(\cdot)$ as

$$d_i(\mathbf{p}, \mathbf{p}) = V_{\tilde{c}_i}\big(d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i)\big). \tag{C.9}$$

Since $d_0(\mathbf{p}, \mathbf{p}) + \sum_{i\in N} d_i(\mathbf{p}, \mathbf{p}) = 1$, together with Eq. (C.9), we have that

$$d_0(\mathbf{p}, \mathbf{p}) + \sum_{i\in N} V_{\tilde{c}_i}\big(d_0(\mathbf{p}, \mathbf{p}) \cdot \exp(a_i)\big) = 1. \tag{C.10}$$

We observe that function $V_{\tilde{c}_i}(x)$ is strictly increasing, i.e., v is monotone increasing in x in Eq. (C.8). Hence, the left-hand side of Eq. (C.10) is monotone increasing in $d_0(\mathbf{p}, \mathbf{p})$, and its range clearly contains one as $d_0(\mathbf{p}, \mathbf{p})$ increases from zero to one. So, there exist a unique solution $d_0(\mathbf{p}, \mathbf{p})$ that satisfies Eq. (C.10). Together with Eq. (C.9), we observe that the demand for every product i is uniquely determined from Eq. (C.3). Due to the one-to-one mapping between $\mathbf{p}$ and $\{d_i(\mathbf{p}, \mathbf{p})\}_{i\in N}$, we conclude that there must exist a unique price vector that satisfies both Eqs. (C.9) and (C.10), equivalently Eq. (C.3). Below, we denote this unique solution as $\mathbf{p}^s$.

Next, we show that $\mathbf{p}^s$ is an SNE. Since Eq. (C.5) arises from the first-order condition for NE, we know $\mathbf{p}^s$ is a stationary point. Then, to prove $\mathbf{p}^s$ is indeed an SNE, it suffices to show that for all $i \in N$

$$\lim_{\Delta p_i \to 0} \frac{\Pi_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big) - \Pi_i\big((p_i - \Delta p_i, , \mathbf{p}^s_{-i}), \mathbf{p}^s\big)}{\Delta p_i} \geq 0, \quad \forall p_i \leq p_i^s, \tag{C.11a}$$

$$\lim_{\Delta p_i \to 0} \frac{\Pi_i\big((p_i + \Delta p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big) - \Pi_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)}{\Delta p_i} \leq 0, \quad \forall p_i \geq p_i^s. \tag{C.11b}$$

If Eq. (C.11) holds, then for every firm $i \in N$, the revenue always increases in p_i when $p_i \leq p_i^s$ and decreases in p_i when $p_i \geq p_i^s$, assuming the prices for all other products remain at $\mathbf{p}^s_{-i}$. This implies that firm i can never achieve a higher revenue by deviating from the stationary price p_i^s, and thereby $\mathbf{p}^s$ is an SNE. We compute the left-hand derivative and observe that Eq. (C.11a) is equivalent to

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^+} \leq 0, \quad \forall p_i \leq p_i^s, \tag{C.12}$$

where we use c_i^+ because the $r_i = p_i^s \geq p_i$. It is clear that the left-hand side of Eq. (C.12) is monotone increasing in p_i. Together with $c_i^+ \leq \tilde{c}_i$, we have for all $p_i \leq p_i^s$

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^+} \leq p_i^s \cdot \Big[1 - d_i(\mathbf{p}^s, \mathbf{p}^s)\Big] - \frac{1}{b_i + \tilde{c}_i} = 0, \tag{C.13}$$

where the last equality stems from the fact that $\mathbf{p}^s$ is the unique solution to Eq. (C.3). Similarly, we validate Eq. (C.11b) by showing that

$$p_i \cdot \Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - \frac{1}{b_i + c_i^-} \geq p_i^s \cdot \Big[1 - d_i(\mathbf{p}^s, \mathbf{p}^s)\Big] - \frac{1}{b_i + \tilde{c}_i} = 0, \quad \forall p_i \geq p_i^s, \tag{C.14}$$

as Eq. (C.11b) is equivalent to $p_i\Big[1 - d_i\big((p_i, \mathbf{p}^s_{-i}), \mathbf{p}^s\big)\Big] - 1/(b_i + c_i^-) \geq 0$. Since both conditions in (C.11) are satisfied, we conclude that $\mathbf{p}^s$ is an SNE.

Combining the results in both parts, we finally complete the proof of Proposition 1. □

C.2 Proof of Proposition 2

PROPOSITION 2 **(Restated)**. *In loss-averse and loss-neutral scenarios where SNE(s) always exists, its uniqueness depends on the presence of any loss-averse product. Specifically,*

- *The SNE is unique, i.e., $\mathcal{S}$ is a singleton, if and only if all products are loss-neutral.*
- *Otherwise, with any loss-averse product, there always exists a continuum of SNEs, and $\mathcal{S}$ can be a non-convex set.*

Furthermore, any SNE $\mathbf{p}^\star \in \mathcal{S}$ can be bounded as

$$\frac{1}{b_i + c_i^-} < p_i^\star < \frac{1}{b_i + c_i^+} + \frac{1}{b_i} W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right), \quad \forall i \in N, \tag{C.15}$$

where $W(\cdot)$ is the Lambert W function (see definition in Eq. (C.25)).

Proof of Proposition 2. First, the uniqueness of SNE when all products are loss-neutral directly follows from the characterization of SNE in Eq. (C.1). As $c_i^+ = c_i^- := c_i$ in the loss-neutral scenario, the interval in Eq. (C.1) reduces to a single value, and thus there only exists a unique price vector, denoted by $\mathbf{p}^\star$, such that

$$p_i^\star = \frac{1}{(b_i + c_i)\cdot(1 - d_i(\mathbf{p}^\star, \mathbf{p}^\star))}, \quad \forall i \in N, \tag{C.16}$$

where the uniqueness follows from the same reasoning as Eqs. (C.6) to Eq. (C.10).

Next, to prove the reverse direction ($\mathcal{S}$ is a singleton only if all products are loss-neutral), we show that in the presence of any loss-averse product, there always exist infinitely many SNEs that form a continuum. Without loss of generality, suppose consumers are loss-averse towards product $i_0 \in N$, i.e., $c_{i_0}^+ < c_{i_0}^-$. Then, going back to Eq. (C.3), it suffices to show that for two different $\tilde{c}_{i_0,1}, \tilde{c}_{i_0,2} \in [c_{i_0}^+, c_{i_0}^-]$, the pseudo sensitivities $(\tilde{c}_{i_0,1}, \tilde{\mathbf{c}}_{-i_0})$ and $(\tilde{c}_{i_0,2}, \tilde{\mathbf{c}}_{-i_0})$ produce two different SNEs. Note that we use $\tilde{\mathbf{c}}_{-i}$ to denote the vector $(\tilde{c}_j)_{j\neq i}$.

Let $\tilde{\mathbf{p}}^{\star,1}$ be the SNE that satisfies Eq. (C.3) with pseudo sensitivities $(\tilde{c}_{i_0,1}, \tilde{\mathbf{c}}_{-i_0})$, and $\tilde{\mathbf{p}}^{\star,2}$ be the SNE with pseudo sensitivities $(\tilde{c}_{i_0,2}, \tilde{\mathbf{c}}_{-i_0})$. Below, we show that $\tilde{\mathbf{p}}^{\star,1} \neq \tilde{\mathbf{p}}^{\star,2}$. Suppose by contradiction that $\tilde{\mathbf{p}}^{\star,1} = \tilde{\mathbf{p}}^{\star,2}$, which implies that $d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})$. Since the two SNEs share the same pseudo sensitivities for all products except i_0, we deduce from Eq. (C.9) that for all $i \neq i_0$

$$d_i(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = V_{\tilde{c}_i}(d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1})\cdot\exp(a_i)) = V_{\tilde{c}_i}(d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})\cdot\exp(a_i)) = d_i(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}). \tag{C.17}$$

Since $d_0(\mathbf{p},\mathbf{r}) + \sum_{i\in N} d_i(\mathbf{p},\mathbf{r}) = 1$, it also holds that

$$\begin{aligned} d_{i_0}(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) = 1 - d_0(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) - \sum_{i\neq i_0} d_i(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1}) &= 1 - d_0(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}) - \sum_{i\neq i_0} d_i(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}) \\ &= d_{i_0}(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2}). \end{aligned} \tag{C.18}$$

Together with Eq. (C.3), Eq. (C.18) indicates that

$$b_i + \tilde{c}_{i_0,1} = \frac{1}{\tilde{p}_{i_0}^{\star,1}\big(1 - d_{i_0}(\tilde{\mathbf{p}}^{\star,1}, \tilde{\mathbf{p}}^{\star,1})\big)} = \frac{1}{\tilde{p}_{i_0}^{\star,2}\big(1 - d_{i_0}(\tilde{\mathbf{p}}^{\star,2}, \tilde{\mathbf{p}}^{\star,2})\big)} = b_i + \tilde{c}_{i_0,2}, \tag{C.19}$$

which contradicts with the assumption that $\tilde{c}_{i_0,1} \neq \tilde{c}_{i_0,2}$. Therefore, we must have $\tilde{\mathbf{p}}^{\star,1} \neq \tilde{\mathbf{p}}^{\star,2}$. Since $[c_i^+, c_i^-]$ is a continuous interval, we conclude that there exist infinitely many SNEs in the presence of any loss-averse product. Finally, it is clear from Eqs. (C.3) to (C.10) that the dependency of the SNE price $\tilde{\mathbf{p}}^\star$ on the pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ is continuous. Thus, the set of SNEs must form a continuous area, i.e., $\mathcal{S}$ is a continuum. The non-convexity of set $\mathcal{S}$ has already been confirmed by Figure 1.

Below, we show the boundedness of set $\mathcal{S}$. By performing a transformation on the relation in Eq. (C.1), we obtain the following inequalities for any $i \in N$ and $\mathbf{p} \in \mathcal{S}$:

$$p_i(b_i + c_i^-) \geq 1 + \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\neq i}\exp(a_k - b_k p_k)}, \quad p_i(b_i + c_i^+) \leq 1 + \frac{\exp(a_i - b_i p_i)}{1 + \sum_{k\neq i}\exp(a_k - b_k p_k)}. \tag{C.20}$$

Then, it immediately follows that

$$\frac{1}{b_i + c_i^-} < p_i < \frac{1 + \exp(a_i - b_i p_i)}{b_i + c_i^+}, \quad \forall i \in N, \ \forall \mathbf{p} \in \mathcal{S}. \tag{C.21}$$

Now, we derive the upper bound in Eq. (C.15) from the second inequality in Eq. (C.21). Since the quantity on the right-hand side of Eq. (C.21) is monotone decreasing in p_i, any price that satisfies Eq. (C.21) must be upper-bounded by the unique solution y_i to the following equation

$$y_i = \frac{1 + \exp(a_i - b_i y_i)}{b_i + c_i^+}. \tag{C.22}$$

Define $x_i := -b_i/(b_i + c_i^+) + b_i y_i$. Then, one can easily verify that Eq. (C.22) can be converted into

$$x_i \exp(x_i) = \frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right), \tag{C.23}$$

which implies that

$$x_i = W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right), \tag{C.24}$$

where $W(\cdot)$ is known as the Lambert W function (Weisstein 2002). For any value $z \geq 0$, $W(z)$ is defined to be the unique real solution w to the equation

$$w \cdot \exp\big(w\big) = z. \tag{C.25}$$

Hence, we have that

$$p_i < y_i = \frac{1}{b_i + c_i^+} + \frac{1}{b_i} W\left(\frac{b_i}{b_i + c_i^+} \exp\left(a_i - \frac{b_i}{b_i + c_i^+}\right)\right), \quad \forall i \in N, \ \forall \mathbf{p} \in \mathcal{S}. \tag{C.26}$$

Together with the lower bound provided in Eq. (C.21), this completes the proof. □

C.3 Proof of Proposition 3

Proposition 3 **(Restated)**. *The set of SNEs can be equivalently expressed as* $\mathcal{S} = \{\mathbf{p} \in \mathcal{P}^n \mid \widetilde{\kappa}(\mathbf{p}, \mathbf{p}) = 0\}$, *where* $\widetilde{\kappa}(\cdot, \cdot)$ *is the metric defined in Eq. (17).*

Proof of Proposition 3. By Eq. (C.1) and the definition of functions $D_i^+(\mathbf{p}, \mathbf{p}), D_i^-(\mathbf{p}, \mathbf{p})$ in Eq. (18), we observe that the set of SNE(s) can be equivalently written as

$$\mathcal{S} = \left\{\mathbf{p} \in \mathcal{P}^n \;\middle|\; D_i^+(\mathbf{p}, \mathbf{p}) \geq 0, \ D_i^-(\mathbf{p}, \mathbf{p}) \leq 0, \ \forall i \in N\right\}. \tag{C.27}$$

Therefore, since $D_i^+(\mathbf{p}, \mathbf{r})$ is consistently greater than $D_i^-(\mathbf{p}, \mathbf{r})$ in the loss-averse scenario, we can derive that

$$\begin{aligned} \widetilde{\kappa}(\mathbf{p}, \mathbf{p}) = 0 \quad &\Leftrightarrow \quad \sum_{i \in N} \mathtt{dist}\Big(0, \mathtt{Hull}\big\{D_i^-(\mathbf{p}, \mathbf{p}), D_i^+(\mathbf{p}, \mathbf{p})\big\}\Big) = 0 \\ &\Leftrightarrow \quad \mathtt{dist}\big(0, I_i(\mathbf{p})\big) = 0, \quad \forall i \in N \\ &\Leftrightarrow \quad D_i^+(\mathbf{p}, \mathbf{p}) \geq 0, \ D_i^-(\mathbf{p}, \mathbf{p}) \leq 0, \quad \forall i \in N, \end{aligned} \tag{C.28}$$

where we use the notation $I_i(\mathbf{p})$ to denote the interval $[D_i^-(\mathbf{p}, \mathbf{p}), D_i^+(\mathbf{p}, \mathbf{p})]$. This completes the proof of the proposition. □

Appendix D Proof of Theorem 1

Theorem 1 **(Restated)**. *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Then, their price and reference price paths converge to the unique stationary Nash equilibrium.*

Proof of Theorem 1. We will leverage the metrics $\kappa(\cdot)$ and $\kappa_\epsilon(\cdot)$ defined in Eq. (11). It is clear that

$$\kappa_\epsilon(\mathbf{p}) \leq \kappa(\mathbf{p}) \leq \kappa_\epsilon(\mathbf{p}) + n\epsilon, \quad \forall \mathbf{p} \in \mathcal{P}^n. \tag{D.1}$$

In our proof, we will show that for every $\epsilon > 0$, it holds that $\lim_{t\to\infty}\kappa(\mathbf{p}^t) \leq \mathcal{O}(\epsilon)$, thereby proving the convergence of the price path $\{\mathbf{p}^t\}_{t\geq 0}$. As the reference price is updated through the exponential smoothing scheme (see Eq. (4)), the convergence of $\{\mathbf{p}^t\}_{t\geq 0}$ also implies the convergence of the reference path $\{\mathbf{r}^t\}_{t\geq 0}$.

Before the proof, we introduce some helpful definitions. Let $G_i(\mathbf{p},\mathbf{r})$ be the scaled partial derivative of the log-revenue, defined as

$$G_i(\mathbf{p},\mathbf{r}) := \frac{1}{b_i + c_i}\cdot\frac{\partial \log\big(\Pi_i(\mathbf{p},\mathbf{r})\big)}{\partial p_i} = \frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1, \quad \forall i \in N. \tag{D.2}$$

For the ease of notation, we denote $\mathcal{P}_i := \{p/(b_i+c_i) \mid p \in \mathcal{P}\}$ as the scaled price range. Then, the price update in Line 5 of Algorithm 1 is equivalent to

$$\frac{p_i^{t+1}}{b_i+c_i} = \mathtt{Proj}_{\mathcal{P}_i}\left(\frac{p_i^t}{b_i+c_i} + \eta^t\frac{D_i^t}{b_i+c_i}\right) = \mathtt{Proj}_{\mathcal{P}_i}\left(\frac{p_i^t}{b_i+c_i} + \eta^t G_i(\mathbf{p}^t,\mathbf{r}^t)\right). \tag{D.3}$$

Let $\mathtt{sign}(\cdot)$ be the sign function defined as

$$\mathtt{sign}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{D.4}$$

An essential observation from Eq. (D.3) is that: if $\mathtt{sign}(p_i^\star - p_i^t)\cdot G_i(\mathbf{p}^t,\mathbf{r}^t) > 0$, we have that $\mathtt{sign}(p_i^\star - p_i^t) = \mathtt{sign}\big(G_i(\mathbf{p}^t,\mathbf{r}^t)\big) = \mathtt{sign}(p_i^{t+1} - p_i^t)$, i.e., the update from p_i^t to p_i^{t+1} is toward the direction of the SNE price $p_i^\star$. Conversely, if $\mathtt{sign}(p_i^\star - p_i^t)\cdot G_i(\mathbf{p}^t,\mathbf{r}^t) < 0$, the update from p_i^t to p_i^{t+1} is deviating from $p_i^\star$. Finally, for every $t \geq 0$, we define

$$N_\epsilon^t := \left\{ i \in N \;\middle|\; \frac{|p_i^\star - p_i^t|}{b_i+c_i} < \epsilon \right\}, \quad \overline{N}_\epsilon^t := N \backslash N_\epsilon^t. \tag{D.5}$$

By definition, N_ϵ^t is the set of products whose prices are close to their SNE prices at period t, and $\overline{N}_\epsilon^t$ is its complement.

Now, we are ready to present the proof. By Lemma EC.2, when $\{\eta^t\}_{t\geq 0}$ is non-increasing and $\lim_{t\to\infty}\eta^t = 0$, the difference between reference price and price converges to zero as t goes to infinity, i.e., $\lim_{t\to\infty}\big(\mathbf{p}^t - \mathbf{r}^t\big) = 0$. Hence, for every $\epsilon > 0$, there exists $T_\epsilon > 0$ such that $\forall t \geq T_\epsilon$, it holds that

$\eta^t M_G < \epsilon$ and $\|\mathbf{p}^t - \mathbf{r}^t\| < \epsilon$, where M_G is an upper bound on $|G_i(\mathbf{p},\mathbf{r})|$ defined in Eq. (L.30). For every $t \geq T_\epsilon$, it follows from the definition of $\overline{N}_\epsilon^{t+1}$ in Eq. (D.5) that

$$\kappa_\epsilon(\mathbf{p}^{t+1}) = \sum_{i\in N} \max\left\{\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i + c_i} - \epsilon, 0\right\} = \sum_{i\in \overline{N}_\epsilon^{t+1}} \left(\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i + c_i} - \epsilon\right). \tag{D.6}$$

For every $i \in \overline{N}_\epsilon^{t+1}$, we have $\left|p_i^\star - p_i^{t+1}\right|/(b_i + c_i) \geq \epsilon$. Then, since

$$\frac{\left|p_i^{t+1} - p_i^t\right|}{b_i + c_i} \leq \frac{\eta^t D_i^t}{b_i + c_i} = \eta^t G_i(\mathbf{p}^t, \mathbf{r}^t) \leq \eta^t M_G < \epsilon, \tag{D.7}$$

it follows that $\mathtt{sign}\left(p_i^\star - p_i^{t+1}\right) = \mathtt{sign}\left(p_i^\star - p_i^t\right)$, and therefore

$$\begin{aligned}
\frac{\left|p_i^\star - p_i^{t+1}\right|}{b_i + c_i} &= \mathtt{sign}\left(p_i^\star - p_i^{t+1}\right) \frac{p_i^\star - p_i^{t+1}}{b_i + c_i} \\
&\leq \mathtt{sign}\left(p_i^\star - p_i^{t+1}\right) \frac{p_i^\star - p_i^t - \eta^t D_i^t}{b_i + c_i} \\
&= \mathtt{sign}\left(p_i^\star - p_i^t\right) \frac{p_i^\star - p_i^t - \eta^t D_i^t}{b_i + c_i} \\
&= \frac{\left|p_i^\star - p_i^t\right|}{b_i + c_i} - \eta^t \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t),
\end{aligned} \tag{D.8}$$

where the inequality is due to the property of the projection operator. We substitute Eq. (D.8) into the right-hand side of Eq. (D.6) to derive that

$$\begin{aligned}
\kappa_\epsilon(\mathbf{p}^{t+1}) &\leq \sum_{i\in \overline{N}_\epsilon^{t+1}} \left[\frac{\left|p_i^\star - p_i^t\right|}{b_i + c_i} - \epsilon - \eta^t \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t)\right] \\
&\leq \sum_{i\in N} \max\left\{\frac{\left|p_i^\star - p_i^t\right|}{b_i + c_i} - \epsilon, 0\right\} - \eta^t \sum_{i\in \overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t) \\
&= \kappa_\epsilon(\mathbf{p}^t) - \eta^t \sum_{i\in \overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t).
\end{aligned} \tag{D.9}$$

Thus, finding a lower bound for the summation term on the right-hand side of Eq. (D.9) is the key to the proof. Based on N_ϵ^{t+1}, we construct a price vector $\widehat{\mathbf{p}}^t$ as follows: $\widehat{p}_i^t = p_i^t$ if $i \in \overline{N}_\epsilon^{t+1}$, and

$\widehat{p}_i^t = p_i^\star$ if $i \in N_\epsilon^{t+1}$. Then, it holds that

$$\begin{aligned}
&\sum_{i\in\overline{N}_\epsilon^{t+1}} \mathtt{sign}\,(p_i^\star - p_i^t)\, G_i(\mathbf{p}^t,\mathbf{r}^t) \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - p_i^t)\, G_i(\mathbf{p}^t,\mathbf{p}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\big\{\big\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\big\|\big\}\,\big\|\mathbf{p}^t-\mathbf{r}^t\big\|\Big] \\
&\overset{(\Delta_1)}{\geq} \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\big\{\big\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\big\|\big\}\,\big\|\mathbf{p}^t-\mathbf{r}^t\big\| - \big|d_i(\mathbf{p}^t,\mathbf{p}^t) - d_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t)\big|\Big] \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}} \Big[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \max_{\mathbf{r}\in\mathcal{P}^n}\big\{\big\|\nabla_{\mathbf{r}} G_i(\mathbf{p}^t,\mathbf{r})\big\|\big\}\,\big\|\mathbf{p}^t-\mathbf{r}^t\big\| - \max_{\mathbf{p}\in\mathcal{P}^n}\big\{\|\nabla_{\mathbf{p}} d_i(\mathbf{p},\mathbf{p})\|\big\}\,\big\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\big\|\Big] \\
&\overset{(\Delta_2)}{\geq} \sum_{i\in\overline{N}_\epsilon^{t+1}} \big[\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \ell_{r,i}\,\big\|\mathbf{p}^t-\mathbf{r}^t\big\| - \ell_{d,i}\,\big\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\big\|\big] \\
&\geq \sum_{i\in N} \mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - \sum_{i\in N}\big[\ell_{r,i}\,\big\|\mathbf{p}^t-\mathbf{r}^t\big\| + \ell_{d,i}\,\big\|\mathbf{p}^t-\widehat{\mathbf{p}}^t\big\|\big]\,,
\end{aligned} \tag{D.10}$$

where inequality (Δ_1) uses the definition of $G_i(\mathbf{p},\mathbf{r})$ in Eq. (D.2). Since $\widehat{p}_i^t = p_i^t$ when $i\in\overline{N}_\epsilon^{t+1}$, it follows that $|G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) - G_i(\mathbf{p}^t,\mathbf{p}^t)| = |d_i(\mathbf{p}^t,\mathbf{p}^t) - d_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t)|$. Note that this difference does not equal to zero in general, since the demand $d_i(\cdot,\cdot)$ depends on the prices and reference prices of all products. Then, step (Δ_2) in Eq. (D.10) applies the Lipschitz continuity of $G_i(\mathbf{p},\cdot)$ and $d_i(\mathbf{p},\mathbf{p})$ from Lemmas EC.4 and EC.5, respectively. Then, the last inequality holds because $\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t) = \mathtt{sign}\,(0) = 0$ for all $i\in N_\epsilon^{t+1}$. Next, using Lemma EC.3, we have that

$$\sum_{i\in N}\mathtt{sign}\,(p_i^\star - \widehat{p}_i^t)\, G_i(\widehat{\mathbf{p}}^t,\widehat{\mathbf{p}}^t) = \mathcal{G}(\widehat{\mathbf{p}}^t) \geq \frac{\kappa(\widehat{\mathbf{p}}^t)}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}. \tag{D.11}$$

To relate Eq. (D.11) with the original inequality in Eq. (D.9), we observe that

$$\begin{aligned}
\kappa(\widehat{\mathbf{p}}^t) &= \sum_{i\in N}\frac{|p_i^\star - \widehat{p}_i^t|}{b_i+c_i} \\
&\overset{(\Delta)}{=} \sum_{i\in\overline{N}_\epsilon^{t+1}}\frac{|p_i^\star - p_i^t|}{b_i+c_i} \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t+1}}\max\left\{\frac{|p_i^\star - p_i^t|}{b_i+c_i}-\epsilon, 0\right\} \\
&\geq \sum_{i\in\overline{N}_\epsilon^{t}}\max\left\{\frac{|p_i^\star - p_i^t|}{b_i+c_i}-\epsilon, 0\right\} - \sum_{i\in\overline{N}_\epsilon^{t}\setminus\overline{N}_\epsilon^{t+1}}\max\left\{\frac{|p_i^\star - p_i^t|}{b_i+c_i}-\epsilon, 0\right\} \\
&= \kappa_\epsilon(\mathbf{p}^t) - \sum_{i\in\overline{N}_\epsilon^{t}\setminus\overline{N}_\epsilon^{t+1}}\max\left\{\frac{|p_i^\star - p_i^t|}{b_i+c_i}-\epsilon, 0\right\},
\end{aligned} \tag{D.12}$$

where equality (Δ) follows from the definition of $\widehat{\mathbf{p}}^t$. Since $i \in \overline{N}_\epsilon^t \setminus \overline{N}_\epsilon^{t+1}$ means that $|p_i^\star - p_i^t|/(b_i + c_i) \geq \epsilon$ and $\left|p_i^\star - p_i^{t+1}\right|/(b_i + c_i) < \epsilon$, we deduce from Eq. (D.7) that $|p_i^\star - p_i^t|/(b_i + c_i) \leq \epsilon + \eta^t G_i(\mathbf{p}^t, \mathbf{r}^t) \leq 2\epsilon$. Thus, Eq. (D.12) further implies that

$$\kappa(\widehat{\mathbf{p}}^t) \geq \kappa_\epsilon(\mathbf{p}^t) - \sum_{i \in \overline{N}_\epsilon^t \setminus \overline{N}_\epsilon^{t+1}} M_G \eta^t \geq \kappa_\epsilon(\mathbf{p}^t) - n\epsilon. \tag{D.13}$$

