Web Appendix

Pricing, Design, and Profitability of Pay-to-Win Add-ons

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1. Proof of Lemma 1

To prove Lemma 1, we first consider the two possible situations where the firm either serves both type-l and type-h players or serves only type-h players. Our goal is to identify the firm's optimal product design (q) and pricing strategy (p) in each situation, and then compare profits to deduce the firm's globally optimal strategy.

1. The firm serves both segments. Because type-h players obtain a greater utility than type-l players, to induce both segments to buy the product, the firm's optimal price is such that $U_l = \theta_l(q + \lambda/2 - \beta/2) - p = 0$. Hence, $p = \theta_l(q + \lambda/2 - \beta/2)$. Next, the firm chooses q to maximize $\pi(q) = p - q^2/2$. To ensure that the price is non-negative, it must be that $q \geq (\beta - \lambda)/2$. Solving the first-order condition, we find that

$$q = \begin{cases} \theta_l & \text{if } \beta \leq \lambda + 2\theta_l, \\ \frac{(\beta-\lambda)}{2} & \text{if } \beta > \lambda + 2\theta_l. \end{cases} \tag{11}$$

Upon plugging q into the expression for the optimal price, we find that

$$p = \begin{cases} \theta_l \left(\theta_l - \frac{\beta-\lambda}{2}\right) & \text{if } \beta \leq \lambda + 2\theta_l, \\ 0 & \text{if } \beta > 2\lambda + \theta_l; \end{cases} \tag{12}$$

It follows that the firm's profit in scenario 1 is

$$\pi_1 = \begin{cases} \frac{\theta_l(\theta_l-(\beta-\lambda))}{2} & \text{if } \beta \leq \lambda + 2\theta_l, \\ -\frac{1}{2}(\frac{\beta-\lambda}{2})^2 & \text{if } \beta > \lambda + 2\theta_l. \end{cases} \tag{13}$$

2. The firm serves only type-h players. The firm's price is such that $U_h = \theta_h(q + \lambda/2 - \beta/2) - p = 0$. Solving the equality, we have $p = \theta_h(q + \lambda/2 - \beta/2)$. The firm chooses q to maximize $\pi(q) = \alpha p - q^2/2$ subject to $q \geq (\beta - \lambda)/2$ to ensure that the price is non-negative. The optimal q and the corresponding p are as follows:

$$q = \begin{cases} \alpha\theta_h & \text{if } \beta \leq \lambda + 2\alpha\theta_h, \\ \frac{\beta-\lambda}{2} & \text{if } \beta > \lambda + 2\alpha\theta_h; \end{cases} \tag{14}$$

$$p = \begin{cases} \theta_h \left(\alpha\theta_h - \frac{\beta-\lambda}{2}\right) & \text{if } \beta \leq \lambda + 2\alpha\theta_h, \\ 0 & \text{if } \beta > \lambda + 2\alpha\theta_h; \end{cases} \tag{15}$$

The firm's profits are given by

$$\pi_2 = \begin{cases} \frac{\alpha\theta_h(\alpha\theta_h-(\beta-\lambda))}{2} & \text{if } \beta \leq \lambda + 2\alpha\theta_h, \\ -\frac{1}{2}(\frac{\beta-\lambda}{2})^2 & \text{if } \beta > \lambda + 2\alpha\theta_h. \end{cases} \tag{16}$$

Optimal Strategy. Equations (13) and (16) reveal that to ensure that the firm can earn positive profits in the market, we need that $\beta < \bar{\beta}$, where $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$: in the rest of our analysis, we assume that this inequality is satisfied. For $\beta < \bar{\beta}$, one can readily show that if $\alpha \le \frac{\theta_l}{\theta_h}$, then $\pi_1 \ge \pi_2$. Hence, on this parameter range, the optimal quality and price are given by the equations (11) and (12). By contrast, if $\frac{\theta_l}{\theta_h} \le \alpha \le 1$, then $\pi_1 < \pi_2$, and the optimal quality and price are as in equations (14) and (15). □

Lemma A1. *If the firm strictly prefers to introduce a pay-to-win add-on, then it must be that, in equilibrium,* $(d_l, d_h) = (B, A)$.

Proof of Lemma A1. We will first consider a pure-strategy equilibrium and then discuss the possibility of a mixed equilibrium. Because type-h players' willingness to pay for the pay-to-win add-on exceeds that of type-l players, we must have $d_h = A$ when the firm introduces the add-on. Given type-l players' decision $d_l \in \{N, B, A\}$, there are three possible situations that may happen in equilibrium: $(d_l, d_h) = (N, A)$, $(d_l, d_h) = (B, A)$, or $(d_l, d_h) = (A, A)$. We will demonstrate that when players' optimal decisions are $(d_l, d_h) = (N, A)$ or $(d_l, d_h) = (A, A)$, the firm does not gain any extra profits from introducing a pay-to-win add-on.

1. $(d_l, d_h) = (A, A)$*.* In this setup, a type-i player's utility from buying just the base product (without the add-on) when all other players buy the firm's product and the add-on is given by $U_i(B|(A,A)) = \theta_i\,(q + \lambda(1-\omega) - \beta\omega) - p$. By contrast, if the player buys the add-on too, then her utility is $U_i(A|(A,A)) = \theta_i\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right) - p - p_a$. Individual rationality (IR) and incentive compatibility (IC) conditions require that the following conditions are met for each player type i.

$$\text{IR}_i: \quad U_i(A|(A,A)) \ge 0, \tag{17}$$

$$\text{IC}_i: \quad U_i(A|(A,A)) \ge U_i(B|(A,A)). \tag{18}$$

Since $\theta_l < \theta_h$, it is easy to see that IR_l and IC_l imply IR_h and IC_h, respectively. Further, if IR_l is slack while IC_l binds, the firm can increase its profits by decreasing p or q. Hence, it must be that IR_l binds, which implies that $p + p_a = \theta_l\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right)$. The IC_l condition reduces to $p_a \le \theta_l(\lambda+\beta)(\omega - \frac{1}{2})$. All price pairs (p, p_a) that satisfy IR_l and IC_l lead to a profit of $\Pi_{AA} = p + p_a - \frac{1}{2}q^2 = \theta_l\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right) - \frac{1}{2}q^2$, where the last equality follows from the IR_l condition. Now, notice that Π_{AA} is the same as what the firm earns in the benchmark without a pay-to-win add-on by setting the base product price equal to $p = \theta_l\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right)$. Therefore, the firm's optimal profits in the benchmark are weakly higher than Π_{AA}.

2. $(d_l, d_h) = (N, A)$*.* When only type-h players buy the firm's product and the add-on, a type-i player's utility from buying only the product (without the add-on) is given by $U_i(B|(N,A)) =$

$\theta_i\left(q+\lambda(1-\omega)-\beta\omega\right)-p$. By contrast, if the player buys add-on too, then her utility is $U_i(A|(N,A))=\theta_i\left(q+\lambda\frac{1}{2}-\beta\frac{1}{2}\right)-p-p_a$. Individual rationality (IR) and incentive compatibility (IC) conditions require that the following conditions are met for each player type i.

$$\text{IR}_h: \quad U_h(A|(N,A))\geq 0, \tag{19}$$

$$\text{IR}_l: \quad \max\{U_l(A|(N,A)),U_l(B|(N,A))\}\leq 0, \tag{20}$$

$$\text{IC}_h: \quad U_h(A|(N,A))\geq U_h(B|(N,A)). \tag{21}$$

The firm chooses p and p_a to maximize its profits $\Pi_{NA}=\alpha(p+p_a)-\frac{1}{2}q^2$ subject to the above inequalities. It is easy to see that IR_h implies $p+p_a\leq\theta_h(q+\lambda\frac{1}{2}-\beta\frac{1}{2})$. Thus, to ensure that IR_h and IC_h are both satisfied while maximizing the revenue $\alpha(p+p_a)$ generated from the high willingness to pay players, the firm can simply set $p_a=0$ and $p=\theta_h(q+\lambda\frac{1}{2}-\beta\frac{1}{2})$. Since $\theta_h>\theta_l$, with these prices, type-l players indeed choose not to buy, i.e., the inequality (20) is satisfied. Plugging the prices into the profit function, we obtain $\Pi_{NA}=\alpha\theta_h(q+\lambda\frac{1}{2}-\beta\frac{1}{2})-\frac{1}{2}q^2$. Notice that Π_{NA} is the same as what the firm earns in the benchmark without a pay-to-win add-on by setting the base product price equal to $p=\theta_h(q+\lambda\frac{1}{2}-\beta\frac{1}{2})$ to target only type-h players. Therefore, the firm's optimal profits in the benchmark are weakly higher than Π_{NA}.

3. $(d_l,d_h)=(B,A)$. As we discussed in the main text, the firm sets prices such that $U_l(B\mid(B,A))=0$ and $U_h(A\mid(B,A))=U_h(B\mid(B,A))$. This leads to the following pricing expressions:

$$\begin{aligned}p^*&=\theta_l(q+\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega+(1-\alpha)\frac{1}{2}))\\ p_a^*&=\theta_h(\beta+\lambda)(\omega-\frac{1}{2}).\end{aligned}$$

The firm maximizes its profit $\pi(q,\omega)=\alpha(p^*+p_a^*)+(1-\alpha)p^*-\frac{1}{2}q^2$, subject to $\omega\in\left[\frac{1}{2},1\right]$ and $q+\lambda\left(\alpha(1-\omega)+(1-\alpha)\frac{1}{2}\right)-\beta\left(\alpha\omega+(1-\alpha)\frac{1}{2}\right)\geq 0$. The last inequality ensures that type-l players obtain non-negative utility. We will separately analyze the following two possible cases based on whether the constraint is non-binding (i.e., $q+\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega+(1-\alpha)\frac{1}{2})>0$) or binding (i.e., $q+\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega+(1-\alpha)\frac{1}{2})=0$).

Case 1): Let's first assume that the optimal q and ω satisfy $q+\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega+(1-\alpha)\frac{1}{2})>0$. In this case, $\frac{\partial\pi}{\partial\omega}=\alpha(\beta+\lambda)(\theta_h-\theta_l)>0$, as $\theta_h>\theta_l$, leading to $\omega^*=1$. Substituting $\omega^*=1$, we obtain $\pi(q)=\frac{1}{2}\left(\alpha(\beta+\lambda)(\theta_h-\theta_l)+\theta_l(\lambda-\beta)-q^2+2\theta_l q\right)$. The profit function $\pi(q)$ is concave in q because $\frac{\partial^2\pi}{\partial q^2}=-1<0$. Applying the first-order condition, we find $q^*=\theta_l$. The condition $q^*+\lambda(\alpha(1-\omega^*)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega^*+(1-\alpha)\frac{1}{2})\geq 0$ is satisfied if and only if $\beta<\hat{\beta}$, where $\hat{\beta}=\frac{2\theta_l+(1-\alpha)\lambda}{1+\alpha}$. Choosing $q^*=\theta_l$ and $\omega^*=1$, the firm obtains a profit

$$\pi_1^*=\frac{\theta_l^2+\theta_l(\lambda-\beta)+\alpha\left(\theta_h-\theta_l\right)\left(\beta+\lambda\right)}{2}. \tag{22}$$

Case 2): Now, let us assume that the constraint is binding, i.e., $q+\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})-\beta(\alpha\omega+(1-\alpha)\frac{1}{2})=0$. Thus, $q^*=-\lambda(\alpha(1-\omega)+(1-\alpha)\frac{1}{2})+\beta(\alpha\omega+(1-\alpha)\frac{1}{2})$. Pluggin this into the profit function, we obtain $\pi(\omega)=\frac{1}{8}\left(4\alpha\theta_h(\beta+\lambda)(-1+2\omega)-(-\beta+\alpha\beta+\lambda+\alpha\lambda-2\alpha(\beta+\lambda)\omega)^2\right)$. Notice that $\pi(\omega)$ is concave in ω since $\frac{\partial^2\pi}{\partial\omega^2}=-\alpha^2(\beta+\lambda)^2<0$. Solving the first-order condition, we find that $\omega^*=\frac{1}{2}+\frac{2\theta_h-(\beta-\lambda)}{2\alpha(\beta+\lambda)}$. Note that $\omega^*\in(\frac{1}{2},1)$ if and only if $\beta<\tilde{\beta}$, where $\tilde{\beta}=\frac{2\theta_h+(1-\alpha)\lambda}{1+\alpha}$. When $\beta\geq\frac{2\theta_h+(1-\alpha)\lambda}{1+\alpha}$, the constraint $\omega\leq 1$ becomes binding, and the firm sets $\omega^*=1$. Using q^* and ω^* from Case 2), we obtain the firm's corresponding profits:

$$\pi_2^*=\begin{cases}\frac{\left(4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2\right)}{8} & \text{if } \beta<\min\left\{\tilde{\beta},\bar{\beta}\right\}\\ \frac{\theta_h(\theta_h+(\lambda-\beta))}{2} & \text{if } \min\left\{\tilde{\beta},\bar{\beta}\right\}\leq\beta<\bar{\beta}\end{cases}. \tag{23}$$

The solutions from Cases 1 and 2 coexist when $\beta<\hat{\beta}$. Comparing π_1 and π_2, we find that $\pi_1^*>\pi_2^*$ when $\beta<\hat{\beta}$, i.e., the global maximizer is the solution in Case 1. When $\beta>\hat{\beta}$, the optimal solution is characterized in Case 2. To summarize, the firm's optimal decisions and corresponding profits are as follows:

- If $0\leq\beta<\hat{\beta}$, then
$$q^*=\theta_l,\quad \omega^*=1\quad \text{and } \pi^*=\frac{\theta_l^2+\theta_l(\lambda-\beta)+\alpha(\theta_h-\theta_l)(\beta+\lambda)}{2}.$$
- If $\hat{\beta}<\beta<\min\{\tilde{\beta},\bar{\beta}\}$, then
$$q^*=\frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2},\quad \omega^*=1\quad \text{and } \pi^*=\frac{4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2}{8}.$$
- If $\min\{\tilde{\beta},\bar{\beta}\}\leq\beta<\bar{\beta}$, then
$$q^*=\theta_h,\quad \omega^*=\frac{1}{2}+\frac{2\theta_h-(\beta-\lambda)}{2\alpha(\beta+\lambda)}\quad \text{and } \pi^*=\frac{\theta_h(\theta_h-(\beta-\lambda))}{2},$$

where $\hat{\beta}=\frac{2\theta_l+(1-\alpha)\lambda}{(1+\alpha)}$ and $\tilde{\beta}=\frac{2\theta_h+(1-\alpha)\lambda}{(1+\alpha)}$.

As we will demonstrate in the proof of Proposition 1, when $(d_l,d_h)=(B,A)$, the firm's equilibrium profits can indeed be strictly higher than in the benchmark without a pay-to-win add-on. For future reference, note that the condition $\beta<\hat{\beta}$ is equivalent to $\lambda>\hat{\lambda}$, where $\hat{\lambda}=\max\{\frac{(1+\alpha)\beta-2\theta_l}{1-\alpha},0\}$. Similarly, $\beta<\tilde{\beta}$ is equivalent to $\lambda>\tilde{\lambda}$, where $\tilde{\lambda}=\max\{\frac{(1+\alpha)\beta-2\theta_h}{1-\alpha},0\}$.

