Online Appendix for: Lemon Cycles

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A Proof of Propositions and Lemmas

A.1 Proof of Lemma 1

Note that the entrepreneur's optimization problem is:

$$y_i \equiv \max_{\ell_i, b_i} \quad z_i \left(k_i - \ell_i + \rho b_i\right) - p\left(b_i - \ell_i\right) = z_i k_i + (\rho z_i - p) b_i + (p - z_i)\ell_i$$

which is a linear function of (ℓ_i, b_i), then we immediately get the policy functions:

$$(b_i, \ell_i) = \begin{cases} (0, k_i), & \text{if } z_i < \underline{z} \\ (0, 0), & \text{if } z_i \in (\underline{z}, \overline{z}) \\ \left(\frac{\phi}{1-\phi} k_i, 0\right) & \text{if } z_i \geq \overline{z} \end{cases}.$$

And the two cutoffs are $(\underline{z}, \bar{z}) \equiv (p, p/\rho)$.

A.2 Proof of Lemma 2

As $z \sim U[0,1]$, equation (7) can be rewritten as:

$$\frac{\phi}{1-\phi} \frac{1-\bar{z}}{\bar{z}} = 1,$$

and the asset quality is given by

$$\rho = \frac{\underline{z}}{\underline{z} + \chi}$$

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so we immediately get $\bar{z} = \phi$, then from $\bar{z} = \underline{z}/\rho$ we get $\underline{z} = \phi - \chi = p$ and $\rho = \frac{\phi - \chi}{\phi}$. If $\phi > \chi$, the capital market exists, then the previous results holds. If $\phi < \chi$, the market collapses, and $p = 0$, ρ =0.

A.3 Proof of Lemma 3

When the capital market exists, capital of firms with $z \leq \underline{z}$ are sold to those with $z \geq \bar{z}$. Capital of firms with $\underline{z} < z < \bar{z}$ will be used on their own, so the aggregate productivity is:

$$Z = (F(\underline{z}) + 1 - F(\bar{z}))\,\mathbb{E}[z|z \geq \bar{z}] + (F(\bar{z}) - F(\underline{z}))\,\mathbb{E}[z|\underline{z} \leq z < \bar{z}]$$

Plug in the results of $\underline{z}$ and $\bar{z}$, and we immediately get:

$$Z = \frac{1 + (1+\chi)\max\{\phi - \chi, 0\}}{2},$$

and the total capital reallocated is related to p or $\underline{z}$.

A.4 Proof of Lemma 4 and Proposition 1

Note that both the productive firms and lemon firms have log utility, so their saving rates are the same and are given by β. For the productive firms, the income in each period is given by:

$$\left[p(\chi) + \max\{z_i - p(\chi), 0\} + \frac{\phi}{1-\phi}\max\{z_i\rho(\chi) - p(\chi), 0\} + (1-\delta)\right] k,$$

and with a saving rate β, the capital stock in the next period is:

$$k' = \beta\left[p(\chi) + \max\{z_i - p(\chi), 0\} + \frac{\phi}{1-\phi}\max\{z_i\rho(\chi) - p(\chi), 0\} + (1-\delta)\right] k.$$

The analysis for the lemon firms are the same, and the income for lemon firms is $[1 - \delta + \mu p(\chi)]x$, so the lemon stock in the next period is given by:

$$x' = \beta[1 - \delta + \mu p(\chi)]x.$$

By integrating the individual firms' decisions we get the aggregate law of motion:

$$K'(\chi) = \beta\left[Z(\chi) - p(\chi)\chi + 1 - \delta\right] K,$$

$$X'(\chi) = \beta[1 - \delta + \mu p(\chi)]X.$$

Then by dividing the two we get the one-equation law of motion that captures the dynamic of the economy:

$$\chi_{t+1} \equiv \Gamma(\chi_t) = \begin{cases} \frac{\mu(\phi-\chi_t)+1-\delta}{(1+(\phi-\chi_t)(1-\chi_t))/2+1-\delta}\chi_t, & \text{if } \chi_t < \phi \\ \frac{1-\delta}{1/2+(1-\delta)}\chi_t, & \text{otherwise.} \end{cases}.$$

