

Pricing Control and Regulation on Online Service Platforms

Online Appendix

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B Proofs

Proof of Proposition 1. In centralized pricing, there exists a single price in the market. The average market price is equivalent to the price set by the platform. Let \bar{p} be the price set and ϕ be the portion of revenue retained. Conditional on participation, an agent with cost c earns

$$\begin{aligned}\pi_c(\bar{p}) &= q(\bar{p})((1 - \phi)\bar{p} - c) \\ &= (1 - \beta\bar{p})((1 - \phi)\bar{p} - c).\end{aligned}$$

Agent earnings are decreasing in cost and therefore there exists a threshold cost, c_h , for which agent with cost c participates if and only if $c \leq c_h$. The highest cost that participates is

$$\pi_{c_h}(\bar{p}) = 0 \implies c_h = c_h(\bar{p}, \phi) = (1 - \phi)\bar{p}.$$

Platform's profit maximization problem is

$$\begin{aligned}\max_{\bar{p}, \phi} \quad \Pi^C(\bar{p}, \phi) &= \phi\bar{p} \int_0^{c_h(\bar{p}, \phi)} q(\bar{p}) dc \\ &= \phi(1 - \phi)\bar{p}^2(1 - \beta\bar{p}).\end{aligned}$$

The objective function is concave in ϕ and maximized at $\phi = 1/2$. Similarly, we have

$$\frac{\partial \Pi^C(\bar{p}, \phi)}{\partial \bar{p}} = \phi(1 - \phi)\bar{p}(2 - 3\beta\bar{p}) \geq 0 \iff \bar{p} \in \left[0, \frac{2}{3\beta}\right].$$

Therefore, the platform's objective is maximized at

$$\begin{aligned}\bar{p} &= \frac{2}{3\beta}, \\ \phi &= \frac{1}{2}.\end{aligned}$$

Plugging in the equilibrium price and commission, the platform's profit is

$$\Pi^C = \phi(1 - \phi)\bar{p}^2(1 - \beta\bar{p}) = \frac{1}{27\beta^2}.$$

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The highest cost that participates is

$$c_h = (1 - \phi)\bar{p} = \frac{1}{3\beta}.$$

The total quantity of customers served in the market is

$$\begin{aligned} Q &= \int_0^{c_h(\bar{p}, \phi)} q(\bar{p}) dc \\ &= (1 - \phi)\bar{p}(1 - \beta\bar{p}) \\ &= \frac{1}{9\beta}. \end{aligned}$$

Agents' total earnings is

$$\begin{aligned} \pi^c &= \int_0^{c_h(\bar{p}, \phi)} q(\bar{p})((1 - \phi)\bar{p} - c) dc \\ &= \frac{1}{2}(1 - \beta\bar{p})\bar{p}^2(1 - \phi)^2 \\ &= \frac{1}{54\beta^2}. \end{aligned}$$

□

Proof of Proposition 2. Consider a broad class of fee structures such that an agent pays the platform (i) a base unit fee, w , per unit served and (ii) a commission, ϕp , per unit served where $\phi < 1$ is the fixed commission rate and p is the agent's price. An agent with demand q pays the platform in total $q(w + \phi p)$. The fixed per-unit fee, w , can be negative, meaning that it is actually a per-unit subsidy.

With this payment structure, we seek to determine the uniqueness of a price equilibrium. Each agent's earnings depend on the average price on the platform, so it is sufficient to consider a price expectations equilibrium in which each agent expects the average price to be \bar{p}_e and all agents select prices based on that expectation that yield \bar{p}_e as the actual average price.

An agent with price expectation \bar{p}_e , cost c and price p expects to earn

$$\begin{aligned} \pi_c(p, \bar{p}_e) &= q(p, \bar{p}_e) ((1 - \phi)p - c - w) \\ &= (1 - \beta\bar{p}_e + \gamma(\bar{p}_e - p)) ((1 - \phi)p - c - w). \end{aligned}$$

Because earnings are strictly concave in price, there exists a unique price, $p^*(c, \bar{p}_e)$, that maximizes the agent's earnings,

$$p^*(c, \bar{p}_e) = \frac{1}{2} \left(\frac{1 - (\beta - \gamma)\bar{p}_e}{\gamma} + \frac{c + w}{1 - \phi} \right). \quad (10)$$

Let the realized average price be \bar{p} . This agent then receives $q(p^*(c, \bar{p}_e), \bar{p})$ demand,

$$q(p^*(c, \bar{p}_e), \bar{p}) = \frac{1}{2} \left(1 + (\beta - \gamma)(\bar{p}_e - 2\bar{p}) - \frac{\gamma(c + w)}{1 - \phi} \right)^+. \quad (11)$$

The realized quantity served is decreasing in cost, c . Define c_0 to be the smallest cost such that the optimal quantity is zero,

$$c_0 = \min\{c : q(p^*(c, \bar{p}_e), \bar{p}) = 0\} = \frac{(1 - \phi)(1 + (\beta - \gamma)(\bar{p}_e - 2\bar{p}))}{\gamma} - w. \quad (12)$$

The realized average price is

$$\bar{p} = \frac{\int_0^{\min\{c_0, 1\}} q(p^*(c, \bar{p}_e), \bar{p}) p^*(c, \bar{p}_e) dc}{\int_0^{\min\{c_0, 1\}} q(p^*(c, \bar{p}_e), \bar{p}) dc}. \quad (13)$$

A price equilibrium exists if $\bar{p}_e = \bar{p}$. We will show that there can exist only one such equilibrium in any market.

From (12), given an equilibrium, i.e., $\bar{p} = \bar{p}_e$, the average price in equilibrium can be expressed in terms of c_0 ,

$$\bar{p} = \frac{\gamma(c_0 + w) + \phi - 1}{(1 - \phi)(\gamma - \beta)}. \quad (14)$$

From (10), (11), (14) and $\bar{p}_e = \bar{p}$, the equilibrium condition (13) can be written as

$$\begin{aligned} 0 &= \frac{\gamma(c_0 + w) + \phi - 1}{(1 - \phi)(\gamma - \beta)} - \frac{\int_0^{\min\{c_0, 1\}} \left(\frac{\gamma(c_0 - c)}{2(1 - \phi)} \right) \left(\frac{c + c_0 + 2w}{2(1 - \phi)} \right) dc}{\int_0^{\min\{c_0, 1\}} \left(\frac{\gamma(c_0 - c)}{2(1 - \phi)} \right) dc} \\ \Leftrightarrow 0 &= \begin{cases} \frac{\gamma(c_0 + w) + \phi - 1}{1 - \phi} - \frac{(2c_0 + 3w)(\gamma - \beta)}{3(1 - \phi)} & , \text{ if } c_0 \leq 1, \\ \frac{\gamma(c_0 + w) + \phi - 1}{1 - \phi} - \frac{(3c_0^2 + 6c_0w - 3w - 1)(\gamma - \beta)}{3(2c_0 - 1)(1 - \phi)} & , \text{ if } c_0 > 1. \end{cases} \end{aligned} \quad (15)$$

The right-hand side of (15) is continuous at $c_0 = 1$. It is also monotonically increasing in c_0 :

$$\begin{aligned} \frac{\partial}{\partial c_0} \left(\frac{\gamma(c_0 + w) + \phi - 1}{1 - \phi} - \frac{(2c_0 + 3w)(\gamma - \beta)}{3(1 - \phi)} \right) &= \frac{2\beta + \gamma}{3(1 - \phi)} > 0 \quad \forall c_0 \leq 1, \\ \frac{\partial}{\partial c_0} \left(\frac{\gamma(c_0 + w) + \phi - 1}{1 - \phi} - \frac{(3c_0^2 + 6c_0w - 3w - 1)(\gamma - \beta)}{3(2c_0 - 1)(1 - \phi)} \right) &= \frac{\gamma + \beta(6(c_0 - 1)c_0 + 2) + 6\gamma(c_0 - 1)c_0}{3(2c_0 - 1)^2(1 - \phi)} > 0 \quad \forall c_0 > 1. \end{aligned}$$

As a result, the right-hand side of (15) can only cross zero once, yielding a unique c_0 .

The unique candidate c_0 in the domain $c_0 \leq 1$ is

$$c_0 = \frac{3(1 - \phi - \beta w)}{2\beta + \gamma}. \quad (16)$$

From (14) and (16), for this candidate c_0 , there exists a unique equilibrium average price,

$$\bar{p} = \frac{2(1 - \phi) + \gamma w}{(1 - \phi)(2\beta + \gamma)}. \quad (17)$$

□

Proof of Proposition 3. The uniqueness of an average price that satisfies consistent expectations equilibrium is guaranteed by Proposition 2. Due to equivalency of the expected average price, \bar{p}_e , and realized average price, \bar{p} , we do not make the distinction between the two throughout the proofs.

In decentralized pricing with commission fees, conditional on participation, an agent with cost c setting a price p earns

$$\pi_c(p, \bar{p}) = (1 - \beta\bar{p} + \gamma(\bar{p} - p))((1 - \phi)p - c).$$

Agent's profit depends on individual price, p , and also the average market price, \bar{p} , which is characterized in the equilibrium. Since each agent is small, the price set by an individual agent does not influence the average market price.

Each agent chooses the price that maximizes own earnings. From (10),

$$p^*(c, \bar{p}) = \frac{1}{2\gamma} \left(1 + (\gamma - \beta)\bar{p} + \frac{\gamma c}{1 - \phi} \right).$$

The agent earns

$$\pi_c(p^*(c, \bar{p}), \bar{p}) = \frac{(1 - \phi - \bar{p}(1 - \phi)(\beta - \gamma) - \gamma c)^2}{4\gamma(1 - \phi)}.$$

Agent earnings are decreasing in cost, c , and therefore there exists a threshold cost, c_h , for which agent with cost c participates if and only if $c \leq c_h$. Define c_0 to be the lowest cost that yields zero profit:

$$\pi_{c_0}(p^*(c_0, \bar{p}), \bar{p}) = 0 \implies c_0 = c_0(\phi) = \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma},$$

Then,

$$c_h = c_h(\phi) = \min\{1, c_0\}.$$

If $c_0 \leq 1$, the average market price is uniquely defined by (17):

$$\bar{p} = \frac{2}{2\beta + \gamma}.$$

The platform's profit-maximization problem is:

$$\begin{aligned}
\max_{\phi} \quad \Pi^{\mathcal{D}}(\phi) &= \phi \bar{p} \int_0^{c_h(\phi)} q(p^*(c, \bar{p}), \bar{p}) dc \\
&= \phi \bar{p} \int_0^{c_0(\phi)} q(p^*(c, \bar{p}), \bar{p}) dc \\
&= \frac{9\gamma(1-\phi)\phi}{2(2\beta+\gamma)^3}.
\end{aligned}$$

The objective function is concave in ϕ and is maximized at $\phi = 1/2$. It yields a profit of

$$\Pi^{\mathcal{D}} = \frac{9\gamma(1-\phi)\phi}{2(2\beta+\gamma)^3} = \frac{9}{8} \left(\frac{\gamma}{(2\beta+\gamma)^3} \right).$$

This is a feasible solution because

$$\begin{aligned}
c_0 &= \frac{(1-\phi)(1+(\gamma-\beta)\bar{p})}{\gamma} \\
&= \frac{3}{4\beta+2\gamma} < 1,
\end{aligned}$$

which follows from the $\beta > 1$ restriction discussed in Section 3.

If $c_0 > 1$, the average price is determined by (15). The platform's profit is

$$\begin{aligned}
\Pi^{\mathcal{D}}(\phi) &= \phi \bar{p} \int_0^{c_h(\phi)} q(p^*(c, \bar{p}), \bar{p}) dc \\
&= \phi \bar{p} \int_0^1 q(p^*(c, \bar{p}), \bar{p}) dc \\
&= \phi \left(\frac{(1+(\gamma-\beta)\bar{p})^2}{4\gamma} - \frac{\gamma}{12(1-\phi)^2} \right) \\
&= \frac{\gamma(3c_0^2-1)\phi}{12(1-\phi)^2}
\end{aligned} \tag{18}$$

where the last equality follows by plugging in (14). We will show that the optimal solution does not reside in any set of ϕ , $c_0 > 1$ that satisfies (15). As a first step, apply implicit function theorem on (15) to get

$$\frac{\partial c_0}{\partial \phi} = -\frac{3(2c_0-1)^2}{\gamma + \beta(6(c_0-1)c_0 + 2) + 6\gamma(c_0-1)c_0} < 0 \quad \forall c_0 > 1.$$

Using the partial derivative, take the total derivative of (18) with respect to ϕ :

$$\begin{aligned}
\frac{d\Pi^{\mathcal{D}}(\phi)}{d\phi} &= \frac{\gamma}{12(\phi-1)^3(\gamma + \beta(6(c_0-1)c_0 + 2) + 6\gamma(c_0-1)c_0)} \left(\gamma(6(1-c_0)c_0 - 1)(3c_0^2 - 1)(\phi + 1) \right. \\
&\quad \left. - 2\beta(9c_0^4 - 9c_0^3 + 3c_0 - 1)(\phi + 1) - 18c_0(1-2c_0)^2(\phi - 1)\phi \right).
\end{aligned}$$

The right-hand side is positive for all $c_0 > 1$ and $1 > \phi > 0$ whenever $\beta > 1$, and therefore

$$\frac{d\Pi^{\mathcal{D}}(\phi)}{d\phi} > 0.$$

This means, in any solution with $c_0 > 1$, the profits are increasing in ϕ . Together with $\partial\phi/\partial c_0 < 0$, this implies that we cannot have an optimal with $c_0 > 1$. With any candidate optimal ϕ satisfying $c_0 > 1$, you could improve the solution by increasing ϕ , which lowers c_0 , creating a contradiction. Hence, the optimal solution resides within the case $c_0 \leq 1$.

