Electronic Companion: Appendix for
Optimal Pricing with a Single Point

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A Preliminaries and properties of Generalized Pareto Distributions

Throughout the paper, whenever a distribution F is defined, and when clear from context, we use q_w to denote $\overline{F}(w)$ to lighten the notation. We also use the generalized inverse of a distribution F in $\mathcal{D}$, defined by $F^{-1}(1-q) := \inf\{v \text{ in } \mathbb{R}^+ \text{ s.t. } F(v) \geq 1-q\}$ for all q in $[0,1]$.

For a distribution with positive density function f on its support $[a,b]$, where $0 \leq a < \infty$ and $a \leq b \leq \infty$, we denote the α-virtual value function for $v \in [a,b]$ by

$$\phi_F^\alpha(v) := (1-\alpha)v - \frac{\overline{F}(v)}{f(v)}.$$

Lemma A-1. *Fix two scalars $\beta > 0$ and $s \geq 0$. The cumulative distribution function $\Gamma_\alpha(\beta(v-s))$ for $v \geq s$ admits a constant α-virtual value function given by*

$$(1-\alpha)s - \frac{1}{\beta}.$$

Proof of Lemma A-1. Let us first explicitly compute the α-virtual value function. The derivative of $\Gamma_\alpha(\beta(v-s))$, for any $v \geq s$, is given by

$$-\beta\left(\Gamma_\alpha(\beta(v-s))\right)^{2-\alpha}.$$

Therefore, the α-virtual value function evaluated at $v \geq s$ is given by

$$\begin{aligned}
(1-\alpha)v - \frac{\Gamma_\alpha(\beta(v-s))}{\beta\left(\Gamma_\alpha(\beta(v-s))\right)^{2-\alpha}} &= (1-\alpha)v - \frac{1}{\beta}\left(\Gamma_\alpha(\beta(v-s))\right)^{\alpha-1} \\
&= (1-\alpha)v - \frac{1}{\beta}\left(1 + (1-\alpha)\Gamma_\alpha^{-1}(\beta(v-s))\right) \\
&= (1-\alpha)v - \frac{1}{\beta} - (1-\alpha)(v-s) \\
&= (1-\alpha)s - \frac{1}{\beta}.
\end{aligned}$$

This completes the proof. □

We will also need the following result derived in Lemma E-1 in Allouah, Bahamou, and Besbes (2022).

Lemma A-2. *Fix $\alpha \in [0,1]$, two scalars $\beta \geq 0$ and $w' \geq 0$. The revenue function $v\Gamma_\alpha(\beta(v-w'))$ for $v \geq w' - \frac{1}{(1-\alpha)\beta}$ is unimodal and attains its maximum at*

$$r = \max\left\{\frac{1-(1-\alpha)\beta w'}{\beta\alpha}, w' - \frac{1}{(1-\alpha)\beta}\right\}.$$

With the following conventions: $\max\{+\infty, v\} = +\infty$, *and* $\max\{-\infty, v\} = v$ *for any real number* v.

B Proofs and auxiliary results for Section 2

Proof of Proposition 1. If q in $\{0,1\}$, then the result follows from Lemma B-1. For q in $(0,1)$, let Ψ a mechanism in $\mathcal{P}$. We know that

$$\lim_{u\to\infty} \bar{\Psi}(u) := 1 - \Psi(u) = 0.$$

Fix $\epsilon > 0$. By definition of the limit, there exists $M \geq w$ such that for any $u \geq M$, we have:

$$\bar{\Psi}(u) \leq \frac{\epsilon}{2}. \tag{B-1}$$

For any integer N, consider the following distribution $F_{\Psi,N}$ defined through its Complementary Cumulative Distribution Function $\overline{F}_{\Psi,N}$:

$$\overline{F}_{\Psi,N}(v) = \begin{cases} 1 & \text{if } v < 0, \\ q & \text{if } v \text{ in } [0,w), \\ \frac{q}{N} & \text{if } v \text{ in } [w, N^2M), \\ 0 & \text{if } v \text{ in } [N^2M, +\infty). \end{cases}$$

Note that $F_{\Psi,N}$ in $\mathcal{G}(w,I)$ and that $F_{\Psi,N}$ represents a three point mass distribution with mass at points 0, w and N^2M.

Note also that $\mathrm{opt}(F_{\Psi,N}) = \max\{qw, qNM\}$. Since $M \geq w$ and $N \geq 1$, we have $\mathrm{opt}(F_{\Psi,N}) = qNM$. Thus the performance of mechanism Ψ is given by

$$\begin{aligned}
R(\Psi, F_{\Psi,N}) &= \frac{1}{qNM}\left(\int_{[0,w)} u\overline{F}_{\Psi,N}(u)d\Psi(u) + \int_{[w,N^2M)} u\overline{F}_{\Psi,N}(u)d\Psi(u) + \int_{[N^2M,+\infty)} u\overline{F}_{\Psi,N}(u)d\Psi(u)\right) \\
&= \frac{1}{NM}\left(\int_{[0,w)} ud\Psi(u) + \frac{1}{N}\int_{[w,N^2M)} ud\Psi(u)\right) \\
&= \frac{1}{NM}\left(\int_{[0,w)} ud\Psi(u) + \frac{1}{N}\int_{[w,M)} ud\Psi(u) + \frac{1}{N}\int_{[M,N^2M)} ud\Psi(u)\right) \\
&\leq \frac{1}{NM}\left(w\Psi(w) + \frac{1}{N}M(\Psi(M) - \Psi(0)) + \frac{1}{N}N^2M(\Psi(N^2M) - \Psi(M))\right) \\
&\leq \frac{w}{NM} + \frac{1}{N^2} + (1 - \Psi(M)),
\end{aligned}$$

where in the last step we use the fact that for any $u \geq 0$ $\Psi(u)$ is in $[0, 1]$.

Let us now choose N large enough such that

$$\frac{w}{NM} + \frac{1}{N^2} \leq \frac{\epsilon}{2},$$

Combining the latter with (B-1), we get

$$R(\Psi, F_{\Psi,N}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Taking $\epsilon \to 0$ concludes the proof. □

B.1 Additional result for the cases q in $\{0, 1\}$

Lemma B-1. *For any mechanism Ψ in $\mathcal{P}$, and any α in $[0, 1]$, if q in $\{0, 1\}$, then*

$$\inf_{F \in \mathcal{F}_\alpha(w,\{q\})} R(\Psi, F) = 0.$$

Proof of Lemma B-1. We will first show the case when $q = 0$ then the case when $q = 1$. For both cases, we will exhibit worst case families of distributions for which the seller cannot achieve a non-trivial guarantee.

For any $r > 0$, let us introduce the following distribution through its Complementary Cumulative Distribution Function:

$$\overline{F}_r(v) = \begin{cases} 1 & \text{if } v \text{ in } [0, r), \\ 0 & \text{if } v \text{ in } [r, +\infty). \end{cases}$$

The latter distribution represents a point mass at r.

Case $q = 0$**.** We have, for any $r < w$, F_r in $\mathcal{F}_\alpha(w, \{0\})$ and $\text{opt}(F_r) = r$. Furthermore, for any mechanism Ψ in $\mathcal{P}$, we have:

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,\{0\})} R(\Psi, F) &\leq \frac{\mathbb{E}_\Psi[p\overline{F}_r(p)]}{\text{opt}(F_r)} = \frac{1}{r}\int_0^r u d\Psi(u) = \frac{1}{r}\int_0^r \int_0^u ds d\Psi(u) \\
&\overset{(a)}{=} \frac{1}{r}\int_0^r \int_s^r d\Psi(u) ds = \frac{1}{r}\int_0^r (\Psi(r) - \Psi(s))\, ds \\
&\leq \Psi(r) - \Psi(0).
\end{aligned}$$

Where in equality (a), we used Fubini–Tonelli theorem as $(s, u) \to 1$ is a non-negative measurable function and $([0, r], d\Psi)$ and $([0, r], dx)$ are σ-finite measure spaces.

The right hand side above converges to zero as $r \to 0^+$ since Ψ in $\mathcal{P} = \mathcal{D}$ and is therefore right continuous. We conclude the case $q = 0$.

Case $q = 1$**.** We have, for any $r > w$, F_r in $\mathcal{F}_\alpha(w, \{1\})$ and $\text{opt}(F_r) = r$. Furthermore, for any mechanism Ψ, we have:

$$\inf_{F \in \mathcal{F}_\alpha(w,\{1\})} R(\Psi, F) \leq \frac{\mathbb{E}_\Psi[p\overline{F}_r(p)]}{\text{opt}(F_r)} = \frac{1}{r}\int_0^r u d\Psi(u) = \frac{1}{r}\int_0^r (\Psi(r) - \Psi(u)) du.$$

Since Ψ in $\mathcal{P}$, we have $\lim_{r\to+\infty} \Psi(r) = 1$, therefore, for any $\epsilon > 0$, there exists $A > 0$ such that:

$$1 - \Psi(r) = |\Psi(r) - 1| < \epsilon \quad \text{if } r \in [A, +\infty). \tag{B-2}$$

Let r in $(A, +\infty)$, we have:

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,\{1\})} R(\Psi, F) &\leq \frac{1}{r}\int_0^r (\Psi(r) - \Psi(u)) du \\
&= \frac{1}{r}\left(\int_{[0,A)} (\Psi(r) - \Psi(u))\, du + \int_{[A,r]} (\Psi(r) - \Psi(u))\, du\right) \\
&\overset{(a)}{\leq} \frac{1}{r}\left(_{[0,A)} du + \int_{[A,r]} (1 - \Psi(u))\, du\right) \\
&= \frac{A}{r} + \frac{1}{r}\int_A^r (1 - \Psi(u))\, du \\
&\overset{(b)}{\leq} \frac{A}{r} + \frac{r - A}{r}\epsilon \overset{(c)}{\leq} \frac{A}{r} + \epsilon,
\end{aligned}$$

where in (a) we use the fact that for any $u \geq 0$, we have $0 \leq \Psi(u) \leq 1$. And in (b) we use (B-2). In (c), we used the fact that $r - A \leq r$.

Hence we conclude that for any $r \geq A/\epsilon$, we get that

$$\inf_{F \in \mathcal{F}_\alpha(w,\{1\})} R(\Psi, F) \leq 2\epsilon.$$

Since ϵ was arbitrary, this completes the proof for the case $q = 1$. □

C Proofs and auxiliary results for Section 3

Lemma C-1. *For any $(s, q_s), (s', q_{s'})$ in $([0, +\infty) \times [0, 1])^2$ such that $s \le s'$ and $q_s \ge q_{s'} > 0$, the distribution $G_{\alpha,t}(\cdot|(s, q_s), (s', q_{s'}))$ with $t \ge s'$, defined in (3.1), belongs to $\mathcal{F}_\alpha(s, q_s) \cap \mathcal{F}_\alpha(s', q_{s'})$ if and only if*

$$q_s \ge \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_{s'}) \frac{s}{s'} \right).$$

Proof of Lemma C-1. Let us show each direction.

$\Longrightarrow$) If the distribution $G_{\alpha,t}(\cdot|(s, q_s), (s', q_{s'}))$ belongs to $\mathcal{F}_\alpha(s, q_s) \cap \mathcal{F}_\alpha(s', q_{s'})$ then $G_{\alpha,t}(\cdot|(s, q_s), (s', q_{s'}))$ is $\alpha-$regular, therefore by Lemma 1 applied to the interval $[0, s']$, we have that

$$q_s = \overline{G}_{\alpha,t}(s|(s, q_s), (s', q_{s'})) \ge \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_{s'}) \frac{s}{s'} \right),$$

and hence the first direction is established. Let us now show the other direction.

$\Longleftarrow$) Suppose now that $q_s \ge \Gamma_\alpha \left(\Gamma_\alpha^{-1}(q_{s'}) \frac{s}{s'} \right)$. By definition of $G_{\alpha,t}(\cdot|(s, q_s), (s', q_{s'}))$, we have:

$$\begin{aligned} \overline{G}_{\alpha,t}(s|(s, q_s), (s', q_{s'})) &= q_s \\ \overline{G}_{\alpha,t}(s'|(s, q_s), (s', q_{s'})) &= q_{s'}. \end{aligned}$$

Therefore, we only have to show that the distribution $G_{\alpha,t}(\cdot|(s, q_s), (s', q_{s'}))$ is α-regular. Using Lemma A-1, the associated α-virtual value function is given by

$$\phi^\alpha_{G_{\alpha,t}(\cdot|(s,q_s),(s',q_{s'}))}(v) = \begin{cases} -\frac{s}{\Gamma_\alpha^{-1}(q_s)} & \text{if } v \text{ in } [0, s], \\ (1-\alpha)s - \frac{s'-s}{\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right)} & \text{if } v \text{ in } (s, s']. \end{cases}$$

Thus the α-virtual value function is piece-wise constant. Now we need to show that $\phi^\alpha_{G_{\alpha,t}(\cdot|(s,q_s),(s',q_{s'}))}(v)$ is non-decreasing. Next, we evaluate the difference between the two constant values that the virtual value function is taking.

$$\begin{aligned}
& (1-\alpha)s - \frac{s'-s}{\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right)} - \left(-\frac{s}{\Gamma_\alpha^{-1}(q_s)}\right) \\
= \quad & s\frac{(1-\alpha)\Gamma_\alpha^{-1}(q_s)\,\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right) + \Gamma_\alpha^{-1}(q_s) + \Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right) - \frac{s'}{s}\Gamma_\alpha^{-1}(q_s)}{\Gamma_\alpha^{-1}(q_s)\,\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right)} \\
\overset{(a)}{=} \quad & s\frac{\Gamma_\alpha^{-1}\left(q_s\frac{q_{s'}}{q_s}\right) - \frac{s'}{s}\Gamma_\alpha^{-1}(q_s)}{\Gamma_\alpha^{-1}(q_s)\,\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right)} \\
= \quad & s\frac{\Gamma_\alpha^{-1}(q_{s'}) - \frac{s'}{s}\Gamma_\alpha^{-1}(q_s)}{\Gamma_\alpha^{-1}(q_s)\,\Gamma_\alpha^{-1}\left(\frac{q_{s'}}{q_s}\right)} \overset{(b)}{\geq} 0,
\end{aligned}$$

where (a) stems from the fact that $\Gamma_\alpha^{-1}(uv) = \Gamma_\alpha^{-1}(u) + \Gamma_\alpha^{-1}(v) + (1-\alpha)\Gamma_\alpha^{-1}(u)\,\Gamma_\alpha^{-1}(v)$ and (b) is due to the fact that by assumption $q_s \geq \Gamma_\alpha\left(\frac{s}{s'}\Gamma_\alpha^{-1}(q_{s'})\right)$ and that the function $\Gamma_\alpha(\cdot)$ is non increasing. This shows that the α-virtual value function of $G_{\alpha,t}(\cdot|(s,q_s),(s',q_{s'}))$ is non decreasing. This concludes the proof. ☐

Lemma C-2. *Let α in $[0,1]$, $w>0$, q in $(0,1)$, and r in $[\underline{r}_\alpha(w,q), w) \cup [w, \overline{r}_\alpha(w,q)]$. Then the optimal price associated with $F_\alpha(\cdot|r,(w,q))$ is given by r.*

Proof of Lemma C-2. We compute the virtual value function for the function $F_\alpha(\cdot|r,(w,q))$. Since the definition of $F_\alpha(\cdot|r,(w,q))$ depends on whether $r<w$ or $r \geq w$, we treat each case separately.

Case 1: $r \in [\underline{r}_\alpha(w,q), w)$**:** By applying Lemma A-1 for the pair $((r,1),(w,q))$, we get the virtual value function at $v \geq r$ satisfies

$$\begin{aligned}
\phi^0_{F_\alpha(\cdot|r,(w,q))}(v) \quad &= \quad \alpha v + (1-\alpha)r - \frac{w-r}{\Gamma_\alpha^{-1}(q)} \\
&\geq \quad r\left(1+\frac{1}{\Gamma_\alpha^{-1}(q)}\right) - \frac{w}{\Gamma_\alpha^{-1}(q)} \\
&= \quad \left(1+\frac{1}{\Gamma_\alpha^{-1}(q)}\right)\left(r - \underline{r}_\alpha(w,q)\right),
\end{aligned}$$

since $r \geq \underline{r}_\alpha(w,q)$, then we conclude that $\phi^0_{F_\alpha(\cdot|r,(w,q))}(v) \geq 0$. Now, since r is the lower support of the distribution $F_\alpha(\cdot|r,(w,q))$ in the case $r<w$ and $F_\alpha(\cdot|r,(w,q))$ is regular, we conclude that necessarily the optimal price is at r.

Case 2: $r \in [w, \overline{r}_\alpha(w,q)]$**:** In this case, we assume $\overline{r}_\alpha(w,q) \geq w$, otherwise the set is empty. Similarly, by applying Lemma A-1 for the pair $((0,1),(w,q))$, we get that the virtual value function at $v<r$ satisfies

$$\phi^0_{F_\alpha(\cdot|r,(w,q))}(v) \quad = \quad \alpha v + \left(0 - \frac{w}{\Gamma_\alpha^{-1}(q)}\right) \;=\; \alpha\left(v - \overline{r}_\alpha(w,q)\right).$$

Since $v < r \leq \overline{r}_\alpha(w,q)$, we conclude that $\phi^0_{F_\alpha(\cdot|r,(w,q))}(v) \leq 0$. Now, since r is the upper support of the distribution $F_\alpha(\cdot|r,(w,q))$ in the case $r \geq w$ and $F_\alpha(\cdot|r,(w,q))$ is regular, we conclude that necessarily the optimal price is given by r.

□

Lemma C-3. *The distribution $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$, defined in Eq. (3.1), belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$ if and only if (r^*,q^*) belongs to $\mathscr{B}_\alpha(w,q)$.*

Proof of Lemma C-3. One direction of the proof is direct. In particular, if the distribution $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$ then by definition we have (r^*,q^*) in $\mathscr{B}_\alpha(w,q)$.

Let us now show the other direction, and suppose that (r^*,q^*) belongs to $\mathscr{B}_\alpha(w,q)$ and let F in $\mathcal{F}_\alpha(w,q)$ be a corresponding distribution with $r_F = r^*$ and $q_F = q^*$. We will first show that $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ belongs to $\mathcal{F}_\alpha$ and that the revenue curve of F is lower bounded by the revenue curve of $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$. In a second step, we will show that the optimal revenue of $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ is achieved at r^*.

Step 1: We separate the cases $r^* < w$ and $r^* \geq w$.

__**Case 1:**__ $r^* < w$. By Lemma 1, note that we have that $q^* = \overline{F}(r^*) \geq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)$. By Lemma C-1, applied to the following parameters $(s,q_s) = (r^*,q^*)$ and $(s',q_{s'}) = (w,q)$, $G_{\alpha,w}(\cdot|(r^*,q^*),(w,q))$ belongs to $\mathcal{F}_\alpha$. Furthermore, by Lemma 1 again, we have that

$$v\overline{F}(v) \quad \geq \quad \begin{cases} v\overline{G}_{\alpha,w}(v|(r^*,q^*),(w,q)) & \text{if } v \in [0,r^*], \\ v\overline{G}_{\alpha,w}(v|(r^*,q^*),(w,q)) & \text{if } v \in (r^*,w]. \end{cases}$$

__**Case 2:**__ $r^* \geq w$. By Lemma 1, we have that $q = \overline{F}(w) \geq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\frac{w}{r^*}\right)$ and hence, by Lemma C-1 applied to the following parameters $(s,q_s) = (w,q)$ and $(s',q_{s'}) = (r^*,q^*)$, $G_{\alpha,r^*}(\cdot|(w,q),(r^*,q^*))$ belongs to $\mathcal{F}_\alpha$. Furthermore, by Lemma 1 again, we have that

$$v\overline{F}(v) \quad \geq \quad \begin{cases} v\overline{G}_{\alpha,r^*}(v|(w,q),(r^*,q^*)) & \text{if } v \text{ in } [0,w], \\ v\overline{G}_{\alpha,r^*}(v|(w,q),(r^*,q^*)) & \text{if } v \text{ in } [w,r^*]. \end{cases}$$

Therefore in both cases, we have that $v\overline{F}(v) \geq v\overline{G}_{\alpha,r^*\vee w}(v|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ for all v in $[0,+\infty)$ and $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ is α-regular.

Step 2: To conclude the proof we will show that the optimal revenue associated with the distribution $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ is achieved at r^*. We will show that by contradiction.

Since $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ is α-regular, then the associated revenue function is unimodal and achieves its maximum at some point r_G in $[0,+\infty)$. Suppose for a moment $r_G\overline{G}_{\alpha,r^*\vee w}(r_G|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q)) > r^*\overline{G}_{\alpha,r^*\vee w}(r^*|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$. Then, using the above lower-bounds, one would have

$$\begin{aligned} r_G\overline{F}(r_G) &\geq r_G\overline{G}_{\alpha,r^*\vee w}(r_G|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q)) \\ &> r^*\overline{G}_{\alpha,r^*\vee w}(r^*|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q)) \\ &= r^*q^* = r_F\overline{F}(r_F), \end{aligned}$$

which would contradict the optimality of r_F. Hence $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$.

□

Lemma C-4. *A pair (r^*,q^*) in $\mathbb{R}_+\times[0,1]$ belongs to $\mathscr{B}_\alpha(w,q)$ if and only if it belongs to $\mathscr{B}_l\cup\mathscr{B}_h$, where*

$$\begin{aligned} \mathscr{B}_l &= \left\{(r^*,q^*) \text{ in } [0,w)\times[0,1]: \ q^* \geq \max\left\{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right),\Gamma_\alpha\left(\frac{1}{\alpha}\right),\frac{q}{\Gamma_\alpha\left(\frac{w}{r^*}-1\right)}\right\}\right\}, \\ \mathscr{B}_h &= \left\{(r^*,q^*) \text{ in } [w,+\infty)\times[0,1]: \ q^* \leq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right), \ q^* \geq q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right)\right\}. \end{aligned}$$

Proof of Lemma C-4. By Lemma C-3, we have that (r^*,q^*) in $\mathscr{B}_\alpha(w,q)$ if and only if the distribution $G_{\alpha,r^*\vee w}(\cdot|(r^*\wedge w, q^*\vee q),(r^*\vee w, q^*\wedge q))$ belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$.

Case 1: Suppose $r^* < w$. By definition, we have that

$$v\overline{G}_{\alpha,w}(v|(r^*,q^*),(w,q)) = \begin{cases} v\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\frac{v}{r^*}\right) & \text{if } v\in[0,r^*], \\ vq^*\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)\frac{v-r^*}{w-r^*}\right) & \text{if } v\in(r^*,w], \\ 0 & \text{if } v\in[w,\infty). \end{cases}$$

By applying Lemma C-1 to the following parameters $(s,q_s) := (r^*,q^*)$ and $(s',q_{s'}) := (w,q)$, we have that $G_{\alpha,w}(\cdot|(r^*,q^*),(w,q))$ belongs to $\mathcal{F}_\alpha(w,q)$ and $\mathcal{F}_\alpha(r^*,q^*)$ if and only if $q^* = \overline{F}(r^*) \geq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)$.