We use the shorthand notation $\lambda := 1/\left(\overline{p}\,\|\mathbf{p}^\star\|_\infty\right)$. Then, by substituting Eq. (D.11) and Eq. (D.13) back into Eq. (D.10), we derive that

$$\begin{aligned}\sum_{i \in \overline{N}_\epsilon^{t+1}} \mathtt{sign}\,(p_i^\star - p_i^t)\, G_i(\mathbf{p}^t, \mathbf{r}^t) &\geq \lambda\left(\kappa_\epsilon(\mathbf{p}^t) - n\epsilon\right) - \sum_{i \in N} \left[\ell_{r,i} \left\|\mathbf{p}^t - \mathbf{r}^t\right\| + \ell_{d,i} \left\|\mathbf{p}^t - \widehat{\mathbf{p}}^t\right\|\right] \\ &\overset{(\Delta)}{\geq} \lambda \kappa_\epsilon(\mathbf{p}^t) - \left(n\lambda + \sum_{i \in N} \ell_{r,i} + 2\sqrt{\sum_{i \in N}(b_i + c_i)^2} \cdot \sum_{i \in N} \ell_{d,i}\right)\epsilon,\end{aligned} \tag{D.14}$$

where inequality (Δ) is due to $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \epsilon$ and

$$\left\|\mathbf{p}^t - \widehat{\mathbf{p}}^t\right\| = \sqrt{\sum_{i \in N_\epsilon^{t+1}} \left(p_i^\star - p_i^t\right)^2} \leq \sqrt{\sum_{i \in N_\epsilon^{t+1}} \left[2(b_i + c_i)\epsilon\right]^2} \leq 2\epsilon \sqrt{\sum_{i \in N}(b_i + c_i)^2}. \tag{D.15}$$

Let $C_\kappa := n\lambda + \sum_{i \in N} \ell_{r,i} + 2\sqrt{\sum_{i \in N}(b_i + c_i)^2} \cdot \sum_{i \in N} \ell_{d,i}$. By combining Eq. (D.9) with Eq. (D.14), we have that

$$\kappa_\epsilon(\mathbf{p}^{t+1}) \leq (1 - \lambda\eta^t)\kappa_\epsilon(\mathbf{p}^t) + C_\kappa \epsilon \eta^t, \quad \forall t \geq T_\epsilon. \tag{D.16}$$

Therefore, by unrolling the recursion in Eq. (D.16), we deduce that for all $t > T_\epsilon$

$$\begin{aligned}\kappa_\epsilon(\mathbf{p}^t) &\leq \kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \prod_{t'=T_\epsilon}^{t-1} (1 - \lambda\eta^{t'}) + C_\kappa \epsilon \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1} (1 - \lambda\eta^{t''}) \\ &\overset{(\Delta)}{\leq} \kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + C_\kappa \epsilon \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1} (1 - \lambda\eta^{t''}),\end{aligned} \tag{D.17}$$

where we apply the elementary inequality $1 - x \leq \exp(-x)$ in (Δ). We remark that $\prod_{t''=t}^{t-1}(1 - \lambda\eta^{t''}) = 1$ by default. Since $\sum_{t=0}^{\infty} \eta^t = \infty$, the exponential term in Eq. (D.17) clearly converges to zero as $t \to \infty$. Hence, the key is the second term, denoted as $X^{t-1} := \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1}(1 - \lambda\eta^{t''})$. For every $t > T_\epsilon$, we observe that $X^t = \eta^t + (1 - \lambda\eta^t)X^{t-1}$, and thereby

$$\begin{aligned}X^t - \frac{1}{\lambda} &= \eta^t + (1 - \lambda\eta^t)X^{t-1} - \frac{1}{\lambda} \\ &= (1 - \lambda\eta^t)\left(X^{t-1} - \frac{1}{\lambda}\right) \\ &= \left(X^{T_\epsilon} - \frac{1}{\lambda}\right) \prod_{t'=T_\epsilon+1}^{t} (1 - \lambda\eta^{t'}),\end{aligned} \tag{D.18}$$

which implies $X^t - 1/\lambda$ always has the same sign as $X^{T_\epsilon} - 1/\lambda$ and converges to zero as $t \to \infty$. Thus, together with the relation $\kappa(\mathbf{p}) \le \kappa_\epsilon(\mathbf{p}) + n\epsilon$ in Eq. (D.1), Eqs. (D.17) and (D.18) imply that

$$\begin{aligned}\lim_{t\to\infty} \kappa(\mathbf{p}^t) &\le \lim_{t\to\infty} \kappa_\epsilon(\mathbf{p}^t) + n\epsilon \\ &\le \lim_{t\to\infty} \left[\kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + C_\kappa \epsilon X^{t-1}\right] + n\epsilon \\ &= \left(\frac{C_\kappa}{\lambda} + n\right)\epsilon.\end{aligned} \tag{D.19}$$

Since ϵ can take any non-negative value, and both C_κ and λ depend only on the problem parameters, we conclude that $\lim_{t\to\infty} \kappa(\mathbf{p}^t) = 0$. Therefore, both the price and reference paths converge to $\mathbf{p}^\star$, which completes the proof of Theorem 1. □

Appendix E Proof of Theorem 2

Theorem 2 **(Restated)**. *In the loss-neutral scenario, suppose all firms adopt Algorithm 1 with step-sizes* $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ *for* $t \ge 2$. *Then, there exist constants* T_1, C_p, *and* C_r *such that when* $C_\eta > 2\overline{p}^2/\log 2$, *it holds for all* $t > \max\{2T_1, 10\}$ *that*

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \le C_p \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right), \quad \left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 \le C_r \left(\frac{\log t}{t}\right)^2 = \widetilde{\mathcal{O}}\left(\frac{1}{t^2}\right), \tag{E.1}$$

where constants T_1, C_p, *and* C_r *are explicitly defined in Table EC.1.*

Proof of Theorem 2. We note that the choice of $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t \ge 2$ satisfies the step-size condition specified in Theorem 1. Hence, all analyses in the proof of Theorem 1 are applicable. We first prove the convergence rate in terms of the metric $\kappa(\mathbf{p})$ defined in Eq. (11).

By Lemma EC.2, we have $\|\mathbf{p}^t - \mathbf{r}^t\| \le \eta^t C_{rp}$ for all $t \ge T_1$, where the constants C_{rp} and T_1 are defined in Eq. (L.1). Hence, for every $\epsilon > 0$, we can take T_ϵ as the smallest integer greater than T_1 such that $\epsilon > \max\{\eta^{T_\epsilon} M_G, \eta^{T_\epsilon} C_{rp}\}$, so that we have $\eta^t M_G < \epsilon$ and $\|\mathbf{p}^t - \mathbf{r}^t\| < \epsilon$ for every $t \ge T_\epsilon$. Equivalently, T_ϵ is the smallest integer such that

$$\frac{\epsilon}{C_\eta \widehat{C}_{rp}} \ge \frac{\log(T_\epsilon + 1)}{T_\epsilon + 1} \text{ and } T_\epsilon \ge T_1, \tag{E.2}$$

where $\widehat{C}_{rp} := \max\{C_{rp}, M_G\}$. For every $t > T_\epsilon$, we have from Eq. (D.17) that

$$\kappa(\mathbf{p}^t) \le \kappa_\epsilon(\mathbf{p}^t) + n\epsilon \le \kappa_\epsilon(\mathbf{p}^{T_\epsilon}) \exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) + C_\kappa \epsilon X^{t-1} + n\epsilon, \tag{E.3}$$

where we recall that $X^{t-1} = \sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \prod_{t''=t'+1}^{t-1} (1 - \lambda \eta^{t''})$. From the recursion in Eq. (D.18), we observe that if $X^{T_\epsilon} < 1/\lambda$, then $X^t \le 1/\lambda$ for every $t > T_\epsilon$. Otherwise, it still holds that

$$X^{t-1} = \frac{1}{\lambda} + \left(X^{T_\epsilon} - \frac{1}{\lambda}\right) \prod_{t'=T_\epsilon+1}^{t-1} (1 - \lambda \eta^{t'}) \le \frac{1}{\lambda} + \eta^{T_\epsilon} \le \frac{1}{\lambda} + C_\eta < 2C_\eta, \quad \forall t > T_\epsilon, \tag{E.4}$$

where the first inequality is because $X^{T_\epsilon} = \eta^{T_\epsilon}$ by definition, and the last inequality is due to the choice $C_\eta > 2\overline{p}^2/\log 2 > 2/(\lambda \log 2)$, as $\lambda = 1/\left(\overline{p}\,\|\mathbf{p}^\star\|_\infty\right)$. Thus, it remains to control the exponential term on the right-hand side of Eq. (E.3). Using the integration lower bound, we deduce that

$$\sum_{t'=T_\epsilon}^{t-1} \eta^{t'} \geq C_\eta \int_{T_\epsilon}^{t} \frac{\log(t'+1)}{t'+1} dt' = \frac{C_\eta}{2}\left[\log^2(t+1) - \log^2(T_\epsilon+1)\right]. \tag{E.5}$$

Therefore, it follows that for any $t > T_\epsilon$

$$\begin{aligned}
\exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) &\leq \exp\left(-\frac{\lambda C_\eta}{2}\left[\log^2(t+1) - \log^2(T_\epsilon+1)\right]\right) \\
&= \left[\frac{\exp\left(\log^2(T_\epsilon+1)\right)}{\exp\left(\log^2(t+1)\right)}\right]^{\frac{\lambda C_\eta}{2}} \\
&= \left[\frac{(T_\epsilon+1)^{\log(T_\epsilon+1)}}{(t+1)^{\log(t+1)}}\right]^{\frac{\lambda C_\eta}{2}} \\
&= \left(\frac{T_\epsilon+1}{t+1}\right)^{\frac{\lambda C_\eta \log(T_\epsilon+1)}{2}} \cdot \left(\frac{1}{t+1}\right)^{\frac{\lambda C_\eta}{2}\cdot \log\left(\frac{t+1}{T_\epsilon+1}\right)}.
\end{aligned} \tag{E.6}$$

Hence, when $t \geq 2T_\epsilon + 1$, it holds that

$$\begin{aligned}
\exp\left(-\lambda \sum_{t'=T_\epsilon}^{t-1} \eta^{t'}\right) \leq \left(\frac{1}{t+1}\right)^{\frac{\lambda C_\eta}{2}\cdot\log 2} \overset{(\Delta_1)}{\leq} \frac{1}{t+1} &\overset{(\Delta_2)}{\leq} \frac{1}{2(T_\epsilon+1)} \cdot \frac{C_{rp} C_\eta \log(T_\epsilon+1)}{\sqrt{n}(\overline{p} - \underline{p})} \\
&\leq \frac{\epsilon}{2\sqrt{n}(\overline{p}-\underline{p})},
\end{aligned} \tag{E.7}$$

where step (Δ_1) is due to $C_\eta > 2\overline{p}^2/\log 2 > 2/(\lambda \log 2)$, and step (Δ_2) uses the premise that $t \geq 2T_\epsilon + 1$. We note that by the definition of C_{rp} in Eq. (L.1), it is easy to observe that the second fraction in (Δ_2) is clearly greater than one. Finally, the last inequality in Eq. (E.7) applies the definition of T_ϵ in Eq. (E.2). Together, Eqs. (E.3), (E.4), and (E.7) imply that

$$\kappa(\mathbf{p}^t) \leq \left(2C_\kappa C_\eta + n + \frac{\kappa_\epsilon(\mathbf{p}^{T_\epsilon})}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon \leq \left(2C_\kappa C_\eta + n + \frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon, \tag{E.8}$$

where we replace $\kappa_\epsilon(\mathbf{p}^{T_\epsilon})$ by its universal upper bound $M_\kappa := \sum_{i\in N}(\overline{p}-\underline{p})/(b_i+c_i)$. By far, we have shown that $\kappa(\mathbf{p}^t) = \mathcal{O}(\epsilon)$ when $t \geq 2T_\epsilon + 1$. To obtain the convergence rate that explicitly depends on t, we consider the definition of T_ϵ in Eq. (E.2). We claim that it suffices to choose

$$T_\epsilon = \max\left\{T_1, \left\lceil \frac{2C_\eta \widehat{C}_{rp}}{\epsilon} \log\left(\frac{C_\eta \widehat{C}_{rp}}{\epsilon}\right) - 1\right\rceil\right\}. \tag{E.9}$$

To validate that such a choice satisfies the condition in Eq. (E.2), we compute that

$$\frac{\log(T_\epsilon+1)}{T_\epsilon+1} \leq \frac{\log\left(2C_\eta \widehat{C}_{rp}/\epsilon\right) + \log\left(\log\left(C_\eta \widehat{C}_{rp}/\epsilon\right)\right)}{\left(2C_\eta \widehat{C}_{rp}/\epsilon\right)\cdot \log\left(C_\eta \widehat{C}_{rp}/\epsilon\right)} \overset{(\Delta)}{<} \frac{\epsilon}{C_\eta \widehat{C}_{rp}}, \tag{E.10}$$

where step (Δ) uses the inequality $\max_{x>0}\big(\log 2x+\log(\log x)\big)/(2\log x)=1/2+1/e<1$. Thus, Eq. (E.8) holds true as long as

$$t\geq 2T_\epsilon+1\geq \max\left\{2T_1+1,\left\lceil\frac{4C_\eta\widehat{C}_{rp}}{\epsilon}\log\left(\frac{C_\eta\widehat{C}_{rp}}{\epsilon}\right)\right\rceil\right\}. \tag{E.11}$$

We observe the following equivalence

$$t=\left(4C_\eta\widehat{C}_{rp}/\epsilon\right)\cdot\log\left(C_\eta\widehat{C}_{rp}/\epsilon\right)\quad\Leftrightarrow\quad \log\left(C_\eta\widehat{C}_{rp}/\epsilon\right)=W(t/4), \tag{E.12}$$

where $W(\cdot)$ is the Lambert W function defined in Eq. (C.25). Using the lower bound of the Lambert function $W(x)\geq\log x-\log(\log x)$ for all $x\geq e$, we find that for all $t\geq 4e\approx 10.9$

$$\log\left(\frac{C_\eta\widehat{C}_{rp}}{\epsilon}\right)\geq\log\left(\frac{t}{4}\right)-\log\left(\log\left(\frac{t}{4}\right)\right), \tag{E.13}$$

which is equivalent to

$$\epsilon\leq\frac{C_\eta\widehat{C}_{rp}\log(t/4)}{t/4}\leq\frac{4C_\eta\widehat{C}_{rp}\log t}{t}. \tag{E.14}$$

Together with requirements $t\geq 2T_1+1$ and $t\geq 11$, we conclude from the bound in Eq. (E.8) that

$$\kappa(\mathbf{p}^t)\leq\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon\leq\widehat{C}_\kappa\frac{\log t}{t},\quad\forall t>\{2T_1,10\}, \tag{E.15}$$

where we define $\widehat{C}_\kappa:=4C_\eta\widehat{C}_{rp}\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)$. Finally, to obtain the upper bounds in Eq. (13), we observe that for all $t>\{2T_1,10\}$

$$\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2\leq\left(\sum_{i\in N}\max_{k\in N}\{b_k+c_k\}\cdot\frac{|p_i^\star-p_i^t|}{b_i+c_i}\right)^2\leq\max_{i\in N}\left\{(b_i+c_i)^2\right\}\cdot[\kappa(\mathbf{p}^t)]^2\leq C_p\left(\frac{\log t}{t}\right)^2, \tag{E.16}$$

where $C_p:=\max_{i\in N}\{(b_i+c_i)^2\}\cdot(\widehat{C}_\kappa)^2$. For the reference price path, we can similarly deduce that

$$\begin{aligned}\left\|\mathbf{p}^\star-\mathbf{r}^t\right\|^2&=\left\|\mathbf{p}^\star-\mathbf{p}^t+\mathbf{p}^t-\mathbf{r}^t\right\|^2\\&\leq 2\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2+2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2\\&\leq 2C_p\left(\frac{\log t}{t}\right)^2+2\left(\eta^tC_{rp}\right)^2\\&\leq 2\big(C_p+(C_\eta C_{rp})^2\big)\cdot\left(\frac{\log t}{t}\right)^2.\end{aligned} \tag{E.17}$$

The proof of Theorem 2 is completed by letting $C_r:=2\big(C_p+(C_\eta C_{rp})^2\big)$. □

Appendix F Proof of Theorem 3

THEOREM 3 **(Restated)**. *In the loss-neutral scenario, if all firms adopt Algorithm 1 with step-sizes* $\{\eta^t\}_{t\geq 0}$ *satisfying* $\lim_{t\to\infty}\eta^t = 0$ *and* $\sum_{t=0}^{\infty}\eta^t = \infty$*, the dynamic regret of each firm grows in a sublinear rate, i.e.,*

$$\lim_{T\to\infty}\frac{1}{T}\times \text{D-Regret}_i(T) = 0, \quad \forall i\in N. \tag{F.1}$$

Furthermore, if the step-sizes are specified as $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ *for* $t\geq 2$*, there exist constants* T_1 *and* $C_{R,i}$ *such that when* $C_\eta > 2\overline{p}^2/\log 2$*, it holds that*

$$\text{D-Regret}_i(T) \leq \overline{p}\cdot\max\{2T_1, 10\} + 2C_{R,i} = \mathcal{O}(1), \quad \forall T\geq 1,\ \forall i\in N, \tag{F.2}$$

where constants T_1 *and* $C_{R,i}$ *are explicitly defined in Table EC.1.*

Proof of Theorem 3. We begin by demonstrating an auxiliary result on the smoothness of revenue function $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i, which would be useful in the following proof. Note that

$$\begin{aligned}\frac{\partial \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i} &= d_i(\mathbf{p},\mathbf{r}) - (b_i+c_i)p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\\ &= d_i(\mathbf{p},\mathbf{r})\cdot\big[1-(b_i+c_i)p_i\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\big].\end{aligned} \tag{F.3}$$

Then, the second-order derivative of $\Pi_i(\mathbf{p},\mathbf{r})$ with respect to p_i can be computed as follows

$$\begin{aligned}\frac{\partial^2 \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i^2} =& -(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\big[1-(b_i+c_i)p_i\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\big]\\ &+ d_i(\mathbf{p},\mathbf{r})\big[-(b_i+c_i)\big(1-d_i(\mathbf{p},\mathbf{r})\big)-(b_i+c_i)^2p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\big]\\ =& -2(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)+(b_i+c_i)^2p_i\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)^2\\ &-(b_i+c_i)^2p_i\cdot\big(d_i(\mathbf{p},\mathbf{r})\big)^2\big(1-d_i(\mathbf{p},\mathbf{r})\big)\\ =&(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\big(1-d_i(\mathbf{p},\mathbf{r})\big)\cdot\big[-2+(b_i+c_i)p_i\cdot\big(1-2d_i(\mathbf{p},\mathbf{r})\big)\big],\end{aligned} \tag{F.4}$$

and thereby this second-order derivative can be bounded as

$$\left|\frac{\partial^2 \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i^2}\right| \leq \frac{1}{4}(b_i+c_i)\big(2+(b_i+c_i)\overline{p}\big) =: h_i, \quad \forall \mathbf{p}\in\mathcal{P}^n,\ \forall \mathbf{r}\in\mathcal{P}^n. \tag{F.5}$$

Hence, we have that $\Pi_i(\mathbf{p},\mathbf{r})$ is h_i-smooth with respect to p_i.

Now, we proceed to prove the theorem. For brevity, we denote the regret of firm i at period t as R_i^t, i.e., $R_i^t := \max_{p_i\in\mathcal{P}}\big\{\Pi_i\big((p_i,\mathbf{p}_{-i}^t),\mathbf{r}^t\big)\big\} - \Pi_i(\mathbf{p}^t,\mathbf{r}^t)$, and therefore the total regret over the entire T periods can be expressed as $\text{D-Regret}_i(T) = \sum_{t=1}^T R_i^t$. Let $p_i^B(\cdot,\cdot)$ be a function defined as $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big) := \arg\max_{p_i\in\mathcal{P}}\big\{\Pi_i\big((p_i,\mathbf{p}_{-i}),\mathbf{r}\big)\big\}$, where $\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})$ is the vector of utilities for all firms other than i, as defined in Eq. (L.41). The function $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ represents the best-response price for firm i that achieve the optimal single-period revenue, given the reference price $\mathbf{r}$ and the price of other products $\mathbf{p}_{-i}$.

We observe that R_i^t can be upper-bounded as follows

$$\begin{aligned}
R_i^t &= \max_{p_i \in \mathcal{P}} \left\{ \Pi_i\big((p_i, \mathbf{p}_{-i}^t), \mathbf{r}^t\big) \right\} - \Pi_i(\mathbf{p}^t, \mathbf{r}^t) \\
&= \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big) - \Pi_i\left(\mathbf{p}^t, \mathbf{r}^t\right) \\
&\overset{(\Delta_1)}{\leq} \frac{\partial \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big)}{\partial p_i} \cdot \left(p_i^t - p_i^{B,t}\right) + \frac{h_i}{2}\left(p_i^{B,t} - p_i^t\right)^2 \\
&\overset{(\Delta_2)}{\leq} \frac{h_i}{2}\left(p_i^{B,t} - p_i^t\right)^2,
\end{aligned} \tag{F.6}$$

where we use the shorthand notation $p_i^{B,t} := p_i^B\big(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\big)$ to denote the best-response price for firm i at period t. The step (Δ_1) utilizes the h_i-smoothness of $\Pi_i(\mathbf{p}, \mathbf{r})$ with respect to p_i, as shown in Eq. (F.5). In step (Δ_2), since $p_i^{B,t}$ is the best-response price, it holds that $\partial \Pi_i\big((p_i^{B,t}, \mathbf{p}_{-i}^t), \mathbf{r}^t\big)/\partial p_i = 0$ by the first-order condition.

In the following part, we evaluate term $\left(p_i^{B,t} - p_i^t\right)^2$ in Eq. (F.6), which can be decomposed as

$$\left(p_i^{B,t} - p_i^t\right)^2 = \left(p_i^{B,t} - p_i^\star + p_i^\star - p_i^t\right)^2 \leq 2\left(p_i^{B,t} - p_i^\star\right)^2 + 2\left(p_i^\star - p_i^t\right)^2, \tag{F.7}$$

where the last step is due to the basic inequality $(x+y)^2 \leq 2x^2 + 2y^2$. Since the reference price at SNE is also equal to $\mathbf{p}^\star$, we can further upper-bound the first term on the right-hand side of Eq. (F.7) as follows

$$\begin{aligned}
\left(p_i^{B,t} - p_i^\star\right)^2 &= \Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&= \Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) \\
&\qquad + p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&\leq 2\Big[p_i^B\left(r_i^t, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right)\Big]^2 \\
&\qquad + 2\Big[p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t)\right) - p_i^B\left(p_i^\star, \mathbf{u}_{-i}(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star)\right)\Big]^2 \\
&\overset{(\Delta_1)}{\leq} 2\Big(\frac{c_i}{b_i + c_i}\Big)^2 (r_i^t - p_i^\star)^2 + 2\overline{p}^2 \left\|\mathbf{u}_{-i}\big(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t\big) - \mathbf{u}_{-i}\big(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star\big)\right\|^2 \\
&\overset{(\Delta_2)}{\leq} 2\Big(\frac{c_i}{b_i + c_i}\Big)^2 (r_i^t - p_i^\star)^2 + 4\overline{p}^2 \Big(\max_{j \neq i}\left\{(b_j + c_j)^2\right\} \left\|\mathbf{p}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 + \max_{j \neq i}\left\{c_j^2\right\} \left\|\mathbf{r}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2\Big),
\end{aligned} \tag{F.8}$$

where in step (Δ_1), we use the $c_i/(b_i + c_i)$-Lipschitz continuity of $p_i^B\big(\cdot, \mathbf{u}_{-i}(\mathbf{p}_{-i}, \mathbf{r}_{-i})\big)$ and the $\overline{p}$-Lipschitz continuity of $p_i^B(r_i, \cdot)$ from Eq. (L.43) in Lemma EC.6. In step (Δ_2), by the definition of utility in Eq. (1), for all $j \in N \backslash \{i\}$, it holds that $u_j(p_j^t, r_j^t) - u_j(p_j^\star, p_j^\star) = -(b_j + c_j) \cdot (p_j^t - p_j^\star) + c_j \cdot (r_j^t - p_j^\star)$. Therefore, it holds that

$$\left\|\mathbf{u}_{-i}\big(\mathbf{p}_{-i}^t, \mathbf{r}_{-i}^t\big) - \mathbf{u}_{-i}\big(\mathbf{p}_{-i}^\star, \mathbf{p}_{-i}^\star\big)\right\|^2 \leq 2\max_{j \neq i}\{(b_j + c_j)^2\} \left\|\mathbf{p}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2 + 2\max_{j \neq i}\{c_j^2\} \left\|\mathbf{r}_{-i}^t - \mathbf{p}_{-i}^\star\right\|^2.$$

Combining Eqs. (F.6) and (F.7), we have that $R_i^t \le h_i\left(p_i^{B,t}-p_i^\star\right)^2 + h_i\left(p_i^\star - p_i^t\right)^2$, where recall that h_i is a constant defined in Eq. (F.5). Then, substituting the term $\left(p_i^{B,t}-p_i^\star\right)^2$ with its bound in Eq. (F.8), R_i^t evolves as

$$\begin{aligned} R_i^t \le & \frac{2h_i c_i^2\cdot(r_i^t-p_i^\star)^2}{(b_i+c_i)^2} + 4h_i\overline{p}^2\Big(\max_{j\neq i}\left\{(b_j+c_j)^2\right\}\left\|\mathbf{p}_{-i}^t-\mathbf{p}_{-i}^\star\right\|^2 + \max_{j\neq i}\left\{c_j^2\right\}\left\|\mathbf{r}_{-i}^t-\mathbf{p}_{-i}^\star\right\|^2\Big) + h_i\left(p_i^\star-p_i^t\right)^2 \\ \le & 2h_i\cdot\max\left\{\left(\frac{c_i}{b_i+c_i}\right)^2,\ 2\overline{p}^2\max_{j\neq i}\left\{c_j^2\right\}\right\}\left\|\mathbf{p}^\star-\mathbf{r}^t\right\|^2 + h_i\cdot\max\left\{4\overline{p}^2\max_{j\neq i}\left\{(b_j+c_j)^2\right\},1\right\}\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2, \end{aligned} \tag{F.9}$$

where we apply the basic inequality $k_1x^2+k_2y^2\le\max\{k_1,k_2\}(x^2+y^2)$ in the last step.