The highest cost that participates is

$$c_h = \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma} = \frac{3}{2(2\beta + \gamma)}.$$

Total quantity of customers served in the market is

$$\begin{aligned} Q &= \int_0^{c_h(\phi)} q(p^*(c, \bar{p}), \bar{p}) dc \\ &= \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})^2}{4\gamma} \\ &= \frac{9}{8} \left(\frac{\gamma}{(2\beta + \gamma)^2} \right). \end{aligned}$$

Agents' total earnings is

$$\begin{aligned} \pi^D &= \int_0^{c_h(\phi)} q(p^*(c, \bar{p}), \bar{p})((1 - \phi)p^*(c, \bar{p}) - c) dc \\ &= \frac{9\gamma(1 - \phi)^2}{4(2\beta + \gamma)^3} \\ &= \frac{9}{16} \left(\frac{\gamma}{(2\beta + \gamma)^3} \right). \end{aligned}$$

□

Proof of Proposition 4. The ratio of platform's profits under decentralized pricing and centralized pricing is:

$$\frac{\Pi^D}{\Pi^C}(\gamma, \beta) = \frac{\frac{9\gamma}{8(2\beta + \gamma)^3}}{\frac{1}{27\beta^2}} = \frac{243\beta^2\gamma}{8(2\beta + \gamma)^3}.$$

The platform prefers decentralized pricing if

$$\begin{aligned} \frac{243\beta^2\gamma}{8(2\beta + \gamma)^3} > 1 &\iff -64\beta^3 + 147\beta^2\gamma - 48\beta\gamma^2 - 8\gamma^3 > 0 \\ &\iff -64 \left(\frac{\beta}{\gamma} \right)^3 + 147 \left(\frac{\beta}{\gamma} \right)^2 - 48 \left(\frac{\beta}{\gamma} \right) - 8 > 0 \\ &\iff 0.539\beta < \gamma < 1.785\beta. \end{aligned}$$

□

Proof of Corollary 1. With both centralized pricing and decentralized pricing, agents' total earnings is equal to half of the platform's:

$$\frac{\pi^{\mathcal{C}}}{\Pi^{\mathcal{C}}} = \frac{\frac{1}{27\beta^2}}{\frac{1}{54\beta^2}} = \frac{1}{2} = \frac{\frac{9}{8} \left(\frac{\gamma}{(2\beta + \gamma)^3} \right)}{\frac{9}{16} \left(\frac{\gamma}{(2\beta + \gamma)^3} \right)} = \frac{\pi^{\mathcal{D}}}{\Pi^{\mathcal{D}}}.$$

Therefore, agents' aggregate welfare is higher with decentralized pricing if and only if platform's profit is higher with decentralized pricing. By Proposition 4, this is true if

$$0.539\beta < \gamma < 1.785\beta.$$

To show that not all agents are better off with decentralization, let $\gamma = 3\beta/2$, $\beta = 2$ and consider the agent with cost $c = 1/10$. Numerical evaluation determines the agent earns $1/45 \approx 0.022$ with centralized pricing, whereas the agent earns less, $24/1225 \approx 0.020$, with decentralized pricing.

□

Proof of Proposition 5 According to the Revelation Principle (Myerson 1981) the set of optimal mechanisms can be found within the set of truth-inducing mechanisms. In such a mechanism, the platform posts a menu of costs, and prices and fees associated with each cost. Each agent reports a cost and is assigned the price and fee associated with that cost, and each agent (i) earns non-negative profits (Individual Rationality), and (ii) prefers to report own cost truthfully given that all other agents do the same (Incentive Compatibility). Following this structure, let $p(c)$ be the price the platform assigns to agent c and let $F(c)$ be the fee collected. Note that $F(c)$ is the total fee and not per-unit, $F(c) = q(p(c), \bar{p})f(c)$. Let $\pi_c(\tilde{c})$ be an agent's earnings with cost c when the agent reports cost \tilde{c} :

$$\pi_c(\tilde{c}) = u(c, p(\tilde{c})) - F(\tilde{c})$$

where

$$u(c, p(\tilde{c})) = (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) (p(\tilde{c}) - c).$$

The Individual Rationality (IR) and the Incentive Compatibility (IC) constraints are

$$\pi_c(c) \geq 0,$$

$$\pi_c(c) \geq \pi_c(\tilde{c}),$$

respectively for all $c \in [0, 1]$, $\tilde{c} \in [0, 1]$.

Because of the IC constraints, agent earnings are strictly decreasing in cost:

$$\begin{aligned}\pi_c(c) &\geq \pi_c(\tilde{c}) = (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) (p(\tilde{c}) - c) - F(\tilde{c}) \\ &> (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) (p(\tilde{c}) - \tilde{c}) - F(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c})\end{aligned}$$

for all $\tilde{c} > c$. This also implies that the agent with the highest cost, $c = 1$, earns zero under the optimal truth-inducing mechanism: if the highest cost agent were to earn a strictly positive amount, then the platform could uniformly increase the fee, $F(\tilde{c})$, for all participating agents, thereby increasing its profit.

We can re-formulate an agent's earnings as

$$\pi_c(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c}) - u(\tilde{c}, p(\tilde{c})) + u(c, p(\tilde{c})).$$

For a pair of agents with costs c, \tilde{c} , IC constraints imply

$$\begin{aligned}\pi_c(c) &\geq \pi_c(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c}) - u(\tilde{c}, p(\tilde{c})) + u(c, p(\tilde{c})), \\ \pi_{\tilde{c}}(\tilde{c}) &\geq \pi_{\tilde{c}}(c) = \pi_c(c) - u(c, p(c)) + u(\tilde{c}, p(c)).\end{aligned}$$

These inequalities can be combined:

$$\begin{aligned}u(\tilde{c}, p(c)) - u(c, p(c)) &\leq \pi_{\tilde{c}}(\tilde{c}) - \pi_c(c) \leq u(\tilde{c}, p(\tilde{c})) - u(c, p(\tilde{c})) \\ \iff \int_c^{\tilde{c}} \frac{\partial u(c_k, p(c))}{\partial c_k} dc_k &\leq \pi_{\tilde{c}}(\tilde{c}) - \pi_c(c) \leq \int_c^{\tilde{c}} \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k.\end{aligned}\tag{19}$$

This inequality has two important consequences. First, ignoring the middle term, Equation (19) implies

$$\begin{aligned}\int_c^{\tilde{c}} \frac{\partial u(c_k, p(c))}{\partial c_k} dc_k &\leq \int_c^{\tilde{c}} \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k, \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\ \iff - \int_c^{\tilde{c}} (1 - \beta\bar{p} + \gamma(\bar{p} - p(c))) dc_k &\leq - \int_c^{\tilde{c}} (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) dc_k, \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\ \iff (1 - \beta\bar{p} + \gamma(\bar{p} - p(c))) (c - \tilde{c}) &\leq (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) (c - \tilde{c}), \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\ \iff 0 \leq (p(c) - p(\tilde{c}))(c - \tilde{c}), \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\ \iff p'(c) \geq 0, \quad \forall c \in [0, 1].\end{aligned}\tag{20}$$

Second, because $u(\tilde{c}, p(c))$ is a continuous function of \tilde{c} , this inequality implies that $\pi_{\tilde{c}}(\tilde{c})$ is also continuous with respect to the Lebesgue measure and thus, almost everywhere differentiable. Furthermore, its derivative is

$$\frac{d\pi_c(c)}{dc} = \frac{\partial u(c, p(c))}{\partial c}.$$

Using the fundamental theorem of calculus:

$$\pi_{\tilde{c}}(\tilde{c}) = \pi_c(c) + \int_c^{\tilde{c}} \frac{d\pi_{c_k}(c_k)}{dc_k} dc_k = \pi_c(c) + \int_c^{\tilde{c}} \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k.$$

Because the highest cost agent earns zero, setting $\tilde{c} = 1$, the equation simplifies to

$$\pi_c(c) = - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k,$$

which means

$$F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k. \quad (21)$$

Our analysis so far established that (20) and (21) are necessary conditions for IR and IC. Now, we will show that they are also sufficient. By establishing an equivalence between the two sets of constraints, we will be able to replace the IR and IC constraints in the platform's optimal mechanism problem with Equations (20) and (21) and therefore convert the problem into a tractable form.

It is straightforward to show that (21) is sufficient to imply IR:

$$\begin{aligned} \pi_c(c) &= u(c, p(c)) - F(c) \\ &= u(c, p(c)) - u(c, p(c)) - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\ &= - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\ &= \int_c^1 (1 - \beta \bar{p} + \gamma(\bar{p} - p(c_k))) dc_k \geq 0 \quad \forall c \in [0, 1]. \end{aligned}$$

Similarly for IC, we have

$$\begin{aligned}
\pi_c(\tilde{c}) - \pi_c(c) &= u(c, p(\tilde{c})) - F(\tilde{c}) - u(c, p(c)) + F(c) \\
&= u(c, p(\tilde{c})) - u(\tilde{c}, p(\tilde{c})) - \int_{\tilde{c}}^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k - u(c, p(c)) + u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= u(c, p(\tilde{c})) - u(\tilde{c}, p(\tilde{c})) - \int_{\tilde{c}}^c \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= \int_{\tilde{c}}^c \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k - \int_{\tilde{c}}^c \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= \int_{\tilde{c}}^c (1 - \beta\bar{p} + \gamma(\bar{p} - p(\tilde{c}))) dc_k - \int_{\tilde{c}}^c (1 - \beta\bar{p} + \gamma(\bar{p} - p(c_k))) dc_k \\
&= \gamma \int_{\tilde{c}}^c (p(c_k) - p(\tilde{c})) dc_k \geq 0, \forall c \in [0, 1], \tilde{c} \in [0, 1].
\end{aligned}$$

Now, we construct the platform's problem. The platform chooses $p(c)$ and $F(c)$ to maximize total profits subject to IR and IC constraints, and the natural restriction that the demand of an agent cannot be negative:

$$\begin{aligned}
&\max_{p(c), F(c)} \int_0^1 F(c) dc \\
&\text{s.t.} \quad \pi_c(c) \geq \pi_c(\tilde{c}), \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
&\quad \pi_c(c) \geq 0, \forall c \in [0, 1] \\
&\quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
&\quad \text{Eq. (2)}.
\end{aligned} \tag{22}$$

Using the equivalence we found earlier, we replace IR and IC constraints with Equations (20) and

(21):

$$\begin{aligned}
& \max_{p(c), F(c)} \int_0^1 F(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k, \forall c \in [0, 1] \\
& \quad \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \text{Eq. (2)} \\
& = \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \text{Eq. (2)}.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\int_0^1 \left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) dc &= \left[\left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) c \right]_0^1 - \int_0^1 \left(-\frac{\partial u(c, p(c))}{\partial c} \right) c dc \\
&= \int_0^1 \frac{\partial u(c, p(c))}{\partial c} c dc.
\end{aligned} \tag{23}$$

The platform's problem converts to

$$\begin{aligned}
& \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \frac{\partial u(c, p(c))}{\partial c} c \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \text{Eq. (2)} \\
& = \max_{p(c)} \int_0^1 \left((1 - \beta \bar{p} + \gamma(\bar{p} - p(c)))(p(c) - c) - (1 - \beta \bar{p} + \gamma(\bar{p} - p(c)))c \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \text{Eq. (2)} \\
& = \max_{p(c)} \int_0^1 (1 - \beta \bar{p} + \gamma(\bar{p} - p(c)))(p(c) - 2c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \text{Eq. (2)}.
\end{aligned} \tag{24}$$

We re-formulate this as a quantity-choice problem. Let $q(c)$ be the demand platform assigns to

agent with cost c . By Equation (1), we have one-to-one equivalence between price and quantity:

$$p(c) = \frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma}.$$

The average price in Equation (2) is

$$\begin{aligned} \bar{p} &= \frac{\int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} \right) dc}{\int_0^1 q(c) dc} \\ &= \frac{\int_0^1 \frac{q(c)(1 - q(c))}{\gamma} dc}{\int_0^1 q(c) dc} + \bar{p} \left(1 - \frac{\beta}{\gamma} \right), \end{aligned}$$

which simplifies to

$$\bar{p} = \frac{\int_0^1 q(c)(1 - q(c)) dc}{\beta \int_0^1 q(c) dc}. \quad (25)$$

Platform's optimal quantity-choice problem is

$$\begin{aligned} &\max_{q(c)} \int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} - 2c \right) dc \\ &\text{s.t. } q'(c) \leq 0, \forall c \in [0, 1] \\ &\quad q(c) \geq 0, \forall c \in [0, 1] \\ &\text{Eq. (25)}. \end{aligned}$$

We can further simplify the objective function. Notice that Equation (25) implies:

$$\bar{p} \int_0^1 q(c) dc = \frac{1}{\beta} \int_0^1 q(c)(1 - q(c)) dc.$$

Using this relationship,

$$\begin{aligned}
\Pi &= \int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} - 2c \right) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\gamma} - 2c \right) dc + \left(\frac{\gamma - \beta}{\gamma} \right) \bar{p} \int_0^1 q(c) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\gamma} - 2c \right) dc + \left(\frac{\gamma - \beta}{\gamma} \right) \frac{1}{\beta} \int_0^1 q(c) (1 - q(c)) dc \\
&= \int_0^1 \left(q(c) \left(\frac{1 - q(c)}{\gamma} - 2c \right) + \left(\frac{\gamma - \beta}{\gamma} \right) \frac{1}{\beta} q(c) (1 - q(c)) \right) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\beta} - 2c \right) dc.
\end{aligned} \tag{26}$$

The problem terms no longer depend on the average price:

$$\begin{aligned}
&\max_{q(c)} \int_0^1 q(c) \left(\frac{1 - q(c)}{\beta} - 2c \right) dc \\
&\text{s.t. } q'(c) \leq 0, \forall c \in [0, 1] \\
&\quad q(c) \geq 0, \forall c \in [0, 1].
\end{aligned} \tag{27}$$

Relaxing the first constraint, we can decompose the problem into individual sub-problems for all agents:

$$\begin{aligned}
&\max_{q(c)} q(c) \left(\frac{1 - q(c)}{\beta} - 2c \right) dc \\
&\text{s.t. } q(c) \geq 0.
\end{aligned}$$

This is the maximization of a simple quadratic function with a linear constraint. The optimal quantity is:

$$q^*(c) = \max \left\{ 0, \frac{1}{2} - \beta c \right\}.$$

The highest cost agent with non-zero quantity is

$$c_h = \min \left\{ 1, \frac{1}{2\beta} \right\} = \frac{1}{2\beta}.$$

The second equality follows from the $\beta > 1$ restriction discussed in Section 3. This solution also satisfies:

$$q^{*'}(c) \leq 0.$$

Therefore, the solution our relaxed problem is also optimal for the platform's optimal mechanism.

The average market price is:

$$\begin{aligned}\bar{p} &= \frac{\int_0^1 q^*(c) (1 - q^*(c)) dc}{\beta \int_0^1 q^*(c) dc} \\ &= \frac{2}{3\beta}.\end{aligned}$$

Other equilibrium characteristics are as follows:

$$\begin{aligned}p^*(c) &= \frac{1 + (\gamma - \beta)\bar{p} - q^*(c)}{\gamma} \\ &= c^2 \left(\frac{\beta(\gamma - 2\beta)}{2\gamma} \right) + \frac{2c}{3} \left(\frac{\beta}{\gamma} - 1 \right) + \frac{1}{24} \left(\frac{5}{\beta} - \frac{2}{\gamma} \right).\end{aligned}$$

The per-unit fee collected is

$$\begin{aligned}f^*(c) &= \frac{F^*(c)}{q^*(c)} = \frac{1}{12} \left(\frac{5}{\beta} - \frac{2}{\gamma} \right) + \left(\frac{\beta}{\gamma} - \frac{1}{2} \right) c \\ &= \left(1 - \frac{\gamma}{2\beta} \right) \left(\frac{2}{3\beta} - \frac{1}{6\gamma} + \frac{c\beta}{\gamma} \right) + \frac{\gamma - \beta}{3\beta^2} \\ &= \left(1 - \frac{\gamma}{2\beta} \right) p^*(c) + \frac{\gamma - \beta}{3\beta^2}.\end{aligned}$$

Because we can map the fee structure of the optimal mechanism to one that is linear in price, Proposition 2 applies also to the optimal mechanism. Therefore, with the optimal mechanism, there exists a unique equilibrium average price.