Furthermore, using Lemma A-2, the revenue function $v\mapsto v\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{v}{w}\right)$ is maximized at $r_1 = r^*/(\alpha\Gamma_\alpha^{-1}(q^*))$ and the revenue function $v\mapsto vq^*\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)\frac{v-r^*}{w-r^*}\right)$ is maximized at

$$r_2 = \frac{1}{\alpha}\left(\frac{w-r^*}{\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)} - (1-\alpha)r^*\right).$$

Thus, when $G_{\alpha,w}(\cdot|(r^*,q^*),(w,q))$ belongs to $\mathcal{F}_\alpha(w,q)$, the optimal revenue associated with $G_{\alpha,w}(\cdot|(r^*,q^*),(w,q))$ is achieved at r^* if and only if $r_2\leq r^*\leq r_1$. We have $r_2\leq r^*$ if and only if:

$$\frac{1}{\alpha}\left(\frac{w-r^*}{\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)} - (1-\alpha)r^*\right) \leq r^* \quad \text{iff} \quad \frac{w-r^*}{\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)} \leq r^* \quad \text{iff} \quad \frac{w}{r^*} - 1 \leq \Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right) \quad \text{iff} \quad q^* \geq \frac{q}{\Gamma_\alpha\left(\frac{w}{r^*}-1\right)},$$

and $r^* \leq r_1$ if and only if:

$$r^* \leq \frac{r^*}{\alpha \Gamma_\alpha^{-1}(q^*)} \quad \text{iff} \quad \Gamma_\alpha^{-1}(q) \leq \frac{1}{\alpha} \quad \text{iff} \quad q^* \geq \Gamma_\alpha\left(\frac{1}{\alpha}\right).$$

Therefore $r_2 \leq r^* \leq r_1$ is equivalent to

$$q^* \geq \max\left\{\Gamma_\alpha\left(\frac{1}{\alpha}\right), \frac{q}{\Gamma_\alpha\left(\frac{w}{r^*} - 1\right)}\right\}.$$

We have established that when $r^* < w$, $G_{\alpha,w}(\cdot|(r^*,q^*),(w,q))$ belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$ if and only if

$$q^* \geq \max\left\{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right), \Gamma_\alpha\left(\frac{1}{\alpha}\right), \frac{q}{\Gamma_\alpha\left(\frac{w}{r^*} - 1\right)}\right\}.$$

We have hence established that when $r^* \leq w$, (r^*, q^*) belongs to $\mathscr{B}_\alpha(w,q)$ if and only if (r^*, q^*) belongs to $\mathscr{B}_l$.

Case 2: Suppose now $r^* \geq w$. By definition, we have that

$$v\overline{G}_{\alpha,r^*}(v|(w,q),(r^*,q^*)) = \begin{cases} v\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{v}{w}\right) & \text{if } v \in [0,w], \\ vq\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)\frac{v-w}{r^*-w}\right) & \text{if } v \in (w, r^*], \\ 0 & \text{if } v \text{ in } (r^*, \infty). \end{cases}$$

By applying Lemma C-1 to the following parameters $(s, q_s) := (w, q)$ and $(s', q_{s'}) := (r^*, q^*)$, we have that $G_{\alpha,r^*}(\cdot|(w,q),(r^*,q^*))$ belongs to $\mathcal{F}_\alpha(w,q)$ and $\mathcal{F}_\alpha(w^*, \{q^*\})$ if and only if $q = \overline{F}(w) \geq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\frac{w}{r^*}\right)$ which can be rewritten as $q^* \leq \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)$.

Using Lemma A-2, the oracle price for the revenue function $v \mapsto vq\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)\frac{v-w}{r^*-w}\right)$ is achieved at

$$r' = \frac{1}{\alpha}\left(\frac{r^* - w}{\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)} - (1-\alpha)w\right).$$

Given that r^* is at the end of its support and that the revenue curve is unimodal, the optimal revenue associated $G_{\alpha,r^*}(\cdot|(w,q),(r^*,q^*))$ is achieved at r^* if and only if $r^* \leq r'$, which, in turn, is true if and only if:

$$r^* \le \frac{1}{\alpha}\left(\frac{r^*-w}{\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)} - (1-\alpha)w\right) \quad \text{iff} \quad \alpha r^* + (1-\alpha)w \le \frac{r^*-w}{\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)}$$

$$\text{iff} \quad \Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right) \le \frac{r^*-w}{w+\alpha(r^*-w)}$$

$$\text{iff} \quad \Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le \frac{q^*}{q} \quad \text{iff} \quad q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le q^*.$$

We have established that when $r^* \ge w$, $G_{\alpha,r^*}(\cdot|(w,q),(r^*,q^*))$ belongs to $\{F \text{ in } \mathcal{F}_\alpha(w,q) : r_F = r^*, q_F = q^*\}$ if and only if

$$q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le q^* \le \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right).$$

In turn, we have hence established that when $r^* \ge w$, (r^*,q^*) in $\mathscr{B}_\alpha(w,q)$ if and only if (r^*,q^*) in $\mathscr{B}_h$. This concludes the proof. □

Lemma C-5. *Let $J_{w,q} = \{r :$ there exists q^* s.t. (r^*,q^*) in $\mathscr{B}_\alpha(w,q)\}$. We have $J_{w,q} = [\underline{r}_\alpha(w,q), w) \cup [w, \overline{r}_\alpha(w,q)]$ and for any r^* in $J_{w,q}$, the function $\widetilde{R}_{r^*,w,q}(\cdot)$ is decreasing in the set $\{q^* : (r^*,q^*)$ in $\mathscr{B}_\alpha(w,q)\}$.*

Proof of Lemma C-5. We will show that $J_{w,q} = [\underline{r}_\alpha(w,q), w) \cup [w, \overline{r}_\alpha(w,q)]$ by analyzing the cases when $r^* < w$ and $r^* \ge w$ separately.

Suppose first that $r^* < w$. In this case, we show that there exists a value q^* such that $(r^*,q^*) \in \mathscr{B}_l$ if and only if $r^* \ge \underline{r}_\alpha(w,q)$.

If there exists $(r^*,q^*) \in \mathscr{B}_l$ then we have y Lemma C-4 that $\frac{q}{\Gamma_\alpha\left(\frac{w}{r^*}-1\right)} \le q^*$ and since $q^* \le 1$, we have $\frac{q}{\Gamma_\alpha\left(\frac{w}{r^*}-1\right)} \le 1$, which implies that $r^* \ge \frac{w}{1+\Gamma_\alpha^{-1}(q)} = \underline{r}_\alpha(w,q)$. Now if $r^* \in [\underline{r}_\alpha(w,q), w)$, note that $(r^*,1) \in \mathscr{B}_l$, as we have seen above that $1 \ge \frac{q}{\Gamma_\alpha\left(\frac{w}{r^*}-1\right)}$. Furthermore, we have

$$1 \ge \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right) \quad \text{and} \quad 1 \ge \Gamma_\alpha\left(\frac{1}{\alpha}\right)$$

since $\Gamma_\alpha(x) \le 1$ for all $x \ge 0$.

Now suppose that $r^* \ge w$. In this case, we show that there exists a value q^* such that $(r^*,q^*) \in \mathscr{B}_h$ if and only if $\overline{r}_\alpha(w,q) \ge w$ (which is equivalent to $q \ge \Gamma_\alpha\left(\frac{1}{\alpha}\right)$) and $r^* \le \overline{r}_\alpha(w,q)$.

If there exists $(r^*, q^*) \in \mathscr{B}_h$ then, by Lemma C-4, we have $q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le q^* \le \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)$. Note that we have

$$
\begin{aligned}
q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right) \quad & \text{iff} \quad q^{\alpha-1}\left(\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right)\right)^{\alpha-1} \ge \left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)\right)^{\alpha-1} \\
& \text{iff} \quad \left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\right)\left(1+\frac{1-\alpha}{\alpha+\frac{w}{r^*-w}}\right) \ge 1+(1-\alpha)\Gamma_\alpha^{-1}(q)\frac{r^*}{w} \\
& \text{iff} \quad \frac{1-\alpha}{\alpha+\frac{1}{\frac{r^*}{w}-1}} + (1-\alpha)\Gamma_\alpha^{-1}(q)\left(1+\frac{1-\alpha}{\alpha+\frac{1}{\frac{r^*}{w}-1}}\right) \ge (1-\alpha)\Gamma_\alpha^{-1}(q)\frac{r^*}{w} \\
& \text{iff} \quad \frac{1}{\alpha+\frac{1}{\frac{r^*}{w}-1}} + \Gamma_\alpha^{-1}(q)\left(1+\frac{1-\alpha}{\alpha+\frac{1}{\frac{r^*}{w}-1}}\right) \ge \Gamma_\alpha^{-1}(q)\frac{r^*}{w} \\
& \text{iff} \quad \frac{1}{\alpha+\frac{1}{\frac{r^*}{w}-1}} + \Gamma_\alpha^{-1}(q)\left(\frac{1-\alpha}{\alpha+\frac{1}{\frac{r^*}{w}-1}} - (\frac{r^*}{w}-1)\right) \ge 0 \\
& \text{iff} \quad \frac{1+\Gamma_\alpha^{-1}(q)\left(1-\alpha-\alpha\frac{r^*}{w}+\alpha\right)}{\alpha+\frac{1}{\frac{r^*}{w}-1}} \ge 0 \\
& \text{iff} \quad \frac{1-\Gamma_\alpha^{-1}(q)\,\alpha\frac{r^*}{w}}{\alpha+\frac{1}{\frac{r^*}{w}-1}} \ge 0 \\
& \text{iff} \quad 1-\Gamma_\alpha^{-1}(q)\,\alpha\frac{r^*}{w} \ge 0 \quad \text{iff} \quad r^* \le \frac{w}{\alpha\Gamma_\alpha^{-1}(q)} = \overline{r}_\alpha(w,q).
\end{aligned}
$$

Additionally, since $r^* \ge w$, the above inequality implies that $\overline{r}_\alpha(w,q) \ge w$ (which in turns implies that $q \ge \Gamma_\alpha\left(\frac{1}{\alpha}\right)$).

Now if $q \ge \Gamma_\alpha\left(\frac{1}{\alpha}\right)$ then $\overline{r}_\alpha(w,q) \ge w$ and therefore, if $r^* \in [w, \overline{r}_\alpha(w,q)]$, we always have that $(r^*, \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)) \in \mathscr{B}_h$, as we showed above that $q\Gamma_\alpha\left(\frac{1}{\alpha+\frac{w}{r^*-w}}\right) \le \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{r^*}{w}\right)$.

Next, we show the $\widetilde{R}_{r^*,w,q}(\cdot)$ monotonicity property by analyzing the cases when $r^* < w$ and $r^* \ge w$ separately.

Suppose first that $r^* < w$. In this case, we have

$$
\begin{aligned}
\widetilde{R}_{r^*,w,q}(q^*) &= \int_0^{r^*} \frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\frac{p}{r^*}\right)}{q^*}d\Psi(p) + \int_{r^*}^{w} \frac{p}{r^*}\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)\frac{p-r^*}{w-r^*}\right)d\Psi(p) \\
&= \int_0^{r^*} \frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\frac{p}{r^*}\right)}{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q^*)\right)}d\Psi(p) + \int_{r^*}^{w} \frac{p}{r^*}\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)\frac{p-r^*}{w-r^*}\right)d\Psi(p) \\
&= \int_0^{r^*} \frac{p}{r^*}\left(\Gamma_\alpha\left(\frac{1-p/r^*}{1/\Gamma_\alpha^{-1}(q^*)+(1-\alpha)}\right)\right)^{-1}d\Psi(p) + \int_{r^*}^{w} \frac{p}{r^*}\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q}{q^*}\right)\frac{p-r^*}{w-r^*}\right)d\Psi(p),
\end{aligned}
$$

where the last equality follows from the fact that for $u \ge v$, $\Gamma_\alpha(u)/\Gamma_\alpha(v) = \Gamma_\alpha((u-v)/(1+(1-\alpha)v))$. Each term on the RHS above is decreasing in q^*, since $\Gamma_\alpha(\cdot)$ is decreasing. Hence $\widetilde{R}_{r^*,w,q}(q^*)$ is decreasing in this case.

Suppose now that $r^* \geq w$ and $q \geq \Gamma_\alpha\left(\frac{1}{\alpha}\right)$ so that the interval $[w, \overline{r}_\alpha(w,q)]$ is non-empty. In this case, we have

$$\begin{aligned}
\widetilde{R}_{r^*,w,q}(q^*) &= \int_0^w \frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{p}{w}\right)}{q^*}d\Psi(p) + \int_w^{r^*}\frac{p}{r^*}\frac{q\,\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)\frac{p-w}{r^*-w}\right)}{q^*}d\Psi(p)\\
&= \int_0^w \frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{p}{w}\right)}{q^*}d\Psi(p) + \int_w^{r^*}\frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)\frac{p-w}{r^*-w}\right)}{\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)\right)}d\Psi(p)\\
&= \int_0^w \frac{p}{r^*}\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{p}{w}\right)}{q^*}d\Psi(p) + \int_w^{r^*}\frac{p}{r^*}\left(\Gamma_\alpha\left(\frac{\frac{r^*-p}{r^*-w}}{1/\Gamma_\alpha^{-1}\left(\frac{q^*}{q}\right)+(1-\alpha)\frac{p-w}{r^*-w}}\right)\right)^{-1}d\Psi(p),
\end{aligned}$$

where the last equality follows from the fact that for $u \geq v$, $\Gamma_\alpha(u)/\Gamma_\alpha(v) = \Gamma_\alpha((u-v)/(1+(1-\alpha)v))$. Each term in the above equality is decreasing in q^*, therefore the result also holds for this case. This concludes the proof. □

D Proofs and auxiliary results for Section 4

We define the following useful notation used throughout this section:

$$\tilde{v}_\alpha := \begin{cases} 1 & \text{if } \alpha = 0,\\ \alpha^{\frac{\alpha}{1-\alpha}} & \text{if } \alpha \text{ in } (0,1),\\ e^{-1} & \text{if } \alpha = 1,\end{cases} \tag{D-1}$$

$$\underline{q}_\alpha := \begin{cases} 0 & \text{if } \alpha = 0,\\ \Gamma_\alpha\left(\frac{1}{\alpha}\right) & \text{if } \alpha \text{ in } (0,1].\end{cases} \tag{D-2}$$

One can easily check that, for any α in $[0,1]$, $\underline{q}_\alpha < \Gamma_\alpha(\tilde{v}_\alpha)$.

Proof of Theorem 2. The proof is divided into three separate parts.

In a first step, we simplify the problem given in (4.1) in Section 4. We show in Proposition D-1 that the seller's posted price has to counter at most 3 worst-case distributions, where two of these are fixed, and the third one is a function of the price selected. In a second step, we analyze the case of regular distributions, and in a third step, we analyze the case of mhr distributions.

Recall the definition of $\underline{q}_\alpha$ introduced in (D-2).

Proposition D-1 (Worst-case Distributions against Deterministic Mechanisms)**.** *For any α in $[0,1]$, we have*

$$\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) = \begin{cases} \sup_{p\in[0,1]} \min\left\{ \frac{Rev\left(p|\overline{F}_\alpha(\cdot|\mu_{\alpha,q}(p),(1,q))\right)}{\text{opt}(F_\alpha(\cdot|\mu_{\alpha,q}(p),(1,q)))}, \frac{Rev(p|\delta_1)}{Rev(1|\delta_1)}, \frac{Rev\left(p|\overline{F}_\alpha(\cdot|\overline{r}_\alpha(1,q),(1,q))\right)}{\text{opt}(F_\alpha(\cdot|\overline{r}_\alpha(1,q),(1,q)))} \right\}, & \text{if } q \in [\underline{q}_\alpha, 1), \\ \sup_{p\in[0,1]} \min\left\{ \frac{Rev\left(p|\overline{F}_\alpha(\cdot|\mu_{\alpha,q}(p),(1,q))\right)}{\text{opt}(F_\alpha(\cdot|\mu_{\alpha,q}(p),(1,q)))}, \frac{Rev(p|\delta_1)}{Rev(1|\delta_1)} \right\}, & \text{if } q \in (0, \underline{q}_\alpha), \end{cases}$$

with

$$\begin{aligned} \mu_{\alpha,q}(p) &= 1 - \frac{\sqrt{\Delta_{\alpha,q}(p)} - \alpha\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}} \\ \Delta_{\alpha,q}(p) &= \left(\alpha\Gamma_\alpha^{-1}(q)(1-p)\right)^2 + 4\Gamma_\alpha^{-1}(q)(1-p)\,q^{\alpha-1}, \end{aligned}$$

In the above result, the initial price w is normalized to 1 without loss of generality. The above establishes that, when restricting attention to deterministic prices, one can restrict attention to worst-case distributions consisting of a GPD distribution with support starting at $\mu_{\alpha,q}(p)$ and truncated at 1, a mass at 1, or a GPD distribution truncated at $\overline{r}_\alpha(1,q)$ (when $q \geq \underline{q}_\alpha$). The proof is deferred to Appendix D.1.

We now leverage the above reduction to explicitly derive optimal deterministic mechanisms against regular and mhr distributions.

Let us introduce the following functions that represent the ratios of the worst families for $p \geq \underline{r}_\alpha(1,q)$

$$\begin{aligned} R_{1,\alpha}(p,q) &:= \frac{Rev\left(p|\overline{F}_\alpha\left(\cdot|\mu_{\alpha,q}(p),(1,q)\right)\right)}{\text{opt}\left(F_\alpha\left(\cdot|\mu_{\alpha,q}(p),(1,q)\right)\right)}, \\ R_{2,\alpha}(p,q) &:= \frac{Rev(p|\delta_1)}{Rev(1|\delta_1)}, \\ R_{3,\alpha}(p,q) &:= \frac{Rev\left(p|\overline{F}_\alpha\left(\cdot|\overline{r}_\alpha(1,q),(1,q)\right)\right)}{\text{opt}\left(F_\alpha\left(\cdot|\overline{r}_\alpha(1,q),(1,q)\right)\right)}, \quad \text{defined only for } q \geq \underline{q}_\alpha. \end{aligned}$$

We next analyze some properties of the above defined ratios. To that end, recall the definitions of $\tilde{v}_\alpha$ introduced in (D-1) and of $\underline{q}_\alpha$ in (D-2).

Lemma D-1. *We have the following properties:*

1. *If $q \in [\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha))$, then there exists $p_{13,\alpha,q}$ in $[\underline{r}_\alpha(1,q), 1]$ such that $R_{1,\alpha}(\cdot,q) \geq R_{3,\alpha}(\cdot,q)$ in $[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]$ and $R_{1,\alpha}(\cdot,q) \leq R_{3,\alpha}(\cdot,q)$ in $[p_{13,\alpha,q}, 1]$. Else, if $q \in [\Gamma_\alpha(\tilde{v}_\alpha), 1]$, then $R_{1,\alpha}(p,q) \geq R_{3,\alpha}(p,q)$ for all p in $[\underline{r}_\alpha(1,q), 1]$.*

2. *For any $q \in (0,1)$, there exists $p_{12,\alpha,q}$ such that $R_{1,\alpha}(\cdot,q) \geq R_{2,\alpha}(\cdot,q)$ in $[\underline{r}_\alpha(1,q), p_{12,\alpha,q}]$ and $R_{1,\alpha}(\cdot,q) \leq R_{2,\alpha}(\cdot,q)$ in $[p_{12,\alpha,q}, 1]$.*

3. *If $q \geq \underline{q}_\alpha$, then we have that $R_{3,\alpha}(\cdot,q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$.*

4. *For α in $\{0,1\}$, $R_{1,\alpha}(\cdot,q)$ is non increasing in $[\underline{r}_\alpha(1,q), 1]$, $R_{2,\alpha}(\cdot,q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$.*

Lemma D-2. *For α in $\{0,1\}$, there exists a unique $\hat{q}_\alpha$ in $[\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$ solution to the equation $p_{13,\alpha,q} = p_{12,\alpha,q}$, and we have for q in $[\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$: $\quad p_{13,\alpha,q} \leq p_{12,\alpha,q} \quad$ if and only if $\quad q \leq \hat{q}_\alpha$. Furthermore, we* *have the following expressions:*

$$\hat{q}_0 = \frac{1}{4}, \quad p_{13,0,q} = 1 - \frac{(1-2q)^2}{1-q}, \quad p_{12,0,q} = 1 - \frac{\left(1-\sqrt{q}\right)^2}{1-q},$$
$$\hat{q}_1 = \hat{q}, \quad p_{13,1,q} = \mu_{1,q}^{-1}\left(W\left(\frac{1}{\log\left(q^{-1}\right)}\right)\right), \quad p_{12,1,q} = \mu_{1,q}^{-1}\left(\frac{1}{W\left(\frac{e}{q}\right)}\right),$$

Where $\hat{q}$ is the unique solution in $[0,1]$ to the equation $W\left(\frac{1}{\log(q^{-1})}\right) W\left(\frac{e}{q}\right) = 1$, W is the Lambert function defined as the inverse of $x \to xe^x$ in $[0,+\infty)$. Numerically $\hat{q} \in [0.52, 0.53]$.

The proof can be found in Appendix D.1. We proceed by analyzing three main cases q in $(0, \hat{q}_\alpha]$, q in $(\hat{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$ and q in $(\Gamma_\alpha(\tilde{v}_\alpha), 1)$.

Below, we fix α in $\{0,1\}$.

Case q in $(0, \hat{q}_\alpha]$:

We analyze the two sub-cases q in $(0, \underline{q}_\alpha)$ and q in $[\underline{q}_\alpha, \hat{q}_\alpha]$, which both will lead the same final result. Note that the first sub-case is empty for $\alpha = 0$.

Sub-case: $\alpha = 1$ and q in $(0, \underline{q}_\alpha)$: In this case, based on proposition D-1, we have:

$$\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) = \max\Bigg\{ \max_{p \in [\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q)\right), \max_{p \in [p_{12,\alpha,q}, 1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q)\right) \Bigg\}.$$

Now let us simplify each term.

- Using Lemma D-1-2, we have that, $R_{1,\alpha}(\cdot, q)$ is above $R_{2,\alpha}(\cdot,q)$ in $[\underline{r}_\alpha(1,q), p_{12,\alpha,q}]$, therefore:

  $$\max_{p \in [\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q)\right) = \max_{p \in [\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} R_{2,\alpha}(p,q) \overset{(a)}{=} R_{2,\alpha}(p_{12,\alpha,q}, q),$$

  where in (a), we used the result in Lemma D-1-4 that states that $R_{2,\alpha}(\cdot,q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$.

- Using Lemma D-1-2, we have that, $R_{1,\alpha}(\cdot,q)$ is below $R_{2,\alpha}(\cdot,q)$ in $[p_{12,\alpha,q}, 1]$, therefore:

  $$\max_{p \in [p_{12,\alpha,q}, 1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q)\right) = \max_{p \in [p_{12,\alpha,q}, 1]} R_{1,\alpha}(p,q) \overset{(b)}{=} R_{1,\alpha}(p_{12,\alpha,q}, q),$$

  where in (b), we used the result in Lemma D-1-4 that states that $R_{1,\alpha}(\cdot,q)$ is non increasing in $[\underline{r}_\alpha(1,q), 1]$.

Therefore, we have

$$\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) = \max\{R_{2,\alpha}(p_{12,\alpha,q}, q), R_{1,\alpha}(p_{12,\alpha,q}, q)\} \overset{(c)}{=} p_{12,\alpha,q},$$

where in (c), we used the fact that, by definition, $R_{2,\alpha}(p_{12,\alpha,q}, q) = R_{1,\alpha}(p_{12,\alpha,q}, q)$. We also note that the value above is achieved at $p = p_{12,\alpha,q}$.

