Electronic Companion

Recommender Systems under Privacy Protection

Can Küçükgül
Rutgers School of Business - Camden, New Jersey 08102
can.kucukgul@rutgers.edu

Özalp Özer
FBA Science, Selling Partner Services, Amazon, Seattle, Washington 98109
Naveen Jindal School of Management, The University of Texas at Dallas, Richardson, Texas 75080
ozalpo@amazon.com

Shouqiang Wang
Naveen Jindal School of Management, The University of Texas at Dallas, Richardson, Texas 75080
Shouqiang.Wang@utdallas.edu

Appendix A: Allocative Recommendation

In this section, we examine the benchmark setting, where users are perfectly informed about the product type and the platform's recommendations are *purely allocative*. Under *purely allocative* recommendation mode, users are able to observe the product type i and the platform's recommendation only controls the product's accessibility. When being presented with a recommended product i (with probability $\sigma(i,\widehat{t})$), the user of type t makes the transaction if and only if the product's value v_i^t exceeds the outside option u. Thus, the user's transaction rate is given by

$$\int_0^1 \mathbb{1}\{u \le v_i^t\}\, du = v_i^t, \tag{A.1}$$

namely the product value to the user. Accordingly, the user t's expected utility from reporting her type as $\widehat{t}$ is given by

$$V^a\left(t,\widehat{t}\,\middle|\,\sigma\right) \equiv \sum_{i\in\{L,R\}} \mathbb{P}[i] \int_0^1 \left[\sigma(i,\widehat{t})\max\{u,v_i^t\} + \left(1-\sigma(i,\widehat{t})\right)u\right] du. \tag{A.2}$$

Product i is accessible with probability $\sigma(i,\widehat{t}_k^a)$ and the transaction rate is given by v_i^t in (A.1). Thus, the platform's problem is

$$\Pi_k^a := \max_{\sigma} \mathbb{E}\left[\pi_i \sigma(i,\widehat{t}_k^a) v_i^t \,\middle|\, \sigma\right] \quad \text{subject to (A.2) and (6)}, \quad \text{for privacy regime } k\in\{U,O,S,N\}, \tag{$\mathcal{P}_k^a$}$$

where the expectation (as well as the following ones) is taken over all product-user type pairs (i,t). The platform decides whether to offer a user the access to a product ($d=1$) or not ($d=0$) by solving ($\mathcal{P}_k^a$). As shown in the following theorem (its proof is relegated to the end of the section), we find that in this case, the privacy regime makes no difference in the platform's optimal recommender policy.

Proposition A.1 (Optimal allocative policy). *Regardless of privacy regime, the platform's optimal* purely allocative *recommender policy adopts the "spam" strategy, i.e., $\sigma_k^a(i,\widehat{t})\equiv 1$ for $k\in\{U,O,S,N\}$, $i\in\{L,R\}$ and $\widehat{t}\in\{\ell,r,n\}$.*

In this benchmark, the platform's recommendation cannot influence users' inference about the product type that is perfectly observable to them. Thus, the transaction rate is independent of the platform's recommender policy (and hence the user's reported type) as shown by (A.1). As a result, the platform does not need to trade off the recommendation rate for the user's transaction rate, and can simply maximize all products'

recommendation rate to achieve the highest revenue, hence a spam strategy. Yet, such a spam strategy also benefits the users by offering them full access to both types of products. In fact, the non-personalized nature of the spam strategy renders the privacy regime completely irrelevant.

The optimal allocative policy sets a meaningful benchmark for the subsequent sections: any deviation from it when the recommendation mode is no longer only allocative must involve reduction of some recommendation rates below the maximal level due to the recommendation's informational role in influencing the user's transaction rate.

Proof of Proposition A.1. Expanding the expectation over all product-user type pairs (i,t), with respect to the probability structure given in Table 1, the platform's problem $(\mathcal{P}_k^a)$ can be written as

$$\max_{\sigma\in[0,1]} \lambda\alpha\pi_L v_L^\ell \sigma(L,\widehat{\ell}) + \lambda(1-\alpha)\pi_R v_R^\ell \sigma(R,\widehat{\ell}) + (1-\lambda)\alpha\pi_L v_L^r \sigma(L,\widehat{r}) + (1-\lambda)(1-\alpha)\pi_R v_R^r \sigma(R,\widehat{r})$$

$$\text{subject to (A.2) and (6)}, \quad \text{for } k\in\{U,O,S,N\}.$$

Then setting (to the extent the privacy mode permits) $\sigma(i,\widehat{t})\equiv 1$ for $i\in\{L,R\}$ and $\widehat{t}\in\{\ell,r,n\}$ would maximize the objective function for privacy modes $\{U,O,S,N\}$. Such a symmetric policy would induce, utilizing (4), $V^m\left(t,\widehat{t}=\ell \,\middle|\, \sigma\right) = V^m\left(t,\widehat{t}=r \,\middle|\, \sigma\right) = V^m\left(t,\widehat{t}=n \,\middle|\, \sigma\right)$ for both users $t\in\{\ell,r\}$. Correspondingly, both users $t\in\{\ell,r\}$ find it optimal to set $\widehat{t}_k^a\equiv t$ for modes $k\in\{U,O,S\}$, and set $\widehat{t}_k^a\equiv n$ for mode $k=N$. Thus, (6) is satisfied. □

Appendix B: Proofs in Section 4

Before presenting the proofs of Section 4, we give supplementary derivations that we utilize both in the main body of the paper and in the remainder of the Appendix. For the purely informative privacy mode, direct application of Bayes rule yields

$$\begin{aligned}
\mathbb{P}\left[i=L \,\middle|\, \widehat{t},d=1,\sigma\right] &= 1-\mathbb{P}\left[i=R \,\middle|\, \widehat{t},d=1,\sigma\right] = \alpha\sigma(L,\widehat{t})/\mathbb{P}\left[d=1 \,\middle|\, \widehat{t},\sigma\right], \quad \text{and}\\
\mathbb{P}\left[i=L \,\middle|\, \widehat{t},d=0,\sigma\right] &= 1-\mathbb{P}\left[i=R \,\middle|\, \widehat{t},d=0,\sigma\right] = \alpha(1-\sigma(L,\widehat{t}))/\mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right],
\end{aligned} \tag{B.1}$$

where

$$\mathbb{P}\left[d=1 \,\middle|\, \widehat{t},\sigma\right] = 1-\mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right] = \alpha\sigma(L,\widehat{t}) + (1-\alpha)\sigma(R,\widehat{t}). \tag{B.2}$$

Proof of Lemma 1. First, we expand in (B.3) and (B.4), both users' utility function (4) and the platform's objective function $(\mathcal{P}_k^m)$, respectively, for a general message space $\mathcal{M}$:

$$\begin{aligned}
V^m\left(t,\widehat{t} \,\middle|\, \sigma\right) &= \frac{1}{2}\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\left(\mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d,\sigma\right]^2+1\right)\right\}\\
&= \frac{1}{2}\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\left(\left[v_L^t\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right] + v_R^t\left(1-\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]\right)\right]^2+1\right)\right\}\\
&= \frac{1}{2}\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\left((v_L^t-v_R^t)^2\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2 + 2v_R^t(v_L^t-v_R^t)\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right] + (v_R^t)^2+1\right)\right\}\\
&= \frac{1}{2} + \frac{(v_R^t)^2}{2} + \alpha v_R^t(v_L^t-v_R^t) + \frac{(v_L^t-v_R^t)^2}{2}\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2\right]
\end{aligned} \tag{B.3}$$

and

$$\max_{\sigma}\ \lambda\left\{\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{\ell},\sigma\right]\mathbb{E}\left[\pi_i\,\middle|\,\widehat{\ell},d,\sigma\right]\mathbb{E}\left[v_i^{\ell}\,\middle|\,\ell,\widehat{\ell},d,\sigma\right]\right\}+(1-\lambda)\left\{\sum_{d\in\mathcal{M}}\mathbb{P}\left[s\,|\,\widehat{r},\sigma\right]\mathbb{E}\left[\pi_i\,|\,\widehat{r},d,\sigma\right]\mathbb{E}\left[v_i^{r}\,|\,r,\widehat{r},d,\sigma\right]\right\}$$

$$\Leftrightarrow\quad \max_{\sigma}\ \lambda\left\{\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{\ell},\sigma\right]\left[(\pi_L-\pi_R)\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]+\pi_R\right]\left[(v_L^{\ell}-v_R^{\ell})\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]+v_R^{\ell}\right]\right\}$$

$$+(1-\lambda)\left\{\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,|\,\widehat{r},\sigma\right]\left[(\pi_L-\pi_R)\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]+\pi_R\right]\left[(v_L^{r}-v_R^{r})\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]+v_R^{r}\right]\right\}$$

$$\Leftrightarrow\quad \max_{\sigma}\ \lambda\pi_R v_R^{\ell}\left\{1+\alpha(\rho+\beta_1)+\rho\beta_1\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]^2\right]\right\}$$

$$+(1-\lambda)\pi_R v_R^{r}\left\{1+\alpha(\rho+\beta_0)+\rho\beta_0\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]^2\right]\right\}$$

$$\Leftrightarrow\quad \max_{\sigma}\ \lambda(v_L^{\ell}-v_R^{\ell})\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]^2\right]+(1-\lambda)(v_L^{r}-v_R^{r})\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]^2\right], \tag{B.4}$$

where $\beta_1=v_L^{\ell}/v_R^{\ell}-1$ and $\beta_0=v_L^{r}/v_R^{r}-1$.

We now establish the following Claim B.1, whose proof is relegated after the proof of Proposition 1.

Claim B.1. *For a σ utilizing a general message space $\mathcal{M}$, $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]\in[\alpha^2,\alpha]$. Further, $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\alpha$ can only be achieved with a full-disclosure policy, while $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\alpha^2$ can only be achieved with a no-disclosure policy.*

Having Claim B.1, we now show that any $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\mu_{\mathcal{M}}\in[\alpha^2,\alpha]$ for a general messaging space $\mathcal{M}$, can be replicated by a policy utilizing binary messaging space $d\in\{0,1\}$, for any pair of $t\in\{\ell,r\}$ and $\widehat{t}\in\{\ell,r,n\}$, and hence, by (B.3) and (B.4), achieve the same payoff performance both for the users and the platform. Let us first rewrite $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]$ for a policy utilizing binary messaging space $d\in\{0,1\}$:

$$\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\mathbb{P}\left[d=1\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d=1,\sigma\right]^2+\mathbb{P}\left[d=0\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d=0,\sigma\right]^2$$

$$=\frac{\alpha^2\sigma(L,\widehat{t})^2}{\alpha\sigma(L,\widehat{t})+(1-\alpha)\sigma(R,\widehat{t})}+\frac{\alpha^2\left[1-\sigma(L,\widehat{t})\right]^2}{\alpha\left[1-\sigma(L,\widehat{t})\right]+(1-\alpha)\left[1-\sigma(R,\widehat{t})\right]}. \tag{B.5}$$

Now, inputting $\left(\sigma(L,\widehat{t}),\sigma(R,\widehat{t})\right)=\left(1,\frac{\alpha(\alpha-\mu_{\mathcal{M}})}{(1-\alpha)\mu_{\mathcal{M}}}\right)$ to the above equality give us $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\mu_{\mathcal{M}}$, which completes the proof. □

Proof of Claim B.1. It is intuitive to think that, user t, who is subjected to the utility function (B.3), would always prefer a more informative policy (bounded above by full-disclosure policy) over a less informative (bounded below by no-disclosure policy) one. Let us rewrite $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]$:

$$\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\right]=\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$$

$$=\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]^2\mathbb{P}\left[L\,\middle|\,\widehat{t},\sigma\right]^2/\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]$$

$$=\alpha^2\sum_{d\in\mathcal{M}}\frac{\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]^2}{\alpha\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]+(1-\alpha)\mathbb{P}\left[d\,\middle|\,R,\widehat{t},\sigma\right]}. \tag{B.6}$$

We first show that $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$ can always be improved if σ is not a full-disclosure policy, i.e., when there exists a message $d_a \in \mathcal{M}$ that has $\mathbb{P}\left[d_a \mid L,\widehat{t},\sigma\right] = q_a^L > 0$ and $\mathbb{P}\left[d_a \mid R,\widehat{t},\sigma\right] = q_a^R > 0$. In such a case, keeping the rest of σ the same, splitting d_a into two separate messages d_b and d_c with $\mathbb{P}\left[d_b \mid L,\widehat{t},\sigma\right] = q_a^L, \mathbb{P}\left[d_b \mid R,\widehat{t},\sigma\right] = 0$, and $\mathbb{P}\left[d_c \mid L,\widehat{t},\sigma\right] = 0, \mathbb{P}\left[d_c \mid R,\widehat{t},\sigma\right] = q_a^R$ would increase $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$. The improvement of the separation in $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$ can be calculated, utilizing (B.6), as

$$\frac{q_a^L}{\alpha} - \frac{(q_a^L)^2}{\alpha q_a^L + (1-\alpha)q_a^R} \propto \alpha(q_a^L)^2 + (1-\alpha)q_a^L q_a^R - \alpha(q_a^L)^2 = (1-\alpha)q_a^L q_a^R > 0.$$

Iteratively following the separation procedure for any message $d_a \in \mathcal{M}$ that has $\mathbb{P}\left[d_a \mid L,\widehat{t},\sigma\right] > 0$ and $\mathbb{P}\left[d_a \mid R,\widehat{t},\sigma\right] > 0$, σ will become a full-disclosure policy. In other words, any message d with $\mathbb{P}\left[d \mid \widehat{t},\sigma\right] > 0$ would perfectly indicate product $i \in \{L,R\}$. In such a policy, (B.6) can be written as

$$\alpha^2 \sum_{d\in\mathcal{M}} \frac{\mathbb{P}\left[d \mid L,\widehat{t},\sigma\right]^2}{\alpha\mathbb{P}\left[d \mid L,\widehat{t},\sigma\right] + (1-\alpha)\mathbb{P}\left[d \mid R,\widehat{t},\sigma\right]} = \alpha^2 \sum_{d\in\mathcal{M}} \frac{\mathbb{P}\left[d \mid L,\widehat{t},\sigma\right]}{\alpha} = \alpha,$$

where the last equality follows as $\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \mid L,\widehat{t},\sigma\right] = 1$. Hence, for a policy σ that is less informative than the full-disclosure policy, we have $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right] < \alpha$. Formally, $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right] = \alpha$ can only be achieved via a full-disclosure policy, which is a direct consequence of $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$ being convex in $\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]$, and the martingale property.

Next, notice also that, by Jensen's Inequality, we have $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right] \geq \mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]\right]^2$. We then rewrite $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]\right]$ as

$$\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]\right] = \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \mid \widehat{t},\sigma\right]\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right] = \sum_{d\in\mathcal{M}} \mathbb{P}\left[L \mid \widehat{t},\sigma\right]\mathbb{P}\left[d \mid L,\widehat{t},\sigma\right] = \alpha \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \mid L,\widehat{t},\sigma\right] = \alpha,$$

which translates into $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]\right]^2 = \alpha^2$. We note that $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right] = \alpha^2$ can indeed be achieved by a no-disclosure (equivalently, no-information) policy, which, for each message d with $\mathbb{P}\left[d \mid \widehat{t},\sigma\right] > 0$, has $\mathbb{P}\left[d \mid L,\widehat{t},\sigma\right] = \mathbb{P}\left[d \mid R,\widehat{t},\sigma\right] = q_d > 0$, where $\sum_{d\in\mathcal{M}} q_d = 1$. Utilizing (B.6), we calculate $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$ for such a policy as

$$\alpha^2 \sum_{d\in\mathcal{M}} \frac{(q_d)^2}{\alpha q_d + (1-\alpha)q_d} = \alpha^2 \sum_{d\in\mathcal{M}} q_d = \alpha^2.$$

It is straightforward to see that $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right] = \alpha^2$ can only be achieved via a no-disclosure policy, which is a direct consequence of $\mathbb{E}_{\mathcal{M}}\left[\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]^2\right]$ being convex in $\mathbb{P}\left[L \mid \widehat{t},d,\sigma\right]$, and the martingale property. This completes the proof. □

Proof of Proposition 1. Under unprotected privacy, we have $\widehat{t}_U^m = t \in \{\ell, r\}$, by definition. Utilizing (B.4), we can rewrite the objective function ($\mathcal{P}_k^m$) for $k = U$ as

$$\max_{\sigma}\ \lambda(v_L^\ell - v_R^\ell)\mathbb{E}_d\left[\mathbb{P}\left[L \,\middle|\, \widehat{\ell} = \ell, d, \sigma\right]^2\right] + (1-\lambda)(v_L^r - v_R^r)\mathbb{E}_d\left[\mathbb{P}\left[L \mid \widehat{r} = r, d, \sigma\right]^2\right].$$

As $v_L^\ell > v_R^\ell$ and $v_L^r < v_R^r$, utilizing Claim B.1, it is optimal for the platform to have $\mathbb{E}_d\left[\mathbb{P}\left[L \,\middle|\, \widehat{\ell} = \ell, d, \sigma\right]^2\right] = \alpha$ and $\mathbb{E}_d\left[\mathbb{P}\left[L \mid \widehat{r} = r, d, \sigma\right]^2\right] = \alpha^2$. Hence, again by Claim B.1 and (B.5), the optimal policy is given by

$$\sigma_U^m(L,\ell) = \sigma_U^m(i,r) = 1 \text{ for } i \in \{L,R\} \quad \text{and} \quad \sigma_U^m(R,\ell) = 0.$$

Under opt-out privacy protection, the platform's problem $(\mathcal{P}_k^m)$ for $k=O$, is a constrained version of the platform's problem $(\mathcal{P}_k^m)$ for $k=U$. Hence, if we are able to show that the platform can have $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]^2\right]=\alpha$ and $\mathbb{E}_d\left[\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]^2\right]=\alpha^2$ incentive compatibly, then we reach an optimal solution. Setting

$$\sigma_O^m(L,\ell)=\sigma_O^m(i,r)=1 \text{ for } i\in\{L,R\} \quad \text{and} \quad \sigma_O^m(R,\ell)=0,$$

and $(\sigma_O^m(L,n),\sigma_O^m(R,n))=(1,1)$, makes, by (B.5), $\alpha=\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell}=\ell,d,\sigma\right]^2\right]=\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell}=n,d,\sigma\right]^2\right]=\alpha^2$ and $\alpha^2=\mathbb{E}_d\left[\mathbb{P}\left[L\,|\,\widehat{r}=r,d,\sigma\right]^2\right]>\mathbb{E}_d\left[\mathbb{P}\left[L\,|\,\widehat{r}=n,d,\sigma\right]^2\right]=\alpha^2$. Thus, by (B.3), $V^m\left(\ell,\widehat{\ell}=\ell\,\middle|\,\sigma_O^m\right)>V^m\left(\ell,\widehat{\ell}=n\,\middle|\,\sigma_O^m\right)$ and $V^m\left(r,\widehat{r}=r\,|\,\sigma_O^m\right)=V^m\left(r,\widehat{r}=n\,|\,\sigma_O^m\right)$. Therefore, we have $\widehat{t}_O^m=t\in\{\ell,r\}$, and the optimal policy is given by

$$\sigma_O^m(i,\widehat{t})=\sigma_U^m(i,\widehat{t}) \quad \text{and} \quad \sigma_O^m(i,n)=\sigma_O^m(i,r) \quad \text{for } i\in\{L,R\},\ \widehat{t}\in\{\ell,r\}.$$

This completes the proof. □

Proof of Proposition 2. Under non-disclosure privacy protection, we have $\widehat{t}_N^m=n$, by definition. Utilizing (B.4), we can rewrite the objective function $(\mathcal{P}_k^m)$ for $k=N$ as

$$\max_{\sigma}\ \left\{\lambda(v_L^\ell-v_R^\ell)+(1-\lambda)(v_L^r-v_R^r)\right\}\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right].$$

Thus, utilizing Claim B.1, it is optimal for the platform to have $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right]=\alpha^2$ if $\beta_\lambda\le 0$, and $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right]=\alpha$ otherwise, where $\beta_\lambda=\left[\lambda v_L^\ell+(1-\lambda)v_L^r\right]/\left[\lambda v_R^\ell+(1-\lambda)v_R^r\right]-1$. Hence, again by Claim B.1 and (B.5), the optimal policy is given by

$$\sigma_N^m(L,n)=1 \quad \text{and} \quad \sigma_N^m(R,n)=\begin{cases}1, & \text{if } \beta_\lambda\le 0,\\ 0, & \text{if } \beta_\lambda>0,\end{cases}$$

Under self-disclosure privacy protection, utilizing the revelation principle and (B.4), we can rewrite the objective function $(\mathcal{P}_k^m)$ for $k=S$ as

$$\begin{aligned}\max_{\sigma}\ & \lambda(v_L^\ell-v_R^\ell)\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{\ell},d,\sigma\right]^2\right]+(1-\lambda)(v_L^r-v_R^r)\mathbb{E}_d\left[\mathbb{P}\left[L\,|\,\widehat{r},d,\sigma\right]^2\right]\\ \text{subject to}\quad & \mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=\ell,d,\sigma\right]^2\right]\ge\max\left\{\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=r,d,\sigma\right]^2\right],\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right]\right\}\\ & \mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=r,d,\sigma\right]^2\right]\ge\max\left\{\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=\ell,d,\sigma\right]^2\right],\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right]\right\},\end{aligned}$$

where the constraint (6) is reflected in the first and second line of constraints, for user ℓ and r, respectively. The constraints dictate the optimal solution to have $\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=\ell,d,\sigma\right]^2\right]=\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=r,d,\sigma\right]^2\right]\ge\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=n,d,\sigma\right]^2\right]$. Hence, by Claim B.1, the optimal solution must have

$$\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=\ell,d,\sigma\right]^2\right]=\mathbb{E}_d\left[\mathbb{P}\left[L\,\middle|\,\widehat{t}=r,d,\sigma\right]^2\right]=\begin{cases}\alpha^2, & \text{if } \beta_\lambda\le 0,\\ \alpha, & \text{if } \beta_\lambda>0,\end{cases}$$

Thus, by (B.5), the optimal policy is given by

$$\sigma_S^m(L,\widehat{t})=1 \quad \text{and} \quad \sigma_S^m(R,\widehat{t})=\begin{cases}1, & \text{if } \beta_\lambda\le 0,\\ 0, & \text{if } \beta_\lambda>0,\end{cases}$$

under which, by (B.3), $V^m\left(t,\widehat{t}=\ell \mid \sigma_S^m\right)=V^m\left(t,\widehat{t}=r \mid \sigma_S^m\right)=V^m\left(t,\widehat{t}=n \mid \sigma_S^m\right)$ for $t\in\{\ell,r\}$, and hence, we have $\widehat{t}_S^m=t\in\{\ell,r\}$. With this result, we ultimately show that restricting the platform to truthful type-sharing inducing policies is without loss of optimality, as the platform attains the payoff that it generates under non-disclosure privacy protection. In other words, we prove that the platform could not have earned a higher profit, utilizing any arbitrary policy which may, in fact, induce mimicking behavior in users. This completes the proof. □

Appendix C: Proofs in Section 5

In this section, similar to Appendix B, before presenting the proofs of Section 5, we give supplementary derivations that we utilize in the remainder of the Appendix. We start by introducing a variable transformation for the platform's recommender policy and reformulate the platform's revenue function (under truthful revelation of the user's type) accordingly.

Remark C.1. For brevity, in Appendix C and D, we denote the probability of recommending the product for each realization of $(i,\widehat{t})$ as $\sigma_i^{\widehat{t}}$, as opposed to $\sigma\left(i,\widehat{t}\right)$, given in (1).

Lemma C.1 (Policy variable transformation). *For any policy σ defined in (1), let*

$$x_L^{\widehat{t}}:=\frac{\alpha\sigma_L^{\widehat{t}}}{\sqrt{\alpha\sigma_L^{\widehat{t}}+(1-\alpha)\sigma_R^{\widehat{t}}}} \quad \text{and} \quad x_R^{\widehat{t}}:=\frac{(1-\alpha)\sigma_R^{\widehat{t}}}{\sqrt{\alpha\sigma_L^{\widehat{t}}+(1-\alpha)\sigma_R^{\widehat{t}}}}, \quad \text{for } \widehat{t}\in\{\ell,r,n\}, \tag{C.1}$$

with $x_L^{\widehat{t}}=x_R^{\widehat{t}}=0$ for $\sigma_L^{\widehat{t}}=\sigma_R^{\widehat{t}}=0$. Then, $x=:\left\{x_i^{\widehat{t}}\geq 0: i\in\{L,R\} \text{ and } \widehat{t}\in\{\ell,r,n\}\right\}$ must satisfy

$$x_L^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})\leq\alpha, \quad \text{and} \tag{$H_\alpha^{\widehat{t}}$}$$

$$x_R^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})\leq 1-\alpha, \quad \text{for } \widehat{t}\in\{\ell,r,n\}. \tag{$H_{1-\alpha}^{\widehat{t}}$}$$

Conversely, for any x satisfying ($H_\alpha^{\widehat{t}}$) and ($H_{1-\alpha}^{\widehat{t}}$), the policy

$$\sigma_L^{\widehat{t}}:=x_L^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})/\alpha \text{ and } \sigma_R^{\widehat{t}}:=x_R^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})/(1-\alpha), \quad \text{for} \quad \widehat{t}\in\{\ell,r,n\} \tag{C.2}$$

is well defined and satisfies (C.1).

Proof of Lemma C.1. First, notice by (C.1) and the definition of x that, $x_i^{\widehat{t}}\geq 0$ for all $i\in\{L,R\}$ and $\widehat{t}\in\{\ell,r,n\}$. Second, for $\widehat{t}\in\{\ell,r,n\}$, by (C.1), we have

$$x_L^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})=\alpha\sigma_L^{\widehat{t}}\overset{\text{by (1)}}{\leq}\alpha \quad \text{and} \quad x_R^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})=(1-\alpha)\sigma_R^{\widehat{t}}\overset{\text{by (1)}}{\leq}1-\alpha.$$

Hence x, which is calculated through (C.1) for any σ defined in (1), satisfies ($H_\alpha^{\widehat{t}}$) and ($H_{1-\alpha}^{\widehat{t}}$). Furthermore, for any $x\geq 0$ satisfying ($H_\alpha^{\widehat{t}}$) and ($H_{1-\alpha}^{\widehat{t}}$), the policy σ defined according to (C.2) is well defined, i.e., satisfies (1), as

$$\sigma_L^{\widehat{t}}=x_L^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})/\alpha\overset{\text{by } (H_\alpha^{\widehat{t}})}{\in}[0,1] \quad \text{and} \quad \sigma_R^{\widehat{t}}=x_R^{\widehat{t}}(x_L^{\widehat{t}}+x_R^{\widehat{t}})/(1-\alpha)\overset{\text{by } (H_{1-\alpha}^{\widehat{t}})}{\in}[0,1], \quad \text{for} \quad \widehat{t}\in\{\ell,r,n\}.$$

It is also straightforward to see that (C.1) holds for such a policy σ constructed through (C.2), completing the proof. □

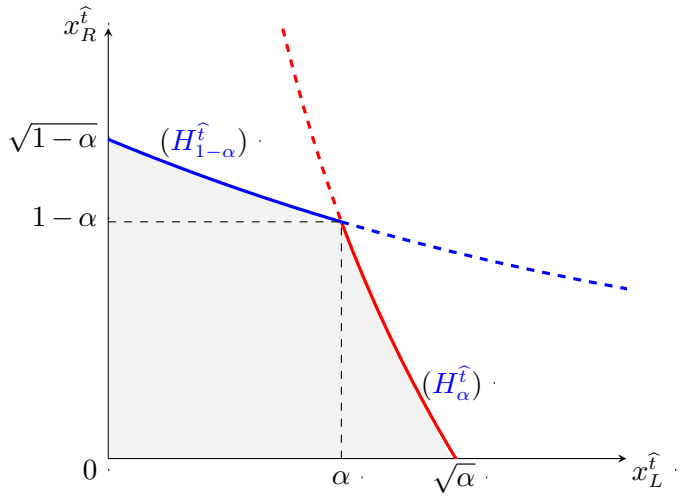


Figure C.1 **The gray-shaded region represents the feasible set of $\left(x_L^{\widehat{t}}, x_R^{\widehat{t}}\right)$ satisfying $(H_{\alpha}^{\widehat{t}})$ and $(H_{1-\alpha}^{\widehat{t}})$, whose respective boundaries are hyperbolas intersecting at $(\alpha, 1-\alpha)$.**

LEMMA C.2 (Reformulation of platform's revenue). *Under the variable transformation in (C.1), the platform's expected revenue given the truthful revelation of user's type (i.e., $\widehat{t}_k^j = t$) can be reformulated as*

$$\begin{aligned}&\mathbb{E}\left[\pi_i \sigma_i^t \mathbb{E}\left[v_i^t \mid t, d=1, \sigma\right] \mid \sigma\right] \\ &\quad = \pi_R[\lambda v_R^{\ell} + (1-\lambda) v_R^{r}] \begin{cases} \lambda' \left[(1+\rho)(1+\beta_1)\left(x_L^{\ell}\right)^2 + (2+\rho+\beta_1) x_L^{\ell} x_R^{\ell} + \left(x_R^{\ell}\right)^2\right] \\ +(1-\lambda')\left[(1+\rho)(1-\beta_r)\left(x_L^{r}\right)^2 + (2+\rho-\beta_r) x_L^{r} x_R^{r} + \left(x_R^{r}\right)^2\right], \end{cases}\end{aligned} \tag{C.3}$$

where

$$\lambda' := \frac{\lambda v_R^{\ell}}{\lambda v_R^{\ell} + (1-\lambda) v_R^{r}} \in [0,1], \quad \rho := \frac{\pi_L}{\pi_R} - 1 > 0, \quad \beta_1 := \frac{v_L^{\ell}}{v_R^{\ell}} - 1 > 0 \quad \textit{and} \quad \beta_r := 1 - \frac{v_L^{r}}{v_R^{r}} \in (0,1]. \tag{C.4}$$

Proof of Lemma C.2. We first utilize the representations of $\mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]$ and $\mathbb{P}\left[i \mid \widehat{t}, d=1, \sigma\right]$ given in Section 3, to express the platform's expected revenue given in $(\mathcal{P}_k^{ma})$. As mentioned above, the expectation is taken over all product-user type pairs (i,t), whose randomness is summarized in Table 1. Accordingly, the platform's expected revenue, with respect to users' optimal reported type $\widehat{t}_k^j \in \{\ell, r, n\}$ for $t \in \{\ell, r\}$, can be expressed as

$$\begin{aligned}&\mathbb{E}\left[\pi_i \sigma_i^{\widehat{t}_k^j} \mathbb{E}\left[v_i^t \mid t, \widehat{t}_k^j, d=1, \sigma\right] \,\middle|\, \sigma\right] = \\ &\quad \lambda\left[\alpha\pi_L \sigma_L^{\widehat{t}_k^j} + (1-\alpha)\pi_R \sigma_R^{\widehat{t}_k^j}\right] \frac{\left[v_L^{\ell}\alpha\sigma_L^{\widehat{t}_k^j} + v_R^{\ell}(1-\alpha)\sigma_R^{\widehat{t}_k^j}\right]}{\alpha\sigma_L^{\widehat{t}_k^j} + (1-\alpha)\sigma_R^{\widehat{t}_k^j}} \\ &\quad + (1-\lambda)\left[\alpha\pi_L \sigma_L^{\widehat{r}_k^j} + (1-\alpha)\pi_R \sigma_R^{\widehat{r}_k^j}\right] \frac{\left[v_L^{r}\alpha\sigma_L^{\widehat{r}_k^j} + v_R^{r}(1-\alpha)\sigma_R^{\widehat{r}_k^j}\right]}{\alpha\sigma_L^{\widehat{r}_k^j} + (1-\alpha)\sigma_R^{\widehat{r}_k^j}}.\end{aligned} \tag{C.5}$$

Therefore, when we condition on $\widehat{t}_k^j = t$ for $t \in \{\ell, r\}$, this expected revenue takes the following form

$$\begin{aligned}&\mathbb{E}\left[\pi_i \sigma_i^t \mathbb{E}\left[v_i^t \mid t, d=1, \sigma\right] \mid \sigma\right] = \\ &\quad \lambda\left[\alpha\pi_L \sigma_L^{\ell} + (1-\alpha)\pi_R \sigma_R^{\ell}\right] \frac{\left[v_L^{\ell}\alpha\sigma_L^{\ell} + v_R^{\ell}(1-\alpha)\sigma_R^{\ell}\right]}{\alpha\sigma_L^{\ell} + (1-\alpha)\sigma_R^{\ell}} + (1-\lambda)\left[\alpha\pi_L \sigma_L^{r} + (1-\alpha)\pi_R \sigma_R^{r}\right] \frac{\left[v_L^{r}\alpha\sigma_L^{r} + v_R^{r}(1-\alpha)\sigma_R^{r}\right]}{\alpha\sigma_L^{r} + (1-\alpha)\sigma_R^{r}}.\end{aligned}$$

Dividing both sides of the above equation to $\pi_R[\lambda v_R^\ell + (1-\lambda)v_R^r]$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\pi_i \sigma_i^t \mathbb{E}\left[v_i^t \mid t, d=1, \sigma\right] \mid \sigma\right] / \{\pi_R[\lambda v_R^\ell + (1-\lambda)v_R^r]\} \\
&= \frac{1}{\lambda v_R^\ell + (1-\lambda)v_R^r} \left\{ \begin{array}{l} \lambda\left[\alpha(1+\rho)\sigma_L^\ell + (1-\alpha)\sigma_R^\ell\right] \frac{\left[v_L^\ell \alpha\sigma_L^\ell + v_R^\ell(1-\alpha)\sigma_R^\ell\right]}{\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell} \\ +(1-\lambda)\left[\alpha(1+\rho)\sigma_L^r + (1-\alpha)\sigma_R^r\right] \frac{\left[v_L^r \alpha\sigma_L^r + v_R^r(1-\alpha)\sigma_R^r\right]}{\alpha\sigma_L^r + (1-\alpha)\sigma_R^r} \end{array} \right. \\
&= \frac{1}{\lambda v_R^\ell + (1-\lambda)v_R^r} \left\{ \begin{array}{l} \lambda v_R^\ell\left[\alpha(1+\rho)\sigma_L^\ell + (1-\alpha)\sigma_R^\ell\right] \frac{\left[\alpha(1+\beta_1)\sigma_L^\ell + (1-\alpha)\sigma_R^\ell\right]}{\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell} \\ +(1-\lambda)v_R^r\left[\alpha(1+\rho)\sigma_L^r + (1-\alpha)\sigma_R^r\right] \frac{\left[\alpha(1-\beta_r)\sigma_L^r + (1-\alpha)\sigma_R^r\right]}{\alpha\sigma_L^r + (1-\alpha)\sigma_R^r} \end{array} \right. \\
&= \underbrace{\frac{\lambda v_R^\ell}{\lambda v_R^\ell + (1-\lambda)v_R^r}}_{\lambda'} \left[(1+\rho)(1+\beta_1)\alpha^2(\sigma_L^\ell)^2 + (2+\rho+\beta_1)\alpha(1-\alpha)\sigma_L^\ell\sigma_R^\ell + (1-\alpha)^2(\sigma_R^\ell)^2\right] / [\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell] \\
&\quad + \underbrace{\frac{(1-\lambda)v_R^r}{\lambda v_R^\ell + (1-\lambda)v_R^r}}_{1-\lambda'} \left[(1+\rho)(1-\beta_r)\alpha^2(\sigma_L^r)^2 + (2+\rho-\beta_r)\alpha(1-\alpha)\sigma_L^r\sigma_R^r + (1-\alpha)^2(\sigma_R^r)^2\right] / [\alpha\sigma_L^r + (1-\alpha)\sigma_R^r] \\
&= \lambda'\left[(1+\rho)(1+\beta_1)(x_L^\ell)^2 + (2+\rho+\beta_1)x_L^\ell x_R^\ell + (x_R^\ell)^2\right] \\
&\quad + (1-\lambda')\left[(1+\rho)(1-\beta_r)(x_L^r)^2 + (2+\rho-\beta_r)x_L^r x_R^r + (x_R^r)^2\right],
\end{aligned}
$$

where the first equality is established through the definition of ρ, the third equality through the definitions of λ', β_1 and β_r, and the last equality through (C.1), completing the proof. □

LEMMA C.3 (Incentive compatibility constraints). *Under the variable transformation in* (C.1)*, the user* $t \in \{\ell, r\}$ *finds it better off to report* t *than to report* $\widehat{t} \in \{\ell, r, n\}$ *if and only if the recommender policy satisfies*

$$(1+\beta_1)x_L^\ell + x_R^\ell \geq (1+\beta_1)x_L^r + x_R^r, \tag{IC$_\ell$}$$

$$(1+\beta_1)x_L^\ell + x_R^\ell \geq (1+\beta_1)x_L^n + x_R^n, \tag{IC$_\ell^n$}$$

$$(1-\beta_r)x_L^r + x_R^r \geq (1-\beta_r)x_L^\ell + x_R^\ell, \quad \textit{and} \tag{IC$_r$}$$

$$(1-\beta_r)x_L^r + x_R^r \geq (1-\beta_r)x_L^n + x_R^n. \tag{IC$_r^n$}$$

Proof of Lemma C.3. First, we rewrite (5), the utility of user $t \in \{\ell, r\}$ from reporting type $\widehat{t}$, as a response to platform's recommender policy σ, as

$$
\begin{aligned}
V^{ma}\left(t, \widehat{t} \mid \sigma\right) &= \mathbb{P}\left[d=1 \mid \widehat{t}, \sigma\right] \int_0^1 \max\left\{u, \mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]\right\} du + \left(1 - \mathbb{P}\left[d=1 \mid \widehat{t}, \sigma\right]\right) \int_0^1 u du \\
&= \mathbb{P}\left[d=1 \mid \widehat{t}, \sigma\right] \left\{ \mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right] \int_0^{\mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]} 1 du - \int_0^{\mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]} u du \right\} + \int_0^1 u du \\
&= \mathbb{P}\left[d=1 \mid \widehat{t}, \sigma\right] \frac{\mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]^2}{2} + \frac{1}{2}.
\end{aligned} \tag{C.6}
$$

Next, utilizing the representations of $\mathbb{E}\left[v_i^t \mid t, \widehat{t}, d=1, \sigma\right]$ and $\mathbb{P}\left[i \mid \widehat{t}, d=1, \sigma\right]$ given in Section 3, we express (C.6) as

$$V^{ma}\left(t, \widehat{t} \mid \sigma\right) = \frac{1}{2}\left\{ \frac{\left[v_L^t \alpha\sigma_L^{\widehat{t}} + v_R^t(1-\alpha)\sigma_R^{\widehat{t}}\right]^2}{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}} + 1 \right\}. \tag{C.7}$$

If part: Having (IC$_\ell$) and (IC$_\ell^n$) satisfied, we first multiply both of their sides by $v_R^\ell/\sqrt{2}$, take the square of both sides, and add $1/2$ to each side to show:

$$(1+\beta_1)x_L^\ell + x_R^\ell \geq (1+\beta_1)x_L^{\widehat{t}} + x_R^{\widehat{t}}$$

$$\Leftrightarrow \quad \left\{\frac{v_R^\ell}{\sqrt{2}}[(1+\beta_1)x_L^\ell + x_R^\ell]\right\}^2 + \frac{1}{2} \geq \left\{\frac{v_R^\ell}{\sqrt{2}}[(1+\beta_1)x_L^{\widehat{t}} + x_R^{\widehat{t}}]\right\}^2 + \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{1}{2}\left\{v_L^\ell x_L^\ell + v_R^\ell x_R^\ell\right\}^2 + \frac{1}{2} \geq \left\{v_L^\ell x_L^{\widehat{t}} + v_R^\ell x_R^{\widehat{t}}\right\}^2 + \frac{1}{2}$$

$$\text{by (C.1)} \quad \Leftrightarrow \quad \underbrace{\frac{1}{2}\left\{\frac{v_L^\ell \alpha\sigma_L^\ell + v_R^\ell(1-\alpha)\sigma_R^\ell}{\sqrt{\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell}}\right\}^2 + \frac{1}{2}}_{V^{ma}(\ell,\ell\,|\,\sigma)} \geq \underbrace{\left\{\frac{v_L^\ell \alpha\sigma_L^{\widehat{t}} + v_R^\ell(1-\alpha)\sigma_R^{\widehat{t}}}{\sqrt{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}}}\right\}^2 + \frac{1}{2}}_{V^{ma}\left(\ell,\widehat{t}\,|\,\sigma\right)},$$

for $\widehat{t} \in \{r, n\}$. Hence, if (IC_ℓ) and (IC_ℓ^n) are satisfied, user ℓ finds it better off to report ℓ than to report $\widehat{t} \in \{r, n\}$. Similarly, having (IC_r) and (IC_r^n) satisfied, we first multiply both of their sides by $v_R^r/\sqrt{2}$, take the square of both sides, and add $1/2$ to each side to show:

$$(1-\beta_r)x_L^r + x_R^r \geq (1-\beta_r)x_L^{\widehat{t}} + x_R^{\widehat{t}}$$

$$\Leftrightarrow \quad \left\{\frac{v_R^r}{\sqrt{2}}[(1-\beta_r)x_L^r + x_R^r]\right\}^2 + \frac{1}{2} \geq \left\{\frac{v_R^r}{\sqrt{2}}[(1-\beta_r)x_L^{\widehat{t}} + x_R^{\widehat{t}}]\right\}^2 + \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{1}{2}\left\{v_L^r x_L^r + v_R^r x_R^r\right\}^2 + \frac{1}{2} \geq \left\{v_L^r x_L^{\widehat{t}} + v_R^r x_R^{\widehat{t}}\right\}^2 + \frac{1}{2}$$

$$\text{by (C.1)} \quad \Leftrightarrow \quad \underbrace{\frac{1}{2}\left\{\frac{v_L^r \alpha\sigma_L^r + v_R^r(1-\alpha)\sigma_R^r}{\sqrt{\alpha\sigma_L^r + (1-\alpha)\sigma_R^r}}\right\}^2 + \frac{1}{2}}_{V^{ma}(r,r\,|\,\sigma)} \geq \underbrace{\left\{\frac{v_L^r \alpha\sigma_L^{\widehat{t}} + v_R^r(1-\alpha)\sigma_R^{\widehat{t}}}{\sqrt{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}}}\right\}^2 + \frac{1}{2}}_{V^{ma}\left(r,\widehat{t}\,|\,\sigma\right)},$$

for $\widehat{t} \in \{\ell, n\}$. Hence, if ($\text{IC}_r$) and ($\text{IC}_r^n$) are satisfied, user r finds it better off to report r than to report $\widehat{t} \in \{\ell, n\}$.

Only if part: When user ℓ finds it better off to report ℓ than to report $\widehat{t} \in \{r, n\}$, by (C.7), we must have

$$V^{ma}(\ell,\ell\,|\,\sigma) = \frac{1}{2}\left\{\frac{[v_L^\ell \alpha\sigma_L^\ell + v_R^\ell(1-\alpha)\sigma_R^\ell]^2}{\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell} + 1\right\} \geq \frac{1}{2}\left\{\frac{\left[v_L^\ell \alpha\sigma_L^{\widehat{t}} + v_R^\ell(1-\alpha)\sigma_R^{\widehat{t}}\right]^2}{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}} + 1\right\} = V^{ma}\left(\ell,\widehat{t}\,|\,\sigma\right)$$

$$\Leftrightarrow \quad \frac{[v_L^\ell \alpha\sigma_L^\ell + v_R^\ell(1-\alpha)\sigma_R^\ell]^2}{\alpha\sigma_L^\ell + (1-\alpha)\sigma_R^\ell} \geq \frac{\left[v_L^\ell \alpha\sigma_L^{\widehat{t}} + v_R^\ell(1-\alpha)\sigma_R^{\widehat{t}}\right]^2}{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}}$$

$$\text{by (C.1)} \quad \Leftrightarrow \quad \left[v_L^\ell x_L^\ell + v_R^\ell x_R^\ell\right]^2 \geq \left[v_L^\ell x_L^{\widehat{t}} + v_R^\ell x_R^{\widehat{t}}\right]^2$$

$$\Leftrightarrow \quad \left\{v_R^\ell\left[(1+\beta_1)x_L^\ell + x_R^\ell\right]\right\}^2 \geq \left\{v_R^\ell\left[(1+\beta_1)x_L^{\widehat{t}} + x_R^{\widehat{t}}\right]\right\}^2$$

$$\Leftrightarrow \quad (1+\beta_1)x_L^\ell + x_R^\ell \geq (1+\beta_1)x_L^{\widehat{t}} + x_R^{\widehat{t}},$$

for $\widehat{t} \in \{r, n\}$. Hence, (IC_ℓ) and (IC_ℓ^n) are satisfied whenever user ℓ finds it better off to report ℓ than to report $\widehat{t} \in \{r, n\}$. Similarly, when user r finds it better off to report r than to report $\widehat{t} \in \{\ell, n\}$, by (C.7), we must have

$$V^{ma}(r,r\,|\,\sigma) = \frac{1}{2}\left\{\frac{[v_L^r \alpha\sigma_L^r + v_R^r(1-\alpha)\sigma_R^r]^2}{\alpha\sigma_L^r + (1-\alpha)\sigma_R^r} + 1\right\} \geq \frac{1}{2}\left\{\frac{\left[v_L^r \alpha\sigma_L^{\widehat{t}} + v_R^r(1-\alpha)\sigma_R^{\widehat{t}}\right]^2}{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}} + 1\right\} = V^{ma}\left(r,\widehat{t}\,|\,\sigma\right)$$

$$\Leftrightarrow \quad \frac{[v_L^r \alpha\sigma_L^r + v_R^r(1-\alpha)\sigma_R^r]^2}{\alpha\sigma_L^r + (1-\alpha)\sigma_R^r} \geq \frac{\left[v_L^r \alpha\sigma_L^{\widehat{t}} + v_R^r(1-\alpha)\sigma_R^{\widehat{t}}\right]^2}{\alpha\sigma_L^{\widehat{t}} + (1-\alpha)\sigma_R^{\widehat{t}}}$$

$$\begin{aligned}\text{by (C.1)} \quad &\Leftrightarrow \quad [v_L^r x_L^r + v_R^r x_R^r]^2 \geq \left[v_L^r x_L^{\widehat{t}} + v_R^r x_R^{\widehat{t}}\right]^2 \\ &\Leftrightarrow \quad \{v_R^r\left[(1-\beta_r)x_L^r + x_R^r\right]\}^2 \geq \left\{v_R^r\left[(1-\beta_r)x_L^{\widehat{t}} + x_R^{\widehat{t}}\right]\right\}^2 \\ &\Leftrightarrow \quad (1-\beta_r)x_L^r + x_R^r \geq (1-\beta_r)x_L^{\widehat{t}} + x_R^{\widehat{t}},\end{aligned}$$

for $\widehat{t} \in \{\ell, n\}$. Hence, ($\text{IC}_r$) and ($\text{IC}_r^n$) are satisfied whenever user r finds it better off to report r than to report $\widehat{t} \in \{r, n\}$, completing the proof. □

Claim C.1.

$$\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}} \geq 2, \text{ and decreasing in } g \geq \sqrt{\alpha}(\beta_1+1), \text{ and}$$
$$\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}} \geq 2, \text{ and decreasing in } g \geq \sqrt{\alpha}(\beta_1+1).$$

Proof of Claim C.1. The derivative of $\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}$ and $\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}$ with respect to g are given by

$$\frac{-[4\alpha\beta_1]^2}{g^2[g^2-4\alpha\beta_1]\sqrt{g^2-4\alpha\beta_1}} \overset{\text{as } g\geq\sqrt{\alpha}(\beta_1+1)}{\leq} 0, \text{ and } \quad \frac{-[4(1-\alpha)\beta_1(1+\beta_1)]^2}{g^2[g^2+4(1-\alpha)\beta_1(1+\beta_1)]\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}} \leq 0,$$

respectively, which establishes that both $\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}$ and $\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}$ are decreasing in $g \geq \sqrt{\alpha}(\beta_1+1)$.