The platform earns

$$\begin{aligned}\Pi &= \int_0^1 q^*(c) \left(\frac{1 - q^*(c)}{\beta} - 2c \right) dc \\ &= \frac{1}{24\beta^2}.\end{aligned}$$

Total quantity of customers served in the market is

$$\begin{aligned}Q &= \int_0^1 q^*(c) dc \\ &= \frac{1}{8\beta}.\end{aligned}$$

Agents' total earnings is

$$\begin{aligned}\pi &= \int_0^1 \pi_c(c) dc \\ &= \frac{1}{48\beta^2}.\end{aligned}$$

□

Proof of Proposition 6. With the optimal mechanism, we have

$$\begin{aligned} F^*(c) &= c^2 \left(\frac{\beta(\gamma - 2\beta)}{2\gamma} \right) + \frac{2c}{3} \left(\frac{\beta}{\gamma} - 1 \right) + \frac{1}{24} \left(\frac{5}{\beta} - \frac{2}{\gamma} \right) \\ &= q^*(c) \left(\left(1 - \frac{\gamma}{2\beta} \right) p^*(c) + \frac{\gamma - \beta}{3\beta^2} \right). \end{aligned}$$

There is a unique candidate commission-plus fee structure that may yield the same equilibrium outcome as the optimal mechanism. To show that this structure indeed replicates the optimal mechanism, let the platform set its terms

$$\phi = 1 - \frac{\gamma}{2\beta}, \quad w = \frac{\gamma - \beta}{3\beta^2},$$

and assume agents expect an average price of

$$\bar{p} = \frac{2}{3\beta}.$$

By Equation (10), an agent with cost c posts a price under commission-plus

$$\begin{aligned} p^*(c, \bar{p}) &= \frac{1}{2} \left(\frac{1 - (\beta - \gamma)\bar{p}}{\gamma} + \frac{c + w}{1 - \phi} \right) \\ &= \frac{2}{3\beta} - \frac{1}{6\gamma} + \frac{c\beta}{\gamma}. \end{aligned}$$

The price chosen by any agent is exactly equivalent to the one in the optimal mechanism. Same is true for the fees paid. Because prices and fees are the only decision variables in the setting, then commission-plus with the given terms replicates the optimal mechanism. □

Proof of Proposition 7. With the optimal mechanism, we have

$$\begin{aligned} F^*(c) &= c^2 \left(\frac{\beta(\gamma - 2\beta)}{2\gamma} \right) + \frac{2c}{3} \left(\frac{\beta}{\gamma} - 1 \right) + \frac{1}{24} \left(\frac{5}{\beta} - \frac{2}{\gamma} \right) \\ &= q^*(c) \left(\left(\frac{1}{2\beta} - \frac{1}{\gamma} \right) q^*(c) + \frac{1}{6\beta} + \frac{1}{3\gamma} \right). \end{aligned}$$

There is a unique candidate quantity pricing fee structure that may yield the same equilibrium outcome as the optimal mechanism. To show that this structure indeed replicates the optimal mechanism, let the platform sets its terms

$$\phi = \frac{1}{2\beta} - \frac{1}{\gamma}, \quad w = \frac{1}{6\beta} + \frac{1}{3\gamma}.$$

Conditional on participation, an agent with cost c setting a price p earns

$$\pi_c(p, \bar{p}) = (1 - \beta\bar{p} + \gamma(\bar{p} - p))(p - \phi(1 - \beta\bar{p} + \gamma(\bar{p} - p))) - c - w.$$

Because earnings are strictly concave in price, there exists a unique price, $p^*(c, \bar{p})$, that maximizes the agent's earnings,

$$\begin{aligned} p^*(c, \bar{p}) &= \frac{1 - \bar{p}(\beta - \gamma)(1 + 2\gamma\phi) + \gamma(c + w + 2\phi)}{2\gamma(1 + \gamma\phi)} \\ &= \frac{2}{3\beta} - \frac{1}{6\gamma} + \frac{c\beta}{\gamma}. \end{aligned}$$

The price chosen by any agent is exactly equivalent to the one in the optimal mechanism. Same is true for the fees paid. Because prices and fees are the only decision variables in the setting, then quantity pricing with the given terms replicates the optimal mechanism. □

Proof of Proposition 8. We will re-solve centralized pricing, decentralized pricing, optimal mechanism and commission-plus models under the assumption that agents have a maximum capacity to serve k demand.

Centralized pricing: Without capacity constraints, the number of customers an agent serves under centralized pricing is

$$q(\bar{p}) = 1 - \beta\bar{p} = \frac{1}{3}.$$

With capacity constraints, the capacity will be binding if and only if $k \leq \frac{1}{3}$. When capacity is binding, if platform sets a price \bar{p} , the profit agent with cost c earns is

$$\pi_c(\bar{p}) = k((1 - \phi)\bar{p} - c).$$

Agents participate as long as they earn positive profits. The highest cost that participates, c_h , earns zero

$$\pi_{c_h}(\bar{p}) = 0 \implies c_h = c_h(\bar{p}, \phi) = (1 - \phi)\bar{p}.$$

The platform's profit is:

$$\begin{aligned} \Pi^C(\bar{p}, \phi) &= \int_0^{c_h(\bar{p}, \phi)} k\phi\bar{p} dc \\ &= k\phi(1 - \phi)\bar{p}^2. \end{aligned}$$

The objective is increasing monotonically in \bar{p} . It is optimal for platform to increase the price as long as there is excess capacity. In the optimal, quantity served is equal to the capacity:

$$k = q(\bar{p}) = 1 - \beta\bar{p} \implies \bar{p} = \frac{1 - k}{\beta}.$$

The objective is also concave in ϕ and is maximized at $\phi = 1/2$.

Plugging in the solution, the platform's optimal profit is

$$\Pi^C = \frac{k(1-k)^2}{4\beta^2}.$$

Decentralized pricing with commission: Without capacity constraints, the maximum number of customers an agent serves under decentralized pricing with just commission is

$$q(p^*(0, \bar{p}), \bar{p}) = 1 - \beta\bar{p} + \gamma(\bar{p} - p^*(0, \bar{p})) = \frac{3\gamma}{4\beta + 2\gamma}.$$

With capacity constraints, the capacity will be binding if and only if $k \leq \frac{3\gamma}{4\beta + 2\gamma}$. The agents need to incorporate the capacity constraint in their pricing decision. Profit earned by agent with cost c and price p conditional on participation is

$$\pi_c(p, \bar{p}) = \min\{1 - \beta\bar{p} + \gamma(\bar{p} - p), k\}((1 - \phi)p - c).$$

Agent sets the price that maximizes own profits:

$$\max_p \min\{1 - \beta\bar{p} + \gamma(\bar{p} - p), k\}((1 - \phi)p - c).$$

Without capacity constraints, an agent's optimal price is characterized by Equation (10),

$$p^*(c, \bar{p}) = \frac{1}{2\gamma} \left(1 + (\gamma - \beta)\bar{p} + \frac{c\gamma}{1 - \phi} \right),$$

which gives quantity

$$q(p^*(c, \bar{p}), \bar{p}) = \frac{1}{2} \left(1 + (\gamma - \beta)\bar{p} - \frac{c\gamma}{1 - \phi} \right).$$

Since quantity served is decreasing in cost, there exists a threshold, \tilde{c} , where only the agents costs $c \leq \tilde{c}$ are capacity constrained. The threshold agent has exactly the quantity k :

$$q(p^*(\tilde{c}, \bar{p}), \bar{p}) = k \implies \tilde{c} = \tilde{c}(\phi) = \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p} - 2k)}{\gamma}.$$

If an agent is capacity constrained ($k \leq q(p^*(c, \bar{p}), \bar{p})$), any decrease in price further than the one that matches capacity and demand decreases the agent's margin, but not increase quantity served. Hence, an agent never sets a price lower than the amount that makes capacity binding. The optimal price of an agent with cost $c < \tilde{c}$ is:

$$p^*(\tilde{c}, \bar{p}) = \frac{1 + (\gamma - \beta)\bar{p} - k}{\gamma}.$$

The highest cost agent that participates has 0 demand:

$$q(p^*(c_h, \bar{p}), \bar{p}) = 0 \implies c_h = c_h(\phi) = \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma}.$$

Agents' equilibrium prices are

$$p^*(c, \bar{p}) = \begin{cases} \frac{1 + (\gamma - \beta)\bar{p} - k}{\gamma}, & c \leq \tilde{c}, \\ \frac{1}{2} \left(\frac{1 + (\gamma - \beta)\bar{p}}{\gamma} + \frac{c}{1 - \phi} \right), & \tilde{c} < c \leq c_h. \end{cases}$$

Agents with cost higher than c_h cannot profitably participate.

The average market price is defined in the equilibrium as a weighted average of all prices set in the market. The average price that occurs by the agent's optimal decisions is consistent with their expectation of the average price. By Equation (2):

$$\begin{aligned} \bar{p} &= \frac{\int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) p^*(c, \bar{p}) dc}{\int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) dc} \\ &= \frac{\int_0^{\tilde{c}} k p^*(\tilde{c}, \bar{p}) dc + \int_{\tilde{c}}^{c_h} (1 - \beta\bar{p} + \gamma(\bar{p} - p^*(c, \bar{p}))) p^*(c, \bar{p}) dc}{\int_0^{\tilde{c}} k dc + \int_{\tilde{c}}^{c_h} (1 - \beta\bar{p} + \gamma(\bar{p} - p^*(c, \bar{p}))) dc} \\ &= \frac{3(\bar{p}(\gamma - \beta) + 1)^2 + 6k(\bar{p}(\beta - \gamma) - 1) + 4k^2}{3\gamma(\bar{p}(\gamma - \beta) - k + 1)}. \end{aligned}$$

There is a unique feasible equilibrium average price:

$$\bar{p} = \frac{\sqrt{-12\beta^2 k^2 + 12\beta\gamma k^2 + 9\gamma^2(k-1)^2} + 6\beta(k-1) - 3\gamma(k-1)}{6\beta(\beta - \gamma)}.$$

Platform's profit maximization problem is:

$$\begin{aligned} \max_{\phi} \quad \Pi^{\mathcal{D}} &= \phi \bar{p} \left(\int_0^{\tilde{c}} (1 - \beta\bar{p} + \gamma(\bar{p} - p^*(\tilde{c}, \bar{p}))) dc + \int_{\tilde{c}}^{c_h} (1 - \beta\bar{p} + \gamma(\bar{p} - p^*(c, \bar{p}))) dc \right) \\ &= \frac{k \left(-k\sqrt{-12\beta^2 k^2 + 12\beta\gamma k^2 + 9\gamma^2(k-1)^2} + \sqrt{-12\beta^2 k^2 + 12\beta\gamma k^2 + 9\gamma^2(k-1)^2} + 2\beta k^2 + 3\gamma(k-1)^2 \right)}{6\beta^2 \gamma} (1 - \phi) \phi. \end{aligned}$$

Platform's profit function is concave in ϕ and is maximized at $\phi = 1/2$. Platform's profit is

$$\Pi^{\mathcal{D}} = \frac{k \left(-k\sqrt{-12\beta^2 k^2 + 12\beta\gamma k^2 + 9\gamma^2(k-1)^2} + \sqrt{-12\beta^2 k^2 + 12\beta\gamma k^2 + 9\gamma^2(k-1)^2} + 2\beta k^2 + 3\gamma(k-1)^2 \right)}{24\beta^2 \gamma}.$$

Optimal mechanism: Under the optimal mechanism, the highest number of customers an agent serves is $q^*(0) = 1/2$. Therefore, the capacity is binding if and only if $k \leq \frac{1}{2}$.

Our proof for platform's optimal mechanism for the demand unconstrained model in Proposition 5 is applicable here up until we reach Equation (27). In the presence of demand constraints,

platform's optimal mechanism problem has an additional constraint:

$$\begin{aligned} & \max_{q(c)} \int_0^1 q(c) \left(\frac{1-q(c)}{\beta} - 2c \right) dc \\ & \text{s.t. } q'(c) \leq 0, \forall c \in [0, 1] \\ & \quad q(c) \geq 0, \forall c \in [0, 1] \\ & \quad q(c) \leq k, \forall c \in [0, 1]. \end{aligned}$$

Relaxing the first constraint, we can decompose this into individual sub-problems for all agents:

$$\begin{aligned} & \max_{q(c)} q(c) \left(\frac{1-q(c)}{\beta} - 2c \right) dc \\ & \text{s.t. } q(c) \geq 0 \\ & \quad q(c) \leq k. \end{aligned}$$

This is the maximization of a simple quadratic function with two linear constraints. The optimal quantity is:

$$q^*(c) = \min \left\{ k, \max \left\{ 0, \frac{1}{2} - \beta c \right\} \right\}.$$

The highest cost agent with non-zero quantity is

$$c_h = \min \left\{ 1, \frac{1}{2\beta} \right\} = \frac{1}{2\beta}.$$

The last equality follows from the $\beta > 1$ restriction discussed in Section 3.

This solution also satisfies:

$$q^{*'}(c) \leq 0.$$

Therefore, the solution to the relaxed problem is also optimal for the platform's optimal mechanism problem.

