Online Appendix to "Product Quality and Information Sharing in the Presence of Reviews"

In the online appendix, we present additional results and the proofs of the theoretical results. The appendix is organized as follows. In §EC.1, we present additional theoretical results. In §EC.2, we provide numerical experiment to show the robustness of our results with respect to the linear demand assumption. In §EC.3–EC.6, we provide the proofs of the theoretical results in §3–6 of the main body, respectively. In §EC.7, we present the proofs of auxiliary lemmas.

EC.1. Additional Theoretical Results

EC.1.1. A Sufficient Condition for Nonnegative Demand

The following lemma characterizes a sufficient condition to ensure that the equilibrium demand function in each stage is nonnegative.

LEMMA EC.1 **(Sufficient Condition for Positive Demand)**. *Under a wholesale price contract, there exists a constant α_0 such that d_1 and d_2 in equilibrium are positive with probability one for any $\alpha > \alpha_0$, where*

$$\alpha_0 \geq \max(\bar{q}(3\bar{\theta} - 4\underline{\theta}), \underline{q}(3\bar{\theta} - 4\underline{\theta})) + \bar{c}.$$

Under a commission contract, there exists a constant α_0' such that d_1 and d_2 in equilibrium are positive with probability one for any $\alpha > \alpha_0'$, where

$$\alpha_0' \geq \max(\bar{q}(\bar{\theta} - 2\underline{\theta}), \underline{q}(\bar{\theta} - 2\underline{\theta})) + \bar{c}/(1-u).$$

In the preceding lemma, we characterize the sufficient condition for $d_1 > 0$ in terms of the nominal demand level, α. The key idea in the proof is that demand cannot be negative because the retail price p_1 is sufficiently close to the deterministic counterpart $\bar{p}$ if α is sufficiently large. As is clear from the proof of the lemma, it is straightforward to derive sufficient conditions in terms of other parameters. For instance, one can easily check that p_1 is sufficiently close to $\bar{p}$ if σ_θ and σ_q are sufficiently close to zero such that $\underline{\theta} \approx \bar{\theta} \approx \mu_\theta$ and $\underline{q} \approx \bar{q} \approx q$, in which case the corresponding demand d_1 is always positive.

EC.1.2. Analysis in the Intermediate Regime

In the intermediate regime, where the rate of learning γ is not close to zero or ∞, the firms' profits cannot be expressed in tractable form. Furthermore, the effects of reviews on firms' profits are not monotone with respect to γ; see Figure EC.1. Specifically, as alluded to earlier in (10), the first-stage pricing decisions are affected by reviews only marginally in the slow and fast learning regimes. By contrast, customers' posterior belief is significantly affected by the number of reviews in an intermediate regime, and thus, pricing decisions depend significantly on the learning rate γ. To quantify this effect, recall that $p_1(w_1)$ is the platform's best-response price given a wholesale price w_1 and observe that

$$\frac{\partial p_1(w_1)}{\partial w_1} = \frac{1}{2}\left(1+\frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 b_{\text{I}}}{(d_1\gamma+1)^3}\right]\right)^{-1} \quad \text{for} \quad \text{I} \in \{\text{S}, \text{NS}\}, \tag{EC.1}$$

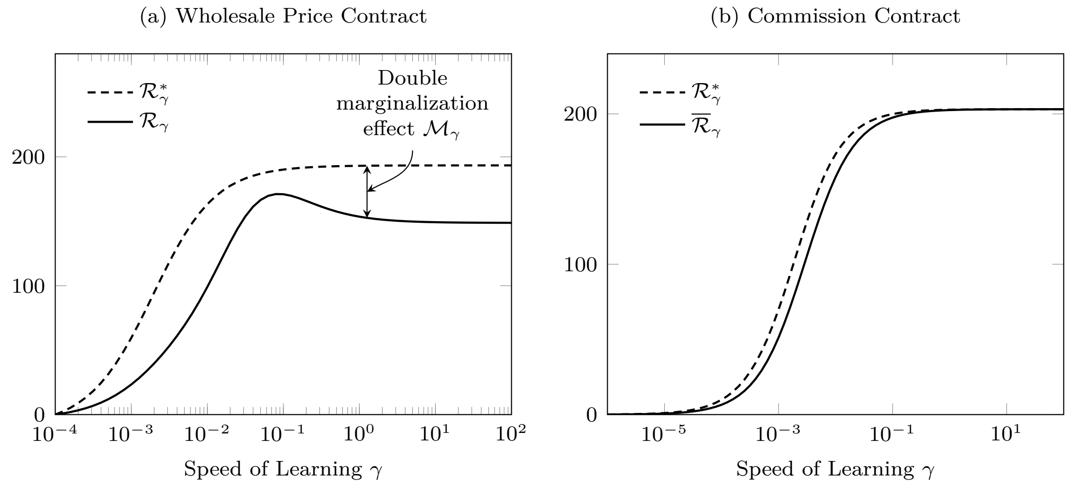
where $b_{\text{S}} := \hat{\theta}(4\theta - 3\hat{\theta})$ and $b_{\text{NS}} := (2\hat{\theta} - \mu_\theta)(4\theta - 2\hat{\theta} - \mu_\theta)$; see the proof of Lemma EC.6 for calculation steps. In the slow or fast learning regimes (as $\gamma \to 0$ or $\gamma \to \infty$), note that $\partial p_1(w_1)/\partial w_1 \to 0.5$; that is, when the wholesale price increases by \$1, the platform would increase its markup by \$0.5. In an intermediate regime, however, it can be confirmed that $\partial p_1(w_1)/\partial w_1 < 0.5$; that is, if the wholesale price increases by \$1, the retail price would increase by *less than* \$0.5. That is, the firms cooperatively set lower prices to attract more reviews in the first stage. This intensified coordination effect that makes the impact of reviews nonmonotone with respect to γ in the intermediate regime.

To visualize the effects of reviews on firms' profits, consider a wholesale price contract. We measure the overall impact of product reviews by $\mathcal{R}_\gamma^* := \Pi^*(q) - \Pi^*(q)|_{\gamma=0}$, where $\Pi^*(q)$ represents the expected profit in the centralized setting. Likewise, let $\mathcal{R}_\gamma := \Pi^{\text{J}}(q) - \Pi^{\text{J}}(q)|_{\gamma=0}$ represent the net effect of reviews on the joint profit in the decentralized setting. Since the double marginalization effect is absent in the centralized setting, $\mathcal{R}_\gamma^*$ measures the positive effects of product reviews on the market, while $\mathcal{M}_\gamma := \mathcal{R}_\gamma^* - \mathcal{R}_\gamma$ captures the loss due to the double marginalization effect in the decentralized setting. Figure EC.1(a) illustrates the overall effects of reviews as functions of $\gamma \in [10^{-4}, 10^2]$. The dotted line represents the positive effect of product reviews, $\mathcal{R}_\gamma^*$, in the benchmark case of the centralized channel. Note that $\mathcal{M}_\gamma \approx 0$ when the learning is slow, whereas $\mathcal{M}_\gamma \gg 0$ if the learning is sufficiently fast. Note that $\mathcal{M}_\gamma$ is nonmonotone with respect to γ; specifically, in an intermediate regime where γ is not too small nor too large, the extent to which reviews impact firms' profits depends significantly on the number of reviews, so the firms tend to cooperatively set a low price, alleviating the double marginalization effect.

Under a commission contract, define $\overline{\mathcal{R}}_\gamma := \bar{\Pi}^{\text{J}}(q) - \bar{\Pi}^{\text{J}}(q)|_{\gamma=0}$ as the net effect of reviews on the joint profit under a commission contract. Figure EC.1(b) depicts the values of $\mathcal{R}_\gamma^*$ and $\overline{\mathcal{R}}_\gamma$ as functions of $\gamma \in [10^{-6}, 10^2]$. Observe that $\mathcal{R}_\gamma^* \approx \overline{\mathcal{R}}_\gamma$ for sufficiently large values of γ; that is, unlike the wholesale price contract, the commission contract achieves the first-best outcome even when reviews are sufficiently informative.

Although the nontrivial effects of reviews make it difficult to characterize explicitly firms' strategies, the following theorem shows that the firms' equilibrium strategy is a threshold type when the market is sufficiently large.

Figure EC.1 Reviews and Double Marginalization Effects



Note. The effects of product reviews on the joint profits, $\mathcal{R}_\gamma$ and $\mathcal{R}^*_\gamma$, in the decentralized and centralized settings, respectively, are plotted as functions of γ. The left and right panels represent the results under the wholesale price and commission contracts, respectively. The common settings for both panels are as follows: $(\text{I}, q, c, \alpha, \phi, \sigma_p) = (\text{S}, 2.3, 165, 700, 0.3, 0.5)$ and the random variable θ is binary and takes values in $\{0, 100\}$ with equal probability. For the commission contract, we set the commission rate $u = 0.2$.

Theorem EC.1 **(Equilibrium Strategy: Intermediate Regime)**. *Fix $\gamma \in (0, \infty)$. Suppose the market is sufficiently large such that $d_1 \geq \delta$ with probability one for some constant $\delta > 0$. Then, firms' equilibrium strategy is a threshold strategy with $(\Phi^\gamma_{\texttt{i}}, \Psi^\gamma_{\texttt{i}})$, where*

$$(\Phi^\gamma_{\texttt{i}}, \Psi^\gamma_{\texttt{i}}) = \left(\frac{2(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma^2_\theta(\bar{q}^2 - \frac{4}{3}\underline{q}^2 - \sigma^2_p)}, \frac{1}{3}(\underline{q}^2 + \sigma^2_p) \right) + \mathcal{O}\left(\frac{1}{\delta}\right)$$

If the minimum demand level δ is sufficiently large, then there will always be some customers who purchase the product even at the lowest possible level of the random variables, that is, even when $\theta = \underline{\theta}$ and $\epsilon_1 = \underline{\epsilon}$. See Lemma EC.1 for sufficient conditions for the existence of the constant $\delta > 0$. Since a sufficient number of reviews are available for any reasonable price in the first stage, the price distortion due to the (dis-)coordination effect becomes small, and the first-stage equilibrium price (p^*_1, w^*_1) is close to that without product reviews, $(\bar{p}, \bar{w})$ in (10).

EC.1.3. Comparative Statics

In the main body of the paper, we analyzed firms' decisions on quality and information sharing and whether customers benefit from the interaction in distinct learning regimes. In this section, we study the marginal impact that a small change in the strength of the propensity/quality signals has on firms' profits and consumer surplus for fixed quality $q \in \{\underline{q}, \bar{q}\}$ and information sharing arrangement $\text{I} \in \{\text{S}, \text{NS}\}$.

Marginal Impact of Propensity Signal. Intuitively, the stronger the propensity signal is, the better the firms can make pricing decisions and thus earn higher profits, which is formalized in the next lemma.

Lemma EC.2 **(Effects of Propensity Signal on Firms' Profits)**. *Fix any $q \in \{\underline{q}, \bar{q}\}$ and consider sufficiently small or large γ. As the propensity signal becomes stronger (i.e., as ϕ increases):*

(a) With information sharing, the manufacturer's profit increases more than the platform's profit does; formally, $\partial \Pi_S^M(q)/\partial \phi > \partial \Pi_S^P(q)/\partial \phi > 0$.

(b) Without information sharing, the platform's profit increases but the manufacturer's profit is essentially constant; formally, $\partial \Pi_{NS}^P(q)/\partial \phi > \partial \Pi_{NS}^M(q)/\partial \phi$ for $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$. Furthermore, $\partial \Pi_{NS}^M(q)/\partial \phi = 0$ for $\gamma = 0$ or $\gamma = \infty$.

For a fixed quality, the preceding lemma implies that the propensity signal—whenever available—helps firms make better-informed pricing decisions, thereby increasing their expected profits. With information sharing, the benefit from the propensity signal is shared between the two firms, whereas without information sharing, the platform receives the benefit exclusively.

The next lemma suggests that a stronger propensity signal harms consumer surplus for a fixed quality.

Lemma EC.3 **(Effects of Propensity Signal on Consumer Surplus)**. *Fix any $I \in \{S, NS\}$ and $q \in \{\underline{q}, \bar{q}\}$. If the learning is sufficiently slow or fast, consumer surplus decreases as the propensity signal becomes stronger under both wholesale price and commission contracts; formally, $\partial \Pi_I^C(q)/\partial \phi < 0$ and $\partial \bar{\Pi}_I^C(q)/\partial \phi < 0$.*

The preceding lemma is intuitive. As the propensity signal becomes stronger, firms can make better-informed pricing decisions: the more the firms know about customers' preferences (i.e., the propensity signal is stronger), the more profits they can earn by adjusting their pricing decisions according to the signal, which in turn reduces consumer surplus.

Marginal Impact of Reviews. As reviews become more informative, firms benefit by adaptively adjusting their pricing decisions according to the information contained in reviews. The overall effects of product reviews on firms' profits are summarized in the following lemma.

Lemma EC.4 **(Effects of Reviews on Firms' Profits)**. *Fix any $I \in \{S, NS\}$ and $q \in \{\underline{q}, \bar{q}\}$. Under either a wholesale price contract or a commission contract, if the learning is slow, the expected profits of the platform and manufacturer strictly increase as reviews become more informative. Formally, there exists a constant $\gamma_1 > 0$ such that $\partial \Pi_I^K(q)/\partial \gamma > \epsilon$ and $\partial \bar{\Pi}_I^K(q)/\partial \gamma > \epsilon$ for all $\gamma < \gamma_1$ and $K \in \{P, M\}$, where ϵ is a positive constant identified in the proof.*

Under a wholesale price contract, we decompose the overall impact of reviews into two main effects: *coordination* and *informational* effects. The coordination effect changes firms' pricing decisions in the first stage. To demonstrate the coordination effect, we show that if customer learning is sufficiently slow (i.e., if γ is sufficiently small), then

$$\frac{\partial p_1^*}{\partial \gamma} < 0, \ \frac{\partial w_1^*}{\partial \gamma} < 0, \ \text{and} \ \frac{\partial d_1^*}{\partial \gamma} > 0. \tag{EC.2}$$

See the proof of Lemma EC.4 for detailed steps for derivation. Namely, as reviews become more informative, the platform and manufacturer cooperatively lower wholesale and retail prices, respectively, and consequently, more reviews become available in equilibrium. The informational effect shifts demand in the second stage. Specifically, firms face higher or lower demand, depending on whether reviews are positive or negative in the second stage. Accordingly, the firms set prices in response to reviews: they increase profits as more positive outcomes become more likely and buffer against losses when more negative outcomes become more likely. Therefore, increased demand variability increases firms' expected profits. Under a commission contract, since the retail price is determined solely by the manufacturer, there exists no coordination effect. Nevertheless, Lemma EC.4 suggests that the firms' profits also increase with γ due to the aforementioned informational effect.

Product reviews may have contrasting effects on consumer surplus in the slow and fast learning regimes. In the slow learning regime, as reviews become more informative, customers can make better-informed purchase decisions, and furthermore, firms discount the retail price to increase the number of reviews in the first stage (the coordination effect), which also benefits customers. Therefore, it is fairly intuitive that the overall effect of reviews on consumer surplus is positive in a slow learning regime, which is formalized in the following lemma.

LEMMA EC.5 **(Effects of Reviews on Consumer Surplus)**. *Fix any* $I \in \{S, NS\}$ *and* $q \in \{\underline{q}, \bar{q}\}$*. Under either a wholesale price contract or a commission contract, if customers' learning is sufficiently slow, consumer surplus increases as reviews become more informative. Formally, there exists a constant* $\gamma_1 > 0$ *such that* $\partial\Pi_I^C(q)/\partial\gamma > \epsilon^C$ *and* $\partial\bar{\Pi}_I^C(q)/\partial\gamma > \epsilon^C$ *for all* $\gamma < \gamma_1$*, where* ϵ^C *is a positive constant identified in the proof.*

To summarize, Lemmas EC.4 and EC.5 suggest that when product quality and information sharing arrangement are exogenous, reviews have a positive impact on firms' profits and consumer surplus. In particular, it is interesting to note that consumers also benefit from reviews even if firms make pricing decisions only to maximize their own profits.

