Appendices for "*Sourcing with Demand Updates*"

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We organize the appendices into seven sections. Appendices A and B provide proofs for results in Sections 3 and 4, respectively. Appendix C provides the proofs for supporting lemmas used in Appendix B. Appendices D, F and G provide supplementary materials for Sections 5 to 7, respectively. Finally, Appendix H derives bounds for the optimal first-stage order S_1^*.

A Proofs for Section 3

This appendix contains the proofs of all lemmas and propositions in Section 3 in the order of their appearance.

Proof of Lemma 1. By standard arguments. □

Proof of Proposition 1. Recall that S_1^*, the unconstraint maximizer of (6), is the root of $f'(S_1) = 0$ by Lemma 1. One can easily verify that

$$f'(S_1) = -c_1 + c_2 - c_2 \int_{-\infty}^{x^*(S_1)} \psi_X(x)dx + \int_{-\infty}^{x^*(S_1)} H'_x(S_1)\psi_X(x)dx, \tag{A-1}$$

where $x^*(S_1)$ is the minimum signal under which a second order needs to be placed, i.e.,

$$x^*(S_1) = \frac{S_1 - \mu_0 - \Psi_D^{-1}\left(\frac{p-c_2}{p-c_v}\right)}{\mu_0^\alpha}, \tag{A-2}$$

by the classic newsvendor solution in the second stage. Also,

$$H'_x(S_1) = p - (p - c_v)\Psi_D(S_1 - \mu_0 - \mu_0^\alpha x).$$

Substitute this expression into (A-1), we get

$$f'(S_1) = -c_1 + c_2 + (p - c_2)\Psi_X(x^*(S_1)) - (p - c_v)\int_{-\infty}^{x^*(S_1)} \Psi_D(S_1 - \mu_0 - \mu_0^\alpha x)\psi_X(x)dx.$$

Since $\frac{df'(S_1)}{dS_1} \leq 0$ by Lemma 1, by the implicit function theorem, to show the desired monotonicities, we only need to show $\frac{df'(S_1)}{dc_2} \geq 0$, $\frac{df'(S_1)}{dp} \geq 0$, $\frac{df'(S_1)}{dc_v} \geq 0$, and $\frac{df'(S_1)}{dc_1} \leq 0$. The last inequality is obvious since $x^*(S_1)$ does not depend on c_1. We now show the first three inequalities. By straightforward algebra, one gets

$$\begin{aligned}\frac{df'(S_1)}{dp} = {} & \Psi_X(x^*(S_1)) + (p - c_2)\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dp} - (p - c_v)\Psi_D(S_1 - \mu_0 - \mu_0^\alpha x^*(S_1))\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dp} \\ & - \int_{-\infty}^{x^*(S_1)} \Psi_D(S_1 - \mu_0 - \mu_0^\alpha x)\psi_X(x)dx\end{aligned}$$

$$= \int_{-\infty}^{x^*(S_1)} (1 - \Psi_D(S_1 - \mu_0 - \mu_0^\alpha x))\psi_X(x)dx \geq 0;$$

$$\begin{aligned}
\frac{df'(S_1)}{dc_2} &= 1 - \Psi_X(x^*(S_1)) + (p - c_2)\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dc_2} \\
&\quad - (p - c_v)\Psi_D(S_1 - \mu_0 - \mu_0^\alpha x^*(S_1))\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dc_2} \\
&= 1 - \Psi_X(x^*(S_1)) \geq 0;
\end{aligned}$$

and

$$\begin{aligned}
\frac{df'(S_1)}{dc_v} &= (p - c_2)\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dc_v} - (p - c_v)\Psi_D(S_1 - \mu_0 - \mu_0^\alpha x^*(S_1))\psi_X(x^*(S_1))\frac{dx^*(S_1)}{dc_v} \\
&\quad + \int_{-\infty}^{x^*(S_1)} \Psi_D(S_1 - \mu_0 - \mu_0^\alpha x)\psi_X(x)dx \\
&= \int_{-\infty}^{x^*(S_1)} \Psi_D(S_1 - \mu_0 - \mu_0^\alpha x)\psi_X(x)dx \geq 0.
\end{aligned}$$

All second equalities above are due to the fact that $\Psi_X(S_1 - \mu_0 - \mu_0^\alpha x^*(S_1)) = \frac{p-c_2}{p-c_v}$. The proposition follows. $\square$

Proof of Corollary 1. The upper and lower bounds are clear from Proposition 1 and the main text. The second order $S_2^*(S_1, x)$ obviously follows from the classic newsvendor solution. We now introduce a few intermediary calculations that will be useful in the later proofs.

With a normal demand distribution, $H_x(S_2)$ takes the form:

$$\begin{aligned}
H_x(S_2) &= p\mathbb{E}\min\{D_x, S_2\} + c_v\mathbb{E}(S_2 - D_x)^+ \\
&= c_v S_2 + (p - c_v)(\mu_0 + \mu_0^\alpha x) - \sigma(p - c_v)L\left(\frac{S_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right), \qquad \text{(A-3)}
\end{aligned}$$

with $L(\cdot)$ the normal loss function, which is decreasing and convex.

Since $H_x(S_2)$ is concave, the unconstrained optimizer of (5), denoted by $S^*(x)$, solves $-c_2 + H_x'(S^*(x)) = 0$, where, in the normal demand case,

$$\begin{aligned}
H_x'(S_2) &= c_v - (p - c_v)L'\left(\frac{S_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right) \\
&= p - (p - c_v)\Phi\left(\frac{S_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right). \qquad \text{(A-4)}
\end{aligned}$$

Thus

$$S^*(x) = (\mu_0 + \mu_0^\alpha x) + \sigma\Phi^{-1}\left(\frac{p - c_2}{p - c_v}\right) = (\mu_0 + \mu_0^\alpha x) + \sigma\kappa_2. \qquad \text{(A-5)}$$

The optimal constraint optimizer of (5), denote by $S_2^*(S_1, x)$, is hence $S_2^*(S_1, x) = S_1 \vee S^*(x)$.

Consequently, according to (A-2), the *minimum* signal $x^*(S_1)$ under which a second-stage order needs to be placed, i.e., the value x that solves $S^*(x) = S_1$, is therefore

$$x^*(S_1) = \frac{S_1 - \mu_0 - \sigma\kappa_2}{\mu_0^\alpha}. \tag{A-6}$$

It follows that

$$S_2^*(S_1, x) = \begin{cases} S_1, & \text{if } x \le x^*(S_1), \\ S^*(x), & \text{if } x > x^*(S_1), \end{cases} \tag{A-7}$$

and hence

$$\frac{dS_2^*(S_1, x)}{dS_1} = \begin{cases} 1 & \text{if } x \le x^*(S_1), \\ 0 & \text{if } x > x^*(S_1). \end{cases}, \quad H_x'(S_2^*(S_1, x)) = \begin{cases} H_x'(S_1) & \text{if } x \le x^*(S_1), \\ c_2 & \text{if } x > x^*(S_1). \end{cases} \tag{A-8}$$

Utilizing these expressions to calculate $f'(S_1)$, we get

$$f'(S_1) = -c_1 + c_2 + \frac{1}{\tau}\int_{-\infty}^{x^*} \left[-c_2 + p - (p - c_v)\Phi\left(\frac{S_1 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\right]\phi(\frac{x}{\tau})dx.$$

□

In the subsequent analysis, we repeatedly need bounds for the hazard function of the standard normal distribution, which are not well known.

Lemma 3. *Let* $h(x) := \frac{\phi(x)}{1-\Phi(x)}, x > 0$ *denote the hazard function of the standard normal distribution. Then*

$$x \le h(x) \le \frac{1}{2}(x + \sqrt{x^2+4}) \le \min\left\{x+1, x+\frac{1}{x}\right\}.$$

The true hazard function as well as the lower and upper bounds are all increasing functions. The second upper bound is sometimes easier to use.

Proof of Lemma 3. The lower bound is well known. To prove the first upper bound, formula 7.1.13 in Abramowitz and Stegun (1964) reads:

$$\frac{1}{x + \sqrt{x^2+2}} < e^{x^2}\int_x^\infty e^{-u^2}du.$$

Substitute x by $t = \sqrt{2}x$ and multiply the inequality by $1/\sqrt{2}$ to get

$$\frac{1}{t + \sqrt{t^2+4}} < \frac{e^{t^2/2}}{\sqrt{2}}\int_{t/\sqrt{2}}^\infty e^{-u^2}du.$$

Change the integration variable to $v = \sqrt{2}u$ to obtain

$$\frac{1}{t+\sqrt{t^2+4}} < \sqrt{\frac{\pi}{2}} e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-v^2/2} dv = \frac{1}{2} \frac{1-\Phi(t)}{\frac{1}{\sqrt{2\pi}} e^{-t^2/2}}.$$

Hence

$$h(t) = \frac{\phi(t)}{1-\Phi(t)} < \frac{1}{2}(t+\sqrt{t^2+4}).$$

The second upper bound is, of course, equivalent to the pair of inequalities

$$x+\sqrt{x^2+4} \leq 2x+2 \Leftrightarrow \sqrt{x^2+4} \leq x+2$$

and

$$x+\sqrt{x^2+4} \leq 2x+\frac{2}{x} \Leftrightarrow \sqrt{x^2+4} \leq x+\frac{2}{x}.$$

Both inequalities can be verified by squaring both sides of each. $\square$

B Proofs for Section 4

In this appendix, we provide the proofs of Theorems 1 to 4 and Lemma 10 in Section 4.

Proof of Theorem 1. Since $f'(S_1^*) = 0$ (see Lemma 1), the implicit function theorem allows us to derive an ordinary differential equation (ODE) for the (unconstrained) initial order S_1^* with respect to the uncertainty coefficient of signal α. In particular,

$$\frac{dS_1^*}{d\alpha} = -\frac{\partial f'(S_1^*)/\partial\alpha}{\partial f'(S_1^*)/\partial S_1^*} = \frac{\mu_0^\alpha \log(\mu_0) \int_{-\infty}^{x^*(S_1^*)} x\phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx}{\int_{-\infty}^{x^*(S_1^*)} \phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx}. \tag{A-9}$$

Apply the changes of variables $y = \mu_0^\alpha x$ and $s(\mu_0) = S_1^*(\mu_0) - \mu_0$:

$$\begin{aligned}
\frac{dS_1^*}{d\alpha} &= \log\mu_0 \frac{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} y\phi\left(\frac{s(\mu_0)-y}{\sigma}\right)\phi\left(\frac{y}{\tau\mu_0^\alpha}\right)dy}{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} \phi\left(\frac{s(\mu_0)-y}{\sigma}\right)\phi\left(\frac{y}{\tau\mu_0^\alpha}\right)dy} \\
&= \log\mu_0 \frac{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} y e^{-\frac{(s(\mu_0)-y)^2/\sigma^2+y^2/(\tau^2\mu_0^{2\alpha})}{2}} dy}{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} e^{-\frac{(s(\mu_0)-y)^2/\sigma^2+y^2/(\tau^2\mu_0^{2\alpha})}{2}} dy} \\
&= \log\mu_0 \frac{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} y f\left(\frac{\tau^2\mu_0^{2\alpha}s(\mu_0)}{\sigma^2+\tau^2\mu_0^{2\alpha}}, \frac{\tau\sigma\mu_0^\alpha}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}, y\right)dy}{\int_{-\infty}^{s(\mu_0)-\sigma\kappa_2} f\left(\frac{\tau^2\mu_0^{2\alpha}s(\mu_0)}{\sigma^2+\tau^2\mu_0^{2\alpha}}, \frac{\tau\sigma\mu_0^\alpha}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}, y\right)dy} \\
&= \log\mu_0 \mathbb{E}\left[y\left(\frac{\tau^2\mu_0^{2\alpha}s(\mu_0)}{\sigma^2+\tau^2\mu_0^{2\alpha}}, \frac{\tau\sigma\mu_0^\alpha}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\right) \middle| y \leq s(\mu_0)-\sigma\kappa_2\right]
\end{aligned}$$

$$= \log \mu_0 \left[\frac{\tau^2\mu_0^{2\alpha} s(\mu_0)}{\sigma^2 + \tau^2\mu_0^{2\alpha}} - \frac{\tau\sigma\mu_0^\alpha}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}} \cdot \frac{\phi\left(\frac{\tau^2\mu_0^{2\alpha}(\mu_0-z_0)+\sigma^2(s(\mu_0)-\sigma\kappa_2)}{\tau\sigma\mu_0^\alpha\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\right)}{\Phi\left(\frac{\tau^2\mu_0^{2\alpha}(\mu_0-z_0)+\sigma^2(s(\mu_0)-\sigma\kappa_2)}{\tau\sigma\mu_0^\alpha\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\right)} \right]$$

$$= \log \mu_0 \frac{\tau^2\mu_0^{2\alpha} s(\mu_0)}{\sigma^2 + \tau^2\mu_0^{2\alpha}} - \log\mu_0 \frac{\tau\sigma\mu_0^\alpha}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}} \cdot \frac{\phi\left(\frac{\tau^2\mu_0^{2\alpha}(\mu_0-z_0)+\sigma^2(s(\mu_0)-\sigma\kappa_2)}{\tau\sigma\mu_0^\alpha\sqrt{\tau^2\mu_0^{2\alpha}+\sigma^2}}\right)}{\Phi\left(\frac{\tau^2\mu_0^{2\alpha}(\mu_0-z_0)+\sigma^2(s(\mu_0)-\sigma\kappa_2)}{\tau\sigma\mu_0^\alpha\sqrt{\tau^2\mu_0^{2\alpha}+\sigma^2}}\right)}, \tag{A-10}$$

where $f(\mu, \sigma, y)$ is the PDF of normal distribution $\mathcal{N}(\mu, \sigma^2)$, and $y(\mu, \sigma) \sim \mathcal{N}(\mu, \sigma^2)$.

Introducing $z_0 := \mu_0 + \sigma\kappa_2$ to the above equation gives us the desired result.

The boundary condition $S_1^*(-\infty) = \mu_0 + \kappa_1\sigma$ is immediate since S_1^* equals the single stage newsvendor solution when $\alpha = -\infty$. □

Proof of Theorem 2. (a) To prove this part, we first establish the non-asymptotic expression for S_1^* below.

$$S_1^* = \mu_0 + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}} \left[\kappa^* + \log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta \right], \tag{A-11}$$

where $\hat{F}(\cdot)$ is defined in Theorem 1. Then we will bound the term $\log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta$ as $\mu_0 \to \infty$.

To prove (A-11), we need three steps, described by Lemmas 4 to 6 below (their proofs can be found in Appendix C). First, in Lemma 4, we characterize the solution to the ODE in Theorem 1 up to one constant. We then bound the last term of the ODE's solution in Lemma 5, based on which we find the value of this constant in Lemma 6.

Lemma 4. *The solution to ODE (11) is of the form:*

$$S_1^*(\alpha) = \mu_0 + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}} \left[C + \log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta \right], \tag{A-12}$$

where C is a constant independent of α and $\hat{F}(\cdot)$ is defined in Theorem 1.