When the step-sizes $\{\eta^t\}_{t\ge0}$ are non-increasing with $\lim_{t\to\infty}\eta^t=0$ and $\sum_{t=0}^\infty\eta^t=\infty$, we have from Theorem 1 that $\mathbf{p}^t\to\mathbf{p}^\star$ and $\mathbf{r}^t\to\mathbf{p}^\star$. Hence, the dynamic regret grows in a sublinear rate, which completes the proof of Eq. (15).

When the step-sizes are specified as $\eta^t=\frac{C_\eta\log(t+1)}{t+1}$ for $t\ge2$, we can further quantify the regret using the convergence rate in Theorem 2. In the case of $t>\widehat{T}_1$, where $\widehat{T}_1:=\max\{2T_1,10\}$ and T_1 can be found in Table EC.1, we can bound terms $\|\mathbf{p}^\star-\mathbf{p}^t\|$ and $\|\mathbf{p}^\star-\mathbf{r}^t\|$ by Eq. (13). Then, Eq. (F.9) becomes that for all $t>\widehat{T}_1$,

$$R_i^t\le\left[2h_iC_r\cdot\max\left\{\left(\frac{c_i}{b_i+c_i}\right)^2,\ 2\overline{p}^2\max_{j\neq i}\left\{c_j^2\right\}\right\}+h_iC_p\cdot\max\left\{4\overline{p}^2\max_{j\neq i}\left\{(b_j+c_j)^2\right\},1\right\}\right]\left(\frac{\log t}{t}\right)^2. \tag{F.10}$$

We use $C_{R,i}$ to denote the multiple of $(\log t/t)^2$ in Eq. (F.10), i.e.,

$$C_{R,i}:=2h_iC_r\cdot\max\left\{\left(\frac{c_i}{b_i+c_i}\right)^2,\ 2\overline{p}^2\max_{j\neq i}\left\{c_j^2\right\}\right\}+h_iC_p\cdot\max\left\{4\overline{p}^2\max_{j\neq i}\left\{(b_j+c_j)^2\right\},1\right\}. \tag{F.11}$$

In the case of $t\le\widehat{T}_1$, we use the plain bound on R_i^t, i.e.,

$$R_i^t\le\max_{p_i\in\mathcal{P}}\left\{\Pi_i\big((p_i,\mathbf{p}_{-i}^t),\mathbf{r}^t\big)\right\}\le\overline{p}. \tag{F.12}$$

Finally, combining Eqs. (F.10) and (F.12), we are ready to derive the regret bound as follows

$$\begin{aligned} \text{D-Regret}_i(T) &= \sum_{t=1}^{\widehat{T}_1}R_i^t+\sum_{t=\widehat{T}_1+1}^{T}R_i^t \\ &\le\overline{p}\cdot\widehat{T}_1+C_{R,i}\sum_{t=\widehat{T}_1+1}^{T}\left(\frac{\log t}{t}\right)^2 \\ &\le\overline{p}\cdot\widehat{T}_1+C_{R,i}\int_1^\infty\left(\frac{\log t}{t}\right)^2dt \\ &=\overline{p}\cdot\widehat{T}_1+2C_{R,i},\qquad\forall T\ge\widehat{T}_1,\ \forall i\in N. \end{aligned} \tag{F.13}$$

Since Eq. (F.13) has already upper-bounded the regrets during the first $\widehat{T}_1=\max\{2T_1,10\}$ periods, this upper bound also holds for all $1\le T<\widehat{T}_1$, which completes the proof of Theorem 3. □

Appendix G Proof of Theorem 4

THEOREM 4 **(Restated)**. *In the loss-averse scenario, let the step-sizes* $\{\eta^t\}_{t\geq 0}$ *be a non-increasing sequence such that* $\lim_{t\to\infty}\eta^t = 0$ *and* $\sum_{t=0}^{\infty}\eta^t = \infty$*. Then, for any reasonably small* $\epsilon > 0$*, the price and reference price paths generated by Algorithm 2 with the step-sizes* $\{\eta^t\}_{t\geq 0}$ *and threshold* ϵ *converge to a* $\tilde{C}_\kappa\epsilon$*-SNE, where constant* $\tilde{C}_\kappa$ *is explicitly defined in Table EC.2.*

Proof of Theorem 4. As we have mentioned, instead of directly working on $\widetilde{\kappa}(\mathbf{p},\mathbf{r})$, we will leverage the surrogate metric defined in Eq. (21), i.e., $\widetilde{\kappa}(\mathbf{p}) = \sum_{i\in N}\mathtt{dist}\big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p},\mathbf{p}),G_i^+(\mathbf{p},\mathbf{p})\big\}\big)$, where $G_i^-(\mathbf{p},\mathbf{r})$ and $G_i^+(\mathbf{p},\mathbf{r})$ are the scaled true/virtual derivatives defined as

$$G_i^-(\mathbf{p},\mathbf{r}) := \frac{1}{(b_i+c_i^-)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1, \quad G_i^+(\mathbf{p},\mathbf{r}) := \frac{1}{(b_i+c_i^+)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1. \tag{G.1}$$

In the loss-averse scenario, it is clear that $G_i^-(\mathbf{p},\mathbf{r}) \leq G_i^+(\mathbf{p},\mathbf{r})$ for all $i\in N$ with the equality only holds true if $c_i^- = c_i^+$, i.e., the consumer is loss-neutral towards this specific product. Below, we show that $\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t) = \mathcal{O}(\epsilon)$, where ϵ is the pre-specified threshold in Algorithm 2.

Since $D_i^-(\mathbf{p},\mathbf{r}) = (b_i+c_i^-)\cdot G_i^-(\mathbf{p},\mathbf{r})$ and $D_i^+(\mathbf{p},\mathbf{r}) = (b_i+c_i^+)\cdot G_i^+(\mathbf{p},\mathbf{r})$, the pausing criteria in line 5 of Algorithm 2 is equivalent to $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$. Hence, we can classify the relation between p_i^{t+1} and p_i^t into the following three possibilities:

- If $G_i^+(\mathbf{p}^t,\mathbf{r}^t) \geq G_i^-(\mathbf{p}^t,\mathbf{r}^t) \geq \epsilon/(b_i+c_i^-)$, then

$$p_i^{t+1} = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot(w_i^t\cdot D_i^{t,+} + (1-w_i^t)\cdot D_i^{t,-})\right) \geq \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot D_i^{t,-}\right) \geq p_i^t, \tag{G.2}$$

where we recall that $D_i^{t,+} = D_i^+(\mathbf{p}^t,\mathbf{r}^t) \geq D_i^-(\mathbf{p}^t,\mathbf{r}^t) = D_i^{t,-}$ by Eq. (18).

- If $G_i^-(\mathbf{p}^t,\mathbf{r}^t) \leq G_i^+(\mathbf{p}^t,\mathbf{r}^t) \leq -\epsilon/(b_i+c_i^+)$, then

$$p_i^{t+1} = \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot(w_i^t\cdot D_i^{t,+} + (1-w_i^t)\cdot D_i^{t,-})\right) \leq \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t\cdot D_i^{t,+}\right) \leq p_i^t. \tag{G.3}$$

- Otherwise, we must have $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$, and thereby the pausing criterion is triggered, i.e., $p_i^{t+1} = p_i^t$.

We remark that it is also possible for $p_i^{t+1} = p_i^t$ in the first two cases. This happens when p_i^t is on the boundary of $\mathcal{P} = [\underline{p},\overline{p}]$, and the price update is deprecated by the projection operation.

By Lemma EC.2, under the non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ with $\lim_{t\to\infty}\eta^t = 0$, there must exist T_ϵ such that

$$\max_{i\in N}\left\{(b_i+c_i^-)\cdot\tilde{\ell}_{r,i}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|,(b_i+c_i^-)\cdot\eta^t\tilde{\ell}_{p,i}\tilde{M}_G\sqrt{\sum_{k\in N}(b_k+c_k^-)^2}\right\} \leq \frac{\epsilon}{2}, \quad \forall t\geq T_\epsilon, \tag{G.4}$$

where the definitions of $\tilde{\ell}_{r,i}$, $\tilde{\ell}_{p,i}$, and $\tilde{M}_G$ can be found in Table EC.2. Now, for any $t\geq T_\epsilon$, consider the following separation of N:

$$N_-^t := \left\{i\in N | p_i^t < p_i^{t+1}\right\}, \quad N_+^t := \left\{i\in N | p_i^t > p_i^{t+1}\right\}, \quad N_c^t := N\setminus\left(N_-^t\cup N_+^t\right). \tag{G.5}$$

We claim that if N_-^t is not empty, then for any $i \in N_-^t$, it holds that $G_i^-(\mathbf{p}^t, \mathbf{p}^t) \geq \frac{\epsilon}{2(b_i + c_i^-)}$ and $G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) \geq 0$. Specifically, since $p_i^t < p_i^{t+1}$ for any $i \in N_-^t$, we have $G_i^-(\mathbf{p}^t, \mathbf{r}^t) \geq \epsilon/(b_i + c_i^-)$, and thereby

$$\begin{aligned} G_i^-(\mathbf{p}^t, \mathbf{p}^t) &= G_i^-(\mathbf{p}^t, \mathbf{r}^t) + \left[G_i^-(\mathbf{p}^t, \mathbf{p}^t) - G_i^-(\mathbf{p}^t, \mathbf{r}^t)\right] \\ &\geq \frac{\epsilon}{b_i + c_i^-} - \tilde{\ell}_{r,i} \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \\ &\geq \frac{\epsilon}{2(b_i + c_i^-)}, \end{aligned} \tag{G.6}$$

where we use the Lipschitz continuity of $G_i^-(\mathbf{p}, \cdot)$ from Lemma EC.7 and the choice of T_ϵ in Eq. (G.4). In addition, using the Lipschitz continuity of $G_i^-(\mathbf{p}, \mathbf{p})$ with respect to $\mathbf{p}$ from Lemma EC.7, we derive that

$$\begin{aligned} G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) &= G_i^-(\mathbf{p}^t, \mathbf{p}^t) + \left[G_i^-(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) - G_i^-(\mathbf{p}^t, \mathbf{p}^t)\right] \\ &\geq \frac{\epsilon}{2(b_i + c_i^-)} - \tilde{\ell}_{p,i} \left\|\mathbf{p}^{t+1} - \mathbf{p}^t\right\| \\ &\overset{(\Delta_1)}{\geq} \frac{\epsilon}{2(b_i + c_i^-)} - \eta^t \tilde{\ell}_{p,i} \sqrt{\sum_{k \in N} (D_k^t)^2} \\ &\overset{(\Delta_2)}{\geq} \frac{\epsilon}{2(b_i + c_i^-)} - \eta^t \tilde{\ell}_{p,i} \sqrt{\sum_{k \in N} (b_k + c_k^-)^2 (\tilde{M}_G)^2} \\ &\geq 0, \end{aligned} \tag{G.7}$$

where step (Δ_1) follows from the price update rule, and inequality (Δ_2) is because $|D_k^t| \leq \max\left\{(b_k + c_k^-)\left|G_k^-(\mathbf{p}^t, \mathbf{r}^t)\right|, (b_k + c_k^+)\left|G_k^+(\mathbf{p}^t, \mathbf{r}^t)\right|\right\} \leq (b_k + c_k^-)\tilde{M}_G$, using the upper bound $\left|G_i^-(\mathbf{p}, \mathbf{r})\right| \leq \tilde{M}_G$ from Lemma EC.7. Hence, combining Eqs. (G.6) and (G.7) with the fact $G_i^-(\mathbf{p}, \mathbf{r}) \leq G_i^+(\mathbf{p}, \mathbf{r})$, we have that

$$\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'})\big\}\Big) = G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), \quad \forall t' \in \{t, t+1\}, \ \forall i \in N_-^t. \tag{G.8}$$

Similarly, we can show that if $N_+^t \neq \emptyset$, then for any $i \in N_+^t$, it holds that $G_i^+(\mathbf{p}^t, \mathbf{p}^t) \leq -\frac{\epsilon}{2(b_i + c_i^+)}$ and $G_i^+(\mathbf{p}^{t+1}, \mathbf{p}^{t+1}) \leq 0$, and thus

$$\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t'}, \mathbf{p}^{t'}), G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'})\big\}\Big) = -G_i^+(\mathbf{p}^{t'}, \mathbf{p}^{t'}), \quad \forall t' \in \{t, t+1\}, \ \forall i \in N_+^t. \tag{G.9}$$

Next, we assess the improvement $\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t)$ by breaking down the summation over N using the definitions of N_-^t, N_+^t, and N_c^t in Eq. (G.5).

$$
\begin{aligned}
&\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t) \\
&= \sum_{i\in N} \Big[\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}), G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big) - \mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}), G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big] \\
&\overset{(\Delta)}{=} \sum_{i\in N_-^t} \big[G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - G_i^-(\mathbf{p}^{t},\mathbf{p}^{t})\big] + \sum_{i\in N_+^t} \big[G_i^+(\mathbf{p}^{t},\mathbf{p}^{t}) - G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big] \\
&\quad + \sum_{i\in N_c^t} \Big[\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}), G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big) - \mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}), G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big] \\
&= \sum_{i\in N_-^t} \left[\frac{1}{b_i + c_i^-}\left(\frac{1}{p_i^{t+1}} - \frac{1}{p_i^t}\right) + d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right] \\
&\quad + \sum_{i\in N_+^t} \left[\frac{1}{b_i + c_i^+}\left(\frac{1}{p_i^{t}} - \frac{1}{p_i^{t+1}}\right) + d_i(\mathbf{p}^{t},\mathbf{p}^{t}) - d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right] \\
&\quad + \sum_{i\in N_c^t} \Big[\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t+1},\mathbf{p}^{t+1}), G_i^+(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big\}\Big) - \mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p}^{t},\mathbf{p}^{t}), G_i^+(\mathbf{p}^{t},\mathbf{p}^{t})\big\}\Big)\Big],
\end{aligned} \tag{G.10}
$$

where step (Δ) uses the equalities in Eqs. (G.8) and (G.9). Although the right-hand side of Eq. (G.10) seems involved due to the presence of the summation over N_c^t, we make the following key observation: since $p_i^{t+1} = p_i^t$ for all $i \in N_c^t$, the difference between the two distance terms only arises from the change of the demand function (see the definitions in Eq. (G.1)). Based on the relative size of $d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})$ and $d_i(\mathbf{p}^t,\mathbf{p}^t)$ for $i \in N_c^t$, we enlarge sets N_-^t and N_+^t as follows:

$$
\begin{aligned}
\widehat{N}_-^t &:= N_-^t \cup \{i \in N_c^t | d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^t,\mathbf{p}^t) \geq 0\}, \\
\widehat{N}_+^t &:= N_+^t \cup \{i \in N_c^t | d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^t,\mathbf{p}^t) < 0\}.
\end{aligned} \tag{G.11}
$$

By definition, it is clear that $\widehat{N}_-^t \cup \widehat{N}_+^t = N_-^t \cup N_+^t \cup N_c^t = N$. Then, we can further deduce from Eq. (G.10) that

$$
\begin{aligned}
&\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t) \\
&\leq \sum_{i\in N_-^t} \left[\frac{1}{b_i + c_i^-}\left(\frac{1}{p_i^{t+1}} - \frac{1}{p_i^t}\right) + d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right] \\
&\quad + \sum_{i\in N_+^t} \left[\frac{1}{b_i + c_i^+}\left(\frac{1}{p_i^{t}} - \frac{1}{p_i^{t+1}}\right) + d_i(\mathbf{p}^{t},\mathbf{p}^{t}) - d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\right] + \sum_{i\in N_c^t} \left|d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^{t},\mathbf{p}^{t})\right| \\
&= \sum_{i\in N_-^t} \frac{1}{b_i + c_i^-}\left(\frac{1}{p_i^{t+1}} - \frac{1}{p_i^t}\right) + \sum_{i\in N_+^t} \frac{1}{b_i + c_i^+}\left(\frac{1}{p_i^{t}} - \frac{1}{p_i^{t+1}}\right) + \sum_{i\in \widehat{N}_-^t} \big[d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) - d_i(\mathbf{p}^{t},\mathbf{p}^{t})\big] \\
&\quad + \sum_{i\in \widehat{N}_+^t} \big[d_i(\mathbf{p}^{t},\mathbf{p}^{t}) - d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1})\big].
\end{aligned} \tag{G.12}
$$

Since $p_i^{t+1} \geq p_i^t$ for all $i \in \widehat{N}_-^t$ and $p_i^{t+1} \leq p_i^t$ for all $i \in \widehat{N}_+^t$, it holds that

$$\begin{aligned}\sum_{i\in\widehat{N}_-^t} d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) &= \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t+1},p_i^{t+1})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t+1},p_i^{t+1})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t+1},p_j^{t+1})\right)}\\ &\leq \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t+1},p_j^{t+1})\right)}\\ &\leq \frac{\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)}{1+\sum_{i\in\widehat{N}_-^t}\exp\left(u_i(p_i^{t},p_i^{t})\right)+\sum_{j\in\widehat{N}_+^t}\exp\left(u_j(p_j^{t},p_j^{t})\right)} = \sum_{i\in\widehat{N}_-^t} d_i(\mathbf{p}^t,\mathbf{p}^t),\end{aligned} \tag{G.13}$$

where we use the fact that $u_i(p_i,p_i) = a_i - b_i p_i$ is monotone decreasing p_i. Similarly, we have $\sum_{i\in\widehat{N}_+^t} d_i(\mathbf{p}^{t+1},\mathbf{p}^{t+1}) \geq \sum_{i\in\widehat{N}_+^t} d_i(\mathbf{p}^t,\mathbf{p}^t)$. These two relations, together with Eq. (G.12), imply that

$$\begin{aligned}\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t) &\leq \sum_{i\in N_-^t}\frac{1}{b_i+c_i^-}\left(\frac{1}{p_i^{t+1}}-\frac{1}{p_i^t}\right)+\sum_{i\in N_+^t}\frac{1}{b_i+c_i^+}\left(\frac{1}{p_i^t}-\frac{1}{p_i^{t+1}}\right)\\ &\overset{(\Delta)}{=} -\sum_{i\in N_-^t}\frac{1}{p_i^t\cdot p_i^{t+1}}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}-\sum_{i\in N_+^t}\frac{1}{p_i^t\cdot p_i^{t+1}}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\\ &\leq \frac{-1}{\overline{p}^2}\left(\sum_{i\in N_-^t}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}+\sum_{i\in N_+^t}\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right)\\ &\leq \frac{-1}{\overline{p}^2}\max\left\{\max_{i\in N_-^t}\left\{\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right\},\max_{i\in N_+^t}\left\{\frac{\left|p_i^{t+1}-p_i^t\right|}{b_i+c_i^-}\right\}\right\},\end{aligned} \tag{G.14}$$

where we apply the definitions of N_-^t and N_+^t from Eq. (G.5) in (Δ). It is worth noting that, when both N_-^t and N_+^t are empty sets, i.e., $N_c^t = N$, then Eq. (G.14) reduces to $\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t)=0$, since $N_c^t = N$ implies $\mathbf{p}^{t+1}=\mathbf{p}^t$. Therefore, we find that the sequence $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing.

We first consider the situation $N_c^t = N$ and demonstrate that when ϵ is reasonably small, $N_c^t = N$ implies that $\widetilde{\kappa}(\mathbf{p}^t) = \mathcal{O}(\epsilon)$. Note that $p_i^{t+1} = p_i^t$ can only happen for the following two reasons:

- The pausing criterion is triggered for product i, i.e., $G_i^+(\mathbf{p}^t,\mathbf{r}^t) > -\epsilon/(b_i+c_i^+)$ and $G_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon/(b_i+c_i^-)$, and thus no price update occurs.
- The price p_i^t is at the boundary of the feasible range $\mathcal{P}$, i.e., $p_i^t = \overline{p}$ or $p_i^t = \underline{p}$, and the price update is towards the outside direction, which is then deprecated by the projection operator.

Below, we show that for any $t \geq T_\epsilon$ and any reasonably small ϵ, the second scenario cannot happen when $\mathbf{p}^{t+1} = \mathbf{p}^t$. We argue from the reverse direction: if there exists $i_0 \in N$ with $p_{i_0}^t = \overline{p}$, then we must have $\mathbf{p}^{t+1} \neq \mathbf{p}^t$ (the case when $p_{i_0}^t = \underline{p}$ is equivalent). Denote $\xi_{i_0} := \min\left\{\overline{p} - p_{i_0}^\star | \mathbf{p}^\star \in \mathcal{S}\right\}$, i.e., the minimum distance between the SNE set $\mathcal{S}$ and the price upper bound in the i_0-th dimension. We consider the following new separation of N based on $\mathbf{p}^t$:

$$\tilde{N}_-^t := \left\{i\in N | G_i^-(\mathbf{p}^t,\mathbf{p}^t) > 0\right\}, \quad \tilde{N}_+^t := \left\{i\in N | G_i^+(\mathbf{p}^t,\mathbf{p}^t) < 0\right\}, \quad \tilde{N}_c^t = N\backslash\left(\tilde{N}_-^t \cup \tilde{N}_+^t\right). \tag{G.15}$$

Equivalently, we can write $\tilde{N}_c^t = \left\{i\in N | G_i^-(\mathbf{p}^t,\mathbf{p}^t)\le 0 \le G_i^+(\mathbf{p}^t,\mathbf{p}^t)\right\}$. Then, we define the pseudo sensitivities $(\tilde{c}_i)_{i\in N}$ as follows: for $i\in\tilde{N}_-^t$, let $\tilde{c}_i = c_i^-$; for $i\in\tilde{N}_+^t$, let $\tilde{c}_i = c_i^+$; for $i\in\tilde{N}_c^t$, let $\tilde{c}_i$ be the unique value that satisfies

$$\frac{1}{(b_i+\tilde{c}_i)p_i^t} + d_i(\mathbf{p}^t,\mathbf{p}^t) - 1 = 0. \tag{G.16}$$

By the definitions of scaled derivatives in Eq. (G.1), since $G_i^-(\mathbf{p}^t,\mathbf{p}^t)\le 0\le G_i^+(\mathbf{p}^t,\mathbf{p}^t)$ for all $i\in\tilde{N}_c^t$, such a $\tilde{c}_i$ must exist and $\tilde{c}_i\in[c_i^+,c_i^-]$. Given $(\tilde{c}_i)_{i\in N}$, let $\tilde{\mathbf{p}}^\star\in\mathcal{S}$ be the unique SNE that satisfies

$$\tilde{p}_i^\star = \frac{1}{(b_i+\tilde{c}_i)\cdot(1-d_i(\tilde{\mathbf{p}}^\star,\tilde{\mathbf{p}}^\star))},\quad \forall i\in N, \tag{G.17}$$

whose existence is guaranteed by the expression of $\mathcal{S}$ (see Eq. (6) and the proofs below Eq. (C.3)). Next, for every $i\in N$, we further introduce that

$$\tilde{G}_i(\mathbf{p},\mathbf{r}) := \frac{1}{(b_i+\tilde{c}_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1,\quad \tilde{\mathcal{G}}(\mathbf{p}) := \sum_{i\in N}\mathtt{sign}\left(\tilde{p}_i^\star - p_i\right)\tilde{G}_i(\mathbf{p},\mathbf{p}). \tag{G.18}$$