EC.2. Robustness of the Linear Demand Model

In our base model described in §2, demand is assumed to be a linear function of the price and (estimated) quality. In this section, we demonstrate whether the main qualitative conclusions from the base model extend to a class of nonlinear demand models, and for this purpose, we focus on a wholesale price contract. From our base model, we find that the overall effect of social learning can be decomposed into two main effects:

1. *Direct effect* changes the manufacturer's quality decision given an information sharing contract. Formally, recall that the manufacturer produces high-quality products if and only if $\Delta_q\Pi_I^M = \Pi_I^M(\bar{q}) -$

$\Pi_{\text{I}}^{\text{M}}(\underline{q}) > 0$, so $\Delta_{\mathtt{q}}\Pi_{\text{I}}^{\text{M}}$ for $\text{I} \in \{\text{S}, \text{NS}\}$ represents the manufacturer's incentive to improve quality. In this regard, we define

$$\mathtt{DE}(\gamma) := \Delta_{\mathtt{q}}\Pi_{\text{S}}^{\text{M}} - \Delta_{\mathtt{q}}\Pi_{\text{S}}^{\text{M}}|_{\gamma=0} \tag{EC.3}$$

as a measure of the direct effect of social learning. (For the purpose of illustration, we let $\text{I} = \text{S}$ and a similar conclusion can be obtained for $\text{I} = \text{NS}$.) When the demand function is linear, Proposition 2 suggests that the direct effect is absent for perfectly informative reviews; that is, $\mathtt{DE}(\gamma) \to 0$ as $\gamma \to \infty$.

2. *Moderation effect* changes the manufacturer's quality decision through an information sharing contract; that is, if the manufacturer decides not to obtain information from the platform (due to a high information fee), the manufacturer accordingly changes its quality decision. Formally, recall that information sharing is arranged if $\Delta_{\mathtt{i}}\Pi^{\text{P}} = \Pi_{\text{S}}^{\text{P}}(q_{\text{S}}) - \Pi_{\text{NS}}^{\text{P}}(q_{\text{NS}}) > -\rho$, so $\Delta_{\mathtt{i}}\Pi^{\text{P}}$ represents the platform's incentive to share information. We let

$$\mathtt{ME}(\gamma) := \Delta_{\mathtt{i}}\Pi^{\text{P}} - \Delta_{\mathtt{i}}\Pi^{\text{P}}|_{\gamma=0} \tag{EC.4}$$

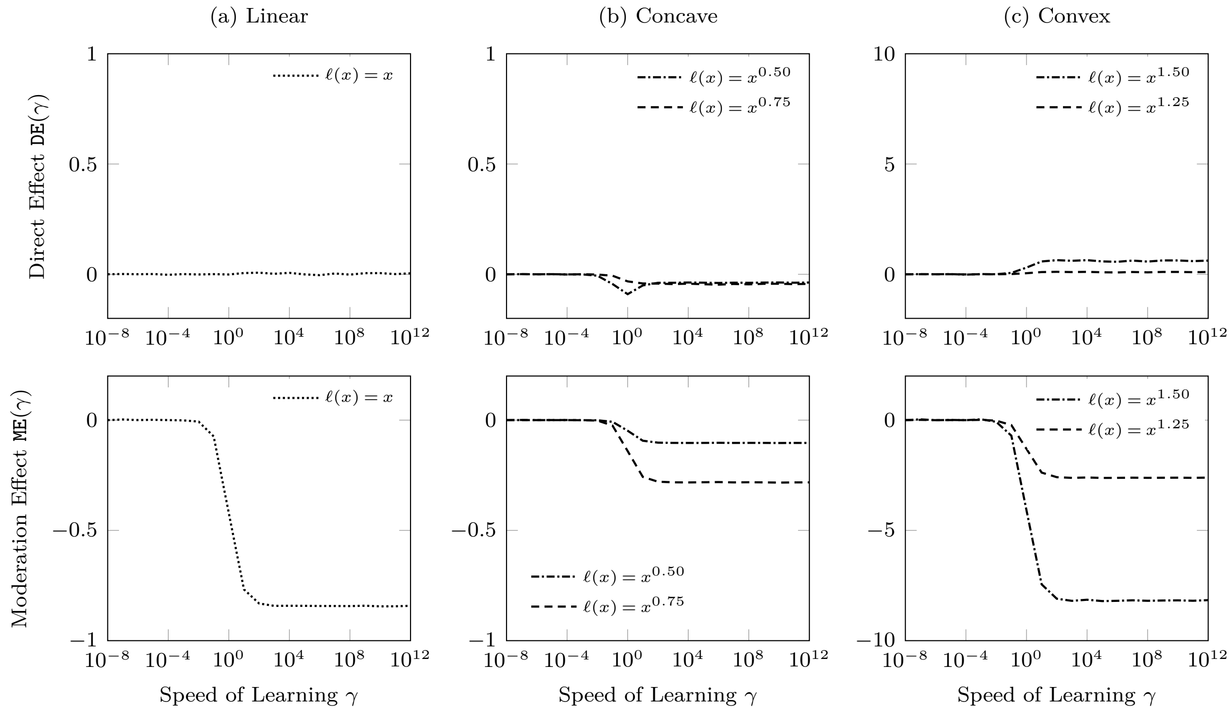
denote the moderation effect of social learning. (One may similarly define the moderation effect in terms of $\Delta_{\mathtt{i}}\Pi^{\text{M}}$, and the same qualitative conclusions hold.) When the demand function is linear, Theorems 1 and 2 suggest that the moderation effect is negative for perfectly informative reviews; that is, $\mathtt{ME}(\gamma) \to \mathtt{ME}(\infty) < 0$ as $\gamma \to \infty$.

To demonstrate the robustness of these results with respect to the linearity assumption, we quantify the direct and moderation effects based on *generalized* linear models in which the demand in stage t is $\tilde{d}_t = \ell(d_t)$ for a nonlinear function ℓ and d_t is given in (1) and (3) for $t = 1$ and $t = 2$, respectively. Specifically, we consider two distinct cases: (a) concave demand, where $\ell(x) = x^z$ with $z \in \{0.50, 0.75\}$ and (b) convex demand, where $\ell(x) = x^z$ with $z \in \{1.25, 1.50\}$. For each γ, the values of $\mathtt{DE}(\gamma)$ and $\mathtt{ME}(\gamma)$ are estimated by taking average over 10^{10} trials of the game and the standard errors of the estimates are at least 10 times smaller than the estimates. The results are illustrated in Figure EC.2. Several comments are in order.

First, consistent with the linear demand case, we observe that the moderation effect is negative in both concave and convex cases. Specifically, observe from bottom panels of Figure EC.2 that $\mathtt{ME}(\gamma) < 0$ for sufficiently large γ. In other words, the platform is less willing to share information (and thus charge a high information fee) when reviews are sufficiently informative, which in turns reduces the manufacturer's incentive to invest in quality. Note that the moderation effect occurs mainly because the impact of double marginalization is aggravated in the presence of reviews, and as is observed in our numerical results, the direction of such an effect does not depend on the convexity/concavity of the demand function.

Second, while the direct effect is null in the linear demand case, we observe that the direct effect can be negative and positive in the cases of concave and convex demand, respectively. Recall that

Figure EC.2 Direct and Moderation Effects of Reviews



Note. The direct effect $\mathtt{DE}(\gamma)$ and the moderation effect $\mathtt{ME}(\gamma)$ are plotted as functions of γ in a log-linear scale, where $\mathtt{DE}(\gamma)$ and $\mathtt{ME}(\gamma)$ are defined in (EC.3) and (EC.4), respectively. The common parameters are $(\underline{q}, \bar{q}, \sigma_p, \underline{c}, \bar{c}, \alpha, \phi) = (1, 1.6, 1, 10, 13, 100, 0.5)$ and $\theta \sim N(5, 3^2)$. The demand for the convex case is larger by an order of magnitude than the linear and concave cases, so the vertical axes for the convex case (c) are accordingly scaled up from those of the linear and concave cases (a) and (b), respectively.

the null direct effect in the linear demand case pertains to the fact that firms' profits depend on the quality signal ($\hat{q}_t$) only through its mean. For the purpose of illustration, let us temporarily consider θ as a deterministic number. Then, it follows that $\mathsf{E}[\ell(d_t)] = \alpha + \theta \mathsf{E}[\hat{q}_t] - p_t = \alpha + \theta q - p_t$ in the linear demand case. However, if the demand function is concave, then $\mathsf{E}[\ell(d_t)] \leq \ell(\mathsf{E}[d_t]) = \ell(\alpha + \theta q - p_t)$ from the Jensen inequality—hence the negative direct effect. Conversely, if the demand function is convex, then $\mathsf{E}[\ell(d_t)] \geq \ell(\mathsf{E}[d_t]) = \ell(\alpha + \theta q - p_t)$ from the (reverse) Jensen inequality—hence the positive direct effect. Nevertheless, we emphasize that the magnitude of the direct effects is not significant compared to that of the moderation effects (that is, $|\mathtt{DE}(\gamma)| \ll |\mathtt{ME}(\gamma)|$), especially when the demand is only mildly nonlinear (e.g., $\ell(x) = x^{0.75}$ or $\ell(x) = x^{1.25}$).

In summary, our numerical experiments based on generalized linear models indicate that our theoretical findings are sufficiently robust against the linearity assumption. For the cases of the commission contract and information shading, similar numerical experiments can be conducted to show the robustness of the linearity assumption, but we omit them to avoid repetition.

EC.3. Proofs for Main Results in §3

Proof of Proposition 1. In the second stage, the optimal retail price is $p_2^* = (\alpha + \hat{\theta}\hat{q}_2 + c)/2$ such that

$$\begin{aligned}\Pi_2^*(q) &= \frac{1}{4}\left(\pi(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[\left(q^2 + \sigma_p^2 \frac{d_1\gamma}{d_1\gamma+1}\right)\hat{\theta}(2\theta - \hat{\theta})\right]\right) \\ &\to \begin{cases} \frac{1}{4}(\pi(q) + \phi\sigma_\theta^2 q^2) & \text{as } \gamma \to 0 \\ \frac{1}{4}(\pi(q) + \phi\sigma_\theta^2(q^2 + \sigma_p^2) & \text{as } \gamma \to \infty, \end{cases}\end{aligned}$$

where the limit follows from the dominated convergence theorem and the fact that $d_1\gamma/(d_1\gamma+1) \to 0$ as $\gamma \to 0$ and $d_1\gamma/(d_1\gamma+1) \to 1$ as $\gamma \to \infty$ almost surely. For the first-stage price, recall that $p_1^* \to \bar{p}_1$ as $\gamma \to 0$ or as $\gamma \to \infty$, where $\bar{p}_1$ is defined in (5). Accordingly, the first-stage profit satisfies

$$\Pi_1^*(q) \to \frac{1}{4}\pi(q) \quad \text{as } \gamma \to 0 \text{ or } \gamma \to \infty.$$

Combining these results, we deduce that the joint firm's expected profit satisfies

$$\Pi^*(q) \to \begin{cases} \frac{1}{4}(2\pi(q) + \phi\sigma_\theta^2 q^2) & \text{as } \gamma \to 0 \\ \frac{1}{4}(2\pi(q) + \phi\sigma_\theta^2(q^2 + \sigma_p^2) + \mu_\theta^2\sigma_p^2) & \text{as } \gamma \to \infty. \end{cases}$$

Next, we show that the first-stage price p_1^*, as well as its derivative with respect to ϕ, is continuous in $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$ for some positive constants $\gamma_1, \gamma_2 < \infty$. To this end, observe from the first order condition for the first-stage price p_1^* that

$$\alpha + \mu_\theta q + c = 2p_1^* + \frac{\sigma_p^2}{4}\mathsf{E}\left[\frac{\gamma\hat{\theta}(2\theta - \hat{\theta})}{(d_1\gamma+1)^2}\right]. \tag{EC.5}$$

The second order condition can be written as

$$\frac{\partial^2}{\partial p_1^*}\left((p_1 - c)(\alpha + \mu_\theta q - p_1) + \Pi_2^*(q)\right)\Big|_{p_1 = p_1^*} = -2 + \frac{\sigma_p^2}{2}\mathsf{E}\left[\frac{\gamma^2\hat{\theta}^2(2\theta - \hat{\theta})}{(d_1\gamma+1)^3}\right],$$

where the interchange of the expectation and derivative is justified by the dominated convergence theorem. Note that $\gamma^2/(d_1\gamma+1)^3 \to 0$ almost surely as $\gamma \to 0$ or as $\gamma \to \infty$, so the second order condition is satisfied for $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$ for some positive constants $\gamma_1, \gamma_2 < \infty$. For $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$, the equilibrium price p_1^* is uniquely defined as a solution to (EC.5); thus, p_1^* is continuous with respect to γ. Last, taking the derivative with respect to ϕ on both sides of (EC.5), we have that

$$0 = 2\frac{\partial p_1^*}{\partial\phi} + \frac{\sigma_p^2}{4}\mathsf{E}\left[\frac{\partial p_1}{\partial\phi}\frac{2\gamma^2\hat{\theta}(2\theta - \hat{\theta})}{(d_1\gamma+1)^3} + \frac{\gamma}{(d_1\gamma+1)^2}\frac{\partial(\hat{\theta}(2\theta - \hat{\theta})))}{\partial\phi}\right],$$

from which it can be verified that $\partial p_1^*/\partial\phi$ is continuous with respect to $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$, which completes the proof of the proposition. □

EC.4. Proofs for Main Results in §4

Proof Preliminaries. We write $f(x) \coloneqq \mathcal{O}(g(x))$ if there exist positive numbers M and x_0 such that $|f(x)| \le M|g(x)|$ for $x \ge x_0$. For simple exposition, we introduce the following notations in the proofs:

$$\begin{aligned} &A_{\text{S},a} = A_{\text{S},b} = \hat{\theta}(4\theta - 3\hat{\theta}), A_{\text{S},c} = (4\theta - 3\hat{\theta})^2 \\ &A_{\text{NS},a} = \mu_\theta(4\theta - 2\hat{\theta} - \mu_\theta), A_{\text{NS},b} = (2\hat{\theta} - \mu_\theta)(4\theta - 2\hat{\theta} - \mu_\theta), A_{\text{NS},c} = (4\theta - 2\hat{\theta} - \mu_\theta)^2 \\ &A_{\text{SS},a} \coloneqq (2\hat{\theta} - \tilde{\theta})(4\theta - 2\hat{\theta} - \tilde{\theta}), A_{\text{SS},b} \coloneqq \tilde{\theta}(4\theta - 2\hat{\theta} - \tilde{\theta}) \end{aligned} \tag{EC.6}$$

For each $\text{I} \in \{\text{S}, \text{NS}, \text{SS}\}$, given q and the first-stage demand d_1, the firms' expected profits and consumer surplus in the second stage are expressed as

$$\begin{aligned} \Pi^{\text{P}}_{\text{I},2} &= \frac{1}{16}\left(\pi(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[A_{\text{I},b}\left(q^2 + \sigma_p^2 \frac{d_1\gamma}{d_1\gamma + 1}\right)\right]\right) \\ \Pi^{\text{M}}_{\text{I},2} &= \frac{1}{8}\left(\pi(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[A_{\text{I},a}\left(q^2 + \sigma_p^2 \frac{d_1\gamma}{d_1\gamma + 1}\right)\right]\right) \\ \Pi^{\text{C}}_{\text{I},2} &= \frac{1}{32}\left(\rho(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[A_{\text{I},c}\left(q^2 + \sigma_p^2 \frac{d_1\gamma}{d_1\gamma + 1}\right)\right]\right). \end{aligned} \tag{EC.7}$$