Lemma 5. *Consider the function $\hat{F}(\theta)$ defined in Theorem 1. Then*

(a) $\log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta \le F^* \log \frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\tau\mu_0^\alpha}$, *where F^* is a constant independent of α and μ_0.*

(b) $\log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta$ *is $O_{\mu_0}(\mu_0^{\beta-\alpha})$ for fixed α and β as $\mu_0 \to \infty$.*

Lemma 6. *In (A-12), $C = \kappa^*$.*

At this point, we have shown (A-11). Part (a) now follows directly using Lemma 5 (c).

(b) It follows from Theorem 2 (a) that $S_1^*(\mu_0) = \mu_0 + \tau\kappa^*\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta)$ where $\tau\kappa^*$ is independent of μ_0. Note from (A-6) that a critical threshold $x^*(\mu_0) := (S_1^*(\mu_0) - \mu_0 - \sigma\kappa_2)/\mu_0^\alpha$ exists such that $S_2^*(S_1^*(\mu_0), x) = S^*(x)$ if $x > x^*(\mu_0)$ and $S_2^*(S_1^*(\mu_0), x) = S_1^*(\mu_0)$ if $x < x^*(\mu_0)$. Indeed $x^*(\mu_0) = \tau\kappa^* + O(\mu_0^{\beta-\alpha})$. We thus have from (A-3)

$$-c_2 S_2^* + H_x(S_2^*) = \begin{cases} -(c_2 - c_v)(\mu_0 + \tau\kappa^*\mu_0^\alpha) + (p - c_v)(\mu_0 + \mu_0^\alpha x) - \sigma(p - c_v)L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x}{\sigma}\right) + O_{\mu_0}(\mu_0^\beta), \\ \hfill \text{if } x < x^*(\mu_0), \\ -(c_2 - c_v)(\mu_0 + \mu_0^\alpha x) + (p - c_v)(\mu_0 + \mu_0^\alpha x) - \sigma(p - c_v)L(\kappa_2) + O_{\mu_0}(\mu_0^\beta), \\ \hfill \text{if } x > x^*(\mu_0). \end{cases} \tag{A-13}$$

Since $L(\cdot)$ is a decreasing function, $0 \le L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x}{\sigma}\right) \le L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x^*}{\sigma}\right) = L(\kappa_2)$ for $x < x^*(\mu_0)$. Thus, (A-13) can be simplified to

$$-c_2 S_2^* + H_x(S_2^*) = \begin{cases} -(c_2 - c_v)(\mu_0 + \tau\kappa^*\mu_0^\alpha) + (p - c_v)(\mu_0 + \mu_0^\alpha x) + O_{\mu_0}(\mu_0^\beta), & \text{if } x < x^*(\mu_0), \\ -(c_2 - c_v)(\mu_0 + \mu_0^\alpha x) + (p - c_v)(\mu_0 + \mu_0^\alpha x) + O_{\mu_0}(\mu_0^\beta), & \text{if } x > x^*(\mu_0). \end{cases}$$

Taking expectations over the distribution of X, we get

$$\begin{aligned} \mathbb{E}_X[-c_2 S_2^* + H_x(S_2^*)] &= (p - c_2)\mu_0 - (c_2 - c_v)\tau\kappa^*\mu_0^\alpha \Phi\left(\frac{x^*(\mu_0)}{\tau}\right) + \frac{1}{\tau}\int_{-\infty}^{x^*(\mu_0)} (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx \\ &\quad - \frac{1}{\tau}\int_{x^*(\mu_0)}^{\infty} (c_2 - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + \frac{1}{\tau}\int_{x^*(\mu_0)}^{\infty} (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + O_{\mu_0}(\mu_0^\beta) \\ &= -(c_2 - c_v)\tau\kappa^*\mu_0^\alpha \Phi(\kappa^*) + \frac{1}{\tau}\int_{-\infty}^{\tau\kappa^*} (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx - \frac{1}{\tau}\int_{\tau\kappa^*}^{\infty} (c_2 - p)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx \\ &\quad + (p - c_2)\mu_0 + O_{\mu_0}(\mu_0^\beta) - \Big\{(c_2 - c_v)\mu_0^\alpha \tau\kappa^* \left[\Phi\left(\frac{x^*(\mu_0)}{\tau}\right) - \Phi(\kappa^*)\right] \\ &\quad + (p - c_v)\frac{1}{\tau}\int_{\tau\kappa^*}^{x^*(\mu_0)} \mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + (c_2 - p)\frac{1}{\tau}\int_{\tau\kappa^*}^{x^*(\mu_0)} \mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx \Big\}. \end{aligned}$$

Since $(x^*(\mu_0) - \tau\kappa^*) = O(\mu_0^{\beta-\alpha})$, one easily verifies that the expression within curled brackets is $O_{\mu_0}(\mu_0^\beta)$. Thus,

$$\begin{aligned} \mathbb{E}_X[-c_2 S_2^* + H_x(S_2^*)] &= (p - c_2)\mu_0 - (c_2 - c_v)\tau\kappa^*\mu_0^\alpha \Phi(\kappa^*) - (p - c_v)\tau\mu_0^\alpha \phi(\kappa^*) - (c_2 - p)\tau\mu_0^\alpha \phi(\kappa^*) + O_{\mu_0}(\mu_0^\beta) \\ &= (p - c_2)\mu_0 - (c_2 - c_v)\tau\kappa^*\mu_0^\alpha \Phi(\kappa^*) - (c_2 - c_v)\tau\mu_0^\alpha \phi(\kappa^*) + O_{\mu_0}(\mu_0^\beta). \end{aligned}$$

Finally, by (5) and (6),

$$f(S_1^*(\mu_0)) = (p - c_1)\mu_0 - (c_2 - c_v)\tau\phi(\kappa^*)\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta), \tag{A-14}$$

and hence

$$\pi^* = (c_2 - c_v)\tau\phi(\kappa^*)\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta).$$

□

Proof of Theorem 3. (a) The proof is again based on three steps, provided by Lemmas 7 to 9 below (their proofs can be found in Appendix C). Note that the characterization of S_1^* in (A-11) (in the proof of Theorem 2) cannot be used in the $\alpha < \beta$ regime, because its last term, in this parameter regime, fails to be bounded. We therefore provide a different characterization of S_1^*, derived from an ODE with respect to μ_0, different from the one in Theorem 1.

Lemma 7. *$S_1^*(\mu_0) = \mu_0 + \sqrt{b^2\mu_0^{2\beta} + \tau^2\mu_0^{2\alpha}}\left[C + \int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu\right]$ for some free parameter C, where*

$$F(\mu) = \frac{(\alpha-\beta)\tau b\mu^{\alpha+\beta-1}}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}} \cdot \frac{\phi(A(\mu))}{\Phi(A(\mu))},$$
$$A(\mu) = \frac{\tau^2\mu^{2\alpha}(\mu - z_0) + b^2\mu^{2\beta}(S_1^*(\mu) - z_0)}{\tau b\mu^{\alpha+\beta}\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}},$$

and $z_0 = \mu + \sigma\kappa_2$ as defined in Theorem 1.

Observe the expression for $S_1^*(\mu)$ in Lemma 7 and $\int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu$. Note first that the integrand is strictly negative since $\alpha - \beta < 0$ while all other factors are strictly positive. This means that $H(\mu_0) := \int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu$ is an strictly increasing function in μ_0, which because of Proposition 1, must be bounded. (Otherwise, by Lemma 7, $S_1^*(\mu_0) = \omega_{\mu_0}(\mu_0^{\beta})$ which contradicts Proposition 1.) This $\lim_{\mu_0\uparrow\infty} H(\mu_0)$ exists and is a finite constant C_2 (independent of μ_0). Moreover, $H(\mu_0) < C_2$ for any μ_0. Let $C_1 := C + C_2$. We first show in Lemma 8 that $C_1 = \kappa_1 > \kappa_2$. Then, we show in Lemma 9 that $\int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu = O_{\mu_0}(\mu_0^{\alpha-\beta})$, hence $C = C_1 = \kappa_1$.

Lemma 8. *The free parameter in Lemma 7 satisfies $C + \int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu = \kappa_1 - o_{\mu_0}(1)$.*

Lemma 9. *$\int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}}d\mu = O_{\mu_0}(\mu_0^{\alpha-\beta})$ when $\alpha < \beta$, where $F(\mu)$ is defined in Lemma 7.*

(b) The result directly follows from the proof of Lemma 8 and Lemma 9. □

Proof of Theorem 4. Consider the ODE of S_1^* w.r.t. μ_0 in (A-23) and let $\alpha = \beta$:

$$\frac{dS_1^*}{d\mu_0} = 1 + \frac{\alpha}{\mu_0} \cdot (S_1^* - \mu_0).$$

The solution to the above ODE is $S_1^* = \mu_0 + \lambda\mu_0^{\alpha}$ where λ is a constant. We next determine the value of λ. The critical threshold $x^*(\mu_0, \lambda) = \lambda - b\kappa_2$, and $S_2^* = S^* = \mu_0 + \mu_0^{\alpha}x + b\mu_0^{\alpha}\kappa_2$ if $x > x^*$ and $S_2^* = S_1^*$ if $x < x^*$. From (A-3) we have

$$-c_2S_2^* + H_x(S_2^*) = \begin{cases} -(c_2-c_v)(\mu_0+\lambda\mu_0^{\alpha}) + (p-c_v)(\mu_0+\mu_0^{\alpha}x) - b\mu_0^{\alpha}(p-c_v)L\left(\frac{\lambda-x}{b}\right), & \text{if } x < x^*(\mu_0,\lambda), \\ -(c_2-c_v)(\mu_0+\mu_0^{\alpha}x+b\mu_0^{\alpha}\kappa_2) + (p-c_v)(\mu_0+\mu_0^{\alpha}x) - b\mu_0^{\alpha}(p-c_v)L(\kappa_2), & \text{if } x > x^*(\mu_0,\lambda). \end{cases} \tag{A-15}$$

Taking expectations of (A-15) over the distribution of X, we get

$$\mathbb{E}_X[-c_2S_2^* + H_x(S_2^*)] = (p-c_2)\mu_0 - (c_2-c_v)b\kappa_2\mu_0^{\alpha} - (c_2-c_v)(\lambda-b\kappa_2)\mu_0^{\alpha}\Phi\left(\frac{\lambda-b\kappa_2}{\tau}\right)$$

$$
\begin{aligned}
&- (c_2 - c_v)\tau\mu_0^\alpha L\left(\frac{\lambda - b\kappa_2}{\tau}\right) - (c_2 - c_v)(\lambda - b\kappa_2)\mu_0^\alpha + (c_2 - c_v)(\lambda - b\kappa_2)\mu_0^\alpha \Phi\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
&- (p - c_v)b\mu_0^\alpha \int_{-\infty}^{\lambda - b\kappa_2} L\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx - (p - c_v)b\mu_0^\alpha L(\kappa_2)\overline{\Phi}\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
=&(p - c_2)\mu_0 - (c_2 - c_v)\lambda\mu_0^\alpha - (c_2 - c_v)\tau\mu_0^\alpha L\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
&- (p - c_v)b\mu_0^\alpha \int_{-\infty}^{\lambda - b\kappa_2} L\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx - (p - c_v)b\mu_0^\alpha L(\kappa_2)\overline{\Phi}\left(\frac{\lambda - b\kappa_2}{\tau}\right).
\end{aligned}
$$

Thus, by (5) and (6),

$$
\begin{aligned}
f(S_1^*(\mu_0)) =&(c_2 - c_1)\mu_0 + (c_2 - c_1)\lambda\mu_0^\alpha + (p - c_2)\mu_0 - (c_2 - c_v)\lambda\mu_0^\alpha - (c_2 - c_v)\tau\mu_0^\alpha L\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
&- (p - c_v)b\mu_0^\alpha \int_{-\infty}^{\lambda - b\kappa_2} L\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx - (p - c_v)b\mu_0^\alpha L(\kappa_2)\overline{\Phi}\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
=&(p - c_1)\mu_0 + (c_v - c_1)\lambda\mu_0^\alpha - (c_2 - c_v)\tau\mu_0^\alpha L\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
&- (p - c_v)b\mu_0^\alpha \int_{-\infty}^{\lambda - b\kappa_2} L\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx - (p - c_v)b\mu_0^\alpha L(\kappa_2)\overline{\Phi}\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
=&(p - c_1)\mu_0 - \Lambda(\lambda)\mu_0^\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda(\lambda) =&(c_1 - c_v)\lambda + (c_2 - c_v)\tau L\left(\frac{\lambda - b\kappa_2}{\tau}\right) \\
&+ (p - c_v)b \int_{-\infty}^{\lambda - b\kappa_2} L\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx + (p - c_v)bL(\kappa_2)\overline{\Phi}\left(\frac{\lambda - b\kappa_2}{\tau}\right).
\end{aligned}
$$

Calculate the derivatives of $\Lambda(\lambda)$:

$$
\Lambda'(\lambda) = (c_1 - c_2) - (p - c_2)\Phi\left(\frac{\lambda - b\kappa_2}{\tau}\right) + (p - c_v)\int_{-\infty}^{\lambda - b\kappa_2} \Phi\left(\frac{\lambda - x}{b}\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx
$$

and

$$
\Lambda''(\lambda) = \frac{p - c_v}{b\tau}\int_{-\infty}^{\lambda - b\kappa_2} \phi\left(\frac{\lambda - x}{b}\right)\phi\left(\frac{x}{\tau}\right)dx > 0.
$$

Therefore $\Lambda(\lambda)$ is convex, $\Lambda'(\lambda)$ an increasing continuous function, with $\lim_{\lambda\downarrow-\infty}\Lambda'(\lambda) = c_v - c_2 < 0$ and $\lim_{\lambda\uparrow\infty}\Lambda'(\lambda) = c_1 - c_v > 0$. Thus $\Lambda'(\lambda)$ has a unique root λ^* and it minimizes $\Lambda(\lambda)$. Part (a) and (b) of Theorem 4 hence follow. □

We also prove the next lemma referred to in Section 4.

Lemma 10. $(c_2 - c_v)\phi(\kappa^*) < (p - c_v)\phi(\kappa_1)$.

Proof of Lemma 10. Since $c_1 > c_v$ from (4), it suffices to show $\frac{c_2 - c_v}{c_1 - c_v}\phi(\kappa^*) < \frac{p - c_v}{c_1 - c_v}\phi(\kappa_1)$. Recall that

$\kappa^* = \Phi^{-1}\left(\frac{c_2-c_1}{c_2-c_v}\right)$ and $\kappa_1 = \Phi^{-1}\left(\frac{p-c_1}{p-c_v}\right)$, from which we get

$$\frac{c_2-c_v}{c_1-c_v} = \frac{1}{1-\Phi(\kappa^*)}, \quad \frac{p-c_v}{c_1-c_v} = \frac{1}{1-\Phi(\kappa_1)}.$$

Therefore, proving $\frac{c_2-c_v}{c_1-c_v}\phi(\kappa^*) < \frac{p-c_v}{c_1-c_v}\phi(\kappa_1)$ is equivalent to proving $\frac{\phi(\kappa^*)}{1-\Phi(\kappa^*)} < \frac{\phi(\kappa_1)}{1-\Phi(\kappa_1)}$. This inequality holds since the hazard function of the standard normal distribution, $\frac{\phi(\cdot)}{1-\Phi(\cdot)}$, is an increasing function, and $\kappa^* < \kappa_1$. □

C Proofs for Lemmas in Appendix B

In this appendix, we provide the proofs for Lemmas 4 to 9 appearing in Appendix B.