Sub-case: q in $[\underline{q}_{\alpha}, \hat{q}_{\alpha}]$: In this case, based on Lemma D-2, we have that $p_{13,\alpha,q} \leq p_{12,\alpha,q}$, thus we have

$$\begin{aligned}\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) \quad = \quad \max\Bigg\{ & \max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \\ & \max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \\ & \max_{p\in[p_{12,\alpha,q}, 1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \Bigg\}.\end{aligned}$$

Now let us simplify each term.

- Using Lemma D-1-1, we have that, for $q \in [\underline{q}_{\alpha}, \hat{q}_{\alpha}] \subseteq [\underline{q}_{\alpha}, \Gamma_\alpha(\tilde{v}_\alpha)]$, $R_{1,\alpha}(\cdot, q)$ is above $R_{3,\alpha}(\cdot, q)$ in $[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]$, therefore:

$$\max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \quad = \quad \max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right).$$

- Using Lemma D-1-1 and Lemma D-1-2, we have that both $R_{3,\alpha}(p,q) \geq R_{1,\alpha}(p,q) \geq R_{2,\alpha}(p,q)$ in $[p_{13,\alpha,q}, p_{12,\alpha,q}]$, therefore:

$$\max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \quad = \quad \max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} R_{2,\alpha}(p,q).$$

- Using Lemma D-1-1 and Lemma D-1-2, we have that both $R_{2,\alpha}(\cdot, q)$ and $R_{3,\alpha}(\cdot, q)$ are above $R_{1,\alpha}(\cdot, q)$ in $[p_{12,\alpha,q}, 1]$, therefore:

$$\max_{p\in[p_{12,\alpha,q}, 1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \quad = \quad \max_{p\in[p_{12,\alpha,q}, 1]} R_{1,\alpha}(p,q).$$

Therefore, we have

$$\begin{aligned}\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) \quad &= \quad \max\Bigg\{ \max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} R_{2,\alpha}(p,q), \\ & \qquad \max_{p\in[p_{12,\alpha,q}, 1]} R_{1,\alpha}(p,q) \Bigg\} \\ &\overset{(a)}{=} \quad \max\Bigg\{ \max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} R_{2,\alpha}(p,q), \max_{p\in[p_{12,\alpha,q}, 1]} R_{1,\alpha}(p,q) \Bigg\},\end{aligned}$$

where in (a), we used the fact that

$$\max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \leq \max_{p\in[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]} R_{2,\alpha}(p,q) \overset{(b)}{\leq} \max_{p\in[p_{13,\alpha,q}, p_{12,\alpha,q}]} R_{2,\alpha}(p,q),$$

and in (b), we used the result in Lemma D-1-4 that states that $R_{2,\alpha}(\cdot, q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$.

Using the latter and also the fact that $R_{1,\alpha}(\cdot, q)$ is non increasing in $[\underline{r}_\alpha(1,q), 1]$ we have

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) &= \max\{R_{2,\alpha}\left(p_{12,\alpha,q}, q\right), R_{1,\alpha}\left(p_{12,\alpha,q}, q\right)\} \\ &\overset{(c)}{=} p_{12,\alpha,q},\end{aligned}$$

where in (c), we used the fact that, by definition, $R_{2,\alpha}\left(p_{12,\alpha,q}, q\right) = R_{1,\alpha}\left(p_{12,\alpha,q}, q\right)$. We also note that the value above is achieved at $p = p_{12,\alpha,q}$.

We conclude that, for q in $(0, \hat{q}_\alpha]$, $\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) = p_{12,\alpha,q}$ and is achieved at $p = p_{12,\alpha,q}$. For $\alpha = 0$, using the expressions in Lemma D-2, we obtain $(0, \hat{q}_\alpha] \overset{\alpha=0}{=} (0, \frac{1}{4}]$ and:

$$\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) \overset{\alpha=0}{=} p_{12,0,q} = 1 - \frac{\left(1-\sqrt{q}\right)^2}{1-q}, \quad \text{which is achieved at } p = 1 - \frac{\left(1-\sqrt{q}\right)^2}{1-q}.$$

For $\alpha = 1$, using the expressions in Lemma D-2, we obtain $(0, \hat{q}_\alpha] \overset{\alpha=1}{=} (0, \hat{q}]$ and:

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) &\overset{\alpha=1}{=} p_{12,1,q} = \mu_{1,q}^{-1}\left(\frac{1}{W\left(\frac{e}{q}\right)}\right), \quad \text{which is achieved at } p = \mu_{1,q}^{-1}\left(\frac{1}{W\left(\frac{e}{q}\right)}\right) \\ &= 1 - \frac{1}{\log\left(q^{-1}\right)}\left(W\left(\frac{e}{q}\right) + \frac{1}{W\left(\frac{e}{q}\right)} - 2\right) := \beta_q\left(\frac{e}{q}\right).\end{aligned}$$

<u>**Case q in $(\hat{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$**</u> : In this case, based on Lemma D-2, we have that $p_{12,\alpha,q} \leq p_{13,\alpha,q}$, thus we have

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) = \max\Bigg\{ &\max_{p\in[\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \\ &\max_{p\in[p_{12,\alpha,q}, p_{13,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \\ &\max_{p\in[p_{13,\alpha,q}, 1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right)\Bigg\}.\end{aligned}$$

Now let us simplify each term.

- Using Lemma D-1-1, we have that, for $q \in (\hat{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)] \subseteq [\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$, $R_{1,\alpha}(\cdot, q) \geq R_{3,\alpha}(\cdot, q)$ in $[\underline{r}_\alpha(1,q), p_{13,\alpha,q}]$, therefore:

$$\max_{p\in[\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) = \max_{p\in[\underline{r}_\alpha(1,q), p_{12,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right).$$

- Using Lemma D-1-1 and Lemma D-1-2, we have that both $R_{2,\alpha}(p,q) \geq R_{1,\alpha}(p,q) \geq R_{3,\alpha}(p,q)$ in $[p_{12,\alpha,q}, p_{13,\alpha,q}]$, therefore:

$$\max_{p\in[p_{12,\alpha,q}, p_{13,\alpha,q}]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) = \max_{p\in[p_{12,\alpha,q}, p_{13,\alpha,q}]} R_{3,\alpha}(p,q).$$

- Using Lemma D-1-1 and Lemma D-1-2, we have that both $R_{2,\alpha}(\cdot, q)$ and $R_{3,\alpha}(\cdot, q)$ are above $R_{1,\alpha}(\cdot, q)$ in $[p_{13,\alpha,q}, 1]$, therefore:

$$\max_{p\in[p_{13,\alpha,q},1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) = \max_{p\in[p_{13,\alpha,q},1]} R_{1,\alpha}(p,q).$$

Therefore, we have

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) &= \max\Bigg\{\max_{p\in[\underline{r}_\alpha(1,q),p_{12,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right), \max_{p\in[p_{12,\alpha,q},p_{13,\alpha,q}]} R_{3,\alpha}(p,q),\\ &\qquad \max_{p\in[p_{13,\alpha,q},1]} R_{1,\alpha}(p,q)\Bigg\}\\ &\overset{(a)}{=} \max\left\{\max_{p\in[p_{12,\alpha,q},p_{13,\alpha,q}]} R_{3,\alpha}(p,q), \max_{p\in[p_{13,\alpha,q},1]} R_{1,\alpha}(p,q)\right\},\end{aligned}$$

where in (a), we used the fact that

$$\max_{p\in[\underline{r}_\alpha(1,q),p_{12,\alpha,q}]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \le \max_{p\in[\underline{r}_\alpha(1,q),p_{12,\alpha,q}]} R_{3,\alpha}(p,q) \overset{(b)}{\le} \max_{p\in[p_{12,\alpha,q},p_{13,\alpha,q}]} R_{3,\alpha}(p,q),$$

and in (b), we used the result in Lemma D-1-3 that states that $R_{3,\alpha}(\cdot, q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$. Therefore, using the property Lemma D-1-4, we have, additionally, that $R_{1,\alpha}(\cdot, q)$ is non increasing in $[\underline{r}_\alpha(1,q), 1]$ and therefore

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) &= \max\{R_{3,\alpha}(p_{13,\alpha,q}, q), R_{1,\alpha}(p_{13,\alpha,q}, q)\}\\ &\overset{(c)}{=} R_{3,\alpha}(p_{13,\alpha,q}, q), \quad \text{which is achieved at } p = p_{13,\alpha,q}.\end{aligned}$$

where in (c), we used the fact that $R_{3,\alpha}(p_{13,\alpha,q}, q) = R_{1,\alpha}(p_{13,\alpha,q}, q)$.
For $\alpha = 0$, using the expressions in Lemma D-2, we obtain $(\hat{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)] \overset{\alpha=0}{=} (\frac{1}{4}, \frac{1}{2}]$ and:

$$\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) \overset{\alpha=0}{=} R_{3,0}(p_{13,0,q}, q) = \frac{3-4q}{4(1-q)}, \quad \text{which is achieved at } p_{13,0,q} = \tfrac{q(3-4q)}{1-q}.$$

For $\alpha = 1$, using the expressions in Lemma D-2, we obtain $(\hat{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)] \overset{\alpha=1}{=} (\hat{q}, e^{-e^{-1}}]$ and:

$$\begin{aligned}\mathcal{R}\left(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)\right) &\overset{\alpha=1}{=} R_{3,1}(p_{13,1,q}, q) = \mu_{1,q}^{-1}\left(W\left(\frac{1}{\log(q^{-1})}\right)\right) e\log\left(q^{-1}\right) e^{-\log(q^{-1})\mu_{1,q}^{-1}\left(W\left(\frac{1}{\log(q^{-1})}\right)\right)} := \rho(q)\\ &\qquad \text{which is achieved at } p = \mu_{1,q}^{-1}\left(W\left(\tfrac{1}{\log(q^{-1})}\right)\right) = \beta_q\left(\tfrac{1}{\log(q^{-1})}\right).\end{aligned}$$

Case q in $(\Gamma_\alpha(\tilde{v}_\alpha), 1)$: Using Lemma D-1-1, we have that for q in $(\Gamma_\alpha(\tilde{v}_\alpha), 1)$, $R_{1,\alpha}(\cdot, q)$ is above $R_{3,\alpha}(\cdot, q)$ in $[\underline{r}_\alpha(1,q), 1]$. Therefore, we have

$$\begin{aligned}
\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) &= \max_{p \in [\underline{r}_\alpha(1,q),1]} \min\left(R_{1,\alpha}(p,q), R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \\
&= \max_{p \in [\underline{r}_\alpha(1,q),1]} \min\left(R_{2,\alpha}(p,q), R_{3,\alpha}(p,q)\right) \\
&\overset{(a)}{=} \min\left(R_{2,\alpha}(1,q), R_{3,\alpha}(1,q)\right) \\
&= \min\left(1, R_{3,\alpha}(1,q)\right) \overset{(b)}{=} R_{3,\alpha}(1,q).
\end{aligned}$$

In (a), we used the results in Lemma D-1-3 that states that $R_{3,\alpha}(\cdot, q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$ and Lemma D-1-4, that states that $R_{2,\alpha}(\cdot, q)$ is non decreasing in $[\underline{r}_\alpha(1,q), 1]$ as in this case $q \geq \Gamma_\alpha(\tilde{v}_\alpha) \geq \underline{q}_\alpha$. In (b), we used the fact that $R_{3,\alpha}(1,q) \leq 1$.

For $\alpha = 0$, using the expressions in Lemma D-2, we obtain $(\Gamma_\alpha(\tilde{v}_\alpha), 1) \overset{\alpha=0}{=} (\frac{1}{2}, 1)$ and:

$$\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w, \{q\})) \overset{\alpha=0}{=} R_{3,0}(1,q) = 1 - q, \quad \text{which is achieved at } p = 1.$$

For $\alpha = 1$, using the expressions in Lemma D-2, we obtain $(\Gamma_\alpha(\tilde{v}_\alpha), 1) \overset{\alpha=1}{=} (e^{-e^{-1}}, 1)$ and:

$$\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) \overset{\alpha=1}{=} R_{3,1}(1,q) = eq \log\left(q^{-1}\right), \quad \text{which is achieved at } p = 1.$$

This completes the proof of Theorem 2. □

D.1 Proofs of auxiliary results

Proof of Proposition D-1. Following the reasoning in Section 4 in Eq.(4.1), we have that:

$$\begin{aligned}
&\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) \\
= &\sup_{p \in [0,w]} \min\left\{ \min_{r \in [\underline{r}_\alpha(w,q), p)} \frac{p\overline{F}_\alpha(p|r,(w,q))}{r}, \frac{p}{w}, \min_{r \in [w, \bar{r}_\alpha(w,q)]} \frac{p\overline{F}_\alpha(p|r,(w,q))}{\operatorname{opt}(F_\alpha(\cdot|r,(w,q)))} \right\} \\
= &\sup_{\frac{p}{w} \in [0,1]} \min\left\{ \inf_{\frac{r}{w} \in [\underline{r}_\alpha(1,q), \frac{p}{w})} \frac{\frac{p}{w}\overline{F}_\alpha(p/w|r/w,(1,q))}{\frac{r}{w}}, \frac{p}{w}, \inf_{\frac{r}{w} \in [1, \bar{r}_\alpha(1,q)]} \frac{\frac{p}{w}\overline{F}_\alpha(p/w|r/w,(1,q))}{\frac{r}{w}\overline{F}_\alpha(r/w|r/w,(1,q))} \right\} \\
\overset{(a)}{=} &\sup_{\tilde{p} \in [0,1]} \min\left\{ \inf_{\tilde{r} \in [\underline{r}_\alpha(1,q), \tilde{p})} \frac{\tilde{p}\overline{F}_\alpha(\tilde{p}|\tilde{r},(1,q))}{\tilde{r}}, \tilde{p}, \inf_{\tilde{r} \in [1, \bar{r}_\alpha(1,q)]} \frac{\tilde{p}\overline{F}_\alpha(\tilde{p}|\tilde{r},(1,q))}{\tilde{r}\overline{F}_\alpha(\tilde{r}|\tilde{r},(1,q))} \right\},
\end{aligned}$$

where in (a) we used two changes of variables to remove the dependency on w, namely $\tilde{p} = p/w$ and $\tilde{r} = r/w$. Note that when $q < \underline{q}_\alpha$, the last term in the brackets does not affect the worst-case. Thus we conclude that

$$\mathcal{R}(\mathcal{P}_d, \mathcal{F}_\alpha(w,q)) = \sup_{p \in [0,1]} \min\left\{ \inf_{r \in [\underline{r}_\alpha(1,q), p)} \frac{p\overline{F}_\alpha(p|r,(1,q))}{r}, p, \inf_{r \in [1, \bar{r}_\alpha(1,q)]} \frac{p\overline{F}_\alpha(p|r,(1,q))}{r\overline{F}_\alpha(r|r,(1,q))} \right\}. \quad \text{(D-3)}$$

For each (normalized) price p in $[0,1]$, we have three terms that determine the worst case performance. We analyze each term separately. The second term is just the identity stemming from nature selecting a point mass at 1. We next analyze the first and third terms with the brackets in (D-3).

Third term. The third term is only present if $q \geq \underline{q}_\alpha$ (ensuring that $[1, \overline{r}_\alpha(1,q)] \neq \emptyset$). In this case, for any p in $[0,1]$ and , the third term can be shown to be equal to

$$\inf_{r \in [1, \overline{r}_\alpha(1,q)]} \frac{p\overline{F}_\alpha\left(p|r,(1,q)\right)}{r\overline{F}_\alpha\left(r|r,(1,q)\right)} = \frac{p\overline{F}_\alpha\left(p|\overline{r}_\alpha(1,q),(1,q)\right)}{\sup_{r\in[1,\overline{r}_\alpha(1,q)]} r\overline{F}_\alpha\left(r|r,(1,q)\right)}.$$

Indeed, fix $q \geq \underline{q}_\alpha$. For any $r \in [1, \overline{r}_\alpha(1,q)]$, $\overline{F}_\alpha\left(p|r,(1,q)\right) = \Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(q\right)p\right) = \overline{F}_\alpha\left(p|\overline{r}_\alpha(1,q),(1,q)\right)$. By Lemma A-2 applied to $\beta := \Gamma_\alpha^{-1}\left(q\right)$ and $w' := 0$, we have that the function $v \to v\overline{F}_\alpha\left(v|\overline{r}_\alpha(1,q),(1,q)\right)$ is maximized at $\overline{r}_\alpha(1,q)$ thus it achieves its maximum at $\overline{r}_\alpha(1,q)$ on the interval $[1, \overline{r}_\alpha(1,q)]$. Hence we get that

$$\begin{aligned}\inf_{r \in [1, \overline{r}_\alpha(1,q)]} \frac{p\overline{F}_\alpha\left(p|r,(1,q)\right)}{r\overline{F}_\alpha\left(r|r,(1,q)\right)} &= \inf_{r \in [1, \overline{r}_\alpha(1,q)]} \frac{p\overline{F}_\alpha\left(p|\overline{r}_\alpha(1,q),(1,q)\right)}{r\overline{F}_\alpha\left(r|\overline{r}_\alpha(1,q)\,(1,q)\right)} = \frac{p\overline{F}_\alpha\left(p|\overline{r}_\alpha(1,q),(1,q)\right)}{\sup_{r\in[1,\overline{r}_\alpha(1,q)]} r\overline{F}_\alpha\left(r|\overline{r}_\alpha(1,q)\,(1,q)\right)} \\ &= \frac{p\overline{F}_\alpha\left(p|\overline{r}_\alpha(1,q),(1,q)\right)}{\mathrm{opt}(F_\alpha\left(\cdot|\overline{r}_\alpha(1,q),(1,q)\right))}. \end{aligned} \tag{D-4}$$

One can easily check that

$$\mathrm{opt}(F_\alpha\left(\cdot|\overline{r}_\alpha(1,q),(1,q)\right)) \quad = \quad \lim_{v \to \overline{r}_\alpha(1,q)} v\overline{F}_\alpha\left(v|\overline{r}_\alpha(1,q),(1,q)\right) = \frac{\tilde{v}_\alpha}{\Gamma_\alpha^{-1}\left(q\right)},$$

where $\tilde{v}_\alpha$ was defined in (D-1).

First term. For any p in $[0,1]$, the first term in (D-3) can be rewritten as

$$\inf_{r \in [\underline{r}_\alpha(1,q),p)} \frac{p\overline{F}_\alpha\left(p|r,(1,q)\right)}{r} = \inf_{r \in [\underline{r}_\alpha(1,q),p)} \Phi(r),$$

where

$$\Phi(r) := \frac{p}{r}\Gamma_\alpha\left(\Gamma_\alpha^{-1}\left(q\right)\frac{p-r}{1-r}\right).$$

We study the function $\Phi(\cdot)$ and by analyzing its derivative, determine exactly where its minimum is achieved. In particular, we will establish the following claim. On $[\underline{r}_\alpha(1,q),p)$, the function $\Phi(\cdot)$ is minimized at

$$\mu_{\alpha,q}\left(p\right) \quad = \quad 1 - \frac{\sqrt{\left(\alpha\Gamma_\alpha^{-1}\left(q\right)(1-p)\right)^2 + 4\Gamma_\alpha^{-1}\left(q\right)(1-p)\,q^{\alpha-1}} - \alpha\Gamma_\alpha^{-1}\left(q\right)(1-p)}{2q^{\alpha-1}}.$$

For any p in $[0,1]$, at any r in $[\underline{r}_\alpha(1,q),p)$, $\Phi(\cdot)$ is differentiable with derivative given by

$$
\begin{aligned}
\frac{d\Phi}{dr}(r) &= -\frac{p}{r^2}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right) - \frac{p}{r}\left(\Gamma_\alpha^{-1}(q)\frac{-(1-r)+(p-r)}{(1-r)^2}\right)\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{2-\alpha} \\
&= -\frac{p}{r^2}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right) - \frac{p}{r}\left(\Gamma_\alpha^{-1}(q)\frac{-(1-p)}{(1-r)^2}\right)\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{2-\alpha} \\
&= -\frac{p}{r^2(1-r)^2}\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{2-\alpha}\left[\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{-(1-\alpha)}(1-r)^2 - \Gamma_\alpha^{-1}(q)\,r(1-p)\right] \\
&= -\frac{p}{r^2(1-r)^2}\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{2-\alpha}\left[\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)(1-r)^2 - \Gamma_\alpha^{-1}(q)\,r(1-p)\right] \\
&= -\frac{p}{r^2(1-r)^2}\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-r}{1-r}\right)\right)^{2-\alpha}\left[(1-r)^2 + (1-\alpha)\Gamma_\alpha^{-1}(q)(p-r)(1-r) - \Gamma_\alpha^{-1}(q)\,r(1-p)\right].
\end{aligned}
$$

Note that the sign of the derivative of Φ is determined by that of the quadratic

$$
\begin{aligned}
\varphi(r) &:= -\left[(1-r)^2 + (1-\alpha)\Gamma_\alpha^{-1}(q)(p-r)(1-r) - \Gamma_\alpha^{-1}(q)\,r(1-p)\right] \\
&= -\Bigg[\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\right)(1-r)^2 - (1-\alpha)\Gamma_\alpha^{-1}(q)(1-p)(1-r) \\
&\quad +\Gamma_\alpha^{-1}(q)(1-r)(1-p) - \Gamma_\alpha^{-1}(q)(1-p)\Bigg] \\
&= -\left[\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\right)(1-r)^2 + \alpha\Gamma_\alpha^{-1}(q)(1-p)(1-r) - \Gamma_\alpha^{-1}(q)(1-p)\right] \\
&= -\left[q^{\alpha-1}(1-r)^2 + \alpha\Gamma_\alpha^{-1}(q)(1-p)(1-r) - \Gamma_\alpha^{-1}(q)(1-p)\right].
\end{aligned}
$$

Let

$$
\Delta_{\alpha,q}(p) = \left(\alpha\Gamma_\alpha^{-1}(q)(1-p)\right)^2 + 4q^{\alpha-1}\Gamma_\alpha^{-1}(q)(1-p).
$$

The above is positive and hence the quadratic $\varphi(r)$ admits two roots given by

$$
\begin{aligned}
r_1 &= 1 + \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) + \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}}, \\
r_2 &= 1 + \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) - \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}}.
\end{aligned}
$$

It is clear that $r_1 \geq 1$. We next establish that r_2 belongs to $[\underline{r}_\alpha(1,q), p]$.

$$
\begin{aligned}
p - r_2 &= p - 1 - \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) - \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}} \\
&= \frac{-(2q^{\alpha-1} + \alpha\Gamma_\alpha^{-1}(q))(1-p) + \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}} \\
&= \frac{-\left(4q^{2(\alpha-1)} + \left(\alpha\Gamma_\alpha^{-1}(q)\right)^2 + 4q^{\alpha-1}\alpha\Gamma_\alpha^{-1}(q)\right)(1-p)^2 + \left(\alpha\Gamma_\alpha^{-1}(q)(1-p)\right)^2 + 4q^{\alpha-1}\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}((2q^{\alpha-1} + \alpha\Gamma_\alpha^{-1}(q))(1-p) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&= 4(1-p)q^{\alpha-1}\frac{-\left(q^{(\alpha-1)} + \alpha\Gamma_\alpha^{-1}(q)\right)(1-p) + \Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}((2q^{\alpha-1} + \alpha\Gamma_\alpha^{-1}(q))(1-p) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&= 4(1-p)q^{\alpha-1}\frac{-(1 + \Gamma_\alpha^{-1}(q))(1-p) + \Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}((2q^{\alpha-1} + \alpha\Gamma_\alpha^{-1}(q))(1-p) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&= 4(1-p)q^{\alpha-1}\Gamma_\alpha^{-1}(q)\frac{p/\underline{r}_\alpha(1,q) - 1}{2q^{\alpha-1}((2q^{\alpha-1} + \alpha\Gamma_\alpha^{-1}(q))(1-p) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows since $p \geq \underline{r}_\alpha(1,q)$.