We note that, since $\tilde{c}_i\in[c_i^+,c_i^-]$, it always holds $G_i^-(\mathbf{p},\mathbf{r})\le\tilde{G}_i(\mathbf{p},\mathbf{r})\le G_i^+(\mathbf{p},\mathbf{r})$. By Lemma EC.8, $\tilde{\mathcal{G}}(\mathbf{p}^t)$ satisfies that

$$\tilde{\mathcal{G}}(\mathbf{p}^t) = \sum_{i\in N}\mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t,\mathbf{p}^t) \ge \frac{1}{\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty}\cdot\sum_{i\in N}\frac{|\tilde{p}_i^\star - p_i^t|}{b_i+\tilde{c}_i} \ge \frac{\xi_{i_0}}{(b_{i_0}+\tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty}, \tag{G.19}$$

where the last inequality holds because $\left|\tilde{p}_{i_0}^\star - p_{i_0}^t\right| = \left|\tilde{p}_{i_0}^\star - \overline{p}\right| \ge \min\left\{\overline{p} - p_{i_0}^\star | \mathbf{p}^\star\in\mathcal{S}\right\} = \xi_{i_0}$. Hence, by the definition of $\tilde{\mathcal{G}}(\mathbf{p})$, if

$$\xi_{i_0} \ge \frac{\left[n(b_{i_0}+\tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty\right]}{\min_{i\in N}\left\{b_i+c_i^+\right\}}\cdot\frac{3\epsilon}{2}, \tag{G.20}$$

we can deduce from the lower bound in Eq. (G.19) that

$$\max_{i\in N}\left\{\mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t,\mathbf{p}^t)\right\} \ge \frac{1}{n}\cdot\frac{\xi_{i_0}}{(b_{i_0}+\tilde{c}_{i_0})\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty} \ge \frac{3\epsilon}{2\min_{i\in N}\left\{b_i+c_i^+\right\}}. \tag{G.21}$$

Denote $i_1 := \arg\max_{i\in N}\left\{\mathtt{sign}\left(\tilde{p}_i^\star - p_i^t\right)\tilde{G}_i(\mathbf{p}^t,\mathbf{p}^t)\right\}$. Then, Eq. (G.21) implies that

$$\left|\tilde{G}_{i_1}(\mathbf{p}^t,\mathbf{p}^t)\right| \ge \frac{3\epsilon}{2\min_{i\in N}\left\{b_i+c_i^+\right\}} \ge \frac{3\epsilon}{2(b_{i_1}+c_{i_1}^+)}. \tag{G.22}$$

Since $\tilde{G}_i(\mathbf{p}^t,\mathbf{p}^t)=0$ for all $i\in\tilde{N}_c^t$ by Eq. (G.16), we must have $i_1\in\tilde{N}_-^t\cup\tilde{N}_+^t$. Now, we prove that $p_{i_1}^{t+1}\ne p_{i_1}^t$. Without loss of generality, suppose $i_1\in\tilde{N}_-^t$, i.e., $G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t)>0$. Then, by the definition of $(\tilde{c}_i)_{i\in N}$ above Eq. (G.16), it follows that $\tilde{G}_{i_1}(\mathbf{p}^t,\mathbf{p}^t) = G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t)>0$. Hence, we must have $\mathtt{sign}\left(\tilde{p}_{i_1}^\star - p_{i_1}^t\right)>0$, i.e., $p_{i_1}^t<\tilde{p}_{i_1}^\star$, which implies that $G_{i_1}^+(\mathbf{p}^t,\mathbf{p}^t)\ge G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t)\ge\frac{3\epsilon}{2(b_{i_1}+c_{i_1}^+)}$. In the meantime, since $t\ge T_\epsilon$, we deduce that

$$\begin{aligned}
G_{i_1}^-(\mathbf{p}^t,\mathbf{r}^t) &= G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t) + \left[G_{i_1}^-(\mathbf{p}^t,\mathbf{r}^t) - G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t)\right]\\
&\ge G_{i_1}^-(\mathbf{p}^t,\mathbf{p}^t) - \tilde{\ell}_{r,i_1}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\\
&\ge \frac{3\epsilon}{2(b_{i_1}+c_{i_1}^+)} - \frac{\epsilon}{2(b_{i_1}+c_{i_1}^-)}\\
&\ge \frac{\epsilon}{b_{i_1}+c_{i_1}^-},
\end{aligned} \tag{G.23}$$

where the inequalities follow from the Lipschitz continuity of $G_{i_1}^-(\mathbf{p},\cdot)$ in Lemma EC.7 and the definition of T_ϵ in Eq. (G.4). Therefore, we conclude that the update $p_{i_1}^{t+1} \leftarrow \texttt{Proj}_{\mathcal{P}}\left(p_{i_1}^t + \eta^t \cdot D_{i_1}^t\right)$ is towards the SNE price $\tilde{p}_{i_1}^\star$, i.e., $\texttt{sign}\left(\tilde{p}_{i_1}^\star - p_{i_1}^t\right) = \texttt{sign}\left(p_{i_1}^{t+1} - p_{i_1}^t\right)$, and thereby $p_{i_1}^{t+1} \neq p_i^t$.

Based on the arguments above, we can provide a sufficient condition for the size of ϵ such that $N_c^t = N$ always implies that the pausing criteria are triggered for all products. We define that

$$\xi_i = \min\left\{\overline{p} - p_i, p_i - \underline{p} | \mathbf{p} \in \mathcal{S}\right\}, \quad \forall i \in N, \tag{G.24}$$

which stands for the minimum distance from $\mathcal{S}$ to the boundaries of the feasible region $\mathcal{P}$ in the i-th dimension. Then, the derivations in Eqs. (G.19) to (G.23) imply that it is sufficient to have

$$\epsilon \leq \frac{2\min_{i\in N}\left\{b_i + c_i^+\right\}}{3n\overline{p}\cdot \max_{\mathbf{p}\in\mathcal{S}}\left\{\|\mathbf{p}\|_\infty\right\}} \cdot \min_{i\in N}\left\{\frac{\xi_i}{b_i + c_i^-}\right\}. \tag{G.25}$$

When ϵ satisfies Eq. (G.25), then if there exists any product $i_0 \in N$ with $p_{i_0}^t = \overline{p}$ or $p_{i_0}^t = \underline{p}$, we can always follow the derivations in Eqs. (G.19) to (G.23) to show that $\mathbf{p}^{t+1}$ is different from $\mathbf{p}^t$ for at least one product. We remark that according to Proposition 1, the value of $\max_{\mathbf{p}\in\mathcal{S}}\left\{\|\mathbf{p}\|_\infty\right\}$ has roughly linear (or inversely linear) dependence on problem parameters. Hence, even when $\overline{p}$ is excessively large, the condition in Eq. (G.25) would not be restrictive because $\lim_{\overline{p}\to\infty} \xi_i/\overline{p} = 1$.

By far, from Eqs. (G.15) to (G.25), we have demonstrated that for reasonably small ϵ and $t \geq T_\epsilon$, $N_c^t = N$ will happen only if the pausing criteria are triggered for all products. Below, suppose $N_c^{t_0} = N$ for some $t_0 \geq T_\epsilon$, we show that $\mathbf{p}^{t_0}$ is already close to the set of SNEs. By the definition of $\widetilde{\kappa}(\cdot)$ from Eq. (21), it follows that

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^{t_0}) &= \sum_{i\in N} \texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0},\mathbf{p}^{t_0}), G_i^+(\mathbf{p}^{t_0},\mathbf{p}^{t_0})\big\}\Big) \\
&\overset{(\Delta_1)}{\leq} \sum_{i\in N}\left[\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0},\mathbf{r}^{t_0}), G_i^+(\mathbf{p}^{t_0},\mathbf{r}^{t_0})\big\}\Big) + \frac{\epsilon}{2(b_i + c_i^-)}\right] \\
&\overset{(\Delta_2)}{\leq} \sum_{i\in N}\left(\frac{\epsilon}{b_i + c_i^+} + \frac{\epsilon}{2(b_i + c_i^-)}\right) \\
&\leq \left[\sum_{i\in N}\frac{3}{2(b_i + c_i^+)}\right]\epsilon,
\end{aligned} \tag{G.26}$$

where inequality (Δ_1) applies the Lipschitz properties from Lemma EC.7 and follows the same derivations as Eqs. (G.6) and (G.23). Step (Δ_2) leverages the presumption that pausing criteria are triggered for all products, i.e., $G_i^-(\mathbf{p}^{t_0},\mathbf{r}^{t_0}) < \epsilon/(b_i + c_i^-)$ and $G_i^+(\mathbf{p}^{t_0},\mathbf{r}^{t_0}) > -\epsilon/(b_i + c_i^+)$, and since $c_i^- \geq c_i^+$, we have

$$\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^{t_0},\mathbf{r}^{t_0}), G_i^+(\mathbf{p}^{t_0},\mathbf{r}^{t_0})\big\}\Big) \leq \frac{\epsilon}{b_i + c_i^+}. \tag{G.27}$$

Therefore, since $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing and $\lim_{t\to\infty}\|\mathbf{p}^t-\mathbf{r}^t\|=0$, we conclude that

$$\begin{aligned}
\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t,\mathbf{r}^t) &= \lim_{t\to\infty}\left[\|\mathbf{p}^t-\mathbf{r}^t\|+\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{D_i^-(\mathbf{p}^t,\mathbf{r}^t),D_i^+(\mathbf{p}^t,\mathbf{r}^t)\big\}\Big)\right]\\
&= \lim_{t\to\infty}\left[\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{D_i^-(\mathbf{p}^t,\mathbf{p}^t),D_i^+(\mathbf{p}^t,\mathbf{p}^t)\big\}\Big)\right]\\
&\leq \lim_{t\to\infty}\left[\max_{k\in N}\left\{b_k+c_k^-\right\}\cdot\sum_{i\in N}\mathtt{dist}\Big(0,\mathtt{Hull}\big\{G_i^-(\mathbf{p}^t,\mathbf{p}^t),G_i^+(\mathbf{p}^t,\mathbf{p}^t)\big\}\Big)\right]\\
&= \max_{k\in N}\left\{b_k+c_k^-\right\}\cdot\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t)\\
&\leq \underbrace{\left[\sum_{i\in N}\frac{3\max_{k\in N}\left\{b_k+c_k^-\right\}}{2(b_i+c_i^+)}\right]}_{=:\tilde{C}_\kappa}\epsilon,
\end{aligned}\tag{G.28}$$

where we use the fact that $D_i^\diamond(\mathbf{p},\mathbf{r})=(b_i+c_i^\diamond)G_i^\diamond(\mathbf{p},\mathbf{r})\leq\max_{k\in N}\left\{b_k+c_k^-\right\}\cdot G_i^\diamond(\mathbf{p},\mathbf{r})$ for every $\diamond\in\{+,-\}$ and all $i\in N$. Eq. (G.28) indicates that Algorithm 2 converges to a $\tilde{C}_\kappa\epsilon$-SNE, where we define $\tilde{C}_\kappa:=\sum_{i\in N}3\max_{k\in N}\left\{b_k+c_k^-\right\}/\left[2(b_i+c_i^+)\right]$.

Now, we show by contradiction that there always exists such a period $t_0\geq T_\epsilon$ with $N_c^{t_0}=N$. Suppose $N_-^t\cup N_+^t\neq\emptyset$ for all $t\geq T_\epsilon$. Then, Eq. (G.14) non-trivially holds true throughout the entire horizon of $t\geq T_\epsilon$. For the similar reasoning as Eqs. (G.19) to (G.23), we observe that for every $t\geq T_\epsilon$, there must exist a product $i^t\in N_-^t\cup N_+^t$ such that $\left|p_{i^t}^{t+1}-p_{i^t}^t\right|=\eta^t\left|D_{i^t}^t\right|$, i.e., the projection operation is not in effect. Without loss of generality, assume that $i^t\in N_-^t$. Then, we further deduce from Eq. (G.14) that

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^{t+1})-\widetilde{\kappa}(\mathbf{p}^t) &\leq \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t\left|D_{i^t}^t\right|}{b_{i^t}+c_{i^t}^-}\\
&= \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t\left(w_{i^t}^t\cdot D_{i^t}^{t,+}+(1-w_{i^t}^t)\cdot D_{i^t}^{t,-}\right)}{b_{i^t}+c_{i^t}^-}\\
&\leq \frac{-1}{\overline{p}^2}\cdot\frac{\eta^t D_{i^t}^{t,-}}{b_{i^t}+c_{i^t}^-}\\
&\leq \frac{-\eta^t\epsilon}{\overline{p}^2\cdot(b_{i^t}+c_{i^t}^-)},
\end{aligned}\tag{G.29}$$

where we use the fact that $D_i^{t,+}\geq D_i^{t,-}=(b_i+c_i^-)G_i^-(\mathbf{p}^t,\mathbf{r}^t)\geq\epsilon$ for any $i\in N_-^t$, due to the definition of N_-^t in Eq. (G.5). The same argument applies to the situation where $i^t\in N_+^t$. Thus, by applying a telescoping sum to Eq. (G.29), it follows that for any $t>T_\epsilon$

$$\widetilde{\kappa}(\mathbf{p}^t)\leq\widetilde{\kappa}(\mathbf{p}^{T_\epsilon})-\frac{\left(\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\right)\epsilon}{\overline{p}^2\cdot\max_{i\in N}\left\{b_i+c_i^-\right\}}.\tag{G.30}$$

Since the step-sizes satisfy $\sum_{t=0}^\infty\eta^t=\infty$, we deduce that $\lim_{t\to\infty}\widetilde{\kappa}(\mathbf{p}^t)=-\infty$, which contradicts with the definition of $\widetilde{\kappa}(\cdot)$. Therefore, there must exist $t_0\geq T_\epsilon$ such that $N_c^{t_0}=N$, which, together with Eq. (G.28), completes the proof of Theorem 4. $\square$

Appendix H Proof of Theorem 5

Theorem 5 **(Restated)**. *In the loss-averse scenario, suppose all firms adopt Algorithm 2 with step-sizes $\eta^t = \frac{C_\eta}{\sqrt{t+1}}$ and a reasonably small threshold ϵ, where C_η is some general constant. Then, there exists $\tilde{T} = \mathcal{O}(1/\epsilon^2)$ such that*

$$\widetilde{\kappa}(\mathbf{p}^t, \mathbf{r}^t) \le \left(\frac{1}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N} \frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+} \right)\epsilon, \quad \forall t \ge \tilde{T}, \tag{H.1}$$

where $\widetilde{\kappa}(\cdot)$ is defined in Eq. (17), and constants $\tilde{T}$ and $\tilde{\ell}_{r,i}$ are explicitly defined in Table EC.2.

Proof of Theorem 5. Consider the decreasing step-sizes of the form $\eta^t = C_\eta(t+1)^{-\beta}$ with $\beta \in (0,1]$, which satisfy the conditions $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Hence, all analyses in the proof of Theorem 4 are applicable, and we will use them as the basis for the proof of the convergence rate.

By Lemma EC.2, we have $\|\mathbf{p}^t - \mathbf{r}^t\| \le \eta^t C_{rp,\beta}$ for all $t \ge T_\beta$, where the constants $C_{rp,\beta}$ and T_β are defined in Eq. (L.2). Hence, to satisfy Eq. (G.4), it suffices to choose T_ϵ as the smallest integer greater than T_β such that

$$\eta^{T_\epsilon} \cdot \max_{i\in N}\left\{ (b_i + c_i^-)\cdot\tilde{\ell}_{r,i}C_{rp,\beta},\ (b_i + c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{k\in N}\left(b_k + c_k^-\right)^2} \right\} \le \frac{\epsilon}{2}. \tag{H.2}$$

Denote $\tilde{C}_\beta := \max_{i\in N}\left\{ (b_i + c_i^-)\cdot\tilde{\ell}_{r,i}C_{rp,\beta},\ (b_i + c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{k\in N}\left(b_k + c_k^-\right)^2} \right\}$. Then, since $\eta^{T_\epsilon} = C_\eta(T_\epsilon + 1)^{-\beta}$, we observe from Eq. (H.2) that T_ϵ can be expressed as

$$T_\epsilon := \max\left\{ T_\beta, \left\lceil \left(\frac{2C_\eta\tilde{C}_\beta}{\epsilon} \right)^{\frac{1}{\beta}} \right\rceil - 1 \right\}. \tag{H.3}$$

Next, recall the separation introduced in Eq. (G.5). According to Eq. (G.26), if there exists $t_0 \ge T_\epsilon$ with $N_c^{t_0} = N$, i.e., the pausing criteria are triggered for all products, it follows that for all $t \ge t_0$

$$\begin{aligned}
\widetilde{\kappa}(\mathbf{p}^t, \mathbf{r}^t) &= \|\mathbf{p}^t - \mathbf{r}^t\| + \sum_{i\in N}\texttt{dist}\Big(0, \texttt{Hull}\big\{D_i^-(\mathbf{p}^t, \mathbf{r}^t), D_i^+(\mathbf{p}^t, \mathbf{r}^t)\big\}\Big) \\
&\overset{(\Delta_1)}{\le} \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\cdot\sum_{i\in N}\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^t, \mathbf{r}^t), G_i^+(\mathbf{p}^t, \mathbf{r}^t)\big\}\Big) \\
&\overset{(\Delta_2)}{\le} \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\cdot\sum_{i\in N}\left[\texttt{dist}\Big(0, \texttt{Hull}\big\{G_i^-(\mathbf{p}^t), G_i^+(\mathbf{p}^t)\big\}\Big) + \frac{\epsilon}{2(b_i + c_i^-)}\right] \\
&= \eta^t C_{rp,\beta} + \max_{k\in N}\left\{b_k + c_k^-\right\}\left(\widetilde{\kappa}(\mathbf{p}^t) + \sum_{i\in N}\frac{\epsilon}{2(b_i + c_i^-)}\right) \\
&\overset{(\Delta_3)}{\le} \frac{\epsilon}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \left[\sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+}\right]\epsilon \\
&= \left(\frac{1}{2\max_{i\in N}\left\{(b_i + c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k + c_k^-\right\}}{b_i + c_i^+} \right)\epsilon,
\end{aligned} \tag{H.4}$$

where in step (Δ_1), we leverage the bound $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \eta^t C_{rp,\beta}$ for all $t \geq T_\beta$ and use the relation $D_i^\diamond(\mathbf{p},\mathbf{r}) = (b_i + c_i^\diamond) G_i^\diamond(\mathbf{p},\mathbf{r}) \leq \max_{k\in N}\left\{b_k + c_k^-\right\} \cdot G_i^\diamond(\mathbf{p},\mathbf{r})$ for every $\diamond \in \{+,-\}$. Next, step (Δ_2) uses the Lipschitz continuity of $G_i^\diamond(\mathbf{p},\cdot)$ in a similar manner as Eq. (G.6). In step (Δ_3), we first apply the upper bound $\eta^t C_{rp,\beta} \leq \eta^{T_\epsilon} C_{rp,\beta} \leq \epsilon/(2\max_{i\in N}\{(b_i+c_i^-)\tilde{\ell}_{r,i}\})$ from Eq. (H.2). Then, since $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\geq T_\epsilon}$ is non-increasing from Eq. (G.14), we have

$$\widetilde{\kappa}(\mathbf{p}^t) \leq \widetilde{\kappa}(\mathbf{p}^{t_0}) \leq \left[\sum_{i\in N} \frac{3}{2(b_i + c_i^+)}\right]\epsilon, \tag{H.5}$$

where the last inequality follows from Eq. (G.26). Thus, Eq. (H.4) shows that Algorithm 2 converges to an $\left(\frac{1}{2\max_{i\in N}\left\{(b_i+c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k+c_k^-\right\}}{b_i+c_i^+}\right)\epsilon$-SNE after t_0 iterations.

Therefore, it remains to determine when would such a period t_0 occur. Let $\tilde{T}$ be some integer greater than T_ϵ such that

$$\frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) > n\tilde{M}_G. \tag{H.6}$$

Since $\sum_{t=0}^{\infty}\eta^t = \infty$, the existence of $\tilde{T}$ is guaranteed. We observe that $t_0 \leq \tilde{T}$ must hold true, otherwise, we can deduce from Eq. (G.30) that

$$\widetilde{\kappa}(\mathbf{p}^{\tilde{T}}) \leq \widetilde{\kappa}(\mathbf{p}^{T_\epsilon}) - \frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) \overset{(\Delta)}{\leq} n\tilde{M}_G - \frac{\epsilon}{\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}\left(\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t\right) < 0, \tag{H.7}$$

where we apply the upper bound $|G_i^\diamond(\mathbf{p},\mathbf{r})| \leq \tilde{M}_G$ for $\diamond \in \{+,-\}$ from Lemma EC.7 in step (Δ) to derive that $\widetilde{\kappa}(\mathbf{p}) = \sum_{i\in N}\mathtt{dist}\Big(0, \mathtt{Hull}\big\{G_i^-(\mathbf{p},\mathbf{p}), G_i^+(\mathbf{p},\mathbf{p})\big\}\Big) \leq n\tilde{M}_G$. Since $\widetilde{\kappa}(\cdot)$ is a non-negative metric, Eq. (H.7) is a clear contradiction.

Next, we compute $\tilde{T}$ under the step-size choice $\eta^t = C_\eta(t+1)^{-\beta}$ and determine the optimal value of β. Using the integration lower bound, we have that

$$\sum_{t=T_\epsilon}^{\tilde{T}-1}\eta^t \geq C_\eta\int_{T_\epsilon}^{\tilde{T}}(t+1)^{-\beta}dt = \frac{C_\eta}{1-\beta}\left[(\tilde{T}+1)^{1-\beta} - (T_\epsilon+1)^{1-\beta}\right]. \tag{H.8}$$

Hence, by Eqs. (H.6) and (H.8), we can choose $\tilde{T}$ to be any positive integer such that

$$\frac{C_\eta}{1-\beta}\left[(\tilde{T}+1)^{1-\beta} - (T_\epsilon+1)^{1-\beta}\right] > \frac{n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{\epsilon}, \tag{H.9}$$

which is further equivalent to

$$\begin{aligned}\tilde{T} &\geq \left[\frac{(1-\beta)n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{C_\eta\epsilon} + (T_\epsilon+1)^{1-\beta}\right]^{\frac{1}{1-\beta}} \\ &= \left[\frac{(1-\beta)n\tilde{M}_G\overline{p}^2 \cdot \max_{i\in N}\left\{b_i + c_i^-\right\}}{C_\eta\epsilon} + \left(\max\left\{T_\beta+1, \left\lceil\left(\frac{2C_\eta\tilde{C}_\beta}{\epsilon}\right)^{\frac{1}{\beta}}\right\rceil\right\}\right)^{1-\beta}\right]^{\frac{1}{1-\beta}},\end{aligned} \tag{H.10}$$

where we substitute in the expression of T_ϵ from Eq. (H.3). We observe that the quantity on the right-hand side of Eq. (H.10) has the order $\mathcal{O}\left(\epsilon^{\frac{-1}{1-\beta}} + \epsilon^{\frac{-1}{\beta}}\right)$ and attains its minimum when $\beta = 1/2$. Therefore, combining Eqs. (H.4) and (H.10), we conclude that under the step-size choice $\eta^t = C_\eta/\sqrt{t+1}$, Algorithm 2 achieves an $\left(\frac{1}{2\max_{i\in N}\left\{(b_i+c_i^-)\tilde{\ell}_{r,i}\right\}} + \sum_{i\in N}\frac{2\max_{k\in N}\left\{b_k+c_k^-\right\}}{b_i+c_i^+}\right)$ ϵ-SNE in $\tilde{T} = \mathcal{O}(1/\epsilon^2)$ iterations, where

$$\tilde{T} := \left[\frac{n\tilde{M}_G\overline{p}^2\cdot\max_{i\in N}\left\{b_i+c_i^-\right\}}{2C_\eta\epsilon} + \sqrt{\max\left\{T_{1/2}+1, \left\lceil\left(\frac{2C_\eta\tilde{C}_{1/2}}{\epsilon}\right)^2\right\rceil\right\}}\right]^2. \tag{H.11}$$

In particular, we use $T_{1/2}$ and $\tilde{C}_{1/2}$ to denote the previously defined constants T_β and $\tilde{C}_\beta$ in the special case of $\beta = 1/2$. By Eq. (L.2) and the definition of $\tilde{C}_\beta$ below Eq. (H.2), we have that

$$\begin{aligned}
T_{1/2} &= \left\lceil\frac{2\cdot(3+\alpha^2)-4}{4-(3+\alpha^2)}\right\rceil = \left\lceil\frac{2+2\alpha^2}{1-\alpha^2}\right\rceil, \\
\tilde{C}_{1/2} &= \max_{i\in N}\left\{(b_i+c_i^-)\cdot\tilde{\ell}_{r,i}\tilde{C}_{rp},\ (b_i+c_i^-)\cdot\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{i\in N}\left(b_i+c_i^-\right)^2}\right\}, \\
\tilde{C}_{rp} &= \max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})\sqrt{T_{1/2}+1}}{C_\eta}\right\}.
\end{aligned} \tag{H.12}$$

This completes the proof of Theorem 5. □

Appendix I Proof of Theorem 6

Theorem 6 **(Restated)**. *Suppose that each firm $i\in N$ takes its own non-increasing step-sizes $\{\eta_i^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta_i^t = 0$ and $\sum_{t=0}^\infty \eta_i^t = \infty$. Then, it follows that:*

- *In the loss-neutral scenario, the price and reference price paths generated by Algorithm 1 converge to the unique SNE, where the convergence rate is determined by the slowest decay rate among the step-size sequences.*
- *In the loss-averse scenario, the price and reference price paths generated by Algorithm 2 with threshold ϵ converge to an $\mathcal{O}(\epsilon)$-SNE, where the convergence rate is determined by both the slowest and fastest decay rates among the step-size sequences.*

Proof of Theorem 6. The proof is built upon the current proofs for Theorems 1, 2, 4, and 5.