Proof of Lemma 4. We first verify that for some constant $\tilde{C}$ independent of α:

$$s(\alpha) = S_1^*(\alpha) - \mu_0 = \tilde{C}\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}} - \log\mu_0\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\int_{-\infty}^{\alpha}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta. \tag{A-16}$$

Compute its derivative with respect to α:

$$\begin{aligned}
\frac{ds(\alpha)}{d\alpha} &= \frac{\tau^2\mu_0^{2\alpha}\log\mu_0}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\left(\tilde{C} - \log\mu_0\int_{-\infty}^{\alpha}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta\right) - \log\mu_0\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\frac{\hat{F}(\alpha)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}} \\
&= \frac{\tau^2\mu_0^{2\alpha}\log\mu_0}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\left(\frac{s(\alpha)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}\right) - \log\mu_0\hat{F}(\alpha) \\
&= \log\mu_0\left(\frac{\tau^2\mu_0^{2\alpha}s(\alpha)}{\sigma^2+\tau^2\mu_0^{2\alpha}} - \hat{F}(\alpha)\right),
\end{aligned}$$

which is identical to (11). Note that (A-16) can be rewritten as

$$s(\alpha) = \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\left[\tilde{C} - \log\mu_0\int_{-\infty}^{\infty}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta\right] + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\log\mu_0\int_{\alpha}^{\infty}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta.$$

The statement follows by letting $C := \tilde{C} - \log\mu_0\int_{-\infty}^{\infty}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta$, a constant independent of α, where it remains to be shown that $\int_{-\infty}^{\infty}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta < \infty$.

We first show that there exists a θ^* such that $\hat{A}(\theta) \geq \frac{0.5(\kappa_1-\kappa_2)\sigma\mu_0^{-\theta}}{\tau}$ for $\theta < \theta^*$. As $\theta \downarrow -\infty$, no information is obtained in the second stage, and it is therefore optimal to procure everything in the first stage. Thus,

$$\lim_{\theta\downarrow-\infty}[S_1^*(\theta) - z_0] = (\mu_0+\kappa_1\sigma) - (\mu_0+\kappa_2\sigma) = (\kappa_1-\kappa_2)\sigma,$$

where $\kappa_1 - \kappa_2 = \Phi^{-1}\left(\frac{p-c_1}{p-c_v}\right) - \Phi^{-1}\left(\frac{p-c_2}{p-c_v}\right) > 0$. Note that

$$\lim_{\theta\downarrow-\infty} \hat{A}(\theta) = \lim_{\theta\downarrow-\infty} \frac{\tau^2\mu_0^{2\theta}(\mu_0 - z_0) + \sigma^2(S_1^* - z_0)}{\tau\sigma\mu_0^\theta\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} = \lim_{\theta\downarrow-\infty}\left[-\frac{\tau\mu_0^\theta\kappa_2}{\sigma} + \frac{\sigma(\kappa_1-\kappa_2)}{\tau\mu_0^\theta}\right] \geq \frac{0.5\sigma(\kappa_1-\kappa_2)}{\tau\mu_0^\theta}.$$

Hence there exists a θ^* such that for all $\theta < \theta^*$, $\hat{A}(\theta) \geq \frac{0.5(\kappa_1-\kappa_2)\sigma\mu_0^{-\theta}}{\tau}$.
In view of Lemma 5 (whose proof is independent of this), it suffices to show that $\int_{-\infty}^{\alpha}\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta < \infty$. By Lemma 5, for θ small enough, $\hat{A}(\theta)$ is bounded from *below* by a positive function that is proportional to $\mu_0^{-\theta}$, and therefore $\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))} \geq 2\phi(\hat{A}(\theta))$ is bounded from above by an integrable function. The same therefore applies to $\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}$. □

Proof of Lemma 5. By Proposition 1, $\mu_0 - S_1^* \leq -\kappa^*\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}$. Thus, $\mu_0 - S_1^* + \sigma\kappa_2 \leq \sigma\kappa_2 - \kappa^*\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}$ and for $\theta > \alpha$:

$$\begin{aligned}
-\hat{A}(\theta) &= \kappa_2 + \frac{\sigma}{\tau^2\mu_0^{2\theta}}(\mu_0 - S_1^* + \sigma\kappa_2) \leq \kappa_2 + \frac{\sigma}{\tau^2\mu_0^{2\theta}}[\sigma\kappa_2 - \kappa^*\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}] \\
&\leq \kappa_2 + \frac{\sigma}{\tau^2\mu_0^{2\theta}}[\sigma|\kappa_2| + |\kappa^*|\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}] \leq \kappa_2 + \frac{\sigma^2|\kappa_2|}{\tau^2\mu_0^{2\alpha}} + |\kappa^*|\frac{\sigma}{\tau\mu_0^\alpha}\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}{\tau\mu_0^\alpha} \\
&\leq \kappa_2 + \frac{\sigma^2|\kappa_2|}{\tau^2\mu_0^{2\alpha}} + |\kappa^*|\frac{\sigma}{\tau\mu_0^\alpha}\sqrt{1+\frac{b^2\mu_0^{2(\beta-\alpha)}}{\tau^2}} \leq \kappa_2 + \frac{b^2\mu_0^{2(\beta-\alpha)}|\kappa_2|}{\tau^2} + |\kappa^*|\frac{\sigma}{\tau\mu_0^\alpha}\sqrt{1+\frac{b^2}{\tau^2}} \\
&\leq \kappa_2 + \frac{b^2\mu_0^{2(\beta-\alpha)}|\kappa_2|}{\tau^2} + |\kappa^*|\frac{b\mu_0^{\beta-\alpha}}{\tau}\sqrt{1+\frac{b^2}{\tau^2}} \leq |\kappa_2| + \frac{b^2|\kappa_2|}{\tau^2} + |\kappa^*|\frac{b}{\tau}\sqrt{1+\frac{b^2}{\tau^2}} \\
&=: \overline{\kappa}_2 > 0.
\end{aligned}$$

Note that

$$\hat{F}(\theta) \leq \sigma\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))}. \tag{A-17}$$

If $\hat{A}(\theta) \geq 0$,

$$\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))} \leq \frac{\phi(0)}{\Phi(0)} = \frac{1/\sqrt{2\pi}}{1/2} = \sqrt{2/\pi}.$$

If $\hat{A}(\theta) < 0$,

$$\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))} = \frac{\phi(-\hat{A}(\theta))}{1-\Phi(-\hat{A}(\theta))} \leq \frac{1}{2}\left(-\hat{A}(\theta) + \sqrt{\hat{A}(\theta)^2+4}\right) \leq \frac{1}{2}\left(\overline{\kappa}_2 + \sqrt{\overline{\kappa}_2^2+4}\right), \tag{A-18}$$

where the first inequality follows from Lemma 3, and the second inequality from the upper bound of the standard normal hazard rate being increasing on the positive half line.

Since either $\hat{A}(\theta) \geq 0$ or $\hat{A}(\theta) < 0$,

$$\hat{F}(\theta) \leq \sigma \max\left\{\sqrt{\frac{2}{\pi}}, \frac{1}{2}\left(\overline{\kappa}_2 + \sqrt{\overline{\kappa}_2^2 + 4}\right)\right\} = \frac{1}{2}\sigma\left(\overline{\kappa}_2 + \sqrt{\overline{\kappa}_2^2 + 4}\right).$$

This equality follows from the function to its left being increasing in $\overline{\kappa}_2$ and equal to $1 > \sqrt{2/\pi}$ for $\overline{\kappa}_2 = 0$.

Note that $\overline{\kappa}_2 = |\kappa_2| + \frac{b^2}{\tau^2}|\kappa_2| + |\kappa^*|\frac{b}{\tau}\sqrt{1 + \frac{b^2}{\tau^2}}$, a constant independent of μ_0 and α, so that $\hat{F}(\theta)$ is bounded by σF^*, where F^* is a constant independent of μ_0 and α.

To bound the integral, apply the transformation of variables: $x = \sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}$. Then

$$\begin{aligned}
\log\mu_0 \int_\alpha^\infty \frac{1}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}}d\theta &= \int_{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}^\infty \frac{1}{x^2 - \sigma^2}dx = \frac{1}{2\sigma}\int_{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}^\infty \left[\frac{1}{x-\sigma} - \frac{1}{x+\sigma}\right]dx \\
&= \frac{1}{2\sigma}\log\frac{x-\sigma}{x+\sigma}\Big|_{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}}^\infty = \frac{1}{2\sigma}\log\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}-\sigma} \\
&= \frac{1}{\sigma}\log\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\tau\mu_0^\alpha}.
\end{aligned}$$

Thus

$$\log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}}d\theta \leq F^* \log\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\tau\mu_0^\alpha}.$$

With $y := \frac{\sigma}{\tau\mu_0^\alpha}$, the log-factor can be written as $\log\left[1 + y + (\sqrt{1+y^2} - 1)\right]$. Since the Taylor expansion of $\sqrt{1+y^2} - 1 = \frac{1}{2}y^2 - \frac{1}{8}y^4 + \frac{1}{16}y^6 + \cdots$, and that of the $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$, we have that

$$\log[1 + y + (\sqrt{1+y^2} - 1)] = O(y) \quad \text{as } y \downarrow 0,$$

with $y = O(\mu_0^{\beta-\alpha})$ when $\mu_0 \to \infty$ (since $\beta < \alpha$).

Asymptotic behavior when $\mu_0 \to \infty$. $\log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta$ goes to zero as $O(\mu_0^{(\beta-\alpha)})$, $\mu_0 \to \infty$. $\square$

Proof of Lemma 6. In this proof, all $O_\alpha(1)$ terms represent functions that are bounded in α.

It follows from Lemma 4 that

$$S_1^*(\alpha) = \mu_0 + \sqrt{\sigma^2 + \tau^2\mu_0^{2\alpha}}\left[C + \log\mu_0 \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}}d\theta\right].$$

Hence,

$$S_1^*(\alpha) = \mu_0 + C\sqrt{\sigma^2 + \tau^2\mu_0^{2\alpha}} + O_\alpha(1), \text{ as } \alpha \uparrow \infty,$$

since by Lemma 5, $\hat{F}(\theta) \leq F^*$ for all $\theta \geq \alpha$, and $\int_\alpha^\infty \frac{1}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} d\theta \leq \int_\alpha^\infty \frac{\mu_0^{-\theta}}{\tau} d\theta = \frac{\mu_0^{-\alpha}}{\tau \log \mu_0}$, which implies

$$\sqrt{\sigma^2 + \tau^2 \mu_0^{2\alpha}} \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2 \mu_0^{2\theta}}} d\theta \leq \frac{F^* \mu_0^{-\alpha}}{\tau \log \mu_0} \sqrt{\sigma^2 + \tau^2 \mu_0^{2\alpha}}$$
$$= \frac{F^* \sqrt{\tau^2 + \sigma^2 \mu_0^{-2\alpha}}}{\tau \log \mu_0} \leq \frac{F^* \sqrt{\tau^2 + \sigma^2}}{\tau \log \mu_0} = O(1) \text{ in } \alpha. \tag{A-19}$$

It follows that $S_1^*(\alpha) = \mu_0 + \lambda \mu_0^\alpha + \tilde{F}(\alpha)$ for some $-\infty < \lambda < \infty$ independent of α and $\tilde{F}(\alpha)$ a bounded function, i.e., $|\tilde{F}(\alpha)| \leq F^{**}$, for some constant F^{**} independent of α.

Note from (A-6) that a critical threshold $x^*(\alpha, \lambda)$ exists such that $S_2^* = S^*(x)$ if $x > x^*(\alpha, \lambda)$ and $S_2^* = S_1^*$ if $x \leq x^*(\alpha, \lambda)$. Indeed $x^*(\alpha, \lambda) = \lambda + \mu_0^{-\alpha}[\tilde{F}(\alpha) - \sigma\kappa_2]$. We thus have from (A-3)

$$\begin{aligned} & -c_2 S_2^* + H_x(S_2^*) \\ &= \begin{cases} -(c_2 - c_v)\lambda\mu_0^\alpha + (p - c_v)\mu_0^\alpha x - \sigma(p - c_v) L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x}{\sigma}\right) - (c_2 - c_v)(\mu_0 + \tilde{F}(\alpha)) + (p - c_v)\mu_0, & \text{if } x \leq x^*(\alpha, \lambda), \\ -(c_2 - c_v)\mu_0^\alpha x + (p - c_v)\mu_0^\alpha x - \sigma(p - c_v) L(\kappa_2) - (c_2 - c_v)(\mu_0 + \sigma\kappa_2) + (p - c_v)\mu_0, & \text{if } x > x^*(\alpha, \lambda). \end{cases} \end{aligned} \tag{A-20}$$

The terms $-(c_2 - c_v)(\mu_0 + \tilde{F}(\alpha)) + (p - c_v)\mu_0$ and $-(c_2 - c_v)(\mu_0 + \sigma\kappa_2) + (p - c_v)\mu_0$ are both bounded in α and independent of x, where $|\tilde{F}(\alpha)| \leq F^{**}$ (F^{**} is independent of α).

Since $L(\cdot)$ is a decreasing function, $0 \leq L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x}{\sigma}\right) \leq L\left(\frac{S_1^* - \mu_0 - \mu_0^\alpha x^*}{\sigma}\right) = L(\kappa_2)$ for $x \leq x^*(\alpha, \lambda)$. Thus, the last three terms in both cases in (A-20) can be bounded by a function $O(1)$ which is independent of x, and (A-20) can be simplified to

$$-c_2 S_2^* + H_x(S_2^*) = \begin{cases} -(c_2 - c_v)\lambda\mu_0^\alpha + (p - c_v)\mu_0^\alpha x + O(1), & \text{if } x \leq x^*(\mu_0, \lambda), \\ -(c_2 - c_v)\mu_0^\alpha x + (p - c_v)\mu_0^\alpha x + O(1), & \text{if } x > x^*(\mu_0, \lambda). \end{cases}$$

Taking expectations over the distribution of X, we get

$$\begin{aligned} \mathbb{E}_X[-c_2 S_2^* + H_x(S_2^*)] &= -(c_2 - c_v)\lambda\mu_0^\alpha \Phi\left(\frac{x^*(\alpha, \lambda)}{\tau}\right) + \frac{1}{\tau}\int_{-\infty}^{x^*(\alpha,\lambda)} (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx \\ &\quad - \frac{1}{\tau}\int_{x^*(\alpha,\lambda)}^\infty (c_2 - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + \frac{1}{\tau}\int_{x^*(\alpha,\lambda)}^\infty (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + O(1) \\ &= -(c_2 - c_v)\lambda\mu_0^\alpha \Phi\left(\frac{\lambda}{\tau}\right) + \frac{1}{\tau}\int_{-\infty}^{\lambda} (p - c_v)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx - \frac{1}{\tau}\int_\lambda^\infty (c_2 - p)\mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + O(1) \\ &\quad - \Big\{(c_2 - c_v)\mu_0^\alpha \lambda \left[\Phi\left(\frac{x^*(\alpha, \lambda)}{\tau}\right) - \Phi\left(\frac{\lambda}{\tau}\right)\right] \\ &\quad + (p - c_v)\frac{1}{\tau}\int_\lambda^{x^*(\alpha,\lambda)} \mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx + (c_2 - p)\frac{1}{\tau}\int_\lambda^{x^*(\alpha,\lambda)} \mu_0^\alpha \phi\left(\frac{x}{\tau}\right) x dx\Big\}, \end{aligned}$$

since $(x^*(\alpha, \lambda) - \lambda) = O(\mu_0^{-\alpha})$, one easily verifies that the expression within curled brackets is

$O(1)$. Thus,

$$\begin{aligned}\mathbb{E}_X[-c_2 S_2^* + H_x(S_2^*)] &= -(c_2-c_v)\lambda\mu_0^\alpha\Phi\left(\frac{\lambda}{\tau}\right) - (p-c_v)\tau\mu_0^\alpha\phi\left(\frac{\lambda}{\tau}\right) - (c_2-p)\tau\mu_0^\alpha\phi\left(\frac{\lambda}{\tau}\right) + O(1)\\ &= -(c_2-c_v)\lambda\mu_0^\alpha\Phi\left(\frac{\lambda}{\tau}\right) - (c_2-c_v)\tau\mu_0^\alpha\phi\left(\frac{\lambda}{\tau}\right) + O(1).\end{aligned}$$

Then

$$f(S_1^{*+}(\alpha)) = M(\lambda)\mu_0^\alpha + O(1), \tag{A-21}$$

where

$$M(\lambda) = (c_2-c_1)\lambda - (c_2-c_v)\lambda\Phi\left(\frac{\lambda}{\tau}\right) - (c_2-c_v)\tau\phi\left(\frac{\lambda}{\tau}\right).$$

By taking its second derivative, $M(\lambda)$ is a strictly concave function and achieves its global maximum at the unique root $\tau\kappa^* = \tau\Phi^{-1}(\frac{c_2-c_1}{c_2-c_v})$ of the equation $\Lambda'(\lambda) = 0$.