For the second part of the proof, notice, for any $g \geq \sqrt{\alpha}(\beta_1+1)$, that both $\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}$ and $\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}$ are in the form of $k + \frac{1}{k}$ for $k > 0$. Then, it is straightforward to see that

$$(k-1)^2 \geq 0 \;\Leftrightarrow\; k^2+1 \geq 2k \;\Leftrightarrow\; \frac{k^2+1}{k} \geq 2 \;\Leftrightarrow\; k + \frac{1}{k} \geq 2,$$

which establishes that both $\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}} \geq 2$ and $\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}} \geq 2$, completing the proof. □

Lemma C.4 (Users' minimum expected utility). *Under a policy σ with $(\sigma_L^n, \sigma_R^n) = (0,0)$, $V^{ma}\left(t, \widehat{t} \,\middle|\, \sigma\right) \geq V^{ma}\left(t, n \,|\, \sigma\right)$ for all $t, \widehat{t} \in \{\ell, r\}$.*

Proof of Lemma C.4. Note that, by (B.2) and Footnote 10, $(\sigma_L^n, \sigma_R^n) = (0,0)$ makes $\mathbb{P}\left[d=1 \,|\, n, \sigma\right] = 0$ and $\mathbb{E}\left[v_i^t \,|\, t, n, d=1, \sigma\right] = 0$, respectively. Thus, as $\mathbb{P}\left[d=1 \,\middle|\, \widehat{t}, \sigma\right] \geq 0$ and $\mathbb{E}\left[v_i^t \,\middle|\, t, \widehat{t}, d=1, \sigma\right] \geq 0$, we have for all $t, \widehat{t} \in \{\ell, r\}$, $V^{ma}\left(t, \widehat{t} \,\middle|\, \sigma\right) \geq V^{ma}\left(t, n \,|\, \sigma\right) = 1/2$ by (C.6), completing the proof. ■

Lemma C.5 (Revelation Principle). *Solving ($\mathcal{P}_k^{ma}$), for both $k \in \{O, S\}$, it is without loss of optimality for the platform to search among policies σ that induce $\widehat{t}_k^{ma} = t$ for all $t \in \{\ell, r\}$.*

Proof of Lemma C.5. Given any arbitrary policy $\tilde{\sigma}$ as defined by (1), let $\tilde{t}^\star = \underset{\widehat{t} \in \{\ell, r, n\}}{\arg\max}\, V^{ma}\left(t, \widehat{t} \,\middle|\, \tilde{\sigma}\right)$ be the optimal type to report for user $t \in \{\ell, r\}$, under $\tilde{\sigma}$. We then define policy σ as

$$\sigma_i^t = \tilde{\sigma}_i^{\tilde{t}^\star} \quad \text{for } t \in \{\ell, r\} \text{ and } i \in \{L, R\}, \quad \text{and } \sigma_L^n = \sigma_R^n = 0. \tag{C.8}$$

By (C.8) and the fact that $\widehat{t}^\star = t$ for user $t \in \{\ell, r\}$ under σ, it is straightforward to see that both $\tilde{\sigma}$ and σ generate the platform same expected profit, calculated by (C.5). In the remainder of the proof, we show user $t \in \{\ell, r\}$ finds it optimal to report her type as t under σ.

First note that, by (C.8) and (1), σ is well-defined. Since $\tilde{t}^\star$ is the optimal type to report under $\tilde{\sigma}$, we have, by (C.8), $V^{ma}(t, t \mid \sigma) = V^{ma}\left(t, \tilde{t}^\star \mid \tilde{\sigma}\right) \geq V^{ma}\left(t, \widehat{t} \mid \tilde{\sigma}\right)$ for all $t \in \{\ell, r\}$ and $\widehat{t} \in \{\ell, r, n\}$. Now, notice also by (C.8) that, for user ℓ we have $V^{ma}(\ell, \ell \mid \sigma) = V^{ma}\left(\ell, \tilde{\ell}^\star \,\middle|\, \tilde{\sigma}\right)$ and $V^{ma}(\ell, r \mid \sigma) = V^{ma}(\ell, \tilde{r}^\star \mid \tilde{\sigma})$, and for user r we have $V^{ma}(r, r \mid \sigma) = V^{ma}(r, \tilde{r}^\star \mid \tilde{\sigma})$ and $V^{ma}(r, \ell \mid \sigma) = V^{ma}\left(r, \tilde{\ell}^\star \,\middle|\, \tilde{\sigma}\right)$. These immediately imply (i) $V^{ma}(t, t \mid \sigma) \geq V^{ma}\left(t, \widehat{t} \,\middle|\, \sigma\right)$ for $t \in \{\ell, r\}$ and $\widehat{t} \in \{\ell, r\}$. Next, by Lemma C.4 we also have (ii) $V^{ma}\left(t, \widehat{t} \,\middle|\, \sigma\right) \geq V^{ma}(t, n \mid \sigma)$ for all $t \in \{\ell, r\}$ and $\widehat{t} \in \{\ell, r, n\}$. Combining (i) and (ii) we immediately have $V^{ma}(t, t \mid \sigma) \geq V^{ma}\left(t, \widehat{t} \,\middle|\, \sigma\right) \geq V^{ma}(t, n \mid \sigma)$ for $t \in \{\ell, r\}$ and $\widehat{t} \in \{\ell, r, n\}$. Therefore, under σ, user $t \in \{\ell, r\}$ optimally sets $\widehat{t}_k^{ma} = t$. This completes the proof. ■

Proof of Proposition 3. By Lemmas C.1 and C.2, the platform's problem ($\mathcal{P}_k^{ma}$) for $k = U$ can be reformulated as

$$\max_{x_L^\ell, x_R^\ell, x_L^r, x_R^r \geq 0} \text{(C.3)}, \quad \text{subject to } (H_\alpha^{\widehat{t}}) \text{ and } (H_{1-\alpha}^{\widehat{t}}) \text{ for } \widehat{t} \in \{\ell, r\}. \tag{C.9}$$

Since the objective function (C.3) is separable in (x_L^ℓ, x_R^ℓ), which appears only in the first term of (C.3), and in (x_L^r, x_R^r), which appears only in the second term of (C.3), (C.9) further reduces to two separate optimization problems:

$$\max_{x_L^\ell, x_R^\ell \geq 0} (1+\rho)(1+\beta_1)\left(x_L^\ell\right)^2 + (2+\rho+\beta_1) x_L^\ell x_R^\ell + \left(x_R^\ell\right)^2, \quad \text{subject to } (H_\alpha^\ell) \text{ and } (H_{1-\alpha}^\ell), \text{ and} \tag{C.10}$$

$$\max_{x_L^r, x_R^r \geq 0} (1+\rho)(1-\beta_r)\left(x_L^r\right)^2 + (2+\rho-\beta_r) x_L^r x_R^r + \left(x_R^r\right)^2, \quad \text{subject to } (H_\alpha^r) \text{ and } (H_{1-\alpha}^r), \tag{C.11}$$

which can be solved individually as shown below.

Solution to (C.10). Notice that the objective function of (C.10) is quadratic in (x_L^ℓ, x_R^ℓ) with all the coefficients being positive. Thus, it is straightforward to see that, in the optimal benchmark policy, we must have at least one of (H_α^ℓ) and $(H_{1-\alpha}^\ell)$ binding, as otherwise the objective value could be improved without violating the constraints (see Figure C.1). For the case when $(H_{1-\alpha}^\ell)$ is binding, we have $x_R^\ell \in [1-\alpha, \sqrt{1-\alpha}]$ and $x_L^\ell = \frac{1-\alpha}{x_R^\ell} - x_R^\ell$. Substituting this relationship reduces the objective function of (C.10) to

$$\left[\frac{1-\alpha}{x_R^\ell} - x_R^\ell\right]^2 (1+\rho)(1+\beta_1) + x_R^\ell \left[\frac{1-\alpha}{x_R^\ell} - x_R^\ell\right](2+\rho+\beta_1) + (x_R^\ell)^2,$$

whose derivative with respect to x_R^ℓ is given by $x_R^\ell \left[-\frac{(1-\alpha)^2}{(x_R^\ell)^4} 2(1+\rho+\beta_1+\rho\beta_1) + 2\rho\beta_1\right] \leq 0$, as $x_R^\ell \leq \sqrt{1-\alpha}$. Hence, if $(H_{1-\alpha}^\ell)$ is binding, we must have $(x_L^\ell, x_R^\ell) = (\alpha, 1-\alpha)$, which immediately implies (H_α^ℓ) must also be binding. Thus it suffices to consider only the case when (H_α^ℓ) is binding. Then, we have $x_L^\ell \in [\alpha, \sqrt{\alpha}]$ and $x_R^\ell = \frac{\alpha}{x_L^\ell} - x_L^\ell$. Substituting this relationship reduces the objective function of (C.10) to

$$(x_L^\ell)^2 (1+\rho)(1+\beta_1) + x_L^\ell \left[\frac{\alpha}{x_L^\ell} - x_L^\ell\right](2+\rho+\beta_1) + \left[\frac{\alpha}{x_L^\ell} - x_L^\ell\right]^2,$$

whose derivative with respect to x_L^ℓ is given by $2\left[-\frac{\alpha^2}{(x_L^\ell)^3} + \rho\beta_1 x_L^\ell\right]$, and hence is increasing in x_L^ℓ, implying that its maximum must be achieved at either $x_L^\ell = \alpha$ or $x_L^\ell = \sqrt{\alpha}$. In the former case, we have $(x_L^\ell, x_R^\ell) = (\alpha, 1-\alpha)$, which, by (C.2), corresponds to $(\sigma_L^\ell, \sigma_R^\ell) = (1, 1)$ and yields an objective value of

$1+\alpha^2\rho\beta_1$ in (C.10). In the latter case, we have $(x_L^\ell, x_R^\ell) = (\sqrt{\alpha}, 0)$, which, by (C.2), corresponds to $(\sigma_L^\ell, \sigma_R^\ell) = (1,0)$ and yields an objective value of $\alpha(1+\rho\beta_1)$ in (C.10). Therefore, $(\sigma_U^{ma}(L,\ell), \sigma_U^{ma}(R,\ell)) = (1,1)$ (resp., $(\sigma_U^{ma}(L,\ell), \sigma_U^{ma}(R,\ell)) = (1,0)$) is optimal if $1+\alpha^2\rho\beta_1 \geq \alpha(1+\rho\beta_1) \Leftrightarrow \alpha\beta_1\rho \leq 1$ (resp., if $\alpha\beta_1\rho > 1$).

Solution to (C.11). Notice that the objective function of (C.11) is quadratic in (x_L^r, x_R^r) with all the coefficients being positive (as $\beta_r \leq 1$). Thus, it is straightforward to see that, in the optimal benchmark policy, we must have at least one of (H_α^r) and $(H_{1-\alpha}^r)$ binding, as otherwise the objective value could be improved without violating the constraints (see Figure C.1). For the case when (H_α^r) is binding, we have $x_L^r \in [\alpha, \sqrt{\alpha}]$ and $x_R^r = \frac{\alpha}{x_L^r} - x_L^r$. Substituting this relationship reduces the objective function of (C.11) to

$$(x_L^r)^2(1+\rho)(1-\beta_r) + x_L^r\left[\frac{\alpha}{x_L^r} - x_L^r\right](2+\rho-\beta_r) + \left[\frac{\alpha}{x_L^r} - x_L^r\right]^2,$$

whose derivative with respect to x_L^r is given by $2\left[-\rho\beta_r x_L^r - \frac{\alpha^2}{(x_L^r)^3}\right] < 0$. Hence, if (H_α^r) is binding, we must have $(x_L^r, x_R^r) = (\alpha, 1-\alpha)$, which, by (C.2), corresponds to $(\sigma_L^r, \sigma_R^r) = (1,1)$. Similarly, for the case when $(H_{1-\alpha}^r)$ is binding, we have $x_R^r \in [1-\alpha, \sqrt{1-\alpha}]$ and $x_L^r = \frac{1-\alpha}{x_R^r} - x_R^r$. Substituting this relationship reduces the objective function of (C.11) to

$$\left[\frac{1-\alpha}{x_R^r} - x_R^r\right]^2(1+\rho)(1-\beta_r) + x_R^r\left[\frac{1-\alpha}{x_R^r} - x_R^r\right](2+\rho-\beta_r) + (x_R^r)^2,$$

whose derivative with respect to x_R^r is given by $2\left[-\rho\beta_r x_R^r - \frac{(1-\alpha)^2}{(x_R^r)^3}(1+\rho)(1-\beta_r)\right] \leq 0$. Hence, if $(H_{1-\alpha}^r)$ is binding, we must have $(x_L^r, x_R^r) = (\alpha, 1-\alpha)$, which, by (C.2), corresponds to $(\sigma_L^r, \sigma_R^r) = (1,1)$. Thus, in either case, the optimal policy must have $(\sigma_U^{ma}(L,\ell), \sigma_U^{ma}(R,\ell)) = (1,1)$. This completes the proof the first part of the proof.

Next, by Lemma C.5, an optimal policy σ_O^{ma} to $(\mathcal{P}_k^{ma})$ for $k = O$ is given by the solution to

$$\begin{array}{ll} \max\limits_{\sigma} & \mathbb{E}\left[\pi_i \sigma_i^t \mathbb{E}\left[v_i^t \mid t, d=1, \sigma\right] \mid \sigma\right] \\ \text{subject to} & V^{ma}(t,t \mid \sigma) \geq V^{ma}(t,n \mid \sigma) \ \text{ for } t \in \{\ell, r\}. \end{array} \tag{C.12}$$

Clearly, $(\mathcal{P}_k^{ma})$ for $k = U$ is a relaxation of (C.12) by ignoring its constraint. Therefore, since σ_U^{ma} is the solution to $(\mathcal{P}_k^{ma})$ for $k = U$ and the objective function of (C.12) only depends on $(\sigma_L^\ell, \sigma_R^\ell, \sigma_L^r, \sigma_R^r)$, it suffices to show $\sigma_O^{ma}(i,\widehat{t}) = \sigma_U^{ma}(i,\widehat{t})$ and $\sigma_O^{ma}(i,n) = \sigma_O^{ma}(i,\ell)$ for $i \in \{L,R\}$, $\widehat{t} \in \{\ell, r\}$ satisfies the ignored constraint.

Utilizing (C.7), it is straightforward to see that user ℓ will be indifferent between reporting her type as ℓ and n under such scenario, i.e., $V^{ma}(\ell,\ell \mid \sigma_O^{ma}) = V^{ma}(\ell,n \mid \sigma_O^{ma})$. When $\alpha\beta_1\rho \leq 1$, this is also the case for user r too, i.e., $V^{ma}(r,r \mid \sigma_O^{ma}) = V^{ma}(r,n \mid \sigma_O^{ma})$, because $\left(\sigma_O^{ma}(L,\widehat{t}), \sigma_O^{ma}(R,\widehat{t})\right) = (1,1)$ for each $\widehat{t}$, by (9). When $\alpha\beta_1\rho > 1$, however, the fact that $\left(\sigma_O^{ma}(L,\widehat{t}), \sigma_O^{ma}(R,\widehat{t})\right) = (1,0)$ for $\widehat{t} \in \{\ell, n\}$ and $(\sigma_O^{ma}(L,r), \sigma_O^{ma}(R,r)) = (1,1)$ will make user $V^{ma}(r,r \mid \sigma_O^{ma}) \geq V^{ma}(r,n \mid \sigma_O^{ma})$, because

$$\begin{aligned} V^{ma}(r,r \mid \sigma_O^{ma}) - V^{ma}(r,n \mid \sigma_O^{ma}) &= [v_L^r\alpha + v_R^r(1-\alpha)]^2 - \alpha(v_L^r)^2 = (1-\alpha)\left[(1-\alpha)(v_R^r)^2 + 2\alpha v_L^r v_R^r - \alpha(v_L^r)^2\right] \\ &= (1-\alpha)\left[(1-\alpha)(v_R^r)^2 + \alpha v_L^r(2v_R^r - v_L^r)\right] \geq 0, \end{aligned}$$

where the inequality follows by the fact that $v_R^r > v_L^r$. This completes second part of the proof. ■

Proof of Proposition 4. By Lemma C.5, an optimal policy σ_S^{ma} to $(\mathcal{P}_k^{ma})$ for $k=O$ is given by the solution to

$$\begin{aligned} \max_{\sigma} \quad & \mathbb{E}\left[\pi_i \sigma_i^t \mathbb{E}\left[v_i^t \mid t, d=1, \sigma\right] \mid \sigma\right] \\ \text{subject to} \quad & V^{ma}(t,t \mid \sigma) \geq V^{ma}\left(t,\widehat{t} \,\middle|\, \sigma\right) \text{ for } t \in \{\ell, r\},\ \widehat{t} \in \{\ell, r, n\}. \end{aligned} \tag{C.13}$$

Clearly, (C.12) is a relaxation of (C.13) by ignoring the constraints $V^{ma}(\ell,\ell \mid \sigma) \geq V^{ma}(\ell, r \mid \sigma)$ and $V^{ma}(r,r \mid \sigma) \geq V^{ma}(r,\ell \mid \sigma)$. Accordingly, in proving Proposition 4, we first point out the parameter regions where σ_O^{ma}, characterized in Proposition 3, satisfies $V^{ma}(\ell,\ell \mid \sigma_O^{ma}) \geq V^{ma}(\ell, r \mid \sigma_O^{ma})$ and $V^{ma}(r,r \mid \sigma_O^{ma}) \geq V^{ma}(r,\ell \mid \sigma_O^{ma})$, hence, is optimal for (C.13). When $\alpha\beta_1\rho \leq 1$, $(\sigma_O^{ma}(L,\ell), \sigma_O^{ma}(R,\ell)) = (\sigma_O^{ma}(L,r), \sigma_O^{ma}(R,r)) = (1,1)$, it is straightforward to see $V^{ma}(\ell,\ell \mid \sigma_O^{ma}) = V^{ma}(\ell, r \mid \sigma_O^{ma})$ and $V^{ma}(r,r \mid \sigma_O^{ma}) = V^{ma}(r,\ell \mid \sigma_O^{ma})$ and, hence, the ignored constraints are satisfied. When $\alpha\beta_1\rho > 1$, on the other hand, $(\sigma_O^{ma}(L,\ell), \sigma_O^{ma}(R,\ell), \sigma_O^{ma}(L,r), \sigma_O^{ma}(R,r)) = (1,0,1,1)$. In this case $V^{ma}(r,r \mid \sigma_O^{ma}) \geq V^{ma}(r,\ell \mid \sigma_O^{ma})$ is always satisfied as, by (C.7)

$$\begin{aligned} V^{ma}(r,r \mid \sigma_O^{ma}) - V^{ma}(r,\ell \mid \sigma_O^{ma}) &= [v_L^r\alpha + v_R^r(1-\alpha)]^2 - \alpha(v_L^r)^2 \\ &= (1-\alpha)\left[(1-\alpha)(v_R^r)^2 + 2\alpha v_L^r v_R^r - \alpha(v_L^r)^2\right] \\ &= (1-\alpha)\left[(1-\alpha)(v_R^r)^2 + \alpha v_L^r(2v_R^r - v_L^r)\right] \geq 0, \end{aligned}$$

where the inequality follows as $v_R^r > v_L^r$. However, $V^{ma}(\ell,\ell \mid \sigma_O^{ma}) \geq V^{ma}(\ell, r \mid \sigma_O^{ma})$ is only satisfied when

$$\begin{aligned} V^{ma}(\ell,\ell \mid \sigma_O^{ma}) - V^{ma}(\ell, r \mid \sigma_O^{ma}) &= \alpha(v_L^\ell)^2 - [v_L^\ell\alpha + v_R^\ell(1-\alpha)]^2 \\ &= (1-\alpha)\left[\alpha(v_L^\ell)^2 - (1-\alpha)(v_R^\ell)^2 - 2\alpha v_L^\ell v_R^\ell\right] \geq 0 \\ &\Leftrightarrow\ \alpha(v_L^\ell - v_R^\ell)^2 \geq (v_R^\ell)^2 \ \Leftrightarrow\ \alpha\beta_1^2 \geq 1, \end{aligned}$$

where the last relation follows by dividing both sides to $(v_R^\ell)^2$ and utilizing the definition of β_1 given in Lemma C.2. Thus, we have $\sigma_S^{ma} \equiv \sigma_O^{ma} = (1,0,1,1,1,0)$ when $\alpha\beta_1 \min\{\beta_1, \rho\} \geq 1$. In particular, $\beta_1 \geq \rho$ implies $\sigma_S^{ma} \equiv \sigma_O^{ma}$.

In the remainder of the proof, we focus on the complementary parametric region, i.e., $\alpha\beta_1^2 < 1 < \alpha\beta_1\rho$. First, utilizing Lemmas C.1-C.3, we reformulate (C.13) into the following optimization problem

$$\Pi^\star = \max_{x \geq 0} \text{ (C.3)}, \quad \text{subject to } (\text{IC}_\ell), (\text{IC}_\ell^n), (\text{IC}_r), (\text{IC}_r^n), (H_\alpha^{\widehat{t}}) \text{ and } (H_{1-\alpha}^{\widehat{t}}) \text{ for } \widehat{t} \in \{\ell, r, n\}, \tag{C.14}$$

whose solution, denoted as $\dot{x} = (\dot{x}_L^\ell, \dot{x}_R^\ell, \dot{x}_L^r, \dot{x}_R^r, \dot{x}_L^n, \dot{x}_R^n)$, can be inverted, through (C.2), to identify the optimal solution to (C.13). Our solution strategy for (C.14) is to first ignore constraints (IC_r), (IC_ℓ^n) and (IC_r^n) (and, hence, the decision variables x_L^n, x_R^n), and solve its relaxed problem:

$$\max_{x_L^\ell, x_R^\ell, x_L^r, x_R^r \geq 0} \text{ (C.3)}, \quad \text{subject to } (\text{IC}_\ell), (H_\alpha^{\widehat{t}}) \text{ and } (H_{1-\alpha}^{\widehat{t}}) \text{ for } \widehat{t} \in \{\ell, r\}. \tag{C.15}$$

Subsequently, we show that the solution to (C.15), together with a constructed (x_L^n, x_R^n), automatically satisfies the ignored constraints (IC_r), (IC_ℓ^n) and (IC_r^n).

To that end, we reformulate (C.15) into two separate parametrized optimization problems. More specifically, for a constant $g \in [\underline{g}, \overline{g}]$ with

$$\underline{g} = \sqrt{\alpha}(\beta_1 + 1) \quad \text{and} \quad \overline{g} = \alpha\beta_1 + 1, \tag{C.16}$$

we define

$$F^{\ell}(g) := \max_{x_L^{\ell}, x_R^{\ell} \geq 0} (1+\rho)(1+\beta_1)\left(x_L^{\ell}\right)^2 + (2+\rho+\beta_1)x_L^{\ell}x_R^{\ell} + \left(x_R^{\ell}\right)^2 \quad \text{subject to} \quad (1+\beta_1)x_L^{\ell} + x_R^{\ell} = g,\ (H_{\alpha}^{\ell}) \text{ and } (H_{1-\alpha}^{\ell}), \tag{C.17}$$

and

$$F^{r}(g) := \max_{x_L^{r}, x_R^{r} \geq 0} (1+\rho)(1-\beta_r)\left(x_L^{r}\right)^2 + (2+\rho-\beta_r)x_L^{r}x_R^{r} + \left(x_R^{r}\right)^2 \quad \text{subject to} \quad (1+\beta_1)x_L^{r} + x_R^{r} \leq g,\ (H_{\alpha}^{r}) \text{ and } (H_{1-\alpha}^{r}), \tag{C.18}$$

whose optimal solutions are denoted as $(x_L^{\ell\star}(g), x_R^{\ell\star}(g))$ and $(x_L^{r\star}(g), x_R^{r\star}(g))$, and characterized by the following Lemmas C.6 and C.7, whose proofs are relegated after the proof of Proposition 4, respectively.

LEMMA C.6. *For a given $g \in [\underline{g}, \overline{g}]$, the optimal solution to (C.17) is given by*

$$\left(x_L^{\ell\star}(g), x_R^{\ell\star}(g)\right) = \begin{cases} \left(\frac{g-\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, \frac{g(\beta_1-1)+(\beta_1+1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}\right), & \text{if } \beta_1 < 1 \text{ or } \beta_1 \geq 1 \text{ and } g > \underline{g}, \\ \left(\frac{g+\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, \frac{g(\beta_1-1)-(\beta_1+1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}\right), & \text{if } \beta_1 \geq 1 \text{ and } g = \underline{g}. \end{cases} \tag{C.19}$$

Furthermore, the corresponding optimal objective value for (C.17) is given by

$$F^{\ell}(g) = g \begin{cases} \frac{g(\rho+\beta_1)-(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, & \text{if } \beta_1 < 1 \text{ or } \beta_1 \geq 1 \text{ and } g > \underline{g}, \\ \frac{g(\rho+\beta_1)+(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, & \text{if } \beta_1 \geq 1 \text{ and } g = \underline{g}. \end{cases} \tag{C.20}$$

LEMMA C.7. *Define*

$$A = \frac{\beta_1^2+\rho\beta_r}{(\beta_1+\beta_r)(\rho-\beta_1)} \in [0,1], \quad B = \frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)} \geq 1,$$

$$\begin{aligned} \left(x_{L_0}^r(g), x_{R_0}^r(g)\right) &= (0, g), \\ \left(x_{L_{1-\alpha}}^r(g), x_{R_{1-\alpha}}^r(g)\right) &= \left(\frac{g(1+2\beta_1)-\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1(1+\beta_1)}, \frac{-g+\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1}\right), \\ \left(x_{L_\alpha}^r(g), x_{R_\alpha}^r(g)\right) &= \left(\frac{g-\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, \frac{g(\beta_1-1)+(\beta_1+1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}\right), \\ \left(x_{L_*}^r(g), x_{R_*}^r(g)\right) &= \left(\frac{g(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)}, \frac{g[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right). \end{aligned} \tag{C.21}$$

Then, for $g \in [\underline{g}, \overline{g}]$, the optimal solution to (C.18) is given as follows

$$\textbf{\textit{If}} \quad \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} \leq 0, \tag{C.22}$$

$$\textbf{\textit{then}} \quad \left(x_L^{r\star}(g), x_R^{r\star}(g)\right) = \begin{cases} \left(x_{L_0}^r(g), x_{R_0}^r(g)\right), & \text{if } \underline{g} \leq g \leq \sqrt{1-\alpha}, \\ \left(x_{L_{1-\alpha}}^r(g), x_{R_{1-\alpha}}^r(g)\right), & \text{if } \sqrt{1-\alpha} \leq g \leq \overline{g}. \end{cases}$$

$$\textbf{\textit{If}} \quad 0 < \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} \leq \frac{\alpha}{\alpha\beta_1+1}, \tag{C.23}$$

$$\textit{then} \quad (x_L^{r\star}(g), x_R^{r\star}(g)) = \begin{cases} \left(x^r_{L_*}(g), x^r_{R_*}(g)\right), & \text{if } \underline{g} \le g < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}, \\ \left(x^r_{L_{1-\alpha}}(g), x^r_{R_{1-\alpha}}(g)\right), & \text{if } \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \le g \le \overline{g}. \end{cases}$$

$$\textit{If} \quad \frac{\alpha}{\alpha\beta_1+1} \le \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} < \frac{1}{\beta_1+1}, \tag{C.24}$$

$$\textit{then} \quad (x_L^{r\star}(g), x_R^{r\star}(g)) = \begin{cases} \left(x^r_{L_*}(g), x^r_{R_*}(g)\right), & \text{if } \underline{g} \le g < \sqrt{\frac{4\alpha\beta_1}{1-A^2}}, \\ \left(x^r_{L_\alpha}(g), x^r_{R_\alpha}(g)\right), & \text{if } \sqrt{\frac{4\alpha\beta_1}{1-A^2}} \le g \le \overline{g}. \end{cases}$$

$$\textit{If} \quad \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} \ge \frac{1}{\beta_1+1}, \tag{C.25}$$

$$\textit{then} \quad (x_L^{r\star}(g), x_R^{r\star}(g)) = \left(x^r_{L_\alpha}(g), x^r_{R_\alpha}(g)\right), \text{ for } \underline{g} \le g \le \overline{g}.$$

Furthermore, the corresponding optimal objective value for (C.18) *is given by*

$$F^r(g) \tag{C.26}$$
$$= \begin{cases} g^2, & \text{for } \left(x^r_{L_0}(g), x^r_{R_0}(g)\right), \\ \frac{g}{2\beta_1^2(1+\beta_1)^2}\left[\begin{matrix} g\left[\beta_1^2(1+\rho-\beta_r-2\rho\beta_r)-\rho\beta_r(1+2\beta_1)\right] \\ +\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}\left[\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)\right] \end{matrix}\right] - \frac{(1-\alpha)(\rho-\beta_1)(\beta_1+\beta_r)}{\beta_1(1+\beta_1)}, & \text{for } \left(x^r_{L_{1-\alpha}}(g), x^r_{R_{1-\alpha}}(g)\right), \\ \frac{g}{2\beta_1^2}\left[g\left[\beta_1^2-\rho\beta_r\right]+\sqrt{g^2-4\alpha\beta_1}\left[\beta_1^2+\rho\beta_r\right]\right] + \frac{\alpha(\rho-\beta_1)(\beta_1+\beta_r)}{\beta_1}, & \text{for } \left(x^r_{L_\alpha}(g), x^r_{R_\alpha}(g)\right), \\ g^2\frac{(\rho+\beta_r)^2}{4(\beta_1+\beta_r)(\rho-\beta_1)}, & \text{for } \left(x^r_{L_*}(g), x^r_{R_*}(g)\right). \end{cases}$$

We utilize Lemmas C.6 and C.7 in establishing the following Lemmas C.8 and C.9, whose proofs are also relegated after the proof of Proposition 4.

LEMMA C.8. *Let $g^\star$ be the optimal solution to:*

$$\max_{g\in[\underline{g},\overline{g}]} \lambda' F^\ell(g) + (1-\lambda')F^r(g). \tag{C.27}$$

Then, $(x_L^{\ell\star}(g^\star), x_R^{\ell\star}(g^\star), x_L^{r\star}(g^\star), x_R^{r\star}(g^\star))$ solves (C.15).

LEMMA C.9. *$\lambda' F^\ell(g) + (1-\lambda')F^r(g)$ is quasi-convex in $g \in [\underline{g},\overline{g}]$. Thus, $g^\star \in \{\underline{g},\overline{g}\}$. In particular,*

$$g^\star \equiv \overline{g} \quad \text{for} \quad \lambda \le \lambda_S^{ma}, \quad \text{and} \quad g^\star \equiv \underline{g} \quad \text{for} \quad \lambda > \lambda_S^{ma},$$

where

$$\lambda_S^{ma} = \frac{v_R^r\left[F^r(\underline{g}) - F^r(\overline{g})\right]}{v_R^\ell\left[F^\ell(\overline{g}) - F^\ell(\underline{g})\right] + v_R^r\left[F^r(\underline{g}) - F^r(\overline{g})\right]} \in [0,1].$$

When $g^\star = \overline{g}$, by Lemmas C.6 and C.7 and Claim C.3, we have $(\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R, \overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R) = (\alpha, 1-\alpha, \alpha, 1-\alpha)$. We construct $(\overset{\star}{x}{}^n_L, \overset{\star}{x}{}^n_R) = (\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R) = (\alpha, 1-\alpha)$ and show that $\left(\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R, \overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R, \overset{\star}{x}{}^n_L, \overset{\star}{x}{}^n_R\right) = (\alpha, 1-\alpha, \alpha, 1-\alpha, \alpha, 1-\alpha)$ satisfies the ignored constraints (IC_r), (IC^n_ℓ) and (IC^n_r), and, hence, is the optimal solution to problem (C.14). It is straightforward to see that $\left(\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R\right) = (\overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R) = (\overset{\star}{x}{}^n_L, \overset{\star}{x}{}^n_R) = (\alpha, 1-\alpha)$ always satisfy (IC_r), (IC^n_ℓ) and (IC^n_r). In this case, by (C.2), we have $(\overset{\star}{\sigma}{}^\ell_L, \overset{\star}{\sigma}{}^\ell_R, \overset{\star}{\sigma}{}^r_L, \overset{\star}{\sigma}{}^r_R, \overset{\star}{\sigma}{}^n_L, \overset{\star}{\sigma}{}^n_R) = (1,1,1,1,1,1)$.

When $g^\star = \underline{g}$, on the other hand, by Lemmas C.6 and C.7 we have $(\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R, \overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R) = (\sqrt{\alpha}, 0, x_L^{r\star}(\underline{g}), x_R^{r\star}(\underline{g}))$. We construct $(\overset{\star}{x}{}^n_L, \overset{\star}{x}{}^n_R) = (\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R) = (\sqrt{\alpha}, 0)$ and show that $\left(\overset{\star}{x}{}^\ell_L, \overset{\star}{x}{}^\ell_R, \overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R, \overset{\star}{x}{}^n_L, \overset{\star}{x}{}^n_R\right) =$

$(\sqrt{\alpha},0,x_L^{r\star}(\underline{g}),x_R^{r\star}(\underline{g}),\sqrt{\alpha},0)$ satisfies the ignored constraints (IC$_r$), (IC$_\ell^n$) and (IC$_r^n$), and, hence, is the optimal solution to problem (C.14). By Claim C.2, we have $x_R^{r\star}(\underline{g})=(1+\beta_1)(\sqrt{\alpha}-x_L^{r\star}(\underline{g}))$. Then, it is easy to see that both (IC$_r$) and (IC$_r^n$) hold as

$$(1-\beta_r)x_L^{r\star}(\underline{g})+x_R^{r\star}(\underline{g})-\sqrt{\alpha}(1-\beta_r)=(x_L^{r\star}(\underline{g})-\sqrt{\alpha})(1-\beta_r)+x_R^{r\star}(\underline{g})=\underbrace{(\sqrt{\alpha}-x_L^{r\star}(\underline{g}))}_{\geq 0}(\beta_1+\beta_r)\geq 0.$$

It is also straightforward to see that $(\overset{\star}{x}{}_L^n,\overset{\star}{x}{}_R^n)=(\overset{\star}{x}{}_L^\ell,\overset{\star}{x}{}_R^\ell)=(\sqrt{\alpha},0)$ always satisfy (IC$_\ell^n$). In this case, by Lemma C.10 and (C.2), we have $(\overset{\star}{\sigma}{}_L^\ell,\overset{\star}{\sigma}{}_R^\ell,\overset{\star}{\sigma}{}_L^r,\overset{\star}{\sigma}{}_R^r,\overset{\star}{\sigma}{}_L^n,\overset{\star}{\sigma}{}_R^n)=(1,0,\breve{\sigma}_L^r,\breve{\sigma}_R^r,1,0)$, where $(\breve{\sigma}_L^r,\breve{\sigma}_R^r)\neq(1,1)$. This completes the proof. ■

Proof of Lemma C.6. First, for a given $g\in[\underline{g},\overline{g}]$, we solve the equality constraint, i.e., $x_R^\ell=g-(1+\beta_1)x_L^\ell$, and ($H_\alpha^\ell$) hyperbola, i.e., $x_R^\ell=\frac{\alpha}{x_L^\ell}-x_L^\ell$, of (C.17) together to calculate their intersection points:

$$x_L^{\ell\mp}(g)=\frac{g\mp\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}\quad\text{and}\quad x_R^{\ell\mp}(g)=\frac{g(\beta_1-1)\pm(\beta_1+1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}. \tag{C.28}$$

The derivative of $x_L^{\ell\mp}(g)$ for the smaller root, i.e., $x_L^{\ell-}(g)=\frac{g-\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$, and the bigger root, i.e., $x_L^{\ell+}(g)=\frac{g+\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$, with respect to g is given by $\frac{1}{2\beta_1}\left[1-\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right]<0$ and $\frac{1}{2\beta_1}\left[1+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right]>0$, respectively. Hence, $x_L^{\ell-}(g)$ (resp., $x_L^{\ell+}(g)$) decreases (resp., increases) in g.

For $\beta_1<1$, by (C.16) and (C.28), we have

$$\left(x_L^{\ell+}(\underline{g}),x_R^{\ell+}(\underline{g})\right)=\left(\frac{\sqrt{\alpha}\left[\beta_1+1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1},\frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right)=\left(\frac{\sqrt{\alpha}}{\beta_1},\frac{\sqrt{\alpha}(\beta_1^2-1)}{\beta_1}\right).$$

Thus, as $x_L^{\ell+}(\underline{g})>\sqrt{\alpha}$ and $x_L^{\ell+}(g)$ increases in g, (i) when $\beta_1<1$, $(x_L^{\ell+}(g),x_R^{\ell+}(g))$ fails to satisfy the constraints of (C.17) for $g\in[\underline{g},\overline{g}]$ (see Figure C.1). By (C.16) and (C.28) we also have

$$\left(x_L^{\ell-}(\underline{g}),x_R^{\ell-}(\underline{g})\right)=\left(\frac{\sqrt{\alpha}\left[\beta_1+1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1},\frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right)=\left(\sqrt{\alpha},0\right),\quad\text{and}$$

$$\left(x_L^{\ell-}(\overline{g}),x_R^{\ell-}(\overline{g})\right)=\left(\frac{\alpha\beta_1+1-\sqrt{(\alpha\beta_1-1)^2}}{2\beta_1},\frac{(\alpha\beta_1+1)(\beta_1-1)+(\beta_1+1)\sqrt{(\alpha\beta_1-1)^2}}{2\beta_1}\right)=(\alpha,1-\alpha).$$

Thus, as $x_L^{\ell-}(g)$ decreases in g, (ii) when $\beta_1<1$, $(x_L^{\ell-}(g),x_R^{\ell-}(g))$ satisfies the constraints of (C.17) for $g\in[\underline{g},\overline{g}]$ (see Figure C.1).

For $\beta_1\geq 1$, by (C.16) and (C.28), we have

$$\left(x_L^{\ell+}(\underline{g}),x_R^{\ell+}(\underline{g})\right)=\left(\frac{\sqrt{\alpha}\left[\beta_1+1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1},\frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right)=\left(\sqrt{\alpha},0\right).$$

Thus, as $x_L^{\ell+}(g)$ increases in g, (iii) when $\beta_1\geq 1$, $(x_L^{\ell+}(g),x_R^{\ell+}(g))$ satisfies the constraints of (C.17) if and only if $g=\underline{g}$ (see Figure C.1). By (C.16) and (C.28) we also have

$$\left(x_L^{\ell-}(\underline{g}),x_R^{\ell-}(\underline{g})\right)=\left(\frac{\sqrt{\alpha}\left[\beta_1+1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1},\frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right)=\left(\frac{\sqrt{\alpha}}{\beta_1},\frac{\sqrt{\alpha}(\beta_1^2-1)}{\beta_1}\right),$$

and

$$\left(x_L^{\ell-}(\overline{g}), x_R^{\ell-}(\overline{g})\right) = \left(\frac{\alpha\beta_1 + 1 - \sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1}, \frac{(\alpha\beta_1 + 1)(\beta_1 - 1) + (\beta_1 + 1)\sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1}\right) = (\alpha, 1-\alpha),$$

where the latter result follows as we have $\alpha\beta_1^2 < 1$, which implies $\alpha\beta_1 < 1$ for $\beta_1 \geq 1$. Thus, (iv) when $\beta_1 \geq 1$, $(x_L^{\ell-}(g), x_R^{\ell-}(g))$ satisfies the constraints of (C.17) for $g \in [\underline{g}, \overline{g}]$ (see Figure C.1). Then, utilizing the equality constraint $(1+\beta_1)x_L^\ell + x_R^\ell = g$ to represent the objective function of (C.17) as $g\,[x_L^\ell(1+\rho) + x_R^\ell]$, it is straightforward to see that (v) $F^{\ell+}(\underline{g})$, the objective value attained by $\left(x_L^{\ell+}(\underline{g}), x_R^{\ell+}(\underline{g})\right) = (\sqrt{\alpha}, 0)$, is higher than $F^{\ell-}(\underline{g})$, the objective value attained by $\left(x_L^{\ell-}(\underline{g}), x_R^{\ell-}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}}{\beta_1}, \frac{\sqrt{\alpha}(\beta_1^2-1)}{\beta_1}\right)$:

$$F^{\ell+}(\underline{g}) = \alpha(1+\beta_1)(1+\rho) \geq \alpha(1+\beta_1)\frac{(\beta_1^2+\rho)}{\beta_1} = F^{\ell-}(\underline{g}), \text{ as } \rho > \beta_1 \geq 1.$$

Lastly, we note that for a given $g \in [\underline{g}, \overline{g}]$, utilizing the equality constraint $x_R^\ell = g - (1+\beta_1)x_L^\ell$, the objective function of (C.17) can be expressed as $g\,[(\rho - \beta_1)x_L^\ell + g]$, which is increasing in x_L^ℓ, as $\rho > \beta_1$. Thus, for $g \in [\underline{g}, \overline{g}]$, by (i)-(iv), we must have (H_α^ℓ) binding, which, by (v), implies the optimality of $(x_L^{\ell-}(g), x_R^{\ell-}(g))$ (resp., $(x_L^{\ell+}(g), x_R^{\ell+}(g))$), defined in (C.28), when $\beta_1 < 1$ or $\beta_1 \geq 1$ and $g > \underline{g}$ (resp., when $\beta_1 \geq 1$ and $g = \underline{g}$). Hence, $(x_L^{\ell\star}(g), x_R^{\ell\star}(g))$ is well-defined for $g \in [\underline{g}, \overline{g}]$, and plugging such $(x_L^{\ell\star}(g), x_R^{\ell\star}(g))$ into the objective function of (C.17) grants us (C.20). This completes the proof. ■

Proof of Lemma C.7. Proving this Lemma we first establish the following claim.

Claim C.2. *The solution to* (C.18) *binds* $(1+\beta_1)x_L^r + x_R^r \leq g$, *i.e.,* $(1+\beta_1)x_L^{r\star}(g) + x_R^{r\star}(g) = g$ *for* $g \in [\underline{g}, \overline{g}]$, *where* $\underline{g}$ *and* $\overline{g}$ *are defined in* (C.16).

Proof of Claim C.2. Notice first that (i) $(x_L^r, x_R^r) = (\alpha, 1-\alpha)$ makes $(1+\beta_1)x_L^r + x_R^r = \alpha\beta_1 + 1 = \overline{g}$, i.e., it satisfies $(1+\beta_1)x_L^r + x_R^r \leq g$ constraint of (C.18) only for $g = \overline{g}$ within our domain of interest $[\underline{g}, \overline{g}]$.

Next, we note that the objective function of (C.18) is quadratic in (x_L^r, x_R^r) with all the coefficients being positive (as $\beta_r \leq 1$). Thus, it is straightforward to see that, in the optimal policy, we must have at least one of $(1+\beta_1)x_L^r + x_R^r \leq g$, (H_α^r) and $(H_{1-\alpha}^r)$ binding, as otherwise the objective value could be improved without violating the constraints (see Figure C.1). Suppose $(1+\beta_1)x_L^{r\star}(g) + x_R^{r\star}(g) < g$, then, we must have either (H_α^r) and $(H_{1-\alpha}^r)$ binding. For the case when (H_α^r) is binding, we have $x_L^r \in [\alpha, \sqrt{\alpha}]$ and $x_R^r = \frac{\alpha}{x_L^r} - x_L^r$. Substituting this relationship reduces the objective function of (C.18) to

$$(x_L^r)^2(1+\rho)(1-\beta_r) + x_L^r\left[\frac{\alpha}{x_L^r} - x_L^r\right](2+\rho-\beta_r) + \left[\frac{\alpha}{x_L^r} - x_L^r\right]^2,$$

whose derivative with respect to x_L^r is given by (ii) $2\left[-\rho\beta_r x_L^r - \frac{\alpha^2}{(x_L^r)^3}\right] < 0$. Similarly, for the case when $(H_{1-\alpha}^r)$ is binding, we have $x_R^r \in [1-\alpha, \sqrt{1-\alpha}]$ and $x_L^r = \frac{1-\alpha}{x_R^r} - x_R^r$. Substituting this relationship reduces the objective function of (C.18) to

$$\left[\frac{1-\alpha}{x_R^r} - x_R^r\right]^2(1+\rho)(1-\beta_r) + x_R^r\left[\frac{1-\alpha}{x_R^r} - x_R^r\right](2+\rho-\beta_r) + (x_R^r)^2,$$

whose derivative with respect to x_R^r is given by (iii) $2\left[-\rho\beta_r x_R^r - \frac{(1-\alpha)^2}{(x_R^r)^3}(1+\rho)(1-\beta_r)\right] \leq 0$.

For a given $g \in [\underline{g}, \overline{g}]$, (i)-(iii) imply that in the case when (H_α^r) is binding (resp., in the case when $(H_{1-\alpha}^r)$ is binding), x_L^r (resp., x_R^r) could be decreased towards α (resp., towards $1-\alpha$) until we have $(1+\beta_1)x_L^r + x_R^r = g$, while improving the objective value (see Figure C.1). Therefore, we must have $(1+\beta_1)x_L^{r\star}(g) + x_R^{r\star}(g) = g$.