Now, we also have

$$
\begin{aligned}
r_2 - \underline{r}_\alpha(1,q) &= 1 + \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) - \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}} - \frac{1}{1 + \Gamma_\alpha^{-1}(q)} \\
&= \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) - \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}} + \Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) \\
&= \frac{\alpha\Gamma_\alpha^{-1}(q)(1-p) + 2q^{\alpha-1}\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) - \sqrt{\Delta_{\alpha,q}(p)}}{2q^{\alpha-1}} \\
&= \frac{(\alpha\Gamma_\alpha^{-1}(q))^2(1-p)^2 + 4q^{2(\alpha-1)}(\Gamma_\alpha^{-1}(q))^2(\underline{r}_\alpha(1,q))^2 + 4q^{\alpha-1}(\Gamma_\alpha^{-1}(q))^2\alpha\underline{r}_\alpha(1,q)(1-p)}{2q^{\alpha-1}(\alpha\Gamma_\alpha^{-1}(q)(1-p) + 2q^{\alpha-1}\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&\quad - \frac{\left(\alpha\Gamma_\alpha^{-1}(q)(1-p)\right)^2 + 4q^{\alpha-1}\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}(\alpha\Gamma_\alpha^{-1}(q)(1-p) + 2q^{\alpha-1}\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&= 4\Gamma_\alpha^{-1}(q)\,q^{\alpha-1}\frac{q^{\alpha-1}\Gamma_\alpha^{-1}(q)(\underline{r}_\alpha(1,q))^2 + \alpha\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q)(1-p) - (1-p)}{2q^{\alpha-1}(\alpha\Gamma_\alpha^{-1}(q)(1-p) + 2q^{\alpha-1}\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&= 4\Gamma_\alpha^{-1}(q)\,q^{\alpha-1}\frac{q^{\alpha-1}\underline{r}_\alpha(1,q)(p - \underline{r}_\alpha(1,q))}{2q^{\alpha-1}(\alpha\Gamma_\alpha^{-1}(q)(1-p) + 2q^{\alpha-1}\Gamma_\alpha^{-1}(q)\,\underline{r}_\alpha(1,q) + \sqrt{\Delta_{\alpha,q}(p)})} \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows since $p \geq \underline{r}_\alpha(1,q)$. Hence we have established that r_2 belongs to $[\underline{r}_\alpha(1,q), p)$, and $r_1 \geq 1 \geq p$.

Now, note that the sign of φ is non-negative on $[r_2, r_1]$ and non-positive on $[0, r_2]$. We deduce that the function Φ is non increasing on $[\underline{r}_\alpha(1,q), r_2]$ and non decreasing on $[r_2, p)$, thus, on $[\underline{r}_\alpha(1,q), p)$, Φ achieves its minimum at $r_2 = \mu_{\alpha,q}(p)$. In other words we have established that for any p in $[0,1]$,

$$\inf_{r\in[\underline{r}_\alpha(1,q),p)} \frac{p\overline{F}_\alpha\left(p|r,(1,q)\right)}{r} = \frac{p\overline{F}_\alpha\left(p|\mu_{\alpha,q}(p),(1,q)\right)}{\mu_{\alpha,q}(p)} = \frac{p\overline{F}_\alpha\left(p|\mu_{\alpha,q}(p),(1,q)\right)}{\mathrm{opt}(F_\alpha\left(\cdot|\mu_{\alpha,q}(p),(1,q)\right))}. \tag{D-5}$$

Combining equations (D-3), (D-4) and (D-5) yields the result. □

Proof of Lemma D-1. We first start by studying the function $p \to \mu_{\alpha,q}(p)$. We have

$$\begin{aligned}
\mu_{\alpha,q}(p) &= 1 - \frac{\sqrt{\left(\alpha\Gamma_\alpha^{-1}(q)(1-p)\right)^2 + 4\Gamma_\alpha^{-1}(q)(1-p)q^{\alpha-1}} - \alpha\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}} \\
&= 1 + \alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}} - \sqrt{\left(\alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}\right)^2 + 2\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}} \\
&= \frac{1-2(1-\alpha)\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}}{1+\alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}} + \sqrt{\left(\alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}\right)^2 + 2\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}}}.
\end{aligned}$$

The numerator of the above ratio $p \to 1-2(1-\alpha)\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}$ is clearly non-decreasing, and the denominator $p \to 1+\alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}} + \sqrt{\left(\alpha\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}\right)^2 + 2\frac{\Gamma_\alpha^{-1}(q)(1-p)}{2q^{\alpha-1}}}$ is clearly non-increasing. Therefore, by composition, $p \to \mu_{\alpha,q}(p)$ is non-decreasing and $\mu_{\alpha,q}(p)$ in $\left[\mu_{\alpha,q}\left(\frac{1}{1+\Gamma_\alpha^{-1}(q)}\right), \mu_{\alpha,q}(1)\right]$ with:

$$\begin{aligned}
\mu_{\alpha,q}\left(\frac{1}{1+\Gamma_\alpha^{-1}(q)}\right) &= 1 + \alpha\frac{\Gamma_\alpha^{-1}(q)^2}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)} - \sqrt{\left(\alpha\frac{\Gamma_\alpha^{-1}(q)^2}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}\right)^2 + 2\frac{\Gamma_\alpha^{-1}(q)^2}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}} \\
&= 1 + \alpha\frac{\Gamma_\alpha^{-1}(q)^2}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)} - \frac{\Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}\sqrt{\alpha^2\Gamma_\alpha^{-1}(q)^2 + 4q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)} \\
&= 1 + \frac{\Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}\left(\alpha\Gamma_\alpha^{-1}(q) - \sqrt{\alpha^2\Gamma_\alpha^{-1}(q)^2 + 4\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\right)\left(1+\Gamma_\alpha^{-1}(q)\right)}\right) \\
&= 1 + \frac{\Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}\left(\alpha\Gamma_\alpha^{-1}(q) - \sqrt{\left(2+(2-\alpha)\Gamma_\alpha^{-1}(q)\right)^2}\right) \\
&= 1 - \frac{\Gamma_\alpha^{-1}(q)}{2q^{\alpha-1}\left(1+\Gamma_\alpha^{-1}(q)\right)}2\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\right) = \frac{1}{1+\Gamma_\alpha^{-1}(q)}. \\
\mu_{\alpha,q}(1) &= 1 + \alpha\frac{\Gamma_\alpha^{-1}(q)(1-1)}{2q^{\alpha-1}} - \sqrt{\left(\alpha\frac{\Gamma_\alpha^{-1}(q)(1-1)}{2q^{\alpha-1}}\right)^2 + 2\frac{\Gamma_\alpha^{-1}(q)(1-1)}{2q^{\alpha-1}}} = 1.
\end{aligned}$$

Therefore $\mu_{\alpha,q}(p)$ in $\left[\frac{1}{1+\Gamma_\alpha^{-1}(q)},1\right]$ and $p\to\mu_{\alpha,q}(p)$ is an increasing function in $\left[\frac{1}{1+\Gamma_\alpha^{-1}(q)},1\right]$ and, for any p in $\left[\frac{1}{1+\Gamma_\alpha^{-1}(q)},1\right]$, its inverse is given by

$$\mu_{\alpha,q}^{-1}(p) := 1-\frac{q^{\alpha-1}(1-p)^2}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-p))}.$$

Next, we will show each point separately.

First point If $q\geq \underline{q}_\alpha$, we have

$$\begin{aligned}
\frac{R_{3,\alpha}(p,q)}{R_{1,\alpha}(p,q)} &= \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\mu_{\alpha,q}(p)\frac{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right)}{\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-\mu_{\alpha,q}(p)}{1-\mu_{\alpha,q}(p)}\right)}\\
&\overset{(a)}{=} \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\mu_{\alpha,q}(p)\,\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-\frac{p-\mu_{\alpha,q}(p)}{1-\mu_{\alpha,q}(p)}}{1+(1-\alpha)\frac{p-\mu_{\alpha,q}(p)}{1-\mu_{\alpha,q}(p)}\Gamma_\alpha^{-1}(q)}\right),
\end{aligned}$$

where in (a), we used the identity $\frac{\Gamma_\alpha(x)}{\Gamma_\alpha(y)}=\Gamma_\alpha\left(\frac{x-y}{1+(1-\alpha)y}\right)$. Let us now focus on simplifying the term (A) inside $\Gamma_\alpha(\cdot)$. We have

$$\begin{aligned}
(A) &= \frac{\Gamma_\alpha^{-1}(q)\,\mu_{\alpha,q}(p)(1-p)}{1-\mu_{\alpha,q}(p)+\Gamma_\alpha^{-1}(q)(1-\alpha)(p-\mu_{\alpha,q}(p))}\\
&\overset{(b)}{=} \frac{\Gamma_\alpha^{-1}(q)\,\mu_{\alpha,q}(p)\frac{q^{\alpha-1}(1-\mu_{\alpha,q}(p))^2}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-\mu_{\alpha,q}(p)))}}{1-\mu_{\alpha,q}(p)+\Gamma_\alpha^{-1}(q)(1-\alpha)\left(1-\mu_{\alpha,q}(p)-\frac{q^{\alpha-1}(1-\mu_{\alpha,q}(p))^2}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-\mu_{\alpha,q}(p)))}\right)}\\
&= \\
&\frac{q^{\alpha-1}\mu_{\alpha,q}(p)(1-\mu_{\alpha,q}(p))}{1-\alpha(1-\mu_{\alpha,q}(p))+(1-\alpha)\left(\Gamma_\alpha^{-1}(q)-\alpha\Gamma_\alpha^{-1}(q)(1-\mu_{\alpha,q}(p))-q^{\alpha-1}(1-\mu_{\alpha,q}(p))\right)}\\
&= \frac{q^{\alpha-1}\mu_{\alpha,q}(p)(1-\mu_{\alpha,q}(p))}{1+(1-\alpha)\Gamma_\alpha^{-1}(q)-(1-\mu_{\alpha,q}(p))\left(\alpha+\alpha(1-\alpha)\Gamma_\alpha^{-1}(q)-(1-\alpha)q^{\alpha-1}\right)}\\
&= \frac{q^{\alpha-1}\mu_{\alpha,q}(p)(1-\mu_{\alpha,q}(p))}{q^{\alpha-1}-(1-\mu_{\alpha,q}(p))q^{\alpha-1}} = 1-\mu_{\alpha,q}(p),
\end{aligned}$$

where in (b), we used that

$$p=\mu_{\alpha,q}^{-1}(\mu_{\alpha,q}(p)) = 1-\frac{q^{\alpha-1}(1-\mu_{\alpha,q}(p))^2}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-\mu_{\alpha,q}(p)))}.$$

Therefore we obtain that:

$$\frac{R_{3,\alpha}(p,q)}{R_{1,\alpha}(p,q)} = \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\mu_{\alpha,q}(p)\,\Gamma_\alpha(1-\mu_{\alpha,q}(p)).$$

Since $p \to \mu_{\alpha,q}(p)$ is non-decreasing and $\Gamma_\alpha(\cdot)$ is non-increasing, then by composition, we have $p \mapsto R_{13,\alpha,q}(p) = \frac{R_{3,\alpha}(p,q)}{R_{1,\alpha}(p,q)}$ is non-decreasing in $[\underline{r}_\alpha(1,q), 1]$ and $\frac{R_{3,\alpha}(p,q)}{R_{1,\alpha}(p,q)}$ in $[R_{13,\alpha,q}(\underline{r}_\alpha(1,q)), R_{13,\alpha,q}(1)]$, we have

$$R_{13,\alpha,q}(\underline{r}_\alpha(1,q)) = \frac{R_{3,\alpha}\left(\frac{1}{1+\Gamma_\alpha^{-1}(q)}, q\right)}{R_{1,\alpha}\left(\frac{1}{1+\Gamma_\alpha^{-1}(q)}, q\right)} = \frac{\frac{\Gamma_\alpha^{-1}(q)}{1+\Gamma_\alpha^{-1}(q)}\Gamma_\alpha\left(\frac{\Gamma_\alpha^{-1}(q)}{1+\Gamma_\alpha^{-1}(q)}\right)}{\tilde{v}_\alpha} \overset{(a)}{\leq} 1,$$

where in (a), we used the fact that the revenue function $x \to x\Gamma_\alpha(x)$ is maximized at $x = \frac{1}{\alpha}$ (with the convention that for $\alpha = 0$, $1/\alpha = \infty$) and the maximum value achieved is $\tilde{v}_\alpha$. Furthermore, we have

$$R_{13,\alpha,q}(1) = \frac{R_{3,\alpha}(1,q)}{R_{1,\alpha}(1,q)} = \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}.$$

Note that $R_{13,\alpha,q}(1) \geq 1$ iff $q \leq \Gamma_\alpha(\tilde{v}_\alpha)$. For q in $(\underline{r}_\alpha(1,q), \Gamma_\alpha(\tilde{v}_\alpha))$, we define $p_{13,\alpha,q} = R_{13,\alpha,q}^{-1}(1)$.

Therefore, we have that when q in $\left[\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)\right)$, then $R_{3,\alpha}(p,q) \leq R_{1,\alpha}(p,q)$ if $p \leq p_{13,\alpha,q}$ and $R_{3,\alpha}(p,q) \geq R_{1,\alpha}(p,q)$ if $p \geq p_{13,\alpha,q}$. And if q in $[\Gamma_\alpha(\tilde{v}_\alpha), 1]$ then $R_{1,\alpha}(p,q) \geq R_{3,\alpha}(p,q)$ for all p in $[\underline{r}_\alpha(1,q), 1]$.

Second point We have

$$\begin{aligned}
\frac{R_{1,\alpha}(p,q)}{R_{2,\alpha}(p,q)} &= \frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\frac{p-\mu_{\alpha,q}(p)}{1-\mu_{\alpha,q}(p)}\right) = \frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\left(1-\frac{1-p}{1-\mu_{\alpha,q}(p)}\right)\right) \\
&\overset{(a)}{=} \frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\left(1-\frac{q^{\alpha-1}(1-\mu_{\alpha,q}(p))}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-\mu_{\alpha,q}(p)))}\right)\right) \\
&= \frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\mu_{\alpha,q}(p)}-\alpha\right)}\right) \\
&= \frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\mu_{\alpha,q}(p)}-\alpha\right)}\right),
\end{aligned}$$

where in (a), we used:

$$p = \mu_{\alpha,q}^{-1}(\mu_{\alpha,q}(p)) = 1 - \frac{q^{\alpha-1}(1-\mu_{\alpha,q}(p))^2}{\Gamma_\alpha^{-1}(q)(1-\alpha(1-\mu_{\alpha,q}(p)))}.$$

Therefore, by composition, we have $p \to R_{12,\alpha,q}(p) = \frac{R_{1,\alpha}(p,q)}{R_{2,\alpha}(p,q)}$ is non-increasing in $[\underline{r}_\alpha(1,q),1]$ and $\frac{R_{1,\alpha}(p,q)}{R_{2,\alpha}(p,q)}$ in $[R_{12,\alpha,q}(1), R_{12,\alpha,q}(\underline{r}_\alpha(1,q))]$, we have

$$\begin{aligned}
R_{12,\alpha,q}(\underline{r}_\alpha(1,q)) &= \frac{1}{\underline{r}_\alpha(1,q)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\underline{r}_\alpha(1,q)} - \alpha\right)}\right) \\
&= \left(1+\Gamma_\alpha^{-1}(q)\right)\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\frac{1+(1-\alpha)\Gamma_\alpha^{-1}(q)}{\Gamma_\alpha^{-1}(q)}}\right) \\
&= \left(1+\Gamma_\alpha^{-1}(q)\right) \geq 1, \\
\lim_{p\to1} R_{12,\alpha,q}(p) &= \lim_{\mu\to1}\frac{1}{\mu}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\mu} - \alpha\right)}\right) = 0.
\end{aligned}$$

We define $p_{12,\alpha,q} = R_{12,\alpha,q}^{-1}(1)$. Therefore, $R_{1,\alpha}(p,q) \leq R_{2,\alpha}(p,q)$ if $p \leq p_{12,\alpha,q}$ and $R_{1,\alpha}(p,q) \geq R_{2,\alpha}(p,q)$ if $p \geq p_{12,\alpha,q}$.

Third point If q belongs to $[\underline{q}_\alpha, 1]$, we have

$$\begin{aligned}
R_{3,\alpha}(p,q) &= \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right) \\
\frac{\partial R_{3,\alpha}(p,q)}{\partial p} &= \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right) - \Gamma_\alpha^{-1}(q)\,p\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right)^{2-\alpha}\right) \\
&= \frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right)\left(1 - \frac{\Gamma_\alpha^{-1}(q)\,p}{1+(1-\alpha)\Gamma_\alpha^{-1}(q)\,p}\right) \\
&= \begin{cases} \frac{\Gamma_\alpha^{-1}(q)^2}{\tilde{v}_\alpha}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right)\left(\frac{\frac{1}{\Gamma_\alpha^{-1}(q)}}{1+(1-\alpha)\Gamma_\alpha^{-1}(q)p}\right) & \text{if } \alpha = 0, \\ \alpha\frac{\Gamma_\alpha^{-1}(q)^2}{\tilde{v}_\alpha}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,p\right)\left(\frac{\overline{r}_\alpha(1,q)-p}{1+(1-\alpha)\Gamma_\alpha^{-1}(q)p}\right) & \text{if } \alpha \in (0,1]. \end{cases}
\end{aligned}$$

Therefore, if q in $[\underline{q}_\alpha, 1]$, then $\overline{r}_\alpha(1,q) \geq 1$ and therefore $R_{3,\alpha}(\cdot,q)$ is non-decreasing in $[\underline{r}_\alpha(1,q),1]$.

Fourth point

Case 1: Regular $\alpha = 0$ In this case, the first function is expressed as follows:

$$\begin{aligned}
R_{1,0}(p,q) &= \frac{p}{\mu_{0,q}(p)\left(1+\left(\frac{1}{q}-1\right)\frac{p-\mu_{0,q}(p)}{1-\mu_{0,q}(p)}\right)} = \frac{pq\left(\mu_{0,q}(p)-1\right)}{\mu_{0,q}(p)\left(p\left(q-1\right)-q+\mu_{0,q}(p)\right)} \\
\text{with } \mu_{0,q}(p) &= 1 - \frac{\sqrt{4\left(q^{-1}-1\right)\left(1-p\right)q^{-1}}}{2q^{-1}} = 1 - \sqrt{\left(1-p\right)\left(1-q\right)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
R_{1,0}(p,q) &= \frac{p}{\left(1-\sqrt{(1-p)(1-q)}\right)\left(1+\left(\frac{1}{q}-1\right)\frac{p-1+\sqrt{(1-p)(1-q)}}{1-1+\sqrt{(1-p)(1-q)}}\right)} \\
&= \frac{p}{\left(1-\sqrt{(1-p)(1-q)}\right)\left(1+\frac{1-q}{q}\frac{p-1+\sqrt{(1-p)(1-q)}}{\sqrt{(1-p)(1-q)}}\right)} \\
&= \frac{p}{\left(1-\sqrt{(1-p)(1-q)}\right)\left(1+\frac{1-q}{q}\left(1-\frac{\sqrt{1-p}}{\sqrt{1-q}}\right)\right)} \\
&= \frac{pq}{\left(1-\sqrt{(1-p)(1-q)}\right)\left(q+1-q-\sqrt{(1-q)(1-p)}\right)} \\
&= \frac{pq}{\left(1-\sqrt{(1-p)(1-q)}\right)^2}.
\end{aligned}$$

We have, for all p in $[q,1]$:

$$\frac{\partial R_{1,0}}{\partial p}(p,q) = -\frac{q\sqrt{1-p}\left(\sqrt{1-q}-\sqrt{1-p}\right)}{(1-p)\left(1-\sqrt{(1-p)(1-q)}\right)^3} \leq 0 \quad \forall p \text{ in } [q,1],$$

and it is easy to see that function $p \to R_2(p,q)$ is non-decreasing.

Case 2: mhr case $\alpha = 1$ For $\alpha = 1$, the first function is expressed as follows:

$$\begin{aligned}
R_{1,1}(p,q) &= \frac{p}{\mu_{1,q}(p)} e^{-\log\left(q^{-1}\right)\frac{p-\mu_{1,q}(p)}{1-\mu_{1,q}(p)}} \\
\text{with } \mu_{1,q}(p) &= 1-\frac{\sqrt{\log\left(q^{-1}\right)(1-p))^2+4\log\left(q^{-1}\right)(1-p)}-\log\left(q^{-1}\right)(1-p)}{2} \\
\text{and } p &= \mu_{1,q}^{-1}\left(\mu_{1,q}(p)\right) = 1-\frac{\left(1-\mu_{1,q}(p)\right)^2}{\log\left(q^{-1}\right)\mu_{1,q}(p)}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
R_{1,1}(p,q) &= \frac{1-\frac{(1-\mu_{1,q}(p))^2}{\log(q^{-1})\mu_{1,q}(p)}}{\mu_{1,q}(p)} e^{-\log\left(q^{-1}\right)\frac{1-\mu_{1,q}(p)-\frac{\left(1-\mu_{1,q}(p)\right)^2}{\log(q^{-1})\mu_{1,q}(p)}}{1-\mu_{1,q}(p)}} \\
&= \left(\frac{1}{\mu_{1,q}(p)}-\frac{1}{\log\left(q^{-1}\right)}\left(\frac{1}{\mu_{1,q}(p)}-1\right)^2\right) e^{-\log\left(q^{-1}\right)+\frac{1}{\mu_{1,q}(p)}-1} \\
&= \frac{q}{e\log\left(q^{-1}\right)}\left(\frac{\log\left(q^{-1}\right)}{\mu_{1,q}(p)}-\frac{1}{\mu_{1,q}(p)^2}+\frac{2}{\mu_{1,q}(p)}-1\right) e^{\frac{1}{\mu_{1,q}(p)}} \\
&= -\frac{q}{e\log\left(q^{-1}\right)}\left(\frac{1}{\mu_{1,q}(p)^2}-\frac{\log\left(q^{-1}\right)+2}{\mu_{1,q}(p)}+1\right) e^{\frac{1}{\mu_{1,q}(p)}} =: \tilde{R}_1\left(\frac{1}{\mu_{1,q}(p)}\right),
\end{aligned}$$

with

$$\tilde{R}_1(x) = -\frac{q}{e\log(q^{-1})}\left(x^2 - \left(2+\log\left(q^{-1}\right)\right)x + 1\right)e^x \quad \text{for } x \text{ in } \left[1, 1+\log\left(q^{-1}\right)\right].$$

On another hand, we have

$$\begin{aligned}
\frac{d\tilde{R}_1(x)}{dx} &= -\frac{q}{e\log(q^{-1})}\left(2x - 2 - \log\left(q^{-1}\right) + x^2 - \left(2+\log\left(q^{-1}\right)\right)x + 1\right)e^x \\
&= -\frac{q}{e\log(q^{-1})}\left(x^2 - \log\left(q^{-1}\right)x - \left(1+\log\left(q^{-1}\right)\right)\right)e^x \\
&= -\frac{q}{e\log(q^{-1})}(x+1)\left(x - \left(1+\log\left(q^{-1}\right)\right)\right)e^x \geq 0 \quad \forall x \text{ in } \left[1, 1+\log\left(q^{-1}\right)\right].
\end{aligned}$$

For all p in $[\frac{1}{1+\log(q^{-1})}, 1]$, we have $\mu_{1,q}(p)$ in $[\frac{1}{1+\log(q^{-1})}, 1]$, therefore $\frac{1}{\mu_{1,q}(p)}$ in $[1, 1+\log\left(q^{-1}\right)]$.