Thus, the choice of $\lambda = \tau\kappa^*$ is optimal, and $S_1^*(\alpha) = \mu_0 + \tau\kappa^*\mu_0^\alpha + O(1)$. □

Proof of Lemma 7. We first derive the ODE of S_1^* w.r.t. μ_0.

$$\begin{aligned}\frac{dS_1^*}{d\mu_0} &= -\frac{\partial f'(S_1^*)/\partial\mu_0}{\partial f'(S_1^*)/\partial S_1^*}\\ &= \frac{p-c_v}{\sigma(\mu_0)\tau}\left[\int_\infty^{x^*(S_1^*)} \frac{(1+\alpha\mu_0^{\alpha-1}x)\sigma(\mu_0) + \sigma'(\mu_0)(S_1^*-\mu_0-\mu_0^\alpha x)}{\sigma(\mu_0)}\phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma(\mu_0)}\right)\phi\left(\frac{x}{\tau}\right)dx\right]\\ &\quad /\left[\frac{p-c_v}{\sigma(\mu_0)\tau}\int_{-\infty}^{x^*(S_1^*)}\phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma(\mu_0)}\right)\phi\left(\frac{x}{\tau}\right)dx\right]\\ &= \left[\int_\infty^{x^*(S_1)}\left(1+\frac{\sigma'(\mu_0)(S_1^*-\mu_0)}{\sigma(\mu_0)} + \left(\frac{\alpha}{\mu_0\log(\mu_0)} - \frac{\sigma'(\mu_0)}{\sigma(\mu_0)\log(\mu_0)}\right)\mu_0^\alpha\log(\mu_0)x\right)\phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma(\mu_0)}\right)\phi\left(\frac{x}{\tau}\right)dx\right]\\ &\quad /\left[\int_{-\infty}^{x^*(S_1^*)}\phi\left(\frac{S_1^*-\mu_0-\mu_0^\alpha x}{\sigma(\mu_0)}\right)\phi\left(\frac{x}{\tau}\right)dx\right]\\ &= 1+\frac{\sigma'(\mu_0)(S_1^*-\mu_0)}{\sigma(\mu_0)} + \left(\frac{\alpha}{\mu_0\log\mu_0} - \frac{\sigma'(\mu_0)}{\sigma(\mu_0)\log\mu_0}\right)\frac{dS_1^*}{d\alpha}\\ &= 1+\frac{\beta(S_1^*-\mu_0)}{\mu_0} + \frac{\alpha-\beta}{\mu_0\log\mu_0}\frac{dS_1^*}{d\alpha}.\end{aligned} \tag{A-22}$$

Plugging in $\frac{dS_1^*}{d\alpha}$ from (11), we can rewrite (A-22) as

$$\frac{dS_1^*}{d\mu_0} = 1 + \frac{\beta b^2\mu_0^{2\beta-1} + \alpha\tau^2\mu_0^{2\alpha-1}}{b^2\mu_0^{2\beta}+\tau^2\mu_0^{2\alpha}}\cdot(S_1^*-\mu_0) - \frac{(\alpha-\beta)\tau b\mu_0^{\alpha+\beta-1}}{\sqrt{b^2\mu_0^{2\beta}+\tau^2\mu_0^{2\alpha}}}\cdot\frac{\phi(A(\mu_0))}{\Phi(A(\mu_0))}, \tag{A-23}$$

where

$$A(\mu_0) = \frac{\tau^2\mu_0^{2\alpha}(\mu_0-z_0) + b^2\mu_0^{2\beta}(S_1^*(\mu_0)-z_0)}{\tau b\mu_0^{\alpha+\beta}\sqrt{b^2\mu_0^{2\beta}+\tau^2\mu_0^{2\alpha}}}. \tag{A-24}$$

In fact, one can verify that the solution to this ODE is

$$S_1^*(\mu_0) = \mu_0 + \sqrt{b_2\mu_0^{2\beta} + \tau^2\mu_0^{2\alpha}}\left[C + \int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta} + \tau^2\mu^{2\alpha}}} d\mu\right],$$

where C is a free parameter independent of μ_0, and

$$F(\mu) = \frac{(\alpha-\beta)\tau b \mu^{\alpha+\beta-1}}{\sqrt{b^2\mu^{2\beta} + \tau^2\mu^{2\alpha}}} \cdot \frac{\phi(A(\mu))}{\Phi(A(\mu))}.$$

□

Proof of Lemma 8. From Lemma 7 and the above argument we know that the optimal first-stage order when $\alpha < \beta$ must be $S_1^*(\mu_0) = \mu_0 + \lambda\mu_0^\beta + o_{\mu_0}(\mu_0^\beta)$ for some λ. The critical threshold is $x^*(\mu_0, \lambda) = (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})$, and $S_2^*(S_1^*(\mu_0), x) = \mu_0 + \mu_0^\alpha x + b\mu_0^\beta\kappa_2$ if $x > x^*$ and $S_2^*(S_1^*(\mu_0), x) = S_1^*(\mu_0)$ if $x \le x^*$. From (A-3) we have

$$-c_2S_2^* + H_x(S_2^*) = \begin{cases} -(c_2 - c_v)(\mu_0 + \lambda\mu_0^\beta) + (p - c_v)\mu_0 - b\mu_0^\beta(p - c_v)L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) + o_{\mu_0}(\mu_0^\beta), & \text{if } x \le x^*(\mu_0, \lambda), \\ -(c_2 - c_v)(\mu_0 + b\mu_0^\beta\kappa_2) + (p - c_v)\mu_0 - b\mu_0^\beta(p - c_v)L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) + o_{\mu_0}(\mu_0^\beta), & \text{if } x > x^*(\mu_0, \lambda). \end{cases} \tag{A-25}$$

Now we show that $\mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) = L(\frac{\lambda}{b}) \wedge L(\kappa_2) + o_{\mu_0}(1)$. From the expression of S_2^*, we have

$$\begin{aligned} L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) &= \begin{cases} L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) & \text{if } x \le (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \\ L(\kappa_2) & \text{if } x > (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \end{cases} \\ &= \begin{cases} L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) & \text{if } x \le (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \\ L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) & \\ \quad + \left(L(\kappa_2) - L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right)\right) & \text{if } x > (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \end{cases} \\ &= \begin{cases} L(\kappa_2) + \left(L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) - L(\kappa_2)\right) & \text{if } x \le (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \\ L(\kappa_2) & \text{if } x > (\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha}), \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) &= \mathbb{E}L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) \\ &\quad + \int_{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})}^{\infty} \left(L(\kappa_2) - L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right)\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx \\ &= L(\kappa_2) + \int_{-\infty}^{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})} \left(L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) - L(\kappa_2)\right)\frac{1}{\tau}\phi\left(\frac{x}{\tau}\right)dx. \end{aligned} \tag{A-26}$$

Let $U \sim \mathcal{N}(0,1)$, independent of X. Note that since $L(y) = \mathbb{E}_U[U-y]^+$,

$$\mathbb{E}_X L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)X\right) = \mathbb{E}_X \mathbb{E}_U \left[U - \frac{\lambda}{b} - o_{\mu_0}(1) - o_{\mu_0}(1)X\right]^+.$$

For any realization of U and X, we have

$$\left[U - \frac{\lambda}{b}\right]^+ - |o_{\mu_0}(1) + o_{\mu_0}(1)X| \leq \left[U - \frac{\lambda}{b} - o_{\mu_0}(1) - o_{\mu_0}(1)X\right]^+ \leq \left[U - \frac{\lambda}{b}\right]^+ + |o_{\mu_0}(1) + o_{\mu_0}(1)X|,$$

which implies that

$$L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)X\right) = \mathbb{E}_U \left[U - \frac{\lambda}{b} - o_{\mu_0}(1) - o_{\mu_0}(1)X\right]^+ = L\left(\frac{\lambda}{b}\right) + o_{\mu_0}(1) + o_{\mu_0}(1)X, \tag{A-27}$$

and hence

$$\mathbb{E}_X L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)X\right) = L\left(\frac{\lambda}{b}\right) + o_{\mu_0}(1). \tag{A-28}$$

If $\frac{\lambda}{b} = \kappa_2$, then

$$\begin{aligned} L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) - L(\kappa_2) &= L\left(\kappa_2 + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) - L(\kappa_2) \\ &= \mathbb{E}_U[U - \kappa_2 - o_{\mu_0}(1) - o_{\mu_0}(1)x]^+ - \mathbb{E}_U[U-\kappa_2]^+ \\ &= o_{\mu_0}(1) + o_{\mu_0}(1)x, \end{aligned}$$

since $-|o_{\mu_0}(1) + o_{\mu_0}(1)x| \leq [U - \kappa_2 - o_{\mu_0}(1) - o_{\mu_0}(1)x]^+ - [U-\kappa_2]^+ \leq |o_{\mu_0}(1) + o_{\mu_0}(1)x|$. By (A-26),

$$\begin{aligned} \mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) &= L(\kappa_2) + \int_{-\infty}^{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})} \left(L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) - L(\kappa_2)\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx \\ &= L(\kappa_2) + \int_{-\infty}^{o_{\mu_0}(\mu_0^{\beta-\alpha})} \left(o_{\mu_0}(1) + o_{\mu_0}(1)x\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx \\ &= L(\kappa_2) + o_{\mu_0}(1). \end{aligned} \tag{A-29}$$

If $\frac{\lambda}{b} > \kappa_2$, then by (A-27),

$$\begin{aligned} &\int_{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})}^{\infty} \left(L(\kappa_2) - L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right)\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx \\ &= \int_{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})}^{\infty} \left(L(\kappa_2) - L\left(\frac{\lambda}{b}\right) - o_{\mu_0}(1) - o_{\mu_0}(1)\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx \end{aligned}$$

$$= \left(L(\kappa_2) - L\left(\frac{\lambda}{b}\right)\right) o_{\mu_0}(1) - \int_{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})}^{\infty} \left(o_{\mu_0}(1) + o_{\mu_0}(1)\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx$$
$$= o_{\mu_0}(1).$$

Thus, combining with (A-26) and (A-28), we get

$$\begin{aligned}
\mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) &= \mathbb{E}L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right) \\
&\quad + \int_{(\lambda - b\kappa_2)\mu_0^{\beta-\alpha} + o_{\mu_0}(\mu_0^{\beta-\alpha})}^{\infty} \left(L(\kappa_2) - L\left(\frac{\lambda}{b} + o_{\mu_0}(1) + o_{\mu_0}(1)x\right)\right) \frac{1}{\tau}\phi\left(\frac{x}{\tau}\right) dx \\
&= L\left(\frac{\lambda}{b}\right) + o_{\mu_0}(1). \qquad \text{(A-30)}
\end{aligned}$$

Similarly, if $\frac{\lambda}{b} < \kappa_2$, we obtain

$$\mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) = L(\kappa_2) + o_{\mu_0}(1). \qquad \text{(A-31)}$$

Combining (A-29), (A-30) and (A-31), since $L(\cdot)$ is a decreasing function, we conclude that

$$\mathbb{E}L\left(\frac{S_2^* - \mu_0 - \mu_0^\alpha x}{b\mu_0^\beta}\right) = L\left(\frac{\lambda}{b}\right) \wedge L(\kappa_2) + o_{\mu_0}(1).$$

Now, taking expectations of (A-25) over the distribution of X, we get

$$\mathbb{E}_X[-c_2 S_2^* + H_x(S_2^*)] = \begin{cases} (p - c_2)\mu_0 - \lambda(c_2 - c_v)\mu_0^\beta - b(p - c_v)L\left(\frac{\lambda}{b}\right)\mu_0^\beta + o_{\mu_0}(\mu_0^\beta), & \text{if } \frac{\lambda}{b} > \kappa_2 \\ (p - c_2)\mu_0 - b\kappa_2(c_2 - c_v)\mu_0^\beta - b(p - c_v)L(\kappa_2)\mu_0^\beta + o_{\mu_0}(\mu_0^\beta), & \text{if } \frac{\lambda}{b} \le \kappa_2. \end{cases}$$

Moreover, by (5) and (6),

$$f(S_1^*(\mu_0)) = (p - c_1)\mu_0 - M(\lambda)\mu_0^\beta + o_{\mu_0}(\mu_0^\beta),$$

where

$$M(\lambda) = \begin{cases} -(c_2 - c_1)\lambda + (c_2 - c_v)\lambda + b(p - c_v)L\left(\frac{\lambda}{b}\right), & \text{if } \lambda > b\kappa_2 \\ -(c_2 - c_1)\lambda + (c_2 - c_v)b\kappa_2 + b(p - c_v)L(\kappa_2), & \text{if } \lambda \le b\kappa_2. \end{cases}$$

Calculate the derivative of $M(\lambda)$:

$$M'(\lambda) = \begin{cases} -c_v + c_1 - (p - c_v)(1 - \Phi(\frac{\lambda}{b})), & \text{if } \lambda > b\kappa_2 \\ -c_2 + c_1, & \text{if } \lambda \le b\kappa_2. \end{cases}$$

Observe that $M(\lambda)$ is convex, with its unique minimizer at $\lambda^* = b\kappa_1 > b\kappa_2$, and

$$M(\lambda^*) = (p - c_v)b\phi(\kappa_1).$$

□

Proof of Lemma 9. From Lemma 8, $C + \int_{\mu_0}^{\infty} \frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}} d\mu = \kappa_1 - o_{\mu_0}(1)$. The integrand of the integral can be written as

$$\frac{F(\mu)}{\sqrt{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}}} = \frac{(\alpha-\beta)\tau b\mu^{\alpha+\beta-1}}{b^2\mu^{2\beta}+\tau^2\mu^{2\alpha}} \cdot \frac{\phi(A(\mu))}{\Phi(A(\mu))}.$$

Since $\kappa_1 > \kappa_2$, then for μ sufficiently large, $A(\mu)$ in Lemma 7 is strictly positive, since $\beta > \alpha$ and $S_1^*(\mu) - z_0 = \sqrt{b^2\mu_0^{2\beta} + \tau^2\mu_0^{2\alpha}}(\kappa_1 - o_{\mu_0}(1)) - \kappa_2 b\mu_0^{\beta} > 0$. Thus, the second factor of the above integrand $0 \le \frac{\phi(A(\mu))}{\Phi(A(\mu))} \le \frac{\phi(0)}{\Phi(0)} = \sqrt{\frac{2}{\pi}}$, where the inequality follows from $\phi(\cdot)$ being decreasing, and $\Phi(\cdot)$ being increasing on the positive half line. Moreover, the first factor of the integrand is $O_{\mu}(\mu^{\alpha-\beta-1})$. It then follows that the integral is $O_{\mu_0}(\mu_0^{\alpha-\beta})$, an exponentially decreasing function.