□

Utilizing Claim C.2, for a given $g \in [\underline{g}, \overline{g}]$, we first solve (C.18) only subject to $(1+\beta_1)x_L^r + x_R^r = g$. In that scenario, the platform's maximization problem becomes

$$\max_{x_L^r} \underbrace{(\beta_1+\beta_r)}_{>0}\underbrace{(\beta_1-\rho)}_{<0}(x_L^r)^2 + g(\rho-\beta_r-2\beta_1)x_L^r + g^2, \tag{C.29}$$

which is a maximization problem of a concave function in x_L^r. Hence, first order condition on the objective function of (C.29) grants us with the unconstrained optimal solution, which we define as

$$\left(x_{L_*}^r(g), x_{R_*}^r(g)\right) = \left(\frac{g(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)}, \frac{g[2(\beta_1+\rho\beta_r)+(\rho-\beta_r)(\beta_1-1)]}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right).$$

However, for a given $g \in [\underline{g}, \overline{g}]$, $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ may violate the remaining constraint(s) of (C.18), namely $x \geq 0$, (H_α^r) and $(H_{1-\alpha}^r)$, i.e., $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ may fall outside the shaded regions of Figure C.2. Utilizing the concavity of the objective function of (C.29), besides $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$, we derive the possible types of optimal $(x_L^{r\star}, x_R^{r\star})$ points:

$$\begin{aligned}
\left(x_{L_{1-\alpha}}^{r\mp}(g), x_{R_{1-\alpha}}^{r\mp}(g)\right) &= \left(\frac{g(1+2\beta_1) \pm \sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1(1+\beta_1)}, \frac{-g \mp \sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1}\right), \\
\left(x_{L_\alpha}^{r\mp}(g), x_{R_\alpha}^{r\mp}(g)\right) &= \left(\frac{g \mp \sqrt{g^2-4\alpha\beta_1}}{2\beta_1}, \frac{g(\beta_1-1) \pm (1+\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}\right), \\
\left(x_{L_0}^r(g), x_{R_0}^r(g)\right) &= (0, g),
\end{aligned} \tag{C.30}$$

where $\left(x_{L_{1-\alpha}}^{r\mp}(g), x_{R_{1-\alpha}}^{r\mp}(g)\right)$ are the intersection points of $(1+\beta_1)x_L^r + x_R^r = g$ line and $(H_{1-\alpha}^r)$ hyperbola, hence, are calculated by solving $x_R^r = g - (1+\beta_1)x_L^r$ and $x_R^r = \frac{\alpha}{x_L^r} - x_L^r$ together. Next, $\left(x_{L_\alpha}^{r\mp}(g), x_{R_\alpha}^{r\mp}(g)\right)$ are the intersection points of $(1+\beta_1)x_L^r + x_R^r = g$ line and (H_α^r) hyperbola, hence, are calculated by solving $x_L^r = \frac{g-x_R^r}{(1+\beta_1)}$ and $x_L^r = \frac{1-\alpha}{x_R^r} - x_R^r$ together. Lastly, $\left(x_{L_0}^r(g), x_{R_0}^r(g)\right)$ is the intersection point of $(1+\beta_1)x_L^r + x_R^r = g$ line and the x_R^r axis, hence, is calculated by solving $x_L^r = 0$ and $x_L^r = \frac{g-x_R^r}{(1+\beta_1)}$ together.

Notice first that

$$x_{R_{1-\alpha}}^{r-}(g) = \frac{-g - \sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1} < 0$$

and, hence, $\left(x_{L_{1-\alpha}}^{r-}(g), x_{R_{1-\alpha}}^{r-}(g)\right)$ violates the nonnegativity constraint $x \geq 0$. Therefore, $\left(x_{L_{1-\alpha}}^{r-}(g), x_{R_{1-\alpha}}^{r-}(g)\right)$ cannot be an optimal solution to (C.18).

Next, we show that $\left(x_{L_\alpha}^{r+}(g), x_{R_\alpha}^{r+}(g)\right)$ cannot be an optimal solution to (C.18) as well. In that, see first the derivative of $x_L^{r\mp}(g)$ for the smaller root, i.e., $x_L^{r-}(g) = \frac{g-\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$, and the bigger root, i.e., $x_L^{r+}(g) = \frac{g+\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$, with respect to g is given by $\frac{1}{2\beta_1}\left[1 - \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right] < 0$ and $\frac{1}{2\beta_1}\left[1 + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right] > 0$, respectively. Hence, $x_L^{r-}(g)$ (resp., $x_L^{r+}(g)$) decreases (resp., increases) in g.

For $\beta_1 < 1$, by (C.16) and (C.30), we have

$$\left(x_L^{r+}(\underline{g}), x_R^{r+}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}\left[\beta_1+1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}, \frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right) = \left(\frac{\sqrt{\alpha}}{\beta_1}, \frac{\sqrt{\alpha}(\beta_1^2-1)}{\beta_1}\right).$$

Thus, as $x_L^{r+}(\underline{g}) > \sqrt{\alpha}$ and $x_L^{r+}(g)$ increases in g, (i) when $\beta_1 < 1$, $(x_L^{r+}(g), x_R^{r+}(g))$ fails to satisfy the constraints of (C.18) for $g \in [\underline{g}, \overline{g}]$ (see Figure C.1). By (C.16) and (C.30) we also have

$$\left(x_L^{r-}(\underline{g}), x_R^{r-}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}\left[\beta_1+1-\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}, \frac{\sqrt{\alpha}(\beta_1+1)\left[\beta_1-1+\sqrt{(\beta_1-1)^2}\right]}{2\beta_1}\right) = \left(\sqrt{\alpha}, 0\right), \quad \text{and}$$

$$(x_L^{r-}(\overline{g}), x_R^{r-}(\overline{g})) = \left(\frac{\alpha\beta_1 + 1 - \sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1}, \frac{(\alpha\beta_1 + 1)(\beta_1 - 1) + (\beta_1 + 1)\sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1} \right) = (\alpha, 1 - \alpha).$$

Thus, as $x_L^{r-}(g)$ decreases in g, (ii) when $\beta_1 < 1$, $(x_L^{r-}(g), x_R^{r-}(g))$ satisfies the constraints of (C.18) for $g \in [\underline{g}, \overline{g}]$ (see Figure C.1).

For $\beta_1 \geq 1$, by (C.16) and (C.30), we have

$$\left(x_L^{r+}(\underline{g}), x_R^{r+}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}\left[\beta_1 + 1 + \sqrt{(\beta_1 - 1)^2}\right]}{2\beta_1}, \frac{\sqrt{\alpha}(\beta_1 + 1)\left[\beta_1 - 1 - \sqrt{(\beta_1 - 1)^2}\right]}{2\beta_1} \right) = \left(\sqrt{\alpha}, 0\right).$$

Thus, as $x_L^{r+}(g)$ increases in g, (iii) when $\beta_1 \geq 1$, $(x_L^{r+}(g), x_R^{r+}(g))$ satisfies the constraints of (C.18) if and only if $g = \underline{g}$ (see Figure C.1). By (C.16) and (C.30) we also have

$$\left(x_L^{r-}(\underline{g}), x_R^{r-}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}\left[\beta_1 + 1 - \sqrt{(\beta_1 - 1)^2}\right]}{2\beta_1}, \frac{\sqrt{\alpha}(\beta_1 + 1)\left[\beta_1 - 1 + \sqrt{(\beta_1 - 1)^2}\right]}{2\beta_1} \right) = \left(\frac{\sqrt{\alpha}}{\beta_1}, \frac{\sqrt{\alpha}(\beta_1^2 - 1)}{\beta_1} \right),$$

and

$$(x_L^{r-}(\overline{g}), x_R^{r-}(\overline{g})) = \left(\frac{\alpha\beta_1 + 1 - \sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1}, \frac{(\alpha\beta_1 + 1)(\beta_1 - 1) + (\beta_1 + 1)\sqrt{(\alpha\beta_1 - 1)^2}}{2\beta_1} \right) = (\alpha, 1 - \alpha),$$

where the latter result follows as we have $\alpha\beta_1^2 < 1$, which implies $\alpha\beta_1 < 1$ for $\beta_1 \geq 1$. Thus, (iv) when $\beta_1 \geq 1$, $(x_L^{r-}(g), x_R^{r-}(g))$ satisfies the constraints of (C.17) for $g \in [\underline{g}, \overline{g}]$ (see Figure C.1). Then, utilizing Claim C.2 (particularly, $(1 + \beta_1)x_L^r + x_R^r = g$) to represent the objective function of (C.18) as $[x_L^r(1 + \rho) + x_R^r]\,[x_L^r(1 - \beta_r) + x_R^r]$, it is straightforward to see that (v) $F^{r+}(\underline{g})$, the objective value attained by $\left(x_L^{r+}(\underline{g}), x_R^{r+}(\underline{g})\right) = (\sqrt{\alpha}, 0)$, is lower than $F^{r-}(\underline{g})$, the objective value attained by $\left(x_L^{r-}(\underline{g}), x_R^{r-}(\underline{g})\right) = \left(\frac{\sqrt{\alpha}}{\beta_1}, \frac{\sqrt{\alpha}(\beta_1^2 - 1)}{\beta_1}\right)$ as we have

$$\begin{aligned} \rho\beta_r(1 - \beta_1^2) \leq \beta_1^2(\beta_1^2 - 1) \quad &\Leftrightarrow \quad \beta_1^2(1 + \rho)(1 - \beta_r) \leq (\beta_1^2 + \rho)(\beta_1^2 - \beta_r) \\ &\Leftrightarrow \quad \underbrace{\alpha(1 + \rho)(1 - \beta_r)}_{F^{r+}(\underline{g})} \leq \underbrace{\frac{\alpha}{\beta_1^2}(\beta_1^2 + \rho)(\beta_1^2 - \beta_r)}_{F^{\ell-}(\underline{g})}, \end{aligned}$$

where the first inequality follows by the fact that $\beta_1 \geq 1$. Subsequently, (i) and (ii), and (iii)-(v) imply that $\left(x_{L_\alpha}^{r+}(g), x_{R_\alpha}^{r+}(g)\right)$ cannot be an optimal solution to (C.18).

Above analysis establish that, for a given $g \in [\underline{g}, \overline{g}]$, the optimal solution to (C.18) is given by one of $\left(x_{L_0}^r(g), x_{R_0}^r(g)\right)$-$\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ defined in (C.21). We note that $\left(x_{L_{1-\alpha}}^r(g), x_{R_{1-\alpha}}^r(g)\right)$, as a function of g, lies on $(H_{1-\alpha}^r)$ (blue thick curve in Figure C.2), hence the subscript $1 - \alpha$. Similarly, $\left(x_{L_\alpha}^r(g), x_{R_\alpha}^r(g)\right)$, as a function of g, lies on (H_α^r) (red thick curve in Figure C.2), hence the subscript α. Lastly, $\left(x_{L_0}^r(g), x_{R_0}^r(g)\right)$, as a function of g, lies on x_R^r axis (black thick line in Figure C.2), hence the subscript 0. Notice further that the expressions of all the above-mentioned point types are independent of λ.

In the second part of the proof, for a given $g \in [\underline{g}, \overline{g}]$, we characterize the conditions determining which of the $\left(x_{L_0}^r(g), x_{R_0}^r(g)\right)$-$\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ defined in (C.21) is optimal for (C.18). Notice first that the signs of both $x_{L_*}^r(g)$ and $x_{R_*}^r(g)$ are independent of g and $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ follows a linear trajectory in $g \in [\underline{g}, \overline{g}]$.

When $\frac{\rho - \beta_r - 2\beta_1}{2(\beta_1 + \beta_r)(\rho - \beta_1)} = \frac{x_{L_*}^r(g)}{g} \leq 0$, we have $x_{L_*}^r(g) \leq 0$ for $g \in [\underline{g}, \overline{g}]$, which corresponds to the condition expressed in (C.22) (see Figure C.2). When $\frac{\rho - \beta_r - 2\beta_1}{2(\beta_1 + \beta_r)(\rho - \beta_1)} = \frac{x_{L_*}^r(g)}{g} \geq \frac{1}{\beta_1 + 1}$, $x_{L_*}^r(g)$ is positive and increasing

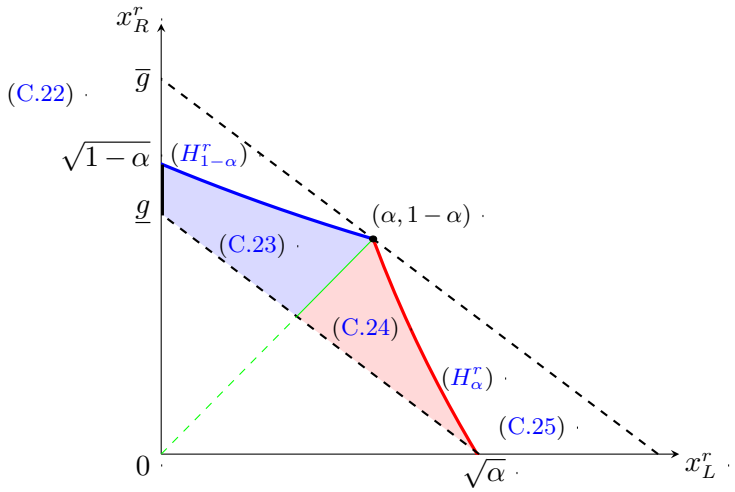


Figure C.2 **Feasible region for (C.18), where (C.22)-(C.25) represents their corresponding parameter conditions expressed in Lemma C.7.**

in $g \in [\underline{g}, \overline{g}]$, and, since $x_{L_*}^r(\underline{g}) \geq \frac{\sqrt{\alpha}(\beta_1+1)}{\beta_1+1} = \sqrt{\alpha}$, we have $x_{L_*}^r(g) \geq \sqrt{\alpha}$ for $g \in [\underline{g}, \overline{g}]$, which corresponds to the condition expressed in (C.25) (see Figure C.2). The remaining parameter region, i.e., $\frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} = \frac{x_{L_*}^r(g)}{g} \in \left(0, \frac{1}{\beta_1+1}\right)$, has $x_{L_*}^r(g)$ non-decreasing in $g \in [\underline{g}, \overline{g}]$, and corresponds to $x_{L_*}^r(\underline{g}) \in (0, \sqrt{\alpha})$. In this region, we further analyze which hyperbola (among (H_α^r) and $(H_{1-\alpha}^r)$), if any, that the linear trajectory of $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ in $g \in [\underline{g}, \overline{g}]$ passes through first. In that, it suffices us to compare $x_{L_*}^r(\overline{g})$ to α, as, by Claim C.2, $(1+\beta_1)x_L^r + x_R^r = \overline{g}$ line cuts through $(x_L^r, x_R^r) = (\alpha, 1-\alpha)$ (higher black dashed line in Figure C.2). Thus, $x_{L_*}^r(\overline{g}) = \frac{(\alpha\beta_1+1)(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)} \geq \alpha$ $\left(\text{resp., } x_{L_*}^r(\overline{g}) = \frac{(\alpha\beta_1+1)(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)} \leq \alpha\right)$ implies the linear trajectory of $\left(x_{L_*}^r(g), x_{R_*}^r(g)\right)$ in $g \in [\underline{g}, \overline{g}]$ is above (resp., below) the green line of Figure C.2 and, hence, passes through (H_α^r) (resp., $(H_{1-\alpha}^r)$) first, corresponds to the condition expressed in (C.24) (resp., (C.23)).

Claim C.3. $\sqrt{1-\alpha} \leq \overline{g}$, $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \leq \overline{g}$ *and* $\sqrt{\frac{4\alpha\beta_1}{1-A^2}} \leq \overline{g}$ *for* (C.22), (C.23) *and* (C.24), *respectively.*

Proof of Claim C.3. Proof of $\sqrt{1-\alpha} \leq \overline{g}$: Notice that $\alpha^2\beta_1^2 + 2\alpha\beta_1 + 1 \geq 1-\alpha$, hence, our claim holds. Proof of $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \leq \overline{g}$: Notice that, by (C.23), we have

$$
\begin{aligned}
& \alpha \geq \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \quad \Leftrightarrow \quad \alpha\beta_1[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)] \geq \beta_1[\rho-\beta_r-2\beta_1] \\
\Leftrightarrow \quad & (\alpha\beta_1+1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)] \geq 2(\beta_1+\beta_r)(\rho-\beta_1) \\
\Leftrightarrow \quad & \frac{(\alpha\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]^2}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \geq 4 \\
\Leftrightarrow \quad & \frac{(\alpha\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]^2}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \geq 4(1-\alpha) + \frac{\alpha(\alpha\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]^2}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
\text{by (C.23)} \Rightarrow \quad & \frac{(\alpha\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]^2}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
& \geq 4(1-\alpha) + \frac{(\alpha\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
\Leftrightarrow \quad & \frac{(\alpha\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)][\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \geq 4(1-\alpha) \\
\text{by (C.21)} \Leftrightarrow \quad & (\alpha\beta_1+1)^2(B-1)(B+1) \geq 4(1-\alpha)\beta_1(\beta_1+1),
\end{aligned}
$$

hence, our claim holds.

Proof of $\sqrt{\frac{4\alpha\beta_1}{1-A^2}} \le \overline{g}$: Notice that, by (C.24), we have

$$\alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \quad \Leftrightarrow \quad \alpha\beta_1\left[\frac{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}\right] < \beta_1\left[\frac{\rho-\beta_r-2\beta_1}{(\beta_1+\beta_r)(\rho-\beta_1)}\right]$$

$$\text{by (C.21)} \ \Leftrightarrow \quad \alpha\beta_1(1+A) < 1-A \quad \Leftrightarrow \quad A^2(1+\alpha\beta_1)^2 < (1-\alpha\beta_1)^2 \quad \Leftrightarrow \quad \frac{4\alpha\beta_1}{1-A^2} < (1+\alpha\beta_1)^2,$$

hence, our claim holds. □

When (C.22) holds, as $x^r_{L_*}(g) \le 0$, we have $(x^{r\star}_L(g), x^{r\star}_R(g)) = (x^r_{L_0}(g), x^r_{R_0}(g))$, if the intersection point of $(1+\beta_1)x^r_L + x^r_R = g$ line and x^r_R axis (which occurs at $x^r_R = g$, see Figure C.2) falls below the intersection point of $(H^r_{1-\alpha})$ hyperbola and x^r_R axis (which occurs at $x^r_R = \sqrt{1-\alpha}$, see Figure C.2). When (C.25) holds, as $x^r_{L_*}(g) \ge \sqrt{\alpha}$, $\big(x^r_{L_*}(g), x^r_{R_*}(g)\big)$ falls to the right-side of (H^r_α) hyperbola for all $g \in [\underline{g}, \overline{g}]$ (see Figure C.2), therefore, $(x^{r\star}_L(g), x^{r\star}_R(g)) = (x^r_{L_\alpha}(g), x^r_{R_\alpha}(g))$. Next, when (C.24) holds, we have $x^r_{L_*}(g) > 0$ and, hence, it is increasing in $g \in [\underline{g}, \overline{g}]$ (see Figure C.2), whereas $x^r_{L_\alpha}(g)$ is decreasing in $g \in [\underline{g}, \overline{g}]$ as its derivative with respect to g is given by $\frac{1}{2\beta_1}\left[1 - \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right] < 0$. Accordingly, we derive the condition for $x^r_{L_*}(g) < x^r_{L_\alpha}(g)$:

$$\frac{g(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)} < \frac{g-\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$$

$$\Leftrightarrow \quad \frac{\sqrt{g^2-4\alpha\beta_1}}{g} < \frac{(\beta_1+\beta_r)(\rho-\beta_1)-\beta_1(\rho-\beta_r-2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)} = \underbrace{\frac{\rho\beta_r+\beta_1^2}{(\beta_1+\beta_r)(\rho-\beta_1)}}_{A} \quad \Leftrightarrow \quad g < \sqrt{\frac{4\alpha\beta_1}{1-A^2}},$$

where $A \ge 0$ as $\rho > \beta_1$, and $A < 1$ as

$$\beta_1 \underbrace{(\rho-\beta_r)}_{>2\beta_1} > 2\beta_1^2 \quad \Leftrightarrow \quad (\beta_1+\beta_r)(\rho-\beta_1) > \rho\beta_r + \beta_1^2.$$

Similarly, when (C.23) holds, we have $x^r_{R_*}(g) > 0$ and, hence, it is non-decreasing in $g \in [\underline{g}, \overline{g}]$ (see Figure C.2), whereas $x^r_{R_{1-\alpha}}(g)$ is decreasing in $g \in [\underline{g}, \overline{g}]$ as its derivative with respect to g is given by $\frac{1}{2\beta_1}\left[\frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}} - 1\right] < 0$. Accordingly, we derive the condition for $x^r_{R_*}(g) < x^r_{R_{1-\alpha}}(g)$:

$$\frac{g[2(\beta_1+\rho\beta_r)+(\rho-\beta_r)(\beta_1-1)]}{2(\beta_1+\beta_r)(\rho-\beta_1)} < \frac{-g+\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{2\beta_1}$$

$$\Leftrightarrow \quad \frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} > \underbrace{\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}}_{B} \quad \Leftrightarrow \quad g < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}},$$

where $B > 1$, as in this region we have

$$\begin{aligned}(1+\beta_1)(\rho-\beta_r-2\beta_1) < 2(\beta_1+\beta_r)(\rho-\beta_1) \quad &\Leftrightarrow \quad \rho-\beta_r-2\beta_1 < \beta_1(\rho-\beta_r)+2\rho\beta_r \\ &\Leftrightarrow \quad \beta_1(\rho-\beta_r) < \beta_1^2(2+\rho-\beta_r)+2\rho\beta_1\beta_r \\ &\Leftrightarrow \quad (\beta_1+\beta_r)(\rho-\beta_1) < \beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1).\end{aligned}$$

Plugging $\big(x^r_{L_0}(g), x^r_{R_0}(g)\big)$-$\big(x^r_{L_*}(g), x^r_{R_*}(g)\big)$ defined in (C.21) into the objective function of (C.18) grants us (C.26). This completes the proof. ■

Proof of Lemma C.8. Proving this Lemma we first establish, by the following two claims, structural properties regarding the platform's optimal recommender policy for problem (C.15).

CLAIM C.4. *(H^ℓ_α) is binding for the solution to (C.15), i.e., $\dot{x}^\ell_L(\dot{x}^\ell_L+\dot{x}^\ell_R)=\alpha$.*

Proof of Claim C.4. Notice first, that the objective function of (C.15), namely (C.3), is quadratic in (x^ℓ_L,x^ℓ_R) with all the coefficients being positive. Notice also that (x^ℓ_L,x^ℓ_R) only appears on the left hand side of (IC_ℓ) and all of their coefficients are positive. Thus, it is straightforward to see that, in the optimal policy, we must have at least one of (H^ℓ_α) and ($H^\ell_{1-\alpha}$) binding, as otherwise the objective value could be improved without violating the constraints (see Figure C.1). For the case when ($H^\ell_{1-\alpha}$) is binding, we have $x^\ell_R\in[1-\alpha,\sqrt{1-\alpha}]$ and $x^\ell_L=\frac{1-\alpha}{x^\ell_R}-x^\ell_R$. Substituting this relationship reduces the first line of (C.3) to

$$\left[\frac{1-\alpha}{x^\ell_R}-x^\ell_R\right]^2(1+\rho)(1+\beta_1)+x^\ell_R\left[\frac{1-\alpha}{x^\ell_R}-x^\ell_R\right](2+\rho+\beta_1)+(x^\ell_R)^2,$$

whose derivative with respect to x^ℓ_R is given by $x^\ell_R\left[-\frac{(1-\alpha)^2}{(x^\ell_R)^4}2(1+\rho+\beta_1+\rho\beta_1)+2\rho\beta_1\right]\le 0$, as $x^\ell_R\le\sqrt{1-\alpha}$. Furthermore, substituting the relationship $x^\ell_R\in[1-\alpha,\sqrt{1-\alpha}]$ and $x^\ell_L=\frac{1-\alpha}{x^\ell_R}-x^\ell_R$ into the left hand side of (IC_ℓ) and taking its derivative with respect to x^ℓ_R we have $-(x^\ell_R)^{-2}(1-\alpha)(1+\beta_1)-\beta_1<0$. Hence, if ($H^\ell_{1-\alpha}$) is binding, we must have $(\dot{x}^\ell_L,\dot{x}^\ell_R)=(\alpha,1-\alpha)$, which immediately implies (H^ℓ_α) must also be binding. Thus, solving (C.15), it suffices for the platform to consider only the case when (H^ℓ_α) is binding. Subsequently, we have $\dot{x}^\ell_L\in[\alpha,\sqrt{\alpha}]$ and $\dot{x}^\ell_R=\frac{\alpha}{\dot{x}^\ell_L}-\dot{x}^\ell_L$. □

CLAIM C.5. *The solution to (C.15) satisfies $(1+\beta_1)\dot{x}^\ell_L+\dot{x}^\ell_R\in[\underline{g},\overline{g}]$, where $\underline{g}$ and $\overline{g}$ are defined in (C.16).*

Proof of Claim C.5. In proving this claim, we make use of Claim C.4 by restricting our search to $x^\ell_L\in[\alpha,\sqrt{\alpha}]$ and taking $x^\ell_R=\frac{\alpha}{x^\ell_L}-x^\ell_L$, which allow us to denote $(1+\beta_1)x^\ell_L+x^\ell_R$ as $g^\circ(x^\ell_L)=\beta_1x^\ell_L+\alpha(x^\ell_L)^{-1}$. Accordingly, to prove our claim, we just need to show $g^\circ(\dot{x}^\ell_L)\in[\underline{g},\overline{g}]$ for $\dot{x}^\ell_L\in[\alpha,\sqrt{\alpha}]$. Notice first that

$$\frac{\partial g^\circ(x^\ell_L)}{\partial x^\ell_L}=-\alpha(x^\ell_L)^{-2}+\beta_1, \tag{C.31}$$

hence, $g^\circ(x^\ell_L)$ is convex in x^ℓ_L.

When $\beta_1<1$, as $\alpha\le 1$, (C.31) is always negative for $x^\ell_L\in[\alpha,\sqrt{\alpha}]$. Hence, for this case we have

$$g^\circ(\sqrt{\alpha})=\underbrace{\sqrt{\alpha}(\beta_1+1)}_{\underline{g}}\le g^\circ(x^\ell_L)\le\underbrace{\alpha\beta_1+1}_{\overline{g}}=g^\circ(\alpha),$$

verifying our claim.

When $\beta_1\ge 1$, notice first by (C.31) that $\arg\min_{x^\ell_L} g^\circ(x^\ell_L)=\sqrt{\frac{\alpha}{\beta_1}}$ and, as $\alpha\beta_1^2<1$, we have $\sqrt{\frac{\alpha}{\beta_1}}\in[\alpha,\sqrt{\alpha}]$. Since $g^\circ(\alpha)=\alpha\beta_1+1>\sqrt{\alpha}(\beta_1+1)=g^\circ(\sqrt{\alpha})$, for this case we have

$$g^\circ(\sqrt{\frac{\alpha}{\beta_1}})=\underbrace{2\sqrt{\alpha\beta_1}}_{<\underline{g}\text{ as }\beta_1\ge1}\le g^\circ(x^\ell_L)\le\underbrace{\alpha\beta_1+1}_{\overline{g}}=g^\circ(\alpha).$$

In the remainder of the proof, we show $g^\circ(\dot{x}^\ell_L)\ge\sqrt{\alpha}(\beta_1+1)=\underline{g}$.

The condition $g^\circ(\alpha)>g^\circ(\sqrt{\alpha})$ immediately implies, since $g^\circ(x^\ell_L)$ is a continuous convex function of $x^\ell_L\in[\alpha,\sqrt{\alpha}]$, the existence of a unique $x^{\ell*}_L\in(\alpha,\sqrt{\alpha})$ such that $g^\circ(x^{\ell*}_L)=\beta_1x^{\ell*}_L+\alpha(x^{\ell*}_L)^{-1}=\sqrt{\alpha}(\beta_1+1)=g^\circ(\sqrt{\alpha})$. Trivially, such $x^{\ell*}_L$ occurs at $\frac{\sqrt{\alpha}}{\beta_1}$. Therefore, for any arbitrary constant $c\in[2\sqrt{\alpha\beta_1},\sqrt{\alpha}(\beta_1+1)]$, we have two distinct x^ℓ_L values x_1 and x_2, such that $\frac{\sqrt{\alpha}}{\beta_1}\le x_1<x_2\le\sqrt{\alpha}$ and $g^\circ(x_1)=g^\circ(x_2)=c$. Note that (i) $g^\circ(x^\ell_L)=c$ implies $(x^\ell_L)^2\beta_1^2+\left(\frac{\alpha}{x^\ell_L}\right)^2=c^2-2\alpha\beta_1$. Notice also that, for such x_1 and x_2, substituting the relationship

$x_R^\ell = \frac{\alpha}{x_L^\ell} - x_L^\ell$ to the objective function of (C.15), namely (C.3), the first line of the objective function can be represented as:

$$(x_L^\ell)^2\beta_1\rho + \left(\frac{\alpha}{x_L^\ell}\right)^2 \underset{\text{by (i)}}{=} c^2 - 2\alpha\beta_1 + (x_L^\ell)^2\beta_1(\rho-\beta_1). \tag{C.32}$$

Thus, as $\rho > \beta_1$, (C.32) is maximized by the bigger of the two values of x_L^ℓ that make $\beta_1 x_L^\ell + \alpha(x_L^\ell)^{-1}$, i.e., the left hand side of (IC_ℓ), same. Lastly, note that $x_L^\ell \in \left[\frac{\sqrt{\alpha}}{\beta_1}, \sqrt{\alpha}\right)$ can never be optimal for (C.15). This is because the derivative of (C.32) with respect to x_L^ℓ, which is given by $2\left[\beta_1\rho x_L^\ell - \frac{\alpha^2}{(x_L^\ell)^3}\right]$, is positive for $x_L^\ell = \arg\min_{x_L^\ell} g^\circ = \sqrt{\frac{\alpha}{\beta_1}}$ and increasing in x_L^ℓ. Hence, we have $g^\circ(\overset{\star}{x}{}_L^\ell) \geq g^\circ(\frac{\sqrt{\alpha}}{\beta_1}) = g^\circ(\sqrt{\alpha}) = \sqrt{\alpha}(\beta_1+1) = \underline{g}$, verifying our claim. □

Notice that in (C.15) and (C.27), (x_L^ℓ, x_R^ℓ) and (x_L^r, x_R^r) are only interacting through (IC_ℓ) and g, respectively. Therefore, Claims C.4 and C.5 imply that $(x_L^{\ell\star}(g^\star), x_R^{\ell\star}(g^\star), x_L^{r\star}(g^\star), x_R^{r\star}(g^\star))$ solves (C.15), as both the objective functions and set of constraints are identical in (C.15) and (C.27). This completes the proof. ■

Proof of Lemma C.9. In this proof, we take $F^\ell(g) = \frac{g(\rho+\beta_1)-(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$ for all $g \in [\underline{g}, \overline{g}]$. We remark that establishing the quasi-convexity of $\lambda' F^\ell(g) + (1-\lambda')F^r(g)$ for such a scenario immediately implies its quasi-convexity for $F^\ell(g)$ defined in (C.20). This is because we are only interested in the domain of $g \in [\underline{g}, \overline{g}]$ and $\frac{g(\rho+\beta_1)+(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1} > \frac{g(\rho+\beta_1)-(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$, as $\rho > \beta_1$. This practice allows us, through Claim C.6, to first analyze the conditions (C.22)-(C.25) in isolation, calculate their respective $\lambda' F^\ell(g) + (1-\lambda')F^r(g)$ utilizing $F^\ell(g) = \frac{g(\rho+\beta_1)-(\rho-\beta_1)\sqrt{g^2-4\alpha\beta_1}}{2\beta_1}$ and (C.26), and show that for a condition, each particular objective function segment is quasi-convex in $g \in [\underline{g}, \overline{g}]$.

Claim C.6. *Each objective function segment expressed below is quasi-convex in $g \in [\underline{g}, \overline{g}]$, where $\underline{g}$ and $\overline{g}$ are defined in (C.16).*

- *When (C.22) holds,*

$$\lambda' F^\ell(g) + (1-\lambda')F^r(g) = \begin{cases} \frac{g}{2\beta_1}\left\{g[\lambda'(\rho+\beta_1) + (1-\lambda')2\beta_1] - \sqrt{g^2-4\alpha\beta_1}[\lambda'(\rho-\beta_1)]\right\}, & \text{if } \underline{g} \leq g \leq \sqrt{1-\alpha}, \\ \frac{g}{2\beta_1^2(1+\beta_1)^2}\left\{\begin{matrix} \lambda'[g\beta_1(1+\beta_1)^2(\rho+\beta_1) - \sqrt{g^2-4\alpha\beta_1}\beta_1(1+\beta_1)^2(\rho-\beta_1)] \\ +(1-\lambda')\left[\begin{matrix} g\{\beta_1^2(1+\rho-\beta_r-2\rho\beta_r) - \rho\beta_r(1+2\beta_1)\} \\ +\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}\{\beta_1^2(1+\rho-\beta_r) + \rho\beta_r(1+2\beta_1)\} \end{matrix}\right] \end{matrix}\right\}, & \text{if } \sqrt{1-\alpha} \leq g \leq \overline{g}. \end{cases} \tag{C.33}$$

- *When (C.23) holds,*

$$\lambda' F^\ell(g) + (1-\lambda')F^r(g) = \begin{cases} \frac{g}{2\beta_1}\left\{g\left[\lambda'(\rho+\beta_1) + (1-\lambda')\frac{\beta_1(\rho+\beta_r)^2}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right] + \sqrt{g^2-4\alpha\beta_1}\lambda'(\beta_1-\rho)\right\}, & \text{if } \underline{g} \leq g < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}, \\ \frac{g}{2\beta_1^2(1+\beta_1)^2}\left\{\begin{matrix} \lambda'[g\beta_1(1+\beta_1)^2(\rho+\beta_1) - \sqrt{g^2-4\alpha\beta_1}\beta_1(1+\beta_1)^2(\rho-\beta_1)] \\ +(1-\lambda')\left[\begin{matrix} g\{\beta_1^2(1+\rho-\beta_r-2\rho\beta_r) - \rho\beta_r(1+2\beta_1)\} \\ +\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}\{\beta_1^2(1+\rho-\beta_r) + \rho\beta_r(1+2\beta_1)\} \end{matrix}\right] \end{matrix}\right\}, & \text{if } \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \leq g \leq \overline{g}, \end{cases} \tag{C.34}$$

  where B is defined in (C.21).

- *When (C.24) holds,*

$$\lambda' F^\ell(g) + (1-\lambda')F^r(g) = \frac{g}{2\beta_1}\begin{cases} g\left[\lambda'(\rho+\beta_1) + (1-\lambda')\frac{\beta_1(\rho+\beta_r)^2}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right] + \sqrt{g^2-4\alpha\beta_1}\lambda'(\beta_1-\rho), & \text{if } \underline{g} \leq g < \sqrt{\frac{4\alpha\beta_1}{1-A^2}}, \\ (g+\sqrt{g^2-4\alpha\beta_1})\left[\beta_1 - \lambda'\rho + \frac{(1-\lambda')\rho\beta_r}{\beta_1}\right] + 2g\left[\lambda'\rho - \frac{(1-\lambda')\rho\beta_r}{\beta_1}\right], & \text{if } \sqrt{\frac{4\alpha\beta_1}{1-A^2}} \leq g \leq \overline{g}, \end{cases} \tag{C.35}$$

  where A is defined in (C.21).

- *When (C.25) holds,*

$$\lambda' F^{\ell}(g) + (1-\lambda') F^{r}(g) = \frac{g}{2\beta_1}\left\{(g+\sqrt{g^2-4\alpha\beta_1})\left[\beta_1 - \lambda'\rho + \frac{(1-\lambda')\rho\beta_r}{\beta_1}\right] + 2g\left[\lambda'\rho - \frac{(1-\lambda')\rho\beta_r}{\beta_1}\right]\right\}, \quad \text{if} \quad \underline{g} \le g \le \overline{g}. \tag{C.36}$$

Proof of Claim C.6. We first examine the objective function segment for $(x^r_{L_0}(g), x^r_{R_0}(g))$, which corresponds to the parameter region of $\underline{g} \le g \le \sqrt{1-\alpha}$ when (C.22) holds, i.e., the first segment of (C.33). Our strategy for proving the quasi-convexity of this function is to take its first order partial derivative, with respect to g, and show that this derivative is first negative then positive in g. Such that,

$$\frac{\partial(\text{first segment of (C.33)})}{\partial g} = \frac{g}{2\beta_1}\left\{\lambda'(\rho+\beta_1) + (1-\lambda')2\beta_1 - \underbrace{\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)}_{\searrow \text{ in } g, \text{ by Claim C.1}}[\lambda' \underbrace{(\rho-\beta_1)}_{>0}]\right\},$$

hence, our claim is established.

Second, we examine the objective function segments for $(x^r_{L_\alpha}(g), x^r_{R_\alpha}(g))$, which correspond to the parameter regions $\underline{g} \le g \le \overline{g}$ when (C.25) holds, i.e., (C.36), and $\sqrt{\frac{4\alpha\beta_1}{1-A^2}} \le g \le \overline{g}$ when (C.24) holds, i.e., the second segment of (C.35). Our strategy for proving the quasi-convexity of this function is, again, to take its first order partial derivative, with respect to g, and show that this derivative is either first negative then positive in g, or positive. Such that,

$$\frac{\partial(\text{C.36})}{\partial g} = \frac{\partial(\text{second segment of (C.35)})}{\partial g} = \frac{g}{2\beta_1}\left\{\underbrace{\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)}_{\ge 2 \text{ and } \searrow \text{ in } g, \text{ by Claim C.1}}\left[\beta_1 - \lambda'\rho + \frac{(1-\lambda')\rho\beta_r}{\beta_1}\right] + 2\left[\beta_1 + \lambda'\rho - (1-\lambda')\frac{\rho\beta_r}{\beta_1}\right]\right\}.$$

Now, if $\beta_1 - \lambda'\rho + \frac{(1-\lambda')\rho\beta_r}{\beta_1} \le 0$, then its straightforward to see that this derivative is first negative then positive in g. If, on the other hand, $\beta_1 - \lambda'\rho + \frac{(1-\lambda')\rho\beta_r}{\beta_1} > 0$, by Claim C.1 we have the derivative positive, irrespective of the sign of $\lambda'\rho - \frac{(1-\lambda')\rho\beta_r}{\beta_1}$, establishing our claim.

Third, we examine the objective function segments for $(x^r_{L_{1-\alpha}}(g), x^r_{R_{1-\alpha}}(g))$, which correspond to the parameter regions of $\sqrt{1-\alpha} \le g \le \overline{g}$ when (C.22) holds, i.e., the second segment of (C.33), and $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \le g \le \overline{g}$ when (C.23) holds, i.e., the second segment of (C.34). Our strategy for proving the quasi-convexity of this function is, again, to take its first order partial derivative, with respect to g, and show that this derivative is either first negative then positive in g, or positive, i.e., exhibit single crossing property. Such that,

$$\frac{\partial(\text{second segment of (C.33)})}{\partial g} = \frac{\partial(\text{second segment of (C.34)})}{\partial g} = \frac{g}{2\beta_1^2(1+\beta_1)^2}\left\{\begin{array}{l} \lambda'\beta_1(1+\beta_1)^2\left[2(\rho+\beta_1) - \left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g} + \frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)(\rho-\beta_1)\right] \\ (1-\lambda')\left[2[\beta_1^2(\rho+1)(1-\beta_r) - \rho\beta_r(1+\beta_1)^2] + \left(\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}\right)[\beta_1^2(\rho+1)(1-\beta_r) + \rho\beta_r(1+\beta_1)^2]\right] \end{array}\right\}. \tag{C.37}$$

Now, notice that the second term of (C.37) (the term with a coefficient of $1-\lambda'$) is always positive by Claim C.1 and the fact that $\beta_r \leq 1$. Therefore, if we also have the first term of (C.37) (the term with a coefficient of λ') always positive, then we have the derivative positive, establishing our claim. We, thus, in the remainder of this proof, focus on and examine the case when the first term of (C.37) is negative. For such analysis, we utilize the signed-ratio monotonicity property of Quah and Strulovici (2012). The authors, in proposition 1, established that for two functions f and h with single crossing property (maximum of one sign change over the function domain), $r_1 f + r_2 h$ also has single crossing property for any nonnegative scalars r_1 and r_2, if and only if, $-h/f$ has signed-ratio monotonicity property, i.e., $-h/f$ is monotonically decreasing in its argument for $h<0$ and $f>0$. Following their convention, we can re-write our (C.37) as follows

$$\frac{\partial(\text{second segment of (C.33)})}{\partial g} = \frac{\partial(\text{second segment of (C.34)})}{\partial g} =$$
$$g\left[\underbrace{\left(\frac{\lambda'(\rho-\beta_1)}{2\beta_1}\right)}_{r_1} h(g) + \underbrace{\left(\frac{(1-\lambda')[\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2]}{2\beta_1^2(1+\beta_1)^2}\right)}_{r_2} f(g)\right],$$

where

$$h(g) = -\frac{\sqrt{g^2-4\alpha\beta_1}}{g} - \frac{g}{\sqrt{g^2-4\alpha\beta_1}} + 2\left(\frac{\rho+\beta_1}{\rho-\beta_1}\right)$$

and

$$f(g) = \frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g} + \frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}} + 2\left(\frac{\beta_1^2(\rho+1)(1-\beta_r)-\rho\beta_r(1+\beta_1)^2}{\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2}\right).$$

Accordingly, we have both r_1 and r_2 nonnegative as $\rho > \beta_1$. Furthermore, by Claim C.1, both $h(g)$ and $f(g)$ have single crossing property. Therefore, we just need to show that $-h(g)/f(g)$ is monotonically decreasing in g. For this purpose, we take the first order partial derivative of $-h(g)/f(g)$ with respect to g, and show that this derivative d is negative. Denoting as $X_1 = \frac{\rho+\beta_1}{\rho-\beta_1} > 1$ and $X_2 = \frac{\beta_1^2(\rho+1)(1-\beta_r)-\rho\beta_r(1+\beta_1)^2}{\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2} > -1$, d is calculated as

$$d = -\left[\frac{g^2}{(1-\alpha)(1+\beta_1)} + 4\beta_1\right]\left\{\frac{\left[\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}+4\beta_1} - \sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}}\right]^2}{+2(X_2+1)\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}}\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}+4\beta_1}}\right\}$$
$$+\left[\frac{g^2}{\alpha}-4\beta_1\right]\left\{\left[\sqrt{\frac{g^2}{\alpha}}-\sqrt{\frac{g^2}{\alpha}-4\beta_1}\right]^2 - 2(X_1-1)\sqrt{\frac{g^2}{\alpha}}\sqrt{\frac{g^2}{\alpha}-4\beta_1}\right\}.$$

Now, notice that $g^2/\alpha - 4\beta_1 \geq \underline{g}^2/\alpha - 4\beta_1 = (\beta_1-1)^2 \geq 0$. Therefore, as the first line of d is negative, $d<0$ when $\left[\sqrt{\frac{g^2}{\alpha}}-\sqrt{\frac{g^2}{\alpha}-4\beta_1}\right]^2 - 2(X_1-1)\sqrt{\frac{g^2}{\alpha}}\sqrt{\frac{g^2}{\alpha}-4\beta_1} < 0$, establishing our claim. We, thus, in the remainder of this proof, focus on the case when $\left[\sqrt{\frac{g^2}{\alpha}}-\sqrt{\frac{g^2}{\alpha}-4\beta_1}\right]^2 - 2(X_1-1)\sqrt{\frac{g^2}{\alpha}}\sqrt{\frac{g^2}{\alpha}-4\beta_1} \geq 0$. In that scenario, taking $(X_1, X_2) = (1,-1)$ gives us (i) a lower bound on the absolute value of the first line, and (ii) an upper bound on the second line of d. Accordingly, it only left for us to show the following inequality holds

$$\left[\frac{g^2}{(1-\alpha)(1+\beta_1)}+4\beta_1\right]\left[\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}+4\beta_1}-\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}}\right]^2 \geq \left[\frac{g^2}{\alpha}-4\beta_1\right]\left[\sqrt{\frac{g^2}{\alpha}}-\sqrt{\frac{g^2}{\alpha}-4\beta_1}\right]^2,$$

for $g \in [\underline{g}, \overline{g}]$. Now, let us define as $a = \sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)} + 4\beta_1}$ and $b = \sqrt{\frac{g^2}{\alpha} - 4\beta_1}$, where both a and b are trivially increasing in $g \in [\underline{g}, \overline{g}]$. Then, we need to show

$$a^2 \underbrace{[a - \sqrt{a^2 - 4\beta_1}]}_{\geq 0}^2 \geq b^2 \underbrace{[\sqrt{b^2 + 4\beta_1} - b]}_{\geq 0}^2 \Rightarrow a[a - \sqrt{a^2 - 4\beta_1}] \geq b[\sqrt{b^2 + 4\beta_1} - b]. \tag{C.38}$$

Notice that $a[a - \sqrt{a^2 - 4\beta_1}]$ is decreasing in a, since its derivative with respect to a is given by $2a - \sqrt{a^2 - 4\beta_1} - \frac{a^2}{\sqrt{a^2 - 4\beta_1}} = -\left[\frac{a}{(a^2-4\beta_1)^{1/4}} - (a^2 - 4\beta_1)^{1/4}\right]^2 < 0$, and $b[\sqrt{b^2 + 4\beta_1} - b]$ is increasing in b, since its derivative with respect to b is given by $-2b + \sqrt{b^2 + 4\beta_1} + \frac{b^2}{\sqrt{b^2+4\beta_1}} = \left[\frac{b}{(b^2+4\beta_1)^{1/4}} - (b^2 + 4\beta_1)^{1/4}\right]^2 > 0$. Therefore, we can sufficiently evaluate (C.38) at $g = \overline{g} = \alpha\beta_1 + 1$, as both a and b are increasing in g. Plugging back the definitions of a and b into (C.38) we have

$$\begin{aligned}
&\frac{g^2}{(1-\alpha)(1+\beta_1)} - \sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)} + 4\beta_1}\sqrt{\frac{g^2}{(1-\alpha)(1+\beta_1)}} \geq \sqrt{\frac{g^2}{\alpha} - 4\beta_1}\sqrt{\frac{g^2}{\alpha}} - \frac{g^2}{\alpha} \\
&\Leftrightarrow \frac{1}{(1-\alpha)(1+\beta_1)}[g^2 - g\sqrt{g^2 + 4(1-\alpha)\beta_1(1+\beta_1)}] \geq \frac{1}{\alpha}[g\sqrt{g^2 - 4\alpha\beta_1} - g^2] \\
&\Leftrightarrow \frac{1}{(1-\alpha)(1+\beta_1)}[g - \sqrt{g^2 + 4(1-\alpha)\beta_1(1+\beta_1)}] \geq \frac{1}{\alpha}[\sqrt{g^2 - 4\alpha\beta_1} - g].
\end{aligned}$$

For $g = \overline{g} = \alpha\beta_1 + 1$, above inequality takes the following form

$$\frac{1}{(1-\alpha)(1+\beta_1)}[\alpha\beta_1 + 1 - \sqrt{(\alpha\beta_1+1)^2 + 4(1-\alpha)\beta_1(1+\beta_1)}] \geq \frac{1}{\alpha}[|\alpha\beta_1 - 1| - \alpha\beta_1 - 1], \tag{C.39}$$

which leads us to separate the analysis into two cases, $\alpha\beta_1 \geq 1$ and $\alpha\beta_1 < 1$. When $\alpha\beta_1 \geq 1$, (C.39) is satisfied as we have

$$\begin{aligned}
&4[1 + \beta_1(1-\alpha)](1-\alpha)(1+\beta_1) \;\geq\; 0 \\
&\Leftrightarrow\; \alpha^2(\alpha\beta_1 - 1)^2 + 4(1 + \beta_1 - \alpha\beta_1)[\alpha^2\beta_1 + (1-\alpha)(1+\beta_1)] \;\geq\; \alpha^2(\alpha\beta_1 - 1)^2 + 4\alpha^2\beta_1(1 + \beta_1 - \alpha\beta_1) \\
&\Leftrightarrow\; [\alpha\beta_1 + 1 + 2(1-\alpha)\beta_1(1+\beta_1)]^2 \;\geq\; (\alpha\beta_1 + 1)^2 + 4(1-\alpha)\beta_1(1+\beta_1) \\
&\Leftrightarrow\; [\alpha(\alpha\beta_1 + 1) + 2(1-\alpha)(1+\beta_1)]^2 \;\geq\; \alpha^2[(\alpha\beta_1+1)^2 + 4(1-\alpha)\beta_1(1+\beta_1)] \\
&\Leftrightarrow\; \frac{\alpha(\alpha\beta_1+1) + 2(1-\alpha)(1+\beta_1)}{\alpha(1-\alpha)(1+\beta_1)} \;\geq\; \sqrt{\frac{(\alpha\beta_1+1)^2}{(1-\alpha)^2(1+\beta_1)^2} + \frac{4\beta_1}{(1-\alpha)(1+\beta_1)}} \\
&\Leftrightarrow\; \frac{\alpha\beta_1+1}{(1-\alpha)(1+\beta_1)} - \sqrt{\frac{(\alpha\beta_1+1)^2}{(1-\alpha)^2(1+\beta_1)^2} + \frac{4\beta_1}{(1-\alpha)(1+\beta_1)}} \;\geq\; -\frac{2}{\alpha}.
\end{aligned}$$

When $\alpha\beta_1 < 1$, on the other hand, (C.39) is also satisfied as we have

$$\begin{aligned}
&4(1-\alpha)\beta_1(1+\beta_1)[\alpha\beta_1 + (1-\alpha)\beta_1(1+\beta_1)] \;\geq\; 0, \\
&\Leftrightarrow\; (\alpha\beta_1+1)^2 + 4(\alpha\beta_1+1)(1-\alpha)\beta_1(1+\beta_1) + 4(1-\alpha)^2\beta_1^2(1+\beta_1)^2 \;\geq\; (\alpha\beta_1+1)^2 + 4(1-\alpha)\beta_1(1+\beta_1) \\
&\Leftrightarrow\; [\alpha\beta_1 + 1 + 2(1-\alpha)\beta_1(1+\beta_1)]^2 \;\geq\; (\alpha\beta_1+1)^2 + 4(1-\alpha)\beta_1(1+\beta_1) \\
&\Leftrightarrow\; \frac{\alpha\beta_1 + 1 + 2(1-\alpha)\beta_1(1+\beta_1)}{(1-\alpha)(1+\beta_1)} \;\geq\; \sqrt{\frac{(\alpha\beta_1+1)^2}{(1-\alpha)^2(1+\beta_1)^2} + \frac{4\beta_1}{(1-\alpha)(1+\beta_1)}} \\
&\Leftrightarrow\; \frac{\alpha\beta_1+1}{(1-\alpha)(1+\beta_1)} - \sqrt{\frac{(\alpha\beta_1+1)^2}{(1-\alpha)^2(1+\beta_1)^2} + \frac{4\beta_1}{(1-\alpha)(1+\beta_1)}} \;\geq\; -2\beta_1,
\end{aligned}$$

establishing our claim.