Under information sharing $\text{I} = \text{S}$ and any $q \in \{\underline{q}, \bar{q}\}$, the firms' expected profits and consumer surplus over the two stages for cases with $\gamma \to 0$ or $\gamma \to \infty$ can be written as

$$\begin{aligned} \Pi^{\text{P}}_{\text{S}} &\to \begin{cases} \frac{1}{16}\left(2\pi(q) + \phi\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1}{16}\left(2\pi(q) + \phi\sigma_\theta^2 q^2 + \sigma_p^2(\mu_\theta^2 + \phi\sigma_\theta^2)\right) \text{ as } \gamma \to \infty \end{cases} \\ \Pi^{\text{M}}_{\text{S}} &\to \begin{cases} \frac{1}{8}\left(2\pi(q) + \phi\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1}{8}\left(2\pi(q) + \phi\sigma_\theta^2 q^2 + \sigma_p^2(\mu_\theta^2 + \phi\sigma_\theta^2)\right) \text{ as } \gamma \to \infty \end{cases} \\ \Pi^{\text{C}}_{\text{S}} &\to \begin{cases} \frac{1}{32}\left(2\pi(q) + \sigma_\theta^2 q^2(32 - 15\phi)\right) \text{ as } \gamma \to 0 \\ \frac{1}{32}\left(2\pi(q) + \sigma_\theta^2 q^2(32 - 15\phi) + \sigma_p^2(\mu_\theta^2 + \sigma_\theta^2(16 - 15\phi))\right) \text{ as } \gamma \to \infty. \end{cases} \end{aligned} \tag{EC.8}$$

In the case without information sharing $\text{I} = \text{NS}$, one may similarly deduce that

$$\begin{aligned} \Pi^{\text{P}}_{\text{NS}} &\to \begin{cases} \frac{1}{16}\left(2\pi(q) + 4\phi\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1}{16}\left(2\pi(q) + 4\phi\sigma_\theta^2 q^2 + \sigma_p^2(\mu_\theta^2 + 4\phi\sigma_\theta^2)\right) \text{ as } \gamma \to \infty \end{cases} \\ \Pi^{\text{M}}_{\text{NS}} &\to \begin{cases} \frac{1}{4}\left(\pi(q)\right) \text{ as } \gamma \to 0 \\ \frac{1}{8}\left(2\pi(q) + \sigma_p^2\mu_\theta^2\right) \text{ as } \gamma \to \infty \end{cases} \\ \Pi^{\text{C}}_{\text{NS}} &\to \begin{cases} \frac{1}{32}\left(2\pi(q) + \sigma_\theta^2 q^2(32 - 12\phi)\right) \text{ as } \gamma \to 0 \\ \frac{1}{32}\left(2\pi(q) + \sigma_\theta^2 q^2(32 - 12\phi) + \sigma_p^2(\mu_\theta^2 + \sigma_\theta^2(16 - 12\phi))\right) \text{ as } \gamma \to \infty. \end{cases} \end{aligned} \tag{EC.9}$$

In the case with information sharing with shading in §5.2, one can deduce that

$$\begin{aligned} \Pi^{\text{P}}_{\text{SS}} &\to \begin{cases} \frac{1}{16}\left(2\pi(q) + \sigma_\theta^2 q^2(4\phi - 3\tilde{\phi})\right) \text{ as } \gamma \to 0 \\ \frac{1}{16}\left(2\pi(q) + \sigma_\theta^2(q^2 + \sigma_p^2)(4\phi - 3\tilde{\phi}) + \sigma_p^2\mu_\theta^2\right) \text{ as } \gamma \to \infty \end{cases} \\ \Pi^{\text{M}}_{\text{SS}} &\to \begin{cases} \frac{1}{8}\left(2\pi(q) + \tilde{\phi}\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1}{8}\left(2\pi(q) + \tilde{\phi}\sigma_\theta^2(q^2 + \sigma_p^2) + \sigma_p^2\mu_\theta^2\right) \text{ as } \gamma \to \infty. \end{cases} \end{aligned} \tag{EC.10}$$

The following lemmas are useful in proving the main results. The proofs for auxiliary lemmas in this section are collected in Appendix EC.7.

LEMMA EC.6 **(Continuity)**. *Fix* $\text{I} \in \{\text{S}, \text{NS}, \text{SS}\}$ *and* $q \in \{\underline{q}, \bar{q}\}$. *There exist positive constants* $\gamma_1, \gamma_2 < \infty$ *such that the following holds for* $\gamma \in (0, \gamma_1) \cup (\gamma_2, \infty)$:

(a) The equilibrium price (p_1, w_1), *as well as its derivative with respect to* ϕ, *is continuous in* γ *and satisfies* $(p_1, w_1) \to (\bar{p}, \bar{w})$ *as* $\gamma \to 0$ *or as* $\gamma \to \infty$, *where* $(\bar{p}, \bar{w})$ *is defined in* (10). *Furthermore, as* $\gamma \to 0$ *or as* $\gamma \to \infty$,

$$\left(\frac{\partial p_1^*}{\partial \phi}, \frac{\partial w_1^*}{\partial \phi}\right) \to (0, 0).$$

(b) The expected profit (or surplus), denoted by $\Pi_{\text{I}}^{\text{K}}$, *for each entity* $\text{K} \in \{\text{P}, \text{M}, \text{J}, \text{C}\}$, *as well as its derivative with respect to* ϕ, *is continuous in* γ *and the limit as* $\gamma \to 0$ *or as* $\gamma \to \infty$.

Proof for Proposition 2. From Lemma EC.6, the expected profits are continuous in γ, so it suffices to prove the results in the limiting cases with $\gamma = 0$ and $\gamma = \infty$. First, consider $\gamma = 0$. Without information sharing, note from (EC.9) that $\Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}}$ is independent of ϕ. Furthermore, since $\pi(\bar{q}) < \pi(\underline{q})$, we have that $\Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}} < 0$ and the manufacturer never improves quality. With information sharing, observe from (EC.8) that $\Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}} < 0$ when $\phi = 0$ since $\pi(\bar{q}) < \pi(\underline{q})$, but it is linearly increasing with ϕ. Define the threshold $\Phi_{\mathtt{q}}^0$ such that if $\phi = \Phi_{\mathtt{q}}^0$, then $\Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}} = 0$, from which we obtain the desired threshold in (8). The threshold is well defined because $\Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}}$ is monotonically increasing with ϕ. Furthermore, it follows from Lemma EC.6 that $\Delta_{\mathtt{q}} \Pi_{\text{I}}^{\text{M}}$ and its derivative with respect to ϕ are continuous with respect to $\gamma < \gamma_1$ for some positive constant γ_1. Hence, the preceding arguments for $\gamma = 0$ extend to $\gamma < \gamma_1$.

Next, consider $\gamma = \infty$. In the case with $\text{I} = \text{S}$, note from (EC.8) and Lemma EC.6 that

$$\begin{aligned} \Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}} &= \frac{1}{4}\Big(\pi(\bar{q}) - \pi(\underline{q})\Big) + \frac{1}{8}\phi \sigma_\theta^2 (\bar{q}^2 - \underline{q}^2) \\ \frac{\partial \Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}}}{\partial \phi} &= \frac{1}{8}\sigma_\theta^2 (\bar{q}^2 - \underline{q}^2). \end{aligned} \tag{EC.11}$$

Therefore, $\Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}}$ is increasing with ϕ and $\Delta_{\mathtt{q}} \Pi_{\text{S}}^{\text{M}} > 0$ if and only if $\phi > \Phi_{\mathtt{q}}$, where $\Phi_{\mathtt{q}}$ is defined in (8). In the case with $\text{I} = \text{NS}$, it can be seen from (EC.8) and (EC.9), as well as from Lemma EC.6, that

$$\begin{aligned} \Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}} &= \frac{1}{4}\Big(\pi(\bar{q}) - \pi(\underline{q})\Big) \\ \frac{\partial \Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}}}{\partial \phi} &= 0. \end{aligned}$$

Since $\pi(\bar{q}) < \pi(\underline{q})$, $\Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}} < 0$ and its derivative with respect to ϕ is zero; that is, $\Delta_{\mathtt{q}} \Pi_{\text{NS}}^{\text{M}} < 0$ for any ϕ. This step completes the proof of the proposition. □

Proof for Theorem 1. From Lemma EC.6, the expected profits are continuous in γ, so it suffices to prove the results in the limiting case with $\gamma = 0$. Suppose first that $\phi \le \Phi_{\mathtt{q}}$. In this case, $q_{\text{S}} = q_{\text{NS}} = \underline{q}$ by Proposition 2. From (12) and (13), it can be confirmed that

$$\Delta_{\mathtt{i}} \Pi^{\text{M}} + \Delta_{\mathtt{i}} \Pi^{\text{P}} = -\frac{1}{8}\phi \sigma_\theta^2 \underline{q}^2.$$

Therefore, there does not exist an information fee $\rho \geq 0$ such that the platform is better off with information sharing (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$), and at the same time, the manufacturer chooses to attain the information (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$). Thus, it is optimal for the platform to charge a sufficiently high information fee $\rho \geq \frac{1}{8}\phi\sigma_\theta^2 \underline{q}^2$ such that information is not shared. These observations imply that when $\phi \leq \Phi_{\mathtt{q}}$, the equilibrium strategy is $(\text{I}^*, q^*) = (\text{NS}, \underline{q})$.

Suppose now that $\phi > \Phi_{\mathtt{q}}$ in which case $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$ by Proposition 2. In this case, the platform must set the information fee ρ such that $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$ implies $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$; that is, if the manufacturer chooses to pay the platform and obtain the information, this decision should also benefit the platform. From (12) and (13), this condition can be written as

$$\rho \geq \frac{(\pi(\underline{q}) - \pi(\bar{q}))\underline{q}^2}{3\bar{q}^2 - 4\underline{q}^2} \quad \text{and} \quad 3\bar{q}^2 > 4\underline{q}^2. \tag{EC.12}$$

If (EC.12) is satisfied, then the manufacturer would opt to acquire information (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$) only when it would produce a high-quality product (i.e., $\Delta_{\mathtt{q}}\Pi^{\text{M}}_{\text{S}} > 0$). To see this, observe that

$$\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0 \iff \phi > \tilde{\Phi}^0_{\mathtt{i}} := \frac{2(\pi(\underline{q}) - \pi(\bar{q})) + 8\rho}{\sigma_\theta^2 \bar{q}^2},$$
$$\Delta_{\mathtt{q}}\Pi^{\text{M}}_{\text{S}} > 0 \iff \phi > \Phi^0_{\mathtt{q}} = \frac{2(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)}.$$

Combined with (EC.12), we establish that

$$\tilde{\Phi}^0_{\mathtt{i}} = \frac{2(\pi(\underline{q}) - \pi(\bar{q})) + 8\rho}{\sigma_\theta^2 \bar{q}^2} \geq \frac{6(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2)} > \Phi_{\mathtt{q}}.$$

Therefore, $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$ implies $\Delta_{\mathtt{q}}\Pi^{\text{M}}_{\text{S}} > 0$.

In summary, if $\phi > \Phi_{\mathtt{q}}$, then the platform should set the information fee that satisfies $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$ and $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$. Using the expressions for $\Delta_{\mathtt{i}}\Pi^{\text{P}}$ and $\Delta_{\mathtt{i}}\Pi^{\text{M}}$ in (12) and (13), respectively, we deduce that the information fee, ρ, must satisfy

$$\frac{(\pi(\underline{q}) - \pi(\bar{q}))\underline{q}^2}{3\bar{q}^2 - 4\underline{q}^2} \leq \rho \leq \frac{1}{8}\left(\phi\sigma_\theta^2\bar{q}^2 - 2(\pi(\underline{q}) - \pi(\bar{q}))\right).$$

Therefore, if ϕ is sufficiently large such that

$$\frac{(\pi(\underline{q}) - \pi(\bar{q}))\underline{q}^2}{3\bar{q}^2 - 4\underline{q}^2} \leq \frac{1}{8}\left(\phi\sigma_\theta^2\bar{q}^2 - 2(\pi(\underline{q}) - \pi(\bar{q}))\right) \iff \phi \geq \frac{6(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2)},$$

then there exists an information fee ρ^* such that the manufacturer chooses $(\text{I}^*, q^*) = (\text{S}, \bar{q})$. Otherwise, for any information fee, the manufacturer would choose $(\text{I}^*, q^*) = (\text{NS}, \underline{q})$. This step completes the proof of the theorem. $\square$

Proof for Theorem 2. From Lemma EC.6, the expected profits are continuous in γ, so it suffices to prove the results in the limiting case with $\gamma = \infty$. Since $\pi(\bar{q}) < \pi(\underline{q})$, we have that $q_{\text{NS}} = \underline{q}$ by Proposition 2. Furthermore, $q_{\text{S}} = \bar{q}$ if and only if $\phi \geq \Phi_{\mathtt{q}}$, where $\Phi_{\mathtt{q}}$ is defined in Proposition 2. We now consider two cases: $\phi \leq \Phi_{\mathtt{q}}$ and $\phi > \Phi_{\mathtt{q}}$.

Case 1: $\phi \leq \Phi_{\mathtt{q}}$. In this case, $q_{\text{S}} = q_{\text{NS}} = \underline{q}$. From (EC.8) and (EC.9), we deduce that $\Delta_{\mathtt{i}}\Pi^{\text{P}}$ satisfies

$$\Delta_{\mathtt{i}}\Pi^{\text{M}} + \Delta_{\mathtt{i}}\Pi^{\text{P}} = -\frac{1}{8}\phi\sigma_\theta^2(\underline{q}^2 + \sigma_p^2). \tag{EC.13}$$

This implies that $\Delta_{\mathtt{i}}\Pi^{\text{P}} < 0$ for any $\phi \in (0,1)$. Thus, there does not exist an information fee such that the platform is better off with information sharing (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$) and the manufacturer chooses to attain the information (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$). Therefore, it is optimal for the platform to charge a sufficiently high information fee $\rho \geq \frac{1}{8}\phi\sigma_\theta^2(\underline{q}^2 + \sigma_p^2)$ such that the information is not shared in equilibrium. These observations imply that when $\phi \leq \Phi_{\mathtt{q}}$, the equilibrium strategy is $(\text{I}^*, q^*) = (\text{NS}, \underline{q})$.

Case 2: $\phi > \Phi_{\mathtt{q}}$. In this case, $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$ by Proposition 2. Observe that

$$\begin{aligned} \Delta_{\mathtt{i}}\Pi^{\text{M}} &= \frac{1}{8}\left(2(\pi(\bar{q}) - \pi(\underline{q})) + \phi\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)\right) - \rho \\ \Delta_{\mathtt{i}}\Pi^{\text{P}} &= \frac{1}{16}\left(2(\pi(\bar{q}) - \pi(\underline{q})) + \phi\sigma_\theta^2(\bar{q}^2 - 4q_{\text{NS}}^2 - 3\sigma_p^2)\right) + \rho. \end{aligned} \tag{EC.14}$$

The platform must set the information fee ρ such that $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$ implies $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$; that is, if the manufacturer chooses to attain the information, this decision should also benefit the platform. This condition can be written as

$$\rho \geq \frac{(\pi(\underline{q}) - \pi(\bar{q}))(\underline{q}^2 + \sigma_p^2)}{3\bar{q}^2 - 4\underline{q}^2 - 3\sigma_p^2} \quad \text{and} \quad 3\bar{q}^2 > 4\underline{q}^2 + 3\sigma_p^2. \tag{EC.15}$$

If (EC.15) is satisfied, then the manufacturer would opt for an information subscription (i.e., $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$) only when it would produce the high-quality product (i.e., $\Delta_{\mathtt{q}}\Pi_{\text{S}}^{\text{M}} > 0$). To see this, observe that

$$\begin{aligned} \Delta_{\mathtt{i}}\Pi^{\text{M}} > 0 &\Longleftrightarrow \phi > \tilde{\Phi}_{\mathtt{i}}^\infty := \frac{2(\pi(\underline{q}) - \pi(\bar{q})) + 8\rho}{\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)}, \\ \Delta_{\mathtt{q}}\Pi_{\text{S}}^{\text{M}} > 0 &\Longleftrightarrow \phi > \Phi_{\mathtt{q}} = \frac{2(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)}. \end{aligned}$$

Combined with (EC.15), we establish that

$$\tilde{\Phi}_{\mathtt{i}}^\infty = \frac{2(\pi(\underline{q}) - \pi(\bar{q})) + 8\rho}{\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)} \geq \frac{6(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2 - \sigma_p^2)} > \Phi_{\mathtt{q}}.$$

Therefore, for the information fee ρ that satisfies (EC.15), $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$ implies $\Delta_{\mathtt{q}}\Pi^{\text{M}} > 0$.