□

D Proofs for Section 5

We prove Proposition 2 in this appendix.

Proof of Proposition 2. (a) Suppose $\alpha < \beta$. By the proof of Theorem 3 (b), we have the expected profit gap under the policy OPT-, like that of the optimal policy, as

$$\pi^{\text{OPT-}}(\mu_0) = (p - c_v)b\phi(\kappa_1)\mu_0^{\beta} + O(\mu_0^{\alpha}), \tag{A-32}$$

since the proof of Theorem 3 (b) shows that (A-32) holds for any policy for which $S_1(\mu_0) = \mu_0 + b\kappa_1\mu_0^{\beta} + O(\mu_0^{\alpha})$, including our policy OPT- as well as the optimal policy OPT. Thus, $\pi^{\text{OPT-}}(\mu_0) - \pi^*(\mu_0) = O(\mu^{\alpha})$. This difference being nonnegative is immediate from the fact that OPT- is a heuristic as opposed to the optimal strategy. The case $\alpha > \beta$ is analogous. When $\alpha = \beta$, the heuristic policy OPT- becomes exact.

(b) When $\alpha < \beta$, OPT- and NV+ are identical, hence the result follows from part (a). When $\alpha > \beta$, note that by Theorem 2 (b) and its proof, we have

$$\pi^{\text{NV+}}(\mu_0) = M(\tau\kappa_1)\mu_0^{\alpha} + O(\mu_0^{\beta}),$$

where

$$M(\tau\kappa_1) := \frac{(p-c_2)(c_1-c_v)}{p-c_v}\tau\kappa_1 + (c_2 - c_v)\tau\phi(\kappa_1).$$

Therefore

$$\pi^{\text{NV+}}(\mu_0) - \pi^*(\mu_0) = \left(\frac{(p-c_2)(c_1-c_v)}{p-c_v} \tau\kappa_1 + (c_2-c_v)\tau\left(\phi(\kappa_1) - \phi(\kappa^*)\right) \right) \mu_0^\alpha + O(\mu_0^\beta)$$

and

$$\frac{\pi^{\text{NV+}}(\mu_0) - \pi^*(\mu_0)}{\pi^*(\mu_0)} = \frac{(p-c_2)(c_1-c_v)\kappa_1}{(p-c_v)(c_2-c_v)\phi(\kappa^*)} + \frac{\phi(\kappa_1)}{\phi(\kappa^*)} - 1 + o_{\mu_0}(1).$$

When $\alpha = \beta$, by the proof of Theorem 4 (b), we know that

$$\pi^{\text{NV+}}(\mu_0) = \Lambda\left(\sqrt{b^2+\tau^2}\kappa_1\right)\mu_0^\alpha,$$

where $\Lambda(\cdot)$ is defined in (13). Also,

$$\pi^*(\mu_0) = \Lambda(\lambda^*)\mu_0^\alpha$$

where λ^* is defined in (12). Therefore, we have

$$\pi^{\text{NV+}}(\mu_0) - \pi^*(\mu_0) = \left(\Lambda\left(\sqrt{b^2+\tau^2}\kappa_1\right) - \Lambda(\lambda^*)\right)\mu_0^\alpha$$

and

$$\frac{\pi^{\text{NV+}}(\mu_0) - \pi^*(\mu_0)}{\pi^*(\mu_0)} = \frac{\Lambda\left(\sqrt{b^2+\tau^2}\kappa_1\right)}{\Lambda(\lambda^*)} - 1.$$

(c) When $\alpha < \beta$, we compare (A-32) with the single-stage problem where the second-stage order is forced to be zero. In this case, the optimal first-stage order reduces to the newsvendor solution $\overline{S}_1 = \mu_0 + \kappa_1\sqrt{b^2\mu_0^{2\beta} + \tau^2\mu_0^{2\alpha}}$ in Proposition 1, which is positive for μ_0 sufficiently large since $\alpha \in [0,1)$. The resulting expected profit

$$\begin{aligned}
f^{\text{NV}}(\mu_0) &= -c_1 S_1^{nv} + \mathbb{E}_x H_x(S_1^{nv}) \\
&= (p-c_1)\mu_0 + (c_v-c_1)\kappa_1 b\mu_0^\beta - b\mu_0^\beta(p-c_v)\mathbb{E}_x L\left(\kappa_1 - \frac{x}{b}\mu_0^{\alpha-\beta} + O_{\mu_0}(\mu_0^{\alpha-\beta})\right) \\
&= (p-c_1)\mu_0 + (c_v-c_1)\kappa_1 b\mu_0^\beta - b\mu_0^\beta(p-c_v)L(\kappa_1) + O_{\mu_0}(\mu_0^\alpha) \\
&= (p-c_1)\mu_0 + (c_v-c_1)\kappa_1 b\mu_0^\beta - b\mu_0^\beta(p-c_v)\phi(\kappa_1) + (c_1-c_v)\kappa_1 b\mu_0^\beta + O_{\mu_0}(\mu_0^\alpha) \\
&= (p-c_1)\mu_0 - b\mu_0^\beta(p-c_v)\phi(\kappa_1) + O_{\mu_0}(\mu_0^\alpha)
\end{aligned}$$

since $L(\kappa_1) = \phi(\kappa_1) - \kappa_1 - \kappa_1\Phi(\kappa_1) = \phi(\kappa_1) - \kappa_1 \cdot \frac{c_1-c_v}{p-c_v}$. Also

$$f^*(\mu_0) = f^{\text{OPT-}}(\mu_0) = (p-c_1)\mu_0 + (c_v-p)b\phi(\kappa_1)\mu_0^\beta + O_{\mu_0}(\mu_0^\alpha).$$

Therefore the profit gap $\pi^{\mathrm{NV}}(\mu_0) = (p-c_1)\mu_0 - f^{\mathrm{NV}}(\mu_0) = b\mu_0^\beta(p-c_v)\phi(\kappa_1) + O_{\mu_0}(\mu_0^\alpha)$, and hence, by Theorem 3 (b),

$$\pi^{\mathrm{NV}}(\mu_0) - \pi^*(\mu_0) = O_{\mu_0}(\mu_0^\alpha),$$

and

$$\frac{\pi^{\mathrm{NV}}(\mu_0) - \pi^*(\mu_0)}{\pi^*(\mu_0)} = o_{\mu_0}(1).$$

When $\alpha > \beta$, the expected profit under the NV strategy is

$$\begin{aligned}
f^{\mathrm{NV}}(\mu_0) &= -c_1 S_1^{nv} + \mathbb{E}_x H_x(S_1^{nv}) \\
&= (p-c_1)\mu_0 + (c_v - c_1)\kappa_1\tau\mu_0^\alpha - b\mu_0^\beta(p-c_v)\mathbb{E}_x L\left(\frac{\kappa_1\sqrt{b^2\mu_0^{2\beta} + \tau^2\mu_0^{2\alpha}} - \mu_0^\alpha x}{b\mu_0^\beta}\right) + O_{\mu_0}(\mu_0^\beta) \\
&= (p-c_1)\mu_0 + (c_v - c_1)\kappa_1\tau\mu_0^\alpha - b\mu_0^\beta(p-c_v)\int_{\kappa_1\tau}^\infty \frac{x-\kappa_1\tau}{b}\cdot\mu_0^{\alpha-\beta}\cdot\frac{1}{\tau}\phi(\frac{x}{\tau})dx + O_{\mu_0}(\mu_0^\beta) \\
&= (p-c_1)\mu_0 + (c_v - c_1)\kappa_1\tau\mu_0^\alpha - (p-c_v)\tau\mu_0^\alpha(\phi(\kappa_1) - \kappa_1 + \kappa_1\Phi(\kappa_1)) + O_{\mu_0}(\mu_0^\beta) \\
&= (p-c_1)\mu_0 + (c_v - c_1)\kappa_1\tau\mu_0^\alpha - (p-c_v)\tau\mu_0^\alpha\left(\phi(\kappa_1) - \kappa_1\cdot\frac{c_1-c_v}{p-c_v}\right) + O_{\mu_0}(\mu_0^\beta) \\
&= (p-c_1)\mu_0 - (p-c_v)\tau\phi(\kappa_1)\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta).
\end{aligned}$$

Therefore the profit gap

$$\pi^{\mathrm{NV}}(\mu_0) = (p-c_v)\tau\phi(\kappa_1)\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta)$$

and hence

$$\pi^{\mathrm{NV}}(\mu_0) - \pi^*(\mu_0) = \left((p-c_v)\phi(\kappa_1) - (c_2-c_v)\phi(\kappa^*)\right)\tau\mu_0^\alpha + O_{\mu_0}(\mu_0^\beta)$$

and

$$\frac{\pi^{\mathrm{NV}}(\mu_0) - \pi^*(\mu_0)}{\pi^*(\mu_0)} = \frac{(p-c_v)\phi(\kappa_1)}{(c_2-c_v)\phi(\kappa^*)} - 1 + o_{\mu_0}(1).$$

When $\alpha = \beta$, the resulting profit gap

$$\begin{aligned}
\pi^{\mathrm{NV}}(\mu_0) &= (p-c_v)\sqrt{b^2+\tau^2}\mu_0^\alpha L(\kappa_1) + (c_1-c_v)\kappa_1\sqrt{b^2+\tau^2}\mu_0^\alpha \\
&= (p-c_v)\sqrt{b^2+\tau^2}\phi(\kappa_1)\mu_0^\alpha.
\end{aligned}$$

Also recall that

$$\pi^*(\mu_0) = \Lambda(\lambda^*)\mu_0^\alpha$$

where λ^* and $\Lambda(\cdot)$ are defined in (12) and (13), respectively. Therefore

$$\pi^{\mathrm{NV}}(\mu_0) - \pi^*(\mu_0) = \left((p - c_v)\sqrt{b^2 + \tau^2}\phi(\kappa_1) - \Lambda(\lambda^*)\right)\mu_0^{\alpha}$$

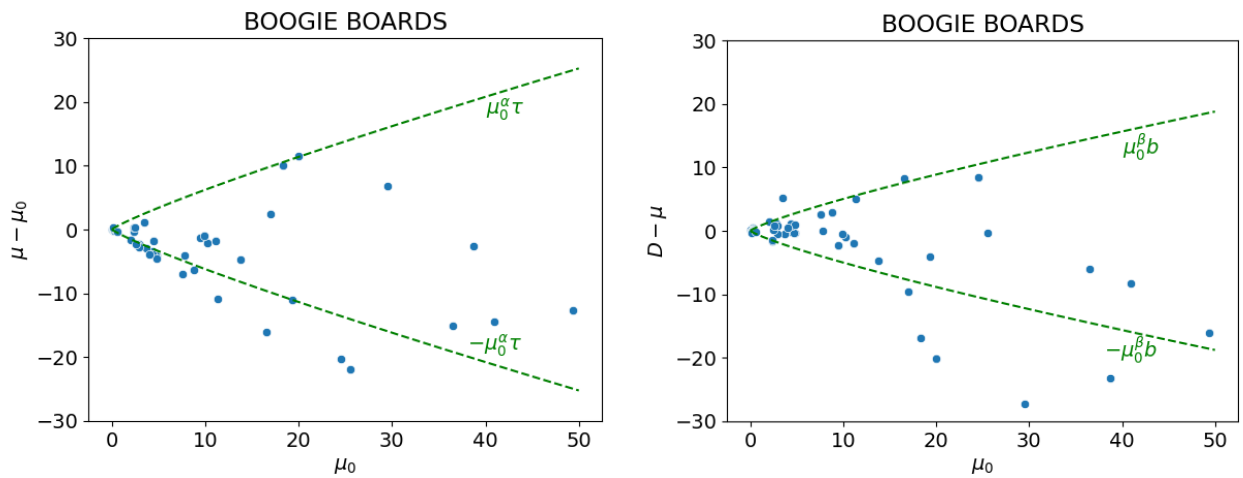
and

$$\frac{\pi^{\mathrm{NV}}\mu_0 - \pi^*(\mu_0)}{\pi^*(\mu_0)} = \frac{(p - c_v)\sqrt{b^2 + \tau^2}\phi(\kappa_1)}{\Lambda(\lambda^*)} - 1.$$

□

E Supplementary Materials for Section 2

We provide supplementary materials, namely Figures A-1 and A-2, for Section 2.



(a) $\mu - \mu_0$ against μ_0

(b) $D - \mu$ against μ_0

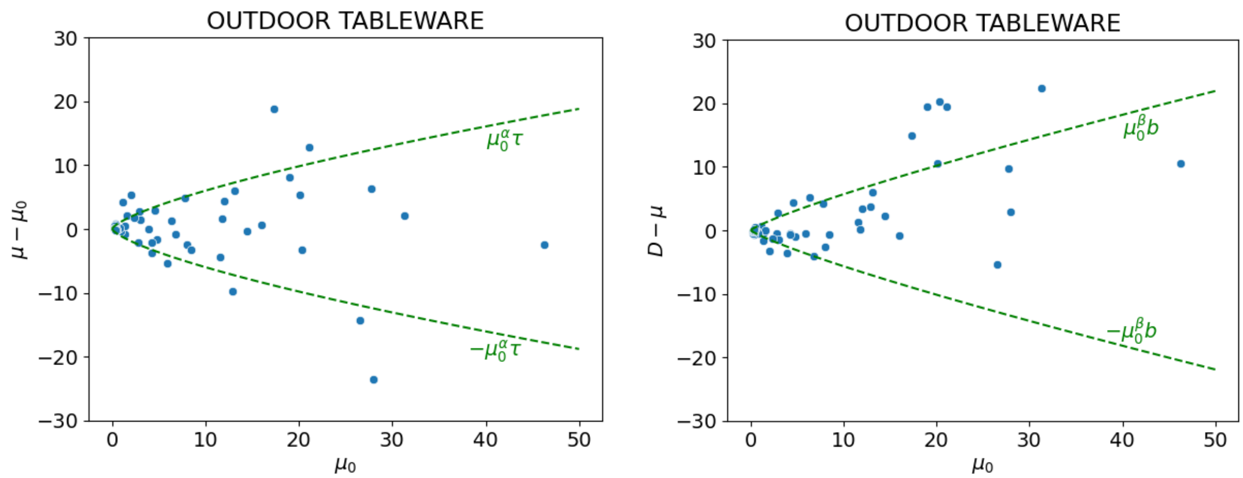
Figure A-1: Forecast adjustment and demand noise vs initial forecast: Boogie Boards ($\hat{\alpha} > \hat{\beta}$)

F Supplementary Materials for Section 6

We provide supplementary materials, namely Tables A-1 and A-2, for Section 6.