The average market price is:

$$\begin{aligned} \bar{p} &= \frac{\int_0^1 q^*(c) (1 - q^*(c)) dc}{\beta \int_0^1 q^*(c) dc} \\ &= \frac{4k^2 - 6k + 3}{3\beta - 3\beta k}. \end{aligned}$$

The platform earns

$$\begin{aligned} \Pi &= \int_0^1 q^*(c) \left(\frac{1-q^*(c)}{\beta} - 2c \right) dc \\ &= \frac{k(4k^2 - 6k + 3)}{12\beta^2}. \end{aligned}$$

Decentralized pricing with commission-plus: Now, we will show that the optimal mechanism can be replicated with decentralized pricing. Let the platform sets its terms,

$$\phi = 1 - \frac{\gamma}{2\beta}, \quad w = \frac{(4k^2 - 6k + 3)(\beta - \gamma)}{6\beta^2(k - 1)},$$

and assume the agents expect an average price of

$$\bar{p} = \frac{4k^2 - 6k + 3}{3\beta(1 - k)}.$$

Without capacity restrictions, an agent with cost c has the following profit-maximization problem:

$$\max_p (1 - \beta\bar{p} + \gamma(\bar{p} - p))((1 - \phi)p - c - w),$$

yielding an optimal price of

$$\begin{aligned} p^*(c, \bar{p}) &= \frac{1}{2} \left(\frac{1 - (\beta - \gamma)\bar{p}}{\gamma} + \frac{c + w}{1 - \phi} \right) \\ &= \frac{3\beta - 6\beta^2c + 6\beta^2ck - 6\gamma + 8\beta k^2 - 8\gamma k^2 - 9\beta k + 12\gamma k}{6\beta\gamma(k - 1)}. \end{aligned} \quad (28)$$

Let \tilde{c} be the highest cost agent that is bounded by capacity:

$$q(p^*(\tilde{c}, \bar{p}), \bar{p}) = k \implies \tilde{c} = \frac{1 - 2t}{2\beta}.$$

All agents with cost less than \tilde{c} set the price that makes their demand exactly equal to k :

$$\begin{aligned} p^*(\tilde{c}, \bar{p}) &= \frac{1 + (\gamma - \beta)\bar{p} - k}{\gamma} \\ &= \frac{-3\gamma + \beta k^2 - 4\gamma k^2 + 6\gamma k}{3\beta\gamma k - 3\beta\gamma}. \end{aligned}$$

Others will set the price in (28). Let c_h be the highest cost agent that can participate with non-negative demand:

$$q(p^*(c_h, \bar{p}), \bar{p}) = 0 \implies c_h = \frac{1}{2\beta}.$$

The realized average price needs to be consistent with expectation:

$$\begin{aligned} \bar{p} &= \frac{\int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) p^*(c, \bar{p}) dc}{\int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) dc} \\ &= \frac{\int_0^{\tilde{c}} k p^*(\tilde{c}, \bar{p}) dc + \int_{\tilde{c}}^{c_h} q(p^*(c, \bar{p}), \bar{p}) p^*(c, \bar{p}) dc}{\int_0^{\tilde{c}} k dc + \int_{\tilde{c}}^{c_h} q(p^*(c, \bar{p}), \bar{p}) dc} \\ &= \frac{4k^2 - 6k + 3}{3\beta(1 - k)}. \end{aligned}$$

The platform earns

$$\begin{aligned}\Pi^{\mathcal{C}^+} &= \int_0^{\tilde{c}} k(\phi p^*(\tilde{c}, \bar{p}) + w) dc + \int_{\tilde{c}}^{c_h} q(p^*(c, \bar{p}), \bar{p})(\phi p^*(c, \bar{p}) + w) dc \\ &= \frac{k(4k^2 - 6k + 3)}{12\beta^2},\end{aligned}$$

same as the optimal mechanism. □

Extension with Other Cost Distributions. We will characterize the equilibrium of the platform's profit-maximizing optimal mechanism and optimal decentralized pricing contract with a commission-plus structure assuming that agents have a probability density function $g(c)$ and cumulative distribution function $G(c)$ with support $c \in [0, 1]$.

Optimal mechanism:

The proof of the optimal mechanism remains unchanged up to the point at which the platform's problem is defined in (22). With a general set of cost distributions, the platform's problem is

$$\begin{aligned}\max_{p(c), F(c)} & \int_0^1 F(c)g(c) dc \\ \text{s.t.} & \quad \pi_c(c) \geq \pi_c(\tilde{c}), \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\ & \quad \pi_c(c) \geq 0, \forall c \in [0, 1] \\ & \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1]\end{aligned}$$

with an additional requirement for the average price to be consistent with agent expectations:

$$\bar{p} = \frac{\int_0^{c_h} q(p(c), \bar{p})p(c)g(c)dc}{\int_0^{c_h} q(p(c), \bar{p})g(c)dc}. \quad (29)$$

Replacing IR and IC constraints with necessary and sufficient conditions in (20) and (21):

$$\begin{aligned}
& \max_{p(c), F(c)} \int_0^1 F(c)g(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k, \forall c \in [0, 1] \\
& \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \text{Eq. (29)} \\
& = \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) g(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \text{Eq. (29)}.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\int_0^1 \left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) g(c) dc &= \left[\left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) G(c) \right]_0^1 - \int_0^1 \left(-\frac{\partial u(c, p(c))}{\partial c} \right) G(c) dc \\
&= \int_0^1 \frac{\partial u(c, p(c))}{\partial c} G(c) dc.
\end{aligned}$$

The platform's problem converts to

$$\begin{aligned}
& \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \frac{\partial u(c, p(c))}{\partial c} \frac{G(c)}{g(c)} \right) g(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \text{Eq. (29)} \\
& = \max_{p(c)} \int_0^1 \left((1 - \beta\bar{p} + \gamma(\bar{p} - p(c)))(p(c) - c) - (1 - \beta\bar{p} + \gamma(\bar{p} - p(c))) \frac{G(c)}{g(c)} \right) g(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \text{Eq. (29)} \\
& = \max_{p(c)} \int_0^1 (1 - \beta\bar{p} + \gamma(\bar{p} - p(c))) \left(p(c) - c - \frac{G(c)}{g(c)} \right) g(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta\bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \text{Eq. (29)}.
\end{aligned}$$

We re-formulate this as a quantity-choice problem. By Equation (1), we have one-to-one equivalence between price and quantity:

$$p(c) = \frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma}.$$

The average price in Equation (29) is

$$\begin{aligned} \bar{p} &= \frac{\int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} \right) g(c) dc}{\int_0^1 q(c)g(c) dc} \\ &= \frac{\int_0^1 \frac{q(c)(1 - q(c))}{\gamma} g(c) dc}{\int_0^1 q(c)g(c) dc} + \bar{p} \left(1 - \frac{\beta}{\gamma} \right), \end{aligned}$$

which simplifies to

$$\bar{p} = \frac{\int_0^1 q(c)(1 - q(c))g(c) dc}{\beta \int_0^1 q(c)g(c) dc}. \quad (30)$$

Platform's optimal quantity-choice problem is

$$\begin{aligned} &\max_{q(c)} \int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} - c - \frac{G(c)}{g(c)} \right) g(c) dc \\ &\text{s.t. } q'(c) \leq 0, \forall c \in [0, 1] \\ &\quad q(c) \geq 0, \forall c \in [0, 1] \\ &\quad \text{Eq. (30)}. \end{aligned}$$

We can further simplify the objective function. Notice that Equation (30) implies:

$$\bar{p} \int_0^1 q(c)g(c) dc = \frac{1}{\beta} \int_0^1 q(c)(1 - q(c))g(c) dc.$$

Using this relationship,

$$\begin{aligned}
\Pi &= \int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} - c - \frac{G(c)}{g(c)} \right) g(c) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\gamma} - c - \frac{G(c)}{g(c)} \right) g(c) dc + \left(\frac{\gamma - \beta}{\gamma} \right) \bar{p} \int_0^1 q(c) g(c) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\gamma} - c - \frac{G(c)}{g(c)} \right) g(c) dc + \left(\frac{\gamma - \beta}{\gamma} \right) \frac{1}{\beta} \int_0^1 q(c) (1 - q(c)) g(c) dc \\
&= \int_0^1 \left(q(c) \left(\frac{1 - q(c)}{\gamma} - c - \frac{G(c)}{g(c)} \right) + \left(\frac{\gamma - \beta}{\gamma} \right) \frac{1}{\beta} q(c) (1 - q(c)) \right) g(c) dc \\
&= \int_0^1 q(c) \left(\frac{1 - q(c)}{\beta} - c - \frac{G(c)}{g(c)} \right) g(c) dc.
\end{aligned}$$

The problem terms no longer depend on the average price:

$$\begin{aligned}
&\max_{q(c)} \int_0^1 q(c) \left(\frac{1 - q(c)}{\beta} - c - \frac{G(c)}{g(c)} \right) g(c) dc \\
&\text{s.t. } q'(c) \leq 0, \forall c \in [0, 1] \\
&\quad q(c) \geq 0, \forall c \in [0, 1].
\end{aligned}$$

Relaxing the first constraint, we can decompose the problem into individual sub-problems for all agents:

$$\begin{aligned}
&\max_{q(c)} q(c) \left(\frac{1 - q(c)}{\beta} - c - \frac{G(c)}{g(c)} \right) dc \\
&\text{s.t. } q(c) \geq 0.
\end{aligned}$$

This is the maximization of a simple quadratic function with a linear constraint. The optimal quantity is:

$$q^*(c) = \max \left\{ 0, \frac{1}{2} - \beta \frac{G(c)}{g(c)} - \beta c \right\}.$$

The solution to this relaxed problem is optimal for the platform's optimal mechanism only if the relaxed constraint is satisfied. That requires the following regularity condition on the cost distribution:

$$\begin{aligned}
\frac{dq^*(c)}{dc} \leq 0 &\iff \frac{d}{dc} \left(\frac{1}{2} - \beta \frac{G(c)}{g(c)} - \beta c \right) \leq 0 \\
&\iff \frac{G(c)g'(c)}{g(c)^2} - 2 \leq 0 \quad \forall c \in [0, 1].
\end{aligned}$$

The highest cost with non-zero quantity does not have a closed form solution, but can be evaluated numerically as an implicit solution to

$$q^*(\hat{c}) = \frac{1}{2} - \beta \frac{G(\hat{c})}{g(\hat{c})} - \beta \hat{c} = 0.$$

The platform's profit under the optimal mechanism is

$$\Pi = \int_0^1 q^*(c) \left(\frac{1 - q^*(c)}{\beta} - c - \frac{G(c)}{g(c)} \right) g(c) dc.$$

Decentralized pricing with commission-plus:

In decentralized pricing with commission-plus, conditional on participation, an agent with cost c setting a price p earns

$$\pi_c(p, \bar{p}) = (1 - \beta \bar{p} + \gamma(\bar{p} - p))((1 - \phi)p - w - c).$$

Each agent chooses the price that maximizes own earnings. From (10),

$$p^*(c, \bar{p}) = \frac{1}{2} \left(\frac{1 - (\beta - \gamma)\bar{p}}{\gamma} + \frac{c + w}{1 - \phi} \right).$$

The agent earns

$$\pi_c(p^*(c, \bar{p}), \bar{p}) = \frac{(1 - \phi + \bar{p}(1 - \phi)(\gamma - \beta) - \gamma(c + w))^2}{4\gamma(1 - \phi)}.$$

Agent earnings are decreasing in cost, c , and therefore there exists a threshold cost, c_h , for which agent with cost c participates if and only if $c \leq c_h$. Define c_0 to be the lowest cost that yields zero profit:

$$\pi_{c_0}(p^*(c_0, \bar{p}), \bar{p}) = 0 \implies c_0 = c_0(\phi, w) = \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma} - w,$$

Then,

$$c_h = c_h(\phi, w) = \min\{1, c_0\}.$$

The average market price is defined by:

$$\begin{aligned} \bar{p} &= \frac{\int_0^{c_h} q(p(c), \bar{p}) p(c) g(c) dc}{\int_0^{c_h} q(p(c), \bar{p}) g(c) dc} \\ &= \frac{\min\left\{1, \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma} - w\right\} \int_0^{\frac{(1 + \bar{p}(\gamma - \beta))^2}{4\gamma} - \frac{\gamma(c + w)^2}{4(1 - \phi)^2}} \left(\frac{(1 + \bar{p}(\gamma - \beta))^2}{4\gamma} - \frac{\gamma(c + w)^2}{4(1 - \phi)^2} \right) g(c) dc}{\min\left\{1, \frac{(1 - \phi)(1 + (\gamma - \beta)\bar{p})}{\gamma} - w\right\} \int_0^{\frac{(1 + \bar{p}(\gamma - \beta))}{2} \left(1 + \bar{p}(\gamma - \beta) - \frac{\gamma(c + w)}{1 - \phi} \right) g(c) dc}}. \end{aligned} \quad (31)$$

The uniqueness of a solution to (31) is not guaranteed, and identifying conditions needed on the cost distribution that ensure uniqueness is nontrivial. As a result, we only work with distributions with known analytical forms. Proposition 2 shows the uniqueness for a uniform distribution. It is also possible to show uniqueness for Beta(2,2), Beta(3,3), Beta(4,4) and Beta(5,5) distributions. This involves following the same steps as in Proposition 2: plugging in $g(c)$, formulating right-hand-side as a piece-wise function of c_0 (rather than \bar{p}) and showing monotonicity. For brevity, we omit the details here.

The platform's profit-maximization problem is:

$$\begin{aligned} \max_{\phi, w} \quad \Pi^{\mathcal{D}}(\phi, w) &= (\phi\bar{p} + w) \int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) g(c) dc \\ &= \frac{1}{2}(\phi\bar{p} + w) \int_0^{c_h} \left(1 + \bar{p}(\gamma - \beta) - \frac{\gamma(c + w)}{1 - \phi} \right) g(c) dc. \end{aligned}$$

There is no closed-form solution to the platform's problem for a general $q(c)$; however, for any given distribution with known analytical form, it is possible to conduct a numerical search across the (ϕ, w) space to identify the platform's optimal profit.