Lastly, we examine the objective function for $(x^r_{L_*}(g), x^r_{R_*}(g))$, which corresponds to the parameter regions of $\underline{g} \le g < \sqrt{\frac{4\alpha\beta_1}{1-A^2}}$ when (C.24) holds, i.e., the first segment of (C.35), and $\underline{g} \le g < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$ when (C.23) holds, i.e., the first segment of (C.34). Our strategy for proving the quasi-convexity of this function is, again, to take its first order partial derivative, with respect to g, and show that this derivative is either first negative then positive in g, or positive. Such that,

$$\frac{\partial(\text{first segment of (C.35)})}{\partial g} = \frac{\partial(\text{first segment of (C.34)})}{\partial g} =$$
$$\frac{g}{2\beta_1}\left\{2\left[\lambda'(\rho+\beta_1)+(1-\lambda')\frac{\beta_1(\rho+\beta_r)^2}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right] - \underbrace{\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)}_{\searrow \text{ in } g, \text{ by Claim C.1}}\lambda'\underbrace{(\rho-\beta_1)}_{>0}\right\},$$

hence, our claim is established. □

Now that we have Claim C.6 established, we next show the global quasi-convexity of $\lambda' F^\ell(\underline{g}) + (1-\lambda')F^r(\underline{g})$ in $g \in [\underline{g}, \overline{g}]$, for the conditions (C.22)-(C.25) with multiple segments, namely (C.22), (C.23) and (C.24). Note that, for condition (C.25), the quasi-convexity of $\lambda' F^\ell(\underline{g}) + (1-\lambda')F^r(\underline{g})$ in $g \in [\underline{g}, \overline{g}]$ is automatically established by Claim C.6.

When (C.22) holds, we first outline the necessary conditions in order to have two objective function segments of (C.33) active, for $g \in [\underline{g}, \overline{g}]$. For that, it must be the case that $\underline{g} < \sqrt{1-\alpha} \Leftrightarrow \underline{g}^2 < 1-\alpha$, otherwise the global quasi-convexity is trivially established, as each objective value segment is shown to be quasi-convex in g by Claim C.6. Now, recall that we have $\alpha\beta_1\rho > 1$. Hence, when $\beta_1 \le 1$, we have $\rho \le 2\beta_1 + \beta_r \le 2 + \beta_r \le 3$ and, therefore, $\alpha \ge \alpha\beta_1 > 1/\rho \ge 1/3$, which implies

$$\underline{g}^2 = \alpha\beta_1^2 + \underbrace{2\alpha\beta_1}_{\ge 2/3} + \underbrace{\alpha}_{\ge 1/3} \ge 1 \ge 1-\alpha.$$

Therefore, in order to have the two objective function segments of (C.33) active, we must have (i) $\beta_1 > 1$ and $\underline{g} = \sqrt{\alpha}(\beta_1+1) < \sqrt{1-\alpha}$, where the latter brings us the condition (ii) $\alpha < \frac{1}{\beta_1^2+2\beta_1+2}$. Condition (ii), together with $\alpha\beta_1\rho > 1$, imply condition (iii) $\rho \ge \beta_1 + \frac{2}{\beta_1} + 2$. Also, (C.22) condition $\beta_1 \ge \frac{\rho-\beta_r}{2}$ together with (iii), imply

$$\beta_1 \ge \frac{\beta_1}{2} + \frac{1}{\beta_1} + 1 - \frac{\beta_r}{2} \ge \frac{\beta_1(\beta_1+1)+2}{2\beta_1} \quad \Leftrightarrow \quad \beta_1^2 \ge \underbrace{\beta_1+2}_{>3 \text{ by (i)}}$$
$$\Rightarrow \quad \text{(iv)} \quad \beta_1^3 \ge 3\beta_1 \quad \underbrace{\ge}_{\substack{\text{by (i) and the}\\ \text{fact that } \beta_r \le 1}} \quad 2\beta_1 + \beta_r \ge \rho.$$

Then, our strategy for proving the global quasi-convexity in this case is to, first, take the first order partial derivative of the second segment of (C.33) with respect to g:

$$\frac{\partial(\text{second segment of (C.33)})}{\partial g} = \tag{C.40}$$
$$\frac{g}{2\beta_1^2(1+\beta_1)^2}\left\{\begin{array}{l}\lambda'\beta_1(1+\beta_1)^2\left[2(\rho+\beta_1) - \left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)(\rho-\beta_1)\right] \\ +(1-\lambda')\left\{+\left(\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g}+\frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}\right)\frac{2\left[\beta_1^2(\rho+1)(1-\beta_r)-\rho\beta_r(1+\beta_1)^2\right]}{\left[\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2\right]}\right\}\end{array}\right.,$$

and then, show that its positive at the switching point between two segments, $g=\sqrt{1-\alpha}$. It is straightforward to see that, in (C.40), the second term that has a coefficient of $(1-\lambda')$ is positive, irrespective of g, as by Claim C.1 and the fact that $\beta_r \le 1$. Next, as $\rho > \beta_1$, by Claim C.1, it suffices us to show that the first line of (C.40) is positive at $g=\underline{g}=\sqrt{\alpha}(\beta_1+1)$, which is equivalent to showing that

$$2\left(\frac{\rho+\beta_1}{\rho-\beta_1}\right) \ge \frac{\sqrt{\alpha(\beta_1+1)^2-4\alpha\beta_1}}{\sqrt{\alpha}(\beta_1+1)} + \frac{\sqrt{\alpha}(\beta_1+1)}{\sqrt{\alpha(\beta_1+1)^2-4\alpha\beta_1}} = \frac{\beta_1-1}{\beta_1+1}+\frac{\beta_1+1}{\beta_1-1}$$
$$\Leftrightarrow \quad (\rho+\beta_1)(\beta_1^2-1) \ge (\rho-\beta_1)(\beta_1^2+1) \quad \Leftrightarrow \quad \beta_1^3 \ge \rho,$$

which follows by (iv). Therefore, when (C.22) holds, we have $\lambda' F^\ell(g)+(1-\lambda')F^r(g)$ quasi-convex in $g\in[\underline{g},\overline{g}]$.

When (C.23) holds, our strategy for proving the global quasi-convexity is to, first, take the first order partial derivatives of both segments of (C.34) with respect to g:

$$\frac{\partial(\text{C.34})}{\partial g} = \frac{g}{2\beta_1^2(1+\beta_1)^2}$$
$$\begin{cases} \left\{ \lambda'\beta_1(1+\beta_1)^2\left[2(\rho+\beta_1)-\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)(\rho-\beta_1)\right] \right. \\ \qquad \left. +(1-\lambda')\frac{[\beta_1(1+\beta_1)(\rho+\beta_r)]^2}{(\beta_1+\beta_r)(\rho-\beta_1)} \right\}, & \text{for } \ \underline{g}\le g < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}, \\ \left\{ \lambda'\beta_1(1+\beta_1)^2\left[2(\rho+\beta_1)-\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)(\rho-\beta_1)\right] \right. \\ \qquad \left. +(1-\lambda')\left[\left(\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g}+\frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}\right)\frac{2[\beta_1^2(\rho+1)(1-\beta_r)-\rho\beta_r(1+\beta_1)^2]}{[\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2]}\right] \right\}, & \text{for } \ \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \le g \le \overline{g}, \end{cases}$$

and then, show that these two partial derivatives are equal at the switching point between two segments, $g=\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$. Notice that, for $g=\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$, we have $\left(\frac{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}{g}+\frac{g}{\sqrt{g^2+4(1-\alpha)\beta_1(1+\beta_1)}}\right) = B+\frac{1}{B} = \frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)} + \frac{(\beta_1+\beta_r)(\rho-\beta_1)}{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}$. Hence, as the first lines of both segments are equal to each other, it suffices for us to show

$$\begin{aligned} \frac{[\beta_1(1+\beta_1)(\rho+\beta_r)]^2}{(\beta_1+\beta_r)(\rho-\beta_1)} &= 2[\beta_1^2(\rho+1)(1-\beta_r)-\rho\beta_r(1+\beta_1)^2]+(\beta_1+\beta_r)(\rho-\beta_1) \\ &\quad + \frac{[\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2]^2}{(\beta_1+\beta_r)(\rho-\beta_1)} \\ \Leftrightarrow \ \beta_1^2(1+\beta_1)^2(\rho+\beta_r)^2 &= [\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2+(\beta_1+\beta_r)(\rho-\beta_1)]^2 \\ &\quad - 4\rho\beta_r(1+\beta_1)^2(\beta_1+\beta_r)(\rho-\beta_1) \\ \Leftrightarrow \ (1+\beta_1)^2[\beta_1^2(\rho+\beta_r)^2+4\rho\beta_r(\beta_1+\beta_r)(\rho-\beta_1)] &= [\beta_1^2(\rho+1)(1-\beta_r)+\rho\beta_r(1+\beta_1)^2+(\beta_1+\beta_r)(\rho-\beta_1)]^2 \\ \Leftrightarrow \ \beta_1^2(\rho-\beta_r)^2+4\rho\beta_r(\rho\beta_1+\rho\beta_r-\beta_1\beta_r) &= [\beta_1(\rho-\beta_r)+2\rho\beta_r]^2, \end{aligned}$$

which trivially follows after straightforward algebraic manipulation. Therefore, when (C.23) holds, we have $\lambda' F^\ell(g)+(1-\lambda')F^r(g)$ quasi-convex in $g\in[\underline{g},\overline{g}]$.

When (C.24) holds, our strategy for proving the global quasi-convexity is to, first, take the first order partial derivatives of both segments of (C.35) with respect to g:

$$\frac{\partial(\text{C.35})}{\partial g} = \frac{g}{2\beta_1}\begin{cases} \left\{ 2\left[\lambda'(\rho+\beta_1)+(1-\lambda')\frac{\beta_1(\rho+\beta_r)^2}{2(\beta_1+\beta_r)(\rho-\beta_1)}\right] \right. \\ \qquad \left. -\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)\lambda'(\rho-\beta_1) \right\}, & \text{for } \ \underline{g}\le g < \sqrt{\frac{4\alpha\beta_1}{1-A^2}}, \\ \left\{ \left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)\left[\beta_1-\lambda'\rho+\frac{(1-\lambda')\rho\beta_r}{\beta_1}\right]+2\left[\beta_1+\lambda'\rho-(1-\lambda')\frac{\rho\beta_r}{\beta_1}\right] \right\}, & \text{for } \ \sqrt{\frac{4\alpha\beta_1}{1-A^2}} \le g \le \overline{g}, \end{cases}$$

and then, show that these two partial derivatives are equal at the switching point between two segments, $g=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}$. Notice that, for $g=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}$, we have $\left(\frac{\sqrt{g^2-4\alpha\beta_1}}{g}+\frac{g}{\sqrt{g^2-4\alpha\beta_1}}\right)=A+\frac{1}{A}$. Hence, it suffices for us to show

$$\begin{aligned}
2\lambda'\beta_1+(1-\lambda')\frac{\beta_1(\rho+\beta_r)^2}{(\beta_1+\beta_r)(\rho-\beta_1)}+\left[A+\frac{1}{A}\right]\lambda'\beta_1 &= \left[A+\frac{1}{A}\right]\left[\beta_1+(1-\lambda')\frac{\rho\beta_r}{\beta_1}\right]+2\left[\beta_1-(1-\lambda')\frac{\rho\beta_r}{\beta_1}\right]\\
\Leftrightarrow\ \frac{(\rho+\beta_r)^2}{(\beta_1+\beta_r)(\rho-\beta_1)}+\frac{2\rho\beta_r}{\beta_1^2}-2 &= \left[1+\frac{\rho\beta_r}{\beta_1^2}\right]\left[A+\frac{1}{A}\right]\\
\Leftrightarrow\ \frac{(\rho+\beta_r)^2}{(\beta_1+\beta_r)(\rho-\beta_1)}-A\left[1+\frac{\rho\beta_r}{\beta_1^2}\right] &= 2\left[1-\frac{\rho\beta_r}{\beta_1^2}\right]+\frac{1}{A}\left[1+\frac{\rho\beta_r}{\beta_1^2}\right]\\
\Leftrightarrow\ \frac{(\rho+\beta_r)^2}{(\beta_1+\beta_r)(\rho-\beta_1)}-\frac{\beta_1^2+\rho\beta_r}{(\beta_1+\beta_r)(\rho-\beta_1)}\left[\frac{\beta_1^2+\rho\beta_r}{\beta_1}\right] &= \frac{2(\beta_1^2-\rho\beta_r)}{\beta_1^2}+\frac{(\beta_1+\beta_r)(\rho-\beta_1)}{\beta_1^2+\rho\beta_r}\left[\frac{\beta_1^2+\rho\beta_r}{\beta_1^2}\right]\\
\Leftrightarrow\ \frac{\beta_1^2(\rho+\beta_r)^2-(\beta_1^2+\rho\beta_r)^2}{(\beta_1+\beta_r)(\rho-\beta_1)} &= 2(\beta_1^2-\rho\beta_r)+(\beta_1+\beta_r)(\rho-\beta_1),
\end{aligned}$$

which trivially follows after straightforward algebraic manipulation. Notice that we use the definition of $A=\frac{\rho\beta_r+\beta_1^2}{(\beta_1+\beta_r)(\rho-\beta_1)}$ in establishing the third relation above. Subsequently, when (C.24) holds, we have $\lambda' F^\ell(g)+(1-\lambda')F^r(g)$ quasi-convex in $g\in[\underline{g},\overline{g}]$. With this result, for all conditions (C.22)-(C.25), the quasi-convexity of $\lambda' F^\ell(\underline{g})+(1-\lambda')F^r(\underline{g})$ in $g\in[\underline{g},\overline{g}]$ is established, hence, $g^\star\in\{\underline{g},\overline{g}\}$.

Now that we have $g^\star\in\{\underline{g},\overline{g}\}$, by Lemma C.8, the solution to (C.15) is given by $(x_L^{\ell\star}(\overline{g}),x_R^{\ell\star}(\overline{g}),x_L^{r\star}(\overline{g}),x_R^{r\star}(\overline{g}))$ (resp., $(x_L^{\ell\star}(\underline{g}),x_R^{\ell\star}(\underline{g}),x_L^{r\star}(\underline{g}),x_R^{r\star}(\underline{g}))$) if $\lambda' F^\ell(\underline{g})+(1-\lambda')F^r(\underline{g})\le\lambda' F^\ell(\overline{g})+(1-\lambda')F^r(\overline{g})$ (resp., if $\lambda' F^\ell(\underline{g})+(1-\lambda')F^r(\underline{g})>\lambda' F^\ell(\overline{g})+(1-\lambda')F^r(\overline{g})$). Accordingly, the platform's optimal strategy is determined by:

$$\begin{aligned}
&\lambda' F^\ell(\underline{g})+(1-\lambda')F^r(\underline{g})\lessgtr\lambda' F^\ell(\overline{g})+(1-\lambda')F^r(\overline{g})\\
\Leftrightarrow\quad &F^r(\underline{g})-F^r(\overline{g})\lessgtr\lambda'\left[F^\ell(\overline{g})-F^\ell(\underline{g})+F^r(\underline{g})-F^r(\overline{g})\right]\\
\Leftrightarrow\quad &\lambda'\lessgtr\frac{F^r(\underline{g})-F^r(\overline{g})}{F^\ell(\overline{g})-F^\ell(\underline{g})+F^r(\underline{g})-F^r(\overline{g})}\\
\text{by (C.4)}\quad\Leftrightarrow\quad &\lambda\lessgtr\underbrace{\frac{v_R^r\left[F^r(\underline{g})-F^r(\overline{g})\right]}{v_R^\ell\left[F^\ell(\overline{g})-F^\ell(\underline{g})\right]+v_R^r\left[F^r(\underline{g})-F^r(\overline{g})\right]}}_{\lambda_S^{ma}}.
\end{aligned}\tag{C.41}$$

We note that $\lambda_S^{ma}\in[0,1]$, as $F^r(g)$ is non-decreasing in $g\in[\underline{g},\overline{g}]$ and, by (C.20), $F^\ell(\overline{g})-F^\ell(\underline{g})=(1-\alpha)(1-\alpha\beta_1\rho)\le 0$, as $\alpha\le 1$ and $\alpha\beta_1\rho>1$. The former condition is straightforward to see as, in (C.18), coefficients of the decision variables are non-negative in the inequality constraint regarding g. This completes the proof. ■

Lemma C.10 (Characterization of $\check{\sigma}_L^r$ and $\check{\sigma}_R^r$).

$$(\check{\sigma}_L^r,\check{\sigma}_R^r)=\begin{cases}
\left(\check{\sigma}_{L_0}^r,\check{\sigma}_{R_0}^r\right), & \text{for (C.22) and } \alpha\le\frac{1}{(\beta_1+1)^2+1},\\
\left(\check{\sigma}_{L_{1-\alpha}}^r,\check{\sigma}_{R_{1-\alpha}}^r\right), & \text{for (C.22) and } \alpha\ge\frac{1}{(\beta_1+1)^2+1}, \text{ and, for (C.23) and } \alpha\ge\frac{4\beta_1}{(\beta_1+1)(B^2-1)+4\beta_1},\\
\left(\check{\sigma}_{L_*}^r,\check{\sigma}_{R_*}^r\right), & \text{for (C.23) and } \alpha<\frac{4\beta_1}{(\beta_1+1)(B^2-1)+4\beta_1}, \text{ and, for (C.24) and } 1-A^2<\frac{4\beta_1}{(\beta_1+1)^2},\\
\left(\check{\sigma}_{L_\alpha}^r,\check{\sigma}_{R_\alpha}^r\right), & \text{for (C.24) and } 1-A^2\ge\frac{4\beta_1}{(\beta_1+1)^2}, \text{ and, for (C.25)},
\end{cases}\tag{C.42}$$

where

$$\left(\check{\sigma}_{L_0}^r,\check{\sigma}_{R_0}^r\right)=\left(0,\frac{\alpha(\beta_1+1)^2}{1-\alpha}\right),$$

$$\left(\check{\sigma}^r_{L_{1-\alpha}}, \check{\sigma}^r_{R_{1-\alpha}}\right) = \left(\sqrt{\frac{\alpha(\beta_1+1)+4(1-\alpha)\beta_1}{4\alpha(\beta_1+1)}} + \frac{\alpha(\beta_1+3)-2}{2\alpha(\beta_1+1)}, 1\right),$$
$$\left(\check{\sigma}^r_{L_*}, \check{\sigma}^r_{R_*}\right) = \frac{(\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{[2(\beta_1+\beta_r)(\rho-\beta_1)]^2}\left(\rho-\beta_r-2\beta_1, \frac{\alpha[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{1-\alpha}\right),$$
$$\left(\check{\sigma}^r_{L_\alpha}, \check{\sigma}^r_{R_\alpha}\right) = \left(1, \frac{\alpha(\beta_1^2-1)\mathbb{1}[\beta_1>1]}{1-\alpha}\right),$$

and A and B are defined as in (C.21).

Proof of Lemma C.10. As $\lambda > \lambda^{ma}_S$, by Lemma C.9, we have $g^\star = \underline{g}$, hence, by Lemmas C.7 and C.8, $(\check{x}^r_L, \check{x}^r_R)$ is given as follows

$$(\check{x}^r_L, \check{x}^r_R) = \begin{cases} \left(x^r_{L_0}(\underline{g}), x^r_{R_0}(\underline{g})\right), & \text{for (C.22) and } \underline{g} \le \sqrt{1-\alpha}, \\ \left(x^r_{L_{1-\alpha}}(\underline{g}), x^r_{R_{1-\alpha}}(\underline{g})\right), & \text{for (C.22) and } \underline{g} \ge \sqrt{1-\alpha}, \text{ and, for (C.23) and } \underline{g} \ge \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}, \\ \left(x^r_{L_*}(\underline{g}), x^r_{R_*}(\underline{g})\right), & \text{for (C.23) and } \underline{g} < \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}, \text{ and, for (C.24) and } \underline{g} < \sqrt{\frac{4\alpha\beta_1}{1-A^2}}, \\ \left(x^r_{L_\alpha}(\underline{g}), x^r_{R_\alpha}(\underline{g})\right), & \text{for (C.24) and } \underline{g} \ge \sqrt{\frac{4\alpha\beta_1}{1-A^2}}, \text{ and, for (C.25).} \end{cases} \tag{C.43}$$

First notice that, as $\underline{g} = \sqrt{\alpha}(\beta_1+1)$, we have

$$\begin{aligned} \underline{g} \lesseqgtr \sqrt{1-\alpha} \quad &\Leftrightarrow \quad \alpha \lesseqgtr \frac{1}{(\beta_1+1)^2+1}, \\ \underline{g} \lesseqgtr \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}} \quad &\Leftrightarrow \quad \alpha \lesseqgtr \frac{4\beta_1}{(\beta_1+1)(B^2-1)+4\beta_1}, \text{ and} \\ \underline{g} \lesseqgtr \sqrt{\frac{4\alpha\beta_1}{1-A^2}} \quad &\Leftrightarrow \quad 1-A^2 \lesseqgtr \frac{4\beta_1}{(\beta_1+1)^2}. \end{aligned} \tag{C.44}$$

Next, by (C.2), we derive $(\check{\sigma}^r_L, \check{\sigma}^r_R)$ utilizing $(\check{x}^r_L, \check{x}^r_R)$.

When $(\check{x}^r_L, \check{x}^r_R) = \left(x^r_{L_0}(\underline{g}), x^r_{R_0}(\underline{g})\right)$, notice first, by (C.21), that

$$x^r_{L_0}(\underline{g}) = 0, \quad \text{and} \quad x^r_{R_0}(\underline{g}) = \sqrt{\alpha}(\beta_1+1) \ge 0. \tag{C.45}$$

Then, utilizing (C.45), let us define $\check{\sigma}^r_{L_0}$ and $\check{\sigma}^r_{R_0}$ as

$$\check{\sigma}^r_{L_0} = \frac{x^r_{L_0}(\underline{g})\left(x^r_{L_0}(\underline{g}) + x^r_{R_0}(\underline{g})\right)}{\alpha} = 0, \quad \text{and} \quad \check{\sigma}^r_{R_0} = \frac{x^r_{R_0}(\underline{g})\left(x^r_{L_0}(\underline{g}) + x^r_{R_0}(\underline{g})\right)}{1-\alpha} = \frac{\alpha(\beta_1+1)^2}{1-\alpha} \le 1,$$

where the inequality follows as, by (C.43) and (C.44), we have $\alpha \le \frac{1}{(\beta_1+1)^2+1}$ for this case. This result, by (C.2), establishes the first line of (C.42).

When $(\check{x}^r_L, \check{x}^r_R) = \left(x^r_{L_{1-\alpha}}(\underline{g}), x^r_{R_{1-\alpha}}(\underline{g})\right)$, notice first, by (C.21), that

$$x^r_{L_{1-\alpha}}(\underline{g}) = \frac{\sqrt{\alpha}(\beta_1+1)(2\beta_1+1) - \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\beta_1(\beta_1+1)} \ge 0, \tag{C.46}$$

where the inequality follows as, by (C.43), we have $\alpha \ge \frac{1}{(\beta_1+1)^2+1}$ for this case. Next, again by (C.21), we have

$$x^r_{R_{1-\alpha}}(\underline{g}) = \frac{-\sqrt{\alpha}(\beta_1+1) + \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\beta_1(\beta_1+1)} > 0, \tag{C.47}$$

where the inequality follows trivially as $\alpha \in (0,1)$. Then, utilizing (C.46) and (C.47), let us define $\check{\sigma}^r_{L_{1-\alpha}}$ and $\check{\sigma}^r_{R_{1-\alpha}}$ as

$$\check{\sigma}^r_{L_{1-\alpha}} = \frac{x^r_{L_{1-\alpha}}(\underline{g})\left(x^r_{L_{1-\alpha}}(\underline{g}) + x^r_{R_{1-\alpha}}(\underline{g})\right)}{\alpha}$$

$$= \left[\frac{\sqrt{\alpha}(\beta_1+1)(2\beta_1+1) - \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\beta_1(\beta_1+1)}\right] \times \left[\frac{\sqrt{\alpha}(\beta_1+1) + \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\alpha(\beta_1+1)}\right]$$

$$= \sqrt{\frac{\alpha(\beta_1+1)+4(1-\alpha)\beta_1}{4\alpha(\beta_1+1)}} + \frac{\alpha(\beta_1+3)-2}{2\alpha(\beta_1+1)} < 1, \tag{C.48}$$

$$\breve{\sigma}^r_{R_{1-\alpha}} = \frac{x^r_{R_{1-\alpha}}(\underline{g})\left(x^r_{L_{1-\alpha}}(\underline{g}) + x^r_{R_{1-\alpha}}(\underline{g})\right)}{1-\alpha}$$

$$= \left[\frac{-\sqrt{\alpha}(\beta_1+1) + \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2(1-\alpha)\beta_1}\right] \times \left[\frac{\sqrt{\alpha}(\beta_1+1) + \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2(\beta_1+1)}\right]$$

$$= \frac{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)-\alpha(\beta_1+1)^2}{4(1-\alpha)\beta_1(\beta_1+1)} = 1. \tag{C.49}$$

Notice that the inequality in C.48 follows as we have

$$\alpha\beta_1^2 < 1 \;\Leftrightarrow\; \alpha(\beta_1^2-1) < 1-\alpha \;\Leftrightarrow\; \alpha(1-\alpha)(\beta_1^2-1) < (1-\alpha)^2$$

$$\Leftrightarrow\; \alpha^2(\beta_1+1)^2 + 4\alpha(1-\alpha)\beta_1(\beta_1+1) < \alpha^2(\beta_1+1)^2 + 4\alpha(1-\alpha)(\beta_1+1) + 4(1-\alpha)^2$$

$$\Leftrightarrow\; \sqrt{\alpha^2(\beta_1+1)^2 + 4\alpha(1-\alpha)\beta_1(\beta_1+1)} < \alpha(\beta_1+1) + 2(1-\alpha)$$

$$\Leftrightarrow\; 2\alpha\beta_1(\beta_1+1)^2 - 4(1-\alpha)\beta_1(\beta_1+1) + 2\beta_1(\beta_1+1)\sqrt{\alpha}\sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)} < 4\alpha\beta_1(\beta_1+1)^2$$

$$\Leftrightarrow\; \left[\frac{\sqrt{\alpha}(\beta_1+1)(2\beta_1+1) - \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\beta_1(\beta_1+1)}\right] \times \left[\frac{\sqrt{\alpha}(\beta_1+1) + \sqrt{\alpha(\beta_1+1)^2+4(1-\alpha)\beta_1(\beta_1+1)}}{2\alpha(\beta_1+1)}\right] < 1.$$

This result, by (C.2), establishes the second line of (C.42).

When $(\overset{\star}{x}^r_L, \overset{\star}{x}^r_R) = \left(x^r_{L_*}(\underline{g}), x^r_{R_*}(\underline{g})\right)$, notice first, by (C.21), that

$$x^r_{L_*}(\underline{g}) = \frac{\sqrt{\alpha}(\beta_1+1)(\rho-\beta_r-2\beta_1)}{2(\beta_1+\beta_r)(\rho-\beta_1)} > 0, \tag{C.50}$$

where the inequality follows as, by (C.43) (particularly, (C.23) and (C.24)), we have $\frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} > 0$ for this case. Next, again by (C.21), we have

$$x^r_{R_*}(\underline{g}) = \frac{\sqrt{\alpha}(\beta_1+1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{2(\beta_1+\beta_r)(\rho-\beta_1)} > 0, \tag{C.51}$$

where the inequality follows as, by (C.21), $\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1) = \frac{(\beta_1+\beta_r)(\rho-\beta_1)(B-1)}{\beta_1} > 0$. Then, utilizing (C.50) and (C.51), let us define $\breve{\sigma}^r_{L_*}$ and $\breve{\sigma}^r_{R_*}$ as

$$\breve{\sigma}^r_{L_*} = \frac{x^r_{L_*}(\underline{g})\left(x^r_{L_*}(\underline{g}) + x^r_{R_*}(\underline{g})\right)}{1-\alpha} = \frac{(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2} < 1, \tag{C.52}$$

$$\breve{\sigma}^r_{R_*} = \frac{x^r_{R_*}(\underline{g})\left(x^r_{L_*}(\underline{g}) + x^r_{R_*}(\underline{g})\right)}{\alpha}$$

$$= \frac{\alpha(\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)][\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{4(1-\alpha)(\beta_1+\beta_r)^2(\rho-\beta_1)^2} < 1. \tag{C.53}$$

Notice that for this case to occur, by (C.43) and (C.44), we must either have (i) $\frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \leq \alpha < \frac{4\beta_1}{(\beta_1+1)(B^2-1)+4\beta_1}$, or (ii) $\alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}$ and $(\beta_1+1)^2(1-A^2) < 4\beta_1$. Therefore, the inequality in (C.52) follows as, for (i), by (C.21), we have

$$\begin{aligned}
&\frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} < \frac{4\beta_1}{(\beta_1+1)(B^2-1)+4\beta_1} \\
\Leftrightarrow\quad & \frac{\beta_1(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)][\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
&\qquad < 4\beta_1 \underbrace{[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}_{=\frac{(\beta_1+\beta_r)(\rho-\beta_1)(B-1)}{\beta_1}>0},
\end{aligned}$$

and, for (ii), again by (C.21), we have

$$\beta_1(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)] < 4\beta_1(\beta_1+\beta_r)^2(\rho-\beta_1)^2.$$

The inequality in (C.53) also follows as, for (i), by (C.21), we have

$$\begin{aligned}
&\alpha(\beta_1+1)(B^2-1) < 4\beta_1(1-\alpha) \\
\Leftrightarrow\quad & \frac{\alpha\beta_1(\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)][\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{(\beta_1+\beta_r)^2(\rho-\beta_1)^2} < 4\beta_1(1-\alpha),
\end{aligned}$$

and, for (ii), by (C.52), we have

$$\begin{aligned}
&\frac{(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2}(1-\alpha) < 1-\alpha \\
\Leftrightarrow\quad & \frac{(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
&\qquad < 1-\alpha+\frac{\alpha(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2} \\
\Rightarrow\quad & \frac{\alpha(\beta_1+1)^2[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]^2}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2} < 1-\alpha+\frac{\alpha(\beta_1+1)^2(\rho-\beta_r-2\beta_1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)]}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2},
\end{aligned}$$

where the last inequality follows by the fact that $\alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}$. This result, by (C.2), establishes the third line of (C.42).

When $(\overset{\star}{x}{}^r_L, \overset{\star}{x}{}^r_R) = \big(x^r_{L_\alpha}(\underline{g}), x^r_{R_\alpha}(\underline{g})\big)$, notice first, by (C.21), that

$$x^r_{L_\alpha}(\underline{g}) = \begin{cases} \frac{\sqrt{\alpha}(\beta_1+1)-\sqrt{\alpha}(1-\beta_1)}{2\beta_1} = \sqrt{\alpha}, & \text{for} \quad \beta_1 < 1 \\ \frac{\sqrt{\alpha}(\beta_1+1)-\sqrt{\alpha}(\beta_1-1)}{2\beta_1} = \frac{\sqrt{\alpha}}{\beta_1}, & \text{for} \quad \beta_1 \geq 1, \end{cases} \tag{C.54}$$

$$x^r_{R_\alpha}(\underline{g}) = \begin{cases} \frac{\sqrt{\alpha}(\beta_1+1)(\beta_1-1)+\sqrt{\alpha}(\beta_1+1)(1-\beta_1)}{2\beta_1} = 0, & \text{for} \quad \beta_1 < 1 \\ \frac{\sqrt{\alpha}(\beta_1+1)(\beta_1-1)+\sqrt{\alpha}(\beta_1+1)(\beta_1-1)}{2\beta_1} = \frac{\sqrt{\alpha}(\beta_1^2-1)}{\beta_1}, & \text{for} \quad \beta_1 \geq 1. \end{cases} \tag{C.55}$$

Then, utilizing (C.54) and (C.55), let us define $\check{\sigma}^r_{L_\alpha}$ and $\check{\sigma}^r_{R_\alpha}$ as

$$\check{\sigma}^r_{L_\alpha} = \frac{x^r_{L_\alpha}(\underline{g})\left(x^r_{L_\alpha}(\underline{g})+x^r_{R_\alpha}(\underline{g})\right)}{\alpha} = 1, \quad \text{and} \quad \check{\sigma}^r_{R_\alpha} = \frac{x^r_{R_\alpha}(\underline{g})\left(x^r_{L_\alpha}(\underline{g})+x^r_{R_\alpha}(\underline{g})\right)}{1-\alpha} = \begin{cases} 0, & \text{for} \quad \beta_1 < 1 \\ \frac{\alpha(\beta_1^2-1)}{1-\alpha}, & \text{for} \quad \beta_1 \geq 1. \end{cases}$$

Notice that, for $\beta_1 \geq 1$, we have $0 \leq \check{\sigma}^r_{R_\alpha} < 1$, where the strict inequality follows by the fact that $\alpha\beta_1^2 < 1$. This result, by (C.2), establishes the last line of (C.42). ■

Proof of Corollary 1. To maintain consistency with the rest of the appendix, in this proof, we use $\beta_r = -\beta_0 = 1 - v_L^r/v_R^r$. We first define $\widetilde{\alpha}$, for $\beta_r + 2\beta_1 < \rho$, as

$$\widetilde{\alpha} = \begin{cases} \min\left\{\frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}, 1\right\}, & \text{if } \beta_r \in \left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}, \rho-2\beta_1\right) \\ 1, & \text{if } \beta_r \le \frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}. \end{cases}$$

By Proposition 4 we have $(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)) = (\check{\sigma}_L^r, \check{\sigma}_R^r)$, where $\check{\sigma}_L^r$ and $\check{\sigma}_R^r$ defined in Lemma C.10.

If part: When $\beta_r \in \left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}, \rho-2\beta_1\right)$, by Lemma C.7, we have

$$0 < \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} < \frac{1}{\beta_1+1}. \tag{C.56}$$

Then, as $\alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}$, we have $\frac{\alpha}{\alpha\beta_1+1} < \frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)}$, which, together with (C.56), imply that (C.24) holds. Therefore, by Lemma C.10, we have

$$(\check{\sigma}_L^r, \check{\sigma}_R^r) = \begin{cases} \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right), & \text{for } 1-A^2 < \frac{4\beta_1}{(\beta_1+1)^2}, \\ \left(\check{\sigma}_{L_\alpha}^r, \check{\sigma}_{R_\alpha}^r\right), & \text{for } 1-A^2 \ge \frac{4\beta_1}{(\beta_1+1)^2}, \end{cases}$$

where $\check{\sigma}_R^r < \check{\sigma}_L^r$ immediately follows, again by Lemma C.10, for $1-A^2 \ge \frac{4\beta_1}{(\beta_1+1)^2}$. For $1-A^2 < \frac{4\beta_1}{(\beta_1+1)^2}$, as $\alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)}$, we have

$$\begin{aligned} & \alpha[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)] - \alpha(\rho-\beta_r-2\beta_1) < (1-\alpha)(\rho-\beta_r-2\beta_1) \\ \Leftrightarrow \quad & \frac{\alpha[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]}{1-\alpha} < \rho-\beta_r-2\beta_1, \end{aligned}$$

which, by Lemma C.10, establishes $\check{\sigma}_R^r < \check{\sigma}_L^r$.

When $\beta_r \le \frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}$, on the other hand, we have $\frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)} \ge \frac{1}{\beta_1+1}$, hence, (C.25) holds. Therefore, by Lemma C.10, we have $(\check{\sigma}_L^r, \check{\sigma}_R^r) = \left(\check{\sigma}_{L_\alpha}^r, \check{\sigma}_{R_\alpha}^r\right)$, which immediately establishes $\check{\sigma}_R^r < \check{\sigma}_L^r$.

Only if part: When $\check{\sigma}_R^r < \check{\sigma}_L^r$, by Lemma C.10, we either have $(\check{\sigma}_L^r, \check{\sigma}_R^r) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$ or $(\check{\sigma}_L^r, \check{\sigma}_R^r) = \left(\check{\sigma}_{L_\alpha}^r, \check{\sigma}_{R_\alpha}^r\right)$. For $(\check{\sigma}_L^r, \check{\sigma}_R^r) = \left(\check{\sigma}_{L_\alpha}^r, \check{\sigma}_{R_\alpha}^r\right)$, by Lemma C.10, we must either have

$$\beta_r \in \left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}, \rho-2\beta_1\right), \ \alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \text{ and } 1-A^2 \ge \frac{4\beta_1}{(\beta_1+1)^2}, \tag{C.57}$$

$$\text{or} \quad \beta_r \le \frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}. \tag{C.58}$$

For $(\check{\sigma}_L^r, \check{\sigma}_R^r) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$, again by Lemma C.10, $\check{\sigma}_R^r < \check{\sigma}_L^r$ implies

$$\beta_r \in \left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}, \rho-2\beta_1\right), \ \alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \text{ and } 1-A^2 \ge \frac{4\beta_1}{(\beta_1+1)^2}. \tag{C.59}$$

Therefore, by (C.57)-(C.59), when $\check{\sigma}_R^r < \check{\sigma}_L^r$, we must either have

$$\beta_r \in \left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}, \rho-2\beta_1\right), \ \alpha < \frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)} \text{ and } 1-A^2 \ge \frac{4\beta_1}{(\beta_1+1)^2},$$
$$\text{or} \quad \beta_r \le \frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}.$$

This completes the proof. ■

Proof of Proposition 5. By Lemmas C.1 and C.2, the platform's problem $(\mathcal{P}_k^{ma})$ for $k = N$ can be reformulated as

$$\max_{x_L^n, x_R^n \ge 0} (C.3), \quad \text{subject to } (H_\alpha^{\widehat{t}}) \text{ and } (H_{1-\alpha}^{\widehat{t}}) \text{ for } \widehat{t} = n,$$

whose objective function can further be refined to

$$\max_{x_L^n, x_R^n \geq 0} \quad (x_L^n)^2(1+\rho)\left\{\lambda'(\beta_1+\beta_r)+1-\beta_r\right\} + x_L^n x_R^n\left\{\lambda'(\beta_1+\beta_r)+2-\beta_r+\rho\right\} + (x_R^n)^2. \tag{C.60}$$

It is then straightforward to see that at least one of $(H_\alpha^{\hat{t}})$ and $(H_{1-\alpha}^{\hat{t}})$ must be binding at the optimal solution, as coefficients of all terms of (C.60) are positive. For the case when $(H_{1-\alpha}^{\hat{t}})$ is binding, we have we have $x_R^n \in [1-\alpha, \sqrt{1-\alpha}]$ and $x_L^n = \frac{1-\alpha}{x_R^n} - x_R^n$. Substituting this relationship reduces the objective function of (C.60) to

$$\left[\frac{1-\alpha}{x_R^n} - x_R^n\right]^2 (1+\rho)\left[\lambda'(\beta_1+\beta_r)+1-\beta_r\right] + x_R^n\left[\frac{1-\alpha}{x_R^n} - x_R^n\right]\left[\lambda'(\beta_1+\beta_r)+2+\rho-\beta_r\right] + (x_R^n)^2,$$

whose derivative with respect to x_R^n is given by $x_R^n\left[-\frac{(1-\alpha)^2}{(x_R^n)^4}2(1+\rho)\left[\lambda'(\beta_1+\beta_r)+1-\beta_r\right] + 2\rho\left[\lambda'(\beta_1+\beta_r)-\beta_r\right]\right]$, which is negative when $\lambda'(\beta_1+\beta_r)-\beta_r \leq 0$. When $\lambda'(\beta_1+\beta_r)-\beta_r > 0$, rewriting the derivative as

$$x_R^n\left[-\frac{(1-\alpha)^2}{(x_R^n)^4}2\left[\lambda'(\beta_1+\beta_r)+1-\beta_r\right] - 2\rho\frac{(1-\alpha)^2}{(x_R^n)^4} + 2\rho\left[\lambda'(\beta_1+\beta_r)-\beta_r\right]\left[1-\frac{(1-\alpha)^2}{(x_R^n)^4}\right]\right],$$

we see that it is also negative, as $x_R^n \leq \sqrt{1-\alpha}$. Thus, the derivative is negative. Hence, if $(H_{1-\alpha}^{\hat{t}})$ is binding, we must have $(x_L^n, x_R^n) = (\alpha, 1-\alpha)$, which implies $(H_\alpha^{\hat{t}})$ must also be binding. Thus it suffices to consider only the case when $(H_\alpha^{\hat{t}})$ is binding. Then, we have $x_L^n \in [\alpha, \sqrt{\alpha}]$ and $x_R^n = \frac{\alpha}{x_L^n} - x_L^n$. Substituting this relationship reduces the objective function of (C.60) to

$$(x_L^n)^2(1+\rho)\left[\lambda'(\beta_1+\beta_r)+1-\beta_r\right] + x_L^n\left[\frac{\alpha}{x_L^n} - x_L^n\right]\left[\lambda'(\beta_1+\beta_r)+2+\rho-\beta_r\right] + \left[\frac{\alpha}{x_L^n} - x_L^n\right]^2,$$

whose derivative with respect to x_L^n is given by $2\left\{\rho x_L^n\left[\lambda'\beta_1 - (1-\lambda')\beta_r\right]\right\}$, and hence, it is either increasing or decreasing in x_L^n, implying that its maximum must be achieved at either $x_L^n = \alpha$ or $x_L^n = \sqrt{\alpha}$. In the former case, we have $(x_L^n, x_R^n) = (\alpha, 1-\alpha)$, and yields an objective value of $1 + \alpha^2\rho\left[\lambda'(\beta_1+\beta_r)-\beta_r\right] + \alpha\left[\lambda'(\beta_1+\beta_r)+1+\rho-\beta_r\right]$ in (C.60). In the latter case, we have $(x_L^n, x_R^n) = (\sqrt{\alpha}, 0)$, which yields an objective value of $\alpha\left[1+\lambda'(\beta_1+\beta_r)(1+\rho)+(1-\beta_r)(1+\rho)\right]$ in (C.60). Therefore, $(x_L^n, x_R^n) = (\alpha, 1-\alpha)$ (resp., $(x_L^n, x_R^n) = (\sqrt{\alpha}, 0)$) is optimal if $(1-\alpha)\left\{\alpha\rho\left[-\lambda'\beta_1 + (1-\lambda')\beta_r\right]\right\} \geq 0$ (resp., if $(1-\alpha)\left\{\alpha\rho\left[-\lambda'\beta_1 + (1-\lambda')\beta_r\right]\right\} < 0$). Plugging in the definitions of λ', β_1 and β_r, the condition becomes: $(x_L^n, x_R^n) = (\alpha, 1-\alpha)$ (resp., $(x_L^n, x_R^n) = (\sqrt{\alpha}, 0)$) is optimal if $\alpha\rho\beta_\lambda \leq 1$ (resp., if $\alpha\rho\beta_\lambda > 1$). We know, by (C.2), $(x_L^n, x_R^n) = (\alpha, 1-\alpha)$ corresponds to $(\sigma_N^{ma}(L,n), \sigma_N^{ma}(R,n)) = (1,1)$ and $(x_L^n, x_R^n) = (\sqrt{\alpha}, 0)$ corresponds to $(\sigma_N^{ma}(L,n), \sigma_N^{ma}(R,n)) = (1,0)$, which completes the first part of the proof.