Therefore, by composition, $p \to R_{1,1}(p,q)$ is non increasing. The function $p \to R_{2,1}(p,q) = p$ is non-decreasing. □

Proof of Lemma D-2. **Case 1: Regular case $\alpha = 0$.** In this case, for q in $(\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)] \overset{\alpha=0}{=} (0, \frac{1}{2}]$, $p_{13,0,q}$ is a solution to the following equation

$$\begin{aligned}
&\frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\mu_{\alpha,q}(p)\,\Gamma_\alpha\left(1-\mu_{\alpha,q}(p)\right) = 1 \\
\text{iff}\quad &\left(\frac{1}{q}-1\right)\frac{1-\sqrt{(1-p)(1-q)}}{1+\sqrt{(1-p)(1-q)}} = 1 \\
\text{iff}\quad &\frac{1-2q}{1-q} = \frac{\sqrt{(1-p)(1-q)}}{1-q} \\
\text{iff}\quad &p = p_{13,0,q} = 1 - \frac{(1-2q)^2}{1-q}.
\end{aligned}$$

and $p_{12,0,q}$ is solution to the following equation

$$\begin{aligned}
&\frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\mu_{\alpha,q}(p)} - \alpha\right)}\right) = 1 \\
\text{iff}\quad &\frac{q}{\left(1-\sqrt{(1-p)(1-q)}\right)^2} = 1 \\
\text{iff}\quad &\sqrt{q} = 1 - \sqrt{(1-p)(1-q)} \\
\text{iff}\quad &\sqrt{(1-p)} = \frac{1-\sqrt{q}}{\sqrt{1-q}} \\
\text{iff}\quad &p = p_{12,0,q} = 1 - \frac{\left(1-\sqrt{q}\right)^2}{1-q}.
\end{aligned}$$

Therefore

$$\begin{aligned} p_{13,0,q} \leq p_{12,0,q} \quad &\text{iff} \quad 1 - \frac{(1-2q)^2}{1-q} \leq 1 - \frac{\left(1-\sqrt{q}\right)^2}{1-q} \\ &\text{iff} \quad (1-2q)^2 \geq (1-\sqrt{q})^2 \\ &\text{iff} \quad 2q \leq \sqrt{q} \\ &\text{iff} \quad q \leq \frac{1}{4} := \hat{q}_0. \end{aligned}$$

Furthermore, note that $\hat{q}_0 \in [\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)]$, since $\underline{q}_\alpha \overset{\alpha=0}{=} 0$ and $\Gamma_\alpha(\tilde{v}_\alpha) \overset{\alpha=0}{=} \frac{1}{2}$.

Case 2: mhr case $\alpha = 1$. In this case, for q in $(\underline{q}_\alpha, \Gamma_\alpha(\tilde{v}_\alpha)] \overset{\alpha=1}{=} \left(e^{-1}, e^{-e^{-1}}\right]$, $p_{13,1,q}$ is solution to the following equation

$$\begin{aligned} &\frac{\Gamma_\alpha^{-1}(q)}{\tilde{v}_\alpha}\mu_{\alpha,q}(p)\,\Gamma_\alpha(1-\mu_{\alpha,q}(p)) \overset{\alpha=1}{=} \log\left(q^{-1}\right) e\mu_{1,q}(p)\, e^{\mu_{1,q}(p)-1} = 1 \\ \text{iff} \quad &\mu_{1,q}(p)\, e^{\mu_{1,q}(p)-1} = \frac{1}{\log(q^{-1})} \\ \text{iff} \quad &p = p_{13,1,q} = \mu_{1,q}^{-1}\left(W\left(\frac{1}{\log(q^{-1})}\right)\right). \end{aligned}$$

And $p_{12,1,q}$ is solution to the following equation

$$\begin{aligned} &\frac{1}{\mu_{\alpha,q}(p)}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q) - \frac{q^{\alpha-1}}{\left(\frac{1}{1-\mu_{\alpha,q}(p)} - \alpha\right)}\right) \overset{\alpha=1}{=} \frac{1}{\mu_{1,q}(p)} e^{\log(q) + \frac{1}{\frac{1}{1-\mu_{1,q}(p)}-1}} = 1 \\ \text{iff} \quad &\frac{1}{\mu_{1,q}(p)} e^{\frac{1}{\mu_{1,q}(p)}} \frac{q}{e} = 1 \\ \text{iff} \quad &\frac{1}{\mu_{1,q}(p)} e^{\frac{1}{\mu_{1,q}(p)}} = \frac{e}{q} \\ \text{iff} \quad &p = p_{12,1,q} = \mu_{1,q}^{-1}\left(\frac{1}{W\left(\frac{e}{q}\right)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} p_{13,1,q} \leq p_{12,1,q} \quad &\text{iff} \quad \mu_{1,q}^{-1}\left(W\left(\frac{1}{\log(q^{-1})}\right)\right) \leq \mu_{1,q}^{-1}\left(\frac{1}{W\left(\frac{e}{q}\right)}\right) \\ &\text{iff} \quad W\left(\frac{1}{\log(q^{-1})}\right) \leq \frac{1}{W\left(\frac{e}{q}\right)} \quad \text{as } \mu_{1,q}^{-1}(.) \text{ is increasing} \\ &\text{iff} \quad W\left(\frac{1}{\log(q^{-1})}\right) W\left(\frac{e}{q}\right) \leq 1. \end{aligned}$$

We now study $q \mapsto g(q) := W\left(\frac{1}{\log(q^{-1})}\right) W\left(\frac{e}{q}\right)$ in $\left(e^{-1}, e^{-e^{-1}}\right]$, we have

$$\frac{dg(q)}{dq} = -\frac{W\left(\frac{e}{q}\right) W\left(-\frac{1}{\log(q)}\right)\left(W\left(\frac{e}{q}\right) + \log(q) W\left(-\frac{1}{\log(q)}\right) + \log(q) + 1\right)}{q \log(q)\left(W\left(\frac{e}{q}\right) + 1\right)\left(W\left(-\frac{1}{\log(q)}\right) + 1\right)}.$$

We next analyze the sign of the derivative.

$$\begin{aligned} sign\left(\frac{dg(q)}{dq}\right) &= sign\left(W\left(\frac{e}{q}\right) + \log(q) W\left(-\frac{1}{\log(q)}\right) + \log(q) + 1\right) = sign(h(q)) \\ \text{with } h(q) &:= W\left(\frac{e}{q}\right) + \log(q) W\left(-\frac{1}{\log(q)}\right) + \log(q) + 1 \\ \frac{dh(q)}{dq} &= \frac{\left(W\left(\frac{e}{q}\right) + 1\right) W\left(-\frac{1}{\log(q)}\right)^2 + W\left(-\frac{1}{\log(q)}\right) + 1}{q\left(W\left(\frac{e}{q}\right) + 1\right)\left(W\left(-\frac{1}{\log(q)}\right) + 1\right)} \geq 0 \quad \forall q \text{ in } \left[e^{-1}, e^{-e^{-1}}\right]. \end{aligned}$$

Therefore $q \mapsto h(q)$ is non-decreasing and we have $h\left(e^{-1}\right) = W\left(e^2\right) - W(1) > 0$. Therefore g is increasing in $\left(e^{-1}, e^{-e^{-1}}\right]$. Furthermore, we have $g\left(e^{-1}\right) = W(1) W\left(e^2\right) < 1$ and $g\left(e^{-e^{-1}}\right) = W\left(e^{1+1/e}\right) > 1$. Therefore there exists a unique $\hat{q}_1$ solution in $\left(e^{-1}, e^{-e^{-1}}\right]$ to the equation $W\left(\frac{1}{\log(q^{-1})}\right) W\left(\frac{e}{q}\right) = 1$.

Therefore, we have, for q in $\left(e^{-1}, e^{-e^{-1}}\right]$

$$p_{13,1,q} \leq p_{12,1,q} \quad \text{iff} \quad g(q) \leq 1 \quad \text{iff} \quad q \leq \hat{q}_1.$$

□

E Proofs and auxiliary results for Section 5

E.1 Proofs and auxiliary results for Section 5.1

Proof of Proposition 2. Fix α in $[0,1]$, Ψ in $\mathcal{P}$, q in $(0,1)$, $N > 1$, and a finite sequence of increasing reals $\mathbb{A} = \{a_i\}_{i=1}^N$ such that $0 < a_1 \leq w \leq a_N$. The proof uses two building blocks associated with uniformly bounding the losses stemming from truncating a mechanism and the losses stemming from local transfers of mass in a mechanism.

Define the two mechanisms Ψ_{a_N} in $\mathcal{P}$ and $\Psi_{\mathbb{A}}$ in $\mathcal{P}_{\mathbb{A}}$ as follows

$$\Psi_{a_N}(v) = \begin{cases} \Psi(v) & \text{if } v \in [0, a_N), \\ 1 & \text{if } v \geq a_N. \end{cases}$$

$$\Psi_{\mathbb{A}}(x) = \begin{cases} 0 & \text{if } x \in [0, a_1). \\ \Psi(a_{i+1}) & \text{if } x \in [a_i, a_{i+1}), \quad \text{for } 1 \leq i \leq N-1, \\ 1 & \text{if } x \in [a_N, \infty). \end{cases}$$

Ψ_{a_N} is a truncated version of Ψ at a_N and $\Psi_{\mathbb{A}}$ is a discretized and truncated version of Ψ.

Let F in $\mathcal{F}_\alpha(w,q)$. Next we analyze $R(\Psi,F) - R(\Psi_{\mathbb{A}},F)$ by decomposing it as follows.

$$R(\Psi,F) - R(\Psi_{\mathbb{A}},F) \quad = \quad R(\Psi,F) - R(\Psi_{a_N},F) + R(\Psi_{a_N},F) - R(\Psi_{\mathbb{A}},F). \tag{E-1}$$

To uniformly bound the maximal losses stemming from truncation $R(\Psi,F) - R(\Psi_{a_N},F)$, we establish the following result, whose proof is deferred to Appendix E.1.1.

Lemma E-1 (Truncation)**.** *Fix a mechanism Ψ in $\mathcal{P}$, $b \geq w$, q in $(0,1)$ and let*

$$\Psi_b(v) \quad = \quad \begin{cases} \Psi(v) & \text{if } v \in [0,b), \\ 1 & \text{if } v \geq b. \end{cases}$$

Then for any distribution F in $\mathcal{F}_\alpha(w,q)$,

$$R(\Psi_b,F) \geq R(\Psi,F) - \frac{1}{q\left(1+(q^{-1}-1)b/w\right)}\mathbf{1}\{b \leq \overline{r}_\alpha(w,q)\}.$$

In particular, the result upper bounds the maximal performance losses that can stem from truncating a pricing mechanism at b.

To uniformly bound the the impact of discretization $R(\Psi_{a_N},F) - R(\Psi_{\mathbb{A}},F)$, we first establish a result (whose proof is deferred to Appendix E.1.1) that bounds the performance losses stemming from transferring mass locally in a mechanism.

Lemma E-2 (Local transfer of mass)**.** *Fix a mechanism Ψ in $\mathcal{P}$, $0 < \epsilon < v$, and let*

$$\Psi_{\epsilon,v}(x) \quad = \quad \begin{cases} \Psi(x) & \text{if } x \in [0, v-\epsilon), \\ \Psi(v) & \text{if } x \in [v-\epsilon, v), \\ \Psi(x) & \text{if } x \geq v. \end{cases}$$

Then, for any distribution F in $\mathcal{F}_\alpha(w,q)$

$$R(\Psi_{\epsilon,v},F) \geq R(\Psi,F) - \frac{\epsilon}{v}\left(\Psi(v) - \Psi(v-\epsilon)\right).$$

Applying lemma E-2 on $(v,\epsilon) = (a_i, a_i - a_{i-1})$, $N-1$ times consecutively for $2 \leq i \leq N$ on the mechanism Ψ_{a_N}, we obtain

$$\begin{aligned} R(\Psi_{a_N},F) - R(\Psi_{\mathbb{A}},F) \quad &\leq \quad \sum_{i=2}^{N} \frac{a_i - a_{i-1}}{a_i}\left(\Psi(a_i) - \Psi(a_{i-1})\right) \\ &\overset{(a)}{\leq} \quad \frac{\Delta(\mathbb{A})}{a_1}\sum_{i=2}^{N}\left(\Psi(a_i) - \Psi(a_{i-1})\right) = \frac{\Delta(\mathbb{A})}{a_1}\left(\Psi(a_N) - \Psi(a_1)\right) \leq \frac{\Delta(\mathbb{A})}{a_1}. \end{aligned}$$

where (a) follows from $a_i - a_{i-1} \leq \sup_i(a_i - a_{i-1}) = \sigma(\mathbb{A})$ and $a_i \geq a_1 > 0$. Using Lemma E-1, we have

$$R(\Psi_b, F) \geq R(\Psi, F) - \frac{1}{q\,(1 + (q^{-1} - 1)a_N/w)} \mathbf{1}\{a_N \leq \overline{r}_\alpha(w, q)\}.$$

Returning to the decomposition in (E-1), we have established

$$R(\Psi, F) - R(\Psi_{\mathbb{A}}, F) \quad \leq \quad \frac{\Delta(\mathbb{A})}{a_1} + \frac{1}{q(1 + (q^{-1} - 1)a_N/w)} \mathbf{1}\{a_N \leq \overline{r}_\alpha(w, q)\}.$$

Noting that the inequality above applies for any F in $\mathcal{F}_\alpha(w, q)$ and that the mechanism $\Psi_{\mathbb{A}}$ does not depend on F, the result follows. □

Proof of Theorem 3. This result is a special case of Theorem 4.

E.1.1 Proofs of auxiliary results

Proof of Lemma E-1. Let $Rev(q) = qF^{-1}(1 - q)$ denote the revenue curve of associated with F in the quantity space. Let r_F denote the optimal oracle price, q_F the corresponding quantity, and recall, from Lemma C-5 that $r_F \leq \overline{r}_\alpha(w, q)$. By definition, we have

$$R(\Psi_b, F) \quad = \quad R(\Psi, F) + \int_b^{+\infty} \frac{Rev(q_b) - Rev(q_x)}{\text{opt}(F)} d\Psi(x).$$

Case 1. Suppose first $b > \overline{r}_\alpha(w, q)$. In this case, then $r_F \leq b$. Given that F is regular, the revenue curve is monotone for $q \leq q_b$, and we have $Rev(q_b) - Rev(q_x)$ for $x \geq b$. We then have

$$R(\Psi_b, F) \geq R(\Psi, F).$$

Case 2. Suppose now that $b \leq \overline{r}_\alpha(w, q)$. In this case, we divide the analysis into two subcases.

Case a). Suppose first that $r_F \leq b$. We have for any $x \geq b \geq r$, by monotonicity of the revenue curve, $Rev(q_b) - Rev(q_x) \geq 0$, and therefore

$$R(\Psi_b, F) \geq R(\Psi, F).$$

Case b). $r_F > b$ We have:

$$\begin{aligned} R(\Psi_b, F) \quad &\geq \quad R(\Psi, F) + \int_b^{+\infty} \frac{Rev(q_b) - Rev(q_x)}{\text{opt}(F)} d\Psi(x) \\ &\geq \quad R(\Psi, F) + \int_b^{+\infty} \left(\frac{Rev(q_b)}{\text{opt}(F)} - 1 \right) d\Psi(x) \\ &\geq \quad R(\Psi, F) + \left(\frac{Rev(q_b)}{\text{opt}(F)} - 1 \right). \end{aligned} \tag{E-1}$$

Recall that by assumption $b \geq w$ and hence $q_b \leq q$. Using concavity of the revenue curve in the quantity space (which follows from regularity of F), we have

$$Rev(q_b) \quad \geq \quad Rev(q_F) + \frac{Rev(q) - Rev(q_F)}{q - q_F}(q_b - q_F).$$

This implies that

$$\frac{Rev(q_b)}{Rev(q_F)} \quad \geq \quad 1 + \left(\frac{Rev(q)}{Rev(q_F)} - 1\right)\frac{q_b - q_F}{q - q_F} \;\geq\; \frac{q - q_b}{q - q_F} + \frac{Rev(q)}{Rev(q_F)}\frac{q_b - q_F}{q - q_F} \;\geq\; \frac{q - q_b}{q - q_F}.$$

Noting that F is regular and using Lemma 1, we have

$$q_b \leq \Gamma_0\left(\Gamma_0^{-1}(q)\,\frac{b}{w}\right) = \frac{1}{1 + (q^{-1} - 1)b/w}.$$

Therefore

$$\frac{Rev(q_b)}{\mathrm{opt}(F)} \quad \geq \quad \frac{q}{q - q_F}\left(1 - \frac{1}{q\left(1 + \left(q^{-1} - 1\right)b/w\right)}\right) \geq 1 - \frac{1}{q\left(1 + \left(q^{-1} - 1\right)b/w\right)}.$$

Returning to (E-1), we deduce

$$R(\Psi_b, F) \geq R(\Psi, F) - \frac{1}{q(1 + (q^{-1} - 1)b/w)}.$$

Combining both cases, the result follows. □

Proof of Lemma E-2. Let r_F denote the optimal oracle price, q_F the corresponding quantity. We have

$$R(\Psi_{\epsilon,v}, F) \quad = \quad R(\Psi, F) + \int_{v-\epsilon}^{v} \frac{Rev(q_{v-\epsilon}) - Rev(q_x)}{\mathrm{opt}(F)} d\Psi(x).$$

Case 1. Suppose $r_F \leq v - \epsilon$. In this case, using the regularity of F and the unimodality of the revenue curve, we have for any $x \geq v - \epsilon \geq r_F$, $Rev(q_{v-\epsilon}) - Rev(q_x) \geq 0$, and

$$R(\Psi_{\epsilon,v}, F) \geq R(\Psi, F).$$

Case 2. Suppose now $v - \epsilon < r_F \leq v$. In this case, we have

$$\begin{aligned}
R(\Psi_{\epsilon,v}, F) \quad &= \quad R(\Psi, F) + \int_{v-\epsilon}^{v} \frac{Rev(q_{v-\epsilon}) - Rev(q_x)}{opt(F)} d\Psi(x) \\
&\geq \quad R(\Psi, F) + \int_{v-\epsilon}^{v} \left(\frac{Rev(q_{v-\epsilon})}{\mathrm{opt}(F)} - 1\right) d\Psi(x) \\
&= \quad R(\Psi, F) + \left(\frac{Rev(q_{v-\epsilon})}{\mathrm{opt}(F)} - 1\right)(\Psi(v) - \Psi(v - \epsilon)).
\end{aligned}$$

In this case, we have $\text{opt}(F) = r_F q_F \leq v q_{v-\epsilon}$. Therefore

$$
\begin{aligned}
R(\Psi_{\epsilon,v}, F) &\geq R(\Psi, F) + \left(\frac{v-\epsilon}{v} - 1\right)(\Psi(v) - \Psi(v-\epsilon)) \\
&\geq R(\Psi, F) - \frac{\epsilon}{v}(\Psi(v) - \Psi(v-\epsilon)).
\end{aligned}
$$

Case 3. Suppose now $v < r_F$. In this case, for any $v - \epsilon \leq x \leq v < r_F$, by monotonicity of the revenue curve, $Rev(q_{v-\epsilon}) - Rev(q_x) \leq 0$, and furthermore, $Rev(q_x) \leq Rev(q_v) \leq \text{opt}(F)$. In turn, we have

$$
\begin{aligned}
R(\Psi_{\epsilon,v}, F) &= R(\Psi, F) + \int_{v-\epsilon}^{v} \frac{Rev(q_{v-\epsilon}) - Rev(q_x)}{\text{opt}(F)} d\Psi(x) \\
&\geq R(\Psi, F) + \int_{v-\epsilon}^{v} \left(\frac{Rev(q_{v-\epsilon})}{Rev(q_v)} - \frac{Rev(q_x)}{Rev(q_v)}\right) d\Psi(x) \\
&\geq R(\Psi, F) + \int_{v-\epsilon}^{v} \left(\frac{Rev(q_{v-\epsilon})}{Rev(q_v)} - 1\right) d\Psi(x) \\
&= R(\Psi, F) + \left(\frac{Rev(q_{v-\epsilon})}{Rev(q_v)} - 1\right)(\Psi(v) - \Psi(v-\epsilon)) \\
&= R(\Psi, F) + \left((1 - \frac{\epsilon}{v})\frac{q_{v-\epsilon}}{q_v} - 1\right)(\Psi(v) - \Psi(v-\epsilon)) \\
&\geq R(\Psi, F) + \left((1 - \frac{\epsilon}{v}) - 1\right)(\Psi(v) - \Psi(v-\epsilon)) \\
&= R(\Psi, F) - \frac{\epsilon}{v}(\Psi(v) - \Psi(v-\epsilon)).
\end{aligned}
$$

Combining the three cases yields the result. □

E.2 Proofs for Section 5.2

Proof of Proposition 3. The proof is divided into two steps. In the first step, we will show the lower bound by analyzing the performance of a specific mechanism. Then in a second step, we will derive the upper through the analysis of a family of hard cases when q is close to 0.

Step 1: Lower bound Let us define the following measure:

$$
d\Psi(u) = \begin{cases} 0 & \text{if } u < wq, \\ \frac{1}{u\log(\frac{1}{q})} & \text{if } u \text{ in } [qw, w) \\ 0 & \text{if } u \geq w. \end{cases}
$$

We have that $\Psi(u)$ is a distribution since

$$
\int_0^\infty d\Psi(u) = \frac{1}{\log(\frac{1}{q})} \int_{wq}^{w} \frac{1}{u} du = 1,
$$

Using Theorem 1 and the fact that $\mathcal{F}_\alpha(w,q) \subseteq \mathcal{F}_0(w,q)$ for any $\alpha \in [0,1]$, we have

$$\begin{aligned}
\inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi,F) &\geq \inf_{F\in\mathcal{F}_0(w,q)} R(\Psi,F) \\
&= \min\Bigg\{ \inf_{x\in[\underline{r}_0(w,q),w)} \frac{1}{\mathrm{opt}(F_0(\cdot|x,(w,q)))} \int_0^\infty u\overline{F}_0(u|x,(w,q))d\Psi(u), \\
&\qquad \inf_{x\in[w,\overline{r}_0(w,q)]} \frac{1}{\mathrm{opt}(F_0(\cdot|x,(w,q)))} \int_0^\infty u\overline{F}_0(u|x,(w,q)))d\Psi(u) \Bigg\} \\
&= \min\Bigg\{ \inf_{x\in[wq,w)} \frac{1}{x}\left[\int_0^x ud\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right], \\
&\qquad \inf_{x\in[w,\infty)} \frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))} \int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) \Bigg\}.
\end{aligned}$$

We will analyze each term separately depending whether x in $[wq,w)$ or x in $[w,\infty)$.