A.5 Proof of Lemma 5

From the law of motion we know that when $\chi_t \geq \phi$ there cannot be any steady state with $\chi^* > 0$, so we only consider the case $\chi^* < \phi$. Then the steady state χ^* is determined by $\Gamma(\chi^*) = \chi^*$, that is:

$$\frac{\mu(\phi-\chi^*)+1-\delta}{(1+(\phi-\chi^*)(1-\chi^*))/2+1-\delta} = 1,$$

which can be rearranged as:

$$\mu(\phi-\chi^*)+1-\delta = (1+(\phi-\chi^*)(1-\chi^*))/2+1-\delta \Rightarrow g(\chi^*) = 0,$$

where $g(\chi) = \chi^2 - (1+\phi-2\mu)\chi + 1 - (2\mu-1)\phi$ is a quadratic equation of χ^*. Note that when $\chi^* = \phi$ we have $g(\phi) = 1 > 0$. To ensure that there is a solution for χ^* we need

$$(\phi+1-2\mu)^2 - 4[1+(1-2\mu)\phi] > 0 \Rightarrow (\phi-2\mu+1)^2 > 4.$$

Obviously, there are two cases. In the first case, we have $\phi-2\mu+1 > 2$ or $\phi > 2\mu-1$. Since we assume that $\mu > 1$ and $\phi < 1$, this cannot be the case. As a result, we need $\phi-2\mu+1 < -2$, or $\phi < 2\mu-3$. In this case, since $\frac{\phi-2\mu+1}{2} < 0$, there will be at most one positive solution for χ^*. To ensure a solution of $0 < \chi^* < \phi$ we just need

$$g(0) = 1-(2\mu-1)\phi < 0,$$

so that $\phi(2\mu-1) > 1$.

In general, we can show that the derivative of $\Gamma(\chi)$ on χ is given by:

$$\Gamma'(\chi^*) = 1 - \frac{\chi^* - \frac{1+\phi}{2} + \mu}{\mu(\phi-\chi^*)+1-\delta}.$$

so as long as $(2\mu+1)\chi^* > \mu(2\phi-1) + 2(1-\delta) + \frac{1+\phi}{2}$, where

$$\chi^* = \frac{1+\phi-2\mu+\sqrt{(\phi-2\mu+1)^2-4}}{2},$$

we will get $\Gamma'(\chi^*) < -1$, where the steady state becomes unstable.

What's more we can prove that the first order derivation of Γ at χ^* satisfies:

$$\frac{\partial \Gamma'(\chi^*)}{\partial \phi} = \frac{\left(\mu\left(\phi-\chi^*\right)+1-\delta\right)\left(\frac{1}{2}-\frac{1-\chi-2\mu}{1+\phi-2\chi-2\mu}\right)-\mu\left(1-\frac{1-\chi-2\mu}{1+\phi-2\chi-2\mu}\right)\left(\frac{1}{2}\left(1+\phi\right)-\chi-\mu\right)}{\left(\mu\left(\phi-\chi^*\right)+1-\delta\right)^2},$$

the numerator is

$$\begin{aligned} & \left(\mu\left(\phi-\chi^*\right)+1-\delta\right)\left(\frac{\frac{1}{2}\left(1+\phi\right)-1+\mu}{1+\phi-2\chi-2\mu}\right)-\mu\left(\frac{\phi-\chi}{1+\phi-2\chi-2\mu}\right)\left(\frac{1}{2}\left(1+\phi\right)-\chi-\mu\right) \\ = & \frac{1}{1+\phi-2\chi-2\mu}\left[\mu\left(\phi-\chi\right)\left(2\mu+\chi-1\right)+\left(1-\delta\right)\left(\frac{1}{2}\left(1+\phi\right)-1+\mu\right)\right], \end{aligned}$$

$\phi\left(2\mu-1\right) > 1$ implies that the term in bracket is positive. Hence we just need to show $(1+\phi-2\chi-2\mu) < 0$ to establish that $\frac{\partial\Gamma'(\chi^*)}{\partial\phi} < 0$. We note that since $\phi < 1$, we have $2\mu - 1 > 1$. Hence

$$1+\phi-2\chi-2\mu < \phi-2\chi-1 < 0.$$

So as ϕ increases $\Gamma'(\chi^*)$ decreases, and when ϕ is so large that $\Gamma'(\chi^*) < -1$, the interior steady state becomes unstable and there may be endogenous cycles.