In summary, if $\phi > \Phi_{\mathtt{q}}$, then the platform should set the information fee such that $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$ and $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$. Using the expressions for $\Delta_{\mathtt{i}}\Pi^{\text{P}}$ and $\Delta_{\mathtt{i}}\Pi^{\text{M}}$ in (EC.14), we deduce that the information fee, ρ, must satisfy

$$\frac{(\pi(\underline{q}) - \pi(\bar{q}))(\underline{q}^2 + \sigma_p^2)}{3\bar{q}^2 - 4\underline{q}^2 - \sigma_p^2} \leq \rho \leq \frac{1}{8}\left(\phi\sigma_\theta^2\bar{q}^2 - 2(\pi(\underline{q}) - \pi(\bar{q}))\right).$$

Therefore, if ϕ is sufficiently large such that

$$\frac{(\pi(\underline{q})-\pi(\bar{q}))(\underline{q}^2+\sigma_p^2)}{3\bar{q}^2-4\underline{q}^2-\sigma_p^2} \le \frac{1}{8}\left(\phi\sigma_\theta^2(\bar{q}^2+\sigma_p^2)-2(\pi(\underline{q})-\pi(\bar{q}))\right) \Longleftrightarrow \phi \ge \frac{6(\pi(\underline{q})-\pi(\bar{q}))}{\sigma_\theta^2(3\bar{q}^2-4\underline{q}^2-\sigma_p^2)},$$

then there exists an information fee ρ^* such that the manufacturer chooses $(\text{I}^*, q^*) = (\text{S}, \bar{q})$. Otherwise, for any information fee, the manufacturer would choose $(\text{I}^*, q^*) = (\text{NS}, \underline{q})$. This step completes the proof of the theorem. $\square$

EC.5. Proofs for Main Results in §5

EC.5.1. Proofs for §5.1

Proof Preliminaries. We write $f(x) := \mathcal{O}(g(x))$ if there exist positive numbers M and x_0 such that $|f(x)| \le M|g(x)|$ for $x \ge x_0$. To simplify, we introduce the following notation in the proofs:

$$\begin{aligned} B_{\text{S},a} &= \hat{\theta}(2\theta-\hat{\theta}), B_{\text{S},b} = (2\theta-\hat{\theta})^2 \\ B_{\text{NS},a} &= \mu_\theta(2\theta-\mu_\theta), B_{\text{NS},b} = (2\theta-\mu_\theta)^2. \end{aligned} \tag{EC.16}$$

Additionally, recall the definition $\pi_u(q) = (\alpha+\mu_\theta q - c_q/(1-u))^2$. It is also useful to define

$$\begin{aligned} \tilde{\pi}_u(q) &:= (\alpha+\mu_\theta q)^2 - \frac{c_q^2}{(1-u)^2} \\ \bar{\pi}(q) &:= (1-u)\pi_u(q) + u\tilde{\pi}_u(q). \end{aligned} \tag{EC.17}$$

For each $\text{I} \in \{\text{S}, \text{NS}\}$, given q and the first-stage demand d_1, the firms' expected profits and consumer surplus in the second stage are expressed as

$$\begin{aligned} \bar{\Pi}_{\text{I},2}^{\text{P}} &= \frac{u}{4}\left(\tilde{\pi}_u(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[B_{\text{I},a}\left(q^2+\sigma_p^2\frac{d_1\gamma}{d_1\gamma+1}\right)\right]\right) \\ \bar{\Pi}_{\text{I},2}^{\text{M}} &= \frac{1-u}{4}\left(\pi_u(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[B_{\text{I},a}\left(q^2+\sigma_p^2\frac{d_1\gamma}{d_1\gamma+1}\right)\right]\right) \\ \bar{\Pi}_{\text{I},2}^{\text{C}} &= \frac{1}{8}\left(\pi_u(q) - \mu_\theta^2 q^2 + \mathsf{E}\left[B_{\text{I},b}\left(q^2+\sigma_p^2\frac{d_1\gamma}{d_1\gamma+1}\right)\right]\right). \end{aligned} \tag{EC.18}$$

Under information sharing $\text{I} = \text{S}$ and any $q \in \{\underline{q}, \bar{q}\}$, the firms' expected profits and consumer surplus over the two stages satisfy

$$\begin{aligned} \bar{\Pi}_{\text{S}}^{\text{P}} &\to \begin{cases} \frac{u}{4}\left(2\tilde{\pi}_u(q) + \phi\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{u}{4}\left(2\tilde{\pi}_u(q) + \phi\sigma_\theta^2 q^2 + \sigma_p^2(\mu_\theta^2+\phi\sigma_\theta^2)\right) \text{ as } \gamma \to \infty \end{cases} \\ \bar{\Pi}_{\text{S}}^{\text{M}} &\to \begin{cases} \frac{1-u}{4}\left(2\pi_u(q) + \phi\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1-u}{4}\left(2\pi_u(q) + \phi\sigma_\theta^2 q^2 + \sigma_p^2(\mu_\theta^2+\phi\sigma_\theta^2)\right) \text{ as } \gamma \to \infty \end{cases} \\ \bar{\Pi}_{\text{S}}^{\text{C}} &\to \begin{cases} \frac{1}{8}\left(2\pi_u(q) + \sigma_\theta^2 q^2(4-3\phi)\right) \text{ as } \gamma \to 0 \\ \frac{1}{8}\left(2\pi_u(q) + \sigma_\theta^2(q^2+\sigma_p^2)(4-3\phi)\right) \text{ as } \gamma \to \infty. \end{cases} \end{aligned} \tag{EC.19}$$

Without information sharing $\text{I} = \text{NS}$ and any $q \in \{\underline{q}, \bar{q}\}$, the firms' expected profits and consumer surplus over the two stages satisfy

$$\bar{\Pi}^{\text{P}}_{\text{NS}} \to \begin{cases} \frac{u}{4}\left(2\tilde{\pi}_u(q)\right) \text{ as } \gamma \to 0 \\ \frac{u}{4}\left(2\tilde{\pi}_u(q) + \mu_\theta^2\sigma_p^2\right) \text{ as } \gamma \to \infty \end{cases}$$
$$\bar{\Pi}^{\text{M}}_{\text{NS}} \to \begin{cases} \frac{1-u}{4}\left(2\pi_u(q)\right) \text{ as } \gamma \to 0 \\ \frac{1-u}{4}\left(2\pi_u(q) + \mu_\theta^2\sigma_p^2\right) \text{ as } \gamma \to \infty \end{cases} \tag{EC.20}$$
$$\bar{\Pi}^{\text{C}}_{\text{NS}} \to \begin{cases} \frac{1}{8}\left(2\pi_u(q) + 4\sigma_\theta^2 q^2\right) \text{ as } \gamma \to 0 \\ \frac{1}{8}\left(2\pi_u(q) + 4\sigma_\theta^2(q^2 + \sigma_p^2)\right) \text{ as } \gamma \to \infty. \end{cases}$$

The following lemma will be useful in proving the main results.

Lemma EC.7 **(Continuity: Commission Contract)**. *Consider a commission contract and fix an information contract* $\text{I} \in \{\text{S}, \text{NS}\}$ *and quality* $q \in \{\underline{q}, \bar{q}\}$. *There exist positive constants* $\gamma_1, \gamma_2 < \infty$ *such that the following holds for* $\gamma \in [0, \gamma_1] \cup [\gamma_2, \infty)$*:*

(a) The equilibrium price p_1^**, as well as its derivative with respect to* ϕ*, is continuous in* γ*. Furthermore, the limits satisfy*

$$p_1^* \to \frac{1}{2}\left(\alpha + \mu_\theta q + \frac{c_1}{1-u}\right) \quad \text{and} \quad \frac{\partial p_1^*}{\partial \phi} \to 0$$

as $\gamma \to 0$ *or as* $\gamma \to \infty$*.*

(b) The expected profit (or surplus), denoted by $\bar{\Pi}^{\text{K}}_{\text{I}}$*, for each entity* $\text{K} \in \{\text{P}, \text{M}, \text{C}\}$*, as well as its derivative with respect to* ϕ*, is continuous in* γ*. Furthermore, the limit of* $\bar{\Pi}^{\text{K}}_{\text{I}}$ *is characterized in* (EC.19) *and* (EC.20)*.*

Proof for Proposition 3. From Lemma EC.7, the expected profits are continuous in γ, so it suffices to prove the results in the limiting cases with $\gamma = 0$ and $\gamma = \infty$.

Without information sharing, for both cases with $\gamma = 0$ and $\gamma = \infty$, we have that

$$\Delta_{\mathsf{q}}\bar{\Pi}^{\text{M}}_{\text{NS}} = \frac{1-u}{4}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})\right),$$

which is independent of the strength of the propensity signal, ϕ. Furthermore, since $\pi_u(\bar{q}) < \pi_u(\underline{q})$, it can be easily checked that $\Delta_{\mathsf{q}}\Pi^{\text{M}}_{\text{NS}} < 0$ and the manufacturer never improves quality. With information sharing, observe that

$$\Delta_{\mathsf{q}}\bar{\Pi}^{\text{M}}_{\text{S}} = \frac{1-u}{4}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})) + \phi\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)\right).$$

Note that Δ_{q} is increasing with ϕ such that the manufacturer would produce the high-quality product if

$$\Delta_{\mathsf{q}}\bar{\Pi}^{\text{M}}_{\text{S}} > 0 \Longleftrightarrow \phi > \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)}.$$

This completes the proof of the proposition. $\square$

Proof for Theorem 3. From Lemma EC.7, the expected profits and their derivative with respect to ϕ are continuous in γ, so it suffices to prove the results in the limiting cases with $\gamma = 0$ and $\gamma = \infty$. Suppose first that $\phi \le \bar{\Phi}_{\mathtt{q}}$. In this case, $q_{\text{S}} = q_{\text{NS}} = \underline{q}$ by Proposition 3. From (EC.19) and (EC.20) it can be verified that

$$\Delta_{\mathtt{i}}\Pi^{\text{M}} + \Delta_{\mathtt{i}}\Pi^{\text{P}} = \begin{cases} \frac{1}{4}\phi\sigma_\theta^2\underline{q}^2 > 0 & \text{for } \gamma = 0 \\ \frac{1}{4}\phi\sigma_\theta^2(\underline{q}^2 + \sigma_p^2) > 0 & \text{for } \gamma = \infty. \end{cases}$$

In both cases with $\gamma = 0$ and $\gamma = \infty$, we deduce that there exists an information fee $\rho \ge 0$ such that both the platform and manufacturers are better off with information sharing; that is, $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$ and $\Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$ for some $\rho \ge 0$. Therefore, when $\phi \le \bar{\Phi}_{\mathtt{q}}$, the equilibrium strategy is $(\text{I}^*, q^*) = (\text{S}, \underline{q})$.

Next, suppose that $\phi > \bar{\Phi}_{\mathtt{q}}$ in which case $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$ by Proposition 3. From (EC.19) and (EC.20), observe that

$$\Delta_{\mathtt{i}}\Pi^{\text{P}} = \begin{cases} \frac{u}{4}\left(2(\tilde{\pi}_u(\bar{q}) - \tilde{\pi}_u(\underline{q})) + \phi\sigma_\theta^2\bar{q}^2\right) & \text{for } \gamma = 0 \\ \frac{u}{4}\left(2(\tilde{\pi}_u(\bar{q}) - \tilde{\pi}_u(\underline{q})) + \phi\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)\right) & \text{for } \gamma = \infty. \end{cases}$$

Note that $\pi_u(\bar{q}) < \pi_u(\underline{q})$ and $u < \bar{u}$ imply that $\tilde{\pi}_u(\bar{q}) > \tilde{\pi}_u(\underline{q})$. Therefore, we deduce that $\Delta_{\mathtt{i}}\Pi^{\text{P}} > 0$ for both cases with $\gamma = 0$ and $\gamma = \infty$ and that the platform is always better off with information sharing. Thus, it suffices to set the information fee ρ such that the manufacturer chooses to attain the information. To see this, observe that

$$\Delta_{\mathtt{i}}\Pi^{\text{M}} = \begin{cases} \frac{1-u}{4}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})) + \phi\sigma_\theta^2\bar{q}^2\right) & \text{for } \gamma = 0 \\ \frac{1-u}{4}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})) + \phi\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)\right) & \text{for } \gamma = \infty. \end{cases}$$

In the case with $\gamma = 0$, the manufacturer will attain the information if $\phi > \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2\bar{q}^2}$. Note that the latter condition is satisfied by the assumption that $\phi > \bar{\Phi}_{\mathtt{q}}$ and the fact that

$$\bar{\Phi}_{\mathtt{q}} = \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)} > \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2\bar{q}^2}.$$

Likewise, in the case with $\gamma = \infty$, the manufacturer will attain the information if $\phi > \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)}$. Note that the latter condition is satisfied by the assumption that $\phi > \bar{\Phi}_{\mathtt{q}}$ and the fact that

$$\bar{\Phi}_{\mathtt{q}} = \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \underline{q}^2)} > \frac{2(\pi_u(\underline{q}) - \pi_u(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 + \sigma_p^2)}.$$

In summary, when $\phi > \bar{\Phi}_{\mathtt{q}}$, the manufacturer always attains the information and chooses to produce a high-quality product; that is, $(\text{I}^*, q^*) = (\text{S}, \bar{q})$. This step completes the proof of the theorem. □

Proof of Proposition 4. Recall from Proposition 2 that $q_{\text{I}}|_{\gamma=0} = q_{\text{I}}|_{\gamma=\infty}$. Therefore, from the expressions in (EC.8) and (EC.9), along with the continuity arguments in Lemma EC.6, the desired result in (16) follows from straightforward algebra and will be omitted. The proof of the theorem is complete. □

Proof of Proposition 5. The desired results immediately follow from the equilibrium outcomes (q^*, I^*) under a wholesale price contract (Proposition 2 and Theorems 1-2) and under a commission contract (Proposition 3 and Theorem 3), as well as the expressions for profits ((EC.8)-(EC.9) and (EC.19)-(EC.20)). We omit the detailed steps. □

EC.5.2. Proofs for §5.2

Proof of Proposition 6. Consider the case without product reviews ($\gamma = 0$). From Lemma EC.6, $\Pi^{\text{K}}_{\text{SS}}$ for $\gamma = 0$ can be obtained by taking the limit as $\gamma \to 0$ in (EC.10). Thus, $\Pi^{\text{P}}_{\text{SS}}$ decreases with $\tilde{\phi}$ with a jump at $\tilde{\phi} = \Phi_{\mathtt{q}}$. Therefore, the optimal shading level should be set at either $\tilde{\phi} = \Phi_{\mathtt{q}}$ or $\tilde{\phi} = 0$. We consider three cases. First, suppose that $\phi > \Phi^0_{\mathtt{i}}$. Then, $\Pi^{\text{P}}_{\text{S}} \geq \Pi^{\text{P}}_{\text{NS}}$ by Theorem 1 such that the platform is better off with information sharing (without shading). If the platform chooses $\tilde{\phi} > \Phi_{\mathtt{q}}$ ($\tilde{\phi} \leq \Phi_{\mathtt{q}}$), then the manufacturer would choose high (low) quality by Proposition 2. If the manufacturer chooses to produce a low-quality product, then the platform would be worse off (Theorem 1). Considering the fact that $\Pi^{\text{P}}_{\text{SS}}$ is decreasing with $\tilde{\phi}$, we deduce that $\Pi^{\text{P}}_{\text{SS}} \geq \Pi^{\text{P}}_{\text{S}}$ for any $\tilde{\phi} \in [\Phi_{\mathtt{q}}, \phi]$, and thus the platform shares information with the optimal shading level $\tilde{\phi}^* = \Phi^0_{\mathtt{i}}$. Second, suppose that $\phi \in (\Phi_{\mathtt{q}}, \Phi^0_{\mathtt{i}}]$, in which case, $\Pi^{\text{P}}_{\text{S}} < \Pi^{\text{P}}_{\text{NS}}$ by Theorem 1. If the platform sets $\tilde{\phi} = \Phi_{\mathtt{q}}$, then the manufacturer will choose high quality such that

$$\Pi^{\text{P}}_{\text{SS}} = \frac{1}{16}\left(2\pi(\bar{q}) + \sigma^2_\theta \bar{q}^2 (4\phi - 3\Phi_{\mathtt{q}})\right), \tag{EC.21}$$

whereas if the platform sets $\tilde{\phi} = 0$, then the manufacturer will choose low quality such that

$$\Pi^{\text{P}}_{\text{SS}} = \frac{1}{16}\left(2\pi(\underline{q}) + 4\sigma^2_\theta \underline{q}^2 \phi\right) = \Pi^{\text{P}}_{\text{NS}}. \tag{EC.22}$$

Comparing (EC.21) and (EC.22) and recalling the definition of $\Phi_{\mathtt{q}}$ in (8), we establish that the platform shares information and that the optimal shading level is $\tilde{\phi}^* = \Phi_{\mathtt{q}}$. Last, suppose that $\phi \in [0, \Phi_{\mathtt{q}}]$, in which case the manufacturer always chooses low quality and the platform does not share information (Theorem 1). It trivially follows that $\tilde{\phi}^* = 0$.