□

Extension with Throughput-Maximization. In the quantity maximizing optimal contract, let $p(c)$ be the price the platform assigns to agent c and $F(c)$ be the fee collected. By Proposition 5, Equations (20) and (21) are necessary and sufficient conditions for agents' IR and IC constraints. In any truth-inducing mechanism, the equilibrium fees charged to the agent who reports a cost c is

$$F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k.$$

The total quantity served in the market is

$$\int_0^1 q(c) dc = \int_0^1 (1 - \beta\bar{p} + \gamma(\bar{p} - p(c))) dc.$$

The optimal truth-inducing contract that maximizes total quantity served subject to non-

negative profit constraint is characterized through the following problem:

$$\begin{aligned}
& \max_{p(c)} \int_0^1 (1 - \beta \bar{p} + \gamma(\bar{p} - p(c))) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k, \forall c \in [0, 1] \\
& \quad \int_0^1 F(c) dc \geq 0 \\
& \text{Eq. (2)} \\
& = \max_{p(c)} \int_0^1 (1 - \beta \bar{p} + \gamma(\bar{p} - p(c))) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad 1 - \beta \bar{p} + \gamma(\bar{p} - p(c)) \geq 0, \forall c \in [0, 1] \\
& \quad \int_0^1 (1 - \beta \bar{p} + \gamma(\bar{p} - p(c)))(p(c) - 2c) dc \geq 0 \\
& \text{Eq. (2)},
\end{aligned}$$

where the equivalence of the constraints follow from Equations (23) and (24). We reformulate the problem with quantities as decision variables:

$$\begin{aligned}
& \max_{q(c)} \int_0^1 q(c) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad q(c) \geq 0, \forall c \in [0, 1] \\
& \quad \int_0^1 q(c) \left(\frac{1 + (\gamma - \beta)\bar{p} - q(c)}{\gamma} - 2c \right) dc \geq 0 \\
& \text{Eq. (25)}.
\end{aligned}$$

Following the same steps as Equation (26), we transform the objective function such that it does not depend on \bar{p} :

$$\begin{aligned}
& \max_{q(c)} \int_0^1 q(c) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad q(c) \geq 0, \forall c \in [0, 1] \\
& \quad \int_0^1 q(c) \left(\frac{1 - q(c)}{\beta} - 2c \right) dc \geq 0.
\end{aligned}$$

To find the solution to this problem, let us look at a related problem, where the platform maximizes

the sum of its profits and total quantity served, with latter weighted by some $\lambda > 0$:

$$\begin{aligned}
& \max_{q(c)} \int_0^1 q(c) \left(\frac{1-q(c)}{\beta} - 2c \right) dc + \lambda \int_0^1 q(c) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad \quad q(c) \geq 0, \forall c \in [0, 1] \\
& = \max_{q(c)} \int_0^1 q(c) \left(\frac{1-q(c)}{\beta} + \lambda - 2c \right) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad \quad q(c) \geq 0, \forall c \in [0, 1]
\end{aligned} \tag{32}$$

Relaxing the first constraint, we can decompose this into individual sub-problems for all agents:

$$\begin{aligned}
& \max_{q(c)} q(c) \left(\frac{1-q(c)}{\beta} + \lambda - 2c \right) dc \\
& \text{s.t.} \quad q(c) \geq 0.
\end{aligned}$$

This is the maximization of a simple quadratic function with a lower-bound. The optimal quantity is:

$$q^*(c) = \max \left\{ 0, \frac{1}{2}(1 + \beta\lambda) - \beta c \right\}.$$

This solution also satisfies:

$$q^{*'}(c) \leq 0.$$

Therefore, the solution to this relaxed problem is also optimal for (32).

The highest cost agent with non-zero quantity is

$$c_h = \min \left\{ 1, \frac{1 + \beta\lambda}{2\beta} \right\}.$$

We will now show that, $(1 + \beta\lambda)/(2\beta) < 1$ for all λ values that yield non-negative profits for the platform as long as $\beta > 1$ restriction discussed in Section 3 holds. To prove by contradiction, assume $(1 + \beta\lambda)/(2\beta) \geq 1$ and $c_h = 1$ in equilibrium. The platform earns

$$\Pi = \int_0^1 q^*(c) \left(\frac{1-q(c)}{\beta} - 2c \right) dc = \frac{1}{12} \left(\beta(4 - 3\lambda^2) + \frac{3}{\beta} - 6 \right).$$

The profit is non-negative only if

$$\lambda \leq \sqrt{\frac{1}{\beta^2} - \frac{2}{\beta} + \frac{4}{3}}.$$

This implies

$$\frac{1 + \beta\lambda}{2\beta} = \frac{1 + \beta \sqrt{\frac{1}{\beta^2} - \frac{2}{\beta} + \frac{4}{3}}}{2\beta} < 1$$

for all feasible $\beta > 1$, creating a contradiction. This means that $(1 + \beta\lambda)/(2\beta) \geq 1$ cannot be true and that we have

$$c_h = \frac{1 + \beta\lambda}{2\beta}.$$

The profits earned by the platform is

$$\Pi = \int_0^{c_h} q^*(c) \left(\frac{1 - q(c)}{\beta} - 2c \right) dc = \frac{(1 + \beta\lambda)^2(1 - 2\beta\lambda)}{24\beta^2}.$$

By definition, Equation (32) maximizes the total quantity served subject to the constraint that the platform earns

$$\Pi = \frac{(1 + \beta\lambda)^2(1 - 2\beta\lambda)}{24\beta^2}.$$

Total throughput-maximizing solution is then the one that yields exactly zero profits for the platform:

$$\Pi = \frac{(1 + \beta\lambda^*)^2(1 - 2\beta\lambda^*)}{24\beta^2} = 0 \iff \lambda^* = \frac{1}{2\beta},$$

which yields the total quantity served of

$$\int_0^{c_h} q^*(c) dc = \frac{(1 + \beta\lambda^*)^2}{8\beta} = \frac{9}{32\beta}.$$

Under this optimal mechanism, the highest cost that participates and the average market price are:

$$\begin{aligned} \bar{p} &= \frac{1}{2\beta}, \\ c_h &= \frac{3}{4\beta}. \end{aligned}$$

Total quantity of customers served in the market is

$$Q = \frac{9}{32\beta}.$$

The fee platform charges to agent with cost c is

$$F^*(c) = \frac{(\gamma - 2\beta)(4\beta c - 3)(4\beta c - 1)}{32\beta\gamma}.$$

We will now show that the platform can implement this mechanism with commission-plus. With decentralized pricing, let the platform sets its terms,

$$\phi = 1 - \frac{\gamma}{2\beta}, \quad w = \frac{\gamma - 2\beta}{4\beta^2},$$

and assume agents expect an average price of

$$\bar{p} = \frac{1}{2\beta}.$$

By Equation (10), an agent with cost c posts a price under decentralized pricing

$$\begin{aligned} p^*(c, \bar{p}) &= \frac{1}{2} \left(\frac{1 - (\beta - \gamma)\bar{p}}{\gamma} + \frac{c + w}{1 - \phi} \right) \\ &= \frac{2\gamma - \beta + 4\beta^2 c}{4\beta\gamma}. \end{aligned}$$

Let c_h be the highest cost agent that can participate with non-negative demand:

$$q(p^*(c_h, \bar{p}), \bar{p}) = 0 \implies c_h = \frac{3}{4\beta}.$$

The realized average price is characterized by Equation (17),

$$\bar{p} = \frac{2(1 - \phi) + \gamma w}{(1 - \phi)(2\beta + \gamma)} = \frac{1}{2\beta}.$$

The platform's profit is

$$\begin{aligned} \Pi^{C^+} &= \int_0^{c_h} q(p^*(c, \bar{p}), \bar{p})(\phi p^*(c, \bar{p}) + w) dc \\ &= 0. \end{aligned}$$

Total quantity served in the market is

$$\begin{aligned} Q &= \int_0^{c_h} q(p^*(c, \bar{p}), \bar{p}) dc \\ &= \frac{9}{32\beta}, \end{aligned}$$

same as the optimal mechanism. □

C Robustness to Straight Average Utility Model

With the quadratic customer utility model and continuum of agents, a representative customer's utility is

$$U(q) = \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{2\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{2\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c)p(c) dc.$$

The finite agent version of this utility model was introduced in [Shubik and Levitan \(1980\)](#) and the linear demand model that arises from this utility function has been used by many authors (e.g.,

Abhishek et al. 2016, Hagi and Wright 2019b, Inderst and Shaffer 2019). Others like Melitz and Ottaviano (2008), Foster et al. (2008), Altomonte et al. (2016) use the continuum agent version above, for which the demand of an agent with positive quantity is

$$q(p(c), Q) = \alpha - \beta Q - \gamma p(c) \tag{33}$$

where

$$Q = \int_0^1 q(p(c), Q) dc.$$

We establish the robustness of our qualitative results to the choice of customer utility model in this document by replicating our analyses using the demand model specified in (33) and observing that our main qualitative results continue to hold. More specifically, we show the following results.

- When platform uses fees that are linear in price, there is a unique centralized pricing equilibrium (C.1.1) and a unique decentralized pricing equilibrium (Section C.1.3).
- With just a simple commission contract, centralized pricing can be better or worse than decentralized pricing depending on the realization of demand parameters (Section C.1.5).
- Platform’s optimal mechanism can be implemented with decentralized pricing using either a commission-plus contract (Section C.1.7) or a quantity discount contract (Section C.1.8).
- Decentralized pricing with commission-plus replicates the optimal mechanism even when the agents have capacity constraints (Section C.1.13) or the platform adopts throughput maximization as its objective (Section C.1.14).

C.1 Analysis

C.1.1 Centralized Pricing

In centralized pricing, there exists a single price in the market. Let p be the price platform sets and ϕ be the portion of revenue retained. Conditional on participation, an agent with cost c earns

$$\pi_c(p) = (\alpha - \beta Q - \gamma p)((1 - \phi)p - c).$$

Agent earnings decrease in cost, and therefore there exists a threshold cost, c_h , for which agent with cost c participates if and only if $c \leq c_h$. The highest cost that participates is

$$\pi_{c_h}(p) = 0 \implies c_h = c_h(p, \phi) = (1 - \phi)p.$$

The total quantity served in the market is

$$\begin{aligned}
Q &= \int_0^{\hat{c}} (\alpha - \beta Q - \gamma p) dc \\
&= \hat{c}(\alpha - \beta Q - \gamma p) \\
&= (1 - \phi)p(\alpha - \beta Q - \gamma p).
\end{aligned}$$

Solving for Q , we get

$$Q = \frac{p(1 - \phi)(\alpha - \gamma p)}{1 + \beta p(1 - \phi)}.$$

Plugging this back into the demand function:

$$q(p) = \frac{\alpha - \gamma p}{1 + \beta(1 - \phi)p}.$$

The demand is non-negative only if $p \leq \alpha/\gamma$.

The platform's profit-maximization problem is:

$$\begin{aligned}
\max_{\phi, p} \quad \Pi^C(p, \phi) &= \phi \int_0^{c_h(\phi)} q(p, Q)p dc \\
&= \frac{p^2(1 - \phi)\phi(\alpha - \gamma p)}{\alpha + \beta p(1 - \phi)}.
\end{aligned}$$

The problem is concave in ϕ within the feasible region $0 \leq \phi \leq 1$, $0 \leq p \leq \alpha/\gamma$, e.g.,

$$\frac{\partial^2 \Pi^C(p, \phi)}{\partial \phi^2} = -\frac{2p^2(1 + \beta p)(\alpha - \gamma p)}{(1 + \beta p(1 - \phi))^3} < 0,$$

and therefore, platform's optimal commission is defined by the first order condition

$$\frac{\partial \Pi^C(p, \phi)}{\partial \phi} = 0 \implies \frac{p^2(\alpha - \gamma p)(\beta p(\phi - 1)^2 - 2\phi + 1)}{(\beta p(\phi - 1) - 1)^2} = 0.$$

This gives a unique feasible solution,

$$\phi^*(p) = \frac{1 + \beta p - \sqrt{1 + \beta p}}{\beta p}.$$

Plugging this back into optimization problem and looking at the total derivative with respect to price, we observe

$$\begin{aligned}
\frac{d\Pi^C(p, \phi^*(p))}{dp} \geq 0 &\iff \frac{\alpha \left(\beta - \frac{\beta}{\sqrt{1 + \beta p}} \right) + \gamma \left(-2\beta p + 3\sqrt{1 + \beta p} - \frac{1}{\sqrt{1 + \beta p}} - 2 \right)}{\beta^2} \geq 0 \\
&\iff p \geq \frac{1}{8\beta} \sqrt{\frac{8\alpha\beta + 9\gamma}{\gamma}} + \frac{4\alpha\beta - 3\gamma}{8\beta\gamma}.
\end{aligned}$$

Hence, platform's profit function is quasi-concave in price and is maximized at

$$p^* = \frac{1}{8\beta} \sqrt{\frac{8\alpha\beta + 9\gamma}{\gamma}} + \frac{4\alpha\beta - 3\gamma}{8\beta\gamma}.$$

Plugging in the optimal decisions, the platform's equilibrium profit is

$$\Pi^C = \frac{8\alpha^2\beta^2 - 9\gamma^{3/2}\sqrt{8\alpha\beta + 9\gamma} + 36\alpha\beta\gamma - 8\alpha\beta\sqrt{\gamma}\sqrt{8\alpha\beta + 9\gamma} + 27\gamma^2}{32\beta^3\gamma}.$$

C.1.2 Decentralized Pricing

With decentralized pricing, agents select their own price to post on the platform. A price equilibrium occurs when all agents select optimal prices given the equilibrium prices of others.

First, in Section C.1.3, we show that the existence and uniqueness of a decentralized pricing equilibrium are guaranteed for a set of contracts where the fees agents pay to the platform are linear in the price they select. Then, in Section C.1.4, we characterize the decentralized pricing equilibrium assuming that the platform uses a commission only contract.

C.1.3 The Existence And Uniqueness Of Decentralized Pricing Equilibrium

Consider a broad class of fee structures such that an agent pays the platform (i) a base unit fee, w , per unit served and (ii) a commission, ϕp , per unit served where $\phi < 1$ is the fixed commission rate and p is the agent's price. An agent with demand q pays the platform in total $q(w + \phi p)$. The fixed per-unit fee, w , can be negative, meaning that it is actually a per-unit subsidy.

With this payment structure, we seek to determine the uniqueness of a price equilibrium. Each agent's earnings depend on the total quantity on the platform, so it is sufficient to consider a pricing equilibrium in which each agent expects the total quantity to be Q_e and all agents select prices based on that expectation that yield Q_e as the actual total quantity served.

An agent with total quantity expectation Q_e , cost c and price p expects to earn

$$\begin{aligned} \pi_c(p, Q_e) &= q(p, Q_e) ((1 - \phi)p - c - w) \\ &= (\alpha - \beta Q_e - \gamma p) ((1 - \phi)p - c - w). \end{aligned}$$

Because earnings are strictly concave in price, there exists a unique price, $p^*(c, Q_e)$, that maximizes the agent's earnings,

$$p^*(c, Q_e) = \frac{(\alpha - \beta Q_e)}{2\gamma} + \frac{c + w}{2(1 - \phi)}. \quad (34)$$

Let the total quantity realized be Q . This agent receives $q(p^*(c, Q_e), Q)$ demand,

$$q(p^*(c, Q_e), Q) = \frac{1}{2} \left(\alpha + \beta(Q_e - 2Q) - \frac{\gamma(c + w)}{1 - \phi} \right)^+. \quad (35)$$

The demand is decreasing in cost, c . Define c_0 to be the smallest cost such that the optimal quantity is zero,

$$c_0 = \min\{c : q(p^*(c, Q_e), Q) = 0\} = \frac{\alpha(1 - \phi) + \beta(1 - \phi)(Q_e - 2Q)}{\gamma} - w. \quad (36)$$

The realized total quantity is

$$Q = \int_0^{\min\{c_0, 1\}} q(p^*(c, Q_e), Q) dc. \quad (37)$$

An equilibrium exists if $Q_e = Q$. We will show that there can exist only one such equilibrium in any market.