4. Mixed Strategy Equilibrium: We will show that when a pay-to-win add-on is introduced, there is no equilibrium where type-h and/or type-l players use a mixed strategy. By contraposition, suppose that such an equilibrium exists. Then, the players must be indifferent between the actions that have a positive probability of being played according to their mixed strategy. If the players attach any positive probability to $d=N$, then the firm can slightly reduce its base price or increase quality to induce players not to choose $d=N$ with any positive probability—doing so will strictly benefit the firm. Hence, let us consider a situation where players mix over B and A. One can readily show that if type-i players are indifferent between A and B, then type-j players strictly prefer one over the other, where $i, j \in \{l, h\}$ and $i \neq j$. Therefore, there are three possible situations. First, type-h players mix over A and B, whereas type-l players choose B. Second, type-h players mix over A and B, whereas type-l players choose N. Third, type-h players choose A, whereas type-l players mix over A and B.

Let us start by analyzing the first situation: type-h players mix over A and B, whereas type-l players choose B. To make type-h players indifferent between A and B, it must be that $p_a = \theta_h(\beta+\lambda)(\omega - \frac{1}{2})$. Let $\sigma_h \in (0,1)$ be the probability with which a type-h player chooses A. Then, by the Law of Large Numbers, the measure of type-h players who will buy the add-on will be $\chi = \alpha\sigma_h$. Note that $\chi \in (0, \alpha)$. The optimal base product price makes type-l players indifferent between buying the base product and not buying it. Specifically,

$$p = \theta_l(q + \lambda(\chi(1-\omega) + (1-\chi)\frac{1}{2}) - \beta(\chi\omega + (1-\chi)\frac{1}{2}))$$

The firm chooses ω and q to maximize its profits $\pi = \chi p_a + p - \frac{1}{2}q^2$. The analysis is identical to that in our main model if we replace α with χ. Solving the firm's profit maximization problem, we find that

$$q^* = \begin{cases} \theta_l & \text{if } 0 \le \beta \le \hat{\beta}(\chi) \\ \frac{(1+\chi)\beta-(1-\chi)\lambda}{2} & \text{if } \hat{\beta}(\chi) < \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\} \\ \theta_h & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \le \beta < \bar{\beta} \end{cases}$$

$$\omega^* = \begin{cases} 1 & \text{if } 0 \le \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\} \\ \frac{1}{2} + \frac{2\theta_h - (\beta-\lambda)}{2\chi(\beta+\lambda)} & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \le \beta < \bar{\beta} \end{cases}$$

where $\hat{\beta}(\chi) = \frac{2\theta_l + (1-\chi)\lambda}{(1+\chi)}$, $\tilde{\beta}(\chi) = \frac{2\theta_h + (1-\chi)\lambda}{(1+\chi)}$ and recall from Lemma 1 that $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$. Using the product design and pricing decisions above, we derive the firm's profits:

$$\pi^*(\chi) = \begin{cases} \frac{{\theta_l}^2 + \theta_l(\lambda-\beta) + \chi(\theta_h-\theta_l)(\beta+\lambda)}{2} & \text{if } 0 \le \beta \le \hat{\beta}(\chi), \\ \frac{4\chi\theta_h(\beta+\lambda) - ((1+\chi)\beta-(1-\chi)\lambda)^2}{8} & \text{if } \hat{\beta}(\chi) < \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\}, \\ \frac{\theta_h(\theta_h - (\beta-\lambda))}{2} & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \le \beta < \bar{\beta}. \end{cases}$$

The function $\pi^*(\chi)$ is continuous in χ and one can readily show that $\frac{\partial\pi}{\partial\chi} \ge 0$ at any $\chi \in (0, \alpha)$. Therefore, the firm is better off when $\chi = \alpha$, i.e., when type-h players choose A with certainty instead of using a mixed strategy. The firm can achieve this by reducing the price of the pay-to-win add-on by an ϵ, inducing all type-h players to strictly prefer choosing $d = A$. It follows that $\sigma_h \in (0,1)$ cannot constitute a mixed strategy equilibrium.

Using a similar proof, we can also rule out the two other candidates for a mixed strategy equilibrium that were mentioned earlier.

□

2. Proof of Proposition 1

Recall the boundary condition $\beta < \bar{\beta}$, where $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$. From Lemma 1, we know that the firm's profits without a pay-to-win add-on are given by

$$\pi^B = \begin{cases} \frac{\theta_l(\theta_l-(\beta-\lambda)}{2} & \text{if } 0 \leq \alpha \leq \frac{\theta_l}{\theta_h}, \\ \frac{\alpha\theta_h(\alpha\theta_h-(\beta-\lambda))}{2} & \text{if } \frac{\theta_l}{\theta_h} \leq \alpha \leq 1. \end{cases}$$

Also recall from Lemma A.1 that selling a pay-to-win add-on may increase the firm's profits only when $(d_l, d_h) = (B, A)$, in which case the profit π^* of the firm is given by (10) in the main text. Namely,

$$\pi^* = \begin{cases} \frac{\left(\theta_l{}^2+\theta_l(\lambda-\beta)+\alpha(\theta_h-\theta_l)(\beta+\lambda)\right)}{2} & \text{if } 0 \leq \beta \leq \hat{\beta}, \\ \frac{\left(4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2\right)}{8} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{\theta_h(\theta_h+(\lambda-\beta))}{2} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta} \end{cases}$$

where $\hat{\beta} = \frac{2\theta_l+(1-\alpha)\lambda}{(1+\alpha)}$ and $\tilde{\beta} = \frac{2\theta_h+(1-\alpha)\lambda}{(1+\alpha)}$.

To prove, Proposition 1, we need to characterize the parameter conditions under which $\pi^* > \pi^B$. We will separately analyze the following two possible cases: i) $0 \leq \alpha \leq \frac{\theta_l}{\theta_h}$, and ii) $\frac{\theta_l}{\theta_h} \leq \alpha \leq 1$.

Case i) $0 \leq \alpha \leq \frac{\theta_l}{\theta_h}$. In this case, $\pi^B = \frac{\theta_l(\theta_l-(\beta-\lambda))}{2}$ and $\bar{\beta} = \lambda + 2\theta_l$.

- If $\beta \in [0, \hat{\beta}]$, then $\pi^* - \pi^B = \frac{\alpha(\beta+\lambda)(\theta_h-\theta_l)}{2} > 0$ due to $\theta_h > \theta_l$.
- If $\beta \in (\hat{\beta}, \min\{\tilde{\beta}, \bar{\beta}\})$, then $\pi^* - \pi^B = -\alpha^2\frac{\theta_h^2}{2} - \alpha\frac{(\theta_l\lambda-(2\theta_h-\theta_l)\beta)}{2} + \frac{\theta_l(\theta_l-(\beta-\lambda)}{2}$. One can readily show that $\frac{d^2(\pi^*-\pi^B)}{d\beta^2} = -\frac{(\alpha+1)^2}{4} < 0$, i.e., the function $\pi^* - \pi^B$ is concave. Further, one can show that $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} > 0$ and $(\pi^* - \pi^B)|_{\beta=\min\{\tilde{\beta},\bar{\beta}\}} > 0$. Concavity of $\pi^* - \pi^B$, together with $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} > 0$ and $(\pi^* - \pi^B)|_{\beta=\min\{\tilde{\beta},\bar{\beta}\}} > 0$, implies that $\pi^* - \pi^B > 0$ for all $\hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}$.
- If $\beta \in [\min\{\tilde{\beta}, \bar{\beta}\}, \bar{\beta})$, then $\pi^* - \pi^B = \frac{(\theta_h-\theta_l)(\theta_h+\theta_l-(\beta-\lambda))}{2}$.[14] Straightforward algebra shows that $\pi^* - \pi^B > 0$ if and only if $\beta < \theta_h + \theta_l + \lambda$. Since $\beta < \bar{\beta} = \lambda + 2\theta_l < \theta_h + \theta_l + \lambda$, it follows that $\pi^* - \pi^B > 0$ for any $\beta \in [\min\{\tilde{\beta}, \bar{\beta}\}, \bar{\beta})$.

Case ii): $\frac{\theta_l}{\theta_h} < \alpha \leq 1$. In this case, $\pi^B = \frac{\alpha\theta_h(\alpha\theta_h-(\beta-\lambda)}{2}$ and $\bar{\beta} = \lambda + 2\alpha\theta_h$.

- If $\beta \in [0, \hat{\beta}]$, then $\pi^* - \pi^B = \frac{-\alpha^2\theta_h^2+\theta_l^2-\beta(\alpha(\theta_l-2\theta_h)+\theta_l)-(\alpha-1)\theta_l\lambda}{2}$. Note that $\frac{d(\pi^*-\pi^B)}{d\beta} = \frac{(\alpha(2\theta_h-\theta_l)-\theta_l)}{2} > 0$. Next, $(\pi^* - \pi^B)|_{\beta=0} = \frac{\theta_l(\theta_l+\lambda)-\alpha\left(\alpha\theta_h^2+\theta_l\lambda c\right)}{2} > 0$ if and only if $\alpha < \alpha_1$, where $\alpha_1 = \frac{\sqrt{\theta_l\left(\theta_l\lambda^2+4\theta_h^2(\theta_l+\lambda)\right)}-\theta_l\lambda}{2\theta_h^2}$. Since $\frac{d(\pi^*-\pi^B)}{d\beta} > 0$, it follows that when $\alpha < \alpha_1$, we have $\pi^* - \pi^B > 0$ for all $\beta \in [0, \hat{\beta}]$. Next, consider $\alpha_1 < \alpha \leq 1$, where we know that $(\pi^* - \pi^B)|_{\beta=0} < 0$. One can show that $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} = -\frac{\alpha^3\theta_h^2+\theta_l^2+\alpha^2\theta_h(\theta_h+2\lambda)+\alpha\left(\theta_l^2-2\theta_h(2\theta_l+\lambda)\right)}{2(\alpha+1)} < 0$ if and only if $\alpha > \alpha_2$, where $\alpha_2 \in (\alpha_1, 1)$ is

[14] Note that the interval $[\min\{\tilde{\beta}, \bar{\beta}\}, \bar{\beta})$ is non-empty if and only if $\tilde{\beta} < \bar{\beta}$, which is equivalent to $\alpha > \frac{\theta_h-\theta_l}{\theta_l+\lambda}$.

implicitly defined by the equality $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} = 0$.[15] Since $\frac{d(\pi^*-\pi^B)}{d\beta} > 0$, having $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} < 0$ and $(\pi^* - \pi^B)|_{\beta=0} < 0$ when $\alpha \in [\alpha_2, 1]$ implies that $\pi^* - \pi^B < 0$ for all $\beta \in [0, \hat{\beta}]$. Finally, when $\alpha \in (\alpha_1, \alpha_2)$, there exists a unique $\beta_a \in (0, \hat{\beta})$ such that $\pi^* - \pi^B < 0$ if $\beta \in [0, \beta_a)$ and $\pi^* - \pi^B > 0$ if $\beta \in (\beta_a, \hat{\beta})$. Solving $\pi^* - \pi^B = 0$, we find that $\beta_a = \frac{\alpha^2\theta_h^2 + \alpha\theta_l\lambda - \theta_l(\theta_l+\lambda)}{2\alpha\theta_h - (\alpha+1)\theta_l}$.

- If $\beta \in (\hat{\beta}, \min\{\tilde{\beta}, \bar{\beta}\})$, then $\pi^* - \pi^B = -\frac{\alpha^2\theta_h^2}{2} - \frac{((\alpha+1)\beta+(\alpha-1)\lambda)^2}{8} + \alpha\beta\theta_h$. Note that $\frac{d^2(\pi^*-\pi^B)}{d\beta^2} = -\frac{(\alpha+1)^2}{4} < 0$, i.e., $\pi^* - \pi^B$ is concave. Further, one can show that $(\pi^* - \pi^B)_{\beta=\min\{\tilde{\beta},\bar{\beta}\}} > 0$. First, when $\alpha \le \alpha_2$, the analysis in the previous paragraph showed that $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} > 0$. Therefore, concavity and continuity of $\pi^* - \pi^B$, together with $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} > 0$ and $(\pi^* - \pi^B)_{\beta=\min\{\tilde{\beta},\bar{\beta}\}} > 0$, suggest that $\pi^* - \pi^B > 0$ for all $\beta \in (\hat{\beta}, \min\{\tilde{\beta}, \bar{\beta}\})$. Second, when $\alpha_2 < \alpha \le 1$, we know from the previous paragraph that $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} < 0$. Concavity and continuity of $\pi^* - \pi^B$, together with $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} < 0$ and $(\pi^* - \pi^B)_{\beta=\min\{\tilde{\beta},\bar{\beta}\}} > 0$, suggest that there exists a unique $\beta_b \in (\hat{\beta}, \min\{\tilde{\beta}, \bar{\beta}\})$ such that $\pi^* - \pi^B < 0$ if $\beta \in [\hat{\beta}, \beta_b)$ and $\pi^* - \pi^B > 0$ if $\beta \in (\beta_b, \min\{\tilde{\beta}, \bar{\beta}\})$. Solving $\pi^* - \pi^B = 0$, we find that $\beta_b = \frac{4\alpha\theta_h - \alpha^2\lambda - 2\sqrt{(1-\alpha)\alpha\theta_h(\alpha(\alpha+3)\theta_h + 2(\alpha+1)\lambda)} + \lambda}{(\alpha+1)^2}$.
- If $\beta \in [\min\{\tilde{\beta}, \bar{\beta}\}, \bar{\beta})$, then $\pi^* - \pi^B = \frac{(1-\alpha)\theta_h(\alpha\theta_h + \theta_h - \beta + \lambda)}{2}$. It is easy to see that $\pi^* - \pi^B > 0$ if and only if $\beta < \alpha\theta_h + \theta_h + \lambda$. Since $\bar{\beta} < \alpha\theta_h + \theta_h + \lambda$, it follows that $\pi^* - \pi^B > 0$ for all $\beta \in [\min\{\tilde{\beta}, \bar{\beta}\}, \bar{\beta})$.

To summarize the above analysis, we found that $\pi^* - \pi^B > 0$ if and only if $\alpha \in [0, \alpha_1)$, or $\alpha \in [\alpha_1, 1]$ and $\beta \in (\beta_1, \bar{\beta}]$, where

$$\beta_1 = \begin{cases} \beta_a \text{ if } \alpha \in [\alpha_1, \alpha_2] \\ \beta_b \text{ if } \alpha \in [\alpha_2, 1] \end{cases} = \begin{cases} \frac{\alpha^2\theta_h^2 + \alpha\theta_l\lambda - \theta_l(\theta_l+\lambda)}{2\alpha\theta_h - (\alpha+1)\theta_l} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{4\alpha\theta_h - \alpha^2\lambda - 2\sqrt{(1-\alpha)\alpha\theta_h(\alpha(\alpha+3)\theta_h + 2(\alpha+1)\lambda)} + \lambda}{(\alpha+1)^2} & \text{if } \alpha \in [\alpha_2, 1]. \end{cases} \tag{24}$$

Note that the above condition on β for the optimality of introducing a pay-to-win add-on can also be expressed in terms of λ. Namely, $\beta > \beta_1$ is equivalent to $\lambda > \lambda_1$, where

$$\lambda_1 = \begin{cases} \lambda_a \text{ if } \alpha \in [\alpha_1, \alpha_2] \\ \lambda_b \text{ if } \alpha \in [\alpha_2, 1] \end{cases} = \begin{cases} \frac{\alpha\theta_h(\alpha\theta_h - 2\beta) + (1+\alpha)\beta\theta_l - \theta_l{}^2}{(1-\alpha)\theta_l} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{\beta(1+\alpha) + 2\sqrt{\alpha\theta h(2\beta + \alpha\theta_h)}}{1-\alpha} & \text{if } \alpha \in [\alpha_2, 1]. \end{cases} \tag{25}$$

□

3. Proof of Lemma 2

Recall that to ensure positive equilibrium sales, we are focusing on the parameter region with $\beta < \bar{\beta}$, where $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$. Also recall that the optimal price in the benchmark without pay-to-win add-on is provided in Lemma 1 and is as follows:

[15] The existence of the unique α_2 is guaranteed because $(\pi^* - \pi^B)|_{\beta=\hat{\beta}}$ is a concave function of α and $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} > 0$ when $\alpha = \alpha_1$ and $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} < 0$ when $\alpha = 1$.

$$p^B = \begin{cases} \theta_l(\theta_l - \frac{\beta-\lambda}{2}) & \text{if } \alpha \leq \frac{\theta_l}{\theta_h}, \\ \theta_h(\alpha\theta_h - \frac{\beta-\lambda}{2}) & \text{if } \alpha \geq \frac{\theta_l}{\theta_h}. \end{cases} \tag{26}$$

Note that the condition $\beta < \bar{\beta}$ entails that $p^B > 0$.