Next, consider the case with perfectly informative reviews ($\gamma = \infty$). By the continuity result of Lemma EC.6, the profit function $\Pi^{\text{P}}_{\text{SS}}$ can be obtained by taking $\gamma \to \infty$. Furthermore, from (EC.10) we can write that

$$\frac{\partial \Pi^{\text{P}}_{\text{SS}}}{\partial \tilde{\phi}} = -\frac{3}{16}\sigma^2_\theta (q^2 + \sigma^2_p).$$

Therefore, $\Pi^{\text{P}}_{\text{SS}}$ decreases with $\tilde{\phi}$ with a jump at $\tilde{\phi} = \Phi_{\mathtt{q}}$. Therefore, the optimal shading level should be set at either $\tilde{\phi} = \Phi_{\mathtt{q}}$ or $\tilde{\phi} = 0$. As in the case without product reviews, we consider three cases. First, suppose that $\phi > \Phi^\infty_{\mathtt{i}}$. Then, $\Pi^{\text{P}}_{\text{S}} \geq \Pi^{\text{P}}_{\text{NS}}$ by Theorem EC.1 such that the platform is better off with information sharing (without shading) than without information sharing. If the platform chooses $\tilde{\phi} > \Phi_{\mathtt{q}}$ ($\tilde{\phi} \leq \Phi_{\mathtt{q}}$), then by Proposition 2 the manufacturer would choose high (low) quality.

If the manufacturer chooses to produce a low-quality product, then the platform would be worse off (Theorem EC.1), and thus, the platform would never choose $\tilde{\phi} \le \Phi_{\mathtt{q}}$. Considering the fact that $\Pi^{\mathrm{P}}_{\mathrm{SS}}$ is decreasing with $\tilde{\phi}$, we deduce that $\Pi^{\mathrm{P}}_{\mathrm{SS}} \ge \Pi^{\mathrm{P}}_{\mathrm{S}}$ for any $\tilde{\phi} \in [\Phi_{\mathtt{q}}, \phi]$, and thus, the platform shares information with the optimal shading level $\tilde{\phi}^* = \Phi_{\mathtt{q}}$. Second, suppose that $\phi \in (\Phi_{\mathtt{q}}, \Phi^{\infty}_{\mathtt{i}}]$, in which case, by Theorem EC.1, $\Pi^{\mathrm{P}}_{\mathrm{S}} < \Pi^{\mathrm{P}}_{\mathrm{NS}}$. If the platform sets $\tilde{\phi} = \Phi_{\mathtt{q}}$, then the manufacturer will choose high quality such that

$$\Pi^{\mathrm{P}}_{\mathrm{SS}} = \frac{1}{16}(2\pi(\bar{q}) + \sigma_\theta^2(q^2 + \sigma_p^2)(4\phi - 3\Phi_{\mathtt{q}}). \tag{EC.23}$$

whereas if the platform sets $\tilde{\phi} = 0$, then the manufacturer will choose low quality such that

$$\Pi^{\mathrm{P}}_{\mathrm{SS}} = \frac{1}{16}(2\pi(\bar{q}) + 4\phi\sigma_\theta^2(q^2 + \sigma_p^2) = \Pi^{\mathrm{P}}_{\mathrm{NS}}. \tag{EC.24}$$

Comparing (EC.23) and (EC.24), we establish that the information sharing is arranged and that the optimal shading level is $\tilde{\phi}^* = \Phi_{\mathtt{q}}$. Last, suppose that $\phi \in [0, \Phi_{\mathtt{q}}]$, in which case the manufacturer always chooses low quality and the platform does not share information (Theorem EC.1). It trivially follows that $\tilde{\phi}^* = 0$. This step completes the proof of the proposition. □

Proof for Corollary 1. Let $\Pi^{\mathrm{K}}_{\mathrm{SS}}$ denote the profit of firm $\mathrm{K} \in \{\mathrm{P}, \mathrm{M}\}$ when the shared information is shaded. In this case, the expected profits are characterized in the asymptotic cases as $\gamma \to 0$ and $\gamma \to \infty$; see (EC.10). Furthermore, from Lemma EC.6, we have that the expected profits are continuous with respect to γ. Hence, it suffices to consider the cases with $\gamma = 0$ and $\gamma = \infty$. Recall that $\Phi^0_{\mathtt{q}} = \Phi^{\infty}_{\mathtt{q}} = \Phi_{\mathtt{q}}$ (Proposition 2). If $\phi \le \Phi_{\mathtt{q}}$, information is not shared in equilibrium and the firms' profits are indifferent to the level of information shading $\tilde{\phi}$. For the remainder of the proof, assume that $\phi > \Phi_{\mathtt{q}}$ in which case $q^* = \bar{q}$.

When $\gamma = 0$, observe that the manufacturer's profit is $\Pi^{\mathrm{M}}_{\mathrm{SS}} = \frac{1}{8}(2\pi(\bar{q}) + \sigma_\theta^2\bar{q}^2\tilde{\phi}^*)$. Hence, under the optimal shading level $\tilde{\phi}^* = \Phi_{\mathtt{q}} < \phi$, the manufacturer's profit is decreased compared to the case without shading. Furthermore, observe that the platform's profit is $\Pi^{\mathrm{M}}_{\mathrm{SS}} = \frac{1}{16}(2\pi(\bar{q}) + \sigma_\theta^2\bar{q}^2(4\phi - 3\tilde{\phi}^*)$. Therefore, the platform's profit is increased compared to the case without shading. Finally, the joint profit is $\Pi^{\mathrm{J}}_{\mathrm{SS}} = \frac{1}{16}(6\pi(\bar{q}) + \sigma_\theta^2\bar{q}^2(4\phi - \tilde{\phi}^*)$, and therefore, compared to the case without shading, the joint profit increases. The proof for $\gamma = \infty$ would follow the same logical steps and is thus omitted. The proof of the corollary is complete. □

EC.6. Proofs for Main Results in §6

Proof for Proposition 7. First, consider a wholesale price contract. By Lemma EC.6, it suffices to consider the two extreme cases with $\gamma = 0$ and $\gamma = \infty$. Fix $\gamma = 0$. From the proof of Theorem 1, we know that information sharing between the firms is arranged if and only if $\Delta_{\mathtt{i}}\Pi^{\mathrm{P}} + \Delta_{\mathtt{i}}\Pi^{\mathrm{M}} > 0$.

Additionally, the manufacturer chooses high quality (i.e., $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$), and further, $\bar{q}^2 - \underline{q}^2 > \Psi_{\mathtt{i}}^0 = \frac{1}{3}\underline{q}^2$ by condition (ii) of Theorem 1. Observe from (EC.8) and (EC.9) that

$$\begin{aligned}\Delta_{\mathtt{i}}\Pi^{\text{C}} &= \frac{1}{16}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\sigma_\theta^2}{32}\left(\bar{q}^2(32-15\phi) - \underline{q}^2(32-12\phi)\right) \\ &\geq \frac{1}{16}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\sigma_\theta^2}{32}\left(17\bar{q}^2 - 20\underline{q}^2\right),\end{aligned} \tag{EC.25}$$

where the inequality follows by replacing ϕ with one. Moreover, from (EC.8) and (EC.9) we have

$$\begin{aligned}\Delta_{\mathtt{i}}\Pi^{\text{P}} + \Delta_{\mathtt{i}}\Pi^{\text{M}} &= \frac{3}{8}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\phi\sigma_\theta^2}{16}\left(3\bar{q}^2 - 4\underline{q}^2\right) \\ &\leq \frac{3}{8}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\sigma_\theta^2}{16}\left(3\bar{q}^2 - 4\underline{q}^2\right),\end{aligned} \tag{EC.26}$$

where the inequality follows by replacing ϕ with one. Combining these observations, it can be seen that

$$6\Delta_{\mathtt{i}}\Pi^{\text{C}} - (\Delta_{\mathtt{i}}\Pi^{\text{P}} + \Delta_{\mathtt{i}}\Pi^{\text{M}}) \geq \sigma_\theta^2\left(3\bar{q}^2 - 4\underline{q}^2\right) > 0, \tag{EC.27}$$

where the last inequality follows from the fact that $\bar{q}^2 - \underline{q}^2 > \Psi_{\mathtt{i}}^0 = \frac{1}{3}\underline{q}^2$ under information sharing (Theorem 1). Thus, we establish that $\Delta_{\mathtt{i}}\Pi^{\text{C}} > 0$.

Next, consider the case with perfectly informative reviews ($\gamma = \infty$) and assume that the information is shared in equilibrium (or equivalently, $\Delta_{\mathtt{i}}\Pi^{\text{P}} + \Delta_{\mathtt{i}}\Pi^{\text{M}} > 0$). From Theorem 2, information sharing implies that the manufacturer improves quality $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$, and further, $\bar{q}^2 - \underline{q}^2 > \Psi_{\mathtt{i}}^\infty = \frac{1}{3}\underline{q}^2 + \frac{1}{3}\sigma_p^2$. Hence, one can write

$$\begin{aligned}\Delta_{\mathtt{i}}\Pi^{\text{C}} &= \frac{1}{16}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\sigma_\theta^2}{32}\left(\bar{q}^2(32-15\phi) - \underline{q}^2(32-12\phi) - 3\phi\sigma_p^2\right) \\ &\geq \frac{1}{16}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{\sigma_\theta^2}{32}\left(17\bar{q}^2 - 20\underline{q}^2 - 3\sigma_p^2\right),\end{aligned} \tag{EC.28}$$

where the inequality follows by replacing ϕ with one. Moreover, from (EC.8) and (EC.9) for $\gamma = \infty$,

$$\begin{aligned}\Delta_{\mathtt{i}}\Pi^{\text{P}} + \Delta_{\mathtt{i}}\Pi^{\text{M}} &= \frac{3}{8}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{1}{16}\left(\phi\sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2 - 3\sigma_p^2)\right) \\ &\leq \frac{3}{8}\left(\pi(\bar{q}) - \pi(\underline{q})\right) + \frac{1}{16}\left(\sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2 - 3\sigma_p^2)\right),\end{aligned} \tag{EC.29}$$

where the inequality follows by replacing ϕ with one. Combining these observations, it can be shown that (EC.27) holds for $\gamma = \infty$. Thus, we deduce that

$$6\Delta_{\mathtt{i}}\Pi^{\text{C}} - (\Delta_{\mathtt{i}}\Pi^{\text{P}} + \Delta_{\mathtt{i}}\Pi^{\text{M}}) \geq \sigma_\theta^2(3\bar{q}^2 - 4\underline{q}^2 + \sigma_p^2) > 0,$$

where the last inequality follows from the condition that $\bar{q}^2 - \underline{q}^2 > \Psi_{\mathtt{i}}^\infty = \frac{1}{3}\underline{q}^2 + \frac{1}{3}\sigma_p^2$ under information sharing (Theorem 2). Hence, we deduce that $\Delta_{\mathtt{i}}\Pi^{\text{C}} > 0$.

Now, consider a commission contract. From Theorems 1 and 2, the equilibrium quality satisfies $q_{\text{S}} = \bar{q} > \underline{q} = q_{\text{NS}}$. From the expressions in (EC.19) and (EC.20), we deduce that

$$\Delta_{\mathtt{i}}\Pi^{\text{C}} = \begin{cases} \frac{1}{8}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})) + \sigma_\theta^2((4-3\phi)\bar{q}^2 - 4\underline{q}^2)\right) & \text{for } \gamma = 0 \\ \frac{1}{8}\left(2(\pi_u(\bar{q}) - \pi_u(\underline{q})) + \sigma_\theta^2((4-3\phi)(\bar{q}^2 + \sigma_p^2) - 4\underline{q}^2)\right) & \text{for } \gamma = \infty, \end{cases}$$

from which we deduce that customers are worse off if and only if $\phi > \bar{\Phi}_{\mathtt{c}}^{\gamma}$ for the threshold $\bar{\Phi}_{\mathtt{c}}^{\gamma}$ defined in the statement of the proposition. This step completes the proof of the proposition. □

Proof of Proposition 8. Consider a wholesale price contract. If conditions (i) and (ii) in the statement of the proposition are satisfied, then $(\text{I}^*, q^*) = (\text{S}, \bar{q})$ for a sufficiently small γ (Theorem 1), whereas $(\text{I}^*, q^*) = (\text{NS}, \underline{q})$ as $\gamma \to \infty$ (Theorem 2). Using the continuity argument in Lemma EC.6, it suffices to consider the limiting cases $\gamma = 0$ and $\gamma = \infty$. Observe from (EC.8) and (EC.9) that

$$\begin{aligned} \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=\infty} - \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=0} &= \Pi_{\text{S}}^{\text{C}}(\bar{q})|_{\gamma=0} - \Pi_{\text{NS}}^{\text{C}}(\underline{q})|_{\gamma=\infty} \\ &= \frac{1}{32}\left(2(\pi(\underline{q}) - \pi(\bar{q})) + \sigma_\theta^2(32-15\phi)(\underline{q}^2 - \bar{q}^2) + \sigma_p^2(\mu_\theta^2 + \sigma_\theta^2(16-15\phi)\right). \end{aligned} \tag{EC.30}$$

From the fact that $\Phi_{\mathtt{i}}^0 \le \phi \le 1$, we deduce that

$$\frac{2(\pi(\underline{q}) - \pi(\bar{q}))}{\sigma_\theta^2(\bar{q}^2 - \frac{4}{3}\underline{q}^2)} \le 1. \tag{EC.31}$$

Combining (EC.31) into (EC.30), we have that

$$\begin{aligned} \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=\infty} - \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=0} &\le \frac{1}{3}\phi\sigma_\theta^2\left(48\bar{q}^2 - 49\underline{q}^2 - 45\sigma_p^2\right) + 32\sigma_\theta^2(\bar{q}^2 - \underline{q}^2) + \sigma_p^2(\mu_\theta^2 + 16\sigma_\theta^2) \\ &\le \frac{1}{3}\sigma_\theta^2\left(-48\bar{q}^2 + 47\underline{q}^2 + 3\sigma_p^2\right) + \sigma_p^2\mu_\theta^2 \\ &\le \frac{1}{3}\sigma_\theta^2\left(-17\underline{q}^2 - 13\sigma_p^2\right) + \sigma_p^2\mu_\theta^2 \\ &= -\frac{17}{3}\sigma_\theta^2\underline{q}^2 + \sigma_p^2\left(\mu_\theta^2 + \frac{13}{3}\sigma_\theta^2\right), \end{aligned}$$

where the first inequality follows from (EC.31), the second follows from condition (ii) and by replacing ϕ with one, and the remainder follows from straightforward algebra. This logic implies that if

$$\sigma_p^2 \le \bar{\sigma}_p^2 := \frac{\frac{17}{3}\sigma_\theta^2\underline{q}^2}{\mu_\theta^2 + \frac{13}{3}\sigma_\theta^2},$$

then $\Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=\infty} - \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=0} \le 0$; that is, customers are worse off with perfectly informative reviews.