Next, we show $\lambda \leq \lambda_S^{ma}$ implies $\alpha\rho\beta_\lambda \leq 1$ for $\alpha\beta_1^2 < 1 < \alpha\rho\beta_1$. In that, we first denote as λ^β the $\lambda \in [0,1]$ value that equates $\beta_\lambda = \frac{1}{\alpha\rho}$. Utilizing the definitions of β_λ, β_1 and β_r, we have

$$\lambda^\beta := \frac{v_R^r - \alpha\rho(v_L^r - v_R^r)}{v_R^r - \alpha\rho(v_L^r - v_R^r) + \alpha\rho(v_L^\ell - v_R^\ell) - v_R^\ell} = \frac{1}{1 + \frac{v_R^\ell}{v_R^r}\frac{(\alpha\rho\beta_1 - 1)}{(1+\alpha\rho\beta_r)}}.$$

Next, utilizing (C.41), we rewrite λ_S^{ma} as

$$\lambda_S^{ma} = \frac{1}{1 + \frac{v_R^\ell}{v_R^r}\frac{\left[F^\ell(\overline{g}) - F^\ell(\underline{g})\right]}{\left[F^r(\underline{g}) - F^r(\overline{g})\right]}}.$$

As noted in the proof of Proposition 4, $F^\ell(\overline{g})-F^\ell(\underline{g})=(1-\alpha)(1-\alpha\beta_1\rho)$ and $F^r(\overline{g})=(1-\alpha\beta_r)(1+\alpha\rho)$, as $(x_L^r,x_R^r)=(\alpha,1-\alpha)$ is the only feasible point that satisfies both $(H_\alpha^{\hat{t}})$ and $(H_{1-\alpha}^{\hat{t}})$ when $g=\overline{g}$. Thus, λ_S^{ma} can be written as

$$\lambda_S^{ma}=\frac{1}{1+\frac{v_R^\ell}{v_R^r}\frac{(1-\alpha)(\alpha\rho\beta_1-1)}{(1-\alpha\beta_r)(1+\alpha\rho)-F^r(\underline{g})}}.$$

Then, notice that it suffices for us to show $\lambda_S^{ma}\leq\lambda^\beta$, in order to establish the claim of: $\lambda\leq\lambda_S^{ma}$ implies $\alpha\rho\beta_\lambda\leq 1$ for $\alpha\beta_1^2<1<\alpha\rho\beta_1$. Thus, in the remainder of the proof, we aim to show $\lambda_S^{ma}\leq\lambda^\beta$ which, utilizing the derivations above, can be represented as $\alpha(1+\rho)(1-\beta_r)\leq F^r(\underline{g})$. We now show that this inequality holds for all four distinct functional forms of $F^r(g)$.

Case 1: $F^r(\underline{g})$ for $\left(x_{L_0}^r(\underline{g}),x_{R_0}^r(\underline{g})\right)$. Here, by (C.26), we have $F^r(\underline{g})=\alpha(\beta_1+1)^2$. The condition $\alpha(1+\rho)(1-\beta_r)\leq\alpha(\beta_1+1)^2\Leftrightarrow\rho-\beta_r-2\beta_1\leq\beta_1^2+\rho\beta_r$ holds, as for $\left(x_{L_0}^r(\underline{g}),x_{R_0}^r(\underline{g})\right)$ to be optimal, by (C.22), we must have $\rho-\beta_r-2\beta_1\leq 0$.

Case 2: $F^r(\underline{g})$ for $\left(x_{L_\alpha}^r(\underline{g}),x_{R_\alpha}^r(\underline{g})\right)$. Here, by (C.26), we have $F^r(\underline{g})=\alpha(1+\rho)(1-\beta_r)$ if $\beta_1<1$, and $F^r(\underline{g})=\alpha/(\beta_1)^2\left\{(1+\rho)(1-\beta_r)+(2+\rho-\beta_r)(\beta_1^2-1)+(\beta_1^2-1)^2\right\}$ if $\beta_1\geq 1$. The condition $\alpha(1+\rho)(1-\beta_r)\leq F^r(\underline{g})$ immediately holds for $\beta_1<1$. For $\beta_1\geq 1$, the condition $(\beta_1^2-1)(1+\rho)(1-\beta_r)\leq(\beta_1^2-1)\left[1+\rho-\beta_r+\beta_1^2\right]$ trivially holds, as $\beta_1\geq 1$.

Case 3: $F^r(\underline{g})$ for $\left(x_{L_*}^r(\underline{g}),x_{R_*}^r(\underline{g})\right)$. Here, by (C.26), we have $F^r(\underline{g})=\alpha(\beta_1+1)^2(\rho+\beta_r)^2/\left[4(\beta_1+\beta_r)(\rho-\beta_1)\right]$. The condition $\alpha(1+\rho)(1-\beta_r)\leq F^r(\underline{g})\Leftrightarrow 4(\beta_1+\beta_r)(\rho-\beta_1)(1+\rho)(1-\beta_r)\leq(\beta_1+1)^2(\rho+\beta_r)^2\Leftrightarrow 0\leq\left[(\beta_1+\beta_r)(\rho+\beta_r)+(1-\beta_r)(2\beta_1+\beta_r-\rho)\right]^2$ holds.

Case 4: $F^r(\underline{g})$ for $\left(x_{L_{1-\alpha}}^r(\underline{g}),x_{R_{1-\alpha}}^r(\underline{g})\right)$. Here, by (C.26), the condition $\alpha(1+\rho)(1-\beta_r)\leq F^r(\underline{g})$ becomes

$$\begin{aligned}
&\alpha\beta_1^2(1+\beta_1)(1+\rho)(1-\beta_r)+2(1-\alpha)\beta_1(\rho-\beta_1)(\beta_1+\beta_r)+\alpha(1+\beta_1^3)\rho\beta_r\\
&\qquad\leq\sqrt{\alpha(1+\beta_1)}\sqrt{\alpha(1+\beta_1)+4(1-\alpha)\beta_1}\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]\\
\propto\quad&2(1-\alpha)\beta_1\left[\frac{(\rho-\beta_1)(\beta_1+\beta_r)}{\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r}\right]^2+\alpha(1+\beta_1)\left[\frac{(\rho-\beta_1)(\beta_1+\beta_r)}{\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r}\right]\\
&\qquad\leq 2\alpha(1+\beta_1)\\
\Leftrightarrow\quad&\frac{2\beta_1(\rho-\beta_1)^2(\beta_1+\beta_r)^2}{(1+\beta_1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]\left\{2\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]-(\rho-\beta_1)(\beta_1+\beta_r)\right\}}\\
&\qquad\leq\frac{\alpha}{1-\alpha}.
\end{aligned}\tag{C.61}$$

Next, notice that in order for $\left(x_{L_{1-\alpha}}^r(\underline{g}),x_{R_{1-\alpha}}^r(\underline{g})\right)$ to be optimal, either constraints of (C.23) and (C.24) must be satisfied. Thus, we now show that (C.61) condition holds for (C.23) and (C.24), separetely.

Case 4.1: $F^r(\underline{g})$ for $\left(x_{L_{1-\alpha}}^r(\underline{g}),x_{R_{1-\alpha}}^r(\underline{g})\right)$ and (C.23) holds. Here, we have (i) $\alpha\geq(\rho\beta_1)^{-1}\Leftrightarrow\alpha/(1-\alpha)\geq(\rho\beta_1-1)^{-1}$, and (ii) $(\rho-\beta_1)<(\beta_1+\beta_r)$. By (i), to establish (C.61), it suffices for us to show

$$\begin{aligned}
&(\rho-\beta_1)(\beta_1+\beta_r)\left\{2\beta_1(\rho\beta_1-1)(\rho-\beta_1)(\beta_1+\beta_r)+(1+\beta_1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]\right\}\\
&\qquad\leq 2(1+\beta_1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]^2.
\end{aligned}$$

Now, by (ii), substituting the left hand side $(\rho-\beta_1)$'s with $(\beta_1+\beta_r)$'s, to establish (C.61), it suffices for us to show

$$2\beta_1(\beta_1+\beta_r)^4(\rho\beta_1-1)\leq 2(\beta_1+\beta_r)^2(\rho\beta_1-1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]$$
$$+2\beta_r(\rho-\beta_1)(1+\beta_1)(\beta_1+\beta_r)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right].$$

Disregarding the right hand side non-negative second term, it suffices for us to show $\beta_1(\beta_1+\beta_r)^2\leq\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r$. As, $\rho>\beta_1$, we can substitute the right hand side $(1+\rho)$ with $(1+\beta_1)$, and see that $\beta_1(\beta_1+\beta_r)^2\leq\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\Leftrightarrow 0\leq\beta_1(1-\beta_r)(\beta_1+\beta_r)$ holds.

Case 4.2: $F^r(\underline{g})$ for $\left(x^r_{L_{1-\alpha}}(\underline{g}),x^r_{R_{1-\alpha}}(\underline{g})\right)$ and (C.24) holds. Here, on top of (i), by (C.24) we also have (iii) $\alpha/(1-\alpha)\geq[\rho-2\beta_1-\beta_r]/[(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)]$. Thus, when the right hand side of (i) is greater than the right hand side of (iii), we have the added condition of (iv) $(\beta_1+\beta_r)(\rho+\rho\beta_1)-(\rho-\beta_1)(\rho\beta_1-\beta_r)\geq 0$; when, on the other hand, the right hand side of (i) is smaller than the right hand side of (iii), we have the added condition of (v) $(\beta_1+\beta_r)(\rho+\rho\beta_1)-(\rho-\beta_1)(\rho\beta_1-\beta_r)<0$.

We first investigate the case where (iv) holds. By (i), to establish (C.61), it suffices for us to show

$$2\beta_1(\rho-\beta_1)^2(\beta_1+\beta_r)^2(\rho\beta_1-1)\leq(1+\beta_1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]$$
$$\times\left\{2\beta_1\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]+(\rho-\beta_1)(\beta_1+\beta_r)\right\}.$$

Disregarding the right hand side non-negative second term, it suffices for us to show $(\rho-\beta_1)^2(\beta_1+\beta_r)^2(\rho\beta_1-1)\leq(1+\beta_1)\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]$. Substituting the right hand side $(1+\rho)$ with ρ, it suffices for us to show $(\rho-\beta_1)^2(\beta_1+\beta_r)^2(\rho\beta_1-1)\leq\rho(1+\beta_1)\left[\beta_1(\beta_1+\beta_r)+(1+\beta_1)\beta_r\right]\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]$. Substituting the right hand side $(1+\beta_1)$ with $(\beta_r+\beta_1)$, it suffices for us to show $(\rho-\beta_1)^2(\rho\beta_1-1)\leq\rho(1+\beta_1)\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]\Leftrightarrow$ $-(\rho-\beta_1)(\rho\beta_1-1)(\beta_1+\beta_r)\leq(1+\rho)\left[(\beta_1+\beta_r)(\rho+\rho\beta_1)-(\rho-\beta_1)(\rho\beta_1-\beta_r)\right]$, which is satisfied, as $\rho>\beta_1,\rho\beta_1>1$ and (iv) holds.

We lastly investigate the case where (v) holds. By (iii), to establish (C.61), it suffices for us to show

$$2\beta_1(\rho-\beta_1)^2(\beta_1+\beta_r)^2\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]$$
$$\leq(1+\beta_1)[\rho-2\beta_1-\beta_r]\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]$$
$$\times\left\{2\beta_1\left[(1+\rho)(\beta_1+\beta_r)-(\rho-\beta_1)(1-\beta_r)\right]+(\rho-\beta_1)(\beta_1+\beta_r)\right\}.$$

Disregarding the right hand side non-negative second term, it suffices for us to show $(\rho-\beta_1)^2(\beta_1+\beta_r)^2\leq(1+\beta_1)[\rho-2\beta_1-\beta_r]\left[\beta_1^2(1+\rho)(1-\beta_r)+(1+\beta_1)^2\rho\beta_r\right]$. Substituting the right hand side $(1+\rho)$ with ρ, it suffices for us to show $(\rho-\beta_1)^2(\beta_1+\beta_r)^2\leq\rho(1+\beta_1)[\rho-2\beta_1-\beta_r]\left[\beta_1(\beta_1+\beta_r)+(1+\beta_1)\beta_r\right]$. Substituting the right hand side $(1+\beta_1)$ with $(\beta_r+\beta_1)$, it suffices for us to show $(\rho-\beta_1)^2\leq\rho(1+\beta_1)[\rho-2\beta_1-\beta_r]\Leftrightarrow$ $0\leq(\rho-\beta_1)(\rho\beta_1+\beta_1)-(\beta_1+\beta_r)(\rho+\rho\beta_1)$, which is satisfied, as $\beta_1>-\beta_r$ and (v) holds. This completes the proof. ■

Appendix D: Proofs in Sections 6 and 7

Proof of Proposition 6. We first rewrite both Π_k^m and Π_k^{ma}, given in $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$, respectively. Namely

$$\begin{aligned}\Pi_k^m =&\lambda\left\{\frac{\left[\pi_L\alpha\sigma_L^{\widehat{\ell}}+\pi_R(1-\alpha)\sigma_R^{\widehat{\ell}}\right]\left[v_L^\ell\alpha\sigma_L^{\widehat{\ell}}+v_R^\ell(1-\alpha)\sigma_R^{\widehat{\ell}}\right]}{\alpha\sigma_L^{\widehat{\ell}}+(1-\alpha)\sigma_R^{\widehat{\ell}}}\right\}\\&+\lambda\left\{\frac{\left[\pi_L\alpha[1-\sigma_L^{\widehat{\ell}}]+\pi_R(1-\alpha)[1-\sigma_R^{\widehat{\ell}}]\right]\left[v_L^\ell\alpha[1-\sigma_L^{\widehat{\ell}}]+v_R^\ell(1-\alpha)[1-\sigma_R^{\widehat{\ell}}]\right]}{\alpha[1-\sigma_L^{\widehat{\ell}}]+(1-\alpha)[1-\sigma_R^{\widehat{\ell}}]}\right\}\\&+(1-\lambda)\left\{\frac{\left[\pi_L\alpha\sigma_L^{\widehat{r}}+\pi_R(1-\alpha)\sigma_R^{\widehat{r}}\right]\left[v_L^r\alpha\sigma_L^{\widehat{r}}+v_R^r(1-\alpha)\sigma_R^{\widehat{r}}\right]}{\alpha\sigma_L^{\widehat{r}}+(1-\alpha)\sigma_R^{\widehat{r}}}\right\}\\&+(1-\lambda)\left\{\frac{\left[\pi_L\alpha[1-\sigma_L^{\widehat{r}}]+\pi_R(1-\alpha)[1-\sigma_R^{\widehat{r}}]\right]\left[v_L^r\alpha[1-\sigma_L^{\widehat{r}}]+v_R^r(1-\alpha)[1-\sigma_R^{\widehat{r}}]\right]}{\alpha[1-\sigma_L^{\widehat{r}}]+(1-\alpha)[1-\sigma_R^{\widehat{r}}]}\right\},\end{aligned}\tag{D.1}$$

and

$$\begin{aligned}\Pi_k^{ma} =&\lambda\left\{\frac{\left[\pi_L\alpha\sigma_L^{\widehat{\ell}}+\pi_R(1-\alpha)\sigma_R^{\widehat{\ell}}\right]\left[v_L^\ell\alpha\sigma_L^{\widehat{\ell}}+v_R^\ell(1-\alpha)\sigma_R^{\widehat{\ell}}\right]}{\alpha\sigma_L^{\widehat{\ell}}+(1-\alpha)\sigma_R^{\widehat{\ell}}}\right\}\\&+(1-\lambda)\left\{\frac{\left[\pi_L\alpha\sigma_L^{\widehat{r}}+\pi_R(1-\alpha)\sigma_R^{\widehat{r}}\right]\left[v_L^r\alpha\sigma_L^{\widehat{r}}+v_R^r(1-\alpha)\sigma_R^{\widehat{r}}\right]}{\alpha\sigma_L^{\widehat{r}}+(1-\alpha)\sigma_R^{\widehat{r}}}\right\}.\end{aligned}\tag{D.2}$$

It is straightforward to see that $\Pi_U^j \geq \Pi_O^j \geq \Pi_S^j \geq \Pi_N^j$ for $j \in \{m, ma\}$, by design. The first two inequalities are established as both $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = U$ are relaxations of $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = O$, and both $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = O$ are relaxations of $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = S$, respectively. The last inequality is established as any feasible solution to $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = N$ is also a feasible solution to $(\mathcal{P}_k^m)$ and $(\mathcal{P}_k^{ma})$ for $k = S$, respectively.

Utilizing (D.1) and (D.2), we further have $\Pi_U^j = \Pi_O^j$ for $j \in \{m, ma\}$, by Propositions 1 and 3, respectively. Utilizing (D.1), we have $\Pi_S^m = \Pi_N^m$, by Proposition 2. It is also straightforward to see that $\Pi_O^{ma} > \Pi_S^{ma}$ if and only if $\alpha\beta_1^2 < 1 < \alpha\beta_1\rho$, by Proposition 4, as $(\mathcal{P}_k^{ma})$ for $k = O$ is a relaxation of $(\mathcal{P}_k^{ma})$ for $k = S$, and σ_O^{ma} is infeasible for $(\mathcal{P}_k^{ma})$ for $k = S$ in this parameter range.

Lastly, $\Pi_O^m > \Pi_S^m$ always hold as, by $(\mathcal{P}_k^m)$ for $k \in \{O, S\}$, utilizing (D.1) and Propositions 1 and 2, we have

$$\Pi_O^m - \Pi_S^m = \begin{cases}\lambda\alpha(1-\alpha)(\pi_L-\pi_R)(v_L^\ell - v_R^\ell) > 0, & \text{for } \ \beta_\lambda \leq 0,\\ (1-\lambda)\alpha(1-\alpha)(\pi_L-\pi_R)(v_R^r - v_L^r) > 0, & \text{for } \ \beta_\lambda > 0,\end{cases}$$

which completes the proof. ■

Proof of Proposition 7. Utilizing (D.1) and (D.2), for privacy regimes $k \in \{U, O\}$, by Propositions 1 and 2, we have

$$\Pi_k^m - \Pi_k^{ma} = \begin{cases}\lambda\alpha(1-\alpha)(\pi_L-\pi_R)(v_L^\ell - v_R^\ell) > 0, & \text{for } \ \alpha\rho\beta_1 \leq 1,\\ \lambda(1-\alpha)\pi_R v_R^\ell > 0, & \text{for } \ \alpha\rho\beta_1 > 1,\end{cases}$$

and again utilizing (D.1) and (D.2), for privacy regimes $k = N$, by Propositions 2 and 5, we have

$$\Pi_N^m - \Pi_N^{ma} = \begin{cases}0, & \text{for } \ \beta_\lambda \leq 0, \alpha\rho\beta_\lambda \leq 1,\\ \alpha(1-\alpha)(\pi_L-\pi_R)[\lambda v_R^\ell + (1-\lambda)v_R^\ell]\beta_\lambda > 0, & \text{for } \ \beta_\lambda > 0, \alpha\rho\beta_\lambda \leq 1,\\ (1-\alpha)\pi_R[\lambda v_R^\ell + (1-\lambda)v_R^\ell] > 0, & \text{for } \ \beta_\lambda > 0, \alpha\rho\beta_\lambda > 1,\end{cases}$$

which completes the proof of the first statement of Proposition 7.

We examine the last statement of Proposition 7 in four cases, enumerated from S1 to S4.

Case S1: When $\beta_1 \geq \rho$ and $\frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} \leq 1$. Here, we have $\alpha\rho\beta_1 > 1$ and $\beta_\lambda \leq 0$. Utilizing (D.1) and (D.2), by Propositions 2 and 4, we have

$$\begin{aligned}
\Pi_S^{ma} - \Pi_S^m &= (1-\alpha)\pi_R \left\{\alpha\rho(1-\lambda)(v_R^r - v_L^r) - \lambda v_R^\ell\right\} \\
(\text{as } \beta_\lambda \leq 0) \quad &\geq (1-\alpha)\pi_R\lambda \left\{\alpha\rho(v_L^\ell - v_R^\ell) - v_R^\ell\right\} \\
&= (1-\alpha)\pi_R\lambda v_R^\ell \left\{\alpha\rho\beta_1 - 1\right\} \\
(\text{as } \alpha\rho\beta_1 > 1) \quad &> 0.
\end{aligned}$$

Case S2: When $\beta_1 \geq \rho$ and $\frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} > 1$. Here, we have $\alpha\rho\beta_1 > \frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} > 1$ and $\beta_\lambda > 0$. Utilizing (D.1) and (D.2), by Propositions 2 and 4, we have

$$\begin{aligned}
\Pi_S^{ma} - \Pi_S^m &= (1-\alpha)\pi_R \left\{\alpha\rho(1-\lambda)(v_R^r - v_L^r) - \lambda v_R^\ell\right\} \\
\left(\text{as } \alpha\rho\beta_1 > \frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)}\right) \quad &> (1-\alpha)\pi_R\lambda \left\{(v_L^\ell - v_R^\ell)/\beta_1 - v_R^\ell\right\} \\
&= (1-\alpha)\pi_R\lambda(v_R^\ell - v_R^\ell) = 0.
\end{aligned}$$

Case S3: When $\beta_1 < \rho$ and $\frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} \leq 1$. Here, we have $\alpha\beta_1^2 > 1$ and $\beta_\lambda > 0$. Utilizing (D.1) and (D.2), by Propositions 2 and 4, we have

$$\begin{aligned}
\Pi_S^{ma} - \Pi_S^m &= \lambda(1-\alpha)v_R^\ell(\alpha\rho\beta_1 - 1) \\
(\text{as } \beta_1 < \rho) \quad &> \lambda(1-\alpha)v_R^\ell(\alpha\beta_1^2 - 1) \\
\left(\text{as } \alpha\beta_1^2 > 1\right) \quad &> 0.
\end{aligned}$$

Case S4: When $\beta_1 < \rho$ and $\frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} > 1$. Here, we have $\alpha\beta_1^2 > \frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)} > 1$ and $\beta_\lambda > 0$. Utilizing (D.1) and (D.2), by Propositions 2 and 4, we have

$$\begin{aligned}
\Pi_S^{ma} - \Pi_S^m &= (1-\alpha)\pi_R \left\{\alpha\rho(1-\lambda)(v_R^r - v_L^r) - \lambda v_R^\ell\right\} \\
\left(\text{as } \alpha\rho\beta_1 > \frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)}\right) \quad &> (1-\alpha)\pi_R\lambda \left\{(v_L^\ell - v_R^\ell)/\beta_1 - v_R^\ell\right\} \\
&= (1-\alpha)\pi_R\lambda(v_R^\ell - v_R^\ell) = 0,
\end{aligned}$$

which completes the proof. ■

Proof of Proposition 8. We first rewrite both US_k^m and US_k^{ma}, given in (12), as

$$\begin{aligned}
US_k^m =& \lambda V^m\left(\ell, \widehat{\ell} \,\middle|\, \sigma\right) + (1-\lambda)V^m\left(r, \widehat{r} \,|\, \sigma\right) \\
=& \frac{\lambda}{2}\left\{\frac{\left[v_L^\ell\alpha\sigma_L^{\widehat{\ell}} + v_R^\ell(1-\alpha)\sigma_R^{\widehat{\ell}}\right]^2}{\alpha\sigma_L^{\widehat{\ell}} + (1-\alpha)\sigma_R^{\widehat{\ell}}} + \frac{\left[v_L^\ell\alpha[1-\sigma_L^{\widehat{\ell}}] + v_R^\ell(1-\alpha)[1-\sigma_R^{\widehat{\ell}}]\right]^2}{\alpha[1-\sigma_L^{\widehat{\ell}}] + (1-\alpha)[1-\sigma_R^{\widehat{\ell}}]} + 1\right\}
\end{aligned}$$

$$+\frac{1-\lambda}{2}\left\{\frac{\left[v_L^r\alpha\sigma_L^{\widehat{r}}+v_R^r(1-\alpha)\sigma_R^{\widehat{r}}\right]^2}{\alpha\sigma_L^{\widehat{r}}+(1-\alpha)\sigma_R^{\widehat{r}}}+\frac{\left[v_L^r\alpha[1-\sigma_L^{\widehat{r}}]+v_R^r(1-\alpha)[1-\sigma_R^{\widehat{r}}]\right]^2}{\alpha[1-\sigma_L^{\widehat{r}}]+(1-\alpha)[1-\sigma_R^{\widehat{r}}]}+1\right\}, \tag{D.3}$$

and

$$\begin{aligned}US_k^{ma}=&\lambda V^{ma}\left(\ell,\widehat{\ell}\,\middle|\,\sigma\right)+(1-\lambda)V^{ma}\left(r,\widehat{r}\,|\,\sigma\right)\\=&\frac{\lambda}{2}\left\{\frac{\left[v_L^\ell\alpha\sigma_L^{\widehat{\ell}}+v_R^\ell(1-\alpha)\sigma_R^{\widehat{\ell}}\right]^2}{\alpha\sigma_L^{\widehat{\ell}}+(1-\alpha)\sigma_R^{\widehat{\ell}}}+1\right\}+\frac{1-\lambda}{2}\left\{\frac{\left[v_L^r\alpha\sigma_L^{\widehat{r}}+v_R^r(1-\alpha)\sigma_R^{\widehat{r}}\right]^2}{\alpha\sigma_L^{\widehat{r}}+(1-\alpha)\sigma_R^{\widehat{r}}}+1\right\},\end{aligned} \tag{D.4}$$

For convenience, in the remainder of this proof and the proof of Proposition 9, we drop the 1's at the end of both surplus functions (D.3) and (D.4), and multiply both (D.3) and (D.4) by 2.

Having (D.3) and (D.4), it is straightforward to see, by Propositions 1 and 3, that $US_U^j=US_O^j$ for $j\in\{m,ma\}$, and by Proposition 2, that $US_S^m=US_N^m$.

Next, utilizing (D.3), by Propositions 1 and 2, it is easy to see that $US_O^m>US_S^m$ if and only if $\beta_\lambda\leq 0$ as

$$US_O^m-US_S^m=\begin{cases}\lambda\alpha(1-\alpha)(v_L^\ell-v_R^\ell)^2>0, & \text{for}\quad \beta_\lambda\leq 0,\\ -(1-\lambda)\alpha(1-\alpha)(v_L^r+v_R^r)^2<0, & \text{for}\quad \beta_\lambda>0.\end{cases}$$

For $\alpha\beta_1\min\{\beta_1,\rho\}>1$, utilizing (D.4), by Proposition 4, we immediately have $US_O^{ma}=US_S^{ma}$. In the same region, again by (D.4) and Proposition 4, we have

$$US_S^{ma}-US_N^{ma}=\begin{cases}\lambda(1-\alpha)(v_R^\ell)^2(\alpha\beta_1^2-1)>0, & \text{for}\quad \alpha\rho\beta_\lambda\leq 1,\\ (1-\lambda)(1-\alpha)(v_R^r)^2(1-\alpha\beta_r)>0, & \text{for}\quad \alpha\rho\beta_\lambda>1,\end{cases}$$

where the strict inequalities follow as $\alpha\beta_1\min\{\beta_1,\rho\}>1$ and $\beta_r<1$. Hence, we have also established $US_S^{ma}>US_N^{ma}$. Hence, together with $US_O^{ma}=US_S^{ma}$, we have shown $US_O^{ma}=US_S^{ma}>US_N^{ma}$ for $\alpha\beta_1\min\{\beta_1,\rho\}>1$.

Lastly, we show, for $\alpha\beta_1^2<1<\alpha\beta_1\rho$ and $\lambda>\lambda_S^{ma}$, we have $US_O^{ma}>US_S^{ma}$ if $\sigma_S^{ma}(R,r)<\sigma_S^{ma}(L,r)$. In that, we resort back to the methodology and parameter transformation we devised for Appendix C, established particularly through (C.1) and (C.2). When $\alpha\beta_\ell^2<1<\alpha\beta_\ell\rho$, by Proposition 4, we have $\sigma_S^{ma}\not\equiv\sigma_O^{ma}$. Under such scenario, we have $\sigma_O^{ma}=(1,0,1,1,1,0)$ (which, by (C.1), is equivalent to $x=(\sqrt{\alpha},0,\alpha,1-\alpha,\sqrt{\alpha},0)$) regardless of λ, whereas, when $\lambda\leq\lambda_S^{ma}$, $\sigma_S^{ma}=(1,1,1,1,1,1)$ (which, by (C.1), is equivalent to $\overset{\star}{x}=(\alpha,1-\alpha,\alpha,1-\alpha,\alpha,1-\alpha)$) and, when $\lambda>\lambda_S^{ma}$, by Proposition 4, $\sigma_S^{ma}=(1,0,\check{\sigma}_L^r,\check{\sigma}_R^r,1,0)$, (which, by (C.1) and Lemmas C.7 and C.10, is equivalent to $\overset{\star}{x}=(\sqrt{\alpha},0,x_L^{r\star}(\underline{g}),x_R^{r\star}(\underline{g}),\sqrt{\alpha},0)$). We re-write (D.4) with the aid of Lemma C.1 as

$$US_k^{ma}=\lambda/2\left\{x_L^\ell(1+\beta_1)+x_R^\ell\right\}+(1-\lambda)/2\left\{x_L^r(1-\beta_r)+x_R^r\right\}+1/2. \tag{D.5}$$

For $\lambda>\lambda_S^{ma}$, we focus on the case where $\check{\sigma}_R^r<\check{\sigma}_L^r$. By Corollary 1 (using $\beta_r=-\beta_0=1-v_L^r/v_R^r$), $\check{\sigma}_R^r<\check{\sigma}_L^r$ corresponds to $\beta_r+2\beta_1<\rho$ and $\alpha<\widetilde{\alpha}$, where

$$\widetilde{\alpha}=\begin{cases}\min\left\{\frac{\rho-\beta_r-2\beta_1}{\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)},1\right\}, & \text{if}\quad \beta_r\in\left(\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1},\rho-2\beta_1\right)\\ 1, & \text{if}\quad \beta_r\leq\frac{\rho(1-\beta_1)-2\beta_1}{2(\rho-\beta_1)+\beta_1+1}.\end{cases}$$

Hence, by (C.43) and (C.44), we have

$$(\overset{\star}{x}_L^r,\overset{\star}{x}_R^r)=\begin{cases}\left(x_{L_*}^r(\underline{g}),x_{R_*}^r(\underline{g})\right), & \text{for (C.24) and } \underline{g}<\sqrt{\frac{4\alpha\beta_1}{1-A^2}},\\ \left(x_{L_\alpha}^r(\underline{g}),x_{R_\alpha}^r(\underline{g})\right), & \text{for (C.24) and } \underline{g}\geq\sqrt{\frac{4\alpha\beta_1}{1-A^2}},\ \text{and, for (C.25).}\end{cases}$$

Then, by Lemma C.7 and Claim C.2, $(\dot{x}^r_L, \dot{x}^r_R)$ lie on the dashed line below the red shaded area in Figure C.2. By solving $\dot{x}^r_R = \frac{(1-\alpha)}{\alpha}\dot{x}^r_L$ and $\dot{x}^r_R = (\sqrt{\alpha} - \dot{x}^r_L)(\beta_1+1)$ together for this line, we have (i) $\dot{x}^r_L > \frac{\alpha\sqrt{\alpha}(\beta_1+1)}{1+\alpha\beta_1}$. Utilizing (D.5) we calculate the welfare difference between the policies σ^{ma}_S and σ^{ma}_O :

$$\begin{aligned} US^{ma}_S - US^{ma}_O &= (1-\lambda)/2\left[(1-\beta_r)(\dot{x}^r_L - \alpha) + \dot{x}^r_R - (1-\alpha)\right] \\ \text{by Claim C.2} \quad &= (1-\lambda)/2\left[(\alpha - \dot{x}^r_L)(\beta_1+\beta_r) + (1-\sqrt{\alpha})(\sqrt{\alpha}\beta_1 - 1)\right]. \end{aligned} \tag{D.6}$$

Now, notice that we also have

$$\begin{aligned} & (1-\sqrt{\alpha})(1-\alpha\beta_r)\underbrace{(\sqrt{\alpha}\beta_1 - 1)}_{<0 \text{ as } \alpha\beta_1^2<1} < 0 \\ \Leftrightarrow \quad & \alpha(\beta_1+\beta_r) + (1-\sqrt{\alpha})(\sqrt{\alpha}\beta_1 - 1) < \frac{\alpha\sqrt{\alpha}(\beta_1+1)}{1+\alpha\beta_1}(\beta_1+\beta_r) \\ \text{by (i)} \quad \Rightarrow \quad & \alpha(\beta_1+\beta_r) + (1-\sqrt{\alpha})(\sqrt{\alpha}\beta_1 - 1) < \dot{x}^r_L(\beta_1+\beta_r), \end{aligned}$$

hence, the expression given in (D.6)< 0. This completes the proof. ■

Claim D.1. *For $\alpha\beta_\ell^2 < 1 < \alpha\beta_\ell\rho$ and $\lambda > \lambda^{ma}_S$, the self-disclosure protection may lower the user surplus, i.e., $US^{ma}_S < US^{ma}_O$, even if $\sigma^{ma}_S(R,r) > \sigma^{ma}_S(L,r)$.*

Proof of Claim D.1. To establish Claim D.1 and, equivalently, Footnote 13, we utilize, again, the methodology and parameter transformation we devised for Appendix C, established particularly through (C.1) and (C.2). We focus on the case where $\check{\sigma}^r_R \geq \check{\sigma}^r_L$, and (using $\beta_r = -\beta_0 = 1 - v^r_L/v^r_R$) utilize as sufficient conditions (i) $\rho - \beta_r \leq 2\beta_1$ and (ii) $\alpha < \min\left\{\frac{1}{(\beta_1+1)^2+1}, \left[\frac{\sqrt{4\beta_r+(\beta_1+1)^2}-(\beta_1+1)}{2\beta_r}\right]^2\right\}$. Under these conditions we have, by Lemma C.10, $(\check{\sigma}^r_L, \check{\sigma}^r_R) = (\check{\sigma}^r_{L_0}, \check{\sigma}^r_{R_0}) = \left(0, \frac{\alpha(\beta_1+1)^2}{1-\alpha}\right)$ and, hence, $\sigma^{ma}_S(R,r) > \sigma^{ma}_S(L,r)$.

Notice that here, we have

$$\begin{aligned} & \alpha < \left[\frac{\sqrt{4\beta_r+(\beta_1+1)^2}-(\beta_1+1)}{2\beta_r}\right]^2 \quad \Leftrightarrow \quad \sqrt{\alpha} < \frac{\sqrt{4\beta_r+(\beta_1+1)^2}-(\beta_1+1)}{2\beta_r} \\ \Leftrightarrow \quad & \sqrt{\alpha}\sqrt{\beta_r} + \frac{\beta_1+1}{2\sqrt{\beta_r}} < \sqrt{\frac{4\beta_r+(\beta_1+1)^2}{4\beta_r}} \quad \Leftrightarrow \quad \left[\sqrt{\alpha}\sqrt{\beta_r} + \frac{\beta_1+1}{2\sqrt{\beta_r}}\right]^2 - \frac{(\beta_1+1)^2}{4\beta_r} < 1 \\ \Leftrightarrow \quad & \alpha\beta_r + \sqrt{\alpha}(\beta_1+1) < 1 \quad \Leftrightarrow \quad (1-\lambda)/2\left[\alpha\beta_r + \sqrt{\alpha}(\beta_1+1) - 1\right] < 0 \\ \Leftrightarrow \quad & (1-\lambda)/2\left[-\alpha(1-\beta_r) + \sqrt{\alpha}(\beta_1+1) - (1-\alpha)\right] < 0 \\ \text{by (i)-(ii), (C.21) and (C.43)} \quad \Leftrightarrow \quad & (1-\lambda)/2\left[(1-\beta_r)\left(x^r_{L_0}(\underline{g}) - \alpha\right) + x^r_{R_0}(\underline{g}) - (1-\alpha)\right] < 0, \end{aligned}$$

which, by (D.6), establishes that $US^{ma}_S < US^{ma}_O$, where we used $(\dot{x}^r_L, \dot{x}^r_R) = \left(x^r_{L_0}(\underline{g}), x^r_{R_0}(\underline{g})\right)$ in the last inequality. This completes the proof. ■

Proof of Proposition 9. For privacy regime $k \in \{U, O\}$, utilizing (D.3) and (D.4), by Propositions 1 and 3, we have

$$US^m_k - US^{ma}_k = \begin{cases} \lambda\alpha(1-\alpha)(v^\ell_L - v^\ell_R)^2 > 0, & \text{for } \alpha\rho\beta_1 \leq 1, \\ \lambda(1-\alpha)(v^\ell_R)^2 > 0, & \text{for } \alpha\rho\beta_1 > 1, \end{cases}$$

establishing $US_k^m > US_k^{ma}$ for $k \in \{U,O\}$. For $k=N$, again utilizing (D.3) and (D.4), by Propositions 2 and 5, we have

$$US_N^m - US_N^{ma} = \begin{cases} 0, & \text{for } \beta_\lambda \le 0, \alpha\rho\beta_\lambda \le 1, \\ \alpha(1-\alpha)\{\lambda(v_L^\ell - v_R^\ell)^2 + (1-\lambda)(v_L^r - v_R^r)^2\} > 0, & \text{for } \beta_\lambda > 0, \alpha\rho\beta_\lambda \le 1, \\ (1-\alpha)\{\lambda(v_R^\ell)^2 + (1-\lambda)(v_R^r)^2\} > 0, & \text{for } \beta_\lambda > 0, \alpha\rho\beta_\lambda > 1, \end{cases}$$

establishing $US_N^m \ge US_N^{ma}$. Hence, together with $US_k^m > US_k^{ma}$ for $k \in \{U,O\}$, we have established $US_k^m \ge US_k^{ma}$ for $k \in \{U,O,N\}$.

The last statement of Proposition 9 follows from (D.3) and (D.4), and Propositions 2 and 4. As such, for the parameter range $\alpha\beta_1 \min\{\beta_1,\rho\} > 1 \ge \frac{\lambda(v_L^\ell - v_R^\ell)}{(1-\lambda)(v_R^r - v_L^r)}$, we have $\sigma_S^m = (1,1,1,1,1,1)$ and $\sigma_S^{ma} = (1,0,1,1,1,0)$, which, by (D.3) and (D.4), immediately implies

$$US_S^{ma} - US_S^m = \lambda(1-\alpha)(v_R^\ell)^2(\alpha\beta_1^2 - 1) > 0.$$

This completes the proof. ■

Appendix E: Proofs in Section 8

LEMMA E.1 (Point type sequence of $\check{\sigma}_i^r$ in $\rho = \pi_L/\pi_R - 1$). *For $\alpha \le [(\beta_1+1)^2+1]^{-1}$, $\check{\sigma}_i^r$ point type changes from $\check{\sigma}_{i_0}^r$ to $\check{\sigma}_{i_*}^r$ and then to $\check{\sigma}_{i_\alpha}^r$, as ρ increases; for $\alpha > [(\beta_1+1)^2+1]^{-1}$, $\check{\sigma}_i^r$ point type changes from $\check{\sigma}_{i_{1-\alpha}}^r$ to $\check{\sigma}_{i_*}^r$ and then to $\check{\sigma}_{i_\alpha}^r$, as ρ increases, where expressions of corresponding $\check{\sigma}_i^r$ point types are given in Lemma C.10. Furthermore, both $\check{\sigma}_i^r$, for $i \in \{L,R\}$, are continuous irrespective of the active point type. In particular,*

- *change from $\check{\sigma}_{i_0}^r$ to $\check{\sigma}_{i_*}^r$ occurs when $\rho = 2\beta_1 + \beta_r$, under which $\check{\sigma}_{i_0}^r = \check{\sigma}_{i_*}^r$,*
- *change from $\check{\sigma}_{i_{1-\alpha}}^r$ to $\check{\sigma}_{i_*}^r$ occurs when $\sqrt{\alpha}(\beta_1+1) = \sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$, under which $\check{\sigma}_{i_{1-\alpha}}^r = \check{\sigma}_{i_*}^r$,*
- *change from $\check{\sigma}_{i_*}^r$ to $\check{\sigma}_{i_\alpha}^r$ occurs either when $\rho = \frac{2\beta_1+\beta_r-\beta_1\beta_r}{1-\beta_1-2\beta_r} \ge 0$, or when $\sqrt{\alpha}(\beta_1+1) = \sqrt{\frac{4\alpha\beta_1}{1-A^2}}$, under which $\check{\sigma}_{i_*}^r = \check{\sigma}_{i_\alpha}^r$.*

Proof of Lemma E.1. In proving this lemma, we use Lemmas C.7 and C.10, to utilize the connection between $\check{\sigma}_i^r$ and $\check{x}_i^r$ for $i \in \{L,R\}$, and the 10 claims that we establish below.

Claim 1: $\frac{\rho-\beta_r-2\beta_1}{2(\beta_1+\beta_r)(\rho-\beta_1)}$ is increasing in ρ. In establishing this claim, it suffices for us to take the first order partial derivative of $\frac{\rho-\beta_r-2\beta_1}{(\rho-\beta_1)}$, with respect to ρ, and show that this derivative is positive. Such that,

$$\frac{\partial\left(\frac{\rho-\beta_r-2\beta_1}{\rho-\beta_1}\right)}{\partial\rho} = \frac{1}{(\rho-\beta_1)} - \frac{(\rho-\beta_r-2\beta_1)}{(\rho-\beta_1)^2} \propto \beta_1+\beta_r > 0.$$

Claim 2: $\sqrt{\frac{4\alpha\beta_1}{1-A^2}}$ is decreasing in ρ. We know that $A = \frac{\beta_1^2+\rho\beta_r}{(\beta_1+\beta_r)(\rho-\beta_1)} \in [0,1]$, hence, it suffices for us to show that $\frac{\beta_1^2+\rho\beta_r}{\rho-\beta_1}$ is decreasing in ρ. In that, we take the first order partial derivative of $\frac{\beta_1^2+\rho\beta_r}{\rho-\beta_1}$, with respect to ρ, and show that this derivative is negative. Such that,

$$\frac{\partial\left(\frac{\beta_1^2+\rho\beta_r}{\rho-\beta_1}\right)}{\partial\rho} = \beta_r(\rho-\beta_1)^{-1} - (\rho-\beta_1)^2(\beta_1^2+\rho\beta_r) \propto -\beta_1(\beta_1+\beta_r) < 0.$$

Claim 3: $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$ is increasing in ρ. We know that $B=\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}\geq 1$, hence, it suffices for us to show that $\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\rho-\beta_1)}$ is decreasing in ρ. In that, we take the first order partial derivative of $\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\rho-\beta_1)}$, with respect to ρ, and show that this derivative is negative. Such that,

$$\frac{\partial\left(\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{\rho-\beta_1}\right)}{\partial\rho}=\frac{\beta_1^2+\beta_r(1+2\beta_1)}{(\rho-\beta_1)}-\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\rho-\beta_1)^2}\propto-\beta_1(\beta_1^2+\beta_1\beta_r+\beta_r)<0.$$

Claim 4: For $\rho=2\beta_1+\beta_r$, $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}<\sqrt{\alpha}(\beta_1+1)$ if and only if $\sqrt{1-\alpha}<\sqrt{\alpha}(\beta_1+1)$. For $\rho=2\beta_1+\beta_r$, we have

$$B=\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}=\frac{\beta_1^2(1+2\beta_1)+(2\beta_1+\beta_r)\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)^2}=1+2\beta_1.$$

Accordingly,

$$\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}<\sqrt{\alpha}(\beta_1+1)\Leftrightarrow 4(1-\alpha)\beta_1<\alpha(\beta_1+1)(B^2-1)\Leftrightarrow\sqrt{1-\alpha}<\sqrt{\alpha}(\beta_1+1).$$

Claim 5: For $\rho=\frac{2\beta_1+\beta_r-\alpha\beta_1\beta_r}{1-\alpha\beta_1-2\alpha\beta_r}\geq 0$, $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}>\sqrt{\alpha}(\beta_1+1)$. For $\rho=\frac{2\beta_1+\beta_r-\alpha\beta_1\beta_r}{1-\alpha\beta_1-2\alpha\beta_r}\geq 0$ and $B=\frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}\geq 1$, the inequality $\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}>\sqrt{\alpha}(\beta_1+1)$ can be written as

$$\begin{aligned}
&\alpha(\beta_1+1)(B-1)(B+1)<4(1-\alpha)\beta_1\\
\Leftrightarrow\quad&\alpha(\beta_1+1)^2\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\left[(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)\right]<4(1-\alpha)(\beta_1+\beta_r)^2(\rho-\beta_1)^2\\
\Leftrightarrow\quad&4\alpha(1-\alpha)(\beta_1+1)^2(\beta_1+\beta_r)^2<4(1-\alpha)(1+\alpha\beta_1)^2(\beta_1+\beta_r)^2\quad\Leftrightarrow\quad\alpha\beta_1^2<1,
\end{aligned}$$

which follows from Proposition 4.

Claim 6: For $\rho=\frac{2\beta_1+\beta_r-\alpha\beta_1\beta_r}{1-\alpha\beta_1-2\alpha\beta_r}\geq 0$, $\sqrt{\frac{4\alpha\beta_1}{1-A^2}}>\sqrt{\alpha}(\beta_1+1)$. For $\rho=\frac{2\beta_1+\beta_r-\alpha\beta_1\beta_r}{1-\alpha\beta_1-2\alpha\beta_r}\geq 0$ and $A=\frac{\beta_1^2+\rho\beta_r}{(\beta_1+\beta_r)(\rho-\beta_1)}\in[0,1]$, the inequality $\sqrt{\frac{4\alpha\beta_1}{1-A^2}}>\sqrt{\alpha}(\beta_1+1)$ can be written as

$$\begin{aligned}
&(\beta_1+1)^2(1-A)(1+A)<4\beta_1\\
\Leftrightarrow\quad&(\beta_1+1)^2\left[(\rho-\beta_1)-(\beta_1+\beta_r)\right]\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]<4(\beta_1+\beta_r)^2(\rho-\beta_1)^2\\
\Leftrightarrow\quad&4\alpha(\beta_1+1)^4(\beta_1+\beta_r)^2<4(1+\alpha\beta_1)^2(\beta_1+\beta_r)^4\quad\Leftrightarrow\quad\alpha\beta_1^2<1,
\end{aligned}$$

which follows from Proposition 4.