We first consider the loss-neutral scenario. Since all the step-size sequences $\{\eta_i^t\}_{t\geq 0}$ are non-increasing, there exist a non-increasing sequence $\{\eta^t\}_{t\geq 0}$ and non-decreasing sequences $\left\{C_{\eta,i}^t\right\}_{t\geq 0}$ for every $i\in N$ such that $\eta_i^t = \eta^t C_{\eta,i}^t$ for every $t\geq 0$ and $i\in N$. The sequence $\{\eta^t\}_{t\geq 0}$ approximately measures the smallest step-sizes among all firms, i.e., the sequence with the fastest decay rates. For example, in a two-firm setting where $\eta_1^t = 1/(t+1)$ and $\eta_2^t = 1/(t+1)^2$, we can take $\eta^t = \eta_2^t$,

$C^t_{\eta,1} = t+1$, and $C^t_{\eta,2} = 1$. We note that our proofs for Theorems 1 and 2 are built upon two key results, the inequality in Eq. (D.9) and Lemma EC.3. To accommodate the firm-differentiated step-sizes, we first modify the distance metrics $\kappa(\cdot)$ and $\kappa_\epsilon(\cdot)$, defined in Eq. (11), to the following non-stationary metrics

$$\kappa^t(\mathbf{p}) := \sum_{i\in N} \frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)}, \quad \kappa^t_\epsilon(\mathbf{p}) := \sum_{i\in N} \max\left\{\frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)} - \epsilon, 0\right\}. \tag{I.1}$$

Then, it is easy to verify that Eq. (D.9) holds under the new metric $\kappa_\epsilon(\cdot)$. For Lemma EC.3, we also observe that

$$\mathcal{G}(\mathbf{p}) \geq \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}\cdot\sum_{i\in N}\frac{|p_i^\star - p_i|}{b_i+c_i} \geq \frac{\min_{i\in N}\left\{C^t_{\eta,i}\right\}}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}\sum_{i\in N}\frac{|p_i^\star - p_i|}{C^t_{\eta,i}(b_i+c_i)} = \frac{\min_{i\in N}\left\{C^t_{\eta,i}\right\}}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}\kappa^t(\mathbf{p}). \tag{I.2}$$

Using Eq. (I.2) and following the derivations from Eqs. (D.6) to (D.16), we can deduce that

$$\kappa^t_\epsilon(\mathbf{p}^{t+1}) \leq \left(1 - \min_{i\in N}\left\{C^t_{\eta,i}\right\}\lambda\eta^t\right)\kappa^t_\epsilon(\mathbf{p}^t) + C_\kappa\epsilon\eta^t, \tag{I.3}$$

where $\lambda = 1/\left(\overline{p}\,\|\mathbf{p}^\star\|_\infty\right)$ and C_κ is defined above Eq. (D.16). In the special scenario where $C^t_{\eta,i} \equiv C_{\eta,i}$ for all $t \geq 0$ and $i \in N$, i.e., the step-sizes for any two firms i and j only differ by a fixed multiplier $C_{\eta,i}/C_{\eta,j}$, Eq. (I.3) demonstrates a similar contraction property as Eq. (D.16), and thus the entire proofs for Theorems 1 and 2 can be adapted. We conclude that the price and reference price paths must converge to the unique SNE, where the convergence rate has the same order as the scenario of uniform step-sizes.

For the more general scenario described in Theorem 6, Eq. (I.3) is not a recursion yet, because we have $\kappa^t_\epsilon(\cdot)$ on both sides of the inequality. Nevertheless, since $C^{t+1}_{\eta,i} \geq C^t_{\eta,i}$, it naturally holds from Eq. (I.3) that

$$\kappa^{t+1}_\epsilon(\mathbf{p}^{t+1}) \leq \kappa^t_\epsilon(\mathbf{p}^{t+1}) \leq \left(1 - \min_{i\in N}\left\{C^t_{\eta,i}\right\}\lambda\eta^t\right)\kappa^t_\epsilon(\mathbf{p}^t) + C_\kappa\epsilon\eta^t, \tag{I.4}$$

which implies a similar convergence behavior for $\{\kappa^t(\mathbf{p}^t)\}_{t\geq 0}$ as Eqs. (D.19) and (E.8). The difference lies in the conversion from $\kappa^t(\mathbf{p}^t)$ to $\|\mathbf{p}^\star - \mathbf{p}^t\|$. Similar as Eq. (E.16), we have that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \left(\max_{i\in N}\left\{C^t_{\eta,i}(b_i+c_i)\right\}\right)^2\cdot\left[\kappa^t(\mathbf{p}^t)\right]^2. \tag{I.5}$$

Hence, instead of only differing by a constant multiplier as that in Eq. (E.16), the bound in Eq. (I.5) depends on the non-decreasing sequences $\left\{C^t_{\eta,i}\right\}_{t\geq 0}$. We observe that, although it might hold $\lim_{t\to\infty} C^t_{\eta,i} = \infty$ for some $i \in N$, Eq. (I.5) still implies the convergence of $\{\|\mathbf{p}^\star - \mathbf{p}^t\|\}_{t\geq 0}$ to zero. In fact, since ϵ can be an arbitrary positive number, one consequence of Eq. (I.4) is that $\kappa^t(\mathbf{p}^t) = \mathcal{O}(\eta^t)$ for reasonably large t. This can be seen from the proof of Theorem 2 (from Eqs. (E.3) to (E.15)). Therefore, since $\lim_{t\to\infty}\eta^t_i = \lim_{t\to\infty} C^t_{\eta,i}\eta^t = 0$ for all $i \in N$, we conclude that $\|\mathbf{p}^\star - \mathbf{p}^t\| \to 0$ as

$t \to \infty$ and the convergence rate is dominated by slowest decay rate among the step-size sequences of all firms. This completes the proof of the first part of the theorem.

Now, for the loss-averse scenario, since the convergence is measured by the stationary metrics $\widetilde{\kappa}(\mathbf{p}, \mathbf{r})$ in Eq. (17) and $\widetilde{\kappa}(\mathbf{p})$ in Eq. (21), the extension to the firm-differentiated step-sizes is more straightforward. Firstly, since the step-sizes for all firms decrease to zero as $t \to \infty$, there exists T_ϵ such that

$$\max_{i\in N}\left\{(b_i + c_i^-)\cdot \tilde{\ell}_{r,i}\left\|\mathbf{p}^t - \mathbf{r}^t\right\|, (b_i + c_i^-)\cdot \eta_i^t \tilde{\ell}_{p,i}\tilde{M}_G\sqrt{\sum_{k\in N}(b_k + c_k^-)^2}\right\} \le \frac{\epsilon}{2}, \quad \forall t \ge T_\epsilon. \tag{I.6}$$

By definition, the size of T_ϵ is determined by the slowest decay rate among the step-size sequences. Following the derivations in the proof of Theorem 4 (see Appendix G), we find that for all $t \ge T_\epsilon$, either $\mathbf{p}^t$ is already an $\mathcal{O}(\epsilon)$-SNE, or it holds that

$$\widetilde{\kappa}(\mathbf{p}^{t+1}) - \widetilde{\kappa}(\mathbf{p}^t) \le -\min_{i\in N}\left\{\frac{\eta_i^t}{b_i + c_i^-}\right\}\cdot \frac{\epsilon}{\overline{p}^2}, \tag{I.7}$$

Therefore, the decrease speed of $\{\widetilde{\kappa}(\mathbf{p}^t)\}_{t\ge T_\epsilon}$ is dominated by the fastest decay rate among the step-size sequences. Together, we conclude that Algorithm 2 still converges to an $\mathcal{O}(\epsilon)$-SNE, and the convergence rate is dominated by both the slowest and fastest decay rates among the step-size sequences. This completes the proof of Theorem 6. □

Appendix J Proof of Theorem 7

Theorem 7 **(Restated)**. *Suppose that the firms can only access an inexact first-order oracle such that the errors are uniformly bounded by some $\delta > 0$. Let the step-sizes $\{\eta^t\}_{t\ge 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t = 0$ and $\sum_{t=0}^{\infty}\eta^t = \infty$. Then, the price and reference price paths generated by Algorithm 1 (or Algorithm 2 with threshold ϵ) converge to an $\mathcal{O}(\delta)$-neighborhood of the unique SNE in the loss-neutral scenario (or an $\mathcal{O}(\delta+\epsilon)$-SNE in the loss-averse scenario), where the convergence rate has the same order as the setting of exact first-order oracle.*

Proof of Theorem 7. The proof is based on the existing proofs of Theorems 1, 2, 4, and 5. The extensions for the loss-neutral scenario and the loss-averse scenario essentially follow the same idea. Below, we consider the loss-neutral scenario (Algorithm 1) for illustration.

In the inexact setting, each firm updates its price by a noisy derivative $\widehat{D}_i^t$, where we assume the difference between $\widehat{D}_i^t$ and the true derivative D_i^t is bounded by δ, i.e., $\left|\widehat{D}_i^t - D_i^t\right| < \delta$. Since $D_i^t = (b_i + c_i)G_i(\mathbf{p}^t, \mathbf{r}^t)$ by the definition in Eq. (D.2), we follow the derivations from Eqs. (D.3) to (D.9) to show that

$$\kappa_\epsilon(\mathbf{p}^{t+1}) \le \kappa_\epsilon(\mathbf{p}^t) - \eta^t \sum_{i\in\overline{N}_\epsilon^{t+1}} \mathtt{sign}\left(p_i^\star - p_i^t\right) G_i(\mathbf{p}^t, \mathbf{r}^t) + \eta^t \sum_{i\in N}\frac{\delta}{b_i + c_i}, \tag{J.1}$$

where $\kappa_\epsilon(\cdot)$ is the metric defined in Eq. (11). Then, by applying Eqs. (D.9) to (D.13), we have

$$\kappa_\epsilon(\mathbf{p}^{t+1}) \le (1-\lambda\eta^t)\kappa_\epsilon(\mathbf{p}^t) + \left(C_\kappa\epsilon + \sum_{i\in N}\frac{\delta}{b_i+c_i}\right)\eta^t, \quad \forall t \ge T_\epsilon, \tag{J.2}$$

where the definitions of λ and C_κ can be found in Table EC.1, and T_ϵ is the break point defined above Eq. (D.6). Hence, by unrolling Eq. (J.2) in a similar manner as Eqs. (D.17) to (D.19), we deduce that

$$\lim_{t\to\infty}\kappa(\mathbf{p}^t) \le \left(\frac{C_\kappa}{\lambda}+n\right)\epsilon + \sum_{i\in N}\frac{\delta}{\lambda(b_i+c_i)}. \tag{J.3}$$

Since Eq. (J.3) holds for any $\epsilon>0$, we conclude that the price and reference price paths converge to an $\mathcal{O}(\delta)$-neighborhood of the unique SNE.

Now, in the loss-neutral scenario, suppose all firms adopt the step-sizes $\eta^t = \frac{C_\eta\log(t+1)}{t+1}$ for $t\ge 2$ with $C_\eta > 2\overline{p}^2/\log 2$. Similar as Eq. (E.3) in the proof of Theorem 2, we have from Eq. (J.2) that

$$\kappa(\mathbf{p}^t) \le \kappa_\epsilon(\mathbf{p}^{T_\epsilon})\exp\left(-\lambda\sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\right) + \left(C_\kappa\epsilon + \sum_{i\in N}\frac{\delta}{b_i+c_i}\right)X^{t-1} + n\epsilon, \tag{J.4}$$

where $X^{t-1} = \sum_{t'=T_\epsilon}^{t-1}\eta^{t'}\prod_{t''=t'+1}^{t-1}(1-\lambda\eta^{t''})$. Then, following the same process as Eqs. (E.4) to (E.8), we first find that

$$\kappa(\mathbf{p}^t) \le \left(2C_\kappa C_\eta + n + \frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)\epsilon + \sum_{i\in N}\frac{2C_\eta\delta}{b_i+c_i}, \quad \forall t\ge 2T_\epsilon+1, \tag{J.5}$$

where $M_\kappa = \sum_{i\in N}(\overline{p}-\underline{p})/(b_i+c_i)$ is the universal upper bound on $\kappa(\cdot)$ and T_ϵ is specified by Eq. (E.2). Next, to convert the upper bound in Eq. (J.5) to a bound that explicitly depends on t, we follow Eqs. (E.9) to (E.15) to conclude that

$$\kappa(\mathbf{p}^t) \le \widehat{C}_\kappa\frac{\log t}{t} + \sum_{i\in N}\frac{2C_\eta\delta}{b_i+c_i}, \quad \forall t > \{2T_1, 10\}, \tag{J.6}$$

where $\widehat{C}_\kappa$ and T_1 are defined in Table EC.1. Hence, with inexact first-order oracles, the convergence rate of Algorithm 1 remains the same as the exact setting.

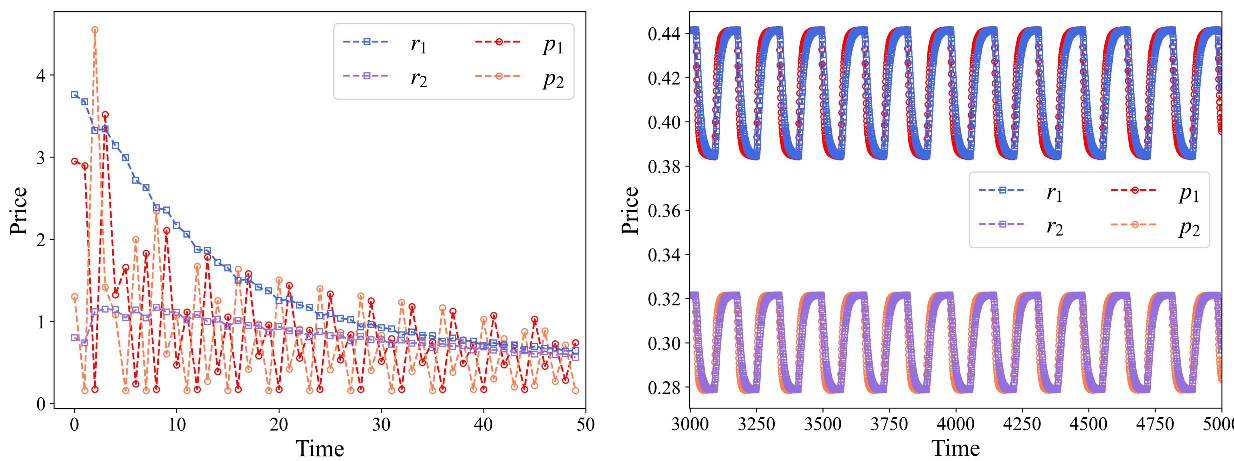
Finally, for loss-averse scenarios, we note that although the inexact first-order oracle affects the evaluation of the true/virtual derivatives, i.e., $D_i^{t,+}$ and $D_i^{t,+}$, a similar analysis as Eqs. (G.2) to (G.29) still holds true if there exists some product i_0 with $D_{i_0}^{t,-} = D_{i_0}^-(\mathbf{p}^t,\mathbf{r}^t) > (b_{i_0}+c_{i_0}^-)\epsilon + \delta$ or $D_{i_0}^{t,+} = D_{i_0}^+(\mathbf{p}^t,\mathbf{r}^t) < -(b_{i_0}+c_{i_0}^+)\epsilon - \delta$, i.e., being at least δ "away" from incurring the pausing criterion. If there does not exist such a product, then we can show that the algorithm has already arrived at an $\mathcal{O}(\delta+\epsilon)$-SNE. $\square$

Appendix K Illustration of OPGA in the Gain-seeking Scenario

In this section, we illustrate the convergence behavior of OPGA (Algorithm 1) in the gain-seeking scenario. Note that with the non-smooth revenue function under gain-seekingness, OPGA is equivalent to the online projected sub-gradient ascent. The reason that we do not plot C-OPGA (Algorithm 2) is due to its similarity with OPGA in gain-seeking scenarios. Recall the true and virtual derivatives $D_i^-(\mathbf{p},\mathbf{r})$ and $D_i^+(\mathbf{p},\mathbf{r})$ defined in Eq. (18). In contrast to the loss-averse scenario, it always holds that $D_i^+(\mathbf{p},\mathbf{r}) < D_i^-(\mathbf{p},\mathbf{r})$ for any gain-seeking product i. Hence, the pausing criterion can be triggered only if $-\epsilon < D_i^+(\mathbf{p}^t,\mathbf{r}^t) < D_i^-(\mathbf{p}^t,\mathbf{r}^t) < \epsilon$, which is unlikely to happen for small ϵ.

Figure EC.1 Cyclic Pattern of OPGA in Gain-seeking Scenario

(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (4.70, 1.55, 4.25, 3.38)$, $(a_2, b_2, c_2^+, c_2^-) = (4.20, 1.15, 5.25, 4.50)$, $(r_1^0, r_2^0) = (3.76, 0.80)$, $(p_1^0, p_2^0) = (2.95, 1.30)$, $\alpha = 0.90$, and $\eta^t = 1/\sqrt{t+1}$.)

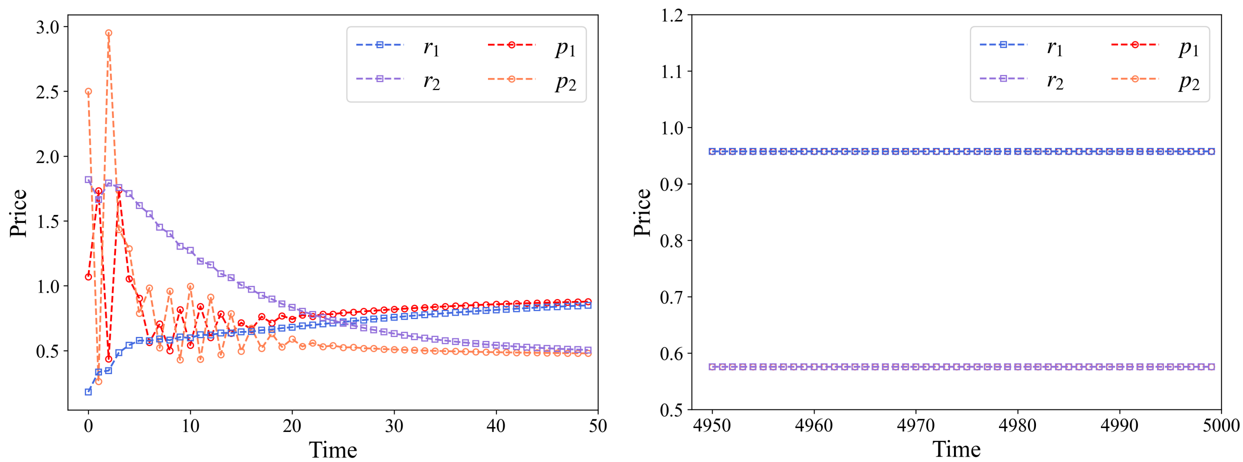


(a) Initial stage. (b) Long-run behavior.

Figures EC.1, EC.2, and EC.3 show the price and reference price paths of OPGA in three gain-seeking scenarios. In Figure EC.1, we observe that the paths oscillate indefinitely without admitting limiting points. Figures EC.2 and EC.3 share the same model parameters and only differ in initial reference prices and prices. Both figures show a convergent pattern in the long run, but have different limiting points. However, we highlight that neither limiting point represents an equilibrium, and both firms can achieve a higher revenue by unilaterally deviating from the limiting point. Indeed, such a convergence results from the monotonicity of price paths when approaching the limiting points. This ensures the effective reference sensitivity stays unchanged during the learning process, thereby leading the paths to converge to the SNE in the loss-neutral scenario. We can verify that the limiting point in Figure EC.2 is the same as the SNE in the loss-neutral scenario with parameters $(a_1, b_1, c_1) = (4.80, 1.30, 1.12)$ and $(a_2, b_2, c_2) = (4.22, 1.70, 1.27)$. The limiting point in Figure EC.3

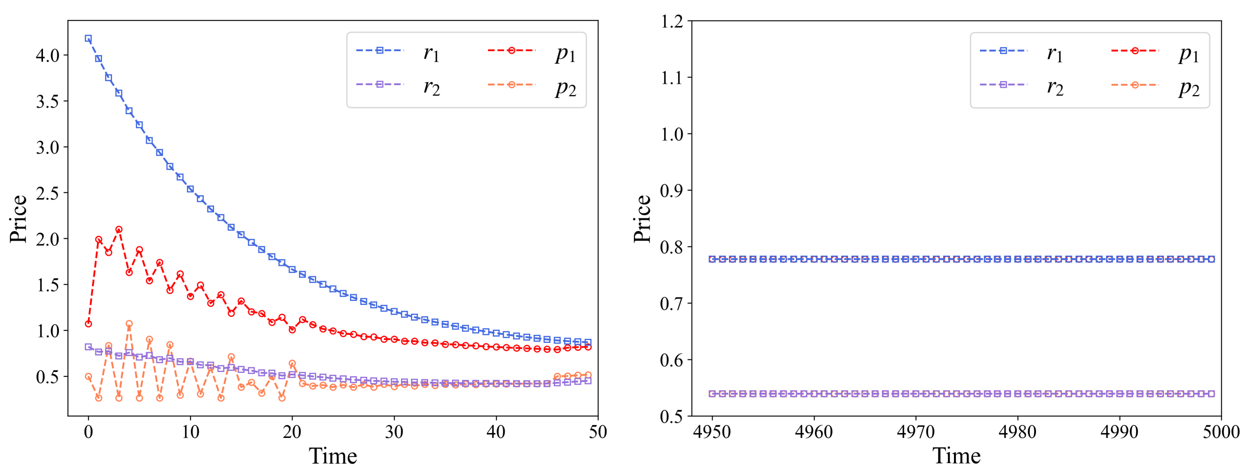
corresponds to the SNE under parameters $(a_1, b_1, c_1) = (4.80, 1.30, 2)$, $(a_2, b_2, c_2) = (4.22, 1.70, 1.27)$. Together, the experiments in Figures EC.1, EC.2, and EC.3 demonstrate that equilibrium and market stability cannot be simultaneously achieved in gain-seeking scenarios.

Figure EC.2 **Convergent Pattern of `OPGA` in Gain-seeking Scenario**
(Parameters: $(a_1, b_1, c_1^+, c_1^-) = (4.80, 1.30, 2.00, 1.12)$, $(a_2, b_2, c_2^+, c_2^-) = (4.22, 1.70, 2.12, 1.27)$, $(r_1^0, r_2^0) = (0.18, 1.82)$ and $(p_1^0, p_2^0) = (1.07, 2.50)$, $\alpha = 0.90$, and $\eta^t = 1/\sqrt{t+1}$.)



(a) Initial stage. (b) Long-run behavior.

Figure EC.3 **Convergent Pattern of `OPGA` in Gain-seeking Scenario with Different Initialization**
(Parameters are the same as Figure EC.2 except for the initializations, which are at $(r_1^0, r_2^0) = (4.18, 0.82)$ and $(p_1^0, p_2^0) = (1.07, 0.50)$.)



(a) Initial stage. (b) Long-run behavior.

Appendix L Supporting Lemmas

L.1 Lemma EC.2

LEMMA EC.2 **(Convergence of Price to Reference Price)**. *Let $\{\mathbf{p}^t\}_{t\geq 0}$ and $\{\mathbf{r}^t\}_{t\geq 0}$ be the price path and reference path generated by Algorithms 1 or 2 with non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$. Then, it holds that $\lim_{t\to\infty}\|\mathbf{p}^t - \mathbf{r}^t\| = 0$. In particular:*

1. *In the loss-neutral scenario, if $\eta^t = \frac{C_\eta \log(t+1)}{t+1}$ for $t\geq 2$, then there exist $C_{rp}, T_1 > 0$, such that $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \eta^t C_{rp}$ for all $t \geq T_1$, where*

$$\begin{aligned} C_{rp} &= \max\left\{\frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta\log(T_1+1)}\right\},\\ T_1 &= \left\lceil \frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}}\right\rceil. \end{aligned} \tag{L.1}$$

2. *In the loss-averse scenario, if $\eta^t = C_\eta (t+1)^{-\beta}$ where $\beta\in(0,1]$, then there exist $C_{rp,\beta}, T_\beta > 0$, such that $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \eta^t C_{rp,\beta}$ for all $t\geq T_\beta$, where*

$$\begin{aligned} C_{rp,\beta} &= \max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_\beta+1)^\beta}{C_\eta}\right\},\\ T_\beta &= \left\lceil \frac{2(3+\alpha^2)^{\frac{1}{2\beta}}-2^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}}-(3+\alpha^2)^{\frac{1}{2\beta}}}\right\rceil. \end{aligned} \tag{L.2}$$

It is worth mentioning that the reason we limit the scope of Lemma EC.2 to the above two special cases is merely because they are sufficient for the proof of our main results.

Proof of Lemma EC.2. We first prove the general convergence result under non-increasing step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t = 0$. Without loss of generality, we focus on the loss-neutral case and consider a trajectory $\{(\mathbf{p}^t,\mathbf{r}^t)\}_{t\geq 0}$ generated by Algorithm 1. The proof is the same for the loss-averse scenario.