By (6)-(9), the price of the base product when selling a pay-to-win add-on is given by

$$p^* = \begin{cases} \theta_l(\theta_l - \frac{((1+\alpha)\beta-(1-\alpha)\lambda)}{2}) & \text{if } 0 \leq \beta < \hat{\beta}, \\ 0 & \text{if } \hat{\beta} \leq \beta \leq \bar{\beta}, \end{cases} \tag{27}$$

where $\hat{\beta} = \frac{2\theta_l+(1-\alpha)\lambda}{(1+\alpha)}$.

To prove Lemma 2, we compare p^B and p^* separately in the following two possible cases: i) $0 \leq \beta < \hat{\beta}$ and ii) $\hat{\beta} \leq \beta \leq \bar{\beta}$.

Case i): If $\hat{\beta} \leq \beta \leq \bar{\beta}$, then, by (27), we have $p^* = 0$. That is, the firm sells the base product as a freemium if the players' sensitivity to losing β is high. Therefore, we have $p^B > p^* = 0$ in this case.

Case ii): If $0 \leq \beta < \hat{\beta}$, then, by (27), we have $p^* = \theta_l(\theta_l - \frac{((1+\alpha)\beta-(1-\alpha)\lambda)}{2})$.

1. If $\alpha \leq \frac{\theta_l}{\theta_h}$, then $p^B = \theta_l(\theta_l - \frac{\beta-\lambda}{2})$ by (26). Since $0 < \alpha < 1$, it is easy to see that $p^* < p^B$.
2. If $\alpha > \frac{\theta_l}{\theta_h}$, then $p^B = \theta_h(\alpha\theta_h - \frac{\beta-\lambda}{2})$ by (26). Further, $\alpha \in [\frac{\theta_l}{\theta_h}, 1)$ implies that $\alpha\theta_h - \frac{\beta-\lambda}{2} > \theta_l - \frac{((1+\alpha)\beta-(1-\alpha)\lambda)}{2}$. The last inequality, together with $\theta_h > \theta_l$, imply that $p^* < p^B$.

To summarize, we showed that $p^* < p^B$. Further, when $\hat{\beta} \leq \beta \leq \bar{\beta}$, we have $p^* = 0$. This finishes the proof of Lemma 2. Note that the condition $\beta \geq \hat{\beta}$ is equivalent to $\lambda \leq \hat{\lambda}$, where $\hat{\lambda} = \max\{\frac{(1+\alpha)\beta-2\theta_l}{1-\alpha}, 0\}$. □

4. Proof of Proposition 2

From Lemma 1 we know that without a pay-to-win add-on, the firm provides quality q^B, where

$$q^B = \max\{\theta_l, \alpha\theta_h\} \tag{28}$$

With a pay-to-win add-on, the firm's optimal quality choice is given by (8):

$$q^* = \begin{cases} \theta_l & \text{if } 0 \leq \beta \leq \hat{\beta}, \\ \frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \theta_h & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta} \end{cases} \tag{29}$$

where $\hat{\beta} = \frac{2\theta_l+(1-\alpha)\lambda}{(1+\alpha)}$ and $\tilde{\beta} = \frac{2\theta_h+(1-\alpha)\lambda}{(1+\alpha)}$.

To prove Proposition 2, we compare q^B and q^* separately in the following three possible cases: i) $0 \leq \beta < \hat{\beta}$, ii) $\hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}$ and iii) $\min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}$.

Case i): If $0 \leq \beta < \hat{\beta}$, then, by (29), we have $q^* = \theta_l$. Using (28), it is easy to see that $q^B \geq q^*$.

Case ii): If $\hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}$, then, by (29), we have $q^* = \frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2}$. If $q^B = \theta_l$ (i.e., $\alpha < \frac{\theta_l}{\theta_h}$), then $q^* > q^B$ because $\beta > \hat{\beta}$. If $q^B = \alpha\theta_h$ (i.e., $\alpha \geq \frac{\theta_l}{\theta_h}$), then $q^* > q^B$ if and only if $\beta > \frac{2\alpha\theta_h+(1-\alpha)\lambda}{(1+\alpha)}$, where $\frac{2\alpha\theta_h+(1-\alpha)\lambda}{(1+\alpha)} < \min\{\tilde{\beta}, \bar{\beta}\}$.

Case iii): If $\min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}$, then $q^* = \theta_h$ by (29). Since $\theta_h > \theta_l$ and $\alpha \in (0,1)$, it follows that $q^* > q^B$.

Define $\beta_2 = \max\{\hat{\beta}, \frac{2\alpha\theta_h+(1-\alpha)\lambda}{(1+\alpha)}\}$. The above analysis proves that if the players' sensitivity to losing is high ($\beta > \beta_2$), then in equilibrium, the firm chooses a higher product quality when selling a pay-to-win add-on than when the firm does not sell the add-on, i.e., $q^* > q^B$. However, if sensitivity to losing is low ($\beta \leq \beta_2$), then the firm chooses a weakly lower quality: $q^* \leq q^B$. Note that the condition $\beta > \beta_2$ is equivalent to $\lambda < \lambda_2$, where $\lambda_2 = \min\{\hat{\lambda}, \frac{(1+\alpha)\beta-2\alpha\theta_h}{1-\alpha}\}$. □

5. Proof of Proposition 3

See the proof of Lemma A1 where we characterize ω^*. Defining $\beta_3 = \min\{\tilde{\beta}, \bar{\beta}\}$ finishes the proof. □.

6. Proof of Proposition 4

Before we proceed to the proof, recall that to ensure positive equilibrium sales, our analysis focuses on $\beta \leq \bar{\beta}$, where $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$. To prove the proposition, we will first obtain the player surplus in the benchmark scenario without a pay-to-win add-on and in the main setting with a pay-to-win add-on. Subsequently, we will compare the player surpluses in these two scenarios.

Without a Pay-to-Win Add-on: The utility of a type-i player from purchasing the product is given by $U_i^B = \theta_i(q^B + \frac{1}{2}\lambda - \frac{1}{2}\beta) - p^B$. Player surplus is defined as $PS^B = \alpha U_h^B + (1-\alpha)U_l^B$. The equilibrium q^B and p^B are provided in Lemma 1. One can readily show that the player surplus in the benchmark case can be described as follows:

$$PS^B = \begin{cases} \alpha(\theta_h - \theta_l)(\theta_l + \frac{\lambda-\beta}{2}) & \text{if } \alpha \leq \frac{\theta_l}{\theta_h}, \\ 0 & \text{if } \alpha \geq \frac{\theta_l}{\theta_h}. \end{cases} \tag{30}$$

With a Pay-to-Win Add-on: Recall that, in equilibrium, $(d_l, d_h) = (B, A)$. The players' equilibrium utilities are as follows: $U_l^* = \theta_l\left(q^* + \lambda\left(\alpha(1-\omega^*) + (1-\alpha)\frac{1}{2}\right) - \beta\left(\alpha\omega^* + (1-\alpha)\frac{1}{2}\right)\right) - p^*$ and $U_h^* = \theta_h(q^* + \lambda\left(\alpha\frac{1}{2} + (1-\alpha)\omega^*\right) - \beta\left(\alpha\frac{1}{2} + (1-\alpha)(1-\omega^*)\right)) - (p^* + p_a^*)$. The equilibrium q^*, ω^*, p^* and p_a^* are characterized in the proof of Lemma A1. One can readily show that

$$PS^* = \begin{cases} \alpha(\theta_h - \theta_l)(\theta_l + \frac{(1-\alpha)\lambda-(1+\alpha)\beta}{2}) & \text{if } 0 \leq \beta \leq \hat{\beta}, \\ 0 & \text{if } \hat{\beta} \leq \beta < \bar{\beta}, \end{cases} \tag{31}$$

where $\hat{\beta} = \frac{2\theta_l + (1-\alpha)\lambda}{(1+\alpha)}$.

Comparing Benchmark with Pay-to-Win Add-on. First, for $\alpha \geq \frac{\theta_l}{\theta_h}$ and $0 < \beta < \hat{\beta}$, it is easy to see that $PS^* = PS^B = 0$. Second, for $\alpha < \frac{\theta_l}{\theta_h}$, we have $PS^* < PS^B$ because $\frac{(1-\alpha)\lambda - (1+\alpha)\beta}{2} < \frac{\lambda - \beta}{2}$. Finally, for $\alpha \geq \frac{\theta_l}{\theta_h}$ and $0 \leq \beta < \hat{\beta}$ (or equivalently, $\lambda > \hat{\lambda}$), we have $PS^* = \alpha(\theta_h - \theta_l)(\theta_l + \frac{(1-\alpha)\lambda - (1+\alpha)\beta}{2}) > PS^B = 0$. Defining $\beta_4 = \hat{\beta}$ and $\lambda_4 = \hat{\lambda}$ completes the proof. □

7. Proof of Proposition 5

We will first prove that the region where $\pi^* - \pi^B > 0$ expands as λ increases. Equivalently, we can show that the region where $\pi^* - \pi^B < 0$ shrinks as λ increases. Define $\Delta\pi = \pi^* - \pi^B$. Recall from Proposition 1 that $\Delta\pi < 0$ if and only if $\alpha_1 < \alpha < 1$ and $\beta < \beta_1$, where $\alpha_1 = \frac{\sqrt{\theta_l\left(\theta_l\lambda^2 + 4\theta_h^2(\theta_l + \lambda)\right)} - \theta_l\lambda}{2\theta_h^2}$. First, note that $\frac{\partial\alpha_1}{\partial\lambda} = \frac{\theta_l\left(2\theta_h^2 + \lambda\theta_l\right)}{2\theta_h^2\sqrt{\theta_l\left(4\theta_h^2(\lambda + \theta_l) + \lambda^2\theta_l\right)}} - \frac{\theta_l}{2\theta_h^2} > 0$, i.e., the interval $(\alpha_1, 1)$ shrinks as λ increases. Second, we can show that $\frac{\partial\beta_1}{\partial\lambda} < 0$. To see this, recall from the proof of Proposition 1 that

$$\beta_1 = \begin{cases} \beta_a \text{ if } \alpha \in [\alpha_1, \alpha_2] \\ \beta_b \text{ if } \alpha \in [\alpha_2, 1] \end{cases} = \begin{cases} \frac{\alpha^2\theta_h^2 + \alpha\theta_l\lambda - \theta_l(\theta_l + \lambda)}{2\alpha\theta_h - (\alpha+1)\theta_l} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{4\alpha\theta_h - \alpha^2\lambda - 2\sqrt{(1-\alpha)\alpha\theta_h(\alpha(\alpha+3)\theta_h + 2(\alpha+1)\lambda)} + \lambda}{(\alpha+1)^2} & \text{if } \alpha \in [\alpha_2, 1], \end{cases}$$

where the cutoff α_2 is implicitly defined by the equality $(\pi^* - \pi^B)_{\beta = \hat{\beta}} = -\frac{\alpha^3\theta_h^2 + \theta_l^2 + \alpha^2\theta_h(\theta_h + 2\lambda) + \alpha\left(\theta_l^2 - 2\theta_h(2\theta_l + \lambda)\right)}{2(\alpha+1)} = 0$. On the interval (α_1, α_2), one can readily show that $\frac{\partial\beta_1}{\partial\lambda} = -\frac{(1-\alpha)\theta_l}{\alpha(2\theta_h - \theta_l) - \theta_l} < 0$. On the interval $(\alpha_2, 1)$, we have $\frac{\partial\beta_1}{\partial\lambda} = -\frac{1-\alpha}{1+\alpha}\left(1 - \frac{2\alpha\theta_h}{\sqrt{(1-\alpha)\alpha\theta_h(2(\alpha+1)\lambda + \alpha(\alpha+3)\theta_h)}}\right) < 0$, where the inequality follows because $2\alpha\theta_h < \sqrt{(1-\alpha)\alpha\theta_h\left(2(\alpha+1)\lambda + \alpha(\alpha+3)\theta_h\right)}$ for any $\alpha \in (\alpha_2, 1)$. It follows that as λ increases, the interval $(0, \beta_1)$ shrinks, making the region where $\pi^* - \pi^B < 0$ smaller.

Next, let us show that as β increases, the region where $\pi^* - \pi^B < 0$ shrinks. As we mentioned above, $\pi^* - \pi^B < 0$ if and only if $\alpha_1 < \alpha < 1$ and $\beta < \beta_1$. Note that $\alpha_1 = \frac{\sqrt{\theta_l\left(\theta_l\lambda^2 + 4\theta_h^2(\theta_l + \lambda)\right)} - \theta_l\lambda}{2\theta_h^2}$ is not a function of β. Further, because β_1 is an increasing function of α (i.e., $\frac{\partial\beta_1}{\partial\alpha} > 0$), it follows that $\beta < \beta_1$ is equivalent to $\alpha > (\beta_1)^{-1}(\beta)$. Since $\frac{\partial\beta_1}{\partial\alpha} > 0$, the Inverse Function Theorem implies that $\frac{\partial(\beta_1)^{-1}}{\partial\beta} > 0$. Hence, as β increases, the interval $((\beta_1)^{-1}(\beta), 1)$ becomes smaller, shrinking the region where the inequality $\pi^* - \pi^B < 0$ holds.

Finally, we will prove that when the firm introduces a pay-to-win add-on, its profits are non-monotone in players' sensitivity to losing, β. Recall that the firm's profits are given by the equation (10) in the main text. That is,

$$\pi^* = \begin{cases} \frac{\left(\theta_l{}^2+\theta_l(\lambda-\beta)+\alpha(\theta_h-\theta_l)(\beta+\lambda)\right)}{2} & \text{if } 0 \le \beta \le \hat{\beta}, \\ \frac{\left(4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2\right)}{8} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta},\bar{\beta}\}, \\ \frac{\theta_h(\theta_h+(\lambda-\beta))}{2} & \text{if } \min\{\tilde{\beta},\bar{\beta}\} \le \beta < \bar{\beta} \end{cases}$$

It is easy to see that $\frac{\partial \pi^*}{\partial \lambda} \ge 0$. Let us show that π^* is non-monotone in β. Before we proceed, note that π^* is continuous and quasi-concave as a function of β.[16] Therefore, π^* is single-peaked.