Claim 7: When $\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}$ and (C.24) holds, $\breve{\sigma}_*^r\equiv\breve{\sigma}_\alpha^r$. We first show that this scenario can only be realized for $\beta_1>1$, as when

$$\begin{aligned}
&\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}\quad\Leftrightarrow\quad\frac{4\beta_1}{1-A^2}=(\beta_1+1)^2\quad\Leftrightarrow\quad4\beta_1=(\beta_1+1)^2(1-A)(1+A)\\
\Leftrightarrow\quad&4(\beta_1+\beta_r)^2(\rho-\beta_1)^2=(\beta_1+1)^2\left[(\rho-\beta_1)-(\beta_1+\beta_r)\right]\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\\
\Leftrightarrow\quad&\left[(\beta_1+\beta_r)(\beta_1^2+\rho)-(\rho-\beta_1)(\beta_1^2-\beta_r)\right]\underbrace{\left[(\beta_1+\beta_r)(1+\rho)-(\rho-\beta_1)(1-\beta_r)\right]}_{>0,\text{ by (C.24)}}=0\\
\Leftrightarrow\quad&\frac{\beta_1^2+\rho}{\beta_1^2-\beta_r}=\frac{\rho-\beta_1}{\beta_1+\beta_r}\quad\underbrace{\Rightarrow}_{\text{by (C.24)}}\quad\frac{\beta_1^2+\rho}{\beta_1^2-\beta_r}<\frac{1+\rho}{1-\beta_r}\quad\Leftrightarrow\quad\beta_1^2>1\quad\Leftrightarrow\quad\beta_1>1.
\end{aligned}$$

Next, notice that $\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}$ can be written as

$$\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4\alpha\beta_1}{1-A^2}}\Leftrightarrow\alpha(\beta_1+1)^2=\frac{4\alpha\beta_1}{1-A^2}\Leftrightarrow\underline{g}^2(1-A^2)=4\alpha\beta_1\Leftrightarrow\underline{g}^2-4\alpha\beta_1=(\underline{g}A)^2\Leftrightarrow\sqrt{\underline{g}^2-4\alpha\beta_1}=\underline{g}A.$$

Plugging this into the expressions of $\check{x}^r_{i_*}(\underline{g})$ and $\check{x}^r_{i_\alpha}(\underline{g})$, for $i\in\{L,R\}$, given in (C.21) of Lemma C.7, it is straightforward to see the equivalence of $\check{x}^r_*$ and $\check{x}^r_\alpha$, and hence, $\check{\sigma}^r_*$ and $\check{\sigma}^r_\alpha$.

Claim 8: When $\rho=\frac{2\beta_1+\beta_r-\beta_1\beta_r}{1-\beta_1-2\beta_r}\geq 0$, $\check{\sigma}^r_*\equiv\check{\sigma}^r_\alpha$. We first show that this scenario can only be realized for $\beta_1<1$, as $\beta_r<1$, which makes the numerator of $\rho=\frac{2\beta_1+\beta_r-\beta_1\beta_r}{1-\beta_1-2\beta_r}$ positive. Next, plugging $\rho=\frac{2\beta_1+\beta_r-\beta_1\beta_r}{1-\beta_1-2\beta_r}$ into the expressions of $\check{\sigma}^r_{i_*}$ and $\check{\sigma}^r_{i_\alpha}$, for $i\in\{L,R\}$, given in Lemma C.10, it is straightforward to see the equivalence $\check{\sigma}^r_*$ and $\check{\sigma}^r_\alpha$.

Claim 9: When $\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$ and (C.23) holds, $\check{\sigma}^r_{1-\alpha}\equiv\check{\sigma}^r_*$. Notice that $\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}$ can be written as

$$\sqrt{\alpha}(\beta_1+1)=\sqrt{\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}}\Leftrightarrow\alpha(\beta_1+1)^2=\frac{4(1-\alpha)\beta_1(1+\beta_1)}{B^2-1}$$
$$\Leftrightarrow\underline{g}^2(B^2-1)=4(1-\alpha)\beta_1(1+\beta_1)\Leftrightarrow\sqrt{\underline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}=\underline{g}B.$$

Plugging this into the expressions of $\check{x}^r_{i_{1-\alpha}}(\underline{g})$ and $\check{x}^r_{i_*}(\underline{g})$, for $i\in\{L,R\}$, given in (C.21) of Lemma C.7, it is straightforward to see the equivalence of $\check{x}^r_{1-\alpha}$ and $\check{x}^r_*$, and hence, $\check{\sigma}^r_{1-\alpha}$ and $\check{\sigma}^r_*$.

Claim 10: When $\rho=2\beta_1+\beta_r$, $\check{\sigma}^r_0\equiv\check{\sigma}^r_*$. Plugging $\rho=2\beta_1+\beta_r$ into the expressions of $\check{\sigma}^r_{i_0}$ and $\check{\sigma}^r_{i_*}$, for $i\in\{L,R\}$, given in Lemma C.10, it is straightforward to see the equivalence $\check{\sigma}^r_0$ and $\check{\sigma}^r_*$.

To bring all our findings together, we now present the following summary. Claim 1 ensures that the conditions (C.22), (C.23), (C.24), and (C.25) hold one after the other, in numeric order, as ρ increases. Claim 2 and 3 establish the monotonicity of point types in ρ for conditions (C.24) and (C.23), respectively. Claim 4 establishes that we cannot have $\check{\sigma}^r_0$ and $\check{\sigma}^r_{1-\alpha}$ in varying ρ, for any possible combination of the remaining model elements. Claims 5 and 6 establish that $\check{\sigma}^r_*$ will always be the active point type, when ρ is at the boundary of the conditions of (C.23) and (C.24). Lastly, Claims 7 to 10 establish the continuity of $\check{\sigma}^r_i$, for $i\in\{L,R\}$, when there is a point type switch between the conditions of (C.22) to (C.23) and (C.24) to (C.25), and within the conditions of (C.23) and (C.24). □

LEMMA E.2 (Monotonicity of Π^{ma}_k in both π_L and π_R). *The platform's revenue function $\Pi^{ma}_k(\pi_L,\pi_R)$ for $k\in\{U,O,S,N\}$, given in ($\mathcal{P}^{ma}_k$), is increasing in both π_L and π_R.*

Proof of Lemma E.2. First, to expand the definition of transaction rates given in (16), we express user t's expected value of the product type, under σ^{ma}_k, upon receiving a signal $d=1$:

$$\mathbb{E}\left[v^t_i\mid t,t,d=1,\sigma^{ma}_k\right]=\frac{v^t_L\alpha\sigma^{ma}_k(L,t)+v^t_R(1-\alpha)\sigma^{ma}_k(R,t)}{\alpha\sigma^{ma}_k(L,t)+(1-\alpha)\sigma^{ma}_k(R,t)},$$

for $k\in\{U,O,S,N\}$ and $t\in\{\ell,r\}$. We can then rewrite (16) as

$$\begin{aligned}tr^{ma}_k\left(i\mid\rho=\pi_L/\pi_R-1\right)=\lambda\sigma^{ma}_k\left(i,\ell\mid\rho=\pi_L/\pi_R-1\right)&\frac{v^\ell_L\alpha\sigma^{ma}_k(L,\ell)+v^\ell_R(1-\alpha)\sigma^{ma}_k(R,\ell)}{\alpha\sigma^{ma}_k(L,\ell)+(1-\alpha)\sigma^{ma}_k(R,\ell)}\\+(1-\lambda)\sigma^{ma}_k\left(i,r\mid\rho=\pi_L/\pi_R-1\right)&\frac{v^r_L\alpha\sigma^{ma}_k(L,r)+v^r_R(1-\alpha)\sigma^{ma}_k(R,r)}{\alpha\sigma^{ma}_k(L,r)+(1-\alpha)\sigma^{ma}_k(R,r)},\end{aligned}\tag{E.1}$$

for $k\in\{U,O,S,N\}$ and $i\in\{L,R\}$, which allows us to write the platform's revenue function $\Pi^{ma}_k(\pi_L,\pi_R)$, given in ($\mathcal{P}^{ma}_k$), as

$$\Pi^{ma}_k(\pi_L,\pi_R)=\alpha\pi_L tr^{ma}_k\left(L\mid\rho=\pi_L/\pi_R-1\right)+(1-\alpha)\pi_R tr^{ma}_k\left(R\mid\rho=\pi_L/\pi_R-1\right)\tag{E.2}$$

for $k \in \{U, O, S, N\}$.

We reformulate the platform's problem given in ($\mathcal{P}_k^{ma}$), including the revenue function (E.2), in Lemmas C.1-C.3, highlighting the corresponding feasible decision variable domain in Figure C.1. Notice then, by Theorem 9.14 of Sundaram (1996), that the platform's revenue function given in (E.2) is continuous in its model parameters π_L and π_R.

It is now straightforward to see that (E.2) for $k \in \{U, O, S, N\}$ is increasing in both π_L and π_R when $tr_k^{ma}(i \mid \rho = \pi_L/\pi_R - 1)$ for $i \in \{L, R\}$, or equivalently, $\sigma_k^{ma}(i,t)$ for $i \in \{L, R\}$ and $t \in \{\ell, r\}$, do not contain the terms π_L and π_R. Propositions 3 and 5 imply immediately that this is indeed the case for $k \in \{U, O, N\}$. For $k = S$, Proposition 4 and Lemma C.10 together imply that $tr_k^{ma}(i \mid \rho = \pi_L/\pi_R - 1)$ for $i \in \{L, R\}$ contain the terms of π_L and π_R only when $\left(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)\right) = \left(\check{\sigma}_L^r, \check{\sigma}_R^r\right) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$.

We thus focus on the scenario for $k = S$ where varying π_L and π_R induce $\left(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)\right) = \left(\check{\sigma}_L^r, \check{\sigma}_R^r\right) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$, and show that the respective segment of the revenue function (E.2) is increasing in both π_L and π_R. Recall that under the variable transformation in (C.1), and with the brevity of denoting the probability of recommending the product for each realization of (i,t) as σ_i^t, as opposed to $\sigma(i,t)$, the platform's expected revenue (E.2) can be reformulated as in (C.3):

$$\begin{aligned}\Pi_S^{ma}(\pi_L, \pi_R) &= \pi_R[\lambda v_R^\ell + (1-\lambda) v_R^r] \left\{ \begin{array}{l} \lambda' \left[(1+\rho)(1+\beta_1)\left(x_L^\ell\right)^2 + (2+\rho+\beta_1) x_L^\ell x_R^\ell + \left(x_R^\ell\right)^2\right] \\ +(1-\lambda')\left[(1+\rho)(1-\beta_r)\left(x_L^r\right)^2 + (2+\rho-\beta_r) x_L^r x_R^r + \left(x_R^r\right)^2\right], \end{array} \right. \\ &= \pi_R[\lambda v_R^\ell + (1-\lambda) v_R^r] \left\{ \begin{array}{l} \lambda' \left[(1+\rho) x_L^\ell + x_R^\ell\right]\left[(1+\beta_1) x_L^\ell + x_R^\ell\right] \\ +(1-\lambda')\left[(1+\rho) x_L^r + x_R^r\right]\left[(1-\beta_r) x_L^r + x_R^r\right]. \end{array} \right. \end{aligned} \tag{E.3}$$

Notice next, by Proposition 4, that when $\left(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)\right) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$, it must be the case that $\left(\sigma_S^{ma}(L,\ell), \sigma_S^{ma}(R,\ell)\right) = (1,0)$, which, by (C.1), corresponds to $(x_L^\ell, x_R^\ell) = (\sqrt{\alpha}, 0)$. The expressions for x_L^r and x_R^r when $\left(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)\right) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$ are also given in the proof of Lemma C.10 by (C.50) and (C.51), respectively. All combined, (E.3) takes the following form:

$$\Pi_S^{ma}(\pi_L, \pi_R) = \pi_R[\lambda v_R^\ell + (1-\lambda) v_R^r](1+\beta_1)\alpha \left\{ \lambda' \pi_L + (1-\lambda') \frac{(\beta_1+1)}{4(\beta_1+\beta_r)} \frac{[\pi_L - \pi_R(1-\beta_r)]^2}{[\pi_L - \pi_R(1+\beta_1)]} \right\}. \tag{E.4}$$

Our strategy for proving the platform's revenue function (E.4) is increasing in π_L is to take its first order partial derivative, with respect to π_L, and show that this derivative is positive. Such that,

$$\begin{aligned} \frac{\partial(\text{E.4})}{\partial \pi_L} &= \lambda' - \frac{[\pi_L - \pi_R(1-\beta_r)]^2}{[\pi_L - \pi_R(1+\beta_1)]^2} + 2\frac{[\pi_L - \pi_R(1-\beta_r)]}{[\pi_L - \pi_R(1+\beta_1)]} \\ &= \lambda' + \frac{[\pi_L - \pi_R(1-\beta_r)]}{[\pi_L - \pi_R(1+\beta_1)]} \left\{ 2 - \frac{[\pi_L - \pi_R(1-\beta_r)]}{[\pi_L - \pi_R(1+\beta_1)]} \right\} \\ &= \lambda' + \frac{(\rho+\beta_r)}{(\rho-\beta_1)} \frac{(\rho-\beta_r-2\beta_1)}{(\rho-\beta_1)} > 0, \end{aligned}$$

where the last inequality follows as, by Proposition 4, (C.23) and (C.24) (last two conditions are necessary for $\left(\sigma_S^{ma}(L,r), \sigma_S^{ma}(R,r)\right) = \left(\check{\sigma}_{L_*}^r, \check{\sigma}_{R_*}^r\right)$), $\rho > \beta_1$ and $\rho - \beta_r - 2\beta_1 > 0$.

Our strategy for proving the platform's revenue function (E.4) is increasing in π_R is to take its first order partial derivative, with respect to π_R, and show that this derivative is positive. In that, as both π_R and all

the terms of (E.4) are nonnegative, it suffices for us to only focus on the part inside the curly brackets. Such that,

$$\begin{aligned}\frac{\partial(\text{E.4})}{\partial \pi_R} &= \frac{[\pi_L-\pi_R(1-\beta_r)]^2}{[\pi_L-\pi_R(1+\beta_1)]^2}(1+\beta_1)-2(1-\beta_r)\frac{[\pi_L-\pi_R(1-\beta_r)]}{[\pi_L-\pi_R(1+\beta_1)]}\\ &= \frac{[\pi_L-\pi_R(1-\beta_r)]}{[\pi_L-\pi_R(1+\beta_1)]}\left\{\frac{[\pi_L-\pi_R(1-\beta_r)]}{[\pi_L-\pi_R(1+\beta_1)]}(1+\beta_1)-2(1-\beta_r)\right\}\\ &\propto (1+\beta_1)\left[\pi_L-\pi_R(1-\beta_r)\right]-2(1-\beta_r)\left[\pi_L-\pi_R(1+\beta_1)\right]\\ &= \pi_R\left[(1+\beta_1)(\rho+\beta_r)-2(1-\beta_r)(\rho-\beta_1)\right]\\ &\propto \rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)=(\beta_1+\beta_r)(\rho-\beta_1)(B-1)/\beta_1\geq 0,\end{aligned}$$

where the last inequality follows as, by Proposition 4, $\rho>\beta_1$ and $B\geq 1$, completing the proof. □

LEMMA E.3 (Monotonicity of $tr_k^{ma}(i\,|\,\rho)$ in $\rho=\pi_L/\pi_R-1$). *For each $k\in\{U,O,S,N\}$, the* total *transaction rate for each product $i\in\{L,R\}$, $tr_k^{ma}(i\,|\,\rho)$, as given by (16) is non-decreasing (resp., non-increasing) in ρ for product $i=L$ (resp., $i=R$). Moreover, for each $k\in\{U,O,S,N\}$ and any ρ, the total transaction rate for product L is always greater than or equal to the total transaction rate for product R, i.e., $tr_k^{ma}(L\,|\,\rho)\geq tr_k^{ma}(R\,|\,\rho)$.*

Proof of Lemma E.3. First, we rewrite the transaction rates of both types of vendors, given in (16) and (E.1), with the brevity of denoting the probability of recommending the product for each realization of (i,t) as σ_i^t, as opposed to $\sigma(i,t)$, and under the variable transformation in (C.1):

$$\begin{aligned}tr_k^{ma}(L\,|\,\rho=\pi_L/\pi_R-1) &= \frac{1}{\alpha}\left\{\lambda v_R^\ell\left[(\beta_1+1)(x_L^\ell)^2+x_L^\ell x_R^\ell\right]+(1-\lambda)v_R^r\left[(1-\beta_r)(x_L^r)^2+x_L^r x_R^r\right]\right\},\\ tr_k^{ma}(R\,|\,\rho=\pi_L/\pi_R-1) &= \frac{1}{(1-\alpha)}\left\{\lambda v_R^\ell\left[(\beta_1+1)x_L^\ell x_R^\ell+(x_R^\ell)^2\right]+(1-\lambda)v_R^r\left[(1-\beta_r)x_L^r x_R^r+(x_R^r)^2\right]\right\},\end{aligned} \tag{E.5}$$

for $k\in\{U,O,S,N\}$. We note that, when calculating $tr_N^{ma}(i\,|\,\rho=\pi_L/\pi_R-1)$, we take $x_i^t=x_i^n$ for $i\in\{L,R\}$ and $t\in\{\ell,r\}$.

Notice next, by Propositions 3-5, for $k\in\{U,O,N\}$ and when $\alpha\beta_1^2\geq 1$ for $k=S$, that, keeping other modeling elements the same, there exists a threshold in ρ on which the optimal policy turn from $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,1,1,1)$ to either $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,0,1,1)$ or $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,0,1,0)$. We now write the corresponding transaction rates of both type of vendors, given in (E.5), under these three potential policies. When the optimal policy for a given ρ is $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,1,1,1)$, by (C.1), $(x_k^{ma}(L,\ell),x_k^{ma}(R,\ell),x_k^{ma}(L,r),x_k^{ma}(R,r))=(\alpha,1-\alpha,\alpha,1-\alpha)$, then the transaction rates are given as follows

$$tr_k^{ma}(R)=tr_k^{ma}(L)=\lambda\left[\alpha v_L^\ell+(1-\alpha)v_R^\ell\right]+(1-\lambda)\left[\alpha v_L^r+(1-\alpha)v_R^r\right].$$

When the optimal policy for a given ρ is $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,0,1,1)$, by (C.1), $(x_k^{ma}(L,\ell),x_k^{ma}(R,\ell),x_k^{ma}(L,r),x_k^{ma}(R,r))=(\sqrt{\alpha},0,\alpha,1-\alpha)$, then the transaction rates are given as follows

$$tr_k^{ma}(R)=(1-\lambda)\left[\alpha v_L^r+(1-\alpha)v_R^r\right]\quad\text{and}\quad tr_k^{ma}(L)=\lambda v_L^\ell+(1-\lambda)\left[\alpha v_L^r+(1-\alpha)v_R^r\right].$$

When the optimal policy for a given ρ is $(\sigma_k^{ma}(L,\ell),\sigma_k^{ma}(R,\ell),\sigma_k^{ma}(L,r),\sigma_k^{ma}(R,r))=(1,0,1,0)$, by (C.1), $(x_k^{ma}(L,\ell),x_k^{ma}(R,\ell),x_k^{ma}(L,r),x_k^{ma}(R,r))=(\sqrt{\alpha},0,\sqrt{\alpha},0)$, then the transaction rates are given as follows

$$tr_k^{ma}(R)=0 \quad \text{and} \quad tr_k^{ma}(L)=\lambda v_L^{\ell}+(1-\lambda)v_L^{r}.$$

It is then straightforward to see that $tr_k^{ma}(R)$ is non-increasing in ρ as $\lambda[\alpha v_L^{\ell}+(1-\alpha)v_R^{\ell}]+(1-\lambda)[\alpha v_L^{r}+(1-\alpha)v_R^{r}]>\max\{(1-\lambda)[\alpha v_L^{r}+(1-\alpha)v_R^{r}],0\}$. We also have $tr_k^{ma}(L)$ non-decreasing in ρ as $\lambda[\alpha v_L^{\ell}+(1-\alpha)v_R^{\ell}]+(1-\lambda)[\alpha v_L^{r}+(1-\alpha)v_R^{r}]<\lambda v_L^{\ell}+(1-\lambda)[\alpha v_L^{r}+(1-\alpha)v_R^{r}]$ (because $v_L^{\ell}>v_R^{\ell}$) and $\lambda[\alpha v_L^{\ell}+(1-\alpha)v_R^{\ell}]+(1-\lambda)[\alpha v_L^{r}+(1-\alpha)v_R^{r}]<\lambda v_L^{\ell}+(1-\lambda)v_L^{r}$ (because for this case to occur, per Proposition 5, we must have $\beta_\lambda>(\alpha\rho)^{-1}>0$, implying $\lambda(v_L^{\ell}-v_R^{\ell})>(1-\lambda)(v_R^{r}-v_L^{r})$). This completes the proof for $k\in\{U,O,N\}$ and when $\alpha\beta_1^2\geq 1$ for $k=S$.

We now exclusively focus on the case of $k=S$ and $\alpha\beta_1^2<1$. By Proposition 4 and the fact that λ_S^{ma} is decreasing in ρ (per Proposition 11), there also exists in this case a threshold in ρ on which the optimal policy turn from $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,1,1,1)$ to $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,0,\check{\sigma}_L^{r},\check{\sigma}_R^{r})$. Unlike the above mentioned scenario for the case of $k\in\{U,O,N\}$ and when $\alpha\beta_1^2\geq 1$ for $k=S$, here, the recommendation rates of the policy observed for sufficiently large values of ρ, i.e., $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,0,\check{\sigma}_L^{r},\check{\sigma}_R^{r})$, also subject to change with ρ, hence the transaction rates of both types of vendors. Thus, in the remainder of the proof, we first establish the monotonicity of transaction rates in ρ within the parameter region where $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,0,\check{\sigma}_L^{r},\check{\sigma}_R^{r})$ is optimal, and then establish that the monotonicity is preserved at the threshold value of ρ, on which the optimal policy turn from $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,1,1,1)$ to $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r))=(1,0,\check{\sigma}_L^{r},\check{\sigma}_R^{r})$.

Notice Proposition 4 and Lemma C.10 together imply that $tr_S^{ma}(i\,|\,\rho=\pi_L/\pi_R-1)$ for $i\in\{L,R\}$ contain the terms of π_L and π_R only when $\left(\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)\right)=\left(\check{\sigma}_L^{r},\check{\sigma}_R^{r}\right)=\left(\check{\sigma}_{L_*}^{r},\check{\sigma}_{R_*}^{r}\right)$. For other functional forms in Lemma C.10, it is straightforward to see the monotonicity of transaction rates in ρ. We thus focus on the scenario where $\left(\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)\right)=\left(\check{\sigma}_L^{r},\check{\sigma}_R^{r}\right)=\left(\check{\sigma}_{L_*}^{r},\check{\sigma}_{R_*}^{r}\right)$. By (C.1), $\left(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell)\right)=(1,0)$ corresponds to $(x_L^{\ell},x_R^{\ell})=(\sqrt{\alpha},0)$. The expressions for x_L^{r} and x_R^{r} when $\left(\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)\right)=\left(\check{\sigma}_{L_*}^{r},\check{\sigma}_{R_*}^{r}\right)$ are also given in the proof of Lemma C.10 by (C.50) and (C.51), respectively. All combined, we have

$$tr_S^{ma}(R\,|\,\rho)\propto(\beta_r-1)(\rho+\beta_r)(\rho-\beta_1)^{-1}+(\beta_1+\beta_r)(1+\rho)(\rho+\beta_r)(\rho-\beta_1)^{-2} \tag{E.6}$$

and

$$tr_S^{ma}(L\,|\,\rho)\propto(\rho-\beta_r-2\beta_1)(\rho+\beta_r)(\rho-\beta_1)^{-2} \tag{E.7}$$

Our strategy for proving (E.6) is decreasing in ρ is to take its first order partial derivative, with respect to ρ, and show that this derivative is negative. Such that,

$$\frac{\partial(\text{E.6})}{\partial\rho}=(1-\beta_r)(\rho-\beta_1)-(\rho+\beta_r)(1+\beta_1)-(1+\rho)(\beta_1+\beta_r)=-2(\beta_1+\beta_r)(1+\rho)<0.$$

Similarly, our strategy for proving (E.7) is increasing in ρ is to take its first order partial derivative, with respect to ρ, and show that this derivative is positive. Such that,

$$\frac{\partial(\text{E.7})}{\partial\rho}=(\rho+\beta_r)(\rho-\beta_1)+(\rho-\beta_r-2\beta_1)(\rho-\beta_1)-2(\rho-\beta_r-2\beta_1)(\rho+\beta_r)=2(\beta_1+\beta_r)>0.$$

This establishes the monotonicity of transaction rates in ρ within the parameter region where $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)) = (1,0,\check{\sigma}_L^r,\check{\sigma}_R^r)$ is optimal.

Lastly, we establish that the monotonicity of transaction rates in ρ is preserved at the threshold value of ρ, on which the optimal policy turn from $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)) = (1,1,1,1)$ to $(\sigma_S^{ma}(L,\ell),\sigma_S^{ma}(R,\ell),\sigma_S^{ma}(L,r),\sigma_S^{ma}(R,r)) = (1,0,\check{\sigma}_L^r,\check{\sigma}_R^r)$. In that, we analyze four distinct functional forms of $(\check{\sigma}_L^r,\check{\sigma}_R^r)$, given in Lemma C.10, in isolation. In particular, we will show that

$$tr_S^{ma}(R \mid \rho, \sigma_S^{ma} = (1,1,1,1)) - tr_S^{ma}(R \mid \rho, \sigma_S^{ma} = (1,0,\check{\sigma}_L^r,\check{\sigma}_R^r)) \geq 0, \tag{E.8}$$

at the ρ value that induces a policy switch, outlined in Proposition 4. This result will immediately imply, by Lemma E.2 and (E.2), that $tr_S^{ma}(L \mid \rho, \sigma_S^{ma} = (1,1,1,1)) - tr_S^{ma}(L \mid \rho, \sigma_S^{ma} = (1,0,\check{\sigma}_L^r,\check{\sigma}_R^r)) \leq 0$.

Case 1: $\sigma_S^{ma} = (1,0,\check{\sigma}_{L_0}^r,\check{\sigma}_{R_0}^r)$. By Lemma C.10 and Lemma C.1, $(x_L^\ell, x_R^\ell, x_L^r, x_R^r) = (\sqrt{\alpha},0,0,\sqrt{\alpha}(\beta_1+1))$. The relevant parameter conditions for this case to arise, again by Lemma C.10, are (i) $\rho \leq \beta_r + 2\beta_1$, (ii) $\alpha(\beta_1+1)^2 \leq 1-\alpha$, and

$$\text{(iii)} \quad \lambda > \lambda_S^{ma} \quad \Leftrightarrow \quad \lambda' > \frac{\alpha(\beta_1+1)^2 - (1-\alpha\beta_r)(1+\alpha\rho)}{(1-\alpha)(1-\alpha\beta_1\rho) + \alpha(\beta_1+1)^2 - (1-\alpha\beta_r)(1+\alpha\rho)}.$$

Notice, by (E.5), (E.8) becomes

$$\begin{aligned} & \lambda v_R^\ell(\alpha\beta_1+1) + (1-\lambda)v_R^r\left[1-\alpha\beta_r-\alpha(\beta_1+1)^2(1-\alpha)^{-1}\right] \geq 0 \\ \Leftrightarrow \quad & \lambda'\alpha\left[(1-\alpha)(\beta_1+\beta_r)+(\beta_1+1)^2\right] \geq \alpha(\beta_1+1)^2 - (1-\alpha)(1-\alpha\beta_r) \end{aligned}$$

Then, by (iii), it suffices for us to show

$$\begin{aligned} & \alpha\left[(1-\alpha)(\beta_1+\beta_r)+(\beta_1+1)^2\right]\left[\alpha(\beta_1+1)^2-(1-\alpha\beta_r)(1+\alpha\rho)\right] \\ & \qquad \leq \left[\alpha(\beta_1+1)^2-(1-\alpha)(1-\alpha\beta_r)\right]\left[(1-\alpha)(1-\alpha\beta_1\rho)+\alpha(\beta_1+1)^2-(1-\alpha\beta_r)(1+\alpha\rho)\right] \\ \Leftrightarrow \quad & \alpha^2\beta_1(\beta_1+1)^2(1+\rho) \leq (1-\alpha\beta_r)\left[(1-\alpha)(\alpha\beta_1\rho-1)+(\alpha\beta_1+1)(1+\alpha\rho)\right] \\ \Leftrightarrow \quad & \alpha\beta_1(\beta_1+1) \leq 1-\alpha\beta_r \end{aligned}$$

which holds as

$$\alpha\beta_1(\beta_1+1) < \alpha(\beta_1+1)^2 \leq 1-\alpha < 1-\alpha\beta_r,$$

where the second inequality follows from (ii), and third inequality follows from $\beta_r < 1$.

Case 2: $\sigma_S^{ma} = (1,0,\check{\sigma}_{L_*}^r,\check{\sigma}_{R_*}^r)$. By Lemma C.10 and Lemma C.1, $(x_L^\ell, x_R^\ell) = (\sqrt{\alpha},0)$ $x_L^r = \sqrt{\alpha}(\beta_1+1)(\rho-\beta_r-2\beta_1)/[2(\beta_1+\beta_r)(\rho-\beta_1)]$, and $x_R^r = \sqrt{\alpha}(\beta_1+1)[\rho(\beta_1+\beta_r)+\beta_r(\rho-\beta_1)-(\rho-\beta_r-2\beta_1)]/[2(\beta_1+\beta_r)(\rho-\beta_1)]$. The relevant parameter conditions for this case to arise, again by Lemma C.10, are (i) $(\rho-\beta_1)(1-\alpha\beta_r) \leq (\beta_1+\beta_r)(1+\alpha\rho)$, (ii) $\alpha\beta_1^2 < 1 < \alpha\beta_1\rho$, and (iii)

$$\lambda > \lambda_S^{ma} \quad \Leftrightarrow \quad \lambda' > \frac{\alpha(\beta_1+1)^2(\rho+\beta_r)^2 - 4(\beta_1+\beta_r)(\rho-\beta_1)(1-\alpha\beta_r)(1+\alpha\rho)}{4(\beta_1+\beta_r)(\rho-\beta_1)[(1-\alpha)(1-\alpha\beta_1\rho)-(1-\alpha\beta_r)(1+\alpha\rho)] + \alpha(\beta_1+1)^2(\rho+\beta_r)^2}.$$

Notice, by (E.5), (E.8) becomes

$$\lambda v_R^\ell(\alpha\beta_1+1) + (1-\lambda)v_R^r\left[1-\alpha\beta_r - \frac{\alpha(\beta_1+1)^2}{(1-\alpha)}\frac{(\rho+\beta_r)\left[(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)\right]}{4(\beta_1+\beta_r)(\rho-\beta_1)^2}\right] \geq 0$$

$$\Leftrightarrow \quad \frac{\lambda'}{1-\lambda'} \geq \frac{-4(1-\alpha)(1-\alpha\beta_r)(\beta_1+\beta_r)(\rho-\beta_1)^2+\alpha(\beta_1+1)^2(\rho+\beta_r)\left[(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)\right]}{4(1-\alpha)(\alpha\beta_1+1)(\beta_1+\beta_r)(\rho-\beta_1)^2}$$

Then, by (iii), it suffices for us to show

$$\begin{aligned}(\beta_1+1)(\rho+\beta_r)\left\{(\alpha\beta_1+1)(\rho-\beta_1)(\rho+\beta_r)+(\alpha\beta_1\rho-1)\left[(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)\right]\right\}& \\ \leq 4(\beta_1+\beta_r)(1-\alpha\beta_r)(1+\rho)(\rho-\beta_1)^2 \qquad \text{(E.9)}&\end{aligned}$$

To show (E.9) holds, we split our analysis into further two scenarios: when (C.23) holds, and when (C.24) holds.

Case 2.1: $\sigma_S^{ma}=(1,0,\check{\sigma}^r_{L_*},\check{\sigma}^r_{R_*})$ and (C.23) holds. Note that for (C.23) to be satisfied, it must be the case that

$$\begin{aligned}&\alpha(\beta_1+1)^2(B^2-1)<4(1-\alpha)\beta_1(\beta_1+1) \quad \Leftrightarrow \\ &\frac{\alpha(\beta_1+1)^2}{(1-\alpha)(\beta_1+\beta_r)}\left\{\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]^2-(\rho-\beta_r-2\beta_1)\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\right\}<4(\beta_1+\beta_r)(\rho-\beta_1)^2\end{aligned}$$

Then, to establish (E.9), it suffices for us to show

$$\begin{aligned}(1-\alpha)(\beta_1+\beta_r)(\rho+\beta_r)\left\{(\rho-\beta_r-2\beta_1)+\alpha\beta_1\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\right\}& \\ \leq \alpha(\beta_1+1)(1-\alpha\beta_r)\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)-(\rho-\beta_r-2\beta_1)\right]&\end{aligned}$$

We also know, for (C.23) to be satisfied, it must be the case that $\alpha\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\geq(\rho-\beta_r-2\beta_1)$, hence, to establish (E.9), it suffices for us to show

$$\begin{aligned}&(1-\alpha)(\beta_1+\beta_r)(\rho+\beta_r)\leq(1-\alpha\beta_r)\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)-(\rho-\beta_r-2\beta_1)\right] \\ \Leftrightarrow \quad &(\rho-\beta_1)(1-\alpha\beta_r)\leq(\beta_1+\beta_r)(1+\alpha\rho)\end{aligned}$$

which, by (C.23), holds.

Case 2.2: $\sigma_S^{ma}=(1,0,\check{\sigma}^r_{L_*},\check{\sigma}^r_{R_*})$ and (C.24) holds. Note that for (C.24) to be satisfied, it must be the case that

$$\begin{aligned}&\alpha(\beta_1+1)^2(1-A^2)<4\alpha\beta_1 \\ \Leftrightarrow \quad &\frac{(\beta_1+1)^2}{(\beta_1+\beta_r)}\left\{\left[(\rho-\beta_1)-(\beta_1+\beta_r)\right]\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]\right\}<4(\beta_1+\beta_r)(\rho-\beta_1)^2\end{aligned}$$

Then, to establish (E.9), it suffices for us to show

$$\begin{aligned}&(\beta_1+\beta_r)(\rho+\beta_r)\left[(\rho-\beta_1)(\alpha\beta_1\beta_r+1)+(\beta_1+\beta_r)(\alpha\beta_1\rho-1)\right] \\ &\qquad\leq(\beta_1+1)(1-\alpha\beta_r)\left[\beta_r(\rho-\beta_1)^2+(\rho-\beta_1)(\beta_1+\beta_r)(\rho-\beta_r)-\rho(\beta_1+\beta_r)^2\right] \\ \Leftrightarrow \quad &\left\{\alpha\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]-(\rho-\beta_r-2\beta_1)\right\}\left\{\beta_1\left[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)\right]+\beta_r(\rho-\beta_r-2\beta_1)\right\}\leq 0 \\ \Leftrightarrow \quad &(\rho-\beta_1)(1-\alpha\beta_r)\geq(\beta_1+\beta_r)(1+\alpha\rho)\end{aligned}$$

which, by (C.24), holds.

Case 3: $\sigma_S^{ma}=(1,0,\check{\sigma}^r_{L_\alpha},\check{\sigma}^r_{R_\alpha})$. By Lemma C.10 and Lemma C.1, for $\beta_1<1$, $(x_L^\ell,x_R^\ell,x_L^r,x_R^r)=(\sqrt{\alpha},0,\sqrt{\alpha},0)$, and for $\beta_1\geq 1$, $(x_L^\ell,x_R^\ell,x_L^r,x_R^r)=(\sqrt{\alpha},0,\sqrt{\alpha}/\beta_1,\sqrt{\alpha}(\beta_1^2-1)/\beta_1)$. The relevant parameter condition for this case to arise, by Proposition 4, is $\alpha\beta_1^2<1<\alpha\beta_1\rho$.

Notice, for $\beta_1 < 1$, by (E.5), that $tr_S^{ma}(R \mid \rho, \sigma_S^{ma}) = 0$, and hence, (E.8) is satisfied. For $\beta_1 \geq 1$, again by (E.5), (E.8) becomes

$$\begin{aligned} & \lambda v_R^{\ell}(\alpha\beta_1 + 1) + (1-\lambda)v_R^r\left[1 - \alpha\beta_r - \alpha(\beta_1^2 - 1)(\beta_1^2 - \beta_r)(1-\alpha)^{-1}\beta_1^{-2}\right] \geq 0 \\ \Leftrightarrow \quad & \lambda'(1-\alpha)\beta_1^2(\alpha\beta_1 + 1) + (1-\lambda')(\beta_1^2 - \alpha\beta_r)(1 - \alpha\beta_1^2) \geq 0 \end{aligned}$$

which holds as $\alpha, \beta_r < 1$, $\beta_1 \geq 1$ and $\alpha\beta_1^2 < 1$.

Case 4: $\sigma_S^{ma} = (1, 0, \check{\sigma}^r_{L_{1-\alpha}}, \check{\sigma}^r_{R_{1-\alpha}})$. To show that (E.8) holds for the ρ value that induces a policy switch, outlined in Proposition 4, from $\sigma_S^{ma} = (1,1,1,1)$ to $\sigma_S^{ma} = (1, 0, \check{\sigma}^r_{L_{1-\alpha}}, \check{\sigma}^r_{R_{1-\alpha}})$, we provide the following chain of arguments. Suppose for a (ρ, λ) pair the policy turns from $\sigma_S^{ma} = (1,1,1,1)$ to $\sigma_S^{ma} = (1, 0, \check{\sigma}^r_{L_{1-\alpha}}, \check{\sigma}^r_{R_{1-\alpha}})$. We know, by Lemma E.1, that there exists a $\rho' \geq \rho$ which turn $\check{\sigma}^r_{i_{1-\alpha}}$ point type to $\check{\sigma}^r_{i_*}$ point type, where we have $\left(\check{\sigma}^r_{L_{1-\alpha}}(\rho'), \check{\sigma}^r_{R_{1-\alpha}}(\rho')\right) = \left(\check{\sigma}^r_{L_*}(\rho'), \check{\sigma}^r_{R_*}(\rho')\right)$, which implies, by Lemma C.1 and (E.5), that $tr_S^{ma}\left(R \,\middle|\, \sigma_S^{ma} = \left(1, 0, \check{\sigma}^r_{L_{1-\alpha}}(\rho'), \check{\sigma}^r_{R_{1-\alpha}}(\rho')\right)\right) = tr_S^{ma}\left(R \,\middle|\, \sigma_S^{ma} = \left(1, 0, \check{\sigma}^r_{L_*}(\rho'), \check{\sigma}^r_{R_*}(\rho')\right)\right)$. Moreover, by (E.5) and the proof of Lemma C.10, we know that

$$\begin{aligned} tr_S^{ma}\left(R \,\middle|\, \sigma_S^{ma} = \left(1, 0, \check{\sigma}^r_{L_{1-\alpha}}(\rho), \check{\sigma}^r_{R_{1-\alpha}}(\rho)\right)\right) &= tr_S^{ma}\left(R \,\middle|\, \sigma_S^{ma} = \left(1, 0, \check{\sigma}^r_{L_{1-\alpha}}(\rho'), \check{\sigma}^r_{R_{1-\alpha}}(\rho')\right)\right) \\ &= tr_S^{ma}\left(R \,\middle|\, \sigma_S^{ma}\left(1, 0, \check{\sigma}^r_{L_*}(\rho'), \check{\sigma}^r_{R_*}(\rho')\right)\right). \end{aligned}$$

We also know that, by Proposition 11, $\lambda_S^{ma}(\rho) \geq \lambda_S^{ma}(\rho')$ and, hence, $\lambda > \lambda_S^{ma}(\rho) \geq \lambda_S^{ma}(\rho')$. Consequently, as we have shown in Case 2 that (E.8) holds for $\check{\sigma}^r_{i_*}$ point type for any such (ρ', λ) pair, it also holds for $\check{\sigma}^r_{i_{1-\alpha}}$ point type for any such (ρ, λ) pair. This completes the proof. □

Proof of Proposition 10. We prove this proposition by establishing the following three claims.

Claim 1: $tr_k^m(L) \geq tr_k^m(R)$ **for** $k \in \{U, O, S, N\}$. First, to expand the definition of transaction rates given in (13), we express user t's expected value of the product type, under σ_k^m, upon receiving a signal d:

$$\mathbb{E}\left[v_i^t \,\middle|\, t, \widehat{t}, d, \sigma_k^m\right] = \begin{cases} \frac{v_L^t \alpha\sigma_k^m(L,\widehat{t}) + v_R^t(1-\alpha)\sigma_k^m(R,\widehat{t})}{\alpha\sigma_k^m(L,\widehat{t}) + (1-\alpha)\sigma_k^m(R,\widehat{t})}, & \text{for} \quad d = 1, \\ \frac{v_L^t \alpha(1-\sigma_k^m(L,\widehat{t})) + v_R^t(1-\alpha)(1-\sigma_k^m(R,\widehat{t}))}{\alpha(1-\sigma_k^m(L,\widehat{t})) + (1-\alpha)(1-\sigma_k^m(R,\widehat{t}))}, & \text{for} \quad d = 0, \end{cases}$$

for $k \in \{U, O, S, N\}$ and $t \in \{\ell, r\}$. Using (13) and the results of Propositions 1 and 2, we can then calculate $tr_k^m(i)$ for $k \in \{U, O, S, N\}$ and $i \in \{L, R\}$, respectively, as follows:

$$tr_k^m(L) = \begin{cases} \lambda\left[\alpha v_L^{\ell} + (1-\alpha)v_R^{\ell}\right] + (1-\lambda)\left[\alpha v_L^r + (1-\alpha)v_R^r\right], & \text{for} \quad k \in \{S, N\} \text{ and } \beta_\lambda \leq 0, \\ \lambda v_L^{\ell} + (1-\lambda)v_L^r, & \text{for} \quad k \in \{S, N\} \text{ and } \beta_\lambda > 0, \\ \lambda v_L^{\ell} + (1-\lambda)\left[\alpha v_L^r + (1-\alpha)v_R^r\right], & \text{for} \quad k \in \{U, O\}, \end{cases} \tag{E.10}$$

and

$$tr_k^m(R) = \begin{cases} \lambda\left[\alpha v_L^{\ell} + (1-\alpha)v_R^{\ell}\right] + (1-\lambda)\left[\alpha v_L^r + (1-\alpha)v_R^r\right], & \text{for} \quad k \in \{S, N\} \text{ and } \beta_\lambda \leq 0, \\ \lambda v_R^{\ell} + (1-\lambda)v_R^r, & \text{for} \quad k \in \{S, N\} \text{ and } \beta_\lambda > 0, \\ \lambda v_R^{\ell} + (1-\lambda)\left[\alpha v_L^r + (1-\alpha)v_R^r\right], & \text{for} \quad k \in \{U, O\}. \end{cases} \tag{E.11}$$

It is then straightforward to see $tr_k^m(L) \geq tr_k^m(R)$ as, for $k \in \{S, N\}$ and $\beta_\lambda \leq 0$ we have $tr_k^m(L) = tr_k^m(R)$, for $k \in \{S, N\}$ and $\beta_\lambda > 0$ we have $tr_k^m(L) - tr_k^m(R) = \lambda(v_L^{\ell} - v_R^{\ell}) + (1-\lambda)(v_L^r - v_R^r) = \beta_\lambda[\lambda v_R^{\ell} + (1-\lambda)v_R^r] > 0$, and for $k \in \{U, O\}$ we have $tr_k^m(L) - tr_k^m(R) = \lambda(v_L^{\ell} - v_R^{\ell}) > 0$. This establishes the claim.

Claim 2: $\pi_R^m \equiv \overline{\pi}_R$**.** First, we rewrite the objective function of the platform given in (14)

$$\Pi_k^m(\pi_L, \pi_R) = \alpha\pi_L tr_k^m(L) + (1-\alpha)\pi_R tr_k^m(R), \tag{E.12}$$

which, as $tr_k^m(i)$ is independent of (π_L, π_R), indicates the platform would aim to increase π_L and π_R as much as possible, while satisfying

$$(\overline{\pi}_L - \pi_L)\, tr_k^m(L) \geq (\overline{\pi}_L - \pi_R)\, tr_k^m(R), \tag{E.13}$$

$$(\overline{\pi}_R - \pi_R)\, tr_k^m(R) \geq (\overline{\pi}_R - \pi_L)\, tr_k^m(L). \tag{E.14}$$

Next, we establish the claim using proof by contradiction. Suppose $\pi_R^m < \overline{\pi}_R$. We examine this scenario in two cases: (i) When (E.14) is strict, and (ii) when (E.14) holds with an equality. On one hand, if (E.14) is strict, then there must exist an $\epsilon \in (0, \overline{\pi}_R - \pi_R^m]$ which satisfies

$$(\overline{\pi}_R - \pi_R^m)\, tr_k^m(R) - \epsilon \geq (\overline{\pi}_R - \pi_L)\, tr_k^m(L).$$

If that is the case, however, $\pi_R^{m\prime} := \pi_R^m + \epsilon$ would also satisfy (E.14) while improving both (E.13) and (E.12). Hence, this contradicts with the optimality of $\pi_R^m < \overline{\pi}_R$. On the other hand, if (E.14) holds with an equality, as $tr_k^m(L) \geq tr_k^m(R)$ and $\pi_R^m < \overline{\pi}_R$, we must have $\pi_L^m \in [\pi_R^m, \overline{\pi}_R)$. If that is the case, however, it is straightforward to see that the pair of $(\pi_L^{m\prime}, \pi_R^{m\prime}) = (\overline{\pi}_R, \overline{\pi}_R)$ would immediately satisfy (E.13) and (E.14) while improving (E.12), as (E.12) increases in both π_L and π_R. Hence, this contradicts with the optimality of $\pi_R^m < \overline{\pi}_R$, and correspondingly, establishes the claim.