First, recall that $D_i^t = (b_i + c_i)\cdot G_i(\mathbf{p}^t,\mathbf{r}^t)$, where $G_i(\mathbf{p},\mathbf{r})$ is the scaled partial derivative of the log-revenue defined in Eq. (D.2). Then, it follows from Eq. (L.30) in Lemma EC.4 that $|D_i^t| \leq (b_i+c_i)M_G$. Since $\{\eta^t\}_{t\geq 0}$ is a non-increasing sequence with $\lim_{t\to\infty}\eta^t = 0$, for any constant $\eta > 0$, there exists $T_\eta \in \mathbb{N}$ such that $|\eta^t D_i^t| \leq \eta$ for every $t \geq T_\eta$ and for all $i\in N$. Thus, it holds that

$$\left|p_i^{t+1} - p_i^t\right| = \left|\mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t D_i^t\right) - p_i^t\right| \leq \left|\eta^t D_i^t\right| \leq \eta, \quad \forall t\geq T_\eta,\ \forall i\in N, \tag{L.3}$$

where the first inequality is due to the property of the projection operator. Then, by the reference price update rule in Eq. (4), it follows that

$$\begin{aligned} \left|p_i^{t+1}-r_i^{t+1}\right| &= \left|p_i^{t+1}-\alpha r_i^t-(1-\alpha)p_i^t\right| \\ &= \left|\left(p_i^{t+1}-p_i^t\right)+\alpha\left(r_i^t-p_i^t\right)\right| \\ &\le \left|p_i^{t+1}-p_i^t\right|+\alpha\left|p_i^t-r_i^t\right| \\ &\le \eta+\alpha\left|p_i^t-r_i^t\right|, \quad \forall t\le T_\eta,\ \forall i\in N, \end{aligned} \tag{L.4}$$

where the last line results from the upper bound in Eq. (L.3). Applying Eq. (L.4) recursively from period t to period T_η, we further derive that

$$\begin{aligned} \left|p_i^{t+1}-r_i^{t+1}\right| &\le \eta\Big(\sum_{\tau=T_\eta}^{t}\alpha^{\tau-T_\eta}\Big)+\alpha^{t+1-T_\eta}\cdot\left|p_i^{T_\eta}-r_i^{T_\eta}\right| \\ &\le \frac{\eta}{1-\alpha}+\alpha^{t+1-T_\eta}\cdot(\overline{p}-\underline{p}), \quad \forall i\in N, \end{aligned} \tag{L.5}$$

which implies that

$$\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|\le\sqrt{n}\left(\frac{\eta}{1-\alpha}+\alpha^{t+1-T_\eta}\cdot(\overline{p}-\underline{p})\right). \tag{L.6}$$

Since η can be arbitrarily close to 0, we have that $\|\mathbf{p}^t-\mathbf{r}^t\|\to 0$ as $t\to\infty$, which completes the proof for the general convergence of price to reference price with non-increasing step-sizes.

In the next part, we consider two specific choices of step-sizes and explicitly quantify the convergence rate. This part of the proof relies on an important recursion, shown as follows

$$\begin{aligned} \left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|^2 &= \left\|\mathbf{p}^{t+1}-\alpha\mathbf{r}^t-(1-\alpha)\mathbf{p}^t\right\|^2 \\ &= \left\|\alpha(\mathbf{p}^t-\mathbf{r}^t)+(\mathbf{p}^{t+1}-\mathbf{p}^t)\right\|^2 \\ &= \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\mathbf{p}^{t+1}-\mathbf{p}^t\right\|^2+2\alpha(\mathbf{p}^t-\mathbf{r}^t)^\top(\mathbf{p}^{t+1}-\mathbf{p}^t) \\ &\overset{(\Delta_1)}{\le} \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\eta^t\mathbf{D}^t\right\|^2+2\alpha\left\|\mathbf{p}^t-\mathbf{r}^t\right\|\left\|\eta^t\mathbf{D}^t\right\| \\ &\overset{(\Delta_2)}{\le} \alpha^2\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\left\|\eta^t\mathbf{D}^t\right\|^2+\frac{1-\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{2\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2 \\ &= \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2, \end{aligned} \tag{L.7}$$

where $\mathbf{D}^t := \left(D_1^t,\dots D_n^t\right)$ in (Δ_1) and the inequality holds due to the Cauchy-Schwarz inequality and the property of the projection operator. Then, step (Δ_2) stems from the inequality of arithmetic and geometric means.

1. **Loss-neutral Scenario.** We first focus on the loss-neutral scenario, where the step-size is specified as $\eta^t=\frac{C_\eta\log(t+1)}{t+1}$ for $t\ge 2$. We adopt an induction-based argument. At some period t, suppose that there exists a constant C_{rp} such that

$$\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2\le\left(\eta^t C_{rp}\right)^2=C_{rp}^2C_\eta^2\left(\frac{\log(t+1)}{t+1}\right)^2. \tag{L.8}$$

Together with Eq. (L.7), we have that at period $t+1$

$$\begin{aligned}
\left\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\right\|^2 &\leq \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t-\mathbf{r}^t\right\|^2+\frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t\mathbf{D}^t\right\|^2 \\
&\leq \frac{1+\alpha^2}{2}\left(\eta^t C_{rp}\right)^2+\frac{1+\alpha^2}{1-\alpha^2}\left(\sum_{i\in N}(b_i+c_i)^2\left[G_i(\mathbf{p}^t,\mathbf{r}^t)\right]^2\right)\cdot(\eta^t)^2 \\
&\overset{(\Delta)}{\leq} \frac{1+\alpha^2}{2}\left(\eta^t C_{rp}\right)^2+\underbrace{\left(\frac{1+\alpha^2}{1-\alpha^2}\cdot M_G^2\sum_{i\in N}(b_i+c_i)^2\right)}_{=:C_D}\cdot(\eta^t)^2 \\
&\leq C_\eta^2\left(\frac{\log(t+1)}{t+1}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right] \\
&= C_\eta^2\left(\frac{\log(t+2)}{t+2}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\cdot\left(\frac{t+2}{t+1}\right)^2\cdot\left(\frac{\log(t+1)}{\log(t+2)}\right)^2 \\
&\leq C_\eta^2\left(\frac{\log(t+2)}{t+2}\right)^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\cdot\left(\frac{t+2}{t+1}\right)^2 \\
&\leq (\eta^{t+1})^2\cdot\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\left(\frac{t+2}{t+1}\right)^2,
\end{aligned}\tag{L.9}$$

where step (Δ) applies the upper bound on function $G_i(\cdot,\cdot)$ in Eq. (L.30) of Lemma EC.4. For the simplicity of notation, we denote the coefficient of $(\eta^t)^2$ in line (Δ) as C_D.

Since our goal is to have $\|\mathbf{p}^{t+1}-\mathbf{r}^{t+1}\|^2\leq(\eta^{t+1}\cdot C_{rp})^2$, based on the inequality in Eq. (L.9), we only need to ensure that

$$\left[\frac{1+\alpha^2}{2}C_{rp}^2+C_D\right]\left(\frac{t+2}{t+1}\right)^2\leq C_{rp}^2 \quad\Leftrightarrow\quad C_{rp}^2\left[\left(\frac{t+1}{t+2}\right)^2-\frac{1+\alpha^2}{2}\right]\geq C_D. \tag{L.10}$$

As $(t+1)/(t+2)$ increases with respect to t, we choose T_1 to be the smallest integer that satisfies

$$\begin{aligned}
\left(\frac{T_1+1}{T_1+2}\right)^2-\frac{1+\alpha^2}{2}\geq\frac{1-\alpha^2}{4} \quad&\Leftrightarrow\quad \left(\frac{T_1+1}{T_1+2}\right)^2\geq\frac{3+\alpha^2}{4} \\
&\Leftrightarrow\quad \left(1+\frac{1}{T_1+1}\right)^2\leq\frac{4}{3+\alpha^2} \\
&\Leftrightarrow\quad T_1=\left\lceil\frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}}\right\rceil.
\end{aligned}\tag{L.11}$$

Given T_1 as specified above, it follows that for all $t\geq T_1$

$$\left(\frac{t+1}{t+2}\right)^2-\frac{1+\alpha^2}{2}\geq\left(\frac{T_1+1}{T_1+2}\right)^2-\frac{1+\alpha^2}{2}\geq\frac{1-\alpha^2}{4}. \tag{L.12}$$

Hence, to ensure Eq. (L.10) hold true for all $t\geq T_1$, it suffices to choose C_{rp} that satisfies

$$C_{rp}\geq 2\sqrt{\frac{C_D}{1-\alpha^2}}, \tag{L.13}$$

where C_D is defined in Eq. (L.7).

Lastly, the base case of the induction requires that $\|\mathbf{p}^{T_1} - \mathbf{r}^{T_1}\| \le C_{rp} C_\eta \frac{\log(T_1+1)}{T_1+1}$. Since $\|\mathbf{p}^{T_1} - \mathbf{r}^{T_1}\| \le \sqrt{n}(\overline{p} - \underline{p})$, it suffices to choose

$$\sqrt{n}(\overline{p} - \underline{p}) \le C_{rp} C_\eta \frac{\log(T_1+1)}{T_1+1} \quad \Leftrightarrow \quad C_{rp} \ge \frac{\sqrt{n}(\overline{p} - \underline{p})(T_1+1)}{C_\eta \log(T_1+1)}. \tag{L.14}$$

Combining the requirements for the base case in Eq. (L.14) and for the induction step in Eq. (L.13), we conclude that it is sufficient to choose

$$C_{rp} = \max\left\{ \frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p} - \underline{p})(T_1+1)}{C_\eta \log(T_1+1)} \right\}, \tag{L.15}$$

which completes the proof for the loss-neutral scenario.

2. **Loss-averse Scenario.** We use a similar induction method to prove the loss-averse case, where the step-size is $\eta^t = C_\eta (t+1)^{-\beta}$ with $\beta \in (0,1]$. At some period t, suppose that there exists a constant $C_{rp,\beta}$ such that

$$\left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 \le (\eta^t C_{rp,\beta})^2 = C_{rp,\beta}^2 C_\eta^2 \cdot (t+1)^{-2\beta}. \tag{L.16}$$

Then, from the recursion in Eq. (L.7), we derive that

$$\begin{aligned}
\left\|\mathbf{p}^{t+1} - \mathbf{r}^{t+1}\right\|^2 &\le \frac{1+\alpha^2}{2}\left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 + \frac{1+\alpha^2}{1-\alpha^2}\left\|\eta^t \mathbf{D}^t\right\|^2 \\
&\overset{(\Delta)}{\le} \frac{1+\alpha^2}{2}(\eta^t C_{rp,\beta})^2 + \underbrace{\left(\frac{1+\alpha^2}{1-\alpha^2}\cdot \tilde{M}_G^2 \sum_{i\in N}(b_i + c_i^-)^2\right)}_{=:\tilde{C}_D} \cdot (\eta^t)^2 \\
&\le \left(\frac{C_\eta}{(t+1)^\beta}\right)^2 \cdot \left[\frac{1+\alpha^2}{2} C_{rp,\beta}^2 + \tilde{C}_D\right] \\
&= \left(\frac{C_\eta}{(t+2)^\beta}\right)^2 \cdot \left(\frac{t+2}{t+1}\right)^{2\beta} \cdot \left[\frac{1+\alpha^2}{2} C_{rp,\beta}^2 + \tilde{C}_D\right] \\
&= (\eta^{t+1})^2 \cdot \left[\frac{1+\alpha^2}{2} C_{rp,\beta}^2 + \tilde{C}_D\right]\left(\frac{t+2}{t+1}\right)^{2\beta},
\end{aligned} \tag{L.17}$$

where we apply the upper bound $|D_i^t| \le \max_{\diamond\in\{+,-\}}\{(b_i + c_i^\diamond) G_i^\diamond(\mathbf{p}^t, \mathbf{r}^t)\} \le (b_i + c_i^-)\tilde{M}_G$ in step (Δ) with the constant $\tilde{M}_G$ coming from Eq. (L.51) of Lemma EC.7. Then, to ensure $\|\mathbf{p}^{t+1} - \mathbf{r}^{t+1}\|^2 \le (\eta^{t+1} \cdot C_{rp,\beta})^2$, we need the following condition to be satisfied

$$\left[\frac{1+\alpha^2}{2} C_{rp,\beta}^2 + \tilde{C}_D\right]\left(\frac{t+2}{t+1}\right)^{2\beta} \le C_{rp,\beta}^2 \quad \Leftrightarrow \quad C_{rp,\beta}^2\left[\left(\frac{t+1}{t+2}\right)^{2\beta} - \frac{1+\alpha^2}{2}\right] \ge \tilde{C}_D. \tag{L.18}$$

Adopting the same approach used for the loss-neutral case described in Eq. (L.11), we choose T_β as the smallest integer such that

$$\left(\frac{T_\beta+1}{T_\beta+2}\right)^{2\beta} - \frac{1+\alpha^2}{2} \ge \frac{1-\alpha^2}{4} \quad \Leftrightarrow \quad T_\beta = \left\lceil \frac{2(3+\alpha^2)^{\frac{1}{2\beta}} - 2^{\frac{1}{\beta}}}{2^{\frac{1}{\beta}} - (3+\alpha^2)^{\frac{1}{2\beta}}} \right\rceil. \tag{L.19}$$

With T_β as specified in Eq. (L.19) and $(t+1)/(t+2)$ increasing in t, it follows that for all $t \geq T_\beta$

$$\left(\frac{t+1}{t+2}\right)^{2\beta} - \frac{1+\alpha^2}{2} \geq \left(\frac{T_\beta+1}{T_\beta+2}\right)^2 - \frac{1+\alpha^2}{2} \geq \frac{1-\alpha^2}{4}. \tag{L.20}$$

The above inequality implies that the condition in Eq. (L.18) is held when $C_{rp,\beta}$ satisfies

$$C_{rp,\beta} \geq 2\sqrt{\frac{\tilde{C}_D}{1-\alpha^2}}, \tag{L.21}$$

where $\tilde{C}_D$ is defined in Eq. (L.17).

Finally, the base case of this induction requires that $\left\|\mathbf{p}^{T_\beta} - \mathbf{r}^{T_\beta}\right\| \leq \frac{C_{rp,\beta}C_\eta}{(T_\beta+1)^\beta}$. Since $\left\|\mathbf{p}^{T_\beta} - \mathbf{r}^{T_\beta}\right\| \leq \sqrt{n}(\overline{p}-\underline{p})$, it suffices to choose

$$\sqrt{n}(\overline{p}-\underline{p}) \leq \frac{C_{rp,\beta}C_\eta}{(T_\beta+1)^\beta} \quad \Leftrightarrow \quad C_{rp,\beta} \geq \frac{\sqrt{n}(\overline{p}-\underline{p})(T_\beta+1)^\beta}{C_\eta}. \tag{L.22}$$

Merging the requirements for $C_{rp,\beta}$ in both Eqs. (L.21) and (L.22), we complete the proof by showing the sufficient condition as follows

$$C_{rp,\beta} = \max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2}, \frac{\sqrt{n}(\overline{p}-\underline{p})(T_\beta+1)^\beta}{C_\eta}\right\}. \tag{L.23}$$

□

L.2 Lemma EC.3

LEMMA EC.3. *In the loss-neutral scenario, define the function $\mathcal{G}(\mathbf{p})$ as*

$$\mathcal{G}(\mathbf{p}) := \sum_{i\in N}\texttt{sign}\,(p_i^\star - p_i)\,G_i(\mathbf{p},\mathbf{p}) = \sum_{i\in N}\texttt{sign}\,(p_i^\star - p_i)\left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right], \tag{L.24}$$

where $\mathbf{p}^\star$ is the unique SNE, $G_i(\mathbf{p},\mathbf{r})$ is the scaled derivative defined in Eq. (D.2), and function $\texttt{sign}(\cdot)$ is defined in Eq. (D.4). Then, it holds that

$$\mathcal{G}(\mathbf{p}) \geq \frac{\kappa(\mathbf{p})}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} = \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}\cdot\sum_{i\in N}\frac{|p_i^\star - p_i|}{b_i+c_i}, \quad \forall \mathbf{p}\in\mathcal{P}^n, \tag{L.25}$$

where $\kappa(\cdot)$ is the weighted ℓ^1-metric function defined in Eq. (11).

Proof of Lemma EC.3. We first consider the following separation of N based on relative size between $\mathbf{p}$ and $\mathbf{p}^\star$:

$$N_1(\mathbf{p}) := \{i\in N | p_i > p_i^\star\}, \quad N_2(\mathbf{p}) := \{i\in N | p_i < p_i^\star\}, \quad N_c(\mathbf{p}) := \{i\in N | p_i = p_i^\star\}. \tag{L.26}$$

Then, since $\texttt{sign}(p_i^\star - p_i) = \texttt{sign}(0) = 0$ for all $i \in N_c(\mathbf{p})$, we rewrite $\mathcal{G}(\mathbf{p})$ to deduce that

$$
\begin{aligned}
\mathcal{G}(\mathbf{p}) &= \sum_{i\in N} \texttt{sign}(p_i^\star - p_i)\, G_i(\mathbf{p},\mathbf{p}) \\
&= \sum_{i\in N_1(\mathbf{p})} \left[1 - d_i(\mathbf{p},\mathbf{p}) - \frac{1}{(b_i+c_i)p_i}\right] + \sum_{i\in N_2(\mathbf{p})} \left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right] \\
&\overset{(\Delta_1)}{=} \sum_{i\in N_1(\mathbf{p})} \left\{\left[1 - d_i(\mathbf{p},\mathbf{p}) - \frac{1}{(b_i+c_i)p_i}\right] - \left[1 - d_i(\mathbf{p}^\star,\mathbf{p}^\star) - \frac{1}{(b_i+c_i)p_i^\star}\right]\right\} \\
&\quad + \sum_{i\in N_2(\mathbf{p})} \left\{\left[\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right] - \left[\frac{1}{(b_i+c_i)p_i^\star} + d_i(\mathbf{p}^\star,\mathbf{p}^\star) - 1\right]\right\} \\
&\overset{(\Delta_2)}{=} \sum_{i\in N_1(\mathbf{p})} \frac{1}{b_i+c_i}\left(\frac{1}{p_i^\star} - \frac{1}{p_i}\right) + \sum_{i\in N_2(\mathbf{p})} \frac{1}{b_i+c_i}\left(\frac{1}{p_i} - \frac{1}{p_i^\star}\right) \\
&\quad + \underbrace{\sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star) - \sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p},\mathbf{p})}_{\geq 0} + \underbrace{\sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}) - \sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star)}_{\geq 0} \\
&\geq \sum_{i\in N} \frac{1}{b_i+c_i} \cdot \frac{|p_i - p_i^\star|}{p_i^\star p_i} \;\geq\; \frac{1}{\overline{p}\,\|\mathbf{p}^\star\|_\infty} \sum_{i\in N} \frac{|p_i - p_i^\star|}{b_i+c_i} \;=\; \frac{\kappa(\mathbf{p})}{\overline{p}\,\|\mathbf{p}^\star\|_\infty}.
\end{aligned}
\tag{L.27}
$$

In step (Δ_1), we introduce two dummy terms, which are equal to zero by Eq. (C.16). To derive step (Δ_2), using the facts that $p_i > p_i^\star$ for $i \in N_1$ and $p_i < p_i^\star$ for $i \in N_2$, we have that

$$
\begin{aligned}
\sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star) &= \frac{\sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i^\star)}{1 + \sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i^\star) + \sum_{i\in N_2(\mathbf{p})} \exp(a_i - b_i p_i^\star) + \sum_{i\in N_c(\mathbf{p})} \exp(a_i - b_i p_i^\star)} \\
&\geq \frac{\sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i)}{1 + \sum_{i\in N_1(\mathbf{p})} \exp(a_i - b_i p_i) + \sum_{i\in N_2(\mathbf{p})} \exp(a_i - b_i p_i) + \sum_{i\in N_c(\mathbf{p})} \exp(a_i - b_i p_i)} \\
&= \sum_{i\in N_1(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}).
\end{aligned}
\tag{L.28}
$$

By similar analysis, we show that the second under-brace term in Eq. (L.27) is also no less than zero, i.e., $\sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p},\mathbf{p}) > \sum_{i\in N_2(\mathbf{p})} d_i(\mathbf{p}^\star,\mathbf{p}^\star)$. This completes the proof of Lemma EC.3. □

L.3 Lemma EC.4

LEMMA EC.4. *In the loss-neutral scenario, let $G_i(\mathbf{p},\mathbf{r})$ be the scaled partial derivative defined in Eq. (D.2). Then, it holds that*

$$
\frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_j} = \begin{cases} -\dfrac{1}{(b_i+c_i)p_i^2} - (b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1 - d_i(\mathbf{p},\mathbf{r})\big) & \textit{if } j = i, \\ (b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}) & \textit{if } j \neq i. \end{cases}
\tag{L.29a}
$$

$$\frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial r_j} = \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial r_j} = \begin{cases} c_i \cdot d_i(\mathbf{p},\mathbf{r}) \cdot \big(1 - d_i(\mathbf{p},\mathbf{r})\big) & \textit{if } j = i, \\ -c_j \cdot d_i(\mathbf{p},\mathbf{r}) \cdot d_j(\mathbf{p},\mathbf{r}) & \textit{if } j \neq i. \end{cases} \tag{L.29b}$$

Meanwhile, $G_i(\mathbf{p},\mathbf{r})$ and its gradient are bounded as follows:

$$\big|G_i(\mathbf{p},\mathbf{r})\big| \leq M_G, \quad \big\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\big\| \leq \ell_{r,i}, \quad \forall \mathbf{p},\mathbf{r} \in \mathcal{P}^n,\ \forall i \in N, \tag{L.30}$$

where the upper bound M_G and the Lipschitz constant $\ell_{r,i}$ are defined as

$$M_G := \max_{i \in N} \left\{ \frac{1}{(b_i + c_i)\underline{p}} \right\} + 1, \quad \ell_{r,i} := \frac{1}{4}\sqrt{c_i^2 + \max_{j \neq i}\{c_j^2\}}. \tag{L.31}$$

Proof of Lemma EC.4. We first verify the partial derivatives in Eqs. (L.29a) and (L.29b):

$$\begin{aligned} \frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_i} &= -\frac{1}{(b_i+c_i)p_i^2} + \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial p_i} \\ &= -\frac{1}{(b_i+c_i)p_i^2} - \frac{(b_i+c_i)\cdot \exp\big(u_i(p_i,r_i)\big)\cdot\Big(1+\sum_{j\neq i}\exp\big(u_j(p_j,r_j)\big)\Big)}{\Big(1+\sum_{k\in N}\exp\big(u_k(p_k,r_k)\big)\Big)^2} \\ &= -\frac{1}{(b_i+c_i)p_i^2} - (b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big). \end{aligned} \tag{L.32}$$

For product $j \neq i$, its partial derivative can be computed as

$$\begin{aligned} \frac{\partial G_i(\mathbf{p},\mathbf{r})}{\partial p_j} &= \frac{\partial d_i(\mathbf{p},\mathbf{r})}{\partial p_j} \\ &= \frac{(b_j+c_j)\cdot\exp\big(u_i(p_i,r_i)\big)\cdot\exp\big(u_j(p_j,r_j)\big)}{\Big(1+\sum_{k\in N}\exp\big(u_k(p_k,r_k)\big)\Big)^2} \\ &= (b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}). \end{aligned} \tag{L.33}$$

Then, the partial derivatives with respect to $\mathbf{r}$, as shown in Eq. (L.29b), can be similarly computed.

In the next part, we show that $G_i(\mathbf{p},\mathbf{r})$ (see its definition in Eq. (D.2)) is bounded for all $\mathbf{p},\mathbf{r} \in \mathcal{P}^n$ and all product $i \in N$.

$$\begin{aligned} \big|G_i(\mathbf{p},\mathbf{r})\big| &= \left|\frac{1}{(b_i+c_i)p_i} + d_i(\mathbf{p},\mathbf{r}) - 1\right| \\ &\leq \left|\frac{1}{(b_i+c_i)p_i}\right| + \big|d_i(\mathbf{p},\mathbf{r}) - 1\big| \\ &\leq \frac{1}{(b_i+c_i)\underline{p}} + 1 \\ &\leq \max_{k\in N}\left\{\frac{1}{(b_k+c_k)\underline{p}}\right\} + 1 =: M_G, \end{aligned} \tag{L.34}$$

where the maximum operation in the last line is to ensure the validity of the bound for all $i \in N$.

Finally, we demonstrate that $\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|$ is also bounded for all $\mathbf{p},\mathbf{r}\in\mathcal{P}^n$ and all product $i\in N$. From Eq. (L.29b), we have that

$$\begin{aligned}
\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|^2 &= \Big(c_i\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\sum_{j\neq i}\Big(-c_j\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r})\Big)^2\\
&\le c_i^2\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\max_{j\neq i}\{c_j^2\}\cdot\big(d_i(\mathbf{p},\mathbf{r})\big)^2\sum_{j\neq i}\big(d_j(\mathbf{p},\mathbf{r})\big)^2\\
&\overset{(\Delta_1)}{\le} c_i^2\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2+\max_{j\neq i}\{c_j^2\}\cdot\Big(d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big)^2\\
&\overset{(\Delta_2)}{\le}\frac{1}{16}\Big(c_i^2+\max_{j\neq i}\{c_j^2\}\Big),
\end{aligned}\tag{L.35}$$

where step (Δ_1) results from the fact that $\sum_{j\neq i}\big(d_j(\mathbf{p},\mathbf{r})\big)^2\le\big(\sum_{j\neq i}d_j(\mathbf{p},\mathbf{r})\big)^2\le\big(1-d_i(\mathbf{p},\mathbf{r})\big)^2$. The inequality (Δ_2) follows from the fact that $x\cdot y\le 1/4$ for any two numbers such that $x,y>0$ and $x+y\le 1$. Therefore, it follows that $\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p},\mathbf{r})\right\|_2\le(1/4)\sqrt{c_i^2+\max_{j\neq i}\{c_j^2\}}=:\ell_{r,i}$. □

L.4 Lemma EC.5

Lemma EC.5. *In the loss-neutral scenario, the revenue and demand function satisfy that*

$$\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|\le\ell_{p,i},\quad \|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|\le\overline{p}\cdot\ell_{r,i},\quad \|\nabla_{\mathbf{p}}d_i(\mathbf{p})\|\le\ell_{d,i},\quad \forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N,\tag{L.36}$$

where $d_i(\mathbf{p}):=d_i(\mathbf{p},\mathbf{p})$, constant $\ell_{r,i}$ is defined in Eq. (L.31)*, and the Lipschitz constants $\ell_{p,i},\ell_{d,i}$ are defined as*

$$\ell_{p,i}:=\frac{1}{4}\sqrt{16+\overline{p}^2\left[(b_i+c_i)^2+\max_{j\neq i}\{(b_j+c_j)^2\}\right]},\quad \ell_{d,i}:=\frac{1}{4}\sqrt{b_i^2+\max_{j\neq i}\{b_j^2\}}.\tag{L.37}$$

Proof of Lemma EC.5. We begin with showing the first bound in Eq. (L.36). Since we have that

$$\frac{\partial\Pi_i(\mathbf{p},\mathbf{r})}{\partial p_j}=\begin{cases} d_i(\mathbf{p},\mathbf{r})-p_i(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big) & \text{if } j=i,\\ p_i(b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r}) & \text{if } j\neq i.\end{cases}\tag{L.38}$$

Using the partial derivatives in Eq. (L.38), we compute that

$$\begin{aligned}
\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|^2 &\le 1+\overline{p}^2\left\{\Big[(b_i+c_i)\cdot d_i(\mathbf{p},\mathbf{r})\cdot\big(1-d_i(\mathbf{p},\mathbf{r})\big)\Big]^2+\sum_{j\neq i}\Big[(b_j+c_j)\cdot d_i(\mathbf{p},\mathbf{r})\cdot d_j(\mathbf{p},\mathbf{r})\Big]^2\right\}\\
&\le 1+\frac{\overline{p}^2}{16}\left[(b_i+c_i)^2+\max_{j\neq i}\{(b_j+c_j)^2\}\right],\quad \forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N,
\end{aligned}$$

where $1/16$ in the last line follows from the same reasoning as Eq. (L.35). Taking the square root on both sides of the above inequality yields the desired upper bound for $\|\nabla_{\mathbf{p}}\Pi_i(\mathbf{p},\mathbf{r})\|$.