When $\frac{\theta_l}{\theta_h-\theta_l} < \alpha < 1$, π^* is increasing on the interval $[0,\hat{\beta})$ because $\frac{\partial \pi^*}{\partial \beta} = \frac{(\alpha\theta_h-(1+\alpha)\theta_l)}{2} > 0$. Further, on the interval $(\min\{\tilde{\beta},\bar{\beta}\},\bar{\beta})$, π^* is decreasing in β because $\frac{\partial \pi^*}{\partial \beta} = -\frac{\theta_h}{2} < 0$. Since π^* is single peaked, it follows that there exists a unique $\beta_5 \in [\hat{\beta}, \min\{\tilde{\beta},\bar{\beta}\}]$ such that $\frac{\partial \pi^*}{\partial \beta} > 0$ when $\beta < \beta_5$ and $\frac{\partial \pi^*}{\partial \beta} < 0$ when $\beta > \beta_5$. Solving $\frac{\partial \pi^*}{\partial \beta} = 0$, we find that $\beta_5 = \frac{\lambda(1-\alpha^2)+2\alpha\theta_h}{(1+\alpha)^2}$. □

8. Analysis with customization of matching probabilities

In the benchmark without a pay-to-win add-on, changing the matching probabilities does not affect the firm's profits, given that all players are equally likely to win against each other. Thus, the equilibrium outcome is the same as in our main model.

Let us now analyze the firm's optimal matching strategy when it introduces a pay-to-win addon. Similar to our main model, the firm sells the add-on to type- h players, whereas type- l players buy the base product. Recall that m is the probability with which a player who owns the add-on is matched against another player owning the add-on. We will characterize the optimal m^*, as well as the optimal pricing and product design decisions of the firm.

If type- h players buy the pay-to-win add-on, then a type- h player is matched against another type- h player with probability $m \in [0,1]$. Because the measure of type- h players is α, it follows that the measure of type- h players who are matched against each other is αm. Hence, $\alpha(1-m)$ measure of type- h players is matched against $\alpha(1-m)$ type-l players. The remaining $1-\alpha-$ $\alpha(1-m)$ type-l players are matched against each other. Using these measures of player combinations matched against each other, we can derive the ex-ante probability of a given match occurring. First, from the perspective of a type-h player, the probability of being matched against another type- h player is given by m, whereas the probability of matching against a type- l player is $1-m$. Second, from the perspective of a type-l player, the probability of being matched with a type- h player is given by $\frac{\alpha(1-m)}{1-\alpha}$, and with the remaining probability $1-\frac{\alpha(1-m)}{1-\alpha}$, the type-$l$ player is matched against another type- l player. Notice that when $m=\alpha$, the match probabilities are the same as in our

[16] Quasi-concavity can be proved in two steps. First, we can show that π^* is piecewise concave on each of the three intervals over which the function is defined. Second, at each of the two kink points (i.e., $\hat{\beta}$ and $\min\{\tilde{\beta},\bar{\beta}\}$), we can show that if the left derivative is negative, then the right derivative is also negative. It follows that the function is quasi-concave.

main model. Thus, $m=\alpha$ represents the situation where each player is equally likely to be matched with any other player. Also note that to ensure $1-\frac{\alpha(1-m)}{1-\alpha} \geq 0, m$ must satisfy $m \geq 2-\frac{1}{\alpha}$ whenever $\alpha>\frac{1}{2}$. In words, when the fraction of type-h players is greater than the fraction of type- l players, m cannot be too low: some type- h players will have to be matched against each other.

We now proceed to solving the firm's profit maximization problem. Given game design decisions q, ω and m, and players' purchase decisions $(d_l, d_h)=(B, A)$, a player who does not have the pay-to-win add-on has an ex-ante chance of winning $\rho_{\text{win}}=(1-\omega) \frac{\alpha(1-m)}{1-\alpha}+\frac{1}{2}(1-\frac{\alpha(1-m)}{1-\alpha}\Big)$. By contrast, if the player has the add-on, then their ex-ante winning probability is $\rho_{win}^{ptw}=\frac{1}{2} m+\omega(1-m)$.

The firm's prices will need to satisfy the players' individual-rationality and incentivecompatibility constraints. Namely,

$$
\begin{aligned}
& U_h(A \mid(B, A)) \geq U_h(B \mid(B, A)) \\
& U_h(A \mid(B, A)) \geq 0 \\
& U_l(B \mid(B, A)) \geq U_l(A \mid(B, A)) \\
& U_l(B \mid(B, A)) \geq 0
\end{aligned}
$$

where $U_i(A \mid(B, A))=\theta_i(q+\lambda \rho_{\text{win}}^{ptw}-\beta\left(1-\rho_{\text{win}}^{ptw}\right))-(p+p_a)$ and $U_i(B \mid(B, A))=\theta_i(q+(\lambda \rho_{\text{win}}-\beta(1-\rho_{\text{win}}))-p$. One can readily show that the optimal prices are such that $U_l(B \mid (B, A))=0$ and $U_h(A \mid(B, A))=U_h(B \mid(B, A))$. Solving these, we find that

$$
\begin{aligned}
& p^{**}=\frac{\theta_l(2 q(1-\alpha)-\beta+\lambda+\alpha(2 \beta-m \beta-m \lambda-2(1-m)(\beta+\lambda) \omega))}{2(1-\alpha)} \\
& p_a^{**}=\frac{(1-m) \theta_h(\beta+\lambda)(2 \omega-1)}{2(1-\alpha)}
\end{aligned}
$$

The firm chooses q, ω and m to maximize $\pi=p^{**}+\alpha p_a^{**}-\frac{1}{2} q^2$, subject to $q \geq 0, \frac{1}{2} \leq \omega \leq 1, \max \left\{0,2-\frac{1}{\alpha}\right\} \leq m \leq 1$, and $q+\lambda \rho_{\text{win}}-\beta(1-\rho_{\text{win}}) \geq 0$. The last inequality ensures that prices are non-negative. We will separately analyze the two possible cases, $q+\lambda \rho_N-\beta(1-\rho_N)>0$ and $q+\lambda \rho_N-\beta(1-\rho_N)=0$, characterizing the solution and corresponding existence conditions. Then, we will compare the solutions to determine the global maximizer.

Case 1) $q+\lambda \rho_N-\beta(1-\rho_N)>0$. One can readily show that $\frac{\partial \pi}{\partial \omega}=\frac{(1-m) \alpha(\theta_h-\theta_l)(\beta+\lambda)}{1-\alpha}>0$. Hence, $\omega^{**}=1$. Next, solving $\left.\frac{\partial \pi}{\partial q}\right|_{\omega=1}=0$, we find that $q^{**}=\theta_l$. Finally, $\left.\frac{\partial \pi}{\partial m}\right|_{(\omega=1, q=\theta_l)}=-\frac{\alpha(\theta_h-\theta_l)(\beta+\lambda)}{2(1-\alpha)}<0$, which implies that $m^{**}=\max \left\{0,2-\frac{1}{\alpha}\right\}$. The corresponding profits are given by $\pi=\frac{\theta_l(\lambda-\beta+\theta_l)+\alpha(\beta \theta_h+\theta_h \lambda-\theta_l(\theta_l+2 \lambda))}{2-2 \alpha}$ when $0 \leq \alpha<\frac{1}{2}$ and $\pi=\frac{\beta(\theta_h-2 \theta_l)+\theta_l^2+\theta_h \lambda}{2}$ when $\frac{1}{2} \leq \alpha \leq 1$. It remains to verify that q^{**}, ω^{**} and m^{**} satisfy $q+\lambda \rho_N-\beta(1-\rho_N)>0$. Straightforward algebra shows that the inequality is satisfied if and only if $\beta<\max \left\{\theta_l, 2 \theta_l+\lambda-2 \alpha\left(\theta_l+\lambda\right)\right\}$. One can verify that the second-order conditions are satisfied.

Case 2) $q+\lambda\rho_N-\beta(1-\rho_N)=0$. Solving the equality and plugging in the expression for ρ_N, we find $q^{**}=\frac{\beta-\lambda-\alpha\beta(2-m)+m\alpha\lambda+2(1-m)\alpha(\beta+\lambda)\omega}{2(1-\alpha)}$. Upon plugging q^{**} into the profit function π and maximizing with respect to m and ω, we obtain the optimal m^{**} and ω^{**} :

- If $0<\beta<\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}$, then $\omega^{**}=1$ and $m^{**}=\max\{0,2-\frac{1}{\alpha}\}$, yielding a profit of $\pi=\frac{-\beta^2+\lambda(-\lambda+4(1-\alpha)\alpha(\theta_h+\lambda))+2\beta(\lambda-2\alpha(\lambda-(1-\alpha)\theta_h))}{8(1-\alpha)^2}$.
- If $\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}<\beta<\bar{\beta}$, then any pair (ω^{**}, m^{**}) is a solution such that $\omega^{**}=\frac{(2-m)\alpha\beta-\beta+2(1-\alpha)\theta_h+(1-m\alpha)\lambda}{2(1-m)\alpha(\beta+\lambda)}$, $\max\left\{0,2-\frac{1}{\alpha}\right\}\leq m^{**}\leq\frac{\beta-2\theta_h(1-\alpha)-\lambda(1-2\alpha)}{\alpha(\beta+\lambda)}$, with a corresponding profit of $\pi=\frac{\theta_h(\theta_h-\beta+\lambda)}{2}$.

Let us now characterize the globally optimal solution. When $\beta<\max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\}$, we need to compare the profits from the two cases to determine the global maximizer. One can readily show that the solution from Case 1) dominates. When $\beta\geq\max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\}$, the solution is as in Case 2). To summarize, the firm's profit-maximizing strategy and corresponding profits are as follows:

$$\omega^{**}=\begin{cases} 1 & \text{if } 0<\beta<\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\} \\ \frac{\left(2-m^{**}\right)\alpha\beta-\beta+2(1-\alpha)\theta_h+\left(1-m^{**}\alpha\right)\lambda}{2(1-m^{**})\alpha(\beta+\lambda)} & \text{if } \max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}\leq\beta<\bar{\beta} \end{cases}$$

$$m^{**}=\begin{cases} \max\left\{0,2-\frac{1}{\alpha}\right\} & \text{if } 0<\beta<\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\} \\ \in\left[\max\left\{0,2-\frac{1}{\alpha}\right\}, \frac{\beta-2\theta_h(1-\alpha)-\lambda(1-2\alpha)}{\alpha(\beta+\lambda)}\right] & \text{if } \max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}\leq\beta<\bar{\beta} \end{cases}$$

$$q^{**}=\begin{cases} \theta_l & \text{if } 0<\beta<\max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\} \\ \min\left\{\beta, \frac{\beta-\lambda+2\alpha\lambda}{2(1-\alpha)}\right\} & \text{if } \max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\}\leq\beta\leq\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\} \\ \theta_h & \text{if } \max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}<\beta<\bar{\beta} \end{cases}$$

$$p^{**}=\begin{cases} \max\left\{\theta_l(\theta_l-\beta), \frac{\theta_l(-\beta+2(1-\alpha)\theta_l+(1-2\alpha)\lambda)}{2(1-\alpha)}\right\} & \text{if } 0<\beta<\max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\} \\ 0 & \text{if } \max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\}\leq\beta<\bar{\beta} \end{cases}$$

$$p_a^{**}=\begin{cases} \min\left\{\frac{\theta_h(\beta+\lambda)}{2\alpha}, \frac{\theta_h(\beta+\lambda)}{2(1-\alpha)}\right\} & \text{if } 0<\beta<\max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\} \\ \frac{\theta_h(\lambda-\beta+2\theta_h)}{2\alpha} & \text{if } \max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}\leq\beta<\bar{\beta} \end{cases}$$

$$\pi^{**}=\begin{cases} \min\left\{\frac{\theta_l(\lambda-\beta+\theta_l)+\alpha(\beta\theta_h+\theta_h\lambda-\theta_l(\theta_l+2\lambda))}{2-2\alpha}, \frac{\beta(\theta_h-2\theta_l)+\theta_l^2+\theta_h\lambda}{2}\right\} & \text{if } 0<\beta<\max\{\theta_l, 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\} \\ \frac{-\beta^2+\lambda(-\lambda-4(-1+\alpha)\alpha(\theta_h+\lambda))+2\beta(\lambda-2\alpha((-1+\alpha)\theta_h+\lambda))}{8(1-\alpha)^2} & \text{if } 0<\alpha<\frac{1}{2} \text{ and } 2\theta_l+\lambda-2\alpha(\theta_l+\lambda)\leq\beta\leq 2\theta_h+\lambda-2\alpha(\theta_h+\lambda) \\ \frac{\theta_h(\lambda+\beta)-\beta^2}{2} & \text{if } \frac{1}{2}<\alpha<1 \text{ and } \theta_l<\beta<\theta_h \\ \frac{\theta_h(\theta_h+\lambda-\beta)}{2} & \text{if } \max\{\theta_h, 2\theta_h+\lambda-2\alpha(\theta_h+\lambda)\}<\beta<\bar{\beta} \end{cases}$$

8.1. Proof of Proposition 6

Recall that without matching, when the firm introduces a pay-to-win add-on, the firm's profits are as follows:

$$\pi^* = \begin{cases} \frac{{\theta_l}^2+\theta_l(\lambda-\beta)+\alpha(\theta_h-\theta_l)(\beta+\lambda)}{2} & \text{if } 0 \le \beta \le \hat{\beta}, \\ \frac{4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2}{8} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{\theta_h(\theta_h-(\beta-\lambda))}{2} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \le \beta < \bar{\beta}. \end{cases}$$

First note that $\tilde{\beta} > \max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\}$, where recall that $\tilde{\beta} = \frac{2\theta_h+(1-\alpha)\lambda}{1+\alpha}$. From the expressions of π^* and π^{**}, it is easy to notice that when $\beta > \tilde{\beta}$, we have $\pi^* = \pi^{**}$. Thus, for $\beta > \tilde{\beta}$, its ability to design players' matching probabilities does not increase the firm's profits. Hence, $m = \alpha$—which corresponds to the situation where each player is equally likely to be matched with any other player—is optimal. Defining $\beta_6 \equiv \tilde{\beta}$ finishes the proof of the first part of the proposition. Note that the condition $\beta < \beta_6$ is equivalent to $\lambda > \lambda_6$, where $\lambda_6 = \max\{\frac{(1+\alpha)\beta-2\theta_h}{1-\alpha}, 0\}$.

Second, for $\beta < \beta_6$, we see that $\pi^* \neq \pi^{**}$. The firm cannot obtain lower profits than π^* because it always has the option of setting $m = \alpha$ and obtaining π^*. Thus, it must be that $\pi^* < \pi^{**}$ when $\beta < \tilde{\beta}$. Let us examine m^{**}. When $0 < \beta < \max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\}$, we have $m^{**} = \max\left\{0, 2 - \frac{1}{\alpha}\right\}$; one can easily show that $\max\left\{0, 2 - \frac{1}{\alpha}\right\} < \alpha$ for any $\alpha \in (0,1)$. When $\max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\} < \beta < \beta_6$, any $m^{**} \in \left[\max\left\{0, 2 - \frac{1}{\alpha}\right\}, \frac{\beta - 2\theta_h(1-\alpha) - \lambda(1-2\alpha)}{\alpha(\beta+\lambda)}\right]$ can be optimal with a corresponding choice of $\omega^{**} = \frac{\left(2-m^{**}\right)\alpha\beta - \beta + 2(1-\alpha)\theta_h + \left(1-m^{**}\alpha\right)\lambda}{2(1-m^{**})\alpha(\beta+\lambda)}$. One can readily show that $\frac{\beta-2\theta_h(1-\alpha)-\lambda(1-2\alpha)}{\alpha(\beta+\lambda)} < \alpha$ for any $\beta < \tilde{\beta}$. Hence, $m^{**} < \alpha$. This proves the second part of the proposition.