Case 1: x in $[wq,w)$ We have

$$\begin{aligned}
&\frac{1}{x}\left[\int_0^x ud\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right] \\
=\;& \frac{1}{x\log(\frac{1}{q})}\left[\int_{wq}^x udu + \int_x^w \overline{G}_{0,w}(u|(x,1),(w,q))du\right] \\
=\;& \frac{1}{x\log(\frac{1}{q})}\left[x - wq + \int_x^w \frac{1}{1+(1/q-1)\frac{u-x}{w-x}}du\right], \\
=\;& \frac{1}{x\log(\frac{1}{q})}\left(x - wq + \left[(w-x)\frac{\log(1+(1/q-1)\frac{u-x}{w-x})}{(1/q-1)}\right]_{u=x}^{u=w}\right), \\
=\;& \frac{1}{x\log(\frac{1}{q})}\left(x - wq + \log(\frac{1}{q})\frac{w-x}{(1/q-1)}\right),
\end{aligned}$$

Hence we get that

$$\begin{aligned}
&\frac{1}{x}\left[\int_0^x ud\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right] \\
=\;& \frac{1}{\log(\frac{1}{q})}\left(1 + \frac{wq}{x}(\frac{\log(q)}{q-1} - 1) - \frac{\log(\frac{1}{q})}{(1/q-1)}\right) \\
\overset{(a)}{\geq}\;& \frac{1}{\log(\frac{1}{q})}\left(1 + \frac{wq}{w}(\frac{\log(q)}{q-1} - 1) - \frac{\log(\frac{1}{q})}{(1/q-1)}\right) = \frac{1-q}{\log(\frac{1}{q})},
\end{aligned}$$

where (a) is due to the fact that $\log(q) \leq q-1 \leq 0$, and $x \leq w$.

Hence we conclude that

$$\frac{1}{x}\left[\int_0^x ud\Psi(u)+\int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right]\geq\frac{1-q}{\log(\frac{1}{q})}. \tag{E-1}$$

Case 2: x in $[w,\infty)$ Let us now analyze the second term, we have

$$\begin{aligned}
&\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u)\\
=\;&\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\left[\int_0^w u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u)+\int_w^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u)\right]\\
=\;&\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))\log(\frac{1}{q})}\int_{wq}^w\frac{1}{1+(\frac{1}{q}-1)\frac{u}{w}}du=\frac{\left[w\log(1+(\frac{1}{q}-1)\frac{u}{w})\right]_{u=wq}^{u=w}}{x\overline{G}_{0,x}(x|(0,1),(w,q))(\frac{1}{q}-1)\log(\frac{1}{q})}\\
=\;&\frac{w}{(\frac{1}{q}-1)\log(\frac{1}{q})}(\log(\frac{1}{q})-\log(2-q))\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\\
=\;&\frac{w}{(\frac{1}{q}-1)}(1-\frac{\log(2-q)}{\log(\frac{1}{q})})\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}.
\end{aligned}$$

The revenue function $x\to x\overline{G}_{0,x}(x|(0,1),(w,q))$ is non-decreasing in $[0,+\infty)$, therefore

$$x\overline{G}_{0,x}(x|(0,1),(w,q))\leq\lim_{x\to+\infty}x\overline{G}_{0,x}(x|(0,1),(w,q))=\frac{w}{(\frac{1}{q}-1)}.$$

Hence

$$\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u)\quad\geq\quad(1-\frac{\log(2-q)}{\log(\frac{1}{q})}), \tag{E-2}$$

By combining (E-1) and (E-1), we get that

$$\inf_{F\in\mathcal{F}_\alpha(w,q)}R(\Psi,F)\geq\min\left(\frac{1-q}{\log(\frac{1}{q})},1-\frac{\log(2-q)}{\log(\frac{1}{q})}\right).$$

For q in $[0,1-\frac{1}{\sqrt{2}}]$, we have

$$\frac{1-q}{\log(\frac{1}{q})}\geq\frac{\frac{1}{\sqrt{2}}}{\log(\frac{1}{q})}\geq\frac{\log(2)}{\log(\frac{1}{q})}$$

and

$$1-\frac{\log(2-q)}{\log(\frac{1}{q})}=\frac{\log(\frac{1}{q(2-q)})}{\log(\frac{1}{q})}=\frac{\log(\frac{1}{1-(1-q)^2})}{\log(\frac{1}{q})}\geq\frac{\log(\frac{1}{1-(\frac{1}{\sqrt{2}})^2})}{\log(\frac{1}{q})}=\frac{\log(2)}{\log(\frac{1}{q})}.$$

Hence we we get that

$$\inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi,F)\geq\frac{\log(2)}{\log(\frac{1}{q})}.$$

This conclude the lower bound.

Step 2: Upper bound Let q in $(0,1)$ and K in $\mathbb{N}^*$. Define $\varepsilon=q^{\frac{1}{K}}$ in $(q,1)$ and $a_k=w\varepsilon^k$ in $[qw,w)$ for $k=1\cdots K$. Consider the family of distributions $F_0(\cdot|a_k,(w,q))$ in $\mathcal{F}_0(w,q)$.

Using Yao's principle (Yao, 1977), we have

$$\begin{aligned}\sup_{\Psi\in\mathcal{P}}\inf_{F\in\mathcal{F}_0(w,q)} R(\Psi,F) &\leq \sup_{p\geq 0}\frac{1}{K}\sum_{i=1}^{K}\frac{p\bar{F}_0(p|a_i,(w,q))}{\text{opt}(F_0(\cdot|a_i,(w,q)))}\\ &= \frac{1}{K}\max_{1\leq k\leq K}\sup_{p\in[a_{k+1},a_k)}\sum_{i=1}^{K}\frac{p\bar{F}_0(p|a_i,(w,q))}{\text{opt}(F_0(\cdot|a_i,(w,q)))}.\end{aligned}\tag{E-3}$$

Now let's analyze the sup on each interval $[a_{k+1},a_k)$. For all $1\leq k\leq K$, the revenue curve associated with $\bar{F}_0(\cdot|a_i,(w,q))$ is monotone non-increasing on $[a_i,w)$ as the optimal reserve price of $F_0(\cdot|a_i,(w,q))$ is a_i.

Furthermore, the revenue curve is convex in $[0,w)$ as we have:

$$\left(p\bar{F}_0(p|a_i,(w,q))\right)' = \begin{cases}1 & \text{if } p\leq a_i\\ \frac{w(1-\frac{a_i}{wq})}{(w-a_i)(1+(\frac{1}{q}-1)\frac{p-a_i}{w-a_i})^2} & \text{if } p<w\\ 0 & \text{if } p\geq w.\end{cases}$$

Therefore the derivative of the revenue function $p\bar{F}_0(p|a_i,(w,q))$ is non-decreasing because $a_i\geq wq$ and $p\mapsto\frac{1}{(w-a_i)(1+(\frac{1}{q}-1)\frac{p-a_i}{w-a_i})^2}$ is non-increasing. Hence the function

$$p\mapsto\sum_{i=1}^{K}\frac{p\bar{F}_0(p|a_i,(w,q))}{\text{opt}(F_0(\cdot|a_i,(w,q)))}$$

is convex on $[a_{k+1},a_k)$. Thus, the sup on an interval must be attained at one of the extreme points of the interval.

Therefore by (E-3), we have that

$$\sup_{\Psi\in\mathcal{P}} \inf_{F\in\mathcal{F}_0(w,q)} R(\Psi,F) \le \sup_{p\ge 0} G(p) \quad = \quad \frac{1}{K}\max_{1\le k\le K}\sum_{i=1}^{K}\frac{a_k\bar{F}_0(a_k|a_i,(w,q))}{\mathrm{opt}(F_0(\cdot|a_i,(w,q)))}. \tag{E-4}$$

Now let us analyze the elementary term, $\frac{a_k\bar{F}_0(a_k|a_i,(w,q))}{\mathrm{opt}(F_0(\cdot|a_i,(w,q)))}$ for any i,k. Note that $\mathrm{opt}(F_0(\cdot|a_i,(w,q)) = a_i$ by Lemma C-4.

There are two cases of interest either $i\le k$ or $i>k$, let us analyze each case separately.

Case 1, $i\le k$: We have $\bar{F}_0(a_k|a_i,(w,q)) = 1$, thus

$$\frac{a_k\bar{F}_{0,a_i,w,q}(a_k)}{\mathrm{opt}(F_{0,a_i,w,q})} = \frac{a_k}{a_i} = \varepsilon^{k-i},$$

which implies that

$$\sum_{i=1}^{k}\frac{a_k\bar{F}_0(a_k|a_i,(w,q))}{\mathrm{opt}(F_0(\cdot|a_i,(w,q)))} = \sum_{i=1}^{k}\varepsilon^{k-i} = \sum_{i=0}^{k-1}\varepsilon^{i} = \frac{1-\varepsilon^k}{1-\varepsilon} \le \frac{1}{1-\varepsilon}.$$

Hence we conclude that

$$\sum_{i=1}^{k}\frac{a_k\bar{F}_0(a_k|a_i,(w,q))}{\mathrm{opt}(F_0(\cdot|a_i,(w,q)))} \le \frac{1}{1-\varepsilon}. \tag{E-5}$$

Case 2, $i\ge k$: We have that

$$\begin{aligned}
\frac{a_k\bar{F}_0(a_k|a_i,(w,q))}{\mathrm{opt}(F_0(\cdot|a_i,(w,q)))} = \frac{a_k}{a_i}\frac{1}{1+(\frac{1}{q}-1)\frac{a_k-a_i}{w-a_i}} \quad &= \quad \varepsilon^{k-i}\frac{1-\varepsilon^i}{1-\varepsilon^i+\frac{1}{q}(\varepsilon^k-\varepsilon^i)-\varepsilon^k+\varepsilon^i} \\
&= \quad \varepsilon^{k-i}\frac{1-\varepsilon^i}{1-\varepsilon^k+\frac{1}{q}(\varepsilon^k-\varepsilon^i)} \\
&= \quad \frac{\varepsilon^{-i}-1}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon^{i-k})} \\
&\le \quad \frac{\varepsilon^{-i}-1}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon)},
\end{aligned}$$

where in the last inequality we used the fact that $\varepsilon\le 1$.

Hence we conclude that

$$\begin{aligned}\sum_{i=k+1}^{K} \frac{a_k \bar{F}_0(a_k|a_i,(w,q))}{\text{opt}(F_0(\cdot|a_i,(w,q)))} &\leq \sum_{i=k+1}^{K} \frac{\varepsilon^{-i}-1}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon)} \\ &\overset{(a)}{\leq} \frac{1}{\frac{1}{\varepsilon}-1} \frac{\frac{1}{\varepsilon^{K+1}}-\frac{1}{\varepsilon^{k+1}}}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon)} \\ &\leq \frac{1}{1-\varepsilon} \frac{\frac{1}{\varepsilon^{K}}-\frac{1}{\varepsilon^{k}}}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon)} \\ &\overset{(b)}{\leq} \frac{1}{1-\varepsilon} \frac{\frac{1}{q}}{(\varepsilon^{-k}-1)+\frac{1}{q}(1-\varepsilon)},\end{aligned}$$

where in (a) we used $\varepsilon^{-i}-1 \leq \varepsilon^{-i}$ and (b) we used the fact that $\varepsilon \geq 0$ and that $\varepsilon^K = q$. From the last inequality we conclude that

$$\sum_{i=k+1}^{K} \frac{a_k \bar{F}_0(a_k|a_i,(w,q))}{\text{opt}(F_0(\cdot|a_i,(w,q)))} \leq \frac{1}{1-\varepsilon} \frac{1}{q(\varepsilon^{-k}-1)+(1-\varepsilon)} \leq \frac{1}{(1-\varepsilon)^2}. \tag{E-6}$$

By combining the last two cases, in particular (E-4), (E-5) and (E-6), we get that

$$\sup_{\Psi \in \mathcal{P}} \inf_{F \in \mathcal{F}_0(w,q)} R(\Psi, F) \leq \frac{1}{K}\left(\frac{1}{1-\varepsilon}+\frac{1}{(1-\varepsilon)^2}\right).$$

By choosing $K = \log(1/q)$, thus $\varepsilon = e^{-1}$, we get:

$$\sup_{\Psi \in \mathcal{P}} \inf_{F \in \mathcal{F}_0(w,q)} R(\Psi, F) \leq \frac{c_2}{\log(1/q)} \text{ with } c_2 = \frac{1}{1-e^{-1}}\left(1+\frac{1}{1-e^{-1}}\right)$$

This concludes the proof. □

Proof of Proposition 4. This proof is divided into two steps. In the first step, we will show the lower bound by analyzing the performance of a specific mechanism. Then in a second step, we will derive the upper through the analysis of a family of hard cases when q is close to 1.

Throughout the proof we will assume that $q \geq 3/4$ since we are interested in the limit when when q is close to 1.

Step 1: Lower bound Let us define the following measure parameterized by $a, b \geq 0$:

$$d\Psi(u) = \begin{cases} a & \text{if } u = w \\ b\frac{(u\overline{G}_{0,u}(u|(0,1),(w,q)))'}{u\overline{G}_{0,u}(u|(0,1),(w,q))} & \text{if } u > w. \end{cases}$$

Note that $d\Psi(u) \geq 0$ since the revenue function $u \to u\overline{G}_{0,u}(u|(0,1),(w,q))d\Psi(u)$ is increasing in $[w, \infty)$. Let us determine the condition on the parameters a and b so that Ψ is a distribution. For that we need the following

$$\int_0^\infty d\Psi(u) = 1,$$

which implies that

$$a + b\log\left(\frac{\lim_{u\to\infty} u\overline{G}_{0,u}(u|(0,1),(w,q))}{w\overline{G}_{0,w}(w|(0,1),(w,q))}\right) = 1.$$

Since $\overline{G}_{0,w}(w|(0,1),(w,q))d\Psi(u) = q$ and $\lim_{u\to\infty} u\overline{G}_{0,u}(u|(0,1),(w,q)) = \frac{w}{\frac{1}{q}-1}$, we get that

$$a + b\log\left(\frac{1}{1-q}\right) = 1.$$

Hence the relation between a and b is as follows

$$b = \frac{1-a}{\log\left(\frac{1}{1-q}\right)} \quad \text{and } a \text{ in } [0,1].$$

Using Theorem 1, we have

$$\begin{aligned}
\inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi,F) &\geq \inf_{F\in\mathcal{F}_0(w,q)} R(\Psi,F) \\
&= \min\Bigg\{\inf_{x\in[wq,w)} \frac{1}{x}\left[\int_0^x u d\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right], \\
&\qquad \inf_{x\in[w,\infty)} \frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u)\Bigg\}.
\end{aligned} \tag{E-7}$$

We will analyze each term separately depending if x in $[wq,w)$ or x in $[w,\infty)$.

Case 1: x in $[wq,w)$ We have

$$\frac{1}{x}\left[\int_0^x u d\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right] \overset{(a)}{=} \frac{aq}{x} \geq aq,$$

where the last inequality is due to the fact that $x \leq w$ and (a) is due to $d\Psi(u) = 0$ for $u < w$, $d\Psi(w) = a$ and $\overline{G}_{0,w}(w|(x,1),(w,q)) = q$.

Hence we conclude that

$$\inf_{x\in[wq,w)} \frac{1}{x}\left[\int_0^x u d\Psi(u) + \int_x^w u\overline{G}_{0,w}(u|(x,1),(w,q))d\Psi(u)\right] \geq aq. \tag{E-8}$$

Case 2: x in $[w,\infty)$ Let us now analyze the second term, we have

$$\begin{aligned}
\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) &= \int_0^w u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) + \int_w^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) \\
&\overset{(a)}{=} aq + b\int_w^x u\overline{G}_{0,x}(u|(0,1),(w,q))\frac{(u\overline{G}_{0,u}(u|(0,1),(w,q)))'}{u\overline{G}_{0,u}(u|(0,1),(w,q))}du \\
&= awq + b\left[u\overline{G}_{0,u}(u|(0,1),(w,q))\right]_w^x \\
&\geq awq + b\left(x\overline{G}_{0,x}(x|(0,1),(w,q)) - wq\right),
\end{aligned}$$

(a) is due to $d\Psi(u) = 0$ for $u < w$, $d\Psi(w) = a$ and $w\overline{G}_{0,w}(w|(0,1),(w,q)) = wq$. Hence we conclude that

$$\begin{aligned}
\frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) &\geq wq\frac{a-b}{x\overline{G}_{0,x}(x|(0,1),(w,q))} + b \\
&\geq (a-b)(1-q) + b,
\end{aligned}$$

where the last inequality we used the fact that $x \to x\overline{G}_{0,x}(x|(0,1),(w,q))$ is non-decreasing in $[w,\infty)$ and that $\lim_{x\to\infty} x\overline{G}_{0,x}(x|(0,1),(w,q)) = \frac{wq}{1-q}$.

Thus we conclude that

$$\inf_{x \text{ in } [w,+\infty)} \frac{1}{x\overline{G}_{0,x}(x|(0,1),(w,q))}\int_0^x u\overline{G}_{0,x}(u|(0,1),(w,q))d\Psi(u) \geq (a-b)(1-q) + b. \tag{E-9}$$

By combining (E-7), (E-8) and (E-9) we get that

$$\inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi, F) \geq \min\{aq, (a-b)(1-q) + b\}.$$

Now let us set

$$a = \frac{q}{2(q-\frac{1}{2})\log\left(\frac{1}{1-q}\right) + q}.$$

Note that a in $[0,1]$ as q in $[3/4,1]$, this also leads to the fact that

$$\begin{aligned}
(a-b)(1-q) + b = aq = \frac{q^2}{2(q-\frac{1}{2})\log\left(\frac{1}{1-q}\right) + q} \overset{(a)}{\geq} \frac{9}{16}\frac{1}{2(1-\frac{1}{2})\log\left(\frac{1}{1-q}\right) + 1} &= \frac{9}{16(\log\left(\frac{1}{1-q}\right) + 1)} \\
&\overset{(b)}{\geq} \frac{9}{32}\frac{1}{\log\left(\frac{1}{1-q}\right)},
\end{aligned}$$

where inequality (a) stems from the fact q in $[3/4,1]$, and in (b) we have used $\log\left(\frac{1}{1-q}\right) \geq 1$.

Hence we we get that

$$\inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi, F) \geq \frac{9}{32} \frac{1}{\log\left(\frac{1}{1-q}\right)}.$$

This conclude the lower bound.

Step 2: Upper bound To show the upper bound we will introduce a family of "hard" cases. We consider the family of distributions $(F_0(\cdot|r,(w,q))$ in $\mathcal{F}_0(w,q))_{r \geq w}$ and the following weight distribution:

$$d\lambda(r) = \begin{cases} 0 & \text{if } r < w \\ \frac{1}{\log\left(\frac{1}{1-q}\right)} \frac{(r\overline{G}_{0,r}(r|(0,1),(w,q)))'}{r\overline{G}_{0,r}(r|(0,1),(w,q))} & \text{if } r \geq w \end{cases}$$

One can verify that $\int_0^\infty d\lambda(r) = 1$ and that $d\lambda(r) \geq 0$. Let us define

$$G(p) = \int_0^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r).$$

Using Yao's principle (Yao, 1977), we have

$$\sup_{\Psi \in \mathcal{P}} \inf_{F \in \mathcal{F}_0(w,q)} R(\Psi, F) \leq \sup_{p \geq 0} G(p) = \sup_{p \geq 0} \int_0^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r). \tag{E-10}$$

Note that

$$\begin{aligned} \sup_{p \geq 0} \int_0^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r) &= \sup_{p \geq 0} \int_w^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r) \\ &= \sup_{p \geq w} \int_w^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r), \end{aligned}$$

where the last equality follows from the fact that $p \mapsto p\overline{F}_0(p|r,(w,q))$ is increasing on $[0,w]$ for any $r \geq w$.

Fix $p \geq w$ and let us analyze the integral term. We have

$$\begin{aligned}
&\int_w^\infty \frac{p\overline{F}_0(p|r,(w,q))}{\mathrm{opt}(F_0(\cdot|r,(w,q)))} d\lambda(r) \\
= \;& \int_p^\infty \frac{p\overline{F}_0(p|r,(w,q))}{r\overline{G}_{0,r}(r|(0,1),(w,q))} \frac{1}{\log\left(\frac{1}{1-q}\right)} \frac{(r\overline{G}_{0,r}(r|(0,1),(w,q)))'}{r\overline{G}_{0,r}(r|(0,1),(w,q))} dr \\
= \;& \frac{1}{\log\left(\frac{1}{1-q}\right)} p\overline{G}_{0,p}(p|(0,1),(w,q)) \int_p^\infty \frac{(r\overline{G}_{0,r}(r|(0,1),(w,q)))'}{(r\overline{G}_{0,r}(r|(0,1),(w,q)))^2} dr \\
= \;& \frac{1}{\log\left(\frac{1}{1-q}\right)} p\overline{G}_{0,p}(p|(0,1),(w,q)) \left(\frac{1}{p\overline{G}_{0,p}(p|(0,1),(w,q))} - \lim_{r\to\infty} \frac{1}{r\overline{G}_{0,r}(r|(0,1),(w,q))} \right) \\
= \;& \frac{1}{\log\left(\frac{1}{1-q}\right)} \left(1 - \left(\frac{1}{q} - 1\right) \frac{p\overline{G}_{0,p}(p|(0,1),(w,q))}{w}\right) \leq \frac{1}{\log\left(\frac{1}{1-q}\right)}.
\end{aligned}$$

By using the last inequality, together with (E-10), we obtain the result. □

F Proofs and auxiliary results for Section 6

Proof of Theorem 4. We aim to show that one can approximate the value of the maximin ratio via lower and upper bounds and we quantify the asymptotic error of this approximation as a function of the grid size $N > 0$.

We will do that in different steps:

- In a first step, we extend previous results to the interval uncertainty case:
  - We will first show in Proposition F-1 that in the interval uncertainty case, we reduce the family of worst case distributions by generalizing Theorem 1.
  - Under such a reduction, we then show in Proposition F-2 that we can still approximate the performance of any mechanism by its discrete version by generalizing Proposition 2.
- In a second step, we derive lower bounds on the maximin ratio in the form of linear programs.
- In a third step, we show that through an appropriate choice of the support of a discrete mechanism, one can approximate the maximin ratio arbitrarily closely through the lower bound.

Step 1. We first reduce the possible set of worst-cases to consider by extending Theorem 1. For that, let us define the following subset of distributions

$$\mathcal{S}_{\alpha,w,q_l,q_h} = \{F_\alpha(\cdot|r,(w,q_l)) : r \text{ in } [\underline{r}_\alpha(w,q_l), w)\} \cup \{F_\alpha(\cdot|r,(w,q_h)) : r \text{ in } [w, \overline{r}_\alpha(w,q_h)]\}. \quad \text{(F-1)}$$

where we use the convention that whenever $\overline{r}_\alpha(w,q_h) < w$, $[w, \overline{r}_\alpha(w,q_h)] := \emptyset$. We have the following result, whose proof is deferred to Appendix F.1.

Proposition F-1. *For any q_l, q_h in $(0,1)^2$ such that $q_l \leq q_h$, and for any subset of mechanisms $\mathcal{P}' \subseteq \mathcal{P}$,*

$$\mathcal{R}(\mathcal{P}', \mathcal{F}_\alpha(w,[q_l,q_h])) = \mathcal{R}(\mathcal{P}', \mathcal{S}_{\alpha,w,q_l,q_h}).$$

In addition, the next proposition generalizes Proposition 2, and its proof is deferred to Appendix F.1.