G Supplementary Materials for Section 7

We provide supplementary materials for the two extensions in Sections 7.1 and 7.2.



(a) $\mu - \mu_0$ against μ_0

(b) $D - \mu$ against μ_0

Figure A-2: Forecast adjustment and demand noise vs initial forecast: Outdoor Tableware ($\hat{\alpha} < \hat{\beta}$)

G.1 The Distribution-Free Approach in Section 7.1

In this appendix, we prove Proposition 3 and present the simplified ODE for determining the optimal first-stage order in (3) when the signal distribution is uniform.

Proof of Proposition 3. We first solve the second-stage problem (15). By the definition of $H_x(S_2)$ (see the base model), for any distribution $\Psi_{D_x} \in \mathcal{G}_x$, the minimand in (15) can be written as

$$\begin{aligned}
h(S_2, x) &:= -c_2 S_2 + H_x(S_2) \\
&= p\mathbb{E}\min\{D_x, S_2\} + c_v \mathbb{E}(S_2 - D_x)^+ - c_2 S_2 \\
&= (p - c_v)(\mu_0 + \mu_0^\alpha x) - [(c_2 - c_v)S_2 + (p - c_v)\mathbb{E}(D_x - S_2)^+] \\
&\geq (p - c_v)(\mu_0 + \mu_0^\alpha x) - \left\{ (c_2 - c_v)S_2 + (p - c_v)\frac{\sqrt{\sigma^2 + (S_2 - \mu_0 - \mu_0^\alpha x)^2} - (S_2 - \mu_0 - \mu_0^\alpha x)}{2} \right\} \\
&=: \tilde{h}(S_2, x),
\end{aligned}$$

where the third line follows from $\mathbb{E}D_x = \mu_0 + \mu_0^\alpha x$ and the inequality follows from Lemma 1 in Gallego and Moon (1993), itself a straightforward application of the Cauchy–Schwarz inequality. Moreover, for any S_2, from Lemma 2 in Gallego and Moon (1993), there exists a distribution $\Psi_{D_x} \in \mathcal{G}_x$ such that $h(S_2, x) = \tilde{h}(S_2, x)$. This means that the maximand in (15) is just $\tilde{h}(S_2, x)$. It can be verified that $\tilde{h}(S_2, x)$ is strictly concave in S_2 and hence has a unique unconstrained maximizer

$$\tilde{S}^*(x) = (\mu_0 + \mu_0^\alpha x) + \sigma\kappa_3, \quad \text{where } \kappa_3 = \frac{1}{2}\left[\sqrt{\frac{p - c_2}{c_2 - c_v}} - \sqrt{\frac{c_2 - c_v}{p - c_2}}\right],$$

see (5) in Gallego and Moon (1993). Thus, the solution to (15) can be obtained by $\tilde{S}_2^*(S_1, x) = S_1 \vee \tilde{S}^*(x)$.

Product line	$\alpha=\beta=0$ π^*	$\alpha=\beta=0$ % gap	$\alpha=\beta=1$ π^*	$\alpha=\beta=1$ % gap	our model π^*
BOARD GAMES	277.29	25.07	259.69	0.81	257.69
BEACH ACCESSORIES	10.28	19.68	8.56	2.17	8.14
BEACH TOWELS	37.26	15.05	68.37	66.86	31.95
BEACHTOYS	74.85	15.11	66.24	2.97	62.41
ACTION FIGURES	168.31	20.44	553.56	264.60	139.17
BOOGIE BOARDS	14.17	26.15	11.50	3.60	10.63
DIE CAST	46.13	5.61	250.36	439.90	43.89
GLOW	27.68	2.56	105.07	228.76	26.95
GOGGLES	25.71	27.92	34.48	44.24	19.73
OTHER GAMES	569.76	33.71	480.94	7.98	445.37
OUTDOOR LIGHTING	18.32	2.07	47.67	105.97	17.85
OUTDOOR TABLEWARE	13.24	17.34	14.99	18.31	11.12
PLANT KITS	47.60	6.79	55.18	14.35	43.71
POOL TOYS	61.81	32.10	263.33	351.96	46.24
PUZZLES	54.88	34.70	68.05	51.74	40.67
SEASONAL PLUSH	160.13	0.08	1291.12	628.83	159.99
SQUISHMALLOWS	1642.66	21.63	1394.35	2.51	1354.68
TOY GUNS	55.12	9.76	634.00	1149.13	50.19
TRADING CARDS	66.60	52.17	48.53	12.20	42.42
WATER PLAY	39.65	39.82	82.81	112.08	29.36

Table A-1: Model comparison ("% gap" is with respect to our model)

Now we solve the first-stage problem. Similar to Lemma 1, one can verify that $\tilde{f}(\cdot)$ is concave. Consider the root of $\tilde{f}(S_1)=0$. Define

$$x^*(S_1)=\frac{S_1-\mu_0-\sigma\kappa_3}{\mu_0^\alpha},$$

we get

$$\tilde{S}_2^*\left(S_1,x\right)=\begin{cases}S_1, & \text{if } x\le x^*\left(S_1\right),\\ \tilde{S}^*(x), & \text{if } x>x^*\left(S_1\right).\end{cases}$$

One can verify that

$$\begin{aligned}\tilde{f}'(S_1)&=c_2-c_1-\frac{d}{dS_1}\mathbb{E}_X\Bigg\{(c_2-c_v)\tilde{S}_2^*(S_1,X)\\&\qquad+(p-c_v)\frac{\sqrt{\sigma^2+(\tilde{S}_2^*(S_1,X)-\mu_0-\mu_0^\alpha x)^2}-(\tilde{S}_2^*(S_1,X)-\mu_0-\mu_0^\alpha x)}{2}\Bigg\}\\&=c_2-c_1-(c_2-c_v)\mathbb{E}_X\frac{d\tilde{S}_2^*(S_1,X)}{dS_1}-\frac{p-c_v}{2}\mathbb{E}_X\left[\left(\frac{\tilde{S}_2^*(S_1,X)-\mu_0-\mu_0^\alpha x}{\sqrt{\sigma^2+(\tilde{S}_2^*(S_1,X)-\mu_0-\mu_0^\alpha x)^2}}-1\right)\frac{d\tilde{S}_2^*(S_1,X)}{dS_1}\right]\end{aligned}$$

Product line	NV π^{NV}	NV % gap	NV+ $\pi^{\text{NV+}}$	NV+ % gap	OPT- $\pi^{\text{OPT-}}$	OPT- % gap	OPT π^*
BOARD GAMES	267.72	12.71	261.27	4.71	261.27	4.71	257.69
BEACH ACCESSORIES	10.74	30.40	9.16	11.97	8.45	2.64	8.14
BEACH TOWELS	38.39	20.38	34.41	7.79	34.41	7.79	31.95
BEACHTOYS	72.03	8.70	66.04	3.15	62.71	1.12	62.41
ACTION FIGURES	162.68	16.14	148.03	6.05	148.86	7.10	139.17
BOOGIE BOARDS	14.48	33.51	12.16	13.27	10.96	1.94	10.63
DIE CAST	47.48	9.07	45.16	3.23	45.16	3.23	43.89
GLOW	32.87	23.12	29.06	8.13	29.06	8.13	26.95
GOGGLES	25.56	27.58	22.02	10.78	20.60	3.80	19.73
OTHER GAMES	555.81	28.91	477.92	9.03	477.92	9.03	445.37
OUTDOOR LIGHTING	21.07	19.77	18.91	6.23	18.91	6.23	17.85
OUTDOOR TABLEWARE	14.03	31.10	12.26	12.27	12.26	12.27	11.12
PLANT KITS	47.70	7.50	45.13	2.62	44.68	1.19	43.71
POOL TOYS	65.72	44.95	53.94	17.57	53.94	17.57	46.24
PUZZLES	52.39	27.63	45.27	10.81	42.80	5.38	40.67
SEASONAL PLUSH	166.07	4.24	161.71	1.18	161.71	1.18	159.99
SQUISHMALLOWS	1619.93	19.86	1424.93	5.18	1424.93	5.18	1354.68
TOY GUNS	60.66	20.77	54.20	7.96	53.37	6.34	50.19
TRADING CARDS	63.76	45.54	50.78	17.88	42.42	0.01	42.42
WATER PLAY	39.22	37.45	33.27	14.94	33.27	14.94	29.36

Table A-2: Performance of heuristic policies (gray rows have $\hat{\alpha} > \hat{\beta}$; "% gap" is with respect to π^*)

$$= c_2 - c_1 - \int_{-\infty}^{x^*(S_1)} \left[(c_2 - c_v) + \frac{p - c_v}{2} \left(\frac{S_1 - \mu_0 - \mu_0^\alpha x}{\sqrt{\sigma^2 + (S_1 - \mu_0 - \mu_0^\alpha x)^2}} - 1 \right) \right] \psi_X(x) dx,$$

where $\psi_X(\cdot)$ denotes the PDF of the distribution of X. Change the integration variable to $z = S_1 - \mu_0 - \mu_0^\alpha x$, i.e.,

$$x = \frac{S_1 - \mu_0 - z}{\mu_0^\alpha}, \quad \frac{dx}{dz} = -\mu_0^{-\alpha},$$

and let

$$R(z) := c_2 - c_v + \frac{p - c_v}{2} \left[\frac{z}{\sqrt{\sigma^2 + z^2}} - 1 \right] = \frac{2c_2 - c_v - p}{2} + \frac{p - c_v}{2} \frac{z}{\sqrt{\sigma^2 + z^2}}.$$

Then

$$\tilde{f}'(S_1) = c_2 - c_1 - \frac{1}{\mu_0^\alpha} \int_{\sigma\kappa_3}^{\infty} R(z) \psi_X \left(\frac{S_1 - \mu_0 - z}{\mu_0^\alpha} \right) dz. \tag{A-33}$$

We thus obtain the following ODE for $\tilde{S}_1^*$ that solves $\tilde{f}'(S_1) = 0$:

$$\frac{d\tilde{S}_1^*}{d\alpha} = -\frac{\frac{\partial f'(S_1)}{\partial \alpha}}{\frac{\partial f'(S_1)}{\partial S_1}}$$

$$= \frac{-\mu_0^\alpha \log \mu_0 \left[\int_{\sigma\kappa_3}^\infty R(z)\psi_X\left(\frac{S_1-\mu_0-z}{\mu_0^\alpha}\right)dz + \frac{1}{\mu_0^\alpha}\int_{\sigma\kappa_3}^\infty R(z)\psi_X'\left(\frac{S_1-\mu_0-z}{\mu_0^\alpha}\right)(S_1-\mu_0-z)dz\right]}{\int_{\sigma\kappa_3}^\infty R(z)\psi_X'\left(\frac{S_1-\mu_0-z}{\mu_0^\alpha}\right)dz}.$$

The boundary condition can be obtained by solving $\tilde{f}'(S_1) = 0$ when $\alpha = -\infty$.

□

We now simplify the ODE in Proposition 3 when the signal X follows a uniform distribution. Assume X is uniform on the interval $[-v, v]$ with density $\psi_X(x) = \frac{1}{2v}$. Rewrite (A-33) as

$$\tilde{f}'(S_1) = c_2 - c_1 - \frac{1}{\mu_0^\alpha}\int_{\sigma\kappa_3}^\infty R(z)\psi_X(x)dz. \tag{A-34}$$

Note that

$$-v \le x \le v \iff -v \le \frac{z - S_1 + \mu_0}{\mu_0^\alpha} \le v \iff S_1 - \mu_0 - v\mu_0^\alpha \le z \le S_1 - \mu_0 + v\mu_0^\alpha.$$

The ODE takes three different forms depending on the location of S_1:

(a) $S_1 - \mu_0 + v\mu_0^\alpha \le \sigma\kappa_3$: In this case, as the entire support of $X \in [-v, v] \Rightarrow z \le \sigma\kappa_3$, i.e., for $z \in (\sigma\kappa_3, \infty)$, the corresponding $x \notin [-v, v]$ and therefore $\psi_X(x) = 0$. On this half line, $\tilde{f}'(S_1) = c_2 - c_1$. Since α and μ_0 do not appear in this equation, $\tilde{S}_1^*$ is a constant in α and μ_0.

(b) $S_1 - \mu_0 - v\mu_0^\alpha \le \sigma\kappa_3 < S_1 - \mu_0 + v\mu_0^\alpha$: In this case, the relevant range for z, where the corresponding value of $X \in [-v, v]$, is $z \in [\sigma\kappa_3, S_1 - \mu_0 + v\mu_0^\alpha]$. We have

$$\begin{aligned}\tilde{f}'(S_1) &= c_2 - c_1 - \frac{1}{\mu_0^\alpha 2v}\int_{\sigma\kappa_3}^{S_1-\mu_0+v\mu_0^\alpha} R(z)dz \\ &= c_2 - c_1 - \frac{2c_2 - c_v - p}{2}\frac{(S_1 - \mu_0 + v\mu_0^\alpha - \sigma\kappa_3)}{\mu_0^\alpha 2v} - \frac{p - c_v}{2}\frac{\sqrt{\sigma^2 + (S_1 - \mu_0 + v\mu_0^\alpha)^2} - \sqrt{\sigma^2 + \sigma^2\kappa_3^2}}{\mu_0^\alpha 2v}.\end{aligned}$$

The ODE now becomes

$$\frac{d\tilde{S}_1^*}{d\alpha} = -\frac{\frac{\partial \tilde{f}'(S_1)}{\partial\alpha}}{\frac{\partial \tilde{f}'(S_1)}{\partial S_1}} = \log\mu_0 \frac{-\frac{\frac{2c_2-c_v-p}{2}(S_1-\mu_0-\sigma\kappa_3)}{2\mu_0^\alpha v} - \frac{p-c_v}{2}\frac{\sqrt{\sigma^2+(S_1-\mu_0+v\mu_0^\alpha)^2}-\sqrt{\sigma^2+\sigma^2\kappa_3^2}}{2\mu_0^\alpha v} + \frac{p-c_v}{2}\frac{(S_1-\mu_0+v\mu_0^\alpha)}{2\sqrt{\sigma^2+(S_1-\mu_0+v\mu_0^\alpha)^2}}}{-\frac{2c_2-c_v-p}{4\mu_0^\alpha v} - \frac{p-c_v}{4\mu_0^\alpha v}\frac{S_1-\mu_0+v\mu_0^\alpha}{\sqrt{\sigma^2+(S_1-\mu_0+v\mu_0^\alpha)^2}}}.$$