From (36), given an equilibrium, i.e., $Q = Q_e$, the total quantity in equilibrium can be expressed in terms of c_0 ,

$$Q = \frac{\alpha(1 - \phi) - \gamma(c_0 + w)}{\beta(1 - \phi)}. \quad (38)$$

From (34), (35), (38) and $Q_e = Q$, the equilibrium condition (37) can be written as

$$\begin{aligned} 0 &= Q - \int_0^{\min\{c_0, 1\}} q(p^*(c, Q), Q) dc \\ &= \frac{\alpha(1 - \phi) - \gamma(c_0 + w)}{\beta(1 - \phi)} - \int_0^{\min\{c_0, 1\}} \frac{1}{2} \left(\alpha - \beta Q - \frac{\gamma(c + w)}{1 - \phi} \right) dc \\ \Leftrightarrow 0 &= \begin{cases} \frac{4\alpha(1 - \phi) - \beta\gamma c_0^2 - 4\gamma(c_0 + w)}{4\beta(1 - \phi)} & , \text{ if } c_0 \leq 1, \\ \frac{\gamma(\beta - 2(\beta + 2)c_0 - 4w) - 4\alpha(\phi - 1)}{4\beta(1 - \phi)} & , \text{ if } c_0 > 1. \end{cases} \end{aligned} \quad (39)$$

The right-hand side of (39) is continuous at $c_0 = 1$. It is also monotonically decreasing in c_0 :

$$\begin{aligned} \frac{\partial}{\partial c_0} \left(\frac{4\alpha(1 - \phi) - \beta\gamma c_0^2 - 4\gamma(c_0 + w)}{4\beta(1 - \phi)} \right) &= -\frac{\gamma(\beta c_0 + 2)}{2\beta(1 - \phi)} < 0 \quad \forall c_0 \leq 1, \\ \frac{\partial}{\partial c_0} \left(\frac{\gamma(\beta - 2(\beta + 2)c_0 - 4w) - 4\alpha(\phi - 1)}{4\beta(1 - \phi)} \right) &= -\frac{(\beta + 2)\gamma}{2\beta(1 - \phi)} < 0 \quad \forall c_0 > 1. \end{aligned}$$

As a result, the right-hand side of (39) only cross zero once, yielding a unique c_0 .

The unique feasible candidate c_0 in the domain $c_0 \leq 1$ is

$$c_0 = \frac{2 \left(\sqrt{\gamma(\gamma + \beta(\alpha(1 - \phi) - \gamma w))} - \gamma \right)}{\beta\gamma}. \quad (40)$$

From (38) and (40), for this candidate c_0 , there exists a unique equilibrium total quantity,

$$Q = \frac{\alpha\beta(1 - \phi) + \gamma(2 - \beta w) - 2\sqrt{\gamma(\gamma + \beta(\alpha(1 - \phi) - \gamma w))}}{\beta^2(1 - \phi)}. \quad (41)$$

C.1.4 Decentralized Pricing With Just A Commission Fee

The analysis in Section C.1.3 shows that the uniqueness of a total quantity that satisfies consistent expectations equilibrium is guaranteed. In decentralized pricing with commission fees, conditional on participation, an agent with cost c setting a price p earns

$$\pi_c(p, Q) = (\alpha - \beta Q - \gamma p)((1 - \phi)p - c).$$

Agent's profit depends on individual price, p , and also the total quantity served in the market, Q , which is characterized in the equilibrium.

Each agent chooses the price that maximizes own earnings. From (34),

$$p^*(c, Q) = \frac{\alpha - \beta Q}{2\gamma} + \frac{c + w}{2(1 - \phi)}.$$

The agent earns

$$\pi_c(p, Q) = \frac{((\alpha - Q\beta)(1 - \phi) - \gamma c)^2}{4\gamma(1 - \phi)}.$$

Agent earnings are decreasing in cost, c , and therefore there exists a threshold cost, c_h , for which agent with cost c participates if and only if $c \leq c_h$. Define c_0 to be the lowest cost that yields zero profit:

$$\pi_{c_0}(p^*(c_0, Q), Q) = 0 \implies c_0 = c_0(\phi) = \frac{(\alpha - Q\beta)(1 - \phi)}{\gamma}.$$

Then,

$$c_h = c_h(\phi) = \min\{1, c_0\}.$$

With $c_0 \leq 1$, the total quantity is uniquely defined by (41):

$$Q = \frac{\alpha\beta(1 - \phi) + \gamma(2 - \beta w) - 2\sqrt{\gamma(\gamma + \beta(\alpha(1 - \phi) - \gamma w))}}{\beta^2(1 - \phi)}.$$

The platform's profit-maximization problem is:

$$\begin{aligned} \max_{\phi} \quad \Pi^{\mathcal{D}}(\phi) &= \phi \int_0^{c_h(\phi)} q(p^*(c, Q), Q) p^*(c, Q) dc \\ &= \frac{4\phi \left(\sqrt{\gamma(\alpha\beta(1 - \phi) + \gamma)} - \gamma \right)^3}{3\beta^3\gamma^2(1 - \phi)^2}. \end{aligned}$$

The profit takes positive values for all $0 < \phi < 1$ and is zero at both boundary points, $\phi = 0$ and $\phi = 1$. Because the profit function is continuous, any maximizer is an interior point and defined by the first order condition:

$$\frac{\partial \Pi^{\mathcal{D}}(\phi)}{\partial \phi} = \frac{2 \left(\gamma - \sqrt{\gamma(\alpha\beta(1 - \phi) + \gamma)} \right)^2 \left(2(\phi + 1) \left(\sqrt{\gamma(\alpha\beta(1 - \phi) + \gamma)} - \gamma \right) - \alpha\beta(\phi - 2)(\phi - 1) \right)}{3\beta^3\gamma(\phi - 1)^3 \sqrt{\gamma(\alpha\beta(1 - \phi) + \gamma)}} = 0. \quad (42)$$

(42) can be shown to yield a unique solution. Note that for any $0 < \phi < 1$,

$$\frac{2\left(\gamma - \sqrt{\gamma(\alpha\beta(1-\phi) + \gamma)}\right)^2}{3\beta^3\gamma(\phi-1)^3\sqrt{\gamma(\alpha\beta(1-\phi) + \gamma)}} < 0.$$

This means that FOC is satisfied when

$$\begin{aligned} & 2(\phi^* + 1) \left(\sqrt{\gamma(\alpha\beta(1 - \phi^*) + \gamma)} - \gamma \right) - \alpha\beta(\phi^* - 2)(\phi^* - 1) = 0 \\ \iff & \frac{2}{\phi^* - 1} \left(\sqrt{\gamma(\alpha\beta(1 - \phi^*) + \gamma)} - \gamma \right) - \alpha\beta \left(\frac{\phi^* - 2}{\phi^* + 1} \right) = 0. \end{aligned} \quad (43)$$

The left-hand side of the equality is monotonically increasing in ϕ^* within the range $\phi^* \in [0, 1]$, and therefore can only cross zero once. Hence, (43) uniquely defines platform's optimal commission, ϕ^* .

C.1.5 Comparison of Centralized and Decentralized Pricing

Equation (43) corresponds to a root of a cubic polynomial. While Cardano's formula can find closed-form expressions for roots of a cubic polynomial, the expressions are length. To compute platform's profit under decentralized pricing and compare it with centralized pricing, we use a computer algebra system and find that there exists a $k \approx 0.149$ such that

$$\Pi^D > \Pi^C \iff \gamma < k\alpha\beta.$$

That is, a platform may prefer centralized or decentralized pricing depending on the competition characteristics of the market. More specifically, the platform's profit is higher with decentralized pricing in markets with low γ , and it is higher with centralized pricing in markets with high γ .

C.1.6 Optimal Mechanism

According to the Revelation Principle, the set of optimal mechanisms can be found within the set of truth-inducing mechanisms. In such a mechanism, the platform posts a menu that maps each cost to a price and a fee. Each agent reports a cost and is assigned the price and fee associated with that cost, and each agent (*i*) earns non-negative profits (Individual Rationality), and (*ii*) prefers to report own cost truthfully given that all other agents do the same (Incentive Compatibility). Following this structure, let $p(c)$ be the price the platform assigns to agent c and let $F(c)$ be the fee collected. Note that $F(c)$ is the total fee and not per-unit, $F(c) = q(p(c), Q)f(c)$. Let $\pi_c(\tilde{c})$ be an agent's earnings with cost c when the agent reports cost \tilde{c} :

$$\pi_c(\tilde{c}) = u(c, p(\tilde{c})) - F(\tilde{c})$$

where

$$u(c, p(\tilde{c})) = (\alpha - \beta Q - \gamma p(\tilde{c})) (p(\tilde{c}) - c).$$

The Individual Rationality (IR) and the Incentive Compatibility (IC) constraints are

$$\begin{aligned}\pi_c(c) &\geq 0, \\ \pi_c(c) &\geq \pi_c(\tilde{c}),\end{aligned}$$

respectively for all $c \in [0, 1]$, $\tilde{c} \in [0, 1]$.

Because of the IC constraints, agent earnings are strictly decreasing in cost:

$$\begin{aligned}\pi_c(c) &\geq \pi_c(\tilde{c}) = (\alpha - \beta Q - \gamma p(\tilde{c})) (p(\tilde{c}) - c) - F(\tilde{c}) \\ &> (\alpha - \beta Q - \gamma p(\tilde{c})) (p(\tilde{c}) - \tilde{c}) - F(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c})\end{aligned}$$

for all $\tilde{c} > c$. This also implies that the agent with the highest cost, $c = 1$, earns zero under the optimal truth-inducing mechanism: if the highest cost agent were to earn a strictly positive amount, then the platform could uniformly increase the fee, $F(\tilde{c})$, for all participating agents, thus increasing its profit.

We can re-formulate an agent's earnings as

$$\pi_c(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c}) - u(\tilde{c}, p(\tilde{c})) + u(c, p(\tilde{c})).$$

For a pair of agents with costs c, \tilde{c} , IC constraints imply

$$\begin{aligned}\pi_c(c) &\geq \pi_c(\tilde{c}) = \pi_{\tilde{c}}(\tilde{c}) - u(\tilde{c}, p(\tilde{c})) + u(c, p(\tilde{c})), \\ \pi_{\tilde{c}}(\tilde{c}) &\geq \pi_{\tilde{c}}(c) = \pi_c(c) - u(c, p(c)) + u(\tilde{c}, p(c)).\end{aligned}$$

These inequalities can be combined:

$$\begin{aligned}u(\tilde{c}, p(c)) - u(c, p(c)) &\leq \pi_{\tilde{c}}(\tilde{c}) - \pi_c(c) \leq u(\tilde{c}, p(\tilde{c})) - u(c, p(\tilde{c})) \\ \iff \int_c^{\tilde{c}} \frac{\partial u(c_k, p(c))}{\partial c_k} dc_k &\leq \pi_{\tilde{c}}(\tilde{c}) - \pi_c(c) \leq \int_c^{\tilde{c}} \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k.\end{aligned}\tag{44}$$

This inequality has two important consequences. First, ignoring the middle term, Equation (44)

implies

$$\begin{aligned}
& \int_c^{\tilde{c}} \frac{\partial u(c_k, p(c))}{\partial c_k} dc_k \leq \int_c^{\tilde{c}} \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k, \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
\iff & - \int_c^{\tilde{c}} (\alpha - \beta Q - \gamma p(c)) dc_k \leq - \int_c^{\tilde{c}} (\alpha - \beta Q - \gamma p(\tilde{c})) dc_k, \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
\iff & (\alpha - \beta Q - \gamma p(c)) (c - \tilde{c}) \leq (\alpha - \beta Q - \gamma p(\tilde{c})) (c - \tilde{c}), \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
\iff & (p(c) - p(\tilde{c})) (c - \tilde{c}) \geq 0, \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
\iff & p'(c) \geq 0, \quad \forall c \in [0, 1].
\end{aligned} \tag{45}$$

Second, because $u(\tilde{c}, p(c))$ is a continuous function of \tilde{c} , this inequality implies that $\pi_{\tilde{c}}(\tilde{c})$ is also continuous with respect to the Lebesgue measure and thus almost everywhere differentiable. Furthermore, its derivative is

$$\frac{d\pi_c(c)}{dc} = \frac{\partial u(c, p(c))}{\partial c}.$$

Using the fundamental theorem of calculus:

$$\pi_{\tilde{c}}(\tilde{c}) = \pi_c(c) + \int_c^{\tilde{c}} \frac{d\pi_{c_k}(c_k)}{dc_k} dc_k = \pi_c(c) + \int_c^{\tilde{c}} \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k.$$

Because the highest cost agent earns zero, setting $\tilde{c} = 1$, the equation simplifies to

$$\pi_c(c) = - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k,$$

which means

$$F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k. \tag{46}$$

Our analysis so far has established that (45) and (46) are necessary conditions for IR and IC. Now, we will show that they are also sufficient. By establishing an equivalence between the two sets of constraints, we will be able to replace the IR and IC constraints in the platform's optimal mechanism problem with Equations (45) and (46) and therefore convert the problem into a tractable form.