Proposition F-2. *Let q_l, q_h in $(0,1)^2$ such that $q_l \leq q_h$. Fix a mechanism Ψ in $\mathcal{P}$, $N > 1$, and any finite sequence of increasing reals $\mathbb{A} = \{a_i\}_{i=0}^N$ such that $a_0 = \underline{r}_\alpha(w, q_l), a_N \geq w$. Then there exists $\Psi_\mathbb{A}$ in $\mathcal{P}_\mathbb{A}$ such that*

$$\inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi_\mathbb{A}, F) \geq \inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) - \frac{\Delta(\mathbb{A})}{\underline{r}_\alpha(w, q_l)} - \frac{\mathbf{1}\{a_N < \overline{r}_\alpha(w, q_h)\}}{q_h(1 + (q_h^{-1} - 1)a_N/w)},$$

where $\Delta(\mathbb{A}) = \sup_i\{a_i - a_{i-1}\}$.

Step 2. Fix an arbitrary sequence of increasing reals $\mathbb{A} = \{a_i\}_{i=0}^{2N+1}$ such that $a_0 = \underline{r}_\alpha(w, q_l)$, $a_{N+1} = w$ and $a_{2N+1} \leq \overline{r}_\alpha(w, q_h)$. Set $a_{2N+2} := \overline{r}_\alpha(w, q_h)$. Note that $\overline{r}_\alpha(w, q_h) = \infty$ when $\alpha = 0$. With some abuse of notation, we will use intervals that include $\overline{r}_\alpha(w, q_h)$. These should be interpreted as open when $\alpha = 0$.

We next develop a lower bound on the maximin ratio $\mathcal{R}(\mathcal{P}_\mathbb{A}, \mathcal{F}_\alpha(w, [q_l, q_h]))$ in the form of a linear program. Fix a mechanism Ψ in $\mathcal{P}_\mathbb{A}$ and denote by $p_0, ..., p_{2N+1}$ the corresponding probabilities. We set $p_{2N+2} := 0$. Using proposition F-1. Then we have:

$$\begin{aligned}
&\inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) \\
= \; & \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l), w)} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_l)) d\Psi(u), \\
& \inf_{x \in [w, \overline{r}_\alpha(w,q_h)]} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_h))) d\Psi(u) \Bigg\} \\
= \; & \min\Bigg\{ \min_{i=0,\cdots,N} \inf_{x \in [a_i, a_{i+1})} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_l)) d\Psi(u), \\
& \min_{i=N+1,\cdots,2N+1} \inf_{x \in [a_i, a_{i+1}]} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_h))) d\Psi(u) \Bigg\}.
\end{aligned}$$

Note that, for $x \in [\underline{r}_\alpha(w, q_l), w)$, $\overline{F}_\alpha(\cdot|x, (w, q_l))$ is non-decreasing in x and that the revenue function $u \mapsto u\overline{F}_\alpha(\cdot|x, (w, q_l))$ is increasing in u on $[0, x)$ and decreasing on (x, w). In addition, note that, for $x \in [w, \overline{r}_\alpha(w, q_h)]$, the revenue function $u \mapsto u\overline{F}_\alpha(u|x, (w, q_h))$ is non-decreasing on $[0, x]$. We let $\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_l))) = \lim_{x \to a_{i+1}^-} \mathrm{opt}(F_\alpha(\cdot|x, (w, q_l)))$ for any $i = 0, \cdots, N$. Hence, we have

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) \; \geq \; & \min\Bigg\{ \min_{i=0,\cdots,N} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|a_i, (w, q_l)) d\Psi(u), \\
& \min_{i=N+1,\cdots,2N+1} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|a_i, (w, q_h)) d\Psi(u) \Bigg\} \\
= \; & \min\Bigg\{ \min_{i=0,\cdots,N} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_l)))} \sum_{j=0}^{2N+1} a_j\overline{F}_\alpha(a_j|a_i, (w, q_l)) p_j, \\
& \min_{i=N+1,\cdots,2N+1} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}, (w, q_h)))} \sum_{j=0}^{2N+1} a_j\overline{F}_\alpha(a_j|a_i, (w, q_h)) p_j \Bigg\}, \quad \text{(F-2)}
\end{aligned}$$

where the equality simply stems from the fact that Ψ in $\mathcal{P}_{\mathbb{A}}$. The problem of maximizing over mechanisms in $\mathcal{P}_{\mathbb{A}}$ is clearly lower bounded by the problem of maximizing the RHS above over $p_0, ..., p_{2N+1}$. The latter problem admits exactly $\mathcal{LP}$-int as its epigraph formulation, and hence we have

$$\mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w, [q_l, q_h])) \quad \geq \quad \underline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}}.$$

Step 3. We next establish that with a proper choice of sequence $\mathbb{A}$, $\underline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}}$ may be arbitrarily close to the maximin ratio $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w, [q_l, q_h]))$. To do so, we will first develop an upper bound on $\mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w, [q_l, q_h]))$. Then, we will construct a particular sequence $\mathbb{A}$ and establish for this sequence, the gap between $\mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w, [q_l, q_h]))$ and $\underline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}}$ is small and that the gap between $\mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w, [q_l, q_h]))$ and $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w, [q_l, q_h]))$ is also small. This will yield the result.

Suppose that $a_0 > 0$. Following the same reasoning as in step 2 above, we may also obtain an upper bound on $\inf_{F \in \mathcal{F}_\alpha(w, [q_l, q_h])} R(\Psi, F)$. Indeed, we have

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w, [q_l, q_h])} R(\Psi, F) \quad &\leq \quad \min \Bigg\{ \min_{i=0,\cdots,N} \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|a_{i+1}^-, (w, q_l)) d\Psi(u), \\
&\qquad \min_{i=N+1,\cdots,2N+1} \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|a_{i+1}^-, (w, q_h)) d\Psi(u) \Bigg\} \\
&= \quad \min \Bigg\{ \min_{i=0,\cdots,N} \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-, (w, q_l)) p_j, \\
&\qquad \min_{i=N+1,\cdots,2N+1} \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-, (w, q_h)) p_j \Bigg\}.
\end{aligned}$$

With $u\overline{F}_\alpha(u|a_{i+1}^-, (w, q_l)) = \lim_{x \to a_{i+1}^-} u\overline{F}_\alpha(u|x, (w, q_l))$ for any $u \geq 0$ and $i = 0, \cdots, 2N+1$. The problem of maximizing over mechanisms in $\mathcal{P}_{\mathbb{A}}$ is clearly upper bounded by the problem of maximizing the RHS above over $p_0, ..., p_{2N+1}$. The epigraph formulation of the latter problem can be written as

$$\begin{aligned}
\overline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}} \quad &= \quad \max_{\mathbf{p}, c} \ c \qquad\qquad (\mathcal{LP}\text{-int-up}) \\
&s.t. \quad \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-, (w, q_l)) p_j \geq c \quad i = 0, ...N, \\
&\qquad \frac{1}{\text{opt}(F_\alpha(\cdot|a_i, (w, q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-, (w, q_h)) p_j \geq c \quad i = N+1, ...2N+1, \\
&\qquad \sum_{j=0}^{2N+1} p_j \leq 1, \quad p_i \geq 0 \quad i = 0, ...2N+1.
\end{aligned}$$

Therefore, we have

$$\mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w, [q_l, q_h])) \quad \leq \quad \overline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}}.$$

Hence, we have established the following.

$$\underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w,[q_l,q_h])) \leq \overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}}.$$

We next quantify the gap $\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}}$ as a function the discretization grid size N for a particular sequence. For $N > 1$, $b = \overline{r}_\alpha(w, q_h)$ if $\alpha \in (0,1]$, $b > w$ if $\alpha = 0$ and η in $(0, \underline{r}_\alpha(w, q_l))$, consider the following finite sequence of prices $\mathbb{A} = \{a_i\}_{i=0}^{2N+1}$ in $[\underline{r}_\alpha(w, q_l), \min\{b, \overline{r}_\alpha(w, q_h)\}]$:

$$a_i \;=\; \begin{cases} \underline{r}_\alpha(w,q_l) + \frac{i}{N}\left((w-\eta) - \underline{r}_\alpha(w,q_l)\right) & \text{if } 0 \leq i \leq N, \\ w + \frac{i-(N+1)}{N}\left(\min\{b, \overline{r}_\alpha(w,q_h)\} - w\right) & \text{if } N+1 \leq i \leq 2N+1. \end{cases}$$

When fixing the probability weights $\mathbf{p}$, let $\underline{c}(\mathbf{p})$ denote the maximum value achievable (as a function of c) in the inner problem in ($\mathcal{LP}$-int). In particular, it can be expressed as the minimum in (F-2).

Let $\mathbf{p}$ correspond be a probability weight vector corresponding to an optimal solution to the upper bound linear program ($\mathcal{LP}$-int-up). We have

$$\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \;\leq\; \overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{c}(\mathbf{p}).$$

We next analyze upper bound the gap $\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{c}(\mathbf{p})$ as a function of the constraints that lead to the minimum value when solving $\underline{c}(\mathbf{p})$.

Case 1: If $\underline{c}(\mathbf{p}) = \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_l))p_j$ for some $0 \leq i \leq N-1$. Then we have

$$\begin{aligned}
\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{c}(\mathbf{p}) \;&=\; \overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_l))p_j \\
&\leq\; \frac{1}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l))p_j \\
&\quad - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_l))p_j \\
&=\; \left(\frac{1}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_l)))} - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\right) \sum_{j=0}^{2N} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l))p_j \\
&\quad + \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))} \sum_{j=0}^{2N+1} a_j \left[\overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l)) - \overline{F}_\alpha(a_j|a_i,(w,q_l))\right] p_j \\
&=\; \left(\frac{a_{i+1}-a_i}{a_i}\right) \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{i+1},(w,q_l))}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_l)))} p_j \\
&\quad + \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_l)))} \sum_{j=i+1}^{N+1} a_j \left[\overline{F}_\alpha(a_j|a_{i+1},(w,q_l)) - \overline{F}_\alpha(a_j|a_i,(w,q_l))\right] p_j,
\end{aligned}$$

where in the last equality, we have used that $a_{i+1} = \text{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_l)))$ for $0 \le i \le N-1$ (cf. Lemma C-2), $\overline{F}_\alpha(\cdot|a_{i+1}^-, (w, q_l)) = \overline{F}_\alpha(\cdot|a_{i+1}, (w, q_l))$ for $0 \le i \le N-1$ and the fact that $\overline{F}_\alpha(\cdot|a_{i+1}, (w, q_l)) = \overline{F}_\alpha(\cdot|a_i, (w, q_l))$ on $[0, a_i]$ and on $(w, +\infty)$. We analyze the two terms on the RHS above separately.

$$\left(\frac{a_{i+1} - a_i}{a_i}\right) \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{i+1}, (w, q_l))}{\text{opt}(F_\alpha(\cdot|a_{i+1}, (w, q_l)))} p_j \;\le\; \left(\frac{a_{i+1} - a_i}{a_i}\right) \sum_{j=0}^{2N+1} p_j \;\le\; \frac{1}{N}\left(\frac{w - \eta - \underline{r}_\alpha(w, q_l)}{\underline{r}_\alpha(w, q_l)}\right),$$

where the first inequality follows from the definition of opt, and the second from the fact that $\mathbf{p}$ belongs to the simplex, from definition and from lower bounding a_i by $a_0 = \underline{r}_\alpha(w, q_l)$.

Now, let for $j = i+1, ..., N$, $g_j(x) = \overline{F}_\alpha(a_j|x, (w, q_l))$. Note that $g_j(\cdot)$ is differentiable in $[a_i, a_j]$ with derivative bounded as follows

$$\begin{aligned} g_j'(x) &= \left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_l)\,\frac{a_j - x}{w - x}\right)\right)' \\ &= \Gamma_\alpha^{-1}(q_l)\,\frac{(w - a_j)}{(w - x)^2}\left(\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_l)\,\frac{a_j - x}{w - x}\right)\right)^{2-\alpha} \le \frac{\Gamma_\alpha^{-1}(q_l)}{w - x} \le \frac{\Gamma_\alpha^{-1}(q_l)}{w - \eta}. \end{aligned}$$

We deduce that

$$\begin{aligned} &\frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}, (w, q_l)))} \sum_{j=i+1}^{N+1} a_j \left[\overline{F}_\alpha(a_j|a_{i+1}, (w, q_l)) - \overline{F}_\alpha(a_j|a_i, (w, q_l))\right] a_j p_j \\ \le\; & \frac{1}{a_{i+1}} \sum_{j=i+1}^{N+1} \frac{\Gamma_\alpha^{-1}(q_l)}{w - \eta}(a_{i+1} - a_i) a_j p_j \\ \le\; & \frac{1}{N}\left(\frac{w - \eta - \underline{r}_\alpha(w, q_l)}{\underline{r}_\alpha(w, q_l)}\right) w \frac{\Gamma_\alpha^{-1}(q_l)}{w - \eta}. \end{aligned}$$

Hence, we have, in this case

$$\overline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}} - \underline{\mathcal{L}}_{\alpha, q_l, q_h, \mathbb{A}} \;\le\; \frac{1}{N}\left(\frac{w - \eta - \underline{r}_\alpha(w, q_l)}{\underline{r}_\alpha(w, q_l)}\right)\left[1 + w\frac{\Gamma_\alpha^{-1}(q_l)}{w - \eta}\right].$$

Case 2: Suppose $\underline{c}(\mathbf{p}) = \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-, (w, q_l)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i, (w, q_l)) p_j$ for $i = N$. In this case, we have

$$\underline{c}(\mathbf{p}) \;=\; \frac{1}{w}\left(\sum_{j=0}^{N} a_j p_j + w q_l p_{N+1}\right),$$

and

$$\begin{aligned}
\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} &\leq \frac{1}{w-\eta}\left(\sum_{j=0}^{N} a_j p_j + wq_l p_{N+1}\right) - \frac{1}{w}\left(\sum_{j=0}^{N} a_j p_j + wq_l p_{N+1}\right) \\
&\leq \left(\frac{1}{w-\eta} - \frac{1}{w}\right)\left(\sum_{j=0}^{N} a_j p_j + wq_l p_{N+1}\right) \\
&\leq \frac{\eta}{w-\eta} w \sum_{j=0}^{N+1} p_j \\
&\leq \frac{\eta w}{w-\eta}.
\end{aligned}$$

Case 3: Suppose $\underline{c}(\mathbf{p}) = \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_h))p_j$ for some $i = N+1, ..., 2N$.

$$\begin{aligned}
\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} &\leq \frac{1}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_h))p_j \\
&\quad - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_h))p_j \\
&= \left(\frac{1}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))} - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))}\right) \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_h))p_j \\
&\quad + \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))} \sum_{j=0}^{2N+1} a_j \left[\overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_h)) - \overline{F}_\alpha(a_j|a_i,(w,q_h))\right] p_j \\
&= \frac{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h))) - \text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))} \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{i+1},(w,q_h))}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))} p_j,
\end{aligned}$$

where in the last equality, we have used that $\overline{F}_\alpha(\cdot|a_{i+1}^-,(w,q_h)) = \overline{F}_\alpha(\cdot|a_i,(w,q_l))$ on $\{a_0,\ldots,a_{i+1}\}$. We analyze the above term on the RHS.

$$\begin{aligned}
&\frac{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h))) - \text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))}{\text{opt}(F_\alpha(\cdot|a_i,(w,q_h)))} \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{i+1},(w,q_h))}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))} p_j \\
&\leq \left(\frac{a_{i+1}\overline{G}_{\alpha,a_{i+1}}(a_{i+1}|(0,1),(w,q_h)) - a_i \overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}{a_i \overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}\right) \sum_{j=0}^{2N+1} p_j \\
&\leq \left(\frac{a_{i+1}\overline{G}_{\alpha,a_{i+1}}(a_{i+1}|(0,1),(w,q_h)) - a_i \overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}{a_i \overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}\right),
\end{aligned}$$

where the first inequality follows from the definition of opt and $\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))$, and the second from the fact that $\mathbf{p}$ belongs to the simplex.

Now, let $g_{\alpha,q_h}(x) = x\overline{G}_{\alpha,x}(x|(0,1),(w,q_h))$. Note that $g_{\alpha,q_h}(\cdot)$ is differentiable in $[w,\overline{r}_\alpha(w,q_h))$ with derivative bounded as follows

$$\begin{aligned}
g'_{\alpha,q_h}(x) &= \left(x\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\right)' \\
&= \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right) - \frac{\Gamma_\alpha^{-1}(q_h)\,x}{w}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)^{2-\alpha} \\
&= \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\left(1 - \frac{\Gamma_\alpha^{-1}(q_h)\,x}{w}\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)^{1-\alpha}\right) \\
&= \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\left(1 - \frac{\Gamma_\alpha^{-1}(q_h)\,x}{w\left(1+(1-\alpha)\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)}\right) \\
&= \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\left(1 - \frac{\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}{1+(1-\alpha)\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}\right) \\
&= \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\frac{1-\alpha\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}{1+(1-\alpha)\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}.
\end{aligned}$$

Therefore, since $x \leq \overline{r}_\alpha(w,q_h) := \frac{w}{\alpha\Gamma_\alpha^{-1}(q_h)}$, we have that

$$|g'_{\alpha,q_h}(x)| = \left|\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\frac{1-\alpha\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}{1+(1-\alpha)\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}\right| = \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q_h)\frac{x}{w}\right)\frac{1-\alpha\Gamma_\alpha^{-1}(q_h)\frac{x}{w}}{1+(1-\alpha)\Gamma_\alpha^{-1}(q_h)\frac{x}{w}} \leq 1.$$

We deduce that

$$\begin{aligned}
\left(\frac{a_{i+1}\overline{G}_{\alpha,a_{i+1}}(a_{i+1}|(0,1),(w,q_h)) - a_i\overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}{a_i\overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))}\right) &\leq \frac{a_{i+1}-a_i}{g_{\alpha,q_h}(a_i)} \\
&\leq \frac{\min\{b,\overline{r}_\alpha(w,q_h)\}-w}{Ng_{\alpha,q_h}(w)} \\
&= \frac{\min\{b,\overline{r}_\alpha(w,q_h)\}-w}{Nwq}.
\end{aligned}$$

Hence, we have, in this case

$$\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \frac{\min\{b,\overline{r}_\alpha(w,q_h)\}-w}{Nwq}.$$

Case 4: Suppose $\underline{c}(\mathbf{p}) = \frac{1}{\text{opt}(F_\alpha(\cdot|a_{i+1},(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_i,(w,q_h))p_j$ for $i = 2N+1$.

$$
\begin{aligned}
& \overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \\
\leq \quad & \frac{1}{\text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{2N+1}^-,(w,q_h))p_j \\
& - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h)))} \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{2N},(w,q_h))p_j \\
= \quad & \left(\frac{1}{\text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))} - \frac{1}{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h)))} \right) \sum_{j=0}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{2N+2}^-,(w,q_h))p_j \\
& + \frac{1}{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h)))} \sum_{j=0}^{2N+1} a_j \left[\overline{F}_\alpha(a_j|a_{2N+2}^-,(w,q_h)) - \overline{F}_\alpha(a_j|a_{2N+1},(w,q_h)) \right] p_j \\
= \quad & \frac{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h))) - \text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))}{\text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))} \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{2N+2}^-,(w,q_h))}{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h)))} p_j,
\end{aligned}
$$

where in the last equality, we have used the fact that $\overline{F}_\alpha(\cdot|a_{2N+2}^-,(w,q_h)) = \overline{F}_\alpha(\cdot|a_{2N+1},(w,q_l))$ on $\{a_0, \cdots, a_{2N+1}\}$.

We analyze the above term on the RHS in two separate cases $\alpha \in (0,1]$ and $\alpha = 0$.

In the case where $\alpha \in (0,1]$, we have

$$
\begin{aligned}
& \frac{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h))) - \text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))}{\text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))} \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a_{2N+2}^-,(w,q_h))}{\text{opt}(F_\alpha(\cdot|a_{2N+2}^-,(w,q_h)))} p_j \\
\leq \quad & \left(\frac{\overline{r}_\alpha(w,q_h)\overline{G}_{\alpha,\overline{r}_\alpha(w,q_h)}(\overline{r}_\alpha(w,q_h)|(0,1),(w,q_h)) - a_{2N+1}\overline{G}_{\alpha,a_{2N+1}}(a_{2N+1}|(0,1),(w,q_h))}{a_{2N+1}\overline{G}_{\alpha,a_{2N+1}}(a_{2N+1}|(0,1),(w,q_h))} \right) \sum_{j=0}^{2N+1} p_j \\
\leq \quad & \left(\frac{\overline{r}_\alpha(w,q_h)\overline{G}_{\alpha,\overline{r}_\alpha(w,q_h)}(\overline{r}_\alpha(w,q_h)|(0,1),(w,q_h)) - a_{2N+1}\overline{G}_{\alpha,a_{2N+1}}(a_{2N+1}|(0,1),(w,q_h))}{a_{2N+1}\overline{G}_{\alpha,a_i}(a_i|(0,1),(w,q_h))} \right) \\
\leq \quad & \frac{g_{\alpha,q_h}(\overline{r}_\alpha(w,q_h)) - g_{\alpha,q_h}(a_{2N+1})}{g_{\alpha,q_h}(a_{2N+1})} \leq \frac{\overline{r}_\alpha(w,q_h) - a_{2N+1}}{g_{\alpha,q_h}(w)} = \frac{\overline{r}_\alpha(w,q_h) - w}{Nwq},
\end{aligned}
$$

where the first inequality follows from the definition of opt, the second from the fact that ı belongs to the simplex, and the fourth from the fact that the derivative $g'_{\alpha,q_h}(\cdot)$ is bounded (established in the previous case).

In the case where $\alpha = 0$, we have

$$
\begin{aligned}
& \frac{\text{opt}(F_\alpha(\cdot|a^-_{2N+2},(w,q_h))) - \text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))}{\text{opt}(F_\alpha(\cdot|a_{2N+1},(w,q_h)))} \sum_{j=0}^{2N+1} \frac{a_j \overline{F}_\alpha(a_j|a^-_{2N+2},(w,q_h))}{\text{opt}(F_\alpha(\cdot|a^-_{2N+2},(w,q_h)))} p_j \\
\leq \ & \frac{\lim_{x\to\infty} \text{opt}(F_0(\cdot|x,(w,q_h))) - \text{opt}(F_0(\cdot|b,(w,q_h)))}{\text{opt}(F_0(\cdot|b,(w,q_h)))} \sum_{j=0}^{2N+1} p_j \\
\leq \ & \frac{\frac{1}{\frac{1}{q_h}-1} - \frac{b}{1+\left(\frac{1}{q_h}-1\right)b}}{\frac{b}{1+\left(\frac{1}{q_h}-1\right)b}} = \frac{1+\left(\frac{1}{q_h}-1\right)b}{\left(\frac{1}{q_h}-1\right)b} - 1 = \frac{1}{\left(\frac{1}{q_h}-1\right)b},
\end{aligned}
$$

where the first inequality follows from the definition of opt, the second from the fact that $\mathbf{p}$ belongs to the simplex, and the definition of $\text{opt}(F_\alpha(\cdot|a^-_{2N+1},(w,q_h)))$. Hence, we have, in this case

$$
\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \frac{q_h}{(1-q_h)b}.
$$

We are now in a position to combine all cases and conclude.