Next, we show the bound for $\|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|$. According to the definition of $G_i(\mathbf{p},\mathbf{r})$ in Eq. (D.2), it holds that $\|\nabla_{\mathbf{r}}G_i(\mathbf{p},\mathbf{r})\|=\|\nabla_{\mathbf{r}}d_i(\mathbf{p},\mathbf{r})\|\le\ell_{r,i}$, where the last inequality stems from Eq. (L.30). Since $\Pi_i(\mathbf{p},\mathbf{r})=p_i\cdot d_i(\mathbf{p},\mathbf{r})$, it follows that $\|\nabla_{\mathbf{r}}\Pi_i(\mathbf{p},\mathbf{r})\|=p_i\,\|\nabla_{\mathbf{r}}d_i(\mathbf{p},\mathbf{r})\|\le\overline{p}\cdot\ell_{r,i}$.

Finally, we derive the last bound in Eq. (L.36). Recall that $d_i(\mathbf{p}) := d_i(\mathbf{p},\mathbf{p})$.

$$\frac{\partial d_i(\mathbf{p})}{\partial p_j} = \begin{cases} -b_i \cdot d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big) & \text{if } j = i, \\ b_j \cdot d_i(\mathbf{p}) \cdot d_j(\mathbf{p}) & \text{if } j \neq i. \end{cases} \tag{L.39}$$

Then, the above partial derivatives indicate that

$$\begin{aligned} \|\nabla_{\mathbf{p}} d_i(\mathbf{p})\|^2 &\leq b_i^2 \Big(d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big)\Big)^2 + \max_{j\neq i}\{b_j^2\} \cdot \big(d_i(\mathbf{p})\big)^2 \sum_{j\neq i} \big(d_j(\mathbf{p})\big)^2 \\ &\leq \frac{1}{16}\big(b_i^2 + \max_{j\neq i}\{b_j^2\}\big), \quad \forall \mathbf{p} \in \mathcal{P}^n,\ \forall i \in N, \end{aligned} \tag{L.40}$$

where the last inequality uses a similar reasoning in Eq. (L.35) again. Lastly, we take the square root of both sides to obtain $\|\nabla_{\mathbf{p}} d_i(\mathbf{p})\| \leq (1/4)\sqrt{b_i^2 + \max_{j\neq i}\{b_j^2\}} =: \ell_{d,i}$. $\square$

L.5 Lemma EC.6

We observe from Eqs. (2) and (3) that the revenue $\Pi_i(\mathbf{p},\mathbf{r})$ depends on $\mathbf{p}_{-i}$ and $\mathbf{r}_{-i}$ through their utility functions. Recall that $\mathbf{p}_{-i} := \big(p_j\big)_{j\in N\setminus\{i\}}$ and $\mathbf{r}_{-i} := \big(r_j\big)_{j\in N\setminus\{i\}}$. We use $\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})$ to denote the vector of utilities for all products except i, i.e.,

$$\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}) := \big(u_1(p_1,r_1),\ldots,u_{i-1}(p_{i-1},r_{i-1}),u_{i+1}(p_{i+1},r_{i+1}),\ldots,u_n(p_n,r_n)\big). \tag{L.41}$$

Given $\mathbf{p}_{-i}$ and $\mathbf{r} = (r_i,\mathbf{r}_{-i})$, we use $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ to denote the best-response price that achieve the optimal single-period revenue for product i, defined as

$$p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big) := \arg\max_{p_i\in\mathcal{P}} \big\{\Pi_i\big((p_i,\mathbf{p}_{-i}),\mathbf{r}\big)\big\} = \arg\max_{p_i\in\mathcal{P}} \big\{p_i \cdot d_i\big((p_i,\mathbf{p}_{-i}),\mathbf{r}\big)\big\}. \tag{L.42}$$

In the following lemma, we demonstrate the Lipschitz continuity of $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$.

Lemma EC.6. *In the loss-neutral scenario, let $p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ be the best-response price for product i given $\mathbf{p}_{-i}$ and $\mathbf{r}$, as defined in Eq. (L.42). Then, it holds that*

$$\left|\frac{\partial p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)}{\partial r_i}\right| \leq \frac{c_i}{b_i + c_i}, \quad \big\|\nabla_{\mathbf{u}_{-i}} p_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)\big\| \leq \overline{p}, \tag{L.43}$$

for all $\mathbf{p}_{-i} \in \mathcal{P}^{n-1}$ and $\mathbf{r} \in \mathcal{P}^n$.

Proof of Lemma EC.6. Given $\mathbf{p}_{-i}$ and $\mathbf{r}$, we use $\Pi_i^B\left(r_i,\mathbf{u}_{-i}\left(\mathbf{p}_{-i},\mathbf{r}_{-i}\right)\right)$ to denote the optimal single-period revenue for product i, defined as $\Pi_i^B\left(r_i,\mathbf{u}_{-i}\left(\mathbf{p}_{-i},\mathbf{r}_{-i}\right)\right) := \max_{p_i\in\mathcal{P}}\big\{\Pi_i\big((p_i,\mathbf{p}_{-i}),\mathbf{r}\big)\big\}$.

Similar as the Part 2 proof of Proposition 1, we can actually show that the first order-condition (i.e., $\partial\Pi_i(\mathbf{p},\mathbf{r})/\partial p_i = 0$) is necessary and sufficient for the best-response price (see Eqs. (C.11) to (C.14) in Appendix C.1). We refer the readers to Theorems 1 and 2 in Guo et al. (2022) for a more detailed discussion. Hence, $\Pi_i^B(\cdot,\cdot)$ is the optimal single-period revenue if and only if the following first-order condition is satisfied

$$\begin{aligned}
&\left.\frac{\partial \Pi_i(\mathbf{p},\mathbf{r})}{\partial p_i}\right|_{p_i=p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))}=0\\
\Leftrightarrow\quad & d_i^B-(b_i+c_i)\cdot p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))\cdot d_i^B\left(1-d_i^B\right)=0\\
\Leftrightarrow\quad & 1=(b_i+c_i)\cdot\left(p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))-\Pi_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))\right)\\
\Leftrightarrow\quad & p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))=\Pi_i^B\left(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\right)+\frac{1}{b_i+c_i},
\end{aligned}\tag{L.44}$$

where we use d_i^B to denote the demand at the best-response price, i.e., $d_i^B:=d_i\big((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\big)$. From Eq. (L.44), we observe that $p_i^B(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i}))$ and $\Pi_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$ only differs by a constant $1/(b_i+c_i)$. Hence, it is equivalent to derive the Lipschitz continuity of $\Pi_i^B\big(r_i,\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})\big)$.

As the information of $\mathbf{p}_{-i}$ and $\mathbf{r}_{-i}$ is already absorbed in their utility functions, we adopt the shorthand notation $\mathbf{u}_{-i}:=\mathbf{u}_{-i}(\mathbf{p}_{-i},\mathbf{r}_{-i})$ and $u_j:=u_j(p_j,r_j)$ for all $j\in N\setminus\{i\}$. In addition, when it is clear from the context, we may also use the simplified notations $p_i^B:=p_i^B(r_i,\mathbf{u}_{-i})$ and $\Pi_i^B:=\Pi_i^B\big(r_i,\mathbf{u}_{-i}\big)$. Using the definition of the revenue function and the relation in Eq. (L.44), we can express Π_i^B using an implicit equation:

$$\begin{aligned}
\Pi_i^B=p_i^Bd_i^B &\overset{(\Delta_1)}{=}\left(\Pi_i^B+\frac{1}{b_i+c_i}\right)\frac{\exp\big(u_i(p_i^B,r_i)\big)}{1+\exp\big(u_i(p_i^B,r_i)\big)+\sum_{j\neq i}\exp\big(u_j\big)}\\
\Leftrightarrow \Pi_i^B\Big(1+\sum_{j\neq i}\exp\big(u_j\big)\Big)&=\frac{1}{b_i+c_i}\exp\big(u_i(p_i^B,r_i)\big)\\
\overset{(\Delta_2)}{\Leftrightarrow}(b_i+c_i)\cdot\Pi_i^B\Big(1+\sum_{j\neq i}\exp\big(u_j\big)\Big)&=\exp\big(a_i-(b_i+c_i)\cdot\Pi_i^B+c_ir_i-1\big),
\end{aligned}\tag{L.45}$$

where the expressions in (Δ_1) and (Δ_2) are obtained by substituting p_i^B with the first-order condition in the last line of Eq. (L.44).

With the goal of computing the Lipschitz coefficients, we use the implicit function theorem to derive the partial derivatives of $\Pi_i^B\big(r_i,\mathbf{u}_{-i}\big)$ with respect to r_i and $\mathbf{u}_{-i}$. To begin with, we first define a function $\Psi(\Pi_i,r_i,\mathbf{u}_{-i})$ as below

$$\Psi\big(\Pi_i,r_i,\mathbf{u}_{-i}\big)=(b_i+c_i)\cdot\Pi_i\cdot\Big(1+\sum_{j\neq i}\exp(u_j)\Big)-\exp\big(a_i-(b_i+c_i)\Pi_i+c_ir_i-1\big).\tag{L.46}$$

Since $\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)=0$ by Eq. (L.45), we apply the implicit function theorem to derive that

$$\begin{aligned}
\frac{\partial\Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial r_i}&=-\frac{\frac{\partial}{\partial r_i}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}{\frac{\partial}{\partial \Pi_i}\Psi\big(\Pi_i^B,r_i,\mathbf{u}_{-i}\big)}\\
&=\frac{c_i\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}{(b_i+c_i)\cdot\big(1+\sum_{j\neq i}\exp(u_j)\big)+(b_i+c_i)\cdot\exp\big(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\big)}.
\end{aligned}\tag{L.47}$$

Hence, we can upper-bound the above partial derivative as detailed below

$$\begin{aligned}\left|\frac{\partial \Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial r_i}\right| &= \left|\frac{c_i \exp\left(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\right)}{(b_i+c_i)\cdot\left(1+\sum_{j\neq i}\exp(u_j)\right)+(b_i+c_i)\cdot\exp\left(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\right)}\right| \\ &\overset{(\Delta)}{=} \left|\frac{c_i\exp\left(u_i(p_i^B,r_i)\right)}{(b_i+c_i)\cdot\left[1+\exp\left(u_i(p_i^B,r_i)\right)+\sum_{j\neq i}\exp(u_j)\right]}\right| \\ &= \frac{c_i}{b_i+c_i}\cdot d_i^B \le \frac{c_i}{b_i+c_i},\end{aligned} \tag{L.48}$$

where in step (Δ), we use the fact that $\Pi_i^B = p_i^B - 1/(b_i+c_i)$ from Eq. (L.44).

Next, to bound the gradient of $\Pi_i^B(r_i,\mathbf{u}_{-i})$ with respect to $\mathbf{u}_{-i}$, we first calculate its partial derivative for product $j\in N\setminus\{i\}$ by the implicit function theorem

$$\begin{aligned}\frac{\partial \Pi_i^B(r_i,\mathbf{u}_{-i})}{\partial u_j} &= -\frac{\frac{\partial}{\partial u_j}\Psi\left(\Pi_i^B,r_i,\mathbf{u}_{-i}\right)}{\frac{\partial}{\partial \Pi_i}\Psi\left(\Pi_i^B,r_i,\mathbf{u}_{-i}\right)} \\ &= \frac{-(b_i+c_i)\cdot\Pi_i^B\cdot\exp(u_j)}{(b_i+c_i)\cdot\left(1+\sum_{k\neq i}\exp(u_k)\right)+(b_i+c_i)\cdot\exp\left(a_i-(b_i+c_i)\Pi_i^B+c_ir_i-1\right)} \\ &= -\Pi_i^B\cdot d_j\left((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\right).\end{aligned} \tag{L.49}$$

Then, we can bound the gradient as follows

$$\left\|\nabla_{\mathbf{u}_{-i}}\Pi_i^B(r_i,\mathbf{u}_{-i})\right\| \le \Pi_i^B\cdot\sqrt{\sum_{j\neq i}\left(d_j\left((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\right)\right)^2} \le \Pi_i^B = p_i^B\cdot d_i^B \le \overline{p}. \tag{L.50}$$

where the last inequality results from the fact that $d_i^B = d_i\left((p_i^B,\mathbf{p}_{-i}),\mathbf{r}\right)\le 1$ and $p_i^B\in[\underline{p},\overline{p}]$. As $p_i^B = \Pi_i^B + 1/(b_i+c_i)$ from Eq. (L.44), calculations in Eqs. (L.48) and (L.49) conclude the proof. ☐

L.6 Lemma EC.7

Lemma EC.7. *In the loss-averse scenario, let $G_i^\diamond(\mathbf{p},\mathbf{r})$ be the scaled true/virtual derivative defined in Eq. (G.1), where $\diamond\in\{+,-\}$. Then, $G_i^\diamond(\mathbf{p},\mathbf{r})$ and its (sub)-gradients are bounded as follows*

$$|G_i^\diamond(\mathbf{p},\mathbf{r})|\le\tilde{M}_G,\ \|\nabla_{\mathbf{r}}G_i^\diamond(\mathbf{p},\mathbf{r})\|\le\tilde{\ell}_{r,i},\ \|\nabla_{\mathbf{p}}G_i^\diamond(\mathbf{p})\|\le\tilde{\ell}_{p,i},\ \forall\diamond\in\{+,-\},\ \forall\mathbf{p},\mathbf{r}\in\mathcal{P}^n,\ \forall i\in N, \tag{L.51}$$

where $G_i^\diamond(\mathbf{p}):=G_i^\diamond(\mathbf{p},\mathbf{p})$. The upper bound $\tilde{M}_G$ and the Lipschitz constants $\tilde{\ell}_{r,i}$, $\tilde{\ell}_{p,i}$ are defined as

$$\tilde{M}_G := \max_{i\in N}\left\{\frac{1}{(b_i+c_i^+)\underline{p}}\right\}+1,\quad \tilde{\ell}_{r,i} := \frac{1}{4}\sqrt{(c_i^-)^2+\max_{j\neq i}\left\{(c_j^-)^2\right\}}, \tag{L.52}$$

$$\tilde{\ell}_{p,i} := \sqrt{\frac{1}{(b_i+c_i^+)^2\underline{p}^4}+\frac{b_i}{2(b_i+c_i^+)\underline{p}^2}+\frac{b_i^2+\max_{j\neq i}\left\{b_j^2\right\}}{16}}. \tag{L.53}$$

Proof of Lemma EC.7. The first two bounds presented in Eq. (L.51) are analogous to their loss-neutral equivalents found in Eq. (L.30). It is straightforward to show that $|G_i^\diamond(\mathbf{p},\mathbf{r})| \le \tilde{M}_G$ and $\|\nabla_{\mathbf{r}} G_i^\diamond(\mathbf{p},\mathbf{r})\| \le \tilde{\ell}_{r,i}$ by similar procedures outlined in Eq. (L.34) and Eq. (L.35), respectively.

Now, we are left to show the last bound in Eq. (L.51). We start with computing the following partial derivative

$$\frac{\partial G_i^\diamond(\mathbf{p})}{\partial p_j} = \begin{cases} -\dfrac{1}{b_i + c_i^\diamond} \cdot \dfrac{1}{p_i^2} - b_i \cdot d_i(\mathbf{p}) \cdot \big(1 - d_i(\mathbf{p})\big) & \text{if } j = i, \\ b_j \cdot d_i(\mathbf{p}) \cdot d_j(\mathbf{p}) & \text{if } j \neq i, \end{cases} \tag{L.54}$$

where we recall that $d_i(\mathbf{p}) := d_i(\mathbf{p},\mathbf{p})$. With the information in Eq. (L.54), we are ready to derive the final bound:

$$\begin{aligned} \|\nabla_{\mathbf{p}} G_i^\diamond(\mathbf{p})\|^2 &\le \frac{1}{(b_i + c_i^\diamond)^2 \underline{p}^4} + \frac{2b_i d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big)}{(b_i + c_i^\diamond)\underline{p}^2} + b_i^2 \Big(d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big)\Big)^2 + \sum_{j \neq i} \Big(b_j d_i(\mathbf{p}) d_j(\mathbf{p})\Big)^2 \\ &\overset{(\Delta)}{\le} \frac{1}{(b_i + c_i^\diamond)^2 \underline{p}^4} + \frac{b_i}{2(b_i + c_i^\diamond)\underline{p}^2} + \frac{b_i^2 + \max_{j \neq i}\{b_j^2\}}{16} \\ &\le \frac{1}{(b_i + c_i^+)^2 \underline{p}^4} + \frac{b_i}{2(b_i + c_i^+)\underline{p}^2} + \frac{b_i^2 + \max_{j \neq i}\{b_j^2\}}{16}, \end{aligned} \tag{L.55}$$

where the second term in step (Δ) follows from the fact that $d_i(\mathbf{p})\big(1 - d_i(\mathbf{p})\big) \le 1/4$, and the last term is derived via the same method used in (Δ_2) of Eq. (L.35). In the last line, we replace $c_i^\diamond$ with c_i^+ to ensure the bound works for both $\diamond \in \{+,-\}$, as $c_i^+ \le c_i^-$ in the loss-averse scenario. Taking the square root of both sides in Eq. (L.55) yields the final result, and this completes the proof of Lemma EC.7. □

L.7 Lemma EC.8

LEMMA EC.8. *In the loss-averse scenario, let $\tilde{\mathbf{p}}^\star$ be the unique SNE that satisfies*

$$\tilde{p}_i^\star = \frac{1}{(b_i + \tilde{c}_i) \cdot (1 - d_i(\tilde{\mathbf{p}}^\star, \tilde{\mathbf{p}}^\star))}, \quad \forall i \in N, \tag{L.56}$$

where $\tilde{c}_i \in [c_i^+, c_i^-]$. Define the function $\tilde{\mathcal{G}}(\mathbf{p})$ as

$$\tilde{\mathcal{G}}(\mathbf{p}) := \sum_{i \in N} \mathtt{sign}\,(\tilde{p}_i^\star - p_i) \left[\frac{1}{(b_i + \tilde{c}_i) p_i} + d_i(\mathbf{p},\mathbf{p}) - 1\right], \tag{L.57}$$

where the function $\mathtt{sign}(\cdot)$ is defined in Eq. (D.4). Then, it holds that

$$\tilde{\mathcal{G}}(\mathbf{p}) \ge \frac{1}{\overline{p}\,\|\tilde{\mathbf{p}}^\star\|_\infty} \cdot \sum_{i \in N} \frac{|\tilde{p}_i^\star - p_i|}{b_i + \tilde{c}_i}, \quad \forall \mathbf{p} \in \mathcal{P}^n. \tag{L.58}$$

Proof of Lemma EC.8. This lemma is the loss-averse version of Lemma EC.3. Its proof follows a similar scheme as that of Lemma EC.3. □

Appendix M Proofs for Local Convergence of OPGA

M.1 Proof of Lemma EC.1

LEMMA EC.1**(Restated).** *In the loss-neutral scenario, define function $\mathcal{H}(\mathbf{p})$ as follows:*

$$\mathcal{H}(\mathbf{p}) := \sum_{i\in N}(p_i^\star - p_i)\cdot \frac{\partial \log\left(\Pi(\mathbf{p},\mathbf{r})\right)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i\in N}\left[\frac{1}{p_i} + (b_i+c_i)\left(d_i(\mathbf{p},\mathbf{p})-1\right)\right](p_i^\star - p_i), \tag{M.1}$$

where $\mathbf{p}^\star$ is the unique SNE. Then, there exist $\gamma>0$ and a open set $U_\gamma \ni \mathbf{p}^\star$ such that

$$\mathcal{H}(\mathbf{p}) \geq \gamma\cdot\|\mathbf{p}-\mathbf{p}^\star\|^2, \quad \forall \mathbf{p}\in U_\gamma. \tag{M.2}$$

Proof of Lemma EC.1. We consider the second-order Taylor expansion of $\mathcal{H}(\mathbf{p})$ at $\mathbf{p}^\star$. For all $\mathbf{p}\in\mathcal{P}^n$, there exists $\widehat{\mathbf{p}}$ on the line segment between $\mathbf{p}$ and $\mathbf{p}^\star$ such that

$$\begin{aligned}\mathcal{H}(\mathbf{p}) &= \mathcal{H}(\mathbf{p}^\star) + \nabla\mathcal{H}(\mathbf{p}^\star)\cdot\left(\mathbf{p}-\mathbf{p}^\star\right) + \frac{1}{2}\left(\mathbf{p}-\mathbf{p}^\star\right)^\top \nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\left(\mathbf{p}-\mathbf{p}^\star\right)\\ &= \nabla\mathcal{H}(\mathbf{p}^\star)\cdot\left(\mathbf{p}-\mathbf{p}^\star\right) + \frac{1}{2}\left(\mathbf{p}-\mathbf{p}^\star\right)^\top \nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\left(\mathbf{p}-\mathbf{p}^\star\right),\end{aligned} \tag{M.3}$$

where the second equality arises from $\mathcal{H}(\mathbf{p}^\star)=0$. We first compute the the gradient $\nabla\mathcal{H}(\mathbf{p}) = (\partial\mathcal{H}(\mathbf{p})/\partial p_1,\ldots,\partial\mathcal{H}(\mathbf{p})/\partial p_n)$, where we adopt the shorthand notation $d_i(\mathbf{p}) := d_i(\mathbf{p},\mathbf{p})$ and use the partial derivative of $d_i(\mathbf{p})$ in Eq. (L.39):

$$\begin{aligned}\frac{\partial\mathcal{H}(\mathbf{p})}{\partial p_i} &= -\left[\frac{1}{p_i} + (b_i+c_i)\big(d_i(\mathbf{p})-1\big)\right] + (p_i^\star - p_i)\left[-\frac{1}{p_i^2} - (b_i+c_i)b_i\cdot d_i(\mathbf{p})\big(1-d_i(\mathbf{p})\big)\right]\\ &\quad + \sum_{j\neq i}(p_j^\star - p_j)\cdot(b_j+c_j)b_i\cdot d_j(\mathbf{p})d_i(\mathbf{p})\\ &= -\left[\frac{1}{p_i} + (b_i+c_i)\big(d_i(\mathbf{p})-1\big)\right] + b_i\cdot d_i(\mathbf{p})\sum_{k\in N}(p_k^\star - p_k)\cdot(b_k+c_k)d_k(\mathbf{p})\\ &\quad - (p_i^\star - p_i)\left[\frac{1}{p_i^2} + (b_i+c_i)b_i\cdot d_i(\mathbf{p})\right].\end{aligned} \tag{M.4}$$

When this partial derivative evaluates at $\mathbf{p}^\star$, the first term becomes $1/p_i^\star + (b_i+c_i)\big(d_i(\mathbf{p}^\star)-1\big)=0$, since $\mathbf{p}^\star$ satisfies the first-order condition in Eq. (C.16). Hence, it follows that $\nabla\mathcal{H}(\mathbf{p}^\star)=0$, and Eq. (M.3) simplifies to $\mathcal{H}(\mathbf{p}) = \frac{1}{2}\left(\mathbf{p}-\mathbf{p}^\star\right)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\left(\mathbf{p}-\mathbf{p}^\star\right)$.