Next, consider a commission contract. From Theorem 3, we deduce that the equilibrium (I^*, q^*) is the same between cases $\gamma = 0$ and $\gamma = \infty$. If $\phi \le \bar{\Phi}_{\mathtt{q}}$, then $(\text{I}^*, q^*) = (\text{S}, \underline{q})$ for both cases with $\gamma = 0$ and $\gamma = \infty$. Thus, using (EC.19), we can write

$$\Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=\infty} - \Pi_{\text{I}^*}^{\text{C}}(q^*)|_{\gamma=0} = \Pi_{\text{S}}^{\text{C}}(\underline{q})|_{\gamma=\infty} - \Pi_{\text{S}}^{\text{C}}(\underline{q})|_{\gamma=0} = \frac{1}{8}\sigma_\theta^2\sigma_p^2(4-3\phi) > 0,$$

where the last inequality follows from the fact that $\phi \le 1$. Likewise, when $\phi > \bar{\Phi}_q$, then $(\text{I}^*, q^*) = (\text{S}, \bar{q})$ for both cases with $\gamma = 0$ and $\gamma = \infty$. Thus, using (EC.20), we can write

$$\Pi^{\text{C}}_{\text{I}^*}(q^*)|_{\gamma=\infty} - \Pi^{\text{C}}_{\text{I}^*}(q^*)|_{\gamma=0} = \Pi^{\text{C}}_{\text{S}}(\bar{q})|_{\gamma=\infty} - \Pi^{\text{C}}_{\text{S}}(\bar{q})|_{\gamma=0} = \frac{1}{2}\sigma_\theta^2\sigma_p^2 > 0.$$

Therefore, we conclude that $\Pi^{\text{C}}_{\text{I}^*}(q^*)|_{\gamma=\infty} - \Pi^{\text{C}}_{\text{I}^*}(q^*)|_{\gamma=0} > 0$ holds for any $\phi \in [0,1)$. This step concludes the proof of the proposition. □

EC.7. Proofs for Auxiliary Results

Proof of Lemma EC.1. First, consider a wholesale price contract. Observe from (9) that the demand function in the second stage is $d_2 = \frac{1}{4}(\alpha + (4\theta - 3\hat{\theta})\hat{q}_2 - c)$. Since $\theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}]$ and $\hat{q}_2 \in [\underline{q}, \bar{q}]$, it follows that $d_2 > 0$ with probability one if $\alpha > a_0 := \max(\bar{q}(3\bar{\theta} - 4\underline{\theta}), \underline{q}(3\bar{\theta} - 4\underline{\theta})) + \bar{c}$. For first-stage demand, dividing by α and taking $\alpha \to \infty$ on both sides of (EC.34), we deduce that $(w_1, p_1) - (\bar{w}, \bar{p}) \to 0$ as $\alpha \to \infty$, where $(\bar{w}, \bar{p})$ is defined in (10). Let $\overline{d}_1 = \frac{1}{4}(\alpha + (4\theta - 3\mu_\theta)q - c)$ be the demand that corresponds to the retail price $\bar{p}$, which is positive with probability one for any $\alpha \ge a_0$. Then, since d_1 is a continuous function of p_1, by the continuous mapping theorem, we deduce that $d_1 - \overline{d}_1 \to 0$ almost surely as $\alpha \to \infty$. Since $\overline{d}_1$ is increasing with α, there exists a constant $\alpha_0 \ge a_0$ such that d_1 is positive with probability one for any $\alpha \ge \alpha_0$.

Next, consider a commission contract. Observe that the demand function in the second stage is $d_2 = \frac{1}{2}(\alpha + (2\theta - 3\hat{\theta})\hat{q}_2 - c/(1-u))$, which is positive for $\alpha > a_0' := \max(\bar{q}(\bar{\theta} - 2\underline{\theta}), \underline{q}(\bar{\theta} - 2\underline{\theta})) + \bar{c}$. Using the same logical steps as in the wholesale price contract, one can deduce that $d_1 - \bar{d}_1' \to 0$ almost surely as $\alpha \to \infty$, where $\bar{d}_1' := \frac{1}{2}(\alpha + (2\theta - \mu_\theta)q - c/(1-u))$. Therefore, there exists $\alpha_0' \ge a_0'$ such that d_1 is positive with probability one for any $\alpha \ge \alpha_0'$. Hence, the proof of the lemma is complete. □

Proof of Lemma EC.2. From Lemma EC.6, the equilibrium prices (p_1^*, w_1^*), as well as their derivatives with respect to ϕ, are continuous with respect to γ. Thus, it suffices to consider the limiting cases with $\gamma = 0$ and $\gamma = \infty$. First, consider the case with information sharing. In this case, observe from (EC.8) that

$$\frac{\partial \Pi^{\text{P}}_{\text{I}}}{\partial \phi} = \frac{\sigma_\theta^2 q^2}{16} \quad \text{and} \quad \frac{\partial \Pi^{\text{M}}_{\text{I}}}{\partial \phi} = \frac{\sigma_\theta^2 q^2}{8}$$

in both cases with $\gamma = 0$ and $\gamma = \infty$. Hence, the desired results in part (a) follows. Next, consider the case without information sharing. In this case, observe from (EC.9) that

$$\frac{\partial \Pi^{\text{P}}_{\text{I}}}{\partial \phi} = \frac{\sigma_\theta^2 (q^2 + \sigma_p^2)}{16} \quad \text{and} \quad \frac{\partial \Pi^{\text{M}}_{\text{I}}}{\partial \phi} = 0$$

in both cases with $\gamma = 0$ and $\gamma = \infty$. Hence, the desired results in part (b) follows. The proof of the lemma is complete. □

Proof of Lemma EC.3. From Lemma EC.6, the equilibrium retail price p_1^*, as well as its derivative with respect to ϕ, is continuous with respect to γ. Thus, it suffices to consider the limiting cases with $\gamma = 0$ and $\gamma = \infty$. First, consider the case of information sharing. From (EC.8) and (EC.19) it can be seen that for $\gamma = 0$,

$$\frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{15\sigma_\theta^2 q^2}{32} \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{3\sigma_\theta^2 q^2}{8}.$$

Likewise, we have that for $\gamma = \infty$,

$$\frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{15\sigma_\theta^2 (q^2 + \sigma_p^2)}{32} \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{3\sigma_\theta^2 (q^2 + \sigma_p^2)}{8}.$$

Next, consider the case without information sharing. From (EC.9) and (EC.20) we have that for $\gamma = 0$,

$$\frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{12\sigma_\theta^2 q^2}{32} \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = 0.$$

Likewise, we have that for $\gamma = \infty$,

$$\frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = -\frac{12\sigma_\theta^2 (q^2 + \sigma_p^2)}{32} \quad \text{and} \quad \frac{\partial \Pi_{\mathrm{S}}^{\mathrm{C}}}{\partial \gamma} = 0.$$

Hence, the desired results follow. □

Proof of Lemma EC.4. In this proof, we consider a wholesale price in *Case 1* and then a commission contract in *Case 2*.

Case 1 (Wholesale Price Contract). From the first order conditions for the best-response retail price, $p_1(w_1) = \arg\max_p \{(p - w_1)(\alpha + \mu_\theta q - p) + \Pi_2^{\mathrm{P}}\}$, and the wholesale price, $w_1 = \arg\max_w \{(w - c)(\alpha + \mu_\theta q - p_1(w)) + \Pi_2^{\mathrm{M}}\}$, we obtain

$$\begin{aligned} \alpha + \mu_\theta q + w_1 &= 2p_1(w_1) + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma A_{\mathrm{I},b}}{(d_1\gamma + 1)^2}\right] \\ \alpha + \mu_\theta q - p_1(w_1) &= p_1'(w_1)\left(w_1 - c + \frac{\sigma_p^2}{8}\mathsf{E}\left[\frac{\gamma A_{\mathrm{I},a}}{(d_1\gamma+1)^2}\right]\right). \end{aligned} \tag{EC.32}$$

Furthermore, taking the derivative on both sides of the first equation of (EC.32) with respect to w_1, we obtain

$$1 = 2p_1'(w_1)\left(1 + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\mathrm{I},b}}{(d_1\gamma + 1)^3}\right]\right). \tag{EC.33}$$

Combining (EC.33) into (EC.32), the first-stage retail price p_1^* in equilibrium is a solution to the following equation:

$$\alpha + \mu_\theta q = g_{\mathrm{I}}(p_1^*) := p_1^* + \frac{2p_1^* - \mu_\theta q - c - \alpha + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma(2A_{\mathrm{I},a} + A_{\mathrm{I},b})}{(d_1\gamma+1)^2}\right]}{2\left(1 + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\mathrm{I},b}}{(d_1\gamma+1)^3}\right]\right)}. \tag{EC.34}$$

Taking the derivative with respect to γ on both sides of equation (EC.34), we deduce that

$$0 = \frac{\partial p_1^*}{\partial \gamma} + \frac{2\frac{\partial p_1^*}{\partial \gamma} + \frac{\sigma_p^2}{16}\mathsf{E}\left[(2A_{\text{I},a} + A_{\text{I},b})\left(\frac{1}{(d_1\gamma+1)^2} - \frac{2\gamma(d_1-\gamma\partial p_1^*/\partial\gamma)}{(d_1\gamma+1)^3}\right)\right]}{2\left(1+\frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\text{I},b}}{(d_1\gamma+1)^3}\right]\right)}$$
$$- \frac{2p_1^* - \mu_\theta q - c - \alpha + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma(2A_{\text{I},a}+A_{\text{I},b})}{(d_1\gamma+1)^2}\right]}{2\left(1+\frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\text{I},b}}{(d_1\gamma+1)^3}\right]\right)}\left(\frac{\sigma_p^2}{8}\mathsf{E}\left[A_{\text{I},b}\left(\frac{2\gamma}{(d_1\gamma+1)^3} - \frac{3\gamma^2(d_1-\gamma\partial p_1^*/\partial\gamma)}{(d_1\gamma+1)^4}\right)\right]\right). \tag{EC.35}$$

Given p_1^*, the first-stage wholesale price w_1 in equilibrium can be readily obtained as

$$w_1^* = 2p_1^* - \alpha - \mu_\theta q + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma A_{\text{I},b}}{(d_1^*\gamma+1)^2}\right]. \tag{EC.36}$$

It can be easily checked from (EC.34) and (EC.36) that $\partial p_1^*/\partial\gamma$ and $\partial w_1^*/\partial\gamma$ are continuous in $\gamma \geq 0$. Therefore, $\partial\Pi_{\text{I}}^{\text{P}}/\partial\gamma$ and $\partial\Pi_{\text{I}}^{\text{M}}/\partial\gamma$ are also continuous in $\gamma \geq 0$. Accordingly, the desired results of the proposition would follow if we show that

$$\lim_{\gamma\to 0}\frac{\partial\Pi_{\text{I}}^{\text{K}}}{\partial\gamma} > 0 \quad \text{and} \lim_{\gamma\to\infty}\frac{\partial\Pi_{\text{I}}^{\text{K}}}{\partial\gamma} = 0$$

for $\text{K} \in \{\text{P},\text{M}\}$ and any $\text{I} \in \{\text{S},\text{NS}\}$.

Case 1(a): $\gamma \to 0$. In (EC.34), taking $\gamma \to 0$ and using the fact that $\gamma^n/(d_1\gamma+1)^{n+1} \to 0$ almost surely for any $n \geq 1$, we establish that

$$\frac{\partial p_1^*}{\partial\gamma} \to -\frac{\sigma_p^2}{64}\mathsf{E}\left[2A_{\text{I},a} + A_{\text{I},b}\right] < 0 \quad \text{as } \gamma \to 0, \tag{EC.37}$$

where the interchange of limit and expectation is justified by the dominated convergence theorem. The inequality in (EC.37) follows from the fact that

$$\mathsf{E}\left[2A_{\text{I},a} + A_{\text{I},b}\right] = \begin{cases} 3\mu_\theta^2 + 3\phi\sigma_\theta^2 & \text{for I} = \text{S} \\ 3\mu_\theta^2 + 4\phi\sigma_\theta^2 & \text{for I} = \text{NS}. \end{cases} \tag{EC.38}$$

Furthermore, taking the derivative with respect to γ on both sides of (EC.36), it can be easily seen that $\partial w_1^*/\partial\gamma$ is continuous with $\gamma \geq 0$, and as $\gamma \to 0$,

$$\frac{\partial w_1^*}{\partial\gamma} \to \frac{\sigma_p^2}{32}\mathsf{E}\left[A_{\text{I},b} - 2A_{\text{I},a}\right] < 0, \tag{EC.39}$$

where the inequality follows from the fact that

$$\mathsf{E}\left[A_{\text{I},a} - 2A_{\text{I},b}\right] = \begin{cases} -\mu_\theta^2 - 3\phi\sigma_\theta^2 & \text{for I} = \text{S} \\ -\mu_\theta^2 - 4\phi\sigma_\theta^2 & \text{for I} = \text{NS}. \end{cases}$$

To show that the expected profit of the platform is increasing with γ, recall that $\Pi^{\text{P}}_{\text{I}} = \Pi^{\text{P}}_{\text{I},1} + \Pi^{\text{P}}_{\text{I},2}$, where the first-stage profit $\Pi^{\text{P}}_{\text{I},1} = (p_1^* - w_1^*)(\alpha + \mu_\theta q - p_1^*)$. Taking the derivative with respect to γ, we obtain that

$$\begin{aligned}
\frac{\partial \Pi^{\text{P}}_{\text{I},1}}{\partial \gamma} &= \frac{\partial}{\partial \gamma}(p_1^* - w_1^*)(\alpha + \mu_\theta q - p_1^*) + (p_1^* - w_1^*)\left(-\frac{\partial p_1^*}{\partial \gamma}\right) \\
&\to \frac{\sigma_p^2}{64}\left(\mathsf{E}[2A_{\text{I},a} - 3A_{\text{I},b}]\frac{(\alpha + \mu_\theta q - c)}{4} + \frac{\alpha + \mu_\theta q - c}{4}\mathsf{E}[2A_{\text{I},a} + A_{\text{I},b}]\right) \\
&= \frac{\sigma_p^2}{128}(\alpha + \mu_\theta q - c)\mathsf{E}[2A_{\text{I},a} - A_{\text{I},b}] \\
&> 0,
\end{aligned} \tag{EC.40}$$

where the limit follows from (EC.37), (EC.39), and (10) and the inequality follows from the fact that

$$\mathsf{E}[2A_{\text{I},a} - A_{\text{I},b}] = \begin{cases} \mu_\theta^2 + \phi\sigma_\theta^2 & \text{for I} = \text{S} \\ \mu_\theta^2 + 4\phi\sigma_\theta^2 & \text{for I} = \text{NS}. \end{cases}$$

To show that the second-stage expected profit for the platform also increases with γ, observe from (EC.7) that

$$\begin{aligned}
\frac{\partial \Pi^{\text{P}}_{\text{I},2}}{\partial \gamma} &= \frac{\sigma_p^2}{16}\mathsf{E}\left[A_{\text{I},b}\frac{d_1^* - \gamma \partial p_1^*/\partial\gamma}{(d_1^*\gamma + 1)^2}\right] \\
&\to \frac{\sigma_p^2}{16}\mathsf{E}\left[(\alpha + \theta q - p_1^*)A_{\text{I},b}\right] \\
&\geq \frac{\sigma_p^2}{16}\mathsf{E}\left[(\alpha + \theta q - p_1^*)\right]\mathsf{E}\left[A_{\text{I},b}\right] \\
&= \frac{\sigma_p^2}{16}\frac{\alpha + \mu_\theta q - c}{4}\mathsf{E}\left[A_{\text{I},b}\right] \\
&> 0,
\end{aligned} \tag{EC.41}$$

where the limit follows as $\gamma \to 0$, the first inequality follows from the fact that both $(\alpha + \theta_q - p_1^*)$ and $A_{\text{I},b}$ are increasing with θ and that $\mathsf{E}[XY] > \mathsf{E}[X]\mathsf{E}[Y]$ if X and Y are positively correlated, and the last inequality follows from the fact that

$$\mathsf{E}[A_{\text{I},b}] = \begin{cases} \mu_\theta^2 + \phi\sigma_\theta^2 & \text{for I} = \text{S} \\ \mu_\theta^2 + 4\phi\sigma_\theta^2 & \text{for I} = \text{NS}. \end{cases}$$

Combining (EC.40) and (EC.41), we deduce that $\lim_{\gamma\to 0}\partial\Pi^{\text{P}}_{\text{I}}/\partial\gamma > 0$. Following the same logical steps, it can be shown that $\lim_{\gamma\to 0}\partial\Pi^{\text{M}}_{\text{I}}/\partial\gamma > 0$; we omit the detailed calculation steps.