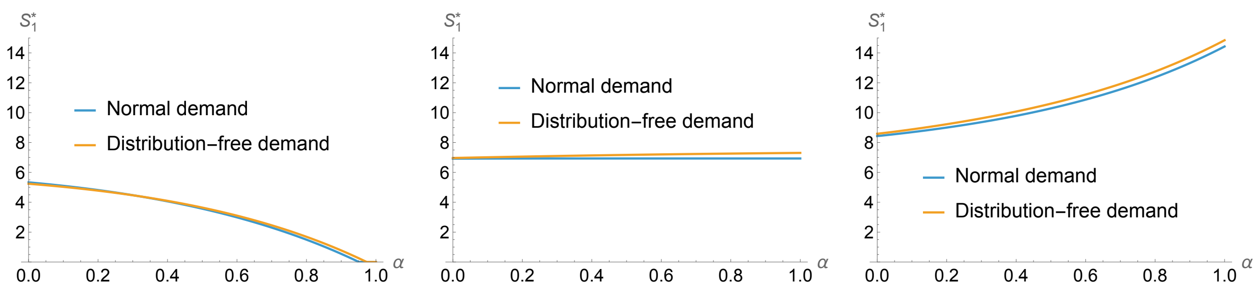
(c) $\sigma\kappa_3 < S_1 - \mu_0 - v\mu_0^\alpha$: In this case,

$$\begin{aligned}\tilde{f}'(S_1) &= c_2 - c_1 - \frac{1}{\mu_0^\alpha 2v}\int_{S_1-\mu_0-v\mu_0^\alpha}^{S_1-\mu_0+v\mu_0^\alpha} R(z)dz \\ &= c_2 - c_1 - \frac{2c_2 - c_v - p}{2} - \frac{p - c_v}{2}\frac{\sqrt{\sigma^2 + (S_1 - \mu_0 + v\mu_0^\alpha)^2} - \sqrt{\sigma^2 + (S_1 - \mu_0 - v\mu_0^\alpha)^2}}{\mu_0^\alpha 2v}.\end{aligned}$$

So the ODE now becomes

$$\frac{d\tilde{S}_1^*}{d\alpha} = -\frac{\frac{\partial \tilde{f}'(S_1)}{\partial \alpha}}{\frac{\partial \tilde{f}'(S_1)}{\partial S_1}} = \log \mu_0 \frac{\frac{\sqrt{\sigma^2+\left(S_1-\mu_0+v\mu_0^\alpha\right)^2}-\sqrt{\sigma^2+\left(S_1-\mu_0-v\mu_0^\alpha\right)^2}}{2\mu_0^\alpha v} - \frac{S_1-\mu_0+v\mu_0^\alpha}{2\sqrt{\sigma^2+\left(S_1-\mu_0+v\mu_0^\alpha\right)^2}} - \frac{S_1-\mu_0-v\mu_0^\alpha}{2\sqrt{\sigma^2+\left(S_1-\mu_0-v\mu_0^\alpha\right)^2}}}{\frac{S_1-\mu_0+v\mu_0^\alpha}{2\mu_0^\alpha v\sqrt{\sigma^2+\left(S_1-\mu_0+v\mu_0^\alpha\right)^2}} - \frac{S_1-\mu_0-v\mu_0^\alpha}{2\mu_0^\alpha v\sqrt{\sigma^2+\left(S_1-\mu_0-v\mu_0^\alpha\right)^2}}}.$$

In our numerical investigations, we have found the distribution-free approach in Proposition 3 to generate values of $\tilde{S}_1^*$ that are quite close to S_1^* which is obtained when the distribution of D_x is known to be normal. As an example, Figure A-3 compares the $\tilde{S}_1^*$ and S_1^* values obtained under the distribution-free approach and under the normal distribution (of D_x) assumption, respectively, for instances with $c_2 = 3, c_v = 1, p = 6, \mu_0 = 5, \sigma = 2$ and $X \sim \mathcal{U}(-3, 3)$. We do this for three values of $c_1 < c_2 = 3$. The figures show that the first stage orders $\tilde{S}_1^*$ and S_1^* under different approaches are indeed close.



(a) $2.5 = c_1 > (c_2 + c_v)/2 : \kappa^* < 0$ **(b)** $2 = c_1 = (c_2 + c_v)/2 : \kappa^* = 0$ **(c)** $1.5 = c_1 < (c_2 + c_v)/2 : \kappa^* > 0$

Figure A-3: $\tilde{S}_1^*(\alpha)$ and $S_1^*(\alpha)$ under uniform signal ($c_v = 1, c_2 = 3, p = 6, \mu_0 = 5, \sigma = 2, v = 3$)

G.2 The Capacitated Model in Section 7.2

In this appendix, we prove Proposition 4 and Lemma 2 in Section 7.2.

Proof of Proposition 4. By standard arguments, e.g., Theorem 1 in Fisher and Raman (1996), the objective function $\hat{f}(S_1)$ in (16) continues to be concave, with maximizer ${\hat{S}_1^*}{}^+ \wedge C_1$ where $\hat{f}'(\hat{S}_1^*) = 0$. Following similar arguments as in the uncapacitated case (see Corollary 1), we get the second-stage unique maximizer being $\hat{S}_2^*(S_1, x) = (S^*(x) \vee S_1) \wedge (S_1 + C_2)$, where (as in the uncapacitated case)

$$S^*(x) = (\mu_0 + \mu_0^\alpha x) + \sigma\kappa_2.$$

Thus,

$$\hat{S}_2^*(S_1, x) = \begin{cases} S_1 & \text{if } x \leq x^*(S_1), \\ S^*(x) & \text{if } x^*(S_1) < x < x^*(S_1 + C_2), \\ S_1 + C_2 & \text{if } x \geq x^*(S_1 + C_2), \end{cases}$$

where

$$x^*(S_1) = (S_1 - \mu_0 - \sigma\kappa_2)/\mu_0^\alpha.$$

Also, $\hat{S}_1^*$ is the unique root of

$$0 = \hat{f}'(S_1) = G(S_1, \alpha) = -c_1 + c_2 + \frac{1}{\tau}\int_{-\infty}^{(S_1-\mu_0-\sigma\kappa_2)/\mu_0^\alpha} \left[-c_2 + p - (p-c_v)\Phi\left(\frac{S_1 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\right]\phi(x/\tau)dx$$
$$+ \frac{1}{\tau}\int_{(S_1+C_2-\mu_0-\sigma\kappa_2)/\mu_0^\alpha}^{\infty} \left[-c_2 + p - (p-c_v)\Phi\left(\frac{S_1 + C_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\right]\phi(x/\tau)dx.$$

As before, we derive an ODE for $\hat{S}_1^*(\alpha)$ by applying the implicit function theorem to this equation:

$$\frac{d\hat{S}_1^*}{d\alpha} = -\frac{\partial G(S_1, \alpha)/\partial\alpha}{\partial G(S_1, \alpha)/\partial S_1}. \tag{A-35}$$

In evaluating the above partial derivatives, while applying the Leibniz integral rule, the following identities allow for significant simplifications. Note that the integrand of the first integral at the integral's upper bound is proportional to

$$-c_2 + p - (p - c_v)\Phi(\kappa_2) = -c_2 + p - (p - c_v)\frac{p - c_2}{p - c_v} = 0.$$

Similarly, the integrand of the second integral at the integral's lower bound equals 0, as well. This implies that

$$\frac{\partial G(S_1, \alpha)}{\partial\alpha} = \frac{\mu_0^\alpha \log\mu_0}{\tau}\frac{p - c_v}{\sigma}\left\{\int_{-\infty}^{x^*(S_1)} x\phi\left(\frac{S_1 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx\right.$$
$$\left. + \int_{x^*(S_1+C_2)}^{\infty} x\phi\left(\frac{S_1 + C_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx\right\},$$
$$\frac{\partial G(S_1, \alpha)}{\partial S_1} = -\frac{1}{\tau}\frac{p - c_v}{\sigma}\left\{\int_{-\infty}^{x^*(S_1)} \phi\left(\frac{S_1 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx\right.$$
$$\left. + \int_{x^*(S_1+C_2)}^{\infty} \phi\left(\frac{S_1 + C_2 - \mu_0 - \mu_0^\alpha x}{\sigma}\right)\phi(x/\tau)dx\right\}.$$

Thus, using (A-35) we obtain the desired ODE in Proposition 4. The boundary condition is obvious from the solution of $\hat{f}'(S_1) = 0$ when $\alpha = -\infty$. □

Proof of Lemma 2. (a) Write the integral as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{C} xe^{-\frac{(\beta x + AB/\beta)^2}{2}} e^{\frac{A^2B^2}{2\beta^2}} e^{-\frac{A^2}{2}} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{A^2}{2(B^2\tau^2+1)}} \int_{-\infty}^{C} x\phi\left(\frac{AB}{\beta} + \beta x\right)dx$$
$$= -e^{-\frac{A^2}{2(B^2\tau^2+1)}}\frac{1}{B^2 + 1/\tau^2}\left\{\phi\left(\frac{AB}{\beta} + \beta C\right) + \frac{AB}{\beta}\Phi\left(\frac{AB}{\beta} + \beta C\right)\right\},$$

since $\int x\phi(a+bx)dx = -\frac{1}{b^2}(\phi(a+bx) + a\Phi(a+bx))$.

(b)

$$\begin{aligned}\int_{-\infty}^{C} \phi(A+Bx)\phi(x/\tau)dx &= \tau\int_{-\infty}^{C/\tau} \phi(A+B\tau y)\phi(y)dy \\ &= \frac{\tau}{\sqrt{B^2\tau^2+1}}\phi\left(\frac{A}{\sqrt{B^2\tau^2+1}}\right) + \Phi\left(\frac{C}{\tau}\sqrt{B^2\tau^2+1} + \frac{AB\tau}{\sqrt{B^2\tau^2+1}}\right),\end{aligned}$$

since $\int \phi(x)\phi(a+bx)dx = \frac{1}{t}\phi(a/t)\Phi(tx+ab/t)$ with $t = \sqrt{1+b^2}$, see Patel and Read (1996). □

H Bounds for S_1^*

In this appendix, we derive two sets of analytical bounds for S_1^* when both D and X are normal distributions. Without loss of generality, we confine ourselves to the case where σ is a constant; that is, $\beta = 0$.

First, Proposition 1 and Lemmas 4 to 6 imply the following bounds (since $F^* > 0$.)

$$\mathrm{LB}_1 := \mu_0 + \kappa^*\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}} \le S_1^* \le \mu_0 + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\left[\kappa^* + F^*\log\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\tau\mu_0^{\alpha}}\right] =: \mathrm{UB}_1. \tag{A-36}$$

Note that the non-constant term within the parentheses above is a monotonically decreasing function of μ_0 since $F^* > 0$ is independent of α and μ_0 and

$$\log\frac{\sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}+\sigma}{\tau\mu_0^{\alpha}} = \log\left[\sqrt{\frac{\sigma^2}{\tau^2\mu_0^{2\alpha}}+1} + \frac{\sigma}{\tau\mu_0^{\alpha}}\right],$$

a function that decreases to 0 as μ_0 grows to ∞.

An alternative set of bounds is provided by Theorem 5 below.

Theorem 5 (Bounds for S_1^*).

$$\begin{aligned}S_1^* \le& \mathrm{UB}_2 := \mu_0 + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\Bigg\{\kappa^* + \sigma\Bigg\{\frac{\kappa_2}{\sigma}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\tau\mu_0^{\alpha}}{\sigma}\right)\right] \\ &+ \frac{\mu_0+\sigma\kappa_2}{\sigma\tau\mu_0^{\alpha}} + \frac{\mu_0+\sigma\kappa_2}{\sigma}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\tau\mu_0^{\alpha}}{\sigma}\right)\right] \\ &+ \frac{1}{\mu_0}\left[\frac{\pi}{2}\left(\sqrt{\frac{\mu_0}{\sigma\kappa_2}+1}-1\right) - \sqrt{\frac{\mu_0}{\sigma\kappa_2}+1}\tan^{-1}\left(\frac{\kappa_2\tau\mu_0^{\alpha}}{\sqrt{\kappa_2^2\sigma^2+\sigma\kappa_2\mu_0}}\right) + \tan^{-1}\left(\frac{\tau\mu_0^{\alpha}}{\sigma}\right)\right]\Bigg\}\Bigg\} \\ S_1^* \ge& \mathrm{LB}_2 := \mu_0 + \sqrt{\sigma^2+\tau^2\mu_0^{2\alpha}}\Bigg\{\kappa^* + \sigma\Bigg\{\frac{1}{\sqrt{2\pi}}\int_{\max\{y_0,\tau\mu_0^{\alpha}\}}^{\infty}\frac{1-\frac{C(y)^2}{2}}{\sigma^2+y^2}dy\end{aligned}$$

$$+\int_{\tau\mu_0^\alpha}^{\max\{y_0,\tau\mu_0^\alpha\}} \frac{\sigma\kappa_2 y^2+\sigma^2(\mu_0+\sigma\kappa_2-\overline{S}_1)}{\sigma y(\sigma^2+y^2)^{3/2}}dy\Bigg\}\Bigg\},$$

where $C(y) := \frac{-\sigma\kappa_2 y^2+\sigma^2(\overline{S}_1-\mu_0-\sigma\kappa_2)}{\sigma y\sqrt{\sigma^2+y^2}}$, *and* y_0 *the unique root of* $\frac{1}{\sqrt{2\pi}}e^{-\frac{C(y)^2}{2}} = -C(y)$. *These integrals can be obtained in closed form.*

Proof. To prove the upper bound UB_2, we need to bound

$$\int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta = \int_\alpha^\infty \frac{\tau\sigma\mu_0^\theta}{\sigma^2+\tau^2\mu_0^{2\theta}}\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))}d\theta.$$

By Lemma 3 and the proof of Lemma 5, $\frac{\phi(\hat{A}(\theta))}{\Phi(\hat{A}(\theta))} \le \overline{\kappa}_2 + \frac{1}{\overline{\kappa}_2}$, where $\overline{\kappa}_2 = \kappa_2+\sigma(\mu_0+\sigma\kappa_2)\tau^{-2}\mu_0^{-2\alpha}$. We first apply a change of variables $y=\tau\mu_0^\theta$. Note that $dy = y\log\mu_0 d\theta \Rightarrow d\theta = \frac{dy}{y\log\mu_0}$. Thus,

$$\begin{aligned}
\int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta &\le \frac{\sigma}{\log\mu_0}\int_{\tau\mu_0^\alpha}^\infty \frac{1}{\sigma^2+y^2}\left[(\kappa_2+\sigma(\mu_0+\sigma\kappa_2)y^{-2}) + \frac{1}{\kappa_2+\sigma(\mu_0+\sigma\kappa_2)y^{-2}}\right]dy \\
&= \frac{\sigma}{\log\mu_0}\int_{\tau\mu_0^\alpha}^\infty \left[\frac{\kappa_2 y^2+\sigma(\mu_0+\sigma\kappa_2)}{y^2(\sigma^2+y^2)} + \frac{y^2}{(\sigma^2+y^2)(\kappa_2 y^2+\sigma(\mu_0+\sigma\kappa_2))}\right]dy \\
&= \frac{\sigma}{\log\mu_0}\Bigg\{\frac{\kappa_2}{\sigma}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\tau\mu_0^\alpha}{\sigma}\right)\right] + \frac{\mu_0+\sigma\kappa_2}{\tau}\int_{\tau\mu_0^\alpha}^\infty\left(\frac{1}{y^2}-\frac{1}{y^2+\sigma^2}\right)dy \\
&\quad + \int_{\tau\mu_0^\alpha}^\infty \frac{y^2}{(\sigma^2+y^2)(\kappa_2 y^2+\sigma(\mu_0+\sigma\kappa_2))}dy\Bigg\} \\
&= \frac{\sigma}{\log\mu_0}\Bigg\{\frac{\kappa_2}{\sigma}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\tau\mu_0^\alpha}{\sigma}\right)\right] + \frac{\mu_0+\sigma\kappa_2}{\sigma\tau\mu_0^\alpha} + \frac{\mu_0+\sigma\kappa_2}{\sigma}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{\tau\mu_0^\alpha}{\sigma}\right)\right] \\
&\quad + \frac{1}{\mu_0}\left[\frac{\pi}{2}\left(\sqrt{\frac{\mu_0}{\sigma\kappa_2}+1}-1\right) - \sqrt{\frac{\mu_0}{\sigma\kappa_2}+1}\tan^{-1}\left(\frac{\kappa_2\tau\mu_0^\alpha}{\sqrt{\kappa_2^2\sigma^2+\sigma\kappa_2\mu_0}}\right) + \tan^{-1}\left(\frac{\tau\mu_0^\alpha}{\sigma}\right)\right]\Bigg\}.
\end{aligned}$$