Below, we aim to show that there exists $\gamma>0$ such that $\nabla^2\mathcal{H}(\mathbf{p}) \succ 2\gamma I_n$ when $\mathbf{p}$ belongs to some neighborhood U_γ of $\mathbf{p}^\star$, where I_n is the $n\times n$ identity matrix. Then, for any $\mathbf{p}\in U_\gamma$, it follows that

$$\mathcal{H}(\mathbf{p}) = \frac{1}{2}\left(\mathbf{p}-\mathbf{p}^\star\right)^\top\nabla^2\mathcal{H}(\widehat{\mathbf{p}})\cdot\left(\mathbf{p}-\mathbf{p}^\star\right) \geq \gamma\left\|\mathbf{p}-\mathbf{p}^\star\right\|^2. \tag{M.5}$$

We first compute the Hessian matrix $\nabla^2\mathcal{H}(\mathbf{p})$ evaluated at $\mathbf{p}^\star$. The second-order partial derivatives can be calculated as follows

$$\begin{aligned}
\frac{\partial^2\mathcal{H}(\mathbf{p}^\star)}{\partial p_i^2} &= \frac{1}{(p_i^\star)^2} + (b_i+c_i)b_i\cdot d_i(\mathbf{p}^\star)\big(1-d_i(\mathbf{p}^\star)\big) - (b_i+c_i)b_i\cdot\big(d_i(\mathbf{p}^\star)\big)^2 + \frac{1}{(p_i^\star)^2} + (b_i+c_i)b_i\cdot d_i(\mathbf{p}^\star)\\
&= \frac{2}{(p_i^\star)^2} + 2(b_i+c_i)b_i\cdot d_i(\mathbf{p}^\star)\big(1-d_i(\mathbf{p}^\star)\big)\\
&\overset{(\Delta)}{=} 2(b_i+c_i)^2\big(1-d_i(\mathbf{p}^\star)\big)^2 + 2(b_i+c_i)b_i\cdot d_i(\mathbf{p}^\star)\big(1-d_i(\mathbf{p}^\star)\big)\\
&= 2(b_i+c_i)\cdot\big(1-d_i(\mathbf{p}^\star)\big)\big(b_i+c_i-c_id_i(\mathbf{p}^\star)\big)\\
&= (b_i+c_i)^2\cdot 2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\frac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right),
\end{aligned} \tag{M.6}$$

were step (Δ) utilizes the first-order condition in Eq. (C.16) and substitutes $1/p_i^\star$ with $(b_i+c_i)\big(1-d_i(\mathbf{p}^\star)\big)$. Similarly, we can compute the second-order cross derivatives as

$$\begin{aligned}
\frac{\partial^2\mathcal{H}(\mathbf{p}^\star)}{\partial p_i\partial p_j} &= -(b_i+c_i)b_j\cdot d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star) - b_i(b_j+c_j)\cdot d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\\
&= -\big[(b_j+c_j)b_i + (b_i+c_i)b_j\big]\cdot d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\\
&= -(b_i+c_i)(b_j+c_j)\cdot\left(\frac{b_i}{b_i+c_i}+\frac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star).
\end{aligned} \tag{M.7}$$

Based on Eqs. (M.6) and Eq. (M.7), we observe that the Hessian matrix can be decomposed as $\nabla^2\mathcal{H}(\mathbf{p}^\star) = AQA$, where A is a diagonal matrix defined as

$$A := \begin{bmatrix} b_1+c_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & b_n+c_n\end{bmatrix},$$

and matrix Q is defined such that its (i,j)-th entry is equal to

$$Q_{ij} = \begin{cases} 2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\dfrac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right) & \text{if } i=j,\\ -\left(\dfrac{b_i}{b_i+c_i}+\dfrac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star) & \text{if } i\neq j.\end{cases} \tag{M.8}$$

Since A is clearly an invertible matrix, demonstrating $\nabla^2\mathcal{H}(\mathbf{p}^\star)$ is positive definite is equivalent to proving Q is positive definite. Then, it suffices to show for any vector $\mathbf{x}$, $\mathbf{x}^\top Q\mathbf{x} > 0$. Without loss of generality, we assume $\|\mathbf{x}\|=1$, and it follows that

$$\begin{aligned}
\mathbf{x}^\top Q\mathbf{x} &= 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\frac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right)-\frac{1}{2}\sum_{j\neq i}x_ix_j\left(\frac{b_i}{b_i+c_i}+\frac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&\geq 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)\left(1-\frac{c_i}{b_i+c_i}d_i(\mathbf{p}^\star)\right)-\frac{1}{2}\sum_{j\neq i}|x_i|\,|x_j|\left(\frac{b_i}{b_i+c_i}+\frac{b_j}{b_j+c_j}\right)d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&> 2\sum_{i\in N}\left[x_i^2\big(1-d_i(\mathbf{p}^\star)\big)^2-\sum_{j\neq i}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right]\\
&= 2\sum_{i\in N}\left\{x_i^2\left[\big(1-d_i(\mathbf{p}^\star)\big)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\sum_{j\in N}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)\right\}\\
&\overset{(\Delta_1)}{=} 2\left\{\sum_{i\in N}x_i^2\left[\big(1-d_i(\mathbf{p}^\star)\big)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2\right\}\\
&\overset{(\Delta_2)}{>} 2\left\{\sum_{i\in N}x_i^2\left[\left(\sum_{j\neq i}d_j(\mathbf{p}^\star)\right)^2+\big(d_i(\mathbf{p}^\star)\big)^2\right]-\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2\right\}\\
&> 2\left\{\sum_{i\in N}x_i^2\cdot\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2-\sum_{i\in N}\big(d_i(\mathbf{p}^\star)\big)^2\right\}=0,
\end{aligned}$$

where step (Δ_1) follows from $\sum_{i\in N}\sum_{j\in N}|x_i|\,|x_j|\,d_i(\mathbf{p}^\star)d_j(\mathbf{p}^\star)=\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2$. In step (Δ_2), we first use the fact that $1-d_i(\mathbf{p}^\star)=d_0(\mathbf{p}^\star)+\sum_{j\neq i}d_j(\mathbf{p}^\star)>\sum_{j\neq i}d_j(\mathbf{p}^\star)$, where $d_0(\mathbf{p}^\star)$ is the no-purchase probability. Then, we use the Cauchy-Schwarz inequality, i.e., $\left(\sum_{i\in N}|x_i|\,d_i(\mathbf{p}^\star)\right)^2\leq\|\mathbf{x}\|^2\sum_{i\in N}\big(d_i(\mathbf{p})\big)^2$.

As a result, we conclude that $\nabla^2\mathcal{H}(\mathbf{p}^\star)$ is positive definite. By the continuity of $\nabla^2\mathcal{H}(\mathbf{p})$, there exists some constant $\gamma>0$ and a open set $U_\gamma\ni\mathbf{p}^\star$ such that $\nabla^2\mathcal{H}(\mathbf{p}^\star)\succ 2\gamma I_n$ for all $\mathbf{p}\in U_\gamma$. Together with Eq. (M.5), this completes the proof of Lemma EC.1. □

M.2 Proof of Proposition EC.1

Proposition EC.1. **(Restated)** *In the loss-neutral scenario, let the step-sizes $\{\eta^t\}_{t\geq 0}$ be a non-increasing sequence such that $\lim_{t\to\infty}\eta^t=0$ and $\sum_{t=0}^\infty\eta^t=\infty$ hold. Then, there exists some neighborhood $\mathcal{B}$ of $\mathbf{p}^\star$ such that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters $\mathcal{B}$ with a sufficiently small step-size, the price path will stay in $\mathcal{B}$ during subsequent periods.*

Furthermore, suppose the step-sizes satisfy $\eta^t=\frac{C_\eta}{t+1}$ for all $t\geq 1$, where C_η is some general constant. Then, the local convergence rate of $\{(\mathbf{p}^t,\mathbf{r}^t)\}_{t\geq 0}$ after the path stays in $\mathcal{B}$ satisfies that

$$\left\|\mathbf{p}^\star-\mathbf{p}^t\right\|^2\leq\mathcal{O}\left(\frac{1}{t}\right),\quad\left\|\mathbf{p}^\star-\mathbf{r}^t\right\|^2\leq\mathcal{O}\left(\frac{1}{t}\right). \tag{M.9}$$

Proof of Proposition EC.1. Let $\{\mathbf{p}^t\}_{t\geq 0}$ be the price path generated by Algorithm 1 with step-sizes $\{\eta^t\}_{t\geq 0}$ such that $\lim_{t\to\infty}\eta^t=0$ and $\sum_{t=0}^\infty\eta^t=\infty$. In the following, we use Lemma EC.1

demonstrate that when the price path $\{\mathbf{p}^t\}_{t\geq 0}$ enters the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0} := \{\mathbf{p} \in \mathcal{P}^n \mid \|\mathbf{p} - \mathbf{p}^\star\| < \epsilon_0\}$ for some sufficiently small $\epsilon_0 > 0$ with small enough step-sizes, the price path will stay in $\mathcal{B}_{\epsilon_0}$ during subsequent periods. In particular, we prove it by induction, where we show that when $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0}$ for some sufficiently large t, then it also holds that $\mathbf{p}^{t+1} \in \mathcal{B}_{\epsilon_0}$. The value of ϵ_0 will be specified later in the proof.

By the update rule of Algorithm 1, it follows that

$$\begin{aligned}
\left|p_i^\star - p_i^{t+1}\right|^2 &= \left|p_i^\star - \mathtt{Proj}_{\mathcal{P}}\left(p_i^t + \eta^t D_i^t\right)\right|^2 \\
&\leq \left|p_i^\star - \left(p_i^t + \eta^t D_i^t\right)\right|^2 \\
&\overset{(\Delta_1)}{=} \left|\left(p_i^\star - p_i^t\right) - \eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{r}^t)\right|^2 \\
&= \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right) \cdot \eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{r}^t) + \left[\eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{r}^t)\right]^2 \\
&= \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right) \cdot \eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{p}^t) + \left[\eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{r}^t)\right]^2 \\
&\quad + 2\left(p_i^\star - p_i^t\right) \cdot \eta^t (b_i + c_i)\left[G_i(\mathbf{p}^t, \mathbf{p}^t) - G_i(\mathbf{p}^t, \mathbf{r}^t)\right] \\
&\overset{(\Delta_2)}{\leq} \left|p_i^\star - p_i^t\right|^2 - 2\left(p_i^\star - p_i^t\right) \cdot \eta^t (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{p}^t) + \left(\eta^t (b_i + c_i) M_G\right)^2 \\
&\quad + 2\left|p_i^\star - p_i^t\right| \cdot \eta^t (b_i + c_i) \cdot \ell_{r,i} \left\|\mathbf{p}^t - \mathbf{r}^t\right\|,
\end{aligned} \tag{M.10}$$

where step (Δ_1) uses the definition of the scaled derivative $G_i(\mathbf{p}, \mathbf{r})$ in Eq. (D.2) and the equivalence that $D_i^t = (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{r}^t)$ from Eq. (D.3). In step (Δ_2), we use $|G_i(\mathbf{p}, \mathbf{r})| \leq M_G$ and the mean value theorem with the fact that $\left\|\nabla_{\mathbf{r}} G_i(\mathbf{p}, \mathbf{r})\right\| \leq \ell_{r,i}$ (see Lemma EC.4).

Let $\mathcal{H}(\mathbf{p})$ be the function defined as

$$\mathcal{H}(\mathbf{p}) := \sum_{i \in N} (p_i^\star - p_i) \cdot \frac{\partial \log\left(\Pi(\mathbf{p}, \mathbf{r})\right)}{\partial p_i}\bigg|_{\mathbf{r}=\mathbf{p}} = \sum_{i \in N} (b_i + c_i) \cdot G_i(\mathbf{p}, \mathbf{p}) \cdot (p_i^\star - p_i). \tag{M.11}$$

Then, by summing Eq. (M.10) over all products $i \in N$, we have that

$$\begin{aligned}
\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t \sum_{i \in N} \left(p_i^\star - p_i^t\right) \cdot (b_i + c_i) G_i(\mathbf{p}^t, \mathbf{p}^t) \\
&\quad + (\eta^t M_G)^2 \sum_{i \in N} (b_i + c_i)^2 + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \sum_{i \in N} \ell_{r,i} (b_i + c_i) \left|p_i^\star - p_i^t\right| \\
&= \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t \mathcal{H}(\mathbf{p}^t) + (\eta^t M_G)^2 \sum_{i \in N} (b_i + c_i)^2 \\
&\quad + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \sum_{i \in N} \ell_{r,i} (b_i + c_i) \left|p_i^\star - p_i^t\right| \\
&\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \Big(2\mathcal{H}(\mathbf{p}^t) - \eta^t \omega_1 - \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big),
\end{aligned} \tag{M.12}$$

where we denote $\omega_1 := M_G^2 \cdot \sum_{i \in N} (b_i + c_i)^2$ and $\omega_2 = 2|\overline{p} - \underline{p}| \cdot \sum_{i \in N} \ell_{r,i} (b_i + c_i)$.

By Lemma EC.1, there exist $\gamma > 0$ and a open set $U_\gamma \ni \mathbf{p}^\star$ such that $\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2$, $\forall \mathbf{p} \in U_\gamma$. Consider $\epsilon_0 > 0$ such that the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0} = \left\{\mathbf{p} \in \mathcal{P}^n \mid \|\mathbf{p} - \mathbf{p}^\star\| < \epsilon_0\right\} \subset U_\gamma$. Furthermore, let T_γ be some period such that for all $t \in T_\gamma$, it holds that

$$\eta^t \left(\eta^t \omega_1 + \sqrt{n}\omega_2(\overline{p} - \underline{p})\right) \leq \frac{\epsilon_0^2}{4} \quad \text{and} \quad \eta^t \omega_1 + \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \leq \frac{\gamma \epsilon_0^2}{2}. \tag{M.13}$$

The existence of such a T_γ follows from the fact that $\lim_{t\to\infty} \eta^t = 0$ and $\lim_{t\to\infty} \|\mathbf{p}^t - \mathbf{r}^t\| = 0$ (see Lemma EC.2). Below, we discuss two cases depending on the location of $\mathbf{p}^t$ in $\mathcal{B}_{\epsilon_0}$.

Case 1. $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0/2} \subset \mathcal{B}_{\epsilon_0}$, i.e., $\|\mathbf{p}^\star - \mathbf{p}^t\| < \epsilon_0/2$.

Since $\mathcal{H}(\mathbf{p}) \geq 0$, $\forall \mathbf{p} \in U_\gamma$ by Lemma EC.1, it follows from Eq. (M.12) and Eq. (M.13) that

$$\begin{aligned}
\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \eta^t \Big(\eta^t \omega_1 + \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big) \\
&\overset{(\Delta)}{\leq} \frac{\epsilon_0^2}{4} + \eta^t \left(\eta^t \omega_1 + \sqrt{n}\omega_2(\overline{p} - \underline{p})\right) \\
&\leq \frac{\epsilon_0^2}{4} + \frac{\epsilon_0^2}{4} < \epsilon_0^2,
\end{aligned} \tag{M.14}$$

where inequality (Δ) is due to $\|\mathbf{p}^t - \mathbf{r}^t\| \leq \sqrt{n}(\overline{p} - \underline{p})$. Eq. (M.14) implies that $\mathbf{p}^{t+1} \in \mathcal{B}_{\epsilon_0}$.

Case 2. $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0} \backslash \mathcal{B}_{\epsilon_0/2}$, i.e., $\|\mathbf{p}^\star - \mathbf{p}^t\| \in [\epsilon_0/2, \epsilon_0)$.

By Lemma EC.1, we have that $\mathcal{H}(\mathbf{p}^t) \geq \gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \geq \gamma\epsilon_0^2/4$. Thus, again by Eq. (M.12) and Eq. (M.13), we have that

$$\begin{aligned}
\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \Big(2\mathcal{H}(\mathbf{p}^t) - \eta^t \omega_1 - \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\Big) \\
&\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t \left(\frac{\gamma\epsilon_0^2}{2} - \eta^t \omega_1 - \omega_2 \left\|\mathbf{p}^t - \mathbf{r}^t\right\|\right) \\
&\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \leq \epsilon_0^2,
\end{aligned} \tag{M.15}$$

which implies $\mathbf{p}^{t+1} \in \mathcal{B}_{\epsilon_0}$. Therefore, we conclude by induction that the price path will stay in the ℓ^2-neighborhood $\mathcal{B}_{\epsilon_0}$.

Next, we proceed to show the local convergence rate in Eq. (M.9). Using the fact that $\mathcal{H}(\mathbf{p}) \geq \gamma \cdot \|\mathbf{p} - \mathbf{p}^\star\|^2$ for all $\mathbf{p} \in U_\gamma$ from Lemma EC.1, we can further derive from Eq. (M.12) that

$$\begin{aligned}
\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\leq \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t\gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1(\eta^t)^2 + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \sum_{i\in N} \ell_{r,i}(b_i + c_i) \left|p_i^\star - p_i^t\right| \\
&\overset{(\Delta_1)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t\gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1(\eta^t)^2 + 2\eta^t \left\|\mathbf{p}^t - \mathbf{r}^t\right\| \cdot \hat{k} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\| \\
&\overset{(\Delta_2)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - 2\eta^t\gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1(\eta^t)^2 + \eta^t \hat{k} \left[\frac{\hat{k}}{\gamma} \left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 + \frac{\gamma}{\hat{k}} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2\right] \\
&\overset{(\Delta_3)}{\leq} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 - \eta^t\gamma \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \omega_1(\eta^t)^2 + \eta^t k \left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2,
\end{aligned} \tag{M.16}$$

where (Δ_1) utilizes the fact that $\sum_{i\in N} \ell_{r,i}(b_i + c_i) \left|p_i^\star - p_i^t\right| \leq \max_{i\in N} \left\{\ell_{r,i}(b_i + c_i)\right\} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|_1 \leq \sqrt{n} \max_{i\in N} \left\{\ell_{r,i}(b_i + c_i)\right\} \left\|\mathbf{p}^\star - \mathbf{p}^t\right\|$, and we define $\hat{k} := \sqrt{n} \max_{i\in N} \left\{\ell_{r,i}(b_i + c_i)\right\}$. Step (Δ_2) follows from the inequality of arithmetic and geometric means, i.e., $2xy \le Ax^2 + (1/A)y^2$ for any constant $A > 0$. The value of constant k in (Δ_4) is given by $k := \hat{k}^2/\gamma = n\big(\max_{i\in N}\{\ell_{r,i}(b_i + c_i)\}\big)^2/\gamma$.

Our goal is to upper-bound the right-hand side of Eq. (M.16). By a similar technique used in Case 1 of Lemma EC.2, we can demonstrate that $\|\mathbf{p}^t - \mathbf{r}^t\|^2 = \mathcal{O}(1/t^2)$ for reasonably large t. Together with the step-sizes of $\eta_t = C_\eta/(t+1)$, we can further bound Eq. (M.16) as

$$\begin{aligned}\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 &\le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \frac{\omega_1 C_\eta^2}{(t+1)^2} + \frac{kC_\eta}{t+1}\cdot\mathcal{O}\left(\frac{1}{t^2}\right)\\ &\le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + \frac{\omega_3}{t(t+1)},\end{aligned} \tag{M.17}$$

where ω_3 in the last line is some constant that satisfies $\frac{\omega_3}{t(t+1)} \ge \frac{\omega_1 C_\eta^2}{(t+1)^2} + \frac{kC_\eta}{t+1}\cdot\mathcal{O}\left(\frac{1}{t^2}\right)$.

Then, we inductively show that $\|\mathbf{p}^\star - \mathbf{p}^t\|^2 = \mathcal{O}(1/t)$ when $\mathbf{p}^t \in \mathcal{B}_{\epsilon_0}$. According to the first part of Proposition EC.1, there exists $T_{\epsilon_0} > 0$ such that $t \in \mathcal{B}_\epsilon$ for every $t \ge T_{\epsilon_0}$. Suppose there exists a constant d_p such that for a fixed period $t \ge T_{\epsilon_0}$, it holds that

$$\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 \le \frac{d_p}{t}. \tag{M.18}$$

To establish the induction, it is sufficient to show that

$$\left\|\mathbf{p}^\star - \mathbf{p}^{t+1}\right\|^2 \le \left(1 - \frac{\gamma C_\eta}{t+1}\right)\frac{d_p}{t} + \frac{\omega_3}{t(t+1)} \le \frac{d_p}{t+1}, \tag{M.19}$$

where the first inequality follows from Eq. (M.17). Then, Eq. (M.19) is further equivalent to

$$(\gamma C_\eta - 1)d_p \ge \omega_3. \tag{M.20}$$

Hence, as long as $\gamma C_\eta - 1 > 0$, there exists $d_p > 0$ such that the induction in Eq. (M.19) holds. For example, when $C_\eta = 2/\gamma$, we can take $d_p = \max\left\{nT_{\epsilon_0}(\overline{p} - \underline{p})^2, \omega_3\right\}$, where the first term in the maximization bracket ensures the base case of the induction. This completes the proof of the local convergence rate for the price path. Finally, the convergence rate of the reference price path can be deduced from the following triangular inequality

$$\left\|\mathbf{p}^\star - \mathbf{r}^t\right\|^2 = \left\|\mathbf{p}^\star - \mathbf{p}^t + \mathbf{p}^t - \mathbf{r}^t\right\|^2 \le 2\left\|\mathbf{p}^\star - \mathbf{p}^t\right\|^2 + 2\left\|\mathbf{p}^t - \mathbf{r}^t\right\|^2 = \mathcal{O}\left(\frac{1}{t}\right), \tag{M.21}$$

which completes the proof of Proposition EC.1. □

Appendix N Summary of Constants

In the following Table EC.1 and Table EC.2, we summarize the definitions of all constants used in the paper, along with references to their initial occurrences.

Table EC.1 Summary of Constants for the Loss-neutral Scenario

Notation	Definition	Location
λ	$1/(\overline{p}\,\\|\mathbf{p}^\star\\|_\infty)$	Eq. (D.14)
M_G	$\max_{i\in N}\left\{\frac{1}{(b_i+c_i)\underline{p}}\right\}+1$	Eq. (L.31)
M_κ	$(\overline{p}-\underline{p})\sum_{i\in N}\frac{1}{b_i+c_i}$	Eq. (E.8)
$\ell_{r,i}$	$\frac{1}{4}\sqrt{c_i^2+\max_{j\neq i}\left\{c_j^2\right\}}$	Eq. (L.31)
$\ell_{d,i}$	$\frac{1}{4}\sqrt{b_i^2+\max_{j\neq i}\left\{b_j^2\right\}}$	Eq. (L.37)
$\ell_{p,i}$	$\frac{1}{4}\sqrt{16+\overline{p}^2\left[(b_i+c_i)^2+\max_{j\neq i}\left\{(b_j+c_j)^2\right\}\right]}$	Eq. (L.37)
T_1	$\left\lceil\frac{2\sqrt{3+\alpha^2}-2}{2-\sqrt{3+\alpha^2}}\right\rceil$	Eq. (L.1)
C_{rp}	$\max\left\{\frac{2M_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i)^2}}{1-\alpha^2},\ \frac{\sqrt{n}(\overline{p}-\underline{p})(T_1+1)}{C_\eta\log(T_1+1)}\right\}$	Eq. (L.1)
$\widehat{C}_{rp}$	$\max\left\{C_{rp},M_G\right\}$	Eq. (E.2)
C_κ	$n\lambda+\sum_{i\in N}\ell_{r,i}+2\sqrt{\sum_{i\in N}(b_i+c_i)^2}\cdot\sum_{i\in N}\ell_{d,i}$	Eq. (D.16)
$\widehat{C}_\kappa$	$4C_\eta\widehat{C}_{rp}\left(2C_\kappa C_\eta+n+\frac{M_\kappa}{2\sqrt{n}(\overline{p}-\underline{p})}\right)$	Eq. (E.15)
C_p	$\max_{i\in N}\left\{(b_i+c_i)^2\right\}\cdot(\widehat{C}_\kappa)^2$	Eq. (E.16)
C_r	$2\big(C_p+(C_{rp}C_\eta)^2\big)$	Eq. (E.17)
h_i	$\frac{1}{4}(b_i+c_i)\big(2+(b_i+c_i)\overline{p}\big)$	Eq. (F.5)
$C_{R,i}$	$h_iC_r\cdot\max\left\{\left(\frac{c_i}{b_i+c_i}\right)^2,\ 2\overline{p}^2\max_{j\neq i}\left\{c_j^2\right\}\right\}$ $+h_iC_p\cdot\max\left\{4\overline{p}^2\max_{j\neq i}\left\{(b_j+c_j)^2\right\},1\right\}$	Eq. (F.11)

Table EC.2 Summary of Constants for the Loss-averse Scenario

Notation	Definition	Location
$\tilde{M}_G$	$\max_{i\in N}\left\{\frac{1}{(b_i+c_i^+)\underline{p}}\right\}+1$	Eq. (L.52)
$\tilde{\ell}_{r,i}$	$\frac{1}{4}\sqrt{(c_i^-)^2+\max_{j\neq i}\left\{(c_j^-)^2\right\}}$	Eq. (L.52)
$\tilde{\ell}_{p,i}$	$\sqrt{\frac{1}{(b_i+c_i^+)^2\underline{p}^4}+\frac{b_i}{2(b_i+c_i^+)\underline{p}^2}+\frac{b_i^2+\max_{j\neq i}\left\{b_j^2\right\}}{16}}$	Eq. (L.53)
$\tilde{C}_\kappa$	$\sum_{i\in N}\frac{3\max_{k\in N}\left\{b_k+c_k^-\right\}}{2(b_i+c_i^+)}$	Eq. (G.28)
$T_{1/2}$	$\left\lceil\frac{2+2\alpha^2}{1-\alpha^2}\right\rceil$	Eq. (H.12)
$\tilde{C}_{rp}$	$\max\left\{\frac{2\tilde{M}_G\sqrt{(1+\alpha^2)\sum_{i\in N}(b_i+c_i^-)^2}}{1-\alpha^2},\frac{\sqrt{n}(\overline{p}-\underline{p})\sqrt{T_{1/2}+1}}{C_\eta}\right\}$	Eq. (H.12)
$\tilde{C}_{1/2}$	$\max_{i\in N}\left\{(b_i+c_i^-)\tilde{\ell}_{r,i}\tilde{C}_{rp},\ (b_i+c_i^-)\tilde{\ell}_{G,i}\tilde{M}_G\sqrt{\sum_{i\in N}\left(b_i+c_i^-\right)^2}\right\}$	Eq. (H.12)
$\tilde{T}$	$\left\lceil\frac{n\tilde{M}_G\overline{p}^2\cdot\max_{i\in N}\left\{b_i+c_i^-\right\}}{2C_\eta\epsilon}+\sqrt{\max\left\{T_{1/2}+1,\left\lceil\left(\frac{2C_\eta\tilde{C}_{1/2}}{\epsilon}\right)^2\right\rceil\right\}}\right\rceil^2$	Eq. (H.11)

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