Case 1(b): $\gamma \to \infty$. From (EC.35), it can be verified that $\partial p_1^*/\partial\gamma \to 0$ as $\gamma \to \infty$. Therefore, from (EC.36) it also follows that $\partial w_1^*/\partial\gamma \to 0$ as $\gamma \to \infty$. Consequently, the first-stage profit of the platform, $\Pi^{\text{P}}_{\text{I},1}$, satisfies

$$\frac{\partial \Pi^{\text{P}}_{\text{I},1}}{\partial \gamma} = \frac{\partial}{\partial \gamma}(p_1^* - w_1^*)(\alpha + \mu_\theta q - p_1^*) + (p_1^* - w_1^*)\left(-\frac{\partial p_1^*}{\partial \gamma}\right) \to 0 \quad \text{as } \gamma \to \infty.$$

Furthermore, the second-stage profit of the platform, $\Pi^{\text{P}}_{\text{I},2}$, satisfies

$$\frac{\partial \Pi^{\text{P}}_{\text{I},2}}{\partial \gamma} = \frac{\sigma_p^2}{16}\mathsf{E}\left[A_{\text{I},b}\frac{d_1^* - \gamma \partial p_1^*/\partial\gamma}{(d_1^*\gamma + 1)^2}\right] \to 0 \quad \text{as } \gamma \to \infty$$

where the limit follows from the fact that $d_1^*/(d_1^*\gamma+1)^2 \to 0$ almost surely and $\partial p_1^*/\partial\gamma \to 0$ as $\gamma\to\infty$. Combining these observations, we deduce that $\partial\Pi_{\text{I}}^{\text{P}}/\partial\gamma \to 0$ as $\gamma\to\infty$. From the same logical steps, it follows that $\partial\Pi_{\text{I}}^{\text{M}}/\partial\gamma \to 0$ as $\gamma\to\infty$. (We omit the detailed steps to avoid repetition.)

Case 2 (Commission Contract). From the first order condition for $p_1^* = \arg\max_p\{((1-u)p-c_q)(\alpha+\mu_\theta q-p)+\bar{\Pi}_{\text{I},2}^{\text{M}}\}$, we obtain

$$\alpha+\mu_\theta q+\frac{c}{1-u} = 2p_1^* + \frac{\sigma_p^2}{4}\mathsf{E}\left[\frac{\gamma B_{\text{I},a}}{(d_1^*\gamma+1)^2}\right].$$

Taking the derivative with respect to γ on both sides of the preceding equation, we deduce that

$$0 = 2\frac{\partial p_1^*}{\partial\gamma} + \frac{\sigma_p^2}{4}\mathsf{E}\left[\frac{B_{\text{I},a}}{(d_1^*\gamma+1)^2} - \frac{2\gamma B_{\text{I},a}(d_1^*-\gamma\frac{\partial p_1^*}{\partial\gamma})}{(d_1^*\gamma+1)^3}\right]. \tag{EC.42}$$

Case 2(a): $\gamma\to 0$. Taking $\gamma\to 0$ on both sides of (EC.42) and after some algebra, we deduce that

$$\frac{\partial p_1^*}{\partial\gamma} \to -\frac{\sigma_p^2}{8}\mathsf{E}[B_{\text{I},a}] < 0,$$

where the inequality follows from the fact that

$$\mathsf{E}[B_{\text{I},a}] = \begin{cases}\mu_\theta^2+\phi\sigma_\theta^2 & \text{for I} = \text{S}\\ \mu_\theta^2 & \text{for I} = \text{NS}.\end{cases} \tag{EC.43}$$

Furthermore, note that $\bar{\Pi}_{\text{I},1}^{\text{M}} = ((1-u)p_1^*-c_q)(\alpha+\mu_\theta q-p_1^*)$. Therefore, one can write that

$$\begin{aligned}\frac{\partial\bar{\Pi}_{\text{I},1}^{\text{M}}}{\partial\gamma} &= (\alpha+\mu_\theta q-p_1^*)(1-u)\frac{\partial p_1^*}{\partial\gamma} + ((1-u)p_1^*-c_q)\left(-\frac{\partial p_1^*}{\partial\gamma}\right)\\ &= \frac{\partial p_1^*}{\partial\gamma}\left((1-u)(\alpha+\mu_\theta q-p_1^*)-(1-u)p_1^*+c\right)\\ &\to 0,\end{aligned} \tag{EC.44}$$

where the limit follows from the fact that $p_1^*\to(\alpha+\mu_\theta q+c_q/(1-u))/2$ as $\gamma\to 0$. Furthermore, observe that

$$\begin{aligned}\frac{\partial\bar{\Pi}_{\text{I},2}^{\text{M}}}{\partial\gamma} &= \frac{(1-u)\sigma_p^2}{4}\mathsf{E}\left[B_{\text{I},a}\frac{d_1^*-\gamma\partial p_1^*/\partial\gamma}{(d_1^*\gamma+1)^2}\right]\\ &\to \frac{(1-u)\sigma_p^2}{4}\mathsf{E}\left[B_{\text{I},a}(\alpha+\theta q-p_1^*)\right]\\ &\geq \frac{(1-u)\sigma_p^2}{4}\mathsf{E}\left[B_{\text{I},a}\right]\mathsf{E}\left[(\alpha+\theta q-p_1^*)\right]\\ &= \frac{(1-u)\sigma_p^2}{4}\mathsf{E}\left[B_{\text{I},a}\frac{\alpha+\mu_\theta q-c/(1-u)}{2}\right]\\ &> 0,\end{aligned} \tag{EC.45}$$

where the limit follows as $\gamma\to 0$, the inequality follows from the fact that both $(\alpha+\theta q-p_1^*)$ and $B_{\text{I},a}$ are increasing with θ and that $\mathsf{E}[XY] > \mathsf{E}[X]\mathsf{E}[Y]$ if X and Y are positively correlated, and the last inequality follows from (EC.43). Combining these results, we establish that $\lim_{\gamma\to 0}\partial\bar{\Pi}_{\text{I}}^{\text{M}}/\partial\gamma >$

0. Following the same logical steps, it can be seen that $\lim_{\gamma\to 0}\partial\bar{\Pi}_{\text{I}}^{\text{P}}/\partial\gamma > 0$; we omit the detailed calculation steps.

Case 2(b): $\gamma\to\infty$. From (EC.42), we deduce that $\partial p_1^*/\partial\gamma\to 0$ as $\gamma\to\infty$. Consequently, from the expressions in (EC.44) and (EC.45), it can be easily seen that $\partial\bar{\Pi}_{\text{I},t}^{\text{M}}/\partial\gamma\to 0$ for both $t=1,2$ as $\gamma\to 0$. Thus, the proof of the lemma is complete. □

Proof of Lemma EC.5. First, consider a wholesale price contract. For $t=1$, observe that

$$\begin{aligned}
\frac{\partial\Pi_{\text{I},1}^{\text{C}}}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\mathsf{E}\left[(\alpha+\theta Q-p_1^*)(\alpha+\theta q-p_1^*)-\frac{(\alpha+\theta q-p_1^*)^2}{2}\right]\\
&= \frac{\partial}{\partial\gamma}\mathsf{E}\left[\frac{(\alpha+\theta q-p_1^*)^2}{2}\right]\\
&= \mathsf{E}\left[\left(-\frac{\partial p_1^*}{\partial\gamma}\right)(\alpha+\theta q-p_1^*)\right]\\
&\to \frac{\sigma_p^2}{256}\mathsf{E}\left[(2A_{\text{I},a}+A_{\text{I},b})\right](\alpha+\theta q-c) \text{ as } \gamma\to 0,
\end{aligned} \tag{EC.46}$$

where the second equality follows from the fact that the random variable $Q\sim N(q,\sigma_p^2)$ is independent of θ, and the limit follows from (EC.37) and is positive from (EC.38). For $t=2$, observe that

$$\begin{aligned}
\frac{\partial\Pi_{\text{I},2}^{\text{C}}}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\mathsf{E}\left[(\alpha+\theta Q-p_2^*)(\alpha+\theta\hat{q}_2-p_2^*)-\frac{(\alpha+\theta\hat{q}_2-p_1^*)^2}{2}\right]\\
&= \frac{\partial}{\partial\gamma}\mathsf{E}\left[\frac{(\alpha+\theta\hat{q}_2-p_2^*)^2}{2}\right]\\
&= \frac{1}{32}\frac{\partial}{\partial\gamma}\mathsf{E}[(\alpha+(4\theta-3\hat{\theta})\hat{q}_2-c)^2]\\
&= \frac{1}{32}\frac{\partial}{\partial\gamma}\left[(\alpha+\mu_\theta q-c)^2-\mu_\theta^2q^2+\mathsf{E}\left[A_{\text{I},c}\left(q^2+\frac{d_1^*\gamma}{d_1^*\gamma+1}\sigma_p^2\right)\right]\right]\\
&\to \frac{\sigma_p^2}{32}\mathsf{E}\left[A_{\text{I},c}(\alpha+\theta q-c)\right],
\end{aligned} \tag{EC.47}$$

where the second equality follows from the fact that the random variable $Q\in N(q,\sigma_p^2)$ and Q is independent of θ, the third equality follows from (9), and the limit follows from the fact that $\partial d_1^*/\partial\gamma = -\partial p_1^*/\partial\gamma$ and (EC.37). The limit in (EC.47) is positive since $A_{\text{I},c}$ defined in (EC.6) and $(\alpha+\theta q-c)$ are increasing with θ and $\mathsf{E}[XY]>\mathsf{E}[X]\mathsf{E}[Y]$ if X and Y are positively correlated. Combining these observations, we deduce that $\partial\Pi_{\text{I}}^{\text{C}}/\partial\gamma$ is strictly positive for sufficiently small γ.

Now, consider a commission contract. For $t=1$, we first derive an expression for $\partial p_1^*/\partial\gamma$. Observe that the optimal price p_1^* in the first stage is a solution p_1 to the equation:

$$(1-u)(\alpha+\mu_\theta q)+c=\bar{g}_{\text{I}}(p_1):=2(1-u)p_1+\frac{1-u}{4}\sigma_p^2\mathsf{E}\left[\frac{B_{\text{I},a}\gamma}{(d_1\gamma+1)^2}\right]. \tag{EC.48}$$

Taking the derivatives with respect to γ on both sides of (EC.48), we deduce that

$$0=2(1-u)\frac{\partial p_1^*}{\partial\gamma}+\frac{1-u}{4}\sigma_p^2\mathsf{E}\left[B_{\text{I},a}\left(\frac{1}{(d_1\gamma+1)^2}-\frac{2(d_1-\gamma\partial p_1^*/\partial\gamma)\gamma}{(d_1\gamma+1)^3}\right)\right].$$

Using the fact that $1/(d_1\gamma+1)^2 \to 0$ and $d_1/(d_1\gamma+1)^3 \to 0$ almost surely as $\gamma \to 0$, we deduce that as $\gamma \to 0$,

$$\frac{\partial p_1^*}{\partial \gamma} \to -\frac{\sigma_p^2}{8}\mathsf{E}[B_{\text{I},a}]. \tag{EC.49}$$

Observe that

$$\begin{aligned}\frac{\partial \bar{\Pi}^{\text{C}}_{\text{I},1}}{\partial \gamma} &= \frac{\partial}{\partial \gamma}\mathsf{E}\left[(\alpha+\theta Q-p_1^*)(\alpha+\theta q-p_1^*)-\frac{(\alpha+\theta q-p_1^*)^2}{2}\right]\\ &= \frac{\partial}{\partial \gamma}\mathsf{E}\left[\frac{(\alpha+\theta q-p_1^*)^2}{2}\right]\\ &= \mathsf{E}\left[\left(-\frac{\partial p_1^*}{\partial \gamma}\right)(\alpha+\theta q-p_1^*)\right]\\ &\to \frac{\sigma_p^2}{32}\mathsf{E}\left[B_{\text{I},a}\right]\left(\alpha+\mu_\theta q-\frac{c}{1-u}\right) \text{ as } \gamma\to 0,\end{aligned} \tag{EC.50}$$

where the limit follows from (EC.49). Note that the limit in (EC.50) is positive because $\mathsf{E}[B_{\text{I},a}]>0$ and $(\alpha+\theta q-c/(1-u))>0$. For $t=2$, observe that

$$\begin{aligned}\frac{\partial \bar{\Pi}^{\text{C}}_{\text{I},2}}{\partial \gamma} &= \frac{\partial}{\partial \gamma}\mathsf{E}\left[(\alpha+\theta Q-p_2^*)(\alpha+\theta \hat{q}_2-p_2^*)-\frac{(\alpha+\theta \hat{q}_2-p_1^*)^2}{2}\right]\\ &= \frac{\partial}{\partial \gamma}\mathsf{E}\left[\frac{(\alpha+\theta \hat{q}_2-p_2^*)^2}{2}\right]\\ &= \frac{1}{32}\frac{\partial}{\partial \gamma}\mathsf{E}\left[\left(\alpha+(4\theta-3\hat{\theta})\hat{q}_2-\frac{c}{1-u}\right)^2\right]\\ &= \frac{1}{32}\frac{\partial}{\partial \gamma}\left[\left(\alpha+\mu_\theta q-\frac{c}{1-u}\right)^2-\mu_\theta^2 q^2+\mathsf{E}\left[B_{\text{I},b}\left(q^2+\frac{d_1^*\gamma}{d_1^*\gamma+1}\sigma_p^2\right)\right]\right]\\ &\to \frac{\sigma_p^2}{32}\mathsf{E}\left[B_{\text{I},b}\left(\alpha+\theta q-\frac{c}{1-u}\right)\right],\end{aligned} \tag{EC.51}$$

where the third equality follows from the fact that $p_2^*=(\alpha+\tilde{\theta}\hat{q}_2+c/(1-u))/2$, and the remaining steps follow from algebra. Note that the limit in (EC.51) is positive because $B_{\text{I},b}$ and $(\alpha+\theta q-c)$ increase with θ with probability one and that $\mathsf{E}[XY]>\mathsf{E}[X]\mathsf{E}[Y]$ when X and Y are positively correlated. Combining these observations, we deduce that $\frac{\partial \bar{\Pi}^{\text{C}}_{\text{I}}}{\partial \gamma}$ is strictly positive for sufficiently small γ. This step completes the proof of the lemma. □