Claim 3: (E.13) must be binding under (π_L^m, π_R^m). We show in Claim 2 that $\pi_R^m \equiv \overline{\pi}_R$ and that $(\pi_L, \pi_R) = (\overline{\pi}_R, \overline{\pi}_R)$ is always a feasible solution for (14). For any $\pi_L > \overline{\pi}_R$, it is straightforward to see (E.14) would continue to hold. Thus, as left hand side of (E.13) is decreasing in π_L, $(\overline{\pi}_L - \overline{\pi}_R)\, tr_k^m(L) \geq (\overline{\pi}_L - \overline{\pi}_R)\, tr_k^m(R)$ and $(\overline{\pi}_L - \overline{\pi}_L)\, tr_k^m(L) = 0 < (\overline{\pi}_L - \overline{\pi}_R)\, tr_k^m(R)$, in maximizing (E.12), the platform must choose π_L^m which binds (E.13), i.e., we must have

$$(\overline{\pi}_L - \pi_L^m)\, tr_k^m(L) = (\overline{\pi}_L - \overline{\pi}_R)\, tr_k^m(R),$$

which can be expressed as

$$\pi_L^m = \overline{\pi}_L - (\overline{\pi}_L - \overline{\pi}_R)\, \frac{tr_k^m(R)}{tr_k^m(L)} \quad \text{for } k \in \{U, O, S, N\}. \tag{E.15}$$

This establishes the claim and, taken together with Claim 2, (E.10) and (E.11), completes the proof of proposition. □

Proof of Proposition 11. The first statement of Proposition 11 immediately follows from Propositions 1 and 2. For recommendation mode ma, the impact of ρ on σ_k^{ma} is also straightforward to see, on one hand, for $k \in \{U, O, N\}$, utilizing Propositions 3 and 5. On the other hand, in order to be able to state that σ_S^{ma} uses less of the spam strategy as ρ increases, it suffices for us to show, whenever defined, λ_S^{ma} (given in Proposition 4) is decreasing in ρ. Utilizing (C.41), let us first rewrite λ_S^{ma} as

$$\lambda_S^{ma} = \frac{1}{1 + \frac{v_R^\ell}{v_R^r} \frac{\left[F^\ell(\overline{g}) - F^\ell(\underline{g})\right]}{\left[F^r(\underline{g}) - F^r(\overline{g})\right]}}.$$

As noted in the proof of Proposition 4, $F^\ell(\overline{g}) - F^\ell(\underline{g}) = (1-\alpha)(1-\alpha\beta_1\rho)$ and $F^r(\overline{g}) = (1-\alpha\beta_r)(1+\alpha\rho)$, as $(x_L^r, x_R^r) = (\alpha, 1-\alpha)$ is the only feasible point that satisfies both $(\widehat{H_\alpha^t})$ and $(\widehat{H_{1-\alpha}^t})$ when $g = \overline{g}$. Thus, λ_S^{ma} can be written as

$$\lambda_S^{ma} = \frac{1}{1 + \frac{v_R^\ell}{v_R^r} \frac{(1-\alpha)(1-\alpha\beta_1\rho)}{F^r(\underline{g}) - (1-\alpha\beta_r)(1+\alpha\rho)}}. \tag{E.16}$$

As $\alpha\beta_1\rho > 1$, in order to show λ_S^{ma} is decreasing in ρ, it suffices for us to show the ratio $F^\rho := (1-\alpha\beta_1\rho)/\left[F^r(\underline{g}) - (1-\alpha\beta_r)(1+\alpha\rho)\right]$ is increasing in ρ. We now show that F^ρ is increasing in ρ by taking its partial derivative, $d_\rho := \partial(F^\rho)/\partial(\rho)$, and showing that $d_\rho \geq 0$ for all four distinct functional forms of $F^r(\underline{g})$ given in (C.26), corresponding to distinct point types. This would suffice as we have shown, in Lemma E.1, that both $\check{\sigma}_i^r$ (and, hence, $\check{x}_i^r$), for $i \in \{L, R\}$, are continuous irrespective of the active point type.

Case 1: $F^r(\underline{g})$ **for** $\left(x_{L_0}^r(\underline{g}), x_{R_0}^r(\underline{g})\right)$**.** Here, by (C.26), we have $F^r(\underline{g}) = \alpha(\beta_1+1)^2$, which makes

$$d_\rho = \frac{-\alpha\beta_1\left[\alpha(\beta_1+1)^2 - (1+\alpha\rho)(1-\alpha\beta_r)\right] - \alpha(\alpha\beta_r - 1)(1-\alpha\beta_1\rho)}{\left[\alpha(\beta_1+1)^2 - (1+\alpha\rho)(1-\alpha\beta_r)\right]^2}$$
$$\propto (1+\beta_1)\left[1-\alpha\beta_1^2 - \alpha\beta_1 - \alpha\beta_r\right] = 1-\alpha-\alpha(\beta_1+1)^2 + \alpha(2+\beta_1-\beta_r) \geq 0,$$

where the last inequality follows by $\beta_r < 1$ and by (C.22) of Lemma C.7. Accordingly, in order for $\left(x_{L_0}^r(\underline{g}), x_{R_0}^r(\underline{g})\right)$ to be optimal, we must have $1-\alpha \geq \alpha(\beta_1+1)^2$.

Case 2.1: $F^r(\underline{g})$ **for** $\left(x_{L_\alpha}^r(\underline{g}), x_{R_\alpha}^r(\underline{g})\right)$ **and** $\beta_1 \leq 1$**.** Here, by (C.26), we have $F^r(\underline{g}) - (1-\alpha\beta_r)(1+\alpha\rho) = -(1-\alpha)(1+\alpha\rho\beta_r)$, which makes

$$d_\rho \propto \frac{-\alpha\beta_1\left[-1-\alpha\rho\beta_r\right] + \alpha\beta_r(1-\alpha\beta_1\rho)}{\left[-1-\alpha\rho\beta_r\right]^2} = \frac{\alpha(\beta_1+\beta_r)}{\left[-1-\alpha\rho\beta_r\right]^2} > 0.$$

Case 2.2: $F^r(\underline{g})$ **for** $\left(x_{L_\alpha}^r(\underline{g}), x_{R_\alpha}^r(\underline{g})\right)$ **and** $\beta_1 > 1$**.** Here, by (C.26), we have $F^r(\underline{g}) - (1-\alpha\beta_r)(1+\alpha\rho) = -(1-\alpha\beta_1^2)\beta_1^{-2}(\beta_1^2 + \alpha\rho\beta_r)$, which makes

$$d_\rho \propto \frac{-\alpha\beta_1\left[-\beta_1^2 - \alpha\rho\beta_r\right] + \alpha\beta_r(1-\alpha\beta_1\rho)}{\left[-\beta_1^2-\alpha\rho\beta_r\right]^2} = \frac{\alpha(\beta_1^3+\beta_r)}{\left[-\beta_1^2-\alpha\rho\beta_r\right]^2} > 0.$$

Case 3: $F^r(\underline{g})$ **for** $\left(x_{L_*}^r(\underline{g}), x_{R_*}^r(\underline{g})\right)$**.** Here, by (C.26), we have $F^r(\underline{g}) = \alpha(\beta_1+1)^2\frac{(\rho+\beta_r)^2}{4(\beta_1+\beta_r)(\rho-\beta_1)}$, which makes

$$d_\rho \propto \quad 4(\beta_1+\beta_r)(1-\alpha\beta_r)(\rho-\beta_1)^2 - (\beta_1+1)(\rho+\beta_r)\left\{(\rho-\beta_1)(1+\alpha\beta_1\beta_r) + (\beta_1+\beta_r)(\alpha\beta_1\rho-1)\right\}. \quad \text{(E.17)}$$

Notice from Lemma C.7, in order for $\left(x_{L_*}^r(\underline{g}), x_{R_*}^r(\underline{g})\right)$ to be optimal, either (C.23) or (C.24) condition must hold. We now separate our analysis into further 2 scenarios where (C.23) and (C.24) holds, respectively.

Case 3.1: $F^r(\underline{g})$ **for** $\left(x_{L_*}^r(\underline{g}), x_{R_*}^r(\underline{g})\right)$ **and** (C.23) **holds.** Notice first, by (E.17), that showing $d_\rho \geq 0$ is equivalent to showing

$$\alpha \leq \frac{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2 - (\beta_1+\beta_r)(\beta_1+1)(\rho+\beta_r)[(\rho-\beta_1)-(\beta_1+\beta_r)]}{4\beta_r(\beta_1+\beta_r)^2(\rho-\beta_1)^2 + \beta_1(\beta_1+\beta_r)(\beta_1+1)(\rho+\beta_r)[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)]}. \quad \text{(E.18)}$$

Notice next, as (C.23) holds, we must have (i)

$$\alpha \geq \frac{(\rho-\beta_1)-(\beta_1+\beta_r)}{\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)},$$

and in order for $\left(x_{L_*}^r(\underline{g}), x_{R_*}^r(\underline{g})\right)$ to be optimal, we also must have (ii)

$$\alpha(\beta_1+1)^2 \leq \frac{4(1-\alpha)\beta_1(1+\beta_1)}{(B-1)(B+1)}$$
$$\Leftrightarrow \quad \alpha \leq \frac{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2 + (\beta_1+1)^2[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)][(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)]},$$

where $B = \frac{\beta_1^2(1+\rho-\beta_r)+\rho\beta_r(1+2\beta_1)}{(\beta_1+\beta_r)(\rho-\beta_1)}$, as defined in Lemma C.7. Straightforwardly, by (i) and (ii), in order for Case 3.1 to be feasible, we need to have (iii)

$$\begin{aligned}
&\frac{(\rho-\beta_1)-(\beta_1+\beta_r)}{\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)} \\
&\quad\le \frac{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2}{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2+(\beta_1+1)^2[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)][(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)]} \\
\Leftrightarrow\quad &(\beta_1+1)^2[(\rho-\beta_1)-(\beta_1+\beta_r)][\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)]\le 4(\beta_1+\beta_r)^2(\rho-\beta_1)^2.
\end{aligned}$$

Now, by (ii), in order to establish (E.18), it suffices for us to show

$$\begin{aligned}
0\le{}&\left[4(\beta_1+\beta_r)^2(\rho-\beta_1)^2\right]^2(1-\beta_r) \\
&+4(\beta_1+\beta_r)^2(\rho-\beta_1)^2(\beta_1+1)\left\{\begin{smallmatrix}(\beta_1+1)[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)][(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)]\\ -(\beta_1+\beta_r)(\rho+\beta_r)[(\beta_1\rho-1)(\beta_1+\beta_r)+(\beta_1\beta_r+1)(\rho-\beta_1)]\end{smallmatrix}\right\} \\
&-(\beta_1+1)^3(\beta_1+\beta_r)(\rho+\beta_r)[\rho-2\beta_1-\beta_r][\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)][(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)].
\end{aligned}$$

By (iii), the above inequality holds, as

$$\begin{aligned}
0\le{}&4(\beta_1+\beta_r)^2(\rho-\beta_1)^2(1-\beta_r) \\
&+(\beta_1+1)\left\{\begin{smallmatrix}(\beta_1+1)[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)][(1+\rho)(\beta_1+\beta_r)-(1-\beta_r)(\rho-\beta_1)]\\ -(\beta_1+\beta_r)(\rho+\beta_r)[(\beta_1\rho-1)(\beta_1+\beta_r)+(\beta_1\beta_r+1)(\rho-\beta_1)]\end{smallmatrix}\right\} \\
&-(\beta_1+1)^2(\beta_1+\beta_r)(\rho+\beta_r)[\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)] \\
\Leftrightarrow\quad 0\le{}&(1-\beta_r)\left\{4(\beta_1+\beta_r)^2(\rho-\beta_1)^2-(\beta_1+1)^2[(\rho-\beta_1)-(\beta_1+\beta_r)][\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)]\right\},
\end{aligned}$$

which establishes $d_\rho \ge 0$ for this case.

Case 3.2: $F^r(\underline{g})$ for $\left(x^r_{L_*}(\underline{g}), x^r_{R_*}(\underline{g})\right)$ and (C.24) holds. Notice first, by (C.24), we must have (iv)

$$(\rho-\beta_1)(1-\alpha\beta_r)\ge(1+\alpha\rho)(\beta_1+\beta_r),$$

and in order for $\left(x^r_{L_*}(\underline{g}), x^r_{R_*}(\underline{g})\right)$ to be optimal, we also must have (v)

$$\begin{aligned}
&\alpha(\beta_1+1)^2\le\frac{4\alpha\beta_1}{(1-A)(1+A)} \\
\Leftrightarrow\quad &4(\beta_1+\beta_r)^2(\rho-\beta_1)^2\ge(\beta_1+1)^2[(\rho-\beta_1)-(\beta_1+\beta_r)][\beta_r(\rho-\beta_1)+\rho(\beta_1+\beta_r)],
\end{aligned}$$

where $A=\frac{\beta_1^2+\rho\beta_r}{(\beta_1+\beta_r)(\rho-\beta_1)}$, as defined in Lemma C.7. Now, by (v), in order to establish $d_\rho\ge 0$, given in (E.17), it suffices for us to show

$$\begin{aligned}
&(\beta_1+\beta_r)(\rho+\beta_r)\left\{(\rho-\beta_1)(1+\alpha\beta_1\beta_r)+(\beta_1+\beta_r)(\alpha\beta_1\rho-1)\right\} \\
&\qquad\le(1-\alpha\beta_r)(\beta_1+1)\left\{\beta_r(\rho-\beta_1)^2-\rho(\beta_1+\beta_r)^2+(\rho-\beta_r)(\rho-\beta_1)(\beta_1+\beta_r)\right\} \\
\Leftrightarrow\quad &(\beta_1+\beta_r)(\rho-\beta_1)\left\{\beta_r(\beta_1+1)(1+\alpha\rho)-(1-\alpha\beta_r)(\beta_1\rho-\beta_r)\right\} \\
&\qquad\le(\rho-\beta_1)^2\beta_r(\beta_1+1)(1-\alpha\beta_r)-(\beta_1+\beta_r)^2(1+\alpha\rho)(\beta_1\rho-\beta_r) \\
\Leftrightarrow\quad &0\le\left[(\rho-\beta_1)\beta_r(\beta_1+1)+(\beta_1+\beta_r)(\beta_1\rho-1)\right]\left[(\rho-\beta_1)(1-\alpha\beta_r)-(1+\alpha\rho)(\beta_1+\beta_r)\right],
\end{aligned}$$

which holds as both terms on the right hand side are non-negative in the above inequality. The first term is non-negative, as we have $\alpha\beta_1\rho>1\Rightarrow\beta_1\rho>1\Rightarrow\beta_1\rho>\beta_r$; and the second term is non-negative by (iv).

Case 4: $F^r(\underline{g})$ **for** $\left(x^r_{L_{1-\alpha}}(\underline{g}), x^r_{R_{1-\alpha}}(\underline{g})\right)$**.** Here, by (C.26), we have

$$F^r(\underline{g}) - (1-\alpha\beta_r)(1+\alpha\rho) = (2\beta_1^2(1+\beta_1)^2)\left\{\begin{array}{c}\frac{(1-\alpha)(\alpha\beta_1^2-1)\left[\rho[\beta_1^2(1-2\beta_r)-\beta_r(1+2\beta_1)]+\beta_1^2(1-\beta_r)\right]}{+\left[\underline{g}\sqrt{\underline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}-\overline{g}\sqrt{\overline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}\right]\left[\rho[\beta_1^2+\beta_r(1+2\beta_1)]+\beta_1^2(1-\beta_r)\right]}\end{array}\right\},$$

where $\underline{g} = \sqrt{\alpha}(\beta_1+1) < \alpha\beta_1+1 = \overline{g}$. This makes

$$d_\rho \propto -\alpha\beta_1^3(1-\beta_r)\left\{(1-\alpha)(\alpha\beta_1^2-1) + \left[\overline{g}\sqrt{\overline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}\right]\right\} - c_\rho.$$

where

$$c_\rho = (1-\alpha)(\alpha\beta_1^2-1)\left[\beta_1^2(1-2\beta_r)-\beta_r(1+2\beta_1)\right] + \left[\underline{g}\sqrt{\underline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}-\overline{g}\sqrt{\overline{g}^2+4(1-\alpha)\beta_1(1+\beta_1)}\right]\left[\beta_1^2+\beta_r(1+2\beta_1)\right]$$

denotes the coefficient of ρ in $\left[F^r(\underline{g})-(1-\alpha\beta_r)(1+\alpha\rho)\right]$. Now, we know the first term of d_ρ is positive, as $\alpha\beta_1^2<1$. Thus, if $c_\rho \le 0$, we immediately have $d_\rho \ge 0$. If, on the other hand, $c_\rho > 0$, then we have $\left[F^r(\underline{g})-(1-\alpha\beta_r)(1+\alpha\rho)\right] < 0$, and increasing in ρ. As $(1-\alpha\beta_1\rho) < 0$ and decreasing in ρ, we also have $(1-\alpha\beta_1\rho)/\left[F^r(\underline{g})-(1-\alpha\beta_r)(1+\alpha\rho)\right]$ is increasing in ρ. This completes the proof.

Proof of $tr^{ma}_k(i\,|\,\rho)$ is non-decreasing (resp., non-increasing) in ρ for product $i=L$ (resp., $i=R$) is provided in Lemma E.3 ■

Proof of Proposition 12. To maintain consistency with the rest of the appendix, in this proof, we use $\beta_r = -\beta_0 = 1 - v^r_L/v^r_R$. First, we rewrite the objective function of the platform given in (17)

$$\Pi^{ma}_k(\pi_L,\pi_R) = \alpha\pi_L tr^{ma}_k(L\,|\,\rho=\pi_L/\pi_R-1) + (1-\alpha)\pi_R tr^{ma}_k(R\,|\,\rho=\pi_L/\pi_R-1), \tag{E.19}$$

which, by Lemma E.2, indicates the platform would aim to increase π_L and π_R as much as possible, while satisfying

$$(\overline{\pi}_L-\pi_L)\,tr^{ma}_k(L\,|\,\rho) \ge (\overline{\pi}_L-\pi_R)\,tr^{ma}_k(R\,|\,\rho), \tag{E.20}$$

$$(\overline{\pi}_R-\pi_R)\,tr^{ma}_k(R\,|\,\rho) \ge (\overline{\pi}_R-\pi_L)\,tr^{ma}_k(L\,|\,\rho). \tag{E.21}$$

Notice that it is without loss of optimality for us to restrict our search to (π_L,π_R) pairs that satisfy $\pi_L \ge \pi_R$. This is due to, again by Lemma E.2, any (π_L,π_R) pair with $\pi_L<\pi_R$ be suboptimal to the pair of $(\pi_L,\pi_R) = (\pi_R,\pi_R)$. In that, we utilize the fact that any (π_L,π_R) pair with $\pi_L=\pi_R$ satisfies both (E.20) and (E.21), which is established by the equivalence of $tr^{ma}_k(L\,|\,\rho=0)$ and $tr^{ma}_k(R\,|\,\rho=0)$, given in (E.5), by Propositions 3 and 5 and (C.1). Consequently, as $\pi^{ma}_R \le \overline{\pi}_R$, we then must have $\pi^{ma}_L \ge \overline{\pi}_R$. Thus, (E.21) will always be satisfied under (π^{ma}_L,π^{ma}_R), as by (E.5), $tr^{ma}_k(R\,|\,\rho=\pi_L/\pi_R-1)\ge 0$ for any ρ.

Notice next, that (E.20) must be binding under (π^{ma}_L,π^{ma}_R). We know that $(\pi_L,\pi_R) = (\overline{\pi}_R,\overline{\pi}_R)$ is always a feasible solution for (E.19). For an arbitrary π_R and $\pi_L > \overline{\pi}_R$, it is straightforward to see (E.21) would continue to hold. As $tr^{ma}_k(R\,|\,\rho)$ is non-increasing and $tr^{ma}_k(L\,|\,\rho)$ is non-decreasing in ρ, for any $\pi_L<\overline{\pi}_L$, we know, by Lemma E.2, that we can improve (E.19) with a small enough increase in π_L, when (E.20) is not binding. As $\pi_R \le \overline{\pi}_R < \overline{\pi}_L$, the platform can only attain $\pi_L = \overline{\pi}_L$ if $tr^{ma}_k(R\,|\,\rho=\pi_L/\pi_R-1)=0$, which

immediately implies (E.20) is binding as well. Hence, in maximizing (E.19), the platform must choose π_L^{ma} which binds (E.20), i.e., we must have

$$(\overline{\pi}_L - \pi_L^{ma})\, tr_k^{ma}(L \mid \rho = \pi_L/\pi_R^{ma} - 1) = (\overline{\pi}_L - \pi_R^{ma})\, tr_k^{ma}(R \mid \rho = \pi_L/\pi_R^{ma} - 1),$$

which can be expressed as

$$\pi_L^{ma} = \overline{\pi}_L - (\overline{\pi}_L - \pi_R^{ma}) \frac{tr_k^{ma}(R \mid \rho = \pi_L^{ma}/\pi_R^{ma} - 1)}{tr_k^{ma}(L \mid \rho = \pi_L^{ma}/\pi_R^{ma} - 1)} \quad \text{for } k \in \{U, O, N\}. \tag{E.22}$$

We also note that $\pi_R^{ma} = \overline{\pi}_R$. To demonstrate, we use proof by contradiction. Suppose $\pi_R^{ma} < \overline{\pi}_R$. We examine this scenario in two cases: When (E.21) is strict, and when (E.21) holds with an equality. Regardless of the case, when $\pi_R^{ma} < \overline{\pi}_R$, we must have (i) $\pi_L > \pi_R^{ma}$, as any pair of (π_L, π_R) values that are both strictly below $\overline{\pi}_R$ would be dominated by $(\pi_L, \pi_R) = (\overline{\pi}_R, \overline{\pi}_R)$. On one hand, if (E.21) holds with an equality, as $tr_k^{ma}(L \mid \rho) \geq tr_k^{ma}(R \mid \rho)$ and $\pi_R^{ma} < \overline{\pi}_R$, we must have $\pi_L^{ma} \in [\pi_R^{ma}, \overline{\pi}_R)$. If that is the case, however, it is straightforward to see that the pair of $(\pi_L^{ma\prime}, \pi_R^{ma\prime}) = (\overline{\pi}_R, \overline{\pi}_R)$ would immediately satisfy (E.20) and (E.21) while improving (E.19), as (E.19) increases in both π_L and π_R. Hence, this contradicts with the optimality of $\pi_R^{ma} < \overline{\pi}_R$. On the other hand, if (E.21) is strict, then there must exist an $\epsilon \in (0, \overline{\pi}_R - \pi_R^{ma}]$ which satisfies

$$(\overline{\pi}_R - \pi_R^{ma})\, tr_k^{ma}(R \mid \rho) - \epsilon \geq (\overline{\pi}_R - \pi_L)\, tr_k^{ma}(L \mid \rho).$$

If that is the case, however, $\pi_R^{ma\prime} := \pi_R^{ma} + \epsilon$ would, as $tr_k^{ma}(R \mid \rho)$ is non-increasing and $tr_k^{ma}(L \mid \rho)$ is non-decreasing in ρ, also satisfy (E.21) while improving both (E.20) and (E.19). Improvement in (E.19) follows directly from Lemma E.2. To see the improvement in (E.20) notice first that, by (E.20) and (i), we must have $tr_k^{ma}(L \mid \rho) > tr_k^{ma}(R \mid \rho)$. With the aid of (E.5), Propositions 3 and 5 illustrate that for $k \in \{U, O, N\}$, there exists a cutoff $\hat{\rho}(k)$ value such that for all $\rho > \hat{\rho}(k)$ we have $tr_k^{ma}(L \mid \rho) = tr_k^{ma}(L \mid \rho)$ and $tr_k^{ma}(R \mid \rho) = tr_k^{ma}(R \mid \rho)$. Thus, a small enough ϵ that will keep $\rho = \pi_L/\pi_R^{ma\prime} - 1 > \hat{\rho}(k)$ will also improve (E.20), which contradicts with the optimality of $\pi_R^{ma} < \overline{\pi}_R$. This completes the proof of the first part of Proposition 12.

For proving the second part of Proposition 12, notice first, by Propositions 3-5, for $k \in \{U, O, N\}$, the optimal policies will either be $(\sigma_k^{ma}(L,\ell), \sigma_k^{ma}(R,\ell), \sigma_k^{ma}(L,r), \sigma_k^{ma}(R,r)) = (1,1,1,1)$ which, by (C.1), correspond to $(x_k^{ma}(L,\ell), x_k^{ma}(R,\ell), x_k^{ma}(L,r), x_k^{ma}(R,r)) = (\alpha, 1-\alpha, \alpha, 1-\alpha)$, or be $(\sigma_k^{ma}(L,\ell), \sigma_k^{ma}(R,\ell), \sigma_k^{ma}(L,r), \sigma_k^{ma}(R,r)) = (1,0,1,1)$ which, by (C.1), correspond to $(x_k^{ma}(L,\ell), x_k^{ma}(R,\ell), x_k^{ma}(L,r), x_k^{ma}(R,r)) = (\sqrt{\alpha}, 0, \alpha, 1-\alpha)$, or be $(\sigma_N^{ma}(L,n), \sigma_N^{ma}(R,n)) = (1,0)$ which, by (C.1), correspond to $(x_N^{ma}(L,n), x_N^{ma}(R,n)) = (\sqrt{\alpha}, 0)$. Using Propositions 3-5 and (E.5), we can then calculate $tr_k^{ma}(i \mid \rho = \pi_L/\overline{\pi}_R - 1)$ for $k \in \{U, O, N\}$ and $i \in \{L, R\}$, respectively, as follows:

$$tr_k^{ma}(L) = \begin{cases} \lambda v_R^{\ell}(\beta_1 + 1) + (1-\lambda) v_R^{r}(1 - \beta_r), & \text{for } k = N \text{ and } \alpha\beta_\lambda\rho > 1, \\ \lambda v_R^{\ell}(\beta_1 + 1) + (1-\lambda) v_R^{r}(1 - \alpha\beta_r), & \text{for } k \in \{U, O\} \text{ and } \alpha\beta_1\rho > 1, \\ \lambda v_R^{\ell}(\alpha\beta_1 + 1) + (1-\lambda) v_R^{r}(1 - \alpha\beta_r), & \text{for } k \in \{U, O\} \text{ and } \alpha\beta_1\rho \leq 1, \text{ and } k = N \text{ and } \alpha\beta_\lambda\rho \leq 1, \end{cases} \tag{E.23}$$

and

$$tr_k^{ma}(R) = \begin{cases} 0, & \text{for } k = N \text{ and } \alpha\beta_\lambda\rho > 1, \\ (1-\lambda) v_R^{r}(1 - \alpha\beta_r), & \text{for } k \in \{U, O\} \text{ and } \alpha\beta_1\rho > 1, \\ \lambda v_R^{\ell}(\alpha\beta_1 + 1) + (1-\lambda) v_R^{r}(1 - \alpha\beta_r), & \text{for } k \in \{U, O\} \text{ and } \alpha\beta_1\rho \leq 1, \text{ and } k = N \text{ and } \alpha\beta_\lambda\rho \leq 1. \end{cases} \tag{E.24}$$

Correspondingly, by plugging (E.23) and (E.24) into (E.22), we have

$$\pi_L^{ma} = \begin{cases} \overline{\pi}_L, & \text{for } k = N \text{ and } \alpha\beta_\lambda\rho > 1, \\ \overline{\pi}_L - (\overline{\pi}_L - \overline{\pi}_R)(1-\nu_R), & \text{for } k \in \{U,O\} \text{ and } \alpha\beta_1\rho > 1, \\ \overline{\pi}_R, & \text{for } k \in \{U,O\} \text{ and } \alpha\beta_1\rho \le 1, \text{ and } k = N \text{ and } \alpha\beta_\lambda\rho \le 1, \end{cases} \tag{E.25}$$

where $\nu_R = \frac{\lambda v_R^\ell(\beta_1+1)}{\lambda v_R^\ell(\beta_1+1)+(1-\lambda)v_R^r(1-\alpha\beta_r)} \in (0,1)$. Then, as $\pi_R^{ma} \equiv \overline{\pi}_R$, (18) follows immediately from Lemma E.2 and the definition of $\overline{\rho}$, by capturing the ρ inside (E.25) as $\rho = \pi_L^{ma}/\pi_R^{ma} - 1$. This completes the proof. □

Proof of Proposition 13. To maintain consistency with the rest of the appendix, in this proof, we use $\beta_r = -\beta_0 = 1 - v_L^r/v_R^r$. First, we rewrite the objective function of the platform given in (17)

$$\Pi_S^{ma}(\pi_L,\pi_R) = \alpha\pi_L tr_S^{ma}(L \mid \rho = \pi_L/\pi_R - 1) + (1-\alpha)\pi_R tr_S^{ma}(R \mid \rho = \pi_L/\pi_R - 1), \tag{E.26}$$

which, by Lemma E.2, indicates the platform would aim to increase π_L and π_R as much as possible, while satisfying

$$(\overline{\pi}_L - \pi_L)\, tr_S^{ma}(L \mid \rho) \ge (\overline{\pi}_L - \pi_R)\, tr_S^{ma}(R \mid \rho), \tag{E.27}$$

$$(\overline{\pi}_R - \pi_R)\, tr_S^{ma}(R \mid \rho) \ge (\overline{\pi}_R - \pi_L)\, tr_S^{ma}(L \mid \rho). \tag{E.28}$$

Notice that it is without loss of optimality for us to restrict our search to (π_L,π_R) pairs that satisfy $\pi_L \ge \pi_R$. This is due to, again by Lemma E.2, any (π_L,π_R) pair with $\pi_L < \pi_R$ be suboptimal to the pair of $(\pi_L,\pi_R) = (\pi_R,\pi_R)$. In that, we utilize the fact that any (π_L,π_R) pair with $\pi_L = \pi_R$ satisfies both (E.27) and (E.28), which is established by the equivalence of $tr_S^{ma}(L \mid \rho = 0)$ and $tr_S^{ma}(R \mid \rho = 0)$, given in (E.5), by 4 and (C.1). Consequently, as $\pi_R^{ma} \le \overline{\pi}_R$, we then must have $\pi_L^{ma} \ge \overline{\pi}_R$. Thus, (E.28) will always be satisfied under (π_L^{ma},π_R^{ma}), as by (E.5), $tr_S^{ma}(R \mid \rho = \pi_L/\pi_R - 1) \ge 0$ for any ρ.

Notice next, that (E.27) must be binding under (π_L^{ma},π_R^{ma}). We know that $(\pi_L,\pi_R) = (\overline{\pi}_R,\overline{\pi}_R)$ is always a feasible solution for (E.26). For an arbitrary π_R and $\pi_L > \overline{\pi}_R$, it is straightforward to see (E.28) would continue to hold. As $tr_S^{ma}(R \mid \rho)$ is non-increasing and $tr_S^{ma}(L \mid \rho)$ is non-decreasing in ρ, for any $\pi_L < \overline{\pi}_L$, we know, by Lemma E.2, that we can improve (E.26) with a small enough increase in π_L, when (E.27) is not binding. As $\pi_R \le \overline{\pi}_R < \overline{\pi}_L$, the platform can only attain $\pi_L = \overline{\pi}_L$ if $tr_S^{ma}(R \mid \rho = \pi_L/\pi_R - 1) = 0$, which immediately implies (E.27) is binding as well. Hence, in maximizing (E.26), the platform must choose π_L^{ma} which binds (E.27), i.e., we must have

$$(\overline{\pi}_L - \pi_L^{ma})\, tr_S^{ma}(L \mid \rho = \pi_L/\pi_R^{ma} - 1) = (\overline{\pi}_L - \pi_R^{ma})\, tr_S^{ma}(R \mid \rho = \pi_L/\pi_R^{ma} - 1).$$

This completes the proof of the first part of Proposition 13.

For proving the second part of Proposition 13, notice first, that $1 > \beta_1 + 2\beta_r$ implies $\beta_1 < 1$, and $\alpha\beta_\lambda\overline{\rho} > 1$ implies $\alpha\beta_1\overline{\rho} > 1$, as $\beta_\lambda \le \beta_1$. Notice also that

$$\alpha\beta_\lambda\overline{\rho} > 1 \;\Leftrightarrow\; \alpha\overline{\rho} > \frac{\lambda v_R^\ell + (1-\lambda)v_R^r}{\lambda(v_L^\ell - v_R^\ell) + (1-\lambda)(v_L^r - v_R^r)} \;\Leftrightarrow\; \lambda > \frac{v_R^r(\alpha\beta_r\overline{\rho}+1)}{v_R^\ell(\alpha\beta_1\overline{\rho}-1) + v_R^r(\alpha\beta_r\overline{\rho}+1)} \;\Leftrightarrow\; \lambda > \lambda_S^{ma}(\overline{\rho})$$

where the last inequality follows from (E.16) as $F^r(\underline{g}) = \alpha(1+\overline{\rho})(1-\beta_r)$ for $\beta_1 < 1$. The conditions stated in Proposition 13 also induce

$$\overline{\rho} \ge (2\beta_1 + \beta_r - \beta_1\beta_r)/(1-\beta_1-2\beta_r) \;\Leftrightarrow\; \frac{\overline{\rho} - \beta_r - 2\beta_1}{2(\beta_1+\beta_r)(\overline{\rho}-\beta_1)} \ge \frac{1}{\beta_1+1}.$$

By Proposition 4, Lemma C.7 and Lemma E.2, that the optimal policy has $\left(1,0,\check{\sigma}^r_{L_\alpha},\check{\sigma}^r_{R_\alpha}\right)$ which, by (C.1), corresponds to $\left(\sqrt{\alpha},0,x^r_{L_\alpha},x^r_{R_\alpha}\right)$. This is because, by Lemma E.2, the platform aims to increase both π_L and π_R as much as possible and, as established by Claim 1 of the proof of Lemma E.1, the conditions (C.22), (C.23), (C.24), and (C.25) hold one after the other, in numeric order, as ρ increases. From the proof of Lemma C.10, by (C.54) and (C.55) in particular, we know that $\left(x^r_{L_\alpha},x^r_{R_\alpha}\right)=(\sqrt{\alpha},0)$ for $\beta_1<1$, which, by (C.1), corresponds to $\left(\sigma^r_{L_\alpha},\sigma^r_{R_\alpha}\right)=(1,0)$. Then, plugging $(x^\ell_L,x^\ell_R,x^r_L,x^r_R)=(\sqrt{\alpha},0,\sqrt{\alpha},0)$ into (E.5) will result in $tr^{ma}_S\left(R\,|\,\overline{\rho}=\overline{\pi}_L/\overline{\pi}_R-1\right)=0$, which, by simultaneously satisfying (E.20) and (E.21), allows the platform to set both profit margins to their respective upper bounds, i.e., $\pi^{ma}_L=\overline{\pi}_L$ and $\pi^{ma}_R=\overline{\pi}_R$. In other words, conditions presented in Proposition 13 ensures the feasibility of $\pi^{ma}_L=\overline{\pi}_L$ and $\pi^{ma}_R=\overline{\pi}_R$, for the platform's problem (E.26). This completes the proof. □

Appendix F: Sufficiency of Binary Messages under Informative-Allocative Recommendation Mode

In this section, we allow the platform to decide whether to make a product accessible to the user and, if accessible, to use *arbitrary* number of messages to recommend the product. More specifically, we still use $d=0$ (as in Section 5) to denote the platform's decision of making the product *not* accessible to the user. When the platform decides to make the product accessible ($d\neq 0$), we allow the message d to be drawn from a general message space $\mathcal{M}\not\ni 0$ of arbitrary cardinality. Altogether, the platform can choose $d\in\{0\}\cup\mathcal{M}\equiv\overline{\mathcal{M}}$. We will use superscript *mma* in the remaining section to denote the informative-allocative recommendation mode with this general message space. The informative-allocative recommendation mode in our base model (Section 5) essentially corresponds to $\mathcal{M}=\{1\}$ and $\overline{\mathcal{M}}=\{0,1\}$.

The platform's recommender policy σ specifies the probability of sending message $d\in\overline{\mathcal{M}}$ for each realization of $\left(i,\widehat{t}\right)$:

$$\sigma(d|i,\widehat{t}):=\mathbb{P}\left[d\,\middle|\,i,\widehat{t},\sigma\right]\in[0,1],\quad \text{for each } \widehat{t}\in\{\ell,r,n\} \text{ and } i\in\{L,R\} \text{ and } d\in\overline{\mathcal{M}}. \tag{F.1}$$

As in Section 3, user $t\in\{\ell,r\}$ expects her utility from choosing privacy setting $\widehat{t}$ to be

$$V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right)\equiv\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\int_0^1\max\left\{u,\mathbb{E}\left[v^t_i\,\middle|\,t,\widehat{t},d,\sigma\right]\right\}du+\mathbb{P}\left[d=0\,\middle|\,\widehat{t},\sigma\right]\int_0^1 udu. \tag{F.2}$$

Thus, given the platform's recommender policy σ under each privacy regime $k\in\{U,O,S,N\}$, the user of type t chooses her optimal privacy setting by maximizing her expected utility:

$$\widehat{t}^{mma}_k\in\arg\max_{\widehat{t}\in\Gamma_k}V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right). \tag{F.3}$$

In anticipation of the user's privacy setting in (F.3), the platform estimates and maximizes its expected revenue by designing an *optimal* recommender policy:

$$\Pi^{mma}_k:=\max_\sigma\ \mathbb{E}\left[\textstyle\sum_{d\in\mathcal{M}}\pi_i\sigma(d|i,\widehat{t}^{mma}_k)\mathbb{E}\left[v^t_i\,\middle|\,t,\widehat{t}^{mma}_k,d,\sigma\right]\,\middle|\,\sigma\right],\quad \text{subject to (F.2) and (F.3)}, \tag{F.4}$$

for privacy regime $k\in\{U,O,S,N\}$.

In the following lemma (whose proof is relegated to Section F.1), we first show that it is sufficient for the platform to consider recommender policies with $|\mathcal{M}|=2$.

LEMMA F.1 (Sufficiency of binary messages). *It is without loss of generality to restrict the message space* $\mathcal{M}$ *to be binary, say* $\mathcal{M}=\{-1,1\}$.

Lemma F.1 essentially shows that it suffices to augment our informative-allocative recommendation mode in Section 5 by at most *one* additional message (when the product is made accessible by the user). As a result, it allows us to reduce the user's expected utility in (F.2) to

$$V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right)=\sum_{d\in\{-1,1\}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\int_0^1\max\left\{u,\mathbb{E}\left[v_i^t\,\middle|\,t,\widehat{t},d,\sigma\right]\right\}du+\mathbb{P}\left[d=0\,\middle|\,\widehat{t},\sigma\right]\int_0^1 udu \tag{F.5}$$

and the platform's problem in (F.4) to

$$\Pi_k^{mma}=\max_{\sigma}\ \mathbb{E}\left[\textstyle\sum_{d\in\{-1,1\}}\pi_i\sigma(d|i,\widehat{t}_k^{mma})\mathbb{E}\left[v_i^t\,\middle|\,t,\widehat{t}_k^{mma},d,\sigma\right]\,\middle|\,\sigma\right],\quad\text{subject to (F.5) and (F.3)},\qquad(\mathcal{P}_k^{mma})$$

for privacy regime $k\in\{U,O,S,N\}$. The corresponding optimal policy is denoted as σ_k^{mma}.

As a key result, the following proposition (whose proof is also relegated to Section F.1) further demonstrates that for $k\in\{U,O,N\}$, the optimal policy σ_k^{mma} always makes the product accessible by users, thus eliminating the usage of $d=0$. Per Lemma 1, a binary message space $\mathcal{M}=\{-1,1\}$ suffices, establishing the equivalence of $(\mathcal{P}_k^{mma})$ to $(\mathcal{P}_k^{m})$. Namely, the *optimal* informative-allocative recommender policy σ_k^{mma} proves to be purely informative, equivalent to σ_k^m for $k\in\{U,O,N\}$.

PROPOSITION F.1 (Equivalence of σ_k^{mma} to σ_k^m for $k\in\{U,O,N\}$). *Under privacy protection modes* $k\in\{U,O,N\}$, *the optimal policy* $\sigma_k^{mma}(0|i,\widehat{t})=0$ *and* $\sigma_k^{mma}(1|i,\widehat{t})=1-\sigma_k^{mma}(-1|i,\widehat{t})=\sigma_k^m(i,\widehat{t})$ *for all* $\widehat{t}\in\{\ell,r,n\}$ *and* $i\in\{L,R\}$.

Proposition F.1 shows that under the three privacy protection modes $k\in\{U,O,N\}$, if the platform could recommend a product with (at least two) different messages (e.g., display different banners), then the platform has no incentive to hide the product from the users. Indeed, under the three privacy protection modes $k\in\{U,O,N\}$, the platform does not face the need to screen the user's type t; the challenge faced by the platform is to convey the product type i to the users. As a result, the platform prefers to use the *costless* recommendation messages $\mathcal{M}$ to signal the product type rather than resorting to $d=0$ which inflicts revenue loss on the platform.

However, under the self-disclosure protection mode $k=S$, the platform is also burdened by the need to screen the user's type t in addition to signaling the product type i to the users. In this case, we demonstrate, through comprehensive numerical studies, that it may be optimal for the platform to strategically withhold the product from the user's access (i.e., $d=0$ with positive probability), causing similar distortions (e.g., *algorithmic discrimination*) as shown in Section 5. In comparison with Figure 1, we combine recommendation rates $\sigma_k^{mma}(-1|i,\widehat{t})$ and $\sigma_k^{mma}(1|i,\widehat{t})$ into $\sigma_k^{mma}(i,\widehat{t}):=\sigma_k^{mma}(-1|i,\widehat{t})+\sigma_k^{mma}(1|i,\widehat{t})$ and hence $\sigma_k^{mma}(0|i,\widehat{t})=1-\sigma_k^{mma}(i,\widehat{t})$ for each $\widehat{t}\in\{\ell,r,n\}$, $i\in\{L,R\}$ and $k\in\{O,S\}$. Proposition F.1 immediately implies that $\sigma_O^{mma}(1|i,\widehat{t})\equiv 1$ for all $\left(i,\widehat{t}\right)$. Figure F.1 shows the parametric regions where the optimal recommender policy under the self-disclosure mode exhibits a *downward distortion* from that under the opt-out model, i.e., $\sigma_S^{mma}(1|i,\widehat{t})\le\sigma_O^{mma}(1|i,\widehat{t})\equiv 1$ with the inequality being strict for some $\left(i,\widehat{t}\right)$. Namely, *algorithmic discrimination* persists under the augmented message space: the platform strategically withholds the

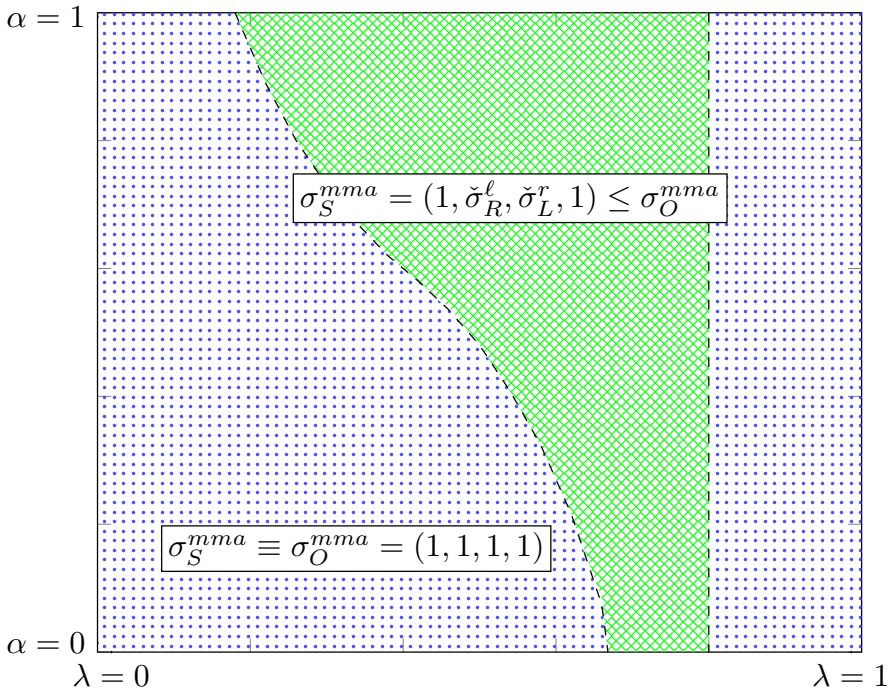


Figure F.1 Distortions of policy σ_S^{mma} from policy σ_O^{mma} with $\sigma_k^{mma} \equiv (\sigma_k^{mma}(L,\ell), \sigma_k^{mma}(R,\ell), \sigma_k^{mma}(L,r), \sigma_k^{mma}(R,r))$ and $\sigma_k^{mma}(i,\widehat{t}) := \sigma_k^{mma}(-1|i,\widehat{t}) + \sigma_k^{mma}(1|i,\widehat{t})$ for $k \in \{O,S\}$. Parametric regions marked in blue dots and green crosshatches represent no ($\sigma_S^{mma} \equiv \sigma_O^{mma}$) and downward ($\sigma_S^{mma} \leq \sigma_O^{mma}$) distortions, respectively, where the inequality is strict ($\sigma_S^{mma}(i,t) \not\equiv \sigma_O^{mma}(i,t)$) for some (i,t).

product's access from users. However, different from our results in Section 5, such a downward distortion *may* now also apply to the more valuable user ℓ, and only their less preferred products (product R for user ℓ and product L for user r) are withheld from each user type.

Finally, we numerically evaluate the gap of the platform's optimal revenue between our base model Π_S^{ma} given by ($\mathcal{P}_k^{ma}$) and the augmented message space Π_S^{mma} given by ($\mathcal{P}_k^{mma}$) under the self-disclosure protection regime. Specifically, we compute the optimality gap ratio $(\Pi_S^{\mathrm{mma}} - \Pi_S^{\mathrm{ma}})/\Pi_S^{\mathrm{mma}}$ by fixing the upper-bounds of ρ and β_1 to $\{2.5, 5, 10, 20\}$, with increments for both parameters of 0.125, 0.25, 0.5 and 1, respectively, while varying α, λ' and β_r (where $\beta_r = -\beta_0$) from 0.1 to 0.9, all with a 0.1 increment. Thus, the total number of instances for each upper-bound pair of ρ and β_1 is $19 \times 19 \times 9 \times 9 \times 9 = 263{,}169$. While the platform necessarily earns higher revenue with the augmented space than in our base model, Table F.1 shows that the optimal informative-allocative policy in our base model generally performs nearly as well as the one with the augmented message space.

F.1. Proofs in Section F

Proof of Lemma F.1. First, we rewrite in (F.6) and (F.7), both users' utility function (F.2) and the platform's objective function (F.4), respectively, for a general message space $\overline{\mathcal{M}}$:

$$V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right) = \sum_{d\in\mathcal{M}} \mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right] \int_0^1 \max\left\{u, \mathbb{E}\left[v_i^t\,\middle|\,t,\widehat{t},d,\sigma\right]\right\} du + \mathbb{P}\left[d=0\,\middle|\,\widehat{t},\sigma\right] \int_0^1 u du$$

Table F.1 Summary statistics of the optimality gap between the optimal informative-allocative policy and optimal general recommendation policy $(\Pi_S^{\mathsf{mma}} - \Pi_S^{\mathsf{ma}})/\Pi_S^{\mathsf{mma}}$ across 263,169 instances of $(\rho, \beta_1, \beta_r, \alpha, \lambda')$.