Third, from the proof of Proposition 1 we know that the cutoff β_1 for introducing a pay-to-win add-on satisfies $\beta_1 < \tilde{\beta}$. As we argued above, for any $\beta < \beta_6 \equiv \tilde{\beta}$, the firm's profits with matching are higher than without matching, i.e., $\pi^{**} > \pi^*$. Therefore, the equality $\pi^{**} > \pi^B$ is satisfied on a larger parameter region than the inequality $\pi^* > \pi^B$.

Fourth, we can see that with matching, the base product is sold at zero price when $\beta > \max\{\theta_l, 2\theta_l + \lambda - 2\alpha(\theta_l + \lambda)\}$, whereas without matching, the price is zero when $\beta > \hat{\beta} = \frac{2\theta_l + (1-\alpha)\lambda}{1+\alpha}$. Because $\hat{\beta} > \max\{\theta_l, 2\theta_l + \lambda - 2\alpha(\theta_l + \lambda)\}$, it follows that with matching, freemium pricing is optimal on a larger parameter region than without matching. Turning to the quality, recall that without matching,

$$q^* = \begin{cases} \theta_l & \text{if } 0 < \beta < \hat{\beta} \\ \frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2} & \text{if } \hat{\beta} \le \beta \le \min\{\tilde{\beta}, \bar{\beta}\} \\ \theta_h & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} < \beta < \bar{\beta} \end{cases}$$

As mentioned earlier, $\max\{\theta_l, 2\theta_l + \lambda - 2\alpha(\theta_l + \lambda)\} < \hat{\beta}$ and $\max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\} < \tilde{\beta}$. When $\beta < \max\{\theta_l, 2\theta_l + \lambda - 2\alpha(\theta_l + \lambda)\}$, we have $q^* = q^{**} = \theta_l$, and when $\beta > \min\{\tilde{\beta}, \bar{\beta}\}$, we have $q^* = q^{**} = \theta_h$. When $\max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\} < \beta < \tilde{\beta}$, we have $q^{**} = \theta_h > \frac{(1+\alpha)\beta - (1-\alpha)\lambda}{2} = q^*$. Further, for $\hat{\beta} < \beta < \max\{\theta_h, 2\theta_h + \lambda - 2\alpha(\theta_h + \lambda)\}$, one can easily show that $\frac{dq^{**}}{d\beta} > \frac{dq^*}{d\beta} > 0$. Since $q^{**} = q^*$ when $\beta = \hat{\beta}$, the inequality $\frac{dq^{**}}{d\beta} > \frac{dq^*}{d\beta} > 0$ implies that $q^{**} > q^*$. This finishes the proof of Proposition 6. □

9. Analysis of negative pricing

We solve the firm's profit optimization problem by removing non-negativity constraints on prices. Note that the benchmark analysis remains unchanged because the firm must charge a non-negative base price to ensure non-negative profits in the scenario without a pay-to-win add-on. Hence, let us analyze the firm's decisions when it introduces the add-on. Similar to the main model, selling a pay-to-win add-on can lead to strictly higher profits only when the firm sells the add-on to type-h players while selling the base game to both type-h and type-l players, i.e., $(d_l, d_h) = (B, A)$.

Denote $f(\omega) = \lambda(\alpha(1-\omega) + (1-\alpha)\frac{1}{2}) - \beta\left(\alpha\omega + (1-\alpha)\frac{1}{2}\right)$. Then, for $i \in \{l, h\}$, we have $U_i(B \mid (B, A)) = \theta_i(q + f(\omega)) - p$ and $U_i(A \mid (B, A)) = \theta_i(q + f(\omega) + (\beta + \lambda)(\omega - \frac{1}{2})) - (p + p_a)$. To ensure that $(d_l, d_h) = (B, A)$, the firm's prices must satisfy the following inequalities:

$$
\begin{aligned}
&U_l(B \mid (B, A)) \geq 0, \\
&U_l(B \mid (B, A)) \geq U_l(A \mid (B, A)), \\
&U_h(A \mid (B, A)) \geq 0, \\
&U_h(A \mid (B, A)) \geq U_h(B \mid (B, A)),
\end{aligned}
$$

The firm's optimal q and ω will satisfy one of the following three cases: $q + f(\omega) > 0$, $q + f(\omega) = 0$, or $q + f(\omega) < 0$. We will characterize the optimal solution in each of these cases and the corresponding existence conditions. Whenever the existence conditions coincide, we will compare the corresponding profits to determine the global optimum.

Case 1: When $q + f(\omega) > 0$, the firm's optimal prices are such that $U_l(B \mid (B, A)) = 0$ and $U_h(A \mid (B, A)) = U_h(B \mid (B, A))$. Solving these equations, we find that

$$p_1^* = \theta_l(q + f(\omega)) \quad \text{and} \quad p_{a_1}^* = \theta_h(\beta + \lambda)(\omega - \frac{1}{2}).$$

Note that $p^* > 0$ since $q + f(\omega) > 0$. The firm chooses q and ω to maximize its profits $\pi(q, \omega) = \alpha(p^* + p_{a_1}^*) + (1-\alpha)p^* - \frac{1}{2}q^2$ subject to $\omega \in \left[\frac{1}{2}, 1\right]$ and $q > \max\{-f(\omega), 0\}$. Solving the firm's profit maximization problem, we find that

$$q_1^* = \theta_l, \quad \omega_1^* = 1,$$

and the corresponding profits are

$$\pi_1^* = \frac{\theta_l^2 + \theta_l(\lambda - \beta) + \alpha(\theta_h - \theta_l)(\beta + \lambda)}{2}.$$

We need to verify that q_1^* and ω_1^* satisfy the constraint $q > \max\{-f(\omega), 0\}$. Straightforward algebra shows the constraint holds if and only if

$$\beta < \hat{\beta} = \frac{2\theta_l + (1-\alpha)\lambda}{1+\alpha}.$$

Case 2: When $q + f(\omega) = 0$, the condition $U_l(B \mid (B, A)) \geq 0$ implies that $p_2^* = 0$. Similar to Case 1 above, the add-on's price is given by $p_{a_2}^* = \theta_h(\beta + \lambda)\left(\omega - \frac{1}{2}\right)$. Solving the firm's profit maximization problem at the product design stage, we find:

$$q_2^* = \begin{cases} \frac{(1+\alpha)\beta - (1-\alpha)\lambda}{2} & \text{if } \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \theta_h & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}. \end{cases}$$

$$\omega_2^* = \begin{cases} 1 & \text{if } \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{1}{2} + \frac{2\theta_h - (\beta - \lambda)}{2\alpha(\beta + \lambda)} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}. \end{cases}$$

where $\tilde{\beta} = \frac{2\theta_h + (1-\alpha)\lambda}{(1+\alpha)}$.

Plugging in the optimal design decisions, the firm's profit is given by:

$$\pi_2^* = \begin{cases} \frac{4\alpha\theta_h(\beta+\lambda) - [(1+\alpha)\beta - (1-\alpha)\lambda]^2}{8}, & \text{if } \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{\theta_h(\theta_h + (\lambda - \beta))}{2}, & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}. \end{cases} \tag{32}$$

One can readily verify that the constraint $q + f(\omega) = 0$ is satisfied by q_2^* and ω_2^*.

Case 3: When $q + f(\omega) < 0$, q and ω need to satisfy $q + f(\omega) + (\beta + \lambda) > 0$ because otherwise, the firm will not be able to earn any positive profits. Also note that $q + f(\omega) < 0$ implies that type-h players obtain a lower utility from buying the base product (without an add-on) than type-l players, i.e., $U_l(B \mid (B, A)) > U_h(B \mid (B, A))$. In view of this, one can show that the optimal prices will be determined by the following two equalities: $U_l(B \mid (B, A)) = 0$ and $U_h(A \mid (B, A)) = 0$. Solving the equations, we find that

$$p_3^* = \theta_l\left(q + f(\omega)\right),$$

$$p_{a_3}^* = \theta_h(q + f(\omega) + (\beta + \lambda)(\omega - \frac{1}{2})) - \theta_l\left(q + f(\omega)\right)$$

Solving the firm's profit maximization problem, we obtain the optimal quality and add-on power:

$$q_3^* = \alpha\theta_h + (1-\alpha)\theta_l, \quad \omega_3^* = 1.$$

One can show that given q_3^* and ω_3^*, the inequality $q+f(\omega)+(\beta+\lambda)>0$ holds for any $\beta<\bar{\beta}$. Further, $q+f(\omega)<0$ is satisfied if and only if

$$\beta \geq \beta_7 := \frac{2(\alpha\theta_h+(1-\alpha)\theta_l)+(1-\alpha)\lambda}{1+\alpha}.$$

Upon plugging q_3^* and ω_3^* into the profit function, we obtain

$$\pi_3^* = \frac{2\alpha(\theta_h-\theta_l)(\theta_l+\lambda)+\theta_l(-\beta+\theta_l+\lambda)-\alpha^2(\theta_h-\theta_l)(\beta-\theta_h+\theta_l+\lambda)}{2}.$$

Global Optimum: Note that $\hat{\beta}<\beta_7$, and thus the solutions from Cases 1 and 2 coexist when $\beta<\hat{\beta}$, while the solutions from Case 2 and Case 3 coexist when $\beta>\beta_7$. Comparing π_1^* and π_2^*, we find that $\pi_1^*>\pi_2^*$ when $\beta<\hat{\beta}$. Similarly, comparing π_2^* and π_3^*, we find that $\pi_3^*>\pi_2^*$ when $\beta>\beta_7$.

To summarize the above analysis, the firm's globally optimal product design decisions and corresponding profits are as follows.

- If $0\leq\beta<\hat{\beta}$, then:
$p^*=\frac{\theta_l(2\theta_l+(1-\alpha)\lambda)-\beta(1+\alpha))}{2}, \quad p_a^*=\frac{1}{2}\theta_h(\beta+\lambda), \quad q^*=\theta_l, \quad \omega^*=1, \quad \pi^*=\frac{\theta_l^2+\theta_l(\lambda-\beta)+\alpha(\theta_h-\theta_l)(\beta+\lambda)}{2}.$

- If $\hat{\beta}<\beta<\min\{\beta_7,\bar{\beta}\}$, then:
$p^*=0, \quad p_a^*=\frac{\theta_h(2\theta_h-\beta+\lambda)}{2\alpha}, \quad q^*=\frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2}, \quad \omega^*=1, \quad \pi^*=\frac{4\alpha\theta_h(\beta+\lambda)-((1+\alpha)\beta-(1-\alpha)\lambda)^2}{8}.$

- If $\min\{\beta_7,\bar{\beta}\}\leq\beta<\bar{\beta}$, then:
$p^*=-\frac{\theta_l(\beta(1+\alpha)-\lambda(1-\alpha)-2(\alpha\theta_h+(1-\alpha)\theta_l)+\theta_l)}{2}, \quad p_a^*=\frac{\left(\beta\theta_l+2\theta_h\theta_l-2\theta_l^2+2\theta_h\lambda-\theta_l\lambda-\alpha(\theta_h-\theta_l)(\beta-2\theta_h+2\theta_l+\lambda)\right)}{2},$
$q^*=\alpha\theta_h+(1-\alpha)\theta_l, \quad \omega^*=1, \quad \pi^*=\frac{2\alpha(\theta_h-\theta_l)(\theta_l+\lambda)+\theta_l(-\beta+\theta_l+\lambda)-\alpha^2(\theta_h-\theta_l)(\beta-\theta_h+\theta_l+\lambda)}{2}.$

where $\hat{\beta}=\frac{2\theta_l+(1-\alpha)\lambda}{(1+\alpha)}, \quad \beta_7=\frac{2(\alpha\theta_h+(1-\alpha)\theta_l)+(1-\alpha)\lambda}{(1+\alpha)}$, and $\bar{\beta}=\lambda+\max\{\theta_l,\alpha\theta_h\}$.

Recall that the players' equilibrium utilities are as follows: $U_l^*=\theta_l\left(q^*+\lambda\left(\alpha(1-\omega^*)+(1-\alpha)\frac{1}{2}\right)-\beta\left(\alpha\omega^*+(1-\alpha)\frac{1}{2}\right)\right)-p^*$ and $U_h^*=\theta_h\left(q^*+\lambda\left(\alpha\frac{1}{2}+(1-\alpha)\omega^*\right)-\beta\left(\alpha\frac{1}{2}+(1-\alpha)(1-\omega^*)\right)\right)-(p^*+p_a^*)$. With the equilibrium q^*, ω^*, p^* and p_a^* are characterized above, one can then show that

$$PS^*=\begin{cases}\alpha(\theta_h-\theta_l)(\theta_l+\frac{(1-\alpha)\lambda-(1+\alpha)\beta}{2}) & \text{if } 0\leq\beta\leq\hat{\beta}.\\ 0 & \text{if } \hat{\beta}\leq\beta<\bar{\beta},\end{cases}$$

9.1. Proof of Proposition 7

The first part of the Proposition directly follows from our characterization of the optimal prices above. Namely, when $\beta > \beta_7$, we have $p^* < 0$.

Turning to the second part of the Proposition, recall that when the firm is constrained to set non-negative prices, the optimal quality and add-on power are given by

$$q^* = \begin{cases} \theta_l & \text{if } 0 \leq \beta \leq \hat{\beta}, \\ \frac{(1+\alpha)\beta-(1-\alpha)\lambda}{2} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \theta_h & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}, \end{cases}$$

$$\omega^* = \begin{cases} 1 & \text{if } 0 \leq \beta \leq \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{1}{2} + \frac{2\theta_h-(\beta-\lambda)}{2\alpha(\beta+\lambda)} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \leq \beta < \bar{\beta}, \end{cases}$$

Since $\beta_7 < \tilde{\beta}$, one can readily show that with negative prices, quality is weakly lower, whereas add-on power is weakly greater than when prices are non-negative.

Finally, as to the last part of the Proposition, the firm is obviously better off when it gains more pricing flexibility than when it is constrained to setting only non-negative prices. Next, recall that with non-negative pricing, player surplus is given by

$$PS^* = \begin{cases} \alpha(\theta_h - \theta_l)(\theta_l + \frac{(1-\alpha)\lambda-(1+\alpha)\beta}{2}) & \text{if } 0 \leq \beta \leq \hat{\beta}. \\ 0 & \text{if } \hat{\beta} \leq \beta < \bar{\beta}, \end{cases}$$

It is easy to see that the above expression is exactly the same as when the firm has a flexibility to set negative prices. Hence, player surplus does not change.