Proof of Lemma EC.6. For part (a), recall definition $g_{\text{I}}(p)$ in (EC.34). We show that $g_{\text{I}}'(p)>0$ for sufficiently small or large values of γ. For any fixed $\gamma>0$, there exists a positive constant $d_{\min}$ such that

$$\frac{\gamma^n}{(d_1\gamma+1)^{n+1}} \le f_n(\gamma) := \frac{\gamma^n}{(d_{\min}\gamma+1)^{n+1}} \text{ for } n=1,2,\ldots \tag{EC.52}$$

with probability one. For any $n \geq 1$, observe that $f_n(\gamma) \to 0$ as $\gamma \to 0$ or as $\gamma \to \infty$. Furthermore, $f_{n+1}(\gamma) = \mathcal{O}(f_n(\gamma))$ for $n \geq 1$. The derivative of $g_{\text{I}}(p)$ can be written as

$$\begin{aligned} g_{\text{I}}'(p) = 1 + &\frac{2 + \frac{2\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2(2A_{\text{I},a}+A_{\text{I},b})}{(d_1\gamma+1)^3}\right]}{2 + \frac{2\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\text{I},b}}{(d_1\gamma+1)^3}\right]} \\ &- \frac{2p - \alpha - \mu_\theta q - c + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma(2A_{\text{I},a}+A_{\text{I},b})}{(d_1\gamma+1)^2}\right]}{4\left(1 + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\text{I},b}}{(d_1\gamma+1)^3}\right]\right)} \left(\frac{6\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^3 B_\iota}{(d_1\gamma+1)^4}\right]\right). \end{aligned} \tag{EC.53}$$

The second term on the right-hand side of the preceding equation is $1 + \mathcal{O}(f_2(\gamma))$ from (EC.52). It is easy to check that $g_{\text{I}}'(p_1) \to 2$ as $\gamma \to 0$ or as $\gamma \to \infty$. Hence, the solution to (EC.34) is unique for sufficiently small or large values of γ. Given p_1^*, the first-stage wholesale price w_1^* satisfies (EC.36). Note that both sides of equations (EC.34) and (EC.36) are continuously differentiable with respect to ϕ, where the interchange of the limit and expectation is justified by the dominated convergence theorem. Therefore, p_1 and w_1, as well as their derivatives with respect to ϕ, are continuous in γ.

To show the limits of the equilibrium prices (p_1^*, w_1^*) and their derivatives with respect to ϕ, we bound the last term on the right-hand side of (EC.53). Observe that

$$\begin{aligned} \mathsf{E}\left|\frac{\gamma^n(2A_{\text{I},a}+A_{\text{I},b})}{(d_1\gamma+1)^{n+1}}\right| &\leq f_n(\gamma)\mathsf{E}|2A_{\text{I},a}+A_{\text{I},b}| \\ \mathsf{E}\left|\frac{\gamma^n A_{\text{I},b}}{(d_1\gamma+1)^{n+1}}\right| &\leq f_n(\gamma)\mathsf{E}|A_{\text{I},b}|, \end{aligned} \tag{EC.54}$$

where the bounds follow from (EC.52). Write $C_{\text{I}} = \max(\mathsf{E}|2A_{\text{I},a}+A_{\text{I},b}|, \mathsf{E}|A_{\text{I},b}|)$ for $\text{I} = \{\text{S}, \text{NS}\}$ and let $C = \max(C_{\text{S}}, C_{\text{NS}})$. Note that $C < \infty$ because θ, $\hat{\theta}$, and μ_θ are bounded random variables. This fact implies that the last term on the right-hand side of (EC.53) is $O(f_2(\gamma))$, and therefore, $g_{\text{I}}'(p) > 0$ for sufficiently small or sufficiently large values of γ.

Since $g_{\text{I}}(p)$ is strictly increasing with p, there exists $\ell > 0$ such that $g_{\text{I}}'(p) \geq \ell$, and thus,

$$|p_1^* - \bar{p}| \leq \ell\, |g_{\text{I}}(p_1^*) - g_{\text{I}}(\bar{p})|\,.$$

It remains to show that $|g_{\text{I}}(p_1^*) - g_{\text{I}}(\bar{p})| = \mathcal{O}(\epsilon(\gamma))$. To this end, define $g_{\text{I}}^\star(p)$ as

$$g_{\text{I}}^\star(p) \coloneqq p + \frac{2p - \alpha - \mu_\theta q - c}{2}$$

and observe that $g_{\text{I}}^\star(\bar{p}) = \alpha + \mu_\theta q$. Additionally, observe that $g_{\text{I}}(p_1^*) = g_{\text{I}}^\star(\bar{p}) = \alpha + \mu_\theta q$ by the definition of $g_{\text{I}}(p_1^*)$ in (EC.34) so that $|g_{\text{I}}(p_1^*) - g_{\text{I}}(\bar{p})| = |g_{\text{I}}^\star(\bar{p}) - g_{\text{I}}(\bar{p})|$. Furthermore, since $\bar{p} = \frac{1}{4}(3\alpha + 3\mu_\theta q + c)$, we have that

$$g_{\text{I}}^\star(\bar{p}) - g_{\text{I}}(\bar{p}) = \frac{\alpha + \mu_\theta q - c}{4} - \frac{\frac{\alpha+\mu_\theta q - c}{2} + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma(2A_{\text{I},a}+A_{\text{I},b})}{\left(\bar{d}_1\gamma+1\right)^2}\right]}{2\left(1 + \frac{\sigma_p^2}{16}\mathsf{E}\left[\frac{\gamma^2 A_{\text{I},b}}{(d_1\gamma+1)^3}\right]\right)} = \mathcal{O}\left(f_2(\gamma)\right),$$

where $\bar{d}_1 := \alpha + \theta q - \bar{p}$ and the last inequality follows from (EC.54). The bound for $|w_1^* - \bar{w}|$ immediately follows from (EC.36).

Furthermore, taking the derivatives on both sides of (EC.34) with respect to ϕ, we deduce that

$$0 = 2\frac{\partial p_1^*}{\partial \phi} + \frac{\sigma_p^2}{4}\mathsf{E}\left[\frac{\partial p_1^*}{\partial \phi}\frac{2\gamma^2\hat{\theta}(2\theta-\hat{\theta})}{(d_1\gamma+1)^3} + \frac{\gamma}{(d_1\gamma+1)^2}\frac{\partial(\hat{\theta}(2\theta-\hat{\theta}))}{\partial \phi}\right].$$

From (EC.52), it is easy to verify that $\partial p_1^*/\partial\phi = \mathcal{O}(f_2(\gamma))$. The bounds for $\partial w_1^*/\partial\phi$ also follow from (EC.36), and we omit the details. This step completes the proof of part (a).

For part (b), we start with the second-stage profits (or surplus). Note that $\Pi_{\mathrm{I},2}^{\mathrm{K}}$ for $\mathrm{I} \in \{\mathrm{S}, \mathrm{NS}\}$ and $\mathrm{K} \in \{\mathrm{P}, \mathrm{M}, \mathrm{C}\}$ can be expressed as

$$\begin{aligned}\Pi_{\mathrm{I},2}^{\mathrm{K}} &= \frac{\bar{\Pi}_{\mathrm{I}}^{\mathrm{K}}}{2} + \mathsf{E}\left[B_{\mathrm{I}}^{\mathrm{K}}\left(q^2 + \frac{d_1\gamma}{d_1\gamma+1}\sigma_p^2\right)\right] \\ &= \frac{\bar{\Pi}^{K}}{2} + \mathsf{E}\left[B_{\mathrm{I}}^{\mathrm{K}}\left(q^2+\sigma_p^2\right)\right] + \mathcal{O}\left(f_2(\gamma)\right),\end{aligned} \tag{EC.55}$$

where $\bar{\Pi}_{\mathrm{I}}^{\mathrm{K}} := \Pi_{\mathrm{I}}^{\mathrm{K}}|_{\gamma=0}$, where $\Pi_{\mathrm{I}}^{\mathrm{K}}|_{\gamma=0}$ is defined in (EC.8) and (EC.9). The constant $B_{\mathrm{I}}^{\mathrm{K}}$ is the term that depends only on θ and s_θ:

$$\begin{aligned}&B_{\mathrm{S}}^{\mathrm{P}} = \frac{\hat{\theta}(4\theta-3\hat{\theta})}{16},\ B_{\mathrm{S}}^{\mathrm{M}} = \frac{\hat{\theta}(4\theta-3\hat{\theta})}{8},\ B_{\mathrm{S}}^{\mathrm{C}} = \frac{(4\theta-3\hat{\theta})^2}{32} \\ &B_{\mathrm{NS}}^{\mathrm{P}} = \frac{(2\hat{\theta}-\mu_\theta)(4\theta-2\hat{\theta}-\mu_\theta)}{16},\ B_{\mathrm{NS}}^{\mathrm{M}} = \frac{\mu_\theta(4\theta-2\hat{\theta}-\mu_\theta)}{8},\ B_{\mathrm{NS}}^{\mathrm{C}} = \frac{(4\theta-2\hat{\theta}-\mu_\theta)^2}{32}.\end{aligned} \tag{EC.56}$$

Additionally, note that $\Pi_{\mathrm{I},1}^{\mathrm{K}}$, being a continuous function of the equilibrium prices (p_1^*, w_1^*) (part (a) of this lemma), satisfies $\Pi_{\mathrm{I},1}^{\mathrm{K}} = \Pi_{\mathrm{I}}^{\mathrm{K}}/2 + \mathcal{O}(f_2(\gamma))$. Therefore, we deduce that $\Pi_{\mathrm{I}}^{\mathrm{K}} = \Pi_{\mathrm{I},1}^{\mathrm{K}} + \Pi_{\mathrm{I},2}^{\mathrm{K}} = \bar{\Pi}_{\mathrm{I}}^{\mathrm{K}} + \mathsf{E}[B_{\mathrm{I}}^{\mathrm{K}}(q^2+\sigma_p^2)] + \mathcal{O}(f_2(\gamma))$. After some standard calculation for $\mathsf{E}[B_{\mathrm{I}}^{\mathrm{K}}]$, one can show that $\bar{\Pi}_{\mathrm{I}}^{\mathrm{K}} + \mathsf{E}[B_{\mathrm{I}}^{\mathrm{K}}(q^2+\sigma_p^2)] = \Pi_{\mathrm{I}}^{\mathrm{K}}|_{\gamma\to\infty}$.

Furthermore, observe from (EC.55) that

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{I},2}^{\mathrm{K}}}{\partial\phi} &= \mathsf{E}\left[\frac{\partial B_{\mathrm{I}}^{\mathrm{K}}}{\partial\phi}\left(q^2+\sigma_p^2\frac{d_1\gamma}{d_1\gamma+1}\right) + B_{\mathrm{I}}^{\mathrm{K}}\frac{-\gamma\frac{\partial p_1^*}{\partial\phi}}{(d_1\gamma+1)^2}\right] \\ &= \mathsf{E}\left[\frac{\partial B_{\mathrm{I}}^{\mathrm{K}}}{\partial\phi}\left(q^2+\sigma_p^2\right) - \frac{\partial B_{\mathrm{I}}^{\mathrm{K}}}{\partial\phi}\frac{\sigma_p^2}{d_1\gamma+1} + B_{\mathrm{I}}^{\mathrm{K}}\frac{-\gamma\frac{\partial p_1^*}{\partial\phi}}{(d_1\gamma+1)^2}\right]\end{aligned} \tag{EC.57}$$

where the interchange of the limit and expectation is justified by the dominated convergence theorem. From parts (a) and (b) of the current lemma, we deduce that $p_1^* = \bar{p} + \mathcal{O}(f_2(\gamma))$ and $\partial p_1^*/\partial\phi = \mathcal{O}(f_2(\gamma))$. Furthermore, note that $d_1 = \bar{d}_1 + \mathcal{O}(f_2(\gamma))$, where $\bar{d}_1 = \alpha + \theta q - \bar{p}$. Therefore, the second and third terms on the right-hand side of (EC.57) are $\mathcal{O}(f_2(\gamma))$. After some straightforward but tedious algebra, it follows that

$$\mathsf{E}\left[\frac{\partial B_{\mathrm{I}}^{\mathrm{K}}}{\partial\phi}\left(q^2+\sigma_p^2\right)\right] = \frac{\partial}{\partial\phi}\left(\Pi_{\mathrm{I}}^{\mathrm{K}}|_{\gamma\to\infty}\right),$$

where $\Pi_{\text{I}}^{\text{K}}|_{\gamma\to\infty}$ is defined in (EC.8) and (EC.9). The calculation steps for the preceding expectation are standard and hence are omitted. Using part (a) of the current lemma once again, we have that $\partial\Pi_{\text{I},1}^{\text{K}}/\partial\phi = \mathcal{O}(f_2(\gamma))$ and the desired result follows from $\partial\Pi_{\text{I}}^{\text{K}}/\partial\phi = \partial\Pi_{\text{I},1}^{\text{K}}/\partial\phi + \partial\Pi_{\text{I},2}^{\text{K}}/\partial\phi$. Hence, the proof of the lemma is complete. □

Proof of Lemma EC.7. The proof of this lemma follows the same logical steps as in the proof of Lemma EC.6, so we only provide a proof sketch.

For part (a), recall the definition of $\bar{g}_{\text{I}}(p)$ in (EC.48) and it can be easily seen that $\bar{g}_{\text{I}}'(p) \to 2(1-u) > 0$ as $\gamma \to 0$ or $\gamma \to \infty$. Therefore, the solution to (EC.48) is unique for sufficiently small or large values of γ. From the same calculation steps as in the proof of Lemma EC.6, it can be easily seen that $p_1^* \to \bar{p}$ as $\gamma \to 0$ or $\gamma \to \infty$; we omit the details to avoid repetition. Furthermore, taking the derivatives on both sides of (EC.48) with respect to ϕ, we deduce that

$$0 = 2(1-u)\frac{\partial p_1^*}{\partial\phi} + \frac{(1-u)\sigma_p^2}{4}\mathsf{E}\left[B_{\text{I},a}\frac{\partial p_1^*}{\partial\phi}\frac{\gamma^2}{(d_1\gamma+1)^3} + \frac{\gamma}{(d_1\gamma+1)^2}\frac{\partial B_{\text{I},a}}{\partial\phi}\right].$$

Recalling the definition $f_n(\gamma) = \gamma^n/(d_{\min}\gamma+1)^{n+1}$, we deduce that $\partial p_1^*/\partial\phi = \mathcal{O}(f_2(\gamma))$. Since $f_n(\gamma) \to 0$ as $\gamma \to 0$ or $\gamma \to \infty$ for any $n \geq 2$, we deduce that $\partial p_1^*/\partial\phi \to 0$ as $\gamma \to 0$ or $\gamma \to \infty$. This step completes the proof of part (a).

For part (b), note that the expected profit $\Pi_{\text{I},1}^{\text{K}}$ is a continuous function of the equilibrium price p_1^*. By part (a) of the lemma, we establish the continuity of $\Pi_{\text{I},1}^{\text{K}}$ and $\partial\Pi_{\text{I},1}^{\text{K}}/\partial\phi$ with respect to γ. Furthermore, observe from (EC.18) that $\Pi_{\text{I},2}^{\text{K}}$ is a continuous function of ϕ. (Here, we use the fact that the continuity of a function is preserved under expectation due to the dominated convergence theorem.) Therefore, the continuity of $\Pi_{\text{I},2}^{\text{K}}$ and $\partial\Pi_{\text{I},2}^{\text{K}}/\partial\phi$ with respect to γ immediately follows. This step completes the proof of the lemma. □