A.6 Proof of Proposition 2

Let $Y\left(\chi,p\right) \equiv Z\left(\chi,p\right) - p\chi = 1/2 - p^2/2 + p - p^2(1-\phi)/2\phi - p\chi\left(1+(1-\phi)/2\phi\right)$, $a = 1-\delta$. Then

$$V\left(\chi\right) = \max_p \frac{1}{1-\beta}\ln\left(Y\left(\chi,p\right)+a\right)+\beta V\left(\chi'\right),$$

subject to:

$$\chi' = \frac{\mu p + a}{Y\left(\chi,p\right)+a}\chi.$$

Then we get

$$\frac{\partial\chi'}{\partial p} = \chi\frac{\mu\left(Y\left(\chi,p\right)+a\right)-\frac{\partial Y}{\partial p}\left(\mu p+a\right)}{\left(Y\left(\chi,p\right)+a\right)^2},$$

and

$$\frac{\partial\chi'}{\partial\chi} = \frac{\mu p+a}{Y\left(\chi,p\right)+a}-\chi\frac{\partial Y}{\partial\chi}\frac{\left(\mu p+a\right)}{\left(Y\left(\chi,p\right)+a\right)^2}.$$

So FOC is:

$$\frac{1}{1-\beta}\frac{\partial Y/\partial p}{Y+a}+\beta V'\left(\chi'\right)\frac{\partial\chi'}{\partial p} = 0.$$

Envelope theorem gives:

$$V'(\chi) = \frac{1}{1-\beta}\frac{\partial Y/\partial\chi}{Y+a} + \beta V'(\chi')\frac{\partial\chi'}{\partial\chi}.$$

In a steady state, $V'(\chi) = V'(\chi')$ hence

$$V'(\chi) = \frac{1}{1-\beta}\frac{\partial Y/\partial\chi}{Y+a}\bigg/\left(1-\beta\frac{\partial\chi'}{\partial\chi}\right).$$

Substitute into FOC, we get:

$$\frac{\partial Y}{\partial p}\left(1-\beta\frac{\partial\chi'}{\partial\chi}\right) + \beta\frac{\partial Y}{\partial\chi}\frac{\partial\chi'}{\partial p} = 0,$$

Hence

$$\begin{aligned}
0 &= \frac{\partial Y}{\partial p}\left(1-\beta\frac{\mu p+a}{Y(\chi,p)+a} + \beta\chi\frac{\partial Y}{\partial\chi}\frac{\mu p+a}{(Y(\chi,p)+a)^2}\right) + \beta\frac{\partial Y}{\partial\chi}\chi\frac{\mu(Y(\chi,p)+a) - \frac{\partial Y}{\partial p}(\mu p+a)}{(Y(\chi,p)+a)^2} \\
&= \frac{\partial Y}{\partial p}\left(1-\beta\frac{\mu p+a}{Y(\chi,p)+a}\right) + \frac{\partial Y}{\partial\chi}\frac{\beta\mu\chi}{Y(\chi,p)+a}
\end{aligned}$$

Now use the fact that

$$\frac{\partial Y}{\partial p} = 1 - p\phi^{-1} - \chi\frac{1+\phi}{2\phi}$$

$$\frac{\partial Y}{\partial\chi} = -p\frac{1+\phi}{2\phi}$$

We have

$$\left(1-p\phi^{-1}-\chi\frac{1+\phi}{2\phi}\right)(Y(\chi,p)+a-\beta(\mu p+a)) = p\frac{1+\phi}{2\phi}\beta\mu\chi$$

$$\left(\phi-p-\chi\frac{1+\phi}{2}\right)(Y(\chi,p)+a-\beta(\mu p+a)) = \beta p\mu\frac{1+\phi}{2}\chi$$

$$(\phi-p)(Y(\chi,p)+a-\beta(\mu p+a)) = \chi\frac{1+\phi}{2}(Y(\chi,p)+(1-\beta)a)$$

$$\begin{aligned}
p &= \phi-\chi\frac{1+\phi}{2}\frac{Y(\chi,p)+(1-\beta)a}{Y(\chi,p)+(1-\beta)a-\beta\mu p} \\
&= \phi-\chi\frac{1+\phi}{2}\left(1+\frac{\beta\mu p}{Y(\chi,p)+(1-\beta)(1-\delta)-\beta\mu p}\right)
\end{aligned}$$

Taking a first-order expansion around the steady state gives rise to the linearized policy function in the proposition. We derive the linearized policy function in Appendix A.7 below. Using implicit function theorem, we can show that the law of motion of the state variable $\chi' = \Gamma(\chi)$ features a positive slope at the steady state, thereby confirming that the planner's solution does not feature cycles.