It is straightforward to show that (46) is sufficient to imply IR:

$$\begin{aligned}
\pi_c(c) &= u(c, p(c)) - F(c) \\
&= u(c, p(c)) - u(c, p(\tilde{c})) - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= - \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= \int_c^1 (\alpha - \beta Q - \gamma p(c_k)) dc_k \geq 0 \quad \forall c \in [0, 1].
\end{aligned}$$

Similarly for IC, we have

$$\begin{aligned}
\pi_c(\tilde{c}) - \pi_c(c) &= u(c, p(\tilde{c})) - F(\tilde{c}) - u(c, p(c)) + F(c) \\
&= u(c, p(\tilde{c})) - u(\tilde{c}, p(\tilde{c})) - \int_{\tilde{c}}^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k - u(c, p(c)) + u(\tilde{c}, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= u(c, p(\tilde{c})) - u(\tilde{c}, p(\tilde{c})) - \int_{\tilde{c}}^c \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= \int_{\tilde{c}}^c \frac{\partial u(c_k, p(\tilde{c}))}{\partial c_k} dc_k - \int_{\tilde{c}}^c \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \\
&= \int_{\tilde{c}}^c (\alpha - \beta Q - \gamma p(\tilde{c})) dc_k - \int_{\tilde{c}}^c (\alpha - \beta Q - \gamma p(c_k)) dc_k \\
&= \gamma \int_{\tilde{c}}^c (p(c_k) - p(\tilde{c})) dc_k \geq 0, \quad \forall c \in [0, 1], \tilde{c} \in [0, 1].
\end{aligned}$$

Now, we construct the platform's problem. The platform chooses $p(c)$ and $F(c)$ to maximize total profits subject to IR and IC constraints, and the natural restriction that the demand of an agent cannot be negative:

$$\begin{aligned}
&\max_{p(c), F(c)} \int_0^1 F(c) dc \\
&\text{s.t.} \quad \pi_c(c) \geq \pi_c(\tilde{c}), \quad \forall c \in [0, 1], \forall \tilde{c} \in [0, 1] \\
&\quad \pi_c(c) \geq 0, \quad \forall c \in [0, 1] \\
&\quad \alpha - \beta Q - \gamma p(c) \geq 0, \quad \forall c \in [0, 1] \\
&\quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc.
\end{aligned}$$

Using the equivalence we found earlier, we replace IR and IC constraints with Equations (45) and (46):

$$\begin{aligned}
& \max_{p(c), F(c)} \int_0^1 F(c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k, \forall c \in [0, 1] \\
& \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\
& \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\
& = \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\
& \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\int_0^1 \left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) dc &= \left[\left(\int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k \right) c \right]_0^1 - \int_0^1 \left(-\frac{\partial u(c, p(c))}{\partial c} \right) c dc \\
&= \int_0^1 \frac{\partial u(c, p(c))}{\partial c} c dc.
\end{aligned} \tag{47}$$

The platform's problem converts to

$$\begin{aligned}
& \max_{p(c)} \int_0^1 \left(u(c, p(c)) + \frac{\partial u(c, p(c))}{\partial c} c \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\
& \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\
& = \max_{p(c)} \int_0^1 \left((\alpha - \beta Q - \gamma p(c))(p(c) - c) - (\alpha - \beta Q - \gamma p(c))c \right) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\
& \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\
& = \max_{p(c)} \int_0^1 (\alpha - \beta Q - \gamma p(c))(p(c) - 2c) dc \\
& \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\
& \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\
& \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc
\end{aligned} \tag{48}$$

We re-formulate this as a quantity-choice problem. Let $q(c)$ be the demand platform assigns to agent with cost c . We have one-to-one equivalence between price and quantity:

$$p(c) = \frac{\alpha - \beta Q - q(c)}{\gamma}.$$

Plugging this back into the platform's problem and expanding the Q expression, we have the following problem

$$\begin{aligned}
& \max_{q(c)} \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c) 2c dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad q(c) \geq 0, \forall c \in [0, 1].
\end{aligned} \tag{49}$$

Relaxing the first constraint, the problem is a specific case of quadratic utility maximization where the prices of the agents are equal to twice their costs, $p(c) = 2c$. For all agents with non-zero assigned quantity, we have

$$q(c) = \frac{\alpha}{2} - \beta Q - \gamma c \tag{50}$$

where

$$Q = \int_0^1 q(c) dc. \tag{51}$$

This solution also satisfies the relaxed constraint. Therefore, the solution in (50) is also optimal for the platform's optimal mechanism.

Solving for (50) and (51) simultaneously, we get one feasible total quantity,

$$Q = \frac{\alpha\beta + 2\gamma - 2\sqrt{\alpha\beta\gamma + \gamma^2}}{2\beta^2}.$$

Other equilibrium characteristics are as follows:

$$\begin{aligned} p^*(c) &= \frac{\alpha - \beta Q - q(c)}{\gamma} \\ &= \frac{\alpha}{2\gamma} + c \\ F^*(c) &= u(c, p^*(c)) + \int_c^1 \frac{\partial u(c_k, p^*(c_k))}{\partial c_k} dc_k \\ &= \frac{\alpha\beta \left(\sqrt{\gamma(\alpha\beta + \gamma)} - \gamma(\beta c + 2) \right) - \gamma \left(\gamma(\beta^2 c^2 + 2\beta c + 2) - 2(\beta c + 1)\sqrt{\gamma(\alpha\beta + \gamma)} \right)}{2\beta^2\gamma}. \end{aligned}$$

With the optimal mechanism, we have

$$\begin{aligned} F^*(c) &= \frac{\alpha\beta \left(\sqrt{\gamma(\alpha\beta + \gamma)} - \gamma(\beta c + 2) \right) - \gamma \left(\gamma(\beta^2 c^2 + 2\beta c + 2) - 2(\beta c + 1)\sqrt{\gamma(\alpha\beta + \gamma)} \right)}{2\beta^2\gamma} \\ &= q^*(c) \left(\frac{1}{2}p^*(c) + \frac{-2\sqrt{\gamma(\alpha\beta + \gamma)} + \alpha\beta + 2\gamma}{4\beta\gamma} \right). \end{aligned}$$

Because we can map the fee structure of the optimal mechanism to one that is linear in price, the uniqueness arguments in Section C.1.3 also apply to the optimal mechanism. Therefore, with the optimal mechanism, there exists a unique equilibrium total quantity.

The platform earns

$$\begin{aligned} \Pi &= \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - 2 \int_0^1 q(c)c dc \\ &= \frac{\left(-2\sqrt{\gamma(\alpha\beta + \gamma)} + 3\alpha\beta + 2\gamma \right) \left(\gamma - \sqrt{\gamma(\alpha\beta + \gamma)} \right)^2}{12\beta^3\gamma^2}. \end{aligned}$$

C.1.7 Replicating The Optimal Mechanism With Commission and Per-unit Fee Contract

With the optimal mechanism, there is a unique candidate commission and per-unit fee structure that may yield the same equilibrium outcome as the optimal mechanism. To show that this structure indeed replicates the optimal mechanism, let the platform sets its terms

$$\phi = \frac{1}{2}, \quad w = \frac{-2\sqrt{\gamma(\alpha\beta + \gamma)} + \alpha\beta + 2\gamma}{4\beta\gamma},$$

and assume agents expect a total quantity of

$$Q = \frac{\alpha\beta + 2\gamma - 2\sqrt{\alpha\beta\gamma + \gamma^2}}{2\beta^2}.$$

By Equation (34), an agent with cost c posts a price under commission-plus

$$\begin{aligned} p^*(c, Q) &= \frac{\alpha - \beta Q}{2\gamma} + \frac{c + w}{2(1 - \phi)} \\ &= \frac{\alpha}{2\gamma} + c. \end{aligned}$$

The price chosen by any agent is exactly equivalent to the one in the optimal mechanism. The same is true for the fees paid. Because prices and fees are the only decision variables in the setting, then commission-plus with the given terms replicates the optimal mechanism.

C.1.8 Replicating The Optimal Mechanism With Quantity Discount Contract

With the optimal mechanism, we have

$$\begin{aligned} F^*(c) &= \frac{\alpha\beta \left(\sqrt{\gamma(\alpha\beta + \gamma)} - \gamma(\beta c + 2) \right) - \gamma \left(\gamma(\beta^2 c^2 + 2\beta c + 2) - 2(\beta c + 1)\sqrt{\gamma(\alpha\beta + \gamma)} \right)}{2\beta^2\gamma} \\ &= q^*(c) \left(-\frac{1}{2\gamma}q^*(c) + \frac{\alpha}{2\gamma} \right). \end{aligned}$$

There is a unique candidate quantity pricing fee structure that may yield the same equilibrium outcome as the optimal mechanism. To show that this structure indeed replicates the optimal mechanism, let the platform sets its terms

$$\phi = -\frac{1}{2\gamma}, \quad w = \frac{\alpha}{2\gamma}.$$

and assume agents expect a total quantity of

$$Q = \frac{\alpha\beta + 2\gamma - 2\sqrt{\alpha\beta\gamma + \gamma^2}}{2\beta^2}.$$

Conditional on participation, an agent with cost c setting a price p earns

$$\pi_c(p, Q) = (\alpha - \beta Q - \gamma p)(p - c - \phi(\alpha - \beta Q - \gamma p) - w).$$

Because earnings are strictly concave in price, there exists a unique price, $p^*(c, Q)$, that maximizes the agent's earnings,

$$\begin{aligned} p^*(c, Q) &= \frac{2\alpha\gamma\phi + \alpha + c\gamma - 2\beta\gamma Q\phi - \beta Q + \gamma w}{2\gamma(\gamma\phi + 1)} \\ &= \frac{\alpha}{2\gamma} + c. \end{aligned}$$

The price chosen by any agent is exactly equivalent to the one in the optimal mechanism. Same is true for the fees paid. Because prices and fees are the only decision variables in the setting, then quantity pricing with the given terms replicates the optimal mechanism.

C.1.9 Capacity Constraints

We will now solve the equilibria of centralized pricing, decentralized pricing, and optimal mechanism under the assumption that agents have a maximum capacity to serve k demand.

C.1.10 Centralized Pricing

Without capacity constraints, the number of customers served by each agent under centralized pricing is

$$q(p, Q) = \frac{\sqrt{\gamma}(2\gamma - \alpha\beta) (-\sqrt{\gamma}\sqrt{8\alpha\beta + 9\gamma} + 4\alpha\beta + 3\gamma)}{\beta (2\alpha\beta (\sqrt{8\alpha\beta + 9\gamma} + 3\sqrt{\gamma}) + \gamma (\sqrt{8\alpha\beta + 9\gamma} + 13\sqrt{\gamma}))}.$$

With capacity constraints, the capacity will be binding if and only if k is smaller than this amount. When capacity is binding, if platform sets a price p , the profit agent with cost c earns is

$$\pi_c(p) = k((1 - \phi)p - c)$$

Agents participate as long as they earn positive profits. The highest cost that participates, c_h , earns zero

$$\pi_{c_h}(p) = 0 \implies c_h = c_h(p, \phi) = (1 - \phi)p.$$

The platform's profit is:

$$\begin{aligned} \Pi^c(p, \phi) &= \int_0^{c_h(p, \phi)} k\phi p \, dc \\ &= k\phi(1 - \phi)p^2. \end{aligned}$$

The objective is monotonically increasing in p . It is optimal for the platform to increase the price as long as there is excess demand. In the optimal, quantity served should equal to the capacity:

$$\begin{aligned} k &= \alpha - \beta Q - \gamma p \\ &= \alpha - \beta k c_h - \gamma p \\ &= \alpha - \beta k(1 - \phi)p - \gamma p \end{aligned}$$

yielding a price

$$p = \frac{\alpha - k}{\gamma + \beta k(1 - \phi)}.$$

Plugging this into the objective,

$$\Pi^c(\phi) = \frac{k(1 - \phi)\phi(k - \alpha)^2}{(\gamma + k\beta(1 - \phi))^2}.$$

Within the feasible set of parameters, we have

$$\begin{aligned}\frac{\partial \Pi^c(\phi)}{\partial \phi} &= \frac{k(k-\alpha)^2(\gamma(2\phi-1) + \beta k(\phi-1))}{(\beta k(\phi-1) - \gamma)^3} \geq 0 \\ \iff \phi &\leq \frac{\gamma + \beta k}{2\gamma + \beta k}.\end{aligned}$$

Therefore, the platform's profit is quasi-concave in ϕ and is maximized at

$$\phi^* = \frac{\gamma + \beta k}{2\gamma + \beta k}.$$

The platform's equilibrium profit is

$$\Pi^c = \frac{k(\alpha - k)^2}{4\gamma(\gamma + \beta k)}.$$

C.1.11 Decentralized Pricing

Without capacity constraints, the highest number of customers an agent serves under decentralized pricing with just commission is

$$q(p^*(0, Q), Q) = \frac{\sqrt{\gamma(\alpha\beta(1 - \phi^*) + \gamma)} - \gamma}{\beta(1 - \phi^*)}.$$

where ϕ^* is given by (43).

With capacity constraints, the capacity will be binding if and only if k is smaller than this amount. The agents need to incorporate the capacity restriction in their pricing decision. Profit earned by agent with cost c and price p conditional on participation is

$$\pi_c(p, Q) = \min\{\alpha - \beta Q + \gamma p, k\}((1 - \phi)p - c).$$

Agent sets the price that maximizes own profits:

$$\max_p \min\{\alpha - \beta Q + \gamma p, k\}((1 - \phi)p - c).$$

Without capacity constraints, an agent's optimal price is characterized by Equation (34),

$$p^*(c, Q) = \frac{\alpha - \beta Q}{2\gamma} + \frac{c}{2(1 - \phi)},$$

which gives quantity

$$q(p^*(c, Q), Q) = \frac{1}{2} \left(\alpha - \beta Q - \frac{\gamma c}{1 - \phi} \right).$$

Since the quantity served is decreasing in cost, there exists a threshold, \tilde{c} , where only the agents with costs $c \leq \tilde{c}$ are constrained in capacity. The threshold agent has exactly the quantity k :

$$q(p^*(\tilde{c}, Q), Q) = k \implies \tilde{c} = \tilde{c}(\phi) = \frac{(1 - \phi)(\alpha - \beta Q - 2k)}{\gamma}.$$

If an agent is capacity constrained, $k \leq q(p^*(c, Q), Q)$, any decrease in price greater than the one that matches capacity and demand decreases the agent's margin, but does not increase the quantity served. Hence, an agent never sets a price lower than the capacity binding level. The optimal price of an agent with cost $c < \tilde{c}$ is:

$$p^*(\tilde{c}, Q) = \frac{\alpha - \beta Q - k}{\gamma}.$$

The highest cost agent that participates has 0 demand:

$$q(p^*(c_h, Q), Q) = 0 \implies c_h = c_h(\phi) = \frac{(1 - \phi)(\alpha - \beta Q)}{\gamma}.$$

Agents' equilibrium prices are

$$p^*(c, Q) = \begin{cases} \frac{\alpha - \beta Q - k}{\gamma}, & c \leq \tilde{c}, \\ \frac{\alpha - \beta Q}{2\gamma} + \frac{c}{2(1 - \phi)}, & \tilde{c} < c \leq c_h. \end{cases}$$

Agents with a cost greater than c_h cannot profitably participate.