10. Analysis of limited add-on supply and Proof of Lemma 3

First, we will use the notation $d_i = A^B$ when player i's strategy is to buy the pay-to-win add-on if it is available and buy only the base product if the add-on is unavailable. Similarly, $d_i = A$ shows that the player chooses not to purchase the base product if the add-on is not available. Second, we assume that a player can simultaneously purchase both the base product and the add-on. Thus, a player can make an informed purchase decision knowing about the availability of the add-on at the time of purchasing the base product.[17]

Using a similar argument as in the proof of Lemma A1, we can show that selling a pay-to-win add-on with limited supply can be profitable only when a fraction of players buy the base product

[17] If players do not observe add-on availability at the time of purchasing the base product but can return the base product for a refund (if they are unable to purchase the pay-to-win add-on), then the results would be similar to the simultaneous purchase model that we analyze.

but not the add-on. Thus, if the firm prefer to introduce a pay-to-win add-on with limited supply, then the firm's pricing and product design decisions must be such that some players prefer to buy the base product without buying the add-on. There are three possible scenarios that can happen in equilibrium: $(d_l, d_h) = (N, A^B)$, $(d_l, d_h) = (B, A)$, and $(d_l, d_h) = (B, A^B)$. We analyze each of these scenarios.

1: $(d_l, d_h) = (N, A^B)$, i.e., type-l players do not buy the base product, whereas all type-h players buy it and χ of them also buy the add-on. A type-h player who is unable to buy the add-on will face an opponent who has the add-on with probability $\frac{\chi}{\alpha}$. Therefore, the player's utility from buying the base product is $U_h(B \mid (N, A^B)) = \theta_h \left(q + \lambda\left(\frac{\chi}{\alpha}(1-\omega) + (1-\frac{\chi}{\alpha})\frac{1}{2}\right) - \beta\left(\frac{\chi}{\alpha}\omega + (1-\frac{\chi}{\alpha})\frac{1}{2}\right)\right) - p$. By contrast, a type-$h$ player who obtains the pay-to-win add-on receives a utility of $U_h(A \mid (N, A^B)) = \theta_h \left(q + \lambda\left(\frac{\chi}{\alpha}\frac{1}{2} + (1-\frac{\chi}{\alpha})\omega\right) - \beta\left(\frac{\chi}{\alpha}\frac{1}{2} + (1-\frac{\chi}{\alpha})(1-\omega)\right)\right) - (p + p_a)$. The firm's optimal prices will satisfy $U_h(B \mid (N, A^B)) = 0$ and $U_h(A \mid (N, A^B)) = U_h(B \mid (N, A^B))$. Solving these equalities, we find that

$$p^* = \frac{\theta_h}{2}(2q - \beta + \lambda + \frac{(\beta+\lambda)\chi(1-2\omega)}{\alpha})$$
$$p_a^* = \theta_h(\beta+\lambda)(\omega - \frac{1}{2}).$$

The firm maximizes its profit $\pi(q, \omega, \chi) = \alpha p^* + \chi p_a^* - \frac{1}{2}q^2$, and plugging the prices into the profit function, we obtain $\pi^*(\omega, q, \chi) = \frac{1}{2}\alpha\theta h(2q - \beta + \lambda) - \theta_h(\beta+\lambda)\chi(2\omega - 1) - \frac{q^2}{2}$. Notice that $\frac{\partial \pi^*(\omega,q,\chi)}{\partial \omega} < 0$, hence the firm sets $\omega^* = 1/2$. Plugging in $\omega^* = \frac{1}{2}$, the profit function simplifies to $\pi = \alpha\theta_h\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right) - \frac{1}{2}q^2$, which is the profit that the firm can earn without introducing a pay-to-win add-on and selling its base product to type-h players at a price $p = \theta_h\left(q + \lambda\frac{1}{2} - \beta\frac{1}{2}\right)$. Therefore, introducing a pay-to-win add-on with limited supply does not benefit the firm.

2: $(d_l, d_h) = (B, A)$, i.e., type-l players buy the base product, whereas a type-h player buys the base product only if the player is able to also purchase the add-on. Note that a type-h player has a higher willingness to pay than a type-l player. Therefore, if a type-l player optimally chooses $d_l = B$, then choosing $d_h = N$ is suboptimal for a type-h player. Hence, $d_h = A$ cannot happen in equilibrium because the player is better off by choosing the strategy $d_h = A^B$ or $d_h = B$.

3: $(d_l, d_h) = (B, A^B)$, i.e., all players buy the base product and χ type-h players also buy the pay-to-win add-on. A type-i player who buys the base product but not the add-on receives a utility $U_i(B \mid (B, A^B)) = \theta_i\left(q + \lambda(\chi(1-\omega) + (1-\chi)\frac{1}{2}) - \beta(\chi\omega + (1-\chi\frac{1}{2})\right) - p$. Those type-$h$ players who obtain the pay-to-win add-on receive a utility $U_h(A \mid ((B, A^B)) = \theta_h\left(q + \lambda(\chi\frac{1}{2} + (1-\chi)\omega) - \beta(\chi\frac{1}{2} + (1-\chi)(1-\omega))\right) - (p + p_a)$. The firm's optimal prices will satisfy $U_l(B \mid (B, A^B)) = 0$ and $U_h(A \mid (B, A^B)) = U_h(B \mid (B, A^B))$. Solving these equalities, we find that

$$p^* = \theta_l(q + \lambda(\chi(1-\omega) + (1-\chi)\frac{1}{2}) - \beta(\chi\omega + (1-\chi)\frac{1}{2}))$$
$$p_a^* = \theta_h(\beta+\lambda)(\omega - \frac{1}{2}).$$

For a given χ, the firm chooses q and ω to maximize its profits $\pi = \chi p_a + p - \frac{1}{2}q^2$. The analysis is identical to that in our main model if we replace α with χ. Solving the firm's profit maximization problem, we find that

$$q^* = \begin{cases} \theta_l & \text{if } 0 \leq \beta \leq \hat{\beta}(\chi), \\ \frac{(1+\chi)\beta-(1-\chi)\lambda}{2} & \text{if } \hat{\beta}(\chi) < \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\}, \\ \theta_h & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \leq \beta < \bar{\beta}, \end{cases} \tag{33}$$

$$\omega^* = \begin{cases} 1 & \text{if } 0 \leq \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\}, \\ \frac{1}{2} + \frac{2\theta_h-(\beta-\lambda)}{2\chi(\beta+\lambda)} & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \leq \beta < \bar{\beta}, \end{cases} \tag{34}$$

where $\hat{\beta}(\chi) = \frac{2\theta_l+(1-\chi)\lambda}{(1+\chi)}$ and $\tilde{\beta}(\chi) = \frac{2\theta_h+(1-\chi)\lambda}{(1+\chi)}$ and recall from Lemma 1 that $\bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\}$. Using the product design and pricing decisions above, we obtain the corresponding profits:

$$\pi^*(\chi) = \begin{cases} \frac{\theta_l{}^2+\theta_l(\lambda-\beta)+\chi(\theta_h-\theta_l)(\beta+\lambda)}{2} & \text{if } 0 \leq \beta \leq \hat{\beta}(\chi), \\ \frac{4\chi\theta_h(\beta+\lambda)-((1+\chi)\beta-(1-\chi)\lambda)^2}{8} & \text{if } \hat{\beta}(\chi) < \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\}, \\ \frac{\theta_h(\theta_h-(\beta-\lambda))}{2} & \text{if } \min\{\tilde{\beta}(\chi), \bar{\beta}\} \leq \beta < \bar{\beta}. \end{cases} \tag{35}$$

The function $\pi(\chi)$ is continuous in χ, and one can readily show that $\frac{\partial \pi}{\partial \chi} \geq 0$ at any $\chi \in (0, \alpha)$.[18] Thus, the firm optimally sets $\chi = \alpha$. Now, recall that in our main model, in equilibrium, α measure of players buy the pay-to-win add-on. Hence, $\chi = \alpha$ suggests that the firm does not gain by limiting the availability of the pay-to-win add-on.

Combining the three cases of the above analysis shows that the firm does not benefit from limiting the supply of a pay-to-win add-on. □

11. Analysis of player heterogeneity in terms of competition sensitivity

Let us assume that players have a similar sensitivity to price, i.e., $\theta_l = \theta_h = \theta$, and normalize $\theta = 1$. Further, players are heterogeneous in their sensitivities to winning and losing. Namely, a fraction γ of players experiences a utility gain (loss) of $\lambda(\beta)$ when they win (lose)—we will call these players type-*hc*. The remaining players are more sensitive to competition, experiencing a utility gain (loss) of $\lambda c(\beta c)$, where $0 < c < 1$; we will refer to these players as type-*lc*. Similar to our main model, we assume that $\beta < \bar{\beta} = \lambda + \max\{\theta_l, \alpha\theta_h\} = \lambda + 1$, which helps ensure that, in equilibrium, the firm

[18] For the first condition, $0 \leq \beta \leq \hat{\beta}(\chi)$, we have $\frac{\partial \pi}{\partial \chi} = \frac{(\theta_h-\theta_l)(\beta+\lambda)}{2} > 0$.And, for the second condition, $\hat{\beta}(\chi) < \beta < \min\{\tilde{\beta}(\chi), \bar{\beta}\}$, we have also $\frac{\partial \pi}{\partial \chi} = \frac{(\beta+\lambda)(2\theta h-\beta(1+\chi)+\lambda(1-\chi))}{4} > 0$.

can have positive sales and earn strictly positive profits. The remaining assumptions are the same as in our main model.

Benchmark without a pay-to-win add-on. The firm will either sell the base product to both segments or only one of the segments. Notice that when $\beta < \lambda$, type-hc players have a higher willingness to pay for the game than type-lc players, whereas when $\beta > \lambda$, the opposite is true.

The analysis is straightforward. We find that the globally optimal decisions, corresponding profits and player surplus are as follows:

- $p^B = 1 - \frac{c(\beta-\lambda)}{2}, q^B = 1$ if and only if either $0 < \gamma \leq -\frac{\lambda}{2} + \frac{1}{2}\sqrt{4+4c\lambda+\lambda^2}$ and $\beta < \lambda$, or $-\frac{\lambda}{2} + \frac{1}{2}\sqrt{4+4c\lambda+\lambda^2} < \gamma < 1$ and $\lambda - \frac{1-\gamma^2}{\gamma-c} < \beta < \lambda$. The firm serves both segments. The corresponding profits and player surplus are

$$\pi^B = \frac{1-c(\beta-\lambda)}{2}, PS^B = \frac{(1-c)\gamma(\lambda-\beta)}{2}.$$

- $p^B = \frac{-\beta+2\gamma+\lambda}{2}, q^B = \gamma$ if and only if $-\frac{\lambda}{2} + \frac{1}{2}\sqrt{4+4c\lambda+\lambda^2} < \gamma < 1$ and $0 < \beta < \lambda - \frac{1-\gamma^2}{\gamma-c}$. The firm serves only type-hc players. The profits and player surplus are

$$\pi^B = \frac{\gamma(-\beta+\gamma+\lambda)}{2}, PS^B = 0.$$

- $p^B = \frac{(2-\beta+\lambda)}{2}, q^B = 1$ if and only if either $0 < \gamma \leq 1-c$ and $\lambda < \beta < \lambda + \frac{(2-\gamma)\gamma}{1-c(1-\gamma)}$, or $1-c < \gamma < 1$ and $\lambda < \beta < \bar{\beta}$. The firm serves both segments. The corresponding profits and player surplus are

$$\pi^B = \frac{(1-\beta+\lambda)}{2}, PS^B = \frac{(1-c)(1-\gamma)(\beta-\lambda)}{2}.$$

- $p^B = 1-\gamma-\frac{c(\beta-\lambda)}{2}, q^B = 1-\gamma$ if and only if $0 < \gamma < 1-c$ and $\lambda + \frac{(2-\gamma)\gamma}{1-c(1-\gamma)} < \beta < \bar{\beta}$.

The firm serves only type-lc players. The corresponding profits and player surplus are

$$\pi^B = \frac{(1-\gamma)(1-\gamma-c(\beta-\lambda))}{2}, PS^B = 0.$$

Analysis with a pay-to-win add-on. Clearly, type-h players have a higher willingness to pay for the add-on. Hence, to profitably sell the pay-to-win add-on, the firm will sell the base product to type- l players, whereas the type- h players will buy the base product and the add-on. The following conditions must be satisfied:

$$\begin{aligned}
\text{IC}_h &: U_h(A \mid (B,A)) \geq U_h(B \mid (B,A)) \\
\text{IR}_h &: U_h(A \mid (B,A)) \geq 0 \\
\text{IC}_l &: U_l(B \mid (B,A)) \geq U_l(A \mid (B,A)) \\
\text{IR}_l &: U_l(B \mid (B,A)) \geq 0
\end{aligned}$$

Note that we can potentially have a situation where a type-hc player obtains a lower utility from choosing B than what a type-lc player would obtain. This feature of the model significantly complicates the analysis, giving rise to many cases that we had to analyze. Specifically, depending on the values of q and ω, the optimal prices p and p_a could fall into one of the following situations: i) IR_l and IC_h bind, or *ii)* IR_l and IR_h bind, or *iii)* IR_h and IC_l bind. The number of the cases is further enhanced based on whether we have a freemium scenario ($p = 0$) or not ($p > 0$). After separately analyzing each possible case and characterizing its existence conditions, we compare the profits across cases to determine the globally optimal pricing and product design decisions, p^*, p_a^*, q^* and ω^*. They are as follows:

- $p^* = 1 - \frac{c(\beta(1+\gamma)-(1-\gamma)\lambda)}{2}, p_a^* = \frac{\beta+\lambda}{2}, q^* = 1$ and $\omega^* = 1$ if and only if $0 < \beta < \frac{\lambda-\gamma\lambda}{1+\gamma}$.