		$\rho, \beta_1 < 2.5$	$\rho, \beta_1 < 5$	$\rho, \beta_1 < 10$	$\rho, \beta_1 < 20$
Average		4.06%	5.42%	4.24%	2.31%
Standard deviation		4.26%	5.63%	5.14%	3.80%
Minimum		0%	0%	0%	0.01%
Maximum		26.15%	32.27%	32.27%	31.76%
Proportion of instances with gap	$\geq 10\%$	10.54%	17.44%	12.67%	5.17%
	$\geq 15\%$	2.95%	7.77%	5.42%	2.27%
	$\geq 20\%$	0.47%	3.14%	2.17%	0.89%

$$
\begin{aligned}
&= \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right] \mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d,\sigma\right]^2 + \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]/2 \\
&= \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right] \left\{v_L^t \mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right] + v_R^t \left(1 - \mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]\right)\right\}^2 + \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]/2 \\
&= (v_R^t)^2 \left(1 - \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\right) + 2v_R^t(v_L^t - v_R^t)\left(\alpha - \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d=0,\sigma\right]\right) \\
&\quad + \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]/2 + (v_L^t - v_R^t)^2 \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right] \mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2 \\
&= 2\alpha v_R^t(v_L^t - v_R^t) + (v_R^t)^2 + \alpha\left[1/2 - (v_R^t)^2 - 2v_R^t(v_L^t - v_R^t)\right]\mathbb{P}\left[d=0 \,\middle|\, L,\widehat{t},\sigma\right] \\
&\quad + (1-\alpha)\left[1/2 - (v_R^t)^2\right]\mathbb{P}\left[d=0 \,\middle|\, R,\widehat{t},\sigma\right] + (v_L^t - v_R^t)^2 \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2 \qquad \text{(F.6)}
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_k^{mma} &= \mathbb{E}\left[\sum_{d\in\mathcal{M}} \pi_i \sigma(d|i,\widehat{t}_k^{mma}) \mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t}_k^{mma},d,\sigma\right] \,\middle|\, \sigma\right] \\
&= \lambda\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{\ell},\sigma\right] \mathbb{E}\left[\pi_i \,\middle|\, \widehat{\ell},d,\sigma\right] \mathbb{E}\left[v_i^\ell \,\middle|\, \ell,\widehat{\ell},d,\sigma\right]\right\} + (1-\lambda)\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{r},\sigma\right] \mathbb{E}\left[\pi_i \,\middle|\, \widehat{r},d,\sigma\right] \mathbb{E}\left[v_i^r \,\middle|\, r,\widehat{r},d,\sigma\right]\right\} \\
&= \lambda\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{\ell},\sigma\right] \left((\pi_L - \pi_R)\mathbb{P}\left[L \,\middle|\, \widehat{\ell},d,\sigma\right] + \pi_R\right)\left((v_L^\ell - v_R^\ell)\mathbb{P}\left[L \,\middle|\, \widehat{\ell},d,\sigma\right] + v_R^\ell\right)\right\} \\
&\quad + (1-\lambda)\left\{\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{r},\sigma\right]\left((\pi_L - \pi_R)\mathbb{P}\left[L \,\middle|\, \widehat{r},d,\sigma\right] + \pi_R\right)\left((v_L^r - v_R^r)\mathbb{P}\left[L \,\middle|\, \widehat{r},d,\sigma\right] + v_R^r\right)\right\} \\
&\propto \lambda\begin{cases} -\alpha\left[\pi_R v_L^\ell + (\pi_L - \pi_R)v_R^\ell\right]\mathbb{P}\left[d=0 \,\middle|\, L,\widehat{\ell},\sigma\right] - (1-\alpha)\pi_R v_R^\ell \mathbb{P}\left[d=0 \,\middle|\, R,\widehat{\ell},\sigma\right] \\ +(\pi_L - \pi_R)(v_L^\ell - v_R^\ell)\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{\ell},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{\ell},d,\sigma\right]^2 \end{cases} \\
&\quad + (1-\lambda)\begin{cases} -\alpha\left[\pi_R v_L^r + (\pi_L - \pi_R)v_R^r\right]\mathbb{P}\left[d=0 \mid L,\widehat{r},\sigma\right] - (1-\alpha)\pi_R v_R^r \mathbb{P}\left[d=0 \mid R,\widehat{r},\sigma\right] \\ +(\pi_L - \pi_R)(v_L^r - v_R^r)\sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{r},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{r},d,\sigma\right]^2 \end{cases} \qquad \text{(F.7)}
\end{aligned}
$$

for each $\widehat{t} \in \{\ell, r, n\}$, $i \in \{L, R\}$ and $k \in \{U, O, S, N\}$, where we utilized

$$
\mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d=0,\sigma\right] + \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right] = \alpha \quad \text{and} \quad \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right] + \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right] = 1,
$$

i.e., martingale and total probability conditions, respectively. Notice in both (F.6) and (F.7), for a general message space $\mathcal{M}$ and a recommender policy σ, where $\sigma(d=0|i,\widehat{t})$ is given for each $\widehat{t}\in\{\ell,r,n\}$ and $i\in\{L,R\}$, the key quantity that determines $V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right)$ and Π_k^{mma} can be written as

$$\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2=\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]^2\mathbb{P}\left[L\,\middle|\,\widehat{t},\sigma\right]^2/\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]$$

$$=\alpha^2\sum_{d\in\mathcal{M}}\frac{\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]^2}{\alpha\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]+(1-\alpha)\mathbb{P}\left[d\,\middle|\,R,\widehat{t},\sigma\right]}. \tag{F.8}$$

We now establish the following Claim F.1, whose proof is relegated after the proof of Lemma F.1.

Claim F.1. *For a σ utilizing a general message space $\mathcal{M}$, $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2\in\left[\alpha^2\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right),\alpha\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right)\right]$. Further, $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2=\alpha\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right)$ can be achieved with a full-disclosure policy, while $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2=\alpha^2\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right)$ can be achieved with a no-disclosure policy.*

Having Claim F.1, we now show that for any arbitrary policy σ, utilizing a general messaging space $\mathcal{M}$, $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2=\mu_{\mathcal{M}}\in\left[\alpha^2\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right),\alpha\left(1-\sigma\left(d=0\,\middle|\,L,\widehat{t}\right)\right)\right]$ can be replicated by a policy $\overline{\sigma}$ utilizing binary messaging space $\{-1,1\}$, for any pair of $t\in\{\ell,r\}$ and $\widehat{t}\in\{\ell,r,n\}$, and hence, by (F.6) and (F.7), achieve the same payoff performance both for the users and the platform. That is, of course, achieved by conditioning on $\left(\overline{\sigma}(0|L,\widehat{t}),\overline{\sigma}(0|R,\widehat{t})\right)=\left(\sigma(0|L,\widehat{t}),\sigma(0|R,\widehat{t})\right)$. Let us first rewrite $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$ for policy $\overline{\sigma}$ utilizing binary messaging space $\{-1,1\}$:

$$\sum_{d\in\{-1,1\}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\overline{\sigma}\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\overline{\sigma}\right]^2=\mathbb{P}\left[d=1\,\middle|\,\widehat{t},\overline{\sigma}\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d=1,\overline{\sigma}\right]^2+\mathbb{P}\left[d=-1\,\middle|\,\widehat{t},\overline{\sigma}\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d=-1,\overline{\sigma}\right]^2$$

$$=\frac{\alpha^2\overline{\sigma}(1|L,\widehat{t})^2}{\alpha\overline{\sigma}(1|L,\widehat{t})+(1-\alpha)\overline{\sigma}(1|R,\widehat{t})}+\frac{\alpha^2\left[1-\overline{\sigma}(0|L,\widehat{t})-\overline{\sigma}(1|L,\widehat{t})\right]^2}{\alpha\left[1-\overline{\sigma}(0|L,\widehat{t})-\overline{\sigma}(1|L,\widehat{t})\right]+(1-\alpha)\left[1-\overline{\sigma}(0|R,\widehat{t})-\overline{\sigma}(1|R,\widehat{t})\right]}.$$

Now, inputting $\left(\overline{\sigma}(1|L,\widehat{t}),\overline{\sigma}(1|R,\widehat{t})\right)=\left(1-\overline{\sigma}(0|L,\widehat{t}),\frac{\alpha(1-\overline{\sigma}(0|L,\widehat{t}))\left[\alpha(1-\overline{\sigma}(0|L,\widehat{t}))-\mu_{\mathcal{M}}\right]}{(1-\alpha)\mu_{\mathcal{M}}}\right)$ to the above equality give us $\sum_{d\in\{-1,1\}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\overline{\sigma}\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\overline{\sigma}\right]^2=\mu_{\mathcal{M}}=\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$, which completes the proof. □

Proof of Claim F.1. We first show that $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$ can always be improved if σ, under message space $\mathcal{M}$, is not a full-disclosure policy, i.e., when there exists a message $d_a\in\mathcal{M}$ that has $\mathbb{P}\left[d_a\,\middle|\,L,\widehat{t},\sigma\right]=q_a^L>0$ and $\mathbb{P}\left[d_a\,\middle|\,R,\widehat{t},\sigma\right]=q_a^R>0$. In such a case, keeping the rest of σ the same, splitting d_a into two separate messages d_b and d_c with $\mathbb{P}\left[d_b\,\middle|\,L,\widehat{t},\sigma\right]=q_a^L,\mathbb{P}\left[d_b\,\middle|\,R,\widehat{t},\sigma\right]=0$, and $\mathbb{P}\left[d_c\,\middle|\,L,\widehat{t},\sigma\right]=0,\mathbb{P}\left[d_c\,\middle|\,R,\widehat{t},\sigma\right]=q_a^R$ would increase $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$. The improvement of the separation in $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]\mathbb{P}\left[L\,\middle|\,\widehat{t},d,\sigma\right]^2$ can be calculated, utilizing (F.8), as

$$\frac{q_a^L}{\alpha}-\frac{(q_a^L)^2}{\alpha q_a^L+(1-\alpha)q_a^R}\propto\alpha(q_a^L)^2+(1-\alpha)q_a^Lq_a^R-\alpha(q_a^L)^2=(1-\alpha)q_a^Lq_a^R>0.$$

Iteratively following the separation procedure for any message $d_a\in\mathcal{M}$ that has $\mathbb{P}\left[d_a\,\middle|\,L,\widehat{t},\sigma\right]>0$ and $\mathbb{P}\left[d_a\,\middle|\,R,\widehat{t},\sigma\right]>0$, σ will become a full-disclosure policy. In other words, any message $d\in\mathcal{M}$ with $\mathbb{P}\left[d\,\middle|\,\widehat{t},\sigma\right]>0$ would perfectly indicate product $i\in\{L,R\}$. In such a policy, (F.8) can be written as

$$\alpha^2\sum_{d\in\mathcal{M}}\frac{\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]^2}{\alpha\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]+(1-\alpha)\mathbb{P}\left[d\,\middle|\,R,\widehat{t},\sigma\right]}=\alpha^2\sum_{d\in\mathcal{M}}\frac{\mathbb{P}\left[d\,\middle|\,L,\widehat{t},\sigma\right]}{\alpha}$$

$$= \alpha \sum_{d\in\mathcal{M}} \frac{\mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]}{\mathbb{P}\left[L \,\middle|\, \widehat{t},\sigma\right]} = \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]$$

$$= \alpha - \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d=0,\sigma\right]$$

$$= \alpha - \mathbb{P}\left[d=0 \,\middle|\, L,\widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},\sigma\right] = \alpha\left(1-\mathbb{P}\left[d=0 \,\middle|\, L,\widehat{t},\sigma\right]\right).$$

Next, we show that $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2$ can always be decreased if σ, under message space $\mathcal{M}$, is not a no-disclosure policy. In that, we show, when there exists two distinct messages $d_b, d_c \in \mathcal{M}$ that have $\mathbb{P}\left[d_b \,\middle|\, L,\widehat{t},\sigma\right] = q_b^L > 0$ and $\mathbb{P}\left[d_c \,\middle|\, R,\widehat{t},\sigma\right] = q_c^R > 0$, $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2$ can be decreased by combining these two messages into one, say message d_a, that has $\mathbb{P}\left[d_a \,\middle|\, L,\widehat{t},\sigma\right] = q_b^L + q_c^L > 0$ and $\mathbb{P}\left[d_a \,\middle|\, R,\widehat{t},\sigma\right] = q_b^R + q_c^R > 0$, keeping rest of σ the same. The decrease of the combination in $\sum_{d\in\mathcal{M}}\mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]^2$ can be calculated, utilizing (F.8), as

$$\frac{(q_b^L+q_c^L)^2)}{\alpha\left(q_b^L+q_c^L\right)+(1-\alpha)\left(q_b^R+q_c^R\right)} - \frac{(q_b^L)^2}{\alpha q_b^L+(1-\alpha)q_b^R} - \frac{(q_c^L)^2}{\alpha q_c^L+(1-\alpha)q_c^R}$$

$$\propto -\left\{q_b^L\left[\alpha q_c^L+(1-\alpha)q_c^R\right] - q_c^L\left[\alpha q_b^L+(1-\alpha)q_b^R\right]\right\}^2 < 0.$$

Iteratively following the combination procedure for any messages $d_b, d_c \in \mathcal{M}$ that have $\mathbb{P}\left[d_b \,\middle|\, L,\widehat{t},\sigma\right] = q_b^L > 0$ and $\mathbb{P}\left[d_c \,\middle|\, R,\widehat{t},\sigma\right] = q_c^R > 0$, σ will become a no-disclosure policy, under $\mathcal{M}$, which can be implemented by using only a singleton. In other words, a message $d \in \mathcal{M}$ would indicate no further inferential value on product type $i \in \{L,R\}$. In such a policy, (F.8) can be written as

$$\alpha^2 \sum_{d\in\mathcal{M}} \frac{\mathbb{P}\left[d \,\middle|\, L,\widehat{t},\sigma\right]^2}{\alpha\mathbb{P}\left[d \,\middle|\, L,\widehat{t},\sigma\right]+(1-\alpha)\mathbb{P}\left[d \,\middle|\, R,\widehat{t},\sigma\right]} = \alpha^2 \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, L,\widehat{t},\sigma\right]$$

$$= \alpha^2 \sum_{d\in\mathcal{M}} \frac{\mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]}{\mathbb{P}\left[L \,\middle|\, \widehat{t},\sigma\right]} = \alpha \sum_{d\in\mathcal{M}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d,\sigma\right]$$

$$= \alpha\left(\alpha - \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},d=0,\sigma\right]\right)$$

$$= \alpha\left(\alpha - \mathbb{P}\left[d=0 \,\middle|\, L,\widehat{t},\sigma\right]\mathbb{P}\left[L \,\middle|\, \widehat{t},\sigma\right]\right) = \alpha^2\left(1-\mathbb{P}\left[d=0 \,\middle|\, L,\widehat{t},\sigma\right]\right).$$

This completes the proof. □

Proof of Proposition F.1. First, we expand in (F.9) and (F.10), both users' utility function (F.5) and the platform's objective function ($\mathcal{P}_k^{mma}$), respectively:

$$V^{mma}\left(t,\widehat{t}\,\middle|\,\sigma\right)$$

$$= \sum_{d\in\{-1,1\}} \mathbb{P}\left[d \,\middle|\, \widehat{t},\sigma\right]\int_0^1 \max\left\{u,\mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d,\sigma\right]\right\}du + \mathbb{P}\left[d=0 \,\middle|\, \widehat{t},\sigma\right]\int_0^1 u du$$

$$= \mathbb{P}\left[d=1 \,\middle|\, \widehat{t},\sigma\right]\int_0^1 \max\left\{u,\mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d=1,\sigma\right]\right\}du$$

$$+ \mathbb{P}\left[d=-1 \,\middle|\, \widehat{t},\sigma\right]\int_0^1 \max\left\{u,\mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d=-1,\sigma\right]\right\}du$$

$$+ \left(1-\mathbb{P}\left[d=1 \,\middle|\, \widehat{t},\sigma\right]-\mathbb{P}\left[d=-1 \,\middle|\, \widehat{t},\sigma\right]\right)\int_0^1 u du$$

$$= \mathbb{P}\left[d=1 \,\middle|\, \widehat{t},\sigma\right] \mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d=1,\sigma\right]^2/2 + \mathbb{P}\left[d=-1 \,\middle|\, \widehat{t},\sigma\right] \mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d=-1,\sigma\right]^2/2 + 1/2$$

$$= \frac{1}{2}\left\{ \frac{\left[v_L^t \alpha\sigma(1|L,\widehat{t}) + v_R^t(1-\alpha)\sigma(1|R,\widehat{t})\right]^2}{\alpha\sigma(1|L,\widehat{t}) + (1-\alpha)\sigma(1|R,\widehat{t})} + \frac{\left[v_L^t \alpha\sigma(-1|L,\widehat{t}) + v_R^t(1-\alpha)\sigma(-1|R,\widehat{t})\right]^2}{\alpha\sigma(-1|L,\widehat{t}) + (1-\alpha)\sigma(-1|R,\widehat{t})} + 1 \right\} \tag{F.9}$$

and

$$\begin{aligned}\Pi_k^{mma} =& \lambda \left\{ \begin{array}{l} \left[\alpha\pi_L\sigma(1|L,\ell) + (1-\alpha)\pi_R\sigma(1|R,\ell)\right] \frac{\alpha v_L^\ell \sigma(1|L,\ell) + (1-\alpha) v_R^\ell \sigma(1|R,\ell)}{\alpha\sigma(1|L,\ell) + (1-\alpha)\sigma(1|R,\ell)} \\ + \left[\alpha\pi_L\sigma(-1|L,\ell) + (1-\alpha)\pi_R\sigma(-1|R,\ell)\right] \frac{\alpha v_L^\ell \sigma(-1|L,\ell) + (1-\alpha) v_R^\ell \sigma(-1|R,\ell)}{\alpha\sigma(-1|L,\ell) + (1-\alpha)\sigma(-1|R,\ell)} \end{array} \right. \\ & + (1-\lambda) \left\{ \begin{array}{l} \left[\alpha\pi_L\sigma(1|L,r) + (1-\alpha)\pi_R\sigma(1|R,r)\right] \frac{\alpha v_L^r \sigma(1|L,r) + (1-\alpha) v_R^r \sigma(1|R,r)}{\alpha\sigma(1|L,r) + (1-\alpha)\sigma(1|R,r)} \\ + \left[\alpha\pi_L\sigma(-1|L,r) + (1-\alpha)\pi_R\sigma(-1|R,r)\right] \frac{\alpha v_L^r \sigma(-1|L,r) + (1-\alpha) v_R^r \sigma(-1|R,r)}{\alpha\sigma(-1|L,r) + (1-\alpha)\sigma(-1|R,r)} \end{array} \right. \end{aligned} \tag{F.10}$$

where we utilized the representations of $\mathbb{E}\left[v_i^t \,\middle|\, t,\widehat{t},d,\sigma\right]$ and $\mathbb{P}\left[i \,\middle|\, \widehat{t},d,\sigma\right]$ given in Section 3. Note also that, in line with (F.1), for any feasible policy σ we have

$$\sigma(1|i,\widehat{t}) + \sigma(-1|i,\widehat{t}) + \sigma(0|i,\widehat{t}) \leq 1, \quad \text{for each } \widehat{t} \in \{\ell,r,n\} \text{ and } i \in \{L,R\}. \tag{F.11}$$

Next, similar to our strategy in Appendix C (particularly with that of Lemma C.1), we introduce a variable transformation for solving the platform's problem in $(\mathcal{P}_k^{mma})$. Such that, for any policy σ defined in (F.1), and for each $\widehat{t} \in \{\ell,r,n\}$ and $d \in \{1,0,-1\}$ we let

$$x(d|L,\widehat{t}) := \frac{\alpha\sigma(d|L,\widehat{t})}{\sqrt{\alpha\sigma(d|L,\widehat{t}) + (1-\alpha)\sigma(d|R,\widehat{t})}} \quad \text{and} \quad x(d|R,\widehat{t}) := \frac{(1-\alpha)\sigma(d|R,\widehat{t})}{\sqrt{\alpha\sigma(d|L,\widehat{t}) + (1-\alpha)\sigma(d|R,\widehat{t})}}, \tag{F.12}$$

with $x(d|L,\widehat{t}) = x(d|R,\widehat{t}) = 0$ for $\sigma(d|L,\widehat{t}) = \sigma(d|R,\widehat{t}) = 0$. Then, $x =: \left\{x(d|i,\widehat{t}) \geq 0 : i \in \{L,R\} \text{ and } \widehat{t} \in \{\ell,r,n\} \text{ and } d \in \{1,0,-1\}\right\}$ must satisfy

$$x(d|L,\widehat{t})\left[x(d|L,\widehat{t}) + x(d|R,\widehat{t})\right] \leq \alpha, \quad \text{and} \tag{$\overline{H}_\alpha^{\widehat{t}}$}$$

$$x(d|R,\widehat{t})\left[x(d|L,\widehat{t}) + x(d|R,\widehat{t})\right] \leq 1-\alpha, \quad \text{for } \widehat{t} \in \{\ell,r,n\} \text{ and } d \in \{1,0,-1\}. \tag{$\overline{H}_{1-\alpha}^{\widehat{t}}$}$$

Conversely, for any x satisfying ($\overline{H}_\alpha^{\widehat{t}}$) and ($\overline{H}_{1-\alpha}^{\widehat{t}}$), the policy

$$\sigma(d|L,\widehat{t}) := x(d|L,\widehat{t})\left[x(d|L,\widehat{t}) + x(d|R,\widehat{t})\right]/\alpha \text{ and } \sigma(d|R,\widehat{t}) := x(d|R,\widehat{t})\left[x(d|L,\widehat{t}) + x(d|R,\widehat{t})\right]/(1-\alpha) \tag{F.13}$$

is well defined and satisfies (F.12). Mathematical breakdown of the above variable transformation is identical to the one provided in the proof of Lemma C.1 and, hence, omitted here for brevity. Similarly, all our results in Lemmas C.2-C.5 about the informative-allocative recommendation mode in our base model extend and apply equally to the general setting of this section. We conclude the proof of this proposition by studying the below three cases, each is exclusively dedicated to a different privacy protection mode $k \in \{U,O,N\}$. In each of the cases we establish that $\sigma_k^{mma}(0|i,\widehat{t}) \equiv 0$, which immediately imply the two display message probabilities are complementary to each other ($\sigma_k^{mma}(1|i,\widehat{t}) \equiv 1 - \sigma_k^{mma}(-1|i,\widehat{t})$ for each $i \in \{L,R\}$ and $\widehat{t} \in \{\ell,r,n\}$), hence, establishing the equivalence of $(\mathcal{P}_k^{mma})$ to $(\mathcal{P}_k^m)$.

Case 1: $k = U$. By Equations (F.9)-(F.13), the platform's problem $(\mathcal{P}_k^{mma})$ for $k = U$ can be reformulated as

$$\max_{x(d|i,\widehat{t}) \geq 0} \lambda' \left\{ \begin{array}{l} (1+\rho)(1+\beta_1)\left[(x(1|L,\ell)^2 + (x(-1|L,\ell)^2\right] \\ +(2+\rho+\beta_1)\left[(x(1|L,\ell)\,(x(1|R,\ell) + (x(-1|L,\ell)\,(x(-1|R,\ell)\right] \\ +\left[(x(1|R,\ell)^2 + (x(-1|R,\ell)^2\right] \end{array} \right.$$

$$+(1-\lambda')\begin{cases}(1+\rho)(1-\beta_r)\left[(x(1|L,r)^2+(x(-1|L,r)^2\right]\\ +(2+\rho-\beta_r)\left[(x(1|L,r)\,(x(1|R,r)+(x(-1|L,r)\,(x(-1|R,r)\right]\\ +\left[(x(1|R,r)^2+(x(-1|R,r)^2\right],\end{cases}$$

$$\text{subject to } (\overline{H}^{\widehat{t}}_{\alpha}) \text{ and } (\overline{H}^{\widehat{t}}_{1-\alpha}) \text{ for } d\in\{1,0,-1\},\ i\in\{L,R\} \text{ and } \widehat{t}\in\{\ell,r\}, \tag{F.14}$$

where

$$\lambda'=\frac{\lambda v_R^{\ell}}{\lambda v_R^{\ell}+(1-\lambda)v_R^{r}}\in[0,1],\quad \rho=\frac{\pi_L}{\pi_R}-1>0,\quad \beta_1=\frac{v_L^{\ell}}{v_R^{\ell}}-1>0\quad\text{and}\quad \beta_r=1-\frac{v_L^{r}}{v_R^{r}}\in(0,1].$$

Suppose, $\sigma_U^{mma}(0|i,\widehat{t})=\epsilon>0$, for an $i\in\{L,R\}$ and $\widehat{t}\in\{\ell,r\}$. Then it must be the case that $\sigma_U^{mma}(1|i,\widehat{t})+\sigma_U^{mma}(-1|i,\widehat{t})\le 1-\epsilon$ which, by (F.13), translates to

$$x(1|i,\widehat{t})\left[x(1|L,\widehat{t})+x(1|R,\widehat{t})\right]+x(-1|i,\widehat{t})\left[x(-1|L,\widehat{t})+x(-1|R,\widehat{t})\right]\le(1-\epsilon)\mathbb{P}[i]<\mathbb{P}[i].$$

Which implies both $x(1|i,\widehat{t})\left[x(1|L,\widehat{t})+x(1|R,\widehat{t})\right]<\mathbb{P}[i]$ and $x(-1|i,\widehat{t})\left[x(-1|L,\widehat{t})+x(-1|R,\widehat{t})\right]<\mathbb{P}[i]$, making both $(\overline{H}^{\widehat{t}}_{\alpha})$ and $(\overline{H}^{\widehat{t}}_{1-\alpha})$ constraints slack. However, this contradicts with the optimality of σ_U^{mma} as both the objective function in (F.14) and $x(d|i,\widehat{t})\left[x(d|L,\widehat{t})+x(d|R,\widehat{t})\right]$ are increasing in $x(d|i,\widehat{t})$ for $d\in\{1,-1\}$, $i\in\{L,R\}$ and $\widehat{t}\in\{\ell,r\}$. Thus, it must be the case that $\sigma_U^{mma}(0|i,\widehat{t})\equiv 0$.

Case 2: $k=O$**.** By Equations (F.9)-(F.13), the platform's problem $(\mathcal{P}_k^{mma})$ for $k=O$ can be reformulated as

$$\max_{x(d|i,\widehat{t})\ge 0}\ \lambda'\begin{cases}(1+\rho)(1+\beta_1)\left[(x(1|L,\ell)^2+(x(-1|L,\ell)^2\right]\\ +(2+\rho+\beta_1)\left[(x(1|L,\ell)\,(x(1|R,\ell)+(x(-1|L,\ell)\,(x(-1|R,\ell)\right]\\ +\left[(x(1|R,\ell)^2+(x(-1|R,\ell)^2\right]\end{cases}$$
$$+(1-\lambda')\begin{cases}(1+\rho)(1-\beta_r)\left[(x(1|L,r)^2+(x(-1|L,r)^2\right]\\ +(2+\rho-\beta_r)\left[(x(1|L,r)\,(x(1|R,r)+(x(-1|L,r)\,(x(-1|R,r)\right]\\ +\left[(x(1|R,r)^2+(x(-1|R,r)^2\right],\end{cases}$$

$$\text{subject to } (\overline{H}^{\widehat{t}}_{\alpha}) \text{ and } (\overline{H}^{\widehat{t}}_{1-\alpha}) \text{ for } d\in\{1,0,-1\},\ i\in\{L,R\} \text{ and } \widehat{t}\in\{\ell,r\},$$

$$[(1+\beta_1)x(1|L,\ell)+x(1|R,\ell)]^2+[(1+\beta_1)x(-1|L,\ell)+x(-1|R,\ell)]^2$$
$$\ge[(1+\beta_1)x(1|L,n)+x(1|R,n)]^2+[(1+\beta_1)x(-1|L,n)+x(-1|R,n)]^2,$$

$$\text{and } [(1-\beta_r)x(1|L,r)+x(1|R,r)]^2+[(1-\beta_r)x(-1|L,r)+x(-1|R,r)]^2$$
$$\ge[(1-\beta_r)x(1|L,n)+x(1|R,n)]^2+[(1-\beta_r)x(-1|L,n)+x(-1|R,n)]^2 \quad \text{for } t\in\{\ell,r\}. \tag{F.15}$$

Clearly, $(\mathcal{P}_k^{ma})$ for $k=U$ (i.e., problem (F.14)) is a relaxation of $(\mathcal{P}_k^{ma})$ for $k=O$ (i.e., problem (F.15)), not considering the last two sets of constraints. In (F.15)), by setting both sets of $x(d|i,n)\equiv 0$ for $d\in\{1,0,-1\}$ and $i\in\{L,R\}$, we allow the platform to attain the expected revenue extracted in (F.14). Hence, by (F.13), without loss of optimality, we have $\sigma_O^{mma}(d|i,n)\equiv 0$ for $d\in\{1,0,-1\}$ and $i\in\{L,R\}$. Thus, the problem reduces to (F.14) which, by the proof of Case 1 above, immediately implies it must be the case that $\sigma_O^{mma}(0|i,\widehat{t})\equiv 0$.

Case 3: $k = N$**.** By Equations (F.9)-(F.13), the platform's problem ($\mathcal{P}_k^{mma}$) for $k = N$ can be reformulated as

$$\max_{x(d|i,n)\geq 0} \lambda' \begin{cases} (1+\rho)(1+\beta_1)\left[(x(1|L,n)^2 + (x(-1|L,n)^2\right] \\ +(2+\rho+\beta_1)\left[(x(1|L,n)\,(x(1|R,n) + (x(-1|L,n)\,(x(-1|R,n)\right] \\ +\left[(x(1|R,n)^2 + (x(-1|R,n)^2\right] \end{cases}$$
$$+ (1-\lambda') \begin{cases} (1+\rho)(1-\beta_r)\left[(x(1|L,n)^2 + (x(-1|L,n)^2\right] \\ +(2+\rho-\beta_r)\left[(x(1|L,n)\,(x(1|R,n) + (x(-1|L,n)\,(x(-1|R,n)\right] \\ +\left[(x(1|R,n)^2 + (x(-1|R,n)^2\right], \end{cases}$$

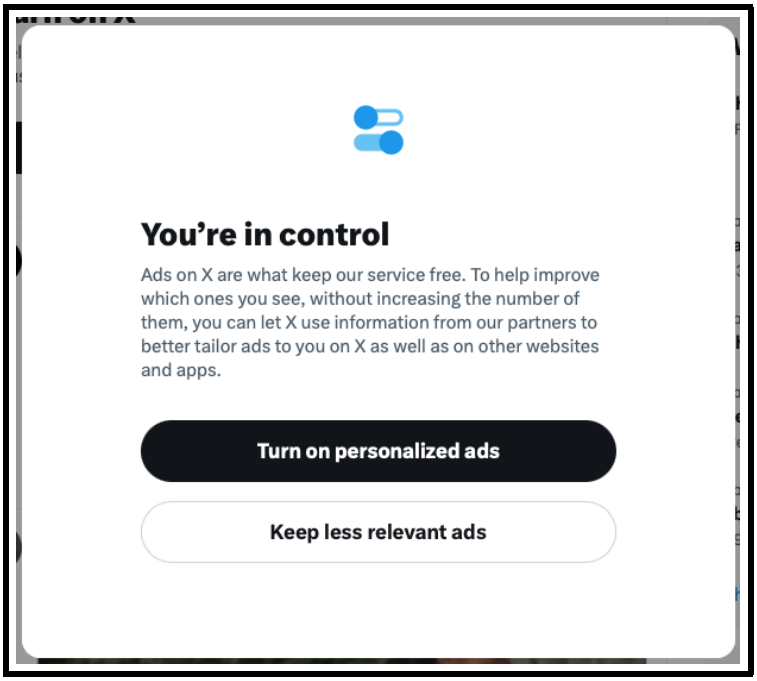
$$\text{subject to } (\widehat{\overline{H}^t_\alpha}) \text{ and } (\widehat{\overline{H}^t_{1-\alpha}}) \text{ for } d \in \{1,0,-1\} \text{ and } i \in \{L,R\}. \tag{F.16}$$

Suppose, $\sigma_N^{mma}(0|i,n) = \epsilon > 0$, for an $i \in \{L,R\}$. Then it must be the case that $\sigma_N^{mma}(1|i,n) + \sigma_N^{mma}(-1|i,n) \leq 1-\epsilon$ which, by (F.13), translates to

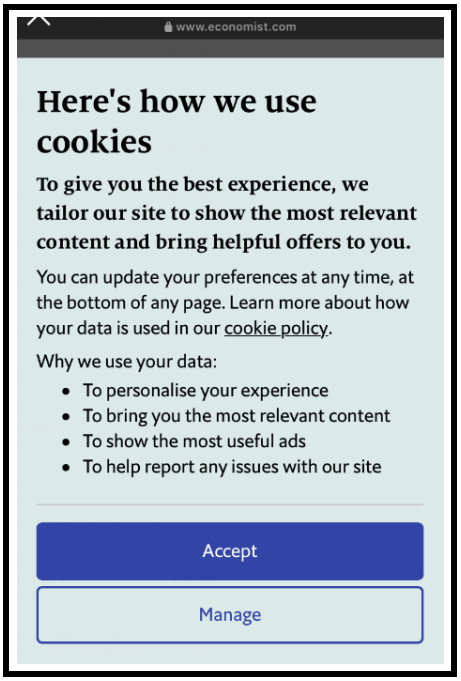
$$x(1|i,n)\,[x(1|L,n) + x(1|R,n)] + x(-1|i,n)\,[x(-1|L,n) + x(-1|R,n)] \leq (1-\epsilon)\mathbb{P}[i] < \mathbb{P}[i].$$

Which implies both $x(1|i,n)\,[x(1|L,n) + x(1|R,n)] < \mathbb{P}[i]$ and $x(-1|i,n)\,[x(-1|L,n) + x(-1|R,n)] < \mathbb{P}[i]$, making both $(\widehat{\overline{H}^t_\alpha})$ and $(\widehat{\overline{H}^t_{1-\alpha}})$ constraints slack. However, this contradicts with the optimality of σ_N^{mma} as both the objective function in (F.16) and $x(d|i,n)\,[x(d|L,n) + x(d|R,n)]$ are increasing in $x(d|i,n)$ for $d \in \{1,-1\}$ and $i \in \{L,R\}$. Thus, it must be the case that $\sigma_N^{mma}(0|i,n) \equiv 0$. This completes the proof of proposition. □

Appendix G: Motivating Examples from Practice



(a) X



(b) Economist.com

Figure G.1 X (formerly Twitter) and the Economist's communication to users about how personal information affects personalization and what may change if the user restricts such use. The interface makes explicit that privacy choices can alter what is shown and how content is targeted, consistent with viewing a platform's recommender policy as specifying whether and how often to recommend items conditional on observed/reported preferences. This figure is intended as a practice-motivating example for the Introduction where we interpret privacy-policy statements as the platform committing to and communicating aspects of its recommender policy. This practice also makes the privacy choice salient and motivates our modeling choice of having private preferences as a user-controlled decision over whether/how to disclose to the platform.

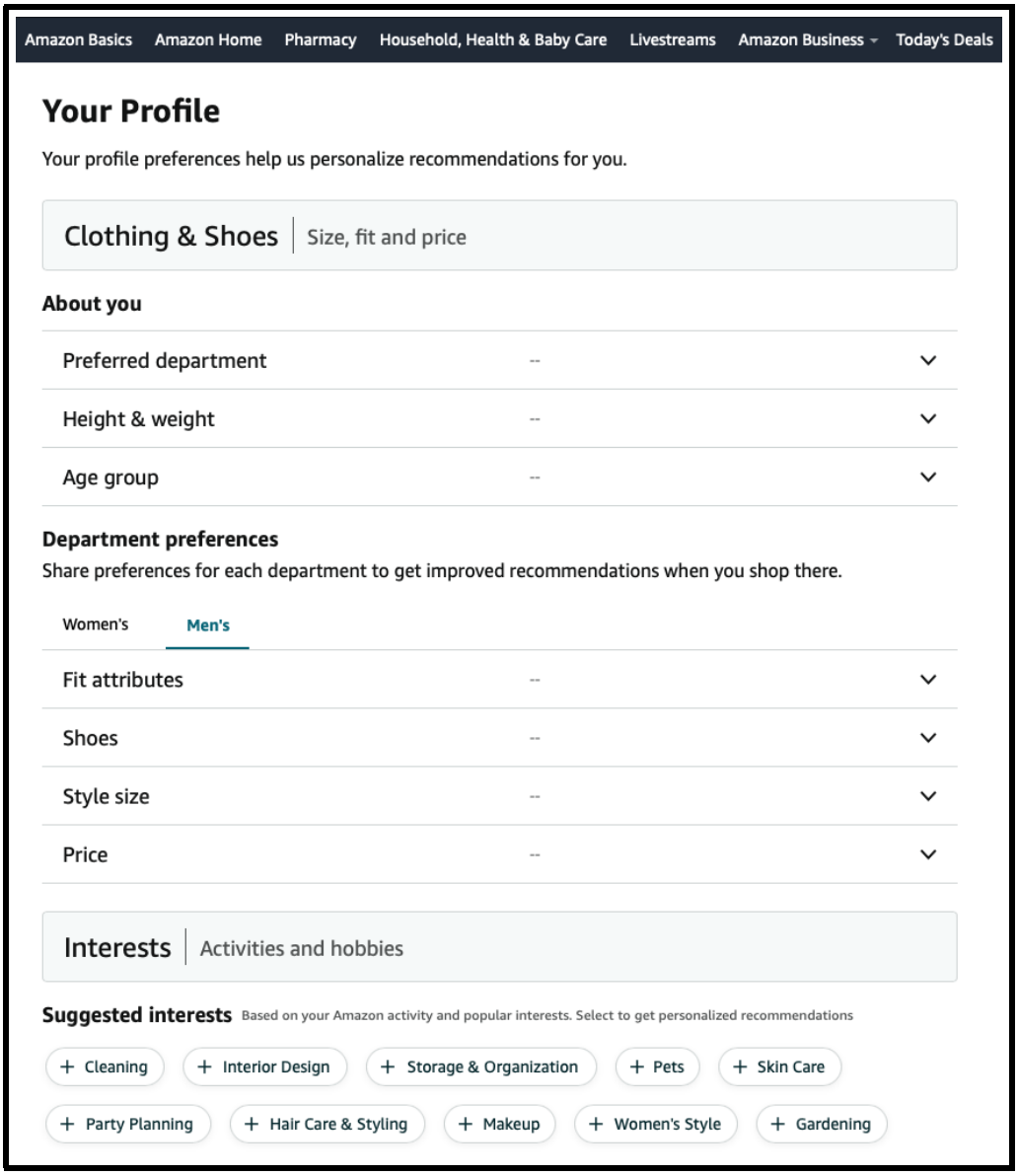


Figure G.2 Amazon's self-disclosure interface that allows users, in full extent, to enter, edit, and remove preference/interest information used for personalization. The screenshot illustrates a self-disclosure mechanism in which the user can disclose her preference to the platform not necessarily truthfully by curating what type/preferences she appears as, and thereby shape what the platform learns and uses for recommendations. This connects to, and motivates our opt-out and self-disclosure benchmarks and the interpretation of user-facing settings as mechanisms that shape what the platform observes or learns about preferences.

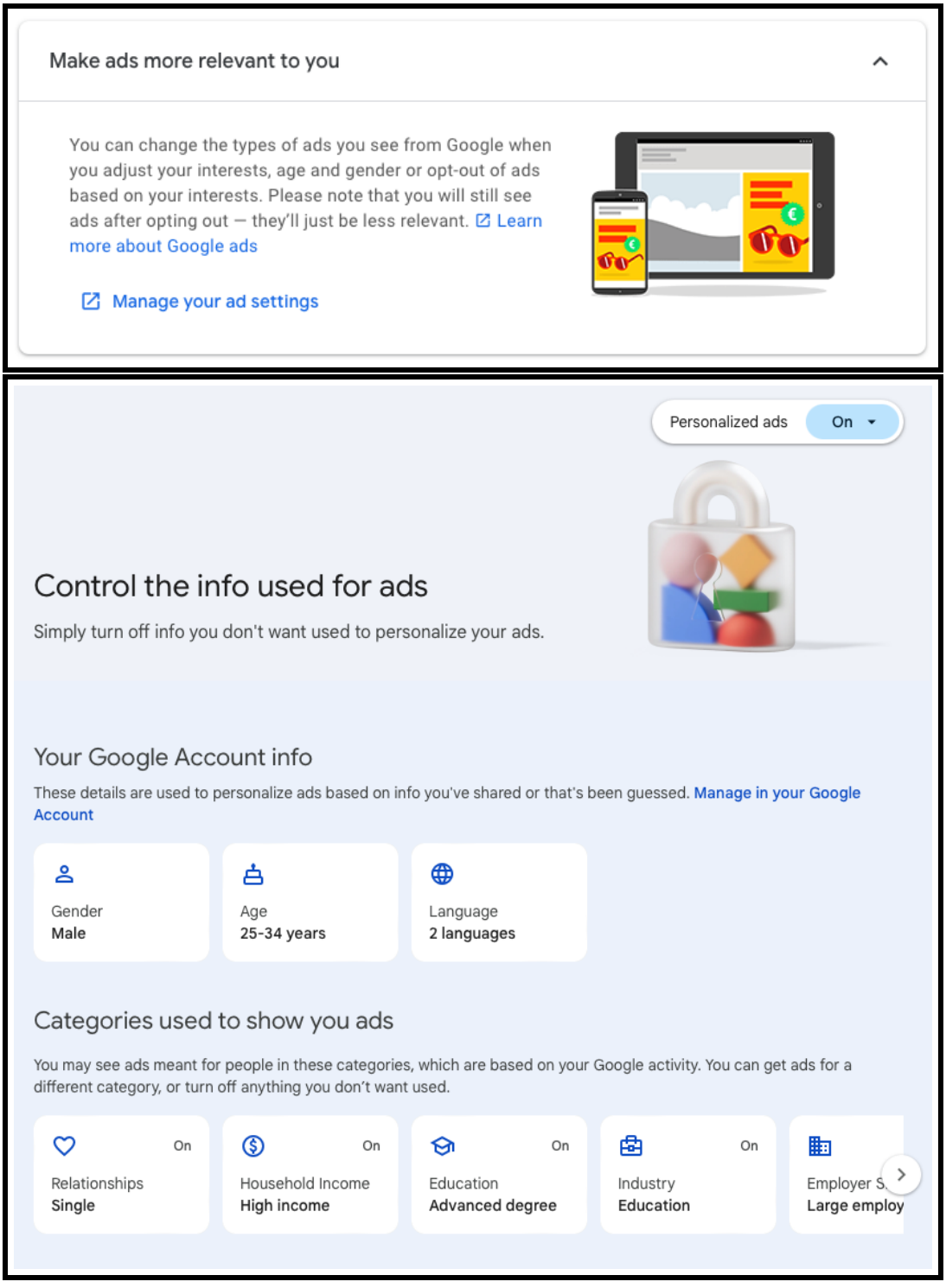


Figure G.3 **Google's communication to users about how personal information affects personalization and what may change if the user restricts such use. The interface makes explicit that privacy choices can alter what is shown and how content is targeted, consistent with viewing a platform's recommender policy as specifying whether and how often to recommend items conditional on observed/reported preferences. This figure is intended as a practice-motivating example for the Introduction where we interpret privacy-policy statements as the platform committing to and communicating aspects of its recommender policy.**

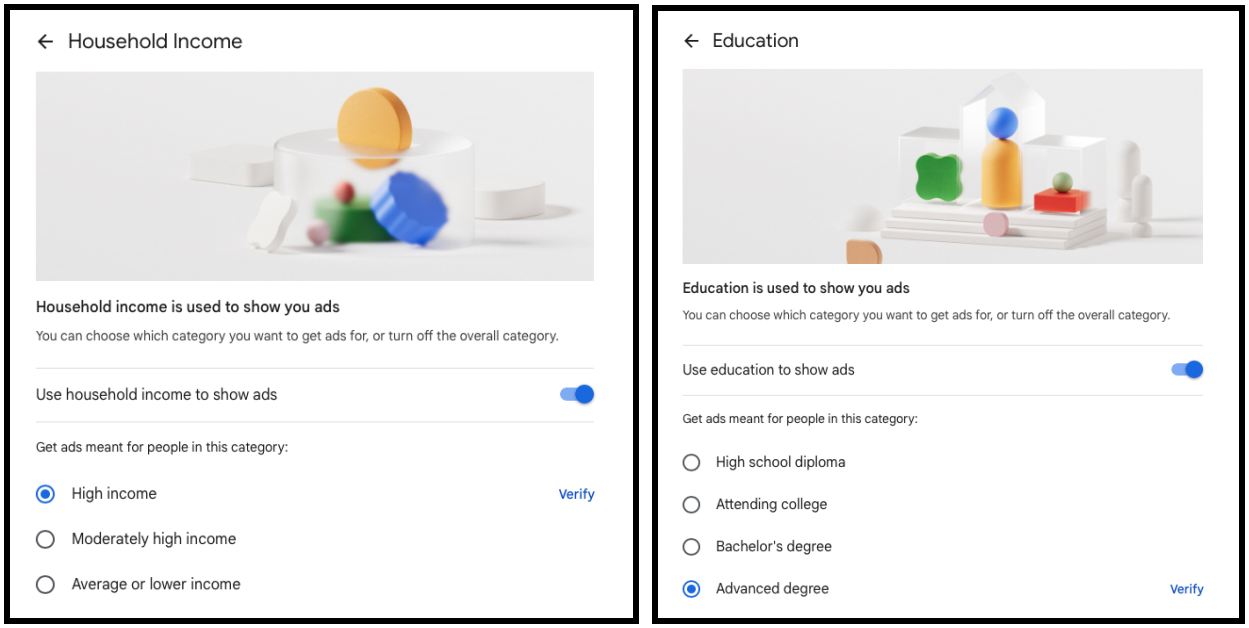


Figure G.4 **Google's self-disclosure interface that allows users to enter, edit, and remove inferred preference/interest categories used for personalization (reachable from the Google Account page shown in Figure G.3). The screenshot illustrates a self-disclosure mechanism in which the user can disclose her preference to the platform not necessarily truthfully by curating what type/preferences she appears as, as well as an option to cease personalized ads entirely. This figure is intended to justify our modeling selection of intermediate privacy regimes between unprotected privacy and non-disclosure protection, i.e., the opt-out protection (choosing whether to share one's true preference) and the self-disclosure protection (choosing what preference signal to provide, potentially strategically).**