If $\alpha \in (0,1]$, we have established that

$$
\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \max\left\{\frac{1}{N}\left(\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{\underline{r}_\alpha(w,q_l)}\right)\left[1+w\frac{\Gamma_\alpha^{-1}(q_l)}{w-\eta}\right], \frac{\eta w}{w-\eta}, \frac{\overline{r}_\alpha(w,q_h)-w}{Nwq}\right\}.
$$

Recall that Proposition F-2 implies that

$$
\mathcal{R}(\mathcal{P}_\mathbb{A},\mathcal{F}_\alpha(w,[q_l,q_h])) \leq \mathcal{R}(\mathcal{P},\mathcal{F}_\alpha(w,[q_l,q_h])) \leq \mathcal{R}(\mathcal{P}_\mathbb{A},\mathcal{F}_\alpha(w,[q_l,q_h])) + \frac{\Delta(\mathbb{A})}{\underline{r}_\alpha(w,q_l)}.
$$

Noting that $\Delta(\mathbb{A}) = \max\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N}, \eta, \frac{\overline{r}_\alpha(w,q_h)-w}{N}\}$, we have

$$
\begin{aligned}
\underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} &\leq \mathcal{R}(\mathcal{P},\mathcal{F}_\alpha(w,[q_l,q_h])) \\
&\leq \mathcal{R}(\mathcal{P}_\mathbb{A},\mathcal{F}_\alpha(w,[q_l,q_h])) + \frac{\Delta(\mathbb{A})}{\underline{r}_\alpha(w,q_l)} \\
&\leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + (\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}}) + \max\left\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N\underline{r}_\alpha(w,q_l)}, \eta, \frac{\overline{r}_\alpha(w,q_h)-w}{N\underline{r}_\alpha(w,q_l)}\right\} \\
&\leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + \max\left\{\frac{1}{N}\left(\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{\underline{r}_\alpha(w,q_l)}\right)\left[1+w\frac{\Gamma_\alpha^{-1}(q_l)}{w-\eta}\right], \frac{\eta w}{w-\eta}, \frac{\overline{r}_\alpha(w,q_h)-w}{Nwq}\right\} \\
&\quad + \max\left\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N\underline{r}_\alpha(w,q_l)}, \eta, \frac{\overline{r}_\alpha(w,q_h)-w}{N\underline{r}_\alpha(w,q_l)}\right\}.
\end{aligned}
$$

By choosing $\eta = \frac{w}{\sqrt{N}}$, we obtain

$$
\underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \mathcal{R}(\mathcal{P},\mathcal{F}_\alpha(w,[q_l,q_h])) \leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + \mathcal{O}(\frac{1}{\sqrt{N}}).
$$

Suppose now $\alpha = 0$. In this case.

$$\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \max\left\{\frac{1}{N}\left(\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{\underline{r}_\alpha(w,q_l)}\right)\left[1+w\frac{\Gamma_\alpha^{-1}(q_l)}{w-\eta}\right], \frac{\eta w}{w-\eta}, \frac{b-w}{Nwq}, \frac{q_h}{(1-q_h)b}\right\}.$$

Using again Proposition F-2 and the fact that $\Delta(\mathbb{A}) = \max\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N}, \eta, \frac{b-w}{N}\}$, we have

$$\begin{aligned}
\underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} &\leq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w,[q_l,q_h])) \\
&\leq \mathcal{R}(\mathcal{P}_{\mathbb{A}}, \mathcal{F}_\alpha(w,[q_l,q_h])) + \frac{\Delta(\mathbb{A})}{\underline{r}_\alpha(w,q_l)} \\
&\leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + (\overline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} - \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}}) + \max\left\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N\underline{r}_\alpha(w,q_l)}, \eta, \frac{b-w}{N\underline{r}_\alpha(w,q_l)}\right\} \\
&\leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + \max\left\{\frac{1}{N}\left(\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{\underline{r}_\alpha(w,q_l)}\right)\left[1+w\frac{\Gamma_\alpha^{-1}(q_l)}{w-\eta}\right], \frac{\eta w}{w-\eta}, \frac{b-w}{Nwq}, \frac{q_h}{(1-q_h)b}\right\} \\
&\quad + \max\left\{\frac{w-\eta-\underline{r}_\alpha(w,q_l)}{N\underline{r}_\alpha(w,q_l)}, \eta, \frac{b-w}{N\underline{r}_\alpha(w,q_l)}\right\}.
\end{aligned}$$

By choosing $\eta = \frac{w}{\sqrt{N}}$ and $b = w\sqrt{N}$, we obatin

$$\underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} \leq \mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w,[q_l,q_h])) \leq \underline{\mathcal{L}}_{\alpha,q_l,q_h,\mathbb{A}} + \mathscr{O}(\frac{1}{\sqrt{N}}).$$

This concludes the proof. □

F.1 Proofs of auxiliary results

Proof of Proposition F-1. First we show that

$$\inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F) = \inf_{q\in[q_l,q_h]} \inf_{F \text{ in } \mathcal{F}_\alpha(w,q)} R(\Psi,F)$$

Let $q \in [q_l, q_h]$ and $F \in \mathcal{F}_\alpha(w,q)$, we have:

$$\begin{aligned}
R(\Psi,F) &\geq \inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F) \\
\implies \inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi,F) &\geq \inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F) \\
\implies \inf_{q\in[q_l,q_h]} \inf_{F\in\mathcal{F}_\alpha(w,q)} R(\Psi,F) &\geq \inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F).
\end{aligned}$$

Let $\epsilon > 0$ and $F_\epsilon \in \mathcal{F}_\alpha(w, [q_l, q_h])$ such that:

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) &\geq R(\Psi, F_\epsilon) - \epsilon \\
\implies \inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) &\geq \inf_{F \in \mathcal{F}_\alpha(w,\{\overline{F}_\epsilon(w)\})} R(\Psi, F) - \epsilon \\
\implies \inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) &\geq \inf_{q \in [q_l,q_h]} \inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi, F) - \epsilon,
\end{aligned}$$

by taking $\epsilon \to 0$, we obtain the desired result.

Let $F \in \mathcal{F}_\alpha(w, [q_l, q_h])$ and $q = \overline{F}(w) \in [q_l, q_h]$, by Theorem 1, we have:

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi, F) &= \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q),w)} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x,(w,q)))} \int_0^\infty u\overline{F}_\alpha(u|x,(w,q))d\Psi(u), \\
&\qquad \inf_{x \in [w,\overline{r}_\alpha(w,q)]} \frac{1}{\mathrm{opt}(F_\alpha(\cdot|x,(w,q)))} \int_0^\infty u\overline{F}_\alpha(u|x,(w,q))d\Psi(u) \Bigg\} \\
&= \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q),w)} \frac{1}{x} \left[\int_{[0,x)} ud\Psi(u) + \int_{[x,w]} u\overline{G}_{\alpha,w}(u|(x,1),(w,q))d\Psi(u) \right], \\
&\qquad \inf_{x \in [w,\overline{r}_\alpha(w,q)]} \frac{1}{x\overline{G}_{\alpha,x}(x|(0,1),(w,q))} \int_{[0,x]} u\overline{G}_{\alpha,x}(u|(0,1),(w,q))d\Psi(u) \Bigg\}.
\end{aligned}$$

Using the non-decreasing monotonicity of the functions $q \to \underline{r}_\alpha(w,q) = \frac{w}{\Gamma_\alpha^{-1}(q)+1}, q \to \overline{r}_\alpha(w,q) = \frac{w}{\alpha\Gamma_\alpha^{-1}(q)}$, we have:

$$\begin{aligned}
\inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi, F) &\geq \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l),w)} \frac{1}{x} \left[\int_{[0,x)} ud\Psi(u) + \int_{[x,w]} u\overline{G}_{\alpha,w}(u|(x,1),(w,q))d\Psi(u) \right], \\
&\qquad \inf_{x \in [w,\overline{r}_\alpha(w,q_h)]} \frac{1}{x\overline{G}_{\alpha,x}(x|(0,1),(w,q))} \int_{[0,x]} u\overline{G}_{\alpha,x}(u|(0,1),(w,q))d\Psi(u) \Bigg\}.
\end{aligned}$$

We have, for fixed (u, x) such that $x \in [\underline{r}_\alpha(w, q_l), w), u \in [x, w)$, the following function is clearly non-decreasing

$$q \to \overline{G}_{\alpha,w}(u|(x,1),(w,q)) = \Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{u-x}{w-x}\right).$$

We have, for fixed (u, x) such that $x \in [w, \overline{r}_\alpha(w, q_h)], u \in [w, x]$, the following function is non-increasing

$$q \to \frac{u\overline{G}_{\alpha,x}(u|(0,1),(w,q))}{x\overline{G}_{\alpha,x}(x|(0,1),(w,q))} = \frac{u\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{u}{w}\right)}{x\Gamma_\alpha\left(\Gamma_\alpha^{-1}(q)\,\frac{x}{w}\right)} = \begin{cases} \frac{u}{x}\left(\frac{u}{x} + \frac{x-u}{x(1+(1-\alpha)\Gamma_\alpha^{-1}(q)\frac{x}{w})}\right)^{\frac{1}{\alpha-1}} & \text{if } \alpha \in [0,1) \\ \frac{u}{x}q^{\frac{u-x}{w}} & \text{if } \alpha = 1. \end{cases}$$

Using the monotonicity of the above functions we get :

$$\inf_{F \text{ in } \mathcal{F}_\alpha(w,q)} R(\Psi, F) \geq \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l),w)} \frac{1}{x}\left[\int_{[0,x)} u d\Psi(u) + \int_{[x,w]} u\overline{G}_{\alpha,w}(u|(x,1),(w,q_l))d\Psi(u)\right],$$
$$\inf_{x \in [w,\overline{r}_\alpha(w,q_h)]} \frac{1}{x\overline{G}_{\alpha,x}(x|(0,1),(w,q))}\int_{[0,x]} u\overline{G}_{\alpha,x}(u|(0,1),(w,q_h))d\Psi(u)\Bigg\}.$$

Since the right hand-side does not depend on q, we take the minimum on q

$$\begin{aligned}\inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) &\geq \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l),w)} \frac{1}{x}\left[\int_{[0,x)]} u d\Psi(u) + \int_{[x,w]} u\overline{G}_{\alpha,w}(u|(x,1),(w,q_l))d\Psi(u)\right],\\
&\qquad \inf_{x \in [w,\overline{r}_\alpha(w,q_h)]} \frac{1}{x\overline{G}_{\alpha,x}(x|(0,1),(w,q))}\int_{[0,x]} u\overline{G}_{\alpha,x}(u|(0,1),(w,q_h))d\Psi(u)\Bigg\}\\
&= \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l),w)} \frac{1}{\operatorname{opt}(F_\alpha(\cdot|x,(w,q_l)))}\int_0^\infty u\overline{F}_\alpha(u|x,(w,q_l))d\Psi(u),\\
&\qquad \inf_{x \in [w,\overline{r}_\alpha(w,q_h)]} \frac{1}{\operatorname{opt}(F_\alpha(\cdot|x,(w,q_h)))}\int_0^\infty u\overline{F}_\alpha(u|x,(w,q_h)))d\Psi(u)\Bigg\}.\end{aligned}$$

This conclues the proof. □

Proof of Proposition F-2. Fix q in $[q_l, q_h]$. Then, using proposition 2, there exists $\Psi_{\mathbb{A}}$ in $\mathcal{P}_{\mathbb{A}}$ such that

$$\inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi_{\mathbb{A}}, F) \geq \inf_{F \in \mathcal{F}_\alpha(w,q)} R(\Psi, F) - \frac{\Delta(\mathbb{A})}{a_1} - \frac{1}{q(1+(q^{-1}-1)a_N/w)}\mathbf{1}\{a_N < \overline{r}_\alpha(w,q)\},$$

where $\Delta(\mathbb{A}) = \sup_i\{a_i - a_{i-1}\}$. We have the following function

$$q \to -\frac{1}{q(1+(q^{-1}-1)a_N/w)} = -\frac{1}{\frac{a_N}{w} + q(1-\frac{a_N}{w})},$$

is non-increasing because $a_N > w$ therefore

$$-\frac{1}{q(1+(q^{-1}-1)a_N/w)} \geq -\frac{1}{q_h(1+(q_h^{-1}-1)a_N/w)},$$

moreover we have $q \to \overline{r}_\alpha(w,q) = \frac{w}{\alpha\Gamma_\alpha^{-1}(q)}$ is non-decreasing, therefore

$$\mathbf{1}\{a_N < \overline{r}_\alpha(w,q)\} \leq \mathbf{1}\{a_N < \overline{r}_\alpha(w,q_h)\}.$$

Hence, since $-\frac{1}{q(1+(q^{-1}-1)a_N/w)} < 0$, we have

$$-\frac{\mathbf{1}\{a_N < \overline{r}_\alpha(w,q)\}}{q(1+(q^{-1}-1)a_N/w)} \geq -\frac{\mathbf{1}\{a_N < \overline{r}_\alpha(w,q_h)\}}{q_h(1+(q_h^{-1}-1)a_N/w)}.$$

Using these lower bounds, we obtain

$$\inf_{F \text{ in } \mathcal{F}_\alpha(w,q)} R(\Psi_{\mathbb{A}}, F) \geq \inf_{F \text{ in } \mathcal{F}_\alpha(w,q)} R(\Psi, F) - \frac{\Delta(\mathbb{A})}{\underline{r}_\alpha(w, q_l)} - \frac{\mathbf{1}\{a_N < \overline{r}_\alpha(w, q_h)\}}{q_h(1 + (q_h^{-1} - 1)a_N/w)},$$

taking the infinimum over q from both sides concludes the proof. □

G Upper bound linear program and implementation parameters

In this section, we show that one can obtain an upper bound on the maxmin ratio $\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w, [q_l, q_h]))$ by solving a linear program. Fix an arbitrary sequence of increasing reals $\mathbb{A} = \{a_i\}_{i=0}^{2N}$ such that $a_0 = 0, a_1 = \underline{r}_\alpha(w, q_l)$, $a_{N+1} = w$ and $a_{2N} \leq \overline{r}_\alpha(w, q_h)$. Set $a_{2N+1} := \overline{r}_\alpha(w, q_h)$.

Fix a mechanism Ψ in $\mathcal{P}$ and denote by $p_{j+1} = \int_{I_j} d\Psi(u)$ where we define the intervals $(I_j)_{j=0,\cdots,2N}$ as follows:

$$I_j = \begin{cases} [a_j, a_{j+1}) & \text{if } 0 \leq j < 2N, \\ [a_{2N}, a_{2N+1}] & \text{if } j = 2N. \end{cases}$$

Using proposition F-1, we have

$$\begin{aligned}
& \inf_{F \in \mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi, F) \\
= \ & \min\Bigg\{ \inf_{x \in [\underline{r}_\alpha(w,q_l), w)} \frac{1}{\text{opt}(F_\alpha(\cdot|x, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_l)) d\Psi(u), \\
& \inf_{x \in [w, \overline{r}_\alpha(w,q_h)]} \frac{1}{\text{opt}(F_\alpha(\cdot|x, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_h))) d\Psi(u) \Bigg\} \\
= \ & \min\Bigg\{ \min_{i=1,\cdots,N} \inf_{x \in [a_i, a_{i+1})} \frac{1}{\text{opt}(F_\alpha(\cdot|x, (w, q_l)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_l)) d\Psi(u), \\
& \min_{i=N+1,\cdots,2N} \inf_{x \in [a_i, a_{i+1}]} \frac{1}{\text{opt}(F_\alpha(\cdot|x, (w, q_h)))} \int_0^\infty u\overline{F}_\alpha(u|x, (w, q_h))) d\Psi(u) \Bigg\}.
\end{aligned}$$

Following the same reasoning as in the proof of Theorem 4, we may also obtain an upper bound on $\inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F)$. Indeed, we have

$$\begin{aligned}\inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F) &\le \min\Bigg\{\min_{i=1,\cdots,N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\int_0^\infty u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_l))d\Psi(u),\\ &\qquad \min_{i=N+1,\cdots,2N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\int_0^\infty u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_h))d\Psi(u)\Bigg\}\\ &= \min\Bigg\{\min_{i=1,\cdots,N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\sum_{j=0}^{2N}\int_{I_j} u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_l))d\Psi(u),\\ &\qquad \min_{i=N+1,\cdots,2N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\sum_{j=0}^{2N}\int_{I_j} u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_h))d\Psi(u)\Bigg\}.\end{aligned}$$

For $x\in[\underline{r}_\alpha(w,q_l),w)$, the revenue function $u\mapsto u\overline{F}_\alpha(\cdot|x,(w,q_l))$ is increasing in u on $[0,x)$ and decreasing on (x,w). In addition, note that, for $x\in[w,\overline{r}_\alpha(w,q_h)]$, the revenue function $u\mapsto u\overline{F}_\alpha(u|x,(w,q_h))$ is non-decreasing on $[0,x]$ and $u\overline{F}_\alpha(u|x,(w,q_h))=0$ for $u>x$. Hence, we have

$$\begin{aligned}\inf_{F\in\mathcal{F}_\alpha(w,[q_l,q_h])} R(\Psi,F) &\le \min\Bigg\{\min_{i=1,\cdots,N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\sum_{j=0}^{2N}\int_{I_j} u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_l))d\Psi(u),\\ &\qquad \min_{i=N+1,\cdots,2N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\sum_{j=0}^{2N}\int_{I_j} u\overline{F}_\alpha(u|a_{i+1}^-,(w,q_h))d\Psi(u)\Bigg\}\\ &\le \min\Bigg\{\min_{i=1,\cdots,N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\Bigg[\sum_{j=0}^{i} a_{j+1}\overline{F}_\alpha(a_{j+1}|a_{i+1}^-,(w,q_l))\int_{I_j}d\Psi(u)+\\ &\qquad \sum_{j=i+1}^{2N} a_j\overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l))\int_{I_j}d\Psi(u)\Bigg],\\ &\qquad \min_{i=N+1,\cdots,2N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\sum_{j=0}^{2N} a_{j+1}\overline{F}_\alpha(a_{j+1}|a_{i+1}^-,(w,q_h))\int_{I_j}d\Psi(u)\Bigg\}\\ &= \min\Bigg\{\min_{i=1,\cdots,N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\Bigg[\sum_{j=1}^{i+1} a_j\overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l))p_j+\\ &\qquad \sum_{j=i+2}^{2N+1} a_{j-1}\overline{F}_\alpha(a_{j-1}|a_{i+1}^-,(w,q_l))p_j\Bigg],\\ &\qquad \min_{i=N+1,\cdots,2N}\frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\sum_{j=1}^{2N+1} a_j\overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_h))p_j\Bigg\}.\end{aligned}$$

The problem of maximizing over mechanisms in $\mathcal{P}$ is clearly upper bounded by the problem of maximizing the RHS above over $p_1, ..., p_{2N+1}$. The epigraph formulation of the latter problem can be written as

$$
\begin{aligned}
\overline{\mathcal{L}_U}_{\alpha,q_l,q_h,\mathbb{A}} \quad = \quad & \max_{\mathbf{p},c} \; c \qquad \text{(G-1)} \\
s.t. \quad & \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_l)))}\left[\sum_{j=1}^{i+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_l))p_j + \sum_{j=i+2}^{2N+1} a_{j-1}\overline{F}_\alpha(a_{j-1}|a_{i+1}^-,(w,q_l))p_j\right] \geq c \\
& i = 1, ...N, \\
& \frac{1}{\mathrm{opt}(F_\alpha(\cdot|a_{i+1}^-,(w,q_h)))}\sum_{j=1}^{2N+1} a_j \overline{F}_\alpha(a_j|a_{i+1}^-,(w,q_h))p_j \geq c \quad i = N+1, ...2N, \\
& \sum_{j=1}^{2N+1} p_j \leq 1, \quad p_i \geq 0 \quad i = 1, ...2N+1.
\end{aligned}
$$

Therefore we obtain that:

$$
\mathcal{R}(\mathcal{P}, \mathcal{F}_\alpha(w,[q_l,q_h])) \leq \overline{\mathcal{L}_U}_{\alpha,q_l,q_h,\mathbb{A}}.
$$

Implementation parameters: For all reported values in the main text, we use the following sequence in the Linear Programs

$$
a_i \quad = \quad \begin{cases} \underline{r}_\alpha(1,q_l) + \frac{i}{N}\left(1-\eta-\underline{r}_\alpha(1,q_l)\right) & \text{if } 0 \leq i \leq N. \\ w + \frac{i-N-1}{N}\left(\min(b,\overline{r}_\alpha(1,q_h))-1\right) & \text{if } N+1 \leq i \leq 2N+1, \end{cases}
$$

with $N = 2500, \eta = 10^{-5}, b = 250$.

H Additional Illustrations of near optimal mechanisms for Section 5

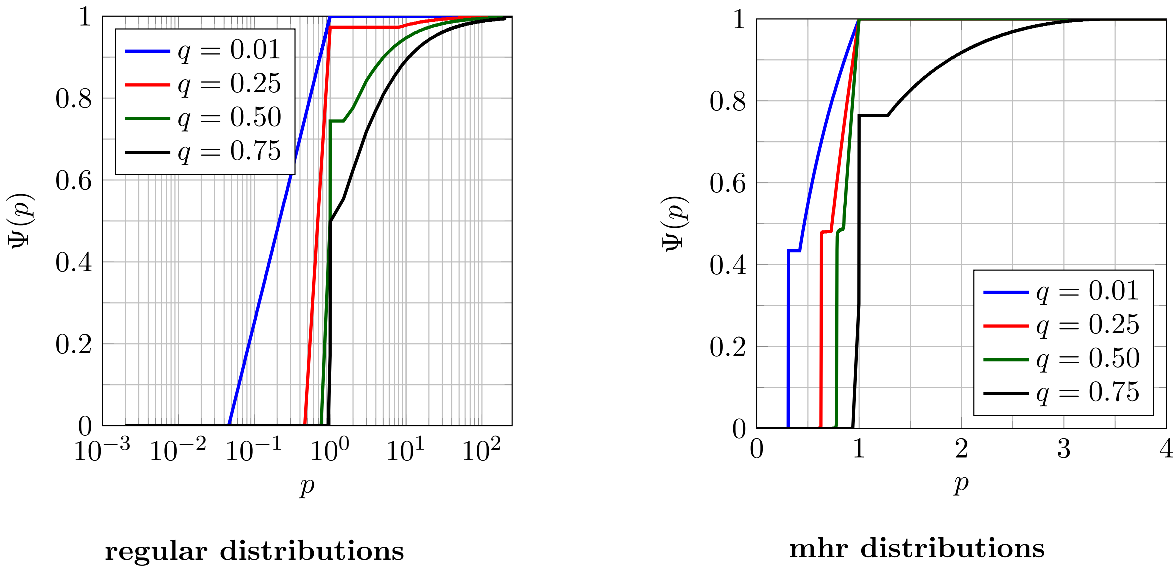

Figure 9: **Illustration of near optimal mechanisms.** The figure depicts near optimal pricing distributions for $w = 1$, q in $\{0.01, 0.25, 0.5, 0.75\}$. The left panel corresponds to regular distributions (plotted using a log scale) and the right panel to mhr distributions (on a regular scale).

References

Yao, A. C. (1977). "Probabilistic computations: Toward a unified measure of complexity". *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*, pp. 222–227.