To prove the lower bound LB_2: If $\hat{A}(\theta) > 0$,

$$\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}} \ge \frac{\tau\sigma\mu_0^\theta}{\sigma^2+\tau^2\mu_0^{2\theta}}\phi(\hat{A}(\theta)) \ge \frac{\tau\sigma\mu_0^\theta}{\sqrt{2\pi}(\sigma^2+\tau^2\mu_0^{2\theta})}\left[1-\frac{\hat{A}^2(\theta)}{2}\right], \tag{A-37}$$

since $0<\Phi(\hat{A}(\theta))<1$ and $e^{-x}\ge 1-x$ where this last inequality is applied with $x=\frac{\hat{A}^2(\theta)}{2}$. Since e^{-x} is a decreasing function, a further lower bound can be obtained by replacing $\hat{A}(\theta)$ by an upper bound $\overline{A}(\theta) = \frac{\tau^2\mu_0^{2\theta}(-\sigma\kappa_2)+\sigma^2(\overline{S}_1-\mu_0-\sigma\kappa_2)}{\tau\sigma\mu_0^\theta\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}$, where $\overline{S}_1$ is given in Proposition 1. Applying the same change of variables $y=\tau\mu_0^\theta \Rightarrow dy = y\log\mu_0 d\theta$,

$$\frac{\hat{F}(\theta)}{\sqrt{\sigma^2+\tau^2\mu_0^{2\theta}}}d\theta \ge \frac{\sigma}{\sqrt{2\pi}(\log\mu_0)(\sigma^2+y^2)}\left[1-\frac{\overline{A}^2(\theta)}{2}\right]dy$$

and

$$\overline{\hat{A}}(\theta) = \frac{-\kappa_2 y^2 + \sigma(\overline{S}_1 - \mu_0 - \sigma\kappa_2)}{y\sqrt{\sigma^2 + y^2}}.$$

Note that the second term translates into a fourth-degree polynomial divided by a sixth-degree polynomial function and is therefore integrable in closed form. (Note that $\overline{S}_1$ is a function of α by Proposition 1, and not of θ.)

If $\hat{A}(\theta) \leq 0$,

$$\frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}} = \frac{\tau\sigma\mu_0^\theta}{\sigma^2 + \tau^2\mu_0^{2\theta}} \frac{\phi(-\hat{A}(\theta))}{1 - \Phi(-\hat{A}(\theta))} \geq -\frac{\hat{A}(\theta)\tau\sigma\mu_0^\theta}{\sigma^2 + \tau^2\mu_0^{2\theta}} \geq -\frac{\overline{\hat{A}}(\theta)\tau\sigma\mu_0^\theta}{\sigma^2 + \tau^2\mu_0^{2\theta}}, \tag{A-38}$$

since $1 - \Phi(x) \leq \phi(x)/x$ for $x > 0$. Apply the same transformation of variables $y = \tau\mu_0^\theta \Rightarrow dy = y \log \mu_0 d\theta$,

$$\frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}} d\theta \geq \frac{\sigma}{\log \mu_0 (\sigma^2 + y^2)} \frac{\kappa_2 y^2 + \sigma(\mu_0 + \sigma\kappa_2 - \overline{S}_1)}{y\sqrt{\sigma^2 + y^2}} dy.$$

Note that both $\int \frac{y}{(\sigma^2+y^2)^{3/2}} dy$ and $\int \frac{dy}{(\sigma^2+y^2)^{3/2} y}$ are analytically integrable:

$$\int \frac{y}{(\sigma^2 + y^2)^{3/2}} dy = -\frac{1}{\sqrt{\sigma^2 + y^2}}, \qquad \int \frac{dy}{(\sigma^2 + y^2)^{3/2} y} = \frac{1}{2\sigma^3} \log \frac{\sqrt{\sigma^2 + y^2} - \sigma}{\sqrt{\sigma^2 + y^2} + \sigma} + \frac{1}{\sigma^2\sqrt{\sigma^2 + y^2}}.$$

(This can be verified by applying the change of variables $z = \sqrt{\sigma^2 + y^2}$.)

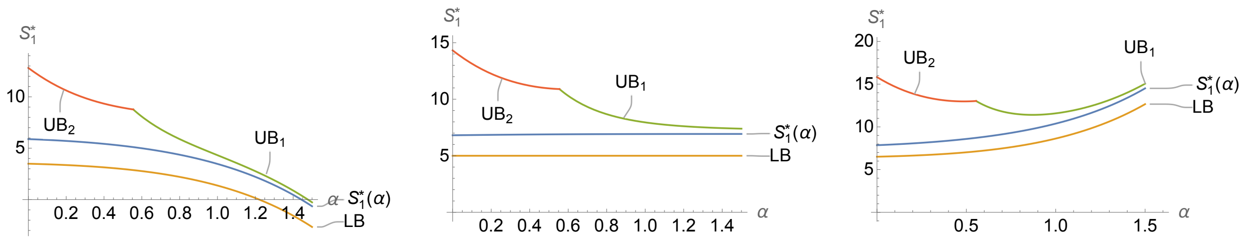
Thus, since $\hat{A}(\theta) \leq 0$ or $\hat{A}(\theta) > 0$, by the combination of (A-37) and (A-38),

$$\begin{aligned} \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}} d\theta &\geq \frac{\sigma}{\log \mu_0} \int_\alpha^\infty \frac{\tau\sigma\mu_0^\theta \min\{-\overline{\hat{A}}(\theta), \phi(\overline{\hat{A}}(\theta))\}}{\sigma^2 + \tau^2\mu_0^{2\theta}} d\theta \\ &= \frac{\sigma}{\log \mu_0} \int_{\tau\mu_0^\alpha}^\infty \frac{\min\{-C(y), \phi(C(y))\}}{\sigma^2 + y^2} dy, \end{aligned} \tag{A-39}$$

where $C(y) := \overline{\hat{A}}(\theta) = \frac{-\sigma\kappa_2 y^2 + \sigma^2(\overline{S}_1 - \mu_0 - \sigma\kappa_2)}{\sigma y\sqrt{\sigma^2+y^2}}$. To analyze for which regions of y the minimum is achieved by the first or second term, let x_0 be the unique root of the equation $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x$ ($x_0 \approx -0.37$). Also since $C(y)$ is a decreasing function of y, there exists *at most* one root y_0 of the equation $C(y) = x_0$, and by (A-39),

$$\begin{aligned} \int_\alpha^\infty \frac{\hat{F}(\theta)}{\sqrt{\sigma^2 + \tau^2\mu_0^{2\theta}}} d\theta &\geq \frac{\sigma}{\log \mu_0} \left\{ \int_{\max\{y_0, \tau\mu_0^\alpha\}}^\infty \frac{\phi(C(y))}{\sigma^2 + y^2} dy + \int_{\tau\mu_0^\alpha}^{\max\{y_0, \tau\mu_0^\alpha\}} \frac{-C(y)}{\sigma^2 + y^2} dy \right\} \\ &\geq \frac{\sigma}{\log \mu_0} \left\{ \frac{1}{\sqrt{2\pi}} \int_{\max\{y_0, \tau\mu_0^\alpha\}}^\infty \frac{1 - \frac{C(y)^2}{2}}{\sigma^2 + y^2} dy + \int_{\tau\mu_0^\alpha}^{\max\{y_0, \tau\mu_0^\alpha\}} \frac{\sigma\kappa_2 y^2 + \sigma^2(\mu_0 + \sigma\kappa_2 - \overline{S}_1)}{\sigma y(\sigma^2 + y^2)^{3/2}} dy \right\}. \end{aligned}$$

Both terms have been shown to be integrable. □



(a) $2.5 = c_1 > (c_2 + c_v)/2 : \kappa^* < 0$ **(b)** $2 = c_1 = (c_2 + c_v)/2 : \kappa^* = 0$ **(c)** $1.5 = c_1 < (c_2 + c_v)/2 : \kappa^* > 0$

Figure A-4: $S_1^*(\alpha)$ with upper and lower bounds ($c_v = 1, c_2 = 3, p = 6, \mu_0 = 5, \sigma = 2, \tau = 1$)

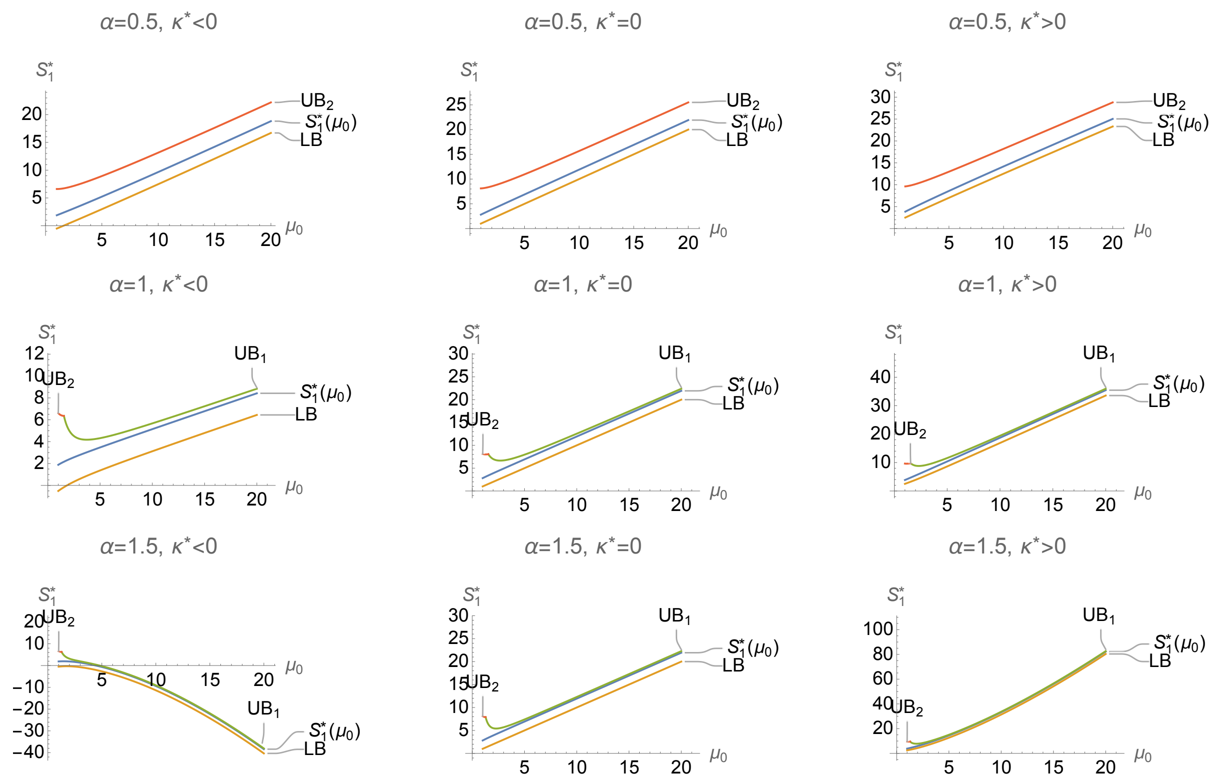


Figure A-5: $S_1^*(\mu_0)$ with upper and lower bounds ($c_v = 1, c_2 = 3, p = 6, \sigma = 2, \tau = 1$)

None of the two upper bounds dominates throughout the parameter space which is why we recommend computing both and determining the maximum of the two values. The first upper bound UB_1 has the advantage of employing the tighter upper bound for the hazard function of the standard normal distribution, see Lemma 3. The second upper bound UB_2 has the advantage of allowing for a tighter upper bound of the function $\hat{F}(\cdot)$ than by a simple constant F^*.

Figure A-4 shows that the combined upper bound $\text{UB} = \min\{\text{UB}_1, \text{UB}_2\}$ is quite accurate, in particular for values of $\alpha \geq 0.5$. As α varies, we have observed that the simpler upper bound UB_1, see (A-36), is best beyond a certain threshold value of α. This threshold is also generally in the proximity of $\alpha = 0.5$. Figure A-4 displays this for three instances which vary in terms of the assumed value of c_1, the initial per-unit procurement price, corresponding with $\kappa^* < 0$ (Figure A-4 (a)), $\kappa^* = 0$ (Figure A-4 (b)) and $\kappa^* > 0$ (Figure A-4 (c)).

These results are confirmed by Figure A-5, where we display the bounds (and the exact value of S_1^*) as a function of the mean μ_0, again for the three sets of cost and price parameters (which vary only in terms of c_1) used above. These correspond with the cases $\kappa^* < 0, \kappa^* = 0$ and $\kappa^* > 0$ in the left, middle and right panel, respectively. Figure A-5 consists of nine plots, with the upper (middle, lower) panel corresponding with $\alpha = 0.5$ ($\alpha = 1$, $\alpha = 1.5$) respectively.

When $\alpha = 0.5$, the second upper bound is uniformly the tighter bound. When $\alpha = 1$ or $\alpha = 1.5$, the first upper bound is the tighter one—and indeed very tight—except for very low values of μ_0. These results are fully consistent with our observations regarding Figure A-4. When $\alpha = 1.5 > 1$ and $\kappa^* < 0$, S_1^* quickly becomes increasingly negative, so that the optimal initial order $S_1^{*+} = 0$. Under these parameter settings, it is optimal to defer the entire procurement to the second stage. Note that in the chosen instances when $\kappa^* < 0$, $\kappa^* = -0.67 > -1$ so that, as shown in Theorem 2 (a), S_1^* continues to be asymptotically linearly increasing. This pattern for $\alpha = 1$ would be reversed only when the initial price c_1 is extremely close to $c_2 = 3$, namely when $c_1 > 2.68$ so that $\kappa^* < -1$.

As far as the lower bounds are concerned, we only display the simpler lower bound $\mathrm{LB}_1 = \mu_0 + \kappa^* \sqrt{\sigma^2 + \tau^2 \mu_0^{2\alpha}}$, see (A-36), as it is very accurate by itself. $\mathrm{LB}_2 \geq \mathrm{LB}_1$, as an even better lower bound, is even more accurate. Displaying LB_2, in addition to LB_1, would clutter most plots as the different graphs sit on top of each other for most cases.

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