The corresponding profits and player surplus are given by

$$\pi^* = \frac{1+\gamma(\beta+\lambda)-c(\beta(1+\gamma)+(-1+\gamma)\lambda)}{2}, PS^* = \frac{(1-c)\gamma((1-\gamma)\lambda-\beta(1+\gamma))}{2}$$

- $p^* = 1 - \frac{c(\beta(1+\gamma)-(1-\gamma)\lambda)}{2}, p_a^* = \frac{c\beta(1+\gamma)+2\lambda-c(1-\gamma)\lambda-\gamma(\beta+\lambda)}{2}, q^* = 1$ and $\omega^* = 1$ if and only if $\frac{\lambda-\gamma\lambda}{1+\gamma} < \beta < \min\left\{\frac{2\lambda-\gamma\lambda}{\gamma}, \frac{2+c\lambda(1-\gamma)}{c(1+\gamma)}, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

$$\pi^* = \frac{1-\gamma(\beta\gamma+(-2+\gamma)\lambda)+c(-1+\gamma)(\beta(1+\gamma)+(-1+\gamma)\lambda)}{2}, PS^* = 0$$

- $p^* = 1 + \lambda - \frac{(c+\gamma)(\beta+\lambda)}{2}, p_a^* = \frac{c(\beta+\lambda)}{2}, q^* = 1$ and $\omega^* = 1$ if and only if $0 < \lambda < \frac{\gamma}{c}$ and $\frac{2\lambda-\gamma\lambda}{\gamma} < \beta < \min\left\{\frac{2+2\lambda-c\lambda-\gamma\lambda}{c+\gamma}, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

$$\pi^* = \frac{1}{2} + \lambda - \frac{(c(1-\gamma)+\gamma)(\beta+\lambda)}{2}, PS^* = \frac{(1-c)(1-\gamma)(\beta\gamma-(2-\gamma)\lambda)}{2}$$

- $p^* = 0, p_a^* = \frac{c\beta(1+\gamma)+2\lambda+c(-1+\gamma)\lambda-\gamma(\beta+\lambda)}{2}, q^* = \frac{c(\beta(1+\gamma)+(-1+\gamma)\lambda)}{2}, \omega^* = 1$ if and only if $\frac{\gamma}{c} < \lambda < \frac{\gamma-\gamma^2+c\gamma^2}{c^2}$ and $\frac{2+c\lambda(1-\gamma)}{c(1+\gamma)} < \beta < \min\left\{\frac{2\lambda-\gamma\lambda}{\gamma}, \bar{\beta}\right\}$, or $\lambda > \frac{\gamma-\gamma^2+c\gamma^2}{c^2}$ and $\frac{2+c\lambda(1-\gamma)}{c(1+\gamma)} < \beta < \min\left\{\frac{2-2(1-c)\gamma+c^2(1-\gamma)\lambda}{c^2(1+\gamma)}, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

$$\pi^* = \frac{-4\gamma(\beta\gamma+(-2+\gamma)\lambda)+4c\gamma(\beta(1+\gamma)+(-1+\gamma)\lambda)-c^2(\beta(1+\gamma)+(-1+\gamma)\lambda)^2}{8}, PS^* = 0$$

- $p^* = 0, p_a^* = \frac{2(1-\gamma)^2-4c(-1+\gamma)\gamma+c^2\left(-\beta+2\gamma^2+\lambda\right)}{2c^2\gamma}, q^* = \frac{1-(1-c)\gamma}{c}, \omega^* = \frac{2-2\gamma+2c\gamma+c^2(\beta(-1+\gamma)+(1+\gamma)\lambda)}{2c^2\gamma(\beta+\lambda)}$ if and only if $\lambda > \frac{\gamma-\gamma^2+c\gamma^2}{c^2}$ and either $0 < \gamma \leq 1-c$ and $\frac{2-2(1-c)\gamma+c^2(1-\gamma)\lambda}{c^2(1+\gamma)} < \beta < \bar{\beta}$, or $1-c < \gamma < 1$ and $\frac{2-2(1-c)\gamma+c^2(1-\gamma)\lambda}{c^2(1+\gamma)} < \beta < \min\left\{\frac{2(1-\gamma)(1-(1-c)\gamma)}{c^2} + \lambda, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

$$\pi^* = \frac{(1-\gamma)^2-2c(-1+\gamma)\gamma+c^2\left(-\beta+\gamma^2+\lambda\right)}{2c^2}, PS^* = 0$$

- $p^* = 0, p_a^* = \frac{c(\beta-\lambda)}{2(1-\gamma)}, q^* = \frac{c(\beta-\lambda)}{2(1-\gamma)}, \omega^* = \frac{\beta(-2+\gamma)+\gamma\lambda}{2(-1+\gamma)(\beta+\lambda)}$ if and only if $1-c<\gamma<1$ and either $\frac{\gamma-\gamma^2+c\gamma^2}{c^2} < \lambda < \frac{\gamma^2}{-1+c+\gamma}$ and $\frac{2(1-\gamma)(1-(1-c)\gamma)}{c^2} + \lambda < \beta \leq \min\left\{\frac{2\lambda-\gamma\lambda}{\gamma}, \bar{\beta}\right\}$, or $\lambda > \frac{\gamma^2}{-1+c+\gamma}$ and $\frac{2(1-\gamma)(1-(1-c)\gamma)}{c^2} + \lambda < \beta < \min\left\{\frac{2(1-\gamma)\gamma}{-1+c+\gamma} + \lambda, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

$$\pi^* = \frac{c(4(1-\gamma)\gamma - c(\beta-\lambda))(\beta-\lambda)}{8(1-\gamma)^2}, PS^* = 0$$

- $p^* = 0, p_a^* = \frac{c(\beta+\lambda)}{2}, q^* = -\lambda + \frac{(c+\gamma)(\beta+\lambda)}{2}, \omega^* = 1$ if and only if $1-c<\gamma<1$ and either $\lambda < \frac{\gamma}{c}$ and $\frac{2+2\lambda-c\lambda-\gamma\lambda}{c+\gamma} < \beta < \min\left\{\frac{2c\gamma+(2-c-\gamma)(-1+c+\gamma)\lambda}{(-1+c+\gamma)(c+\gamma)}, \bar{\beta}\right\}$, or $\frac{\gamma}{c} < \lambda < \frac{\gamma^2}{-1+c+\gamma}$ and $\frac{2\lambda-\gamma\lambda}{\gamma} < \beta < \min\left\{\frac{2c\gamma+(2-c-\gamma)(-1+c+\gamma)\lambda}{(-1+c+\gamma)(c+\gamma)}, \bar{\beta}\right\}$. The corresponding profits and player surplus are given by

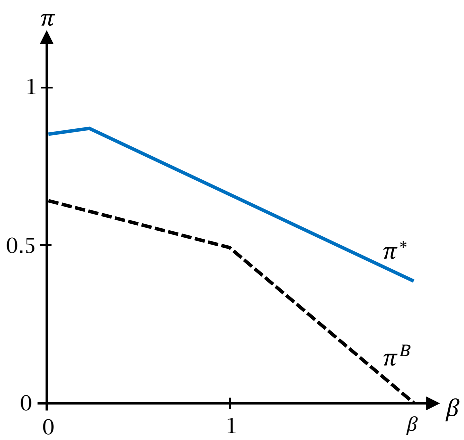
$$\pi^* = \frac{c\gamma(\beta+\lambda)}{2} - \frac{((c+\gamma)(\beta+\lambda)-2\lambda)^2}{8}, PS^* = \frac{(1-c)(1-\gamma)(\beta\gamma+(-2+\gamma)\lambda)}{2}$$

- $p^* = 0, p_a^* = \frac{c(2c\gamma-\beta(-1+c+\gamma)+(-1+c+\gamma)\lambda)}{2(-1+c+\gamma)^2}, q^* = \frac{c\gamma}{-1+c+\gamma}, \omega^* = \frac{2c\gamma+\beta(2-c-\gamma)(1-c-\gamma)-(1-c-\gamma)(c+\gamma)\lambda}{2(1-c-\gamma)^2(\beta+\lambda)}$ if and only if $1-c<\gamma<1$ and either $\lambda < \frac{\gamma^2}{-1+c+\gamma}$ and $\frac{2c\gamma+(2-c-\gamma)(-1+c+\gamma)\lambda}{(-1+c+\gamma)(c+\gamma)} < \beta < \bar{\beta}$, or $\lambda > \frac{\gamma^2}{-1+c+\gamma}$ and $\frac{2(1-\gamma)\gamma}{-1+c+\gamma} + \lambda < \beta < \bar{\beta}$. The corresponding profits and player surplus are given by $\pi^* = \frac{c\gamma(c\gamma-\beta(-1+c+\gamma)+(-1+c+\gamma)\lambda)}{2(1-c-\gamma)^2}, PS^* = \frac{(1-c)c(1-\gamma)(2(-1+\gamma)\gamma+\beta(-1+c+\gamma)-(-1+c+\gamma)\lambda)}{2(1-c-\gamma)^2}$

Without going into the cumbersome technical details of the analysis, below we outline our key findings about the impact of a pay-to-win add-on in terms of profitability and player surplus, as well as firm's decision on pricing and product design.

Profits. Comparing π^* and π^B, one can show that the firm is better off by introducing a pay-to-win add-on, i.e., $\pi^* > \pi^B$ for all $\beta < \bar{\beta}$. Figure 5 graphically illustrates the firm's profits.[19]

Figure 5 Equilibrium Profits with (π^*) and without (π^B) Pay-To-Win Add-On

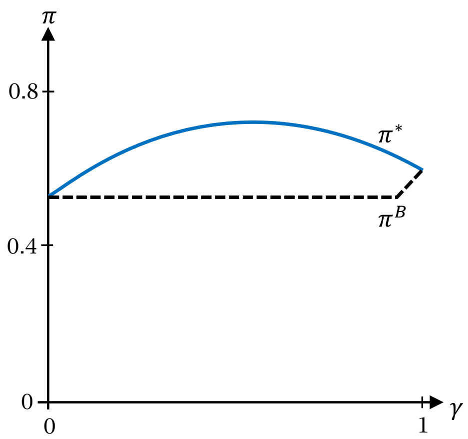


[19] The figure is obtained using the following numerical values: $\lambda = 1, \gamma = 0.6, c = 0.3$.

Notice from Figure 5 that similar to our main model, π^* is non-monotone in β; that is, upon introducing a pay-to-win add-on, the firm's profits may increase when β increases. One can also readily show that profits are monotone increasing in λ.

Next, Figure 6 illustrates the profits as a function of the fraction of type-hc players, γ.[20]

Figure 6 Equilibrium Profits as a Function of γ



We can see above that π^* is non-monotone in γ and starts to decline once the fraction of typehc players becomes too high. The intuition is similar to that in our main model. Specifically, if γ is high, then selling a pay-to-win add-on creates a strong negative externality for type-lc players, making it more difficult/costlier for the firm to maintain these players. Additionally, when a large fraction of players obtains the pay-to-win add-on, the extra surplus created by the add-on is small relative to when the add-on is not sold-this limits the amount of extra profits that the firm can obtain from each type-hc player. However, unlike our main model, where π^B may sometimes exceed π^* at large values of γ, in the current extension, we see that $\pi^* > \pi^B$. This is intuitive. Given that type-lc players' price sensitivity is as low as that of type-hc players, base price tends to be high, giving the firm more room for price reductions to compensate type-lc players' utility loss (due to not having the pay-to-win add-on). Further, type-lc players' low price-sensitivity translates into high willingness to pay for base quality, which means that a small increase in quality can result in a substantial utility improvement for type-lc players. Thus, the firm can maintain type-lc players more easily than when they have high sensitivity to price (as in the main model), making it more attractive for the firm to introduce a pay-to-win add-on.[21]

[20] Figure XYZ is obtained using the following numerical values: $\lambda = 1, \beta = 0.8$, $c = 0.3$.

[21] Note that if the introduction of the add-on entails positive fixed costs, then we obtain a similar no-pay-to-win-add-on region as in our main model, but the size of the region will be smaller.

Prices. We find that, akin to our main model, the introduction of the pay-to-win add-on can lower the price of the base product, and as we can see from the expressions of p^* above, the base product becomes a freemium when β is high. This is consistent with our main model.

Base quality. Similar to our main model (Proposition 2), the introduction of the pay-to-win add-on can lead to an increase in the quality of the base product when players' sensitivity to losing is high. However, we no longer find that quality can decrease when players' sensitivity to losing is low. The reason is that in the current extension, type-lc players' willingness to pay for a marginal increase in quality is as high as that of type-hc players, which gives the firm more incentives to provide high quality.

Power of pay-to-win add-on. Our characterization of the optimal ω^* above reveals that when players' sensitivity to losing is high, the firm may decide to limit the power of the add-on, choosing $\omega^* < 1$. This result is consistent with our main model (Proposition 3).

Player surplus. Recall that in our main model, we find that the introduction of a pay-to-win add-on can increase player surplus (Proposition 4). We find a similar result in this extension. Specifically, in the benchmark without a pay-to-win add-on, when $-\frac{\lambda}{2}+\frac{1}{2}\sqrt{4+4c\lambda+\lambda^2} < \gamma < 1$ and $0 < \beta < \lambda - \frac{1-\gamma^2}{\gamma-c}$, player surplus PS^B is zero. By contrast, with a pay-to-win add-on, if $0 < \beta < \frac{\lambda-\gamma\lambda}{1+\gamma}$, then $PS^* = \frac{(1-c)\gamma((1-\gamma)\lambda-\beta(1+\gamma))}{2} > 0$. Hence, when $-\frac{\lambda}{2}+\frac{1}{2}\sqrt{4+4c\lambda+\lambda^2} < \gamma < 1$ and $\beta < \min\left\{\lambda - \frac{1-\gamma^2}{\gamma-c}, \frac{\lambda-\gamma\lambda}{1+\gamma}\right\}$, we have $PS^* > PS^B$.

In sum, our analysis of the alternative setting-where players *i*) have a similar sensitivity to price, and *ii*) are heterogeneous in their sensitivity to winning and losing-reveals that our key insights related to pricing, base quality, pay-to-win add-on design and profitability continue to hold. A new insight is that when all players have a similar price-sensitivity, the firm has more incentives to introduce a pay-to-win add-on because it is easier to compensate type-lc players' utility loss due to not owning the pay-to-win add-on.

12. Robustness check with $\beta > \lambda$

In this section, we will demonstrate that our main results can continue to hold if we restrict the parameter space by imposing that $\beta > \lambda$. Specifically, let us assume that $\lambda = 0$. Because $\beta > 0$, the requirement that $\beta > \lambda$ is readily satisfied. In what comes next, we will show that our results from section 4 are robust.

Profitability. In the benchmark without a pay-to-win add-on, given any $0 < \beta < \bar{\beta} = \max\{\theta_l, \alpha\theta_h\}$, the firm's profits are given by

$$\pi^B = \begin{cases} \frac{\theta_l(\theta_l-\beta)}{2} & \text{if } 0 \le \alpha \le \frac{\theta_l}{\theta_h}, \\ \frac{\alpha\theta_h(\alpha\theta_h-\beta))}{2} & \text{if } \frac{\theta_l}{\theta_h} \le \alpha \le 1. \end{cases}$$

When the firm introduces a pay-to-win add-on, its profits are

$$\pi^* = \begin{cases} \frac{\theta_l^2 - \theta_l\beta + \alpha(\theta_h - \theta_l)\beta)}{2} & \text{if } 0 \le \beta \le \hat{\beta}, \\ \frac{4\alpha\theta_h\beta - ((1+\alpha)\beta)^2}{8} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{\theta_h(\theta_h - \beta)}{2} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \le \beta < \bar{\beta}. \end{cases}$$

where $\hat{\beta} = \frac{2\theta_l}{(1+\alpha)}$ and $\tilde{\beta} = \frac{2\theta_h}{(1+\alpha)}$.

Comparing π^* and π^B, we find that $\pi^* > \pi^B$ if and only if $\alpha < \alpha_1 = \frac{\theta_l}{\theta_h}$ or $\alpha > \alpha_1$ and $\beta_1 < \beta < \bar{\beta}$, where

$$\beta_1 = \begin{cases} \frac{\alpha^2\theta_h^2 - \theta_l^2}{2\alpha\theta_h - (\alpha+1)\theta_l} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{4\alpha\theta_h - 2\alpha\theta_h\sqrt{(1-\alpha)(\alpha+3)}}{(\alpha+1)^2} & \text{if } \alpha \in [\alpha_2, 1], \end{cases}$$

and $\alpha_2 \in (\alpha_1, 1)$ is the solution to the equation $(\pi^* - \pi^B)|_{\beta=\hat{\beta}} = \frac{1}{2}(-\theta_l^2 + \alpha\theta_h(-\alpha\theta_h + \frac{4\theta_l}{1+\alpha})) = 0$. Hence, our finding in Proposition 1 continues to hold.