The total quantity is defined in the equilibrium:

$$\begin{aligned} Q &= \int_0^{c_h} q(p^*(c, Q), Q) dc \\ &= \int_0^{\tilde{c}} k dc + \int_{\tilde{c}}^{c_h} \frac{1}{2} \left(\alpha - \beta Q - \frac{\gamma c}{1 - \phi} \right) dc \\ &= \frac{k(1 - \phi)(\alpha - \beta Q - k)}{\gamma}. \end{aligned}$$

There is a unique total quantity:

$$Q = \frac{k(1 - \phi)(\alpha - k)}{\gamma + \beta k(1 - \phi)}.$$

Platform's profit maximization problem is:

$$\begin{aligned} \max_{\phi} \quad \Pi^{\mathcal{D}}(\phi) &= \phi \int_0^{c_h} q(p^*(c, Q), Q) p^*(c, Q) dc \\ &= \phi \left(\int_0^{\tilde{c}} k \frac{\alpha - \beta Q - k}{\gamma} dc + \int_{\tilde{c}}^{c_h} \frac{1}{2} \left(\alpha - \beta Q - \frac{\gamma c}{1 - \phi} \right) \left(\frac{\alpha - \beta Q}{2\gamma} + \frac{c}{2(1 - \phi)} \right) dc \right) \\ &= \frac{k(1 - \phi)\phi (3\alpha^2\gamma^2 + \beta^2k^4(\phi - 1)^2 - 2\beta\gamma k^3(\phi - 1) + 4\gamma^2k^2 - 6\alpha\gamma^2k)}{3\gamma^2(\gamma + k(\beta - \beta\phi))^2}. \end{aligned}$$

The profit takes positive values for all $0 < \phi < 1$, $k > 0$ and is zero at both boundary points, $\phi = 0$ and $\phi = 1$. Because the profit function is continuous, any maximizer is an interior point and defined

by the first order condition:

$$\begin{aligned} \frac{\partial \Pi^{\mathcal{D}}(\phi)}{\partial \phi} = \frac{k}{3\gamma^2(\gamma + k(\beta - \beta\phi))^3} & \left(3\alpha^2\gamma^3(1 - 2\phi) + \beta^3k^5(\phi - 1)^3(2\phi - 1) - 3\beta^2\gamma k^4(\phi - 1)^2(2\phi - 1) \right. \\ & \left. + 6\beta\gamma^2k^3(\phi - 1)^2 + 2\gamma^2k^2(3\alpha\beta(\phi - 1) + \gamma(2 - 4\phi)) - 3\alpha\gamma^2k(\alpha\beta(\phi - 1) + \gamma(2 - 4\phi)) \right) = 0, \end{aligned} \quad (52)$$

Equation (52) can be shown to yield a unique solution. Note that for any $0 < \phi < 1$,

$$\frac{k}{3\gamma^2(\gamma + k\beta(1 - \phi))^3} > 0.$$

So, FOC is satisfied when

$$\begin{aligned} 3\alpha^2\gamma^3(1 - 2\phi^*) + \beta^3k^5(\phi^* - 1)^3(2\phi^* - 1) - 3\beta^2\gamma k^4(\phi^* - 1)^2(2\phi^* - 1) + 6\beta\gamma^2k^3(\phi^* - 1)^2 \\ + 2\gamma^2k^2(3\alpha\beta(\phi^* - 1) + \gamma(2 - 4\phi^*)) - 3\alpha\gamma^2k(\alpha\beta(\phi^* - 1) + \gamma(2 - 4\phi^*)) = 0. \end{aligned} \quad (53)$$

The left-hand side of the equality increases monotonically in ϕ^* within the feasible range $\phi^* \in [0, 1]$, $\tilde{c}(\phi^*) \geq 0$, and therefore the expression can only cross zero once. Hence, (53) uniquely defines platform's optimal commission, ϕ^* .

C.1.12 Optimal Mechanism

Without capacity constraints, the highest number of customers an agent serves under the optimal mechanism is

$$q^*(0) = \frac{\sqrt{\gamma(\alpha\beta + \gamma)} - \gamma}{\beta}.$$

The capacity is binding if and only if k is smaller than this amount.

Our proof for the platform's capacity-unconstrained optimal mechanism is applicable up until we reach Equation (49). In the presence of demand constraints, platform's optimal mechanism problem has an additional constraint:

$$\begin{aligned} \max_{q(c)} \quad & \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c) 2c dc \\ \text{s.t.} \quad & q'(c) \leq 0, \forall c \in [0, 1] \\ & q(c) \geq 0, \forall c \in [0, 1] \\ & q(c) \leq k, \forall c \in [0, 1]. \end{aligned}$$

Ignoring the first constraint, this is a quadratic utility maximization problem with capacity constraints in a market where agent prices are equal to twice their cost, $p(c) = 2c$. For all agents

whose quantities are not effected by the two constraints, the assigned optimal quantity is

$$q(c) = \frac{\alpha}{2} - \beta Q - \gamma c.$$

where

$$Q = \int_0^1 q(c) dc \quad (54)$$

For all others, the quantity constraints are binding. That is, the quantity assigned to an agent with cost c is

$$q(c) = \min \left\{ \left(\frac{\alpha}{2} - \beta Q - \gamma c \right)^+, k \right\} \quad (55)$$

This solution also satisfies the relaxed constraint. Therefore, the solution to the relaxed problem is also optimal for the platform's optimal mechanism.

Solving for (54) and (55) simultaneously, we get one feasible total quantity,

$$Q = \frac{k(\alpha - k)}{2(\gamma + \beta k)}.$$

The price assigned to agents with excess capacity is

$$\begin{aligned} p^*(c) &= \frac{\alpha - \beta Q - q(c)}{\gamma} \\ &= \frac{\alpha}{2\gamma} + c, \end{aligned}$$

capacity constrained agents are assigned the price

$$p^*(c) = \frac{(\alpha - k)(2\gamma + \beta k)}{2\gamma(\gamma + \beta k)}.$$

The platform earns

$$\begin{aligned} \Pi &= \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c) 2c dc \\ &= \frac{k(3\alpha^2\gamma + \beta k^3 + 4\gamma k^2 - 6\alpha\gamma k)}{12\gamma^2(\gamma + \beta k)}. \end{aligned}$$

C.1.13 Decentralized Pricing With Commission And Per-Unit Fee

To show that decentralized pricing with commission-plus replicates the optimal mechanism, let the platform sets its terms

$$\phi = \frac{1}{2}, \quad w = \frac{\alpha\beta k - \beta k^2}{4\gamma(\gamma + \beta k)},$$

and assume agents expect a total quantity of

$$Q = \frac{k(\alpha - k)}{2(\gamma + \beta k)}.$$

Without capacity restrictions, an agent with cost c has the following profit-maximization problem:

$$\max_p (\alpha - \beta Q - \gamma p)((1 - \phi)p - c - w),$$

yielding an optimal price of

$$\begin{aligned} p^*(c, Q) &= \frac{\alpha - \beta Q}{2\gamma} + \frac{c + w}{2(1 - \phi)} \\ &= \frac{\alpha}{2\gamma} + c. \end{aligned} \tag{56}$$

Let \tilde{c} be the highest cost agent that is bounded by capacity:

$$q(p^*(\tilde{c}, Q), Q) = k \implies \tilde{c} = \frac{\alpha\gamma - \beta k^2 - 2\gamma k}{2\gamma(\gamma + \beta k)}.$$

All agents with cost less than \tilde{c} set the price that makes their demand exactly equal to k :

$$\begin{aligned} p^*(\tilde{c}, Q) &= \frac{\alpha - \beta Q - k}{\gamma} \\ &= \frac{(\alpha - k)(2\gamma + \beta k)}{2\gamma(\gamma + \beta k)}. \end{aligned}$$

Others will set the price in (56). Let c_h be the highest cost agent that can participate with non-negative demand:

$$q(p^*(c_h, Q), Q) = 0 \implies c_h = \frac{\alpha\gamma + \beta k^2}{2\gamma(\gamma + \beta k)}.$$

The total quantity needs to be consistent with expectation:

$$\begin{aligned} Q &= \int_0^{c_h} q(p^*(c, Q), Q) dc \\ &= \frac{k(\alpha - k)}{2(\gamma + \beta k)}. \end{aligned}$$

The platform earns

$$\begin{aligned} \Pi^{C+} &= \int_0^{\tilde{c}} k(\phi p^*(\tilde{c}, Q) + w) dc + \int_{\tilde{c}}^{c_h} q(p^*(c, Q), Q)(\phi p^*(c, Q) + w) dc \\ &= \frac{k(3\alpha^2\gamma + \beta k^3 + 4\gamma k^2 - 6\alpha\gamma k)}{12\gamma^2(\gamma + \beta k)}, \end{aligned}$$

same as the optimal mechanism.

C.1.14 Throughput Maximization

In the quantity maximizing optimal contract, let $p(c)$ be the price the platform assigns to agent c and $F(c)$ be the fee collected. By Section C.1.6, Equations (45) and (46) are necessary and sufficient conditions for agents' IR and IC constraints. In any truth-inducing mechanism, the equilibrium fees charged to the agent who reports a cost c is

$$F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k.$$

The total quantity served in the market is

$$\int_0^1 q(c) dc = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc.$$

The optimal truth-inducing contract that maximizes total quantity served subject to non-negative profit constraint is characterized through the following problem:

$$\begin{aligned} & \max_{p(c)} \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\ & \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\ & \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\ & \quad F(c) = u(c, p(c)) + \int_c^1 \frac{\partial u(c_k, p(c_k))}{\partial c_k} dc_k, \forall c \in [0, 1] \\ & \quad \int_0^1 F(c) dc \geq 0 \\ & \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\ & = \max_{p(c)} \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc \\ & \text{s.t.} \quad p'(c) \geq 0, \forall c \in [0, 1] \\ & \quad \alpha - \beta Q - \gamma p(c) \geq 0, \forall c \in [0, 1] \\ & \quad \int_0^1 (\alpha - \beta Q - \gamma p(c))(p(c) - 2c) dc \geq 0 \\ & \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc, \end{aligned}$$

where the equivalence of the constraints follow from Equations (47) and (48). We reformulate the problem with quantities as decision variables:

$$\begin{aligned}
& \max_{q(c)} \int_0^1 q(c) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad \quad q(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad \int_0^1 q(c) \left(\frac{\alpha - \beta Q - q(c)}{\gamma} - 2c \right) dc \geq 0 \\
& \quad \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc.
\end{aligned}$$

Let us first look at a related problem, where the platform maximizes a combination of its earning and total quantity served, where the latter is weighted by λ :

$$\begin{aligned}
& \max_{q(c)} \int_0^1 q(c) \left(\frac{\alpha - \beta Q - q(c)}{\gamma} - 2c \right) dc + \lambda \int_0^1 q(c) dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad \quad q(c) \geq 0, \forall c \in [0, 1] \\
& \quad \quad Q = \int_0^1 (\alpha - \beta Q - \gamma p(c)) dc.
\end{aligned}$$

By modifying the value of λ , we can assign different weights to how much platform prioritizes maximizing the quantity served relative to the profit. Specifically, we can recover the quantity maximizing solution by solving for the largest feasible λ^* that makes optimal profit non-negative.

Expanding the Q expression, the problem becomes

$$\begin{aligned}
& \max_{q(c)} \left(\frac{\alpha}{\gamma} + \lambda \right) \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c) 2c dc \\
& \text{s.t.} \quad q'(c) \leq 0, \forall c \in [0, 1] \\
& \quad \quad q(c) \geq 0, \forall c \in [0, 1].
\end{aligned}$$

Relaxing the first constraint, the problem is a specific case of quadratic utility maximization where the prices of the agents are equal to twice their costs, $p(c) = 2c$. For all agents with non-zero assigned quantity, we have

$$q(c) = \frac{\alpha + \lambda\gamma}{2} - \beta Q - \gamma c \tag{57}$$

where

$$Q = \int_0^1 q(c) dc. \tag{58}$$

This solution also satisfies the relaxed constraint. Therefore, the solution in (57) is also optimal for the platform's optimal mechanism.

Solving for (57) and (58) simultaneously, we get one feasible total quantity,

$$Q = \frac{\left(\sqrt{\gamma(\alpha\beta + \beta\gamma\lambda + \gamma)} - \gamma\right)^2}{2\beta^2\gamma}.$$

The total profit earned is

$$\begin{aligned}\Pi &= \frac{\alpha}{\gamma} \int_0^1 q(c) dc - \frac{1}{\gamma} \int_0^1 q(c)^2 dc - \frac{\beta}{\gamma} \left(\int_0^1 q(c) dc \right)^2 - \int_0^1 q(c) 2c dc \\ &= \frac{\left(-2\sqrt{\gamma(\alpha\beta + \beta\gamma\lambda + \gamma)} + 3\alpha\beta + \gamma(2 - 3\beta\lambda)\right) \left(\gamma - \sqrt{\gamma(\alpha\beta + \beta\gamma\lambda + \gamma)}\right)^2}{12\beta^3\gamma^2}.\end{aligned}$$

This expression approaches negative infinity as λ goes to infinity. To get the quantity-maximizing contract, we look for the greatest feasible λ that makes the profit non-negative:

$$\lambda^* = \frac{9\alpha\beta + 8\gamma}{9\beta\gamma} - \frac{2\sqrt{2}}{9\beta} \sqrt{\frac{9\alpha\beta + 8\gamma}{\gamma}}.$$

The total quantity served under the optimal mechanism is

$$Q = \frac{13\gamma + 9\alpha\beta - \sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} - 3\sqrt{\gamma\left(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma\right)}}{9\beta^2}.$$

With the optimal mechanism, we have

$$\begin{aligned}p^*(c) &= \frac{\sqrt{\frac{18\alpha\beta}{\gamma} + 16} - 4}{9\beta} + c \\ F^*(c) &= \frac{1}{54\beta^2} \left(10\sqrt{\gamma\left(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma\right)} \right. \\ &\quad \left. + 2\sqrt{2}\sqrt{(9\alpha\beta + 8\gamma)\left(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma\right)} - 54\gamma \right. \\ &\quad \left. - 3\beta \left(18\alpha + c \left(2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} - 6\sqrt{\gamma\left(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma\right)} + \gamma(9\beta c + 10) \right) \right) \right) \\ &= q^*(c) \left(\frac{1}{2} p^*(c) + \frac{\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} - 3\sqrt{\gamma\left(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma\right)} + 5\gamma}{18\beta\gamma} \right).\end{aligned}$$

Now, let us show that any throughput-maximizing optimal contract can be replicated with a commission-plus contract.

Let the platform sets its terms

$$\phi = \frac{1}{2}, w = \frac{\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} - 3\sqrt{\gamma(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma)} + 5\gamma}{18\beta\gamma},$$

and assume agents expect a total quantity of

$$Q = \frac{13\gamma + 9\alpha\beta - \sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} - 3\sqrt{\gamma(-2\sqrt{2}\sqrt{\gamma(9\alpha\beta + 8\gamma)} + 18\alpha\beta + 17\gamma)}}{9\beta^2}.$$

By Equation (34), an agent with cost c posts a price under commission-plus

$$\begin{aligned} p^*(c, Q) &= \frac{\alpha - \beta Q}{2\gamma} + \frac{c + w}{2(1 - \phi)} \\ &= \frac{\sqrt{\frac{18\alpha\beta}{\gamma} + 16} - 4}{9\beta} + c. \end{aligned}$$

The price chosen by any agent is exactly equivalent to the one in the optimal mechanism. The same is true for the fees paid. Because prices and fees are the only decision variables in the setting, commission-plus with the given terms replicates the optimal mechanism.