Next, let us show that our results from Proposition 5 continue to exist. Because β_1 is an increasing function of α, the inequality $\beta < \beta_1$ is equivalent to $\alpha > (\beta_1)^{-1}(\beta)$. Since $\frac{\partial\beta_1}{\partial\alpha} > 0$, the Inverse Function Theorem implies that $\frac{\partial(\beta_1)^{-1}}{\partial\beta} > 0$. Hence, as β increases, the interval $((\beta_1)^{-1}(\beta), 1)$ becomes smaller, shrinking the region where the inequality $\pi^* - \pi^B < 0$ holds. Consequently, the first part of Proposition 5 continues to apply. Moreover, given that π^* is quasi-concave and attains a unique maximum at $\beta = \beta_5 = \frac{2\alpha\theta_h}{(1+\alpha)^2}$, the second part of Proposition 5 remains valid as well.

Pricing. In the benchmark, base price is given by

$$p^B = \begin{cases} \theta_l\left(\theta_l - \frac{\beta}{2}\right) & \text{if } \alpha \le \frac{\theta_l}{\theta_h}, \\ \theta_h\left(\alpha\theta_h - \frac{\beta}{2}\right) & \text{if } \alpha \ge \frac{\theta_l}{\theta_h}. \end{cases}$$

With a pay-to-in add-on, base price is given by

$$p^* = \begin{cases} \theta_l(\theta_l - \frac{(1+\alpha)\beta}{2}) & \text{if } 0 \le \beta < \hat{\beta}, \\ 0 & \text{if } \hat{\beta} \le \beta \le \bar{\beta}, \end{cases}$$

where $\hat{\beta} = \frac{2\theta_l}{(1+\alpha)}$. Comparing p^B and p^*, it is easy to show that $p^* < p^B$. Moreover, when $\hat{\beta} \le \beta \le \bar{\beta}$, it follows that $p^* = 0$. Therefore, the conclusions of Lemma 2 continue to hold.

Product design decisions. In the benchmark, the equilibrium quality level is given by

$$q^B = \max\{\theta_l, \alpha\theta_h.\}$$

In contrast, when the firm sells a pay-to-win add-on, the equilibrium quality is given by

$$q^* = \begin{cases} \theta_l & \text{if } 0 \le \beta \le \hat{\beta}, \\ \frac{(1+\alpha)\beta}{2} & \text{if } \hat{\beta} < \beta < \min\{\tilde{\beta}, \bar{\beta}\}, \\ \theta_h & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \le \beta < \bar{\beta}. \end{cases}$$

A comparison of q^B and q^* reveals that $q^* > q^B$ when $\beta > \beta_2$, where $\beta_2 = \max\{\frac{2\theta_l}{1+\alpha}, \frac{2\alpha\theta_h}{1+\alpha}\}$. Hence, Proposition 2 continues to hold.

Next, turning to the power of the pay-to-win add-on, in equilibrium, we have

$$\omega^* = \begin{cases} 1 & \text{if } 0 \le \beta \le \min\{\tilde{\beta}, \bar{\beta}\}, \\ \frac{1}{2} + \frac{2\theta_h - \beta}{2\alpha\beta} & \text{if } \min\{\tilde{\beta}, \bar{\beta}\} \le \beta < \bar{\beta}. \end{cases}$$

where $\tilde{\beta} = \frac{2\theta_h}{(1+\alpha)}$. Thus, when players exhibit strong sensitivity to losing ($\beta > \beta_3$), the firm may find it optimal to limit the power of the add-on by choosing $\omega^* < 1$. This proves Proposition 3.

Player surplus. In the benchmark without a pay-to-win add-on, the player's surplus from game is:

$$PS^B = \begin{cases} \alpha\left(\theta_h - \theta_l\right)\left(\theta_l - \frac{\beta}{2}\right) & \text{if } \alpha \le \frac{\theta_l}{\theta_h}, \\ 0 & \text{if } \alpha \ge \frac{\theta_l}{\theta_h}. \end{cases}$$

On the other hand, when the firm offers a pay-to-win add-on, the player's surplus is given by

$$PS^* = \begin{cases} \alpha\left(\theta_h - \theta_l\right)\left(\theta_l - \frac{(1+\alpha)\beta}{2}\right) & \text{if } 0 \le \beta \le \hat{\beta}, \\ 0 & \text{if } \hat{\beta} \le \beta < \bar{\beta}, \end{cases}$$

where $\hat{\beta} = \frac{2\theta_l}{(1+\alpha)}$. Clearly, when $\alpha > \frac{\theta_l}{\theta_h}$ and $\beta < \hat{\beta}$, we have $PS^* > PS^B$, which confirms our finding in Proposition 4.

To summarize, the above analysis shows that our main results continue to hold when we impose the constraint $\beta > \lambda$ on the parameter space.

13. Robustness check with $\beta = \lambda$

In this section, we demonstrate that our main results in Section 4 continue to hold when players have a similar sensitivity to winning and losing, i.e., $\beta = \lambda$. Define $c = \beta = \lambda$.

Profitability. In the benchmark without a pay-to-win add-on, the firm earns

$$\pi^B = \begin{cases} \frac{\theta_l^2}{2} & \text{if } 0 \le \alpha \le \frac{\theta_l}{\theta_h}, \\ \frac{\alpha^2\theta_h^2}{2} & \text{if } \frac{\theta_l}{\theta_h} \le \alpha \le 1, \end{cases}$$

By contrast, upon introducing a pay-to-win add-on, the firm's profits from selling the add-on to type-h players and selling the base product to both type-h and type-l players are given by

$$\pi^* = \begin{cases} \frac{\theta_l^2 + 2\alpha c(\theta_h - \theta_l)}{2} & \text{if } 0 \le c \le \hat{c}, \\ \frac{8\alpha c\theta_h - (2\alpha c)^2}{8} & \text{if } \hat{c} < c < \tilde{c}, \\ \frac{\theta_h^2}{2} & \text{if } c \ge \tilde{c}. \end{cases}$$

Comparing π^* and π^B, we find that $\pi^* > \pi^B$ if and only if $\alpha < \alpha_1 = \frac{\theta_l}{\theta_h}$ or $\alpha > \alpha_1$ and $c > c_1$, where

$$c_1 = \begin{cases} \frac{\alpha^2\theta_h^2-\theta_l^2}{2\alpha(\theta_h-\theta_l)} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{\theta_h+\sqrt{(1-\alpha^2)\theta_h^2}}{\alpha} & \text{if } \alpha \in [\alpha_2, 1], \end{cases}$$

and $\alpha_2 = \frac{\sqrt{2\theta_h-\theta_l}\sqrt{\theta_l}}{\theta_h}$.

Hence, our finding in Proposition 1 about the profitability of introducing a pay-to-win add-on continues to hold.

Pricing. Without a pay-to-win add-on, the base price is given by

$$p^B = \begin{cases} \theta_l^2 & \text{if } 0 \le \alpha \le \frac{\theta_l}{\theta_h}, \\ \alpha\theta_h^2 & \text{if } \frac{\theta_l}{\theta_h} \le \alpha \le 1. \end{cases}$$

With a pay-to-win add-on, the firm charges

$$p^* = \begin{cases} \theta_l(\theta_l - \alpha c) & \text{if } 0 \le c < \hat{c}, \\ 0 & \text{if } c \ge \hat{c}, \end{cases}$$

where $\hat{c} = \frac{\theta_l}{\alpha}$.

One can easily show that $p^* < p^B$. Further, $p^* = 0$ when $c \ge \hat{c}$. Hence, our findings in Lemma 2 continue to hold.

Product design decisions. In the benchmark case, the equilibrium quality levels is

$$q^B = \begin{cases} \theta_l & \text{if } 0 \le \alpha \le \frac{\theta_l}{\theta_h}, \\ \alpha\theta_h & \text{if } \frac{\theta_l}{\theta_h} \le \alpha \le 1. \end{cases}$$

When the firm sells a pay-to-win add-on, the equilibrium quality and add-on strength decisions are

$$q^* = \begin{cases} \theta_l & \text{if } 0 \le c \le \hat{c}, \\ \alpha c & \text{if } \hat{c} < c < \tilde{c}, \\ \theta_h & \text{if } c \ge \tilde{c}, \end{cases} \quad \omega^* = \begin{cases} 1 & \text{if } 0 \le c \le \tilde{c}, \\ \frac{1}{2} + \frac{\theta_h}{2\alpha c} & \text{if } c \ge \tilde{c}. \end{cases}$$

Here, $\hat{c} = \frac{\theta_l}{\alpha}$ and $\tilde{c} = \frac{\theta_h}{\alpha}$.

Comparing q^* and q^B, it is easy to see that if c is above a threshold, then $q^* > q^B$. However, if c is low, then $q^* \le q^B$. These results are consistent with our findings in Proposition 2.

Next, turning to the power of the pay-to-win add-on ω^*, we can see that when c is large, the firm limits the power of the add-on, choosing $\omega^* < 1$. This confirms our result in Proposition 3.

Player surplus. Without a pay-to-win add-on, the players' surplus from the game is:

$$PS^B = \begin{cases} \alpha(\theta_h - \theta_l)\theta_l & \text{if } \alpha < \frac{\theta_l}{\theta_h}, \\ 0 & \text{if } \alpha \ge \frac{\theta_l}{\theta_h}. \end{cases}$$

By contrast, when the firm sells a pay-to-win add-on, the players' surplus is given by

$$PS^* = \begin{cases} \alpha(\theta_h - \theta_l)(\theta_l + \alpha c) & \text{if } 0 \le c \le \hat{c}, \\ 0 & \text{if } c \ge \hat{c}. \end{cases}$$

Clearly, when $\alpha > \frac{\theta_l}{\theta_h}$ and $c < \hat{c}$, we have $PS^* > PS^B$: this is consistent with our finding in Proposition 4.

Effect of c. First, let us show that as c increases, the region where it is optimal to introduce a pay-to-win add-on (i.e., $\pi^* - \pi^B > 0$) expands. Conversely, we can show that the region where $\pi^* - \pi^B < 0$ shrinks as c increases. Recall that $\pi^* - \pi^B < 0$ if and only if $\alpha_1 < \alpha < 1$ and $c < c_1$, where:

$$c_1 = \begin{cases} \frac{\alpha^2\theta_h^2 - \theta_l^2}{2\alpha(\theta_h - \theta_l)} & \text{if } \alpha \in [\alpha_1, \alpha_2], \\ \frac{\theta_h + \sqrt{(1-\alpha^2)\theta_h^2}}{\alpha} & \text{if } \alpha \in [\alpha_2, 1], \end{cases}$$

with $\alpha_1 = \frac{\theta_l}{\theta_h}$ and $\alpha_2 = \frac{\sqrt{2\theta_h - \theta_l}\sqrt{\theta_l}}{\theta_h}$.

Because c_1 increases with α (i.e., $\frac{\partial c_1}{\partial \alpha} > 0$), the inequality $\pi^* - \pi^B < 0$ corresponds to $\alpha > (c_1)^{-1}(c)$. By the Inverse Function Theorem, $\frac{\partial (c_1)^{-1}}{\partial c} > 0$. Hence, as c increases, the inequality $\alpha > (c_1)^{-1}(c)$ is satisfied for a smaller range of parameters. This confirms our finding in the first part of Proposition 5.

Second, let us show that π^* can be non-monotone as a function of c. For a given $c \in [0, \hat{c})$, $\frac{\partial \pi^*}{\partial c} = \frac{\alpha\theta_h - (1+\alpha)\theta_l}{2} > 0$ as long as $\frac{\theta_l}{\theta_h - \theta_l} < \alpha < 1$. Further, for $c \in (\tilde{c}, \bar{c})$, we have $\frac{\partial \pi^*}{\partial c} = -\frac{\theta_h}{2} < 0$. Since π^* is continuous and quasi-concave in c, it is single-peaked. All of the above suggests that there exists a unique $c_5 \in [\hat{c}, \min\{\tilde{c}, \bar{c}\}]$ where π^* peaks. Solving $\frac{\partial \pi^*}{\partial c} = 0$, we find $c_5 = \frac{\theta_h}{1+\alpha}$. Hence, the firm's profits increase with c for $c < c_5$ and decrease for $c > c_5$. These results are consistent with our findings in the second part of Proposition 5.

14. Discussion of Anderson and Dana (2009)

We will check whether the necessary condition for the optimality of selling a product line identified in [AD] (Anderson and Dana 2009) holds in our setting. Specifically, [AD] show that in the traditional product line setting, a necessary condition for selling both high- and low-quality products is that the incremental surplus from higher quality (as a percentage) is increasing in consumer type. That is, $\frac{V_h(q_H) - c_H - V_h(q_L) + c_L}{V_h(q_L) - c_L} > \frac{V_l(q_H) - c_H - V_l(q_L) + c_L}{V_l(q_L) - c_L}$, where the variables q_H and q_L denote the high and low qualities of the products, and c_H and c_L are their respective marginal costs. $V_i(q)$ is defined as type-i consumer's valuation of a product that has quality q, where $i \in \{l, h\}$ and $V_h(q) > V_l(q)$.

Turning to our setting, the base product plus the pay-to-win add-on (A) is the "high-quality" option, whereas the base product without the add-on (B) is the "low-quality" option. Using similar notation as in [AD], define $V_i(d|d_l, d_h) = \theta_i(q + \lambda\rho_{win} - \beta(1 - \rho_{win}))$ as the value that a type-i player obtains from choosing $d \in \{B, A\}$, where d_l and d_h indicate the remaining type-l and type-h players' decisions, respectively.

When the firm sells the pay-to-win add-on, we know that in equilibrium, $d_l = B$ and $d_h = A$. Hence, when evaluating the percentage increase in a type-i player's surplus from buying the high-quality option, we will assume that all other players choose $(d_l, d_h) = (B, A)$. Also, note that in our digital setting, products have zero marginal cost. Therefore, the percentage increase in surplus of a single type-h player (of mass zero) from buying the add-on is given by $\frac{V_h(A|B,A)-V_h(B|B,A)}{V_h(B|B,A)} = \frac{(2\omega-1)(\beta+\lambda)}{2q+\alpha(\beta+\lambda)(1-2\omega)-\beta+\lambda}$. Similarly, for a type-$l$ player, $\frac{V_l(A|B,A)-V_l(B|B,A)}{V_l(B|B,A)} = \frac{(2\omega-1)(\beta+\lambda)}{2q+\alpha(\beta+\lambda)(1-2\omega)-\beta+\lambda}$. Notice that the two expressions are equal to each other, which implies that the necessary condition in [AD] is not satisfied.[22]

To see the intuition, note that in [AD], the firm's alternative to selling both high- and low-quality products is to sell only the high-quality product. The necessary condition in [AD] is obtained by imposing that the profits from selling both products exceed the profits from a) selling only the high-quality product to type-h consumers, and b) selling only the high-quality product to both type-h and type-l consumers. Unlike [AD], in our setting, the high-quality product's value is driven by the number of consumers who buy the low-quality product. If no one buys the low-quality product, then the high-quality one does not provide any additional value on top of the low-quality product. Therefore, if the firm chooses not to sell a product line (i.e., introduce the add-on), the firm's alternative is to sell only the *low*-quality product (rather than the high-quality product as in [AD]). For this reason, in our setting, the firm has a stronger incentive to sell a product line than in [AD] and doing so can be optimal even if the necessary condition identified in [AD] does not hold.

[22] When calculating the percentage change in surplus, we assumed that each player carries zero weight so that a single player's decision d does not influence overall win-loss probabilities. As an alternative approach, we also computed the change in surplus when *all* consumers of type-i buy the high-quality version (i.e., base product plus add-on) instead of the low-quality one (i.e., base product only). Again, we found that the necessary condition in [AD] may not be satisfied when the firm prefers to introduce the pay-to-win add-on.