A.7 With Pareto Weight

In the above analysis, we calculated the social planner's solution when concerning only the welfare of the entrepreneurs. Now consider a more generally setting where the social planner gives a Pareto weight α to the rent seekers, and $1-\alpha$ to the entrepreneurs. The social planner's maximization problem is:

$$V^p(\chi, K) = \max_p \left\{ (1-\alpha)\ln\left[(1-\beta)(Y+a)K\right] + \alpha \ln\left[(1-\beta)\frac{\mu p + a}{\mu}\chi K\right] + \beta V^p(\chi', K') \right\}$$

subject to the two constraints. The first term $\ln[(1-\beta)(Y+a)K]$ corresponds to welfare of the producers, where $Y = Y(\chi, p) \equiv Z(\chi, p) - p\chi$ and $a = 1-\delta$. The term $\ln\left[(1-\beta)\frac{\mu p+a}{\mu}\chi K\right]$ denotes the welfare of the rent seekers. Note that the budget constraint of the rent seekers is:

$$\frac{X' - (1-\delta)X}{\mu} = p(\chi)X - C^L$$

where $X' = \beta\left[\mu p + 1 - \delta\right]X$ so the aggregate consumption of the rent-seekers is $C^L = \frac{\mu p X - \beta[\mu p + 1 - \delta]X + (1-\delta)X}{\mu} =$ $(1-\beta)\frac{\mu p+a}{\mu}\chi K$. Note that here we still find that the value function is log-linear in K, so that $V^p(\chi, K) = v(\chi) + b\ln K$, where b is a constant to be determined. Using this guess-and-verify we can rewite the social planner's value function as (note that here we drop the maximization symbol):

$$\begin{aligned} v(\chi) + b\ln K = \max_p (1-\alpha)\ln(Y+a) + (1-\alpha)\ln K + \alpha\ln\left(\frac{\mu p + a}{\mu}\chi\right) + \alpha \ln K \\ + \beta v(\chi') + \beta b \ln(\beta(Y+a)) + b\beta \ln K + \ln(1-\beta) \end{aligned}$$

Note that we can drop all the constant terms. By comparing coefficients we get that $b = \frac{1}{1-\beta}$, and that $v(\chi)$ satisfies:

$$\begin{aligned} v(\chi) &= \max_p (1-\alpha)\ln(Y+a) + \alpha\ln\left[(\mu p + a)\chi\right] + \frac{\beta}{1-\beta}\ln(Y+a) + \beta v(\chi') \\ &= \max_p \frac{1}{1-\beta}\ln(Y+a) + \alpha\ln\left(\frac{\mu p + a}{Y+a}\chi\right) + \beta v(\chi') \\ &= \max_p \frac{1}{1-\beta}\ln(Y+a) + \alpha\ln\chi' + \beta v(\chi') \end{aligned}$$

The first order condition is:

$$\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial p} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial p} + \beta v'(\chi')\frac{\partial \chi'}{\partial p}$$

Envelope theorem gives:

$$v'(\chi) = \frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial \chi} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial \chi} + \beta v'(\chi')\frac{\partial \chi'}{\partial \chi} \Rightarrow v'(\chi) = \left[\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial \chi} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial \chi}\right] \Big/ \left(1-\beta\frac{\partial \chi'}{\partial \chi}\right)$$

So that the first order condition around the steady state $\chi = \chi'$ is

$$\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial p} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial p} + \beta\frac{\partial \chi'}{\partial p}\left[\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial \chi} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial \chi}\right] \Big/ \left(1-\beta\frac{\partial \chi'}{\partial \chi}\right) = 0$$

$$\left(\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial p} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial p}\right)\left(1-\beta\frac{\partial \chi'}{\partial \chi}\right) + \beta\frac{\partial \chi'}{\partial p}\left[\frac{1}{1-\beta}\frac{1}{Y+a}\frac{\partial Y}{\partial \chi} + \frac{\alpha}{\chi'}\frac{\partial \chi'}{\partial \chi}\right] = 0$$

Note that $\frac{\partial \chi'}{\partial \chi} = \frac{\mu p+a}{Y+a} - \chi\frac{\partial Y}{\partial \chi}\frac{\mu p+a}{(Y+a)^2}$ and $\frac{\partial \chi'}{\partial p} = \chi\frac{\mu(Y+a)-\frac{\partial Y}{\partial p}(\mu p+a)}{(Y+a)^2}$, so we get:

$$\frac{1}{1-\beta}\frac{1}{Y+a}\left[\frac{\partial Y}{\partial p}\left(1-\beta\frac{\mu p+a}{Y+a} + \beta\chi\frac{\partial Y}{\partial \chi}\frac{\mu p+a}{(Y+a)^2}\right) + \beta\frac{\partial Y}{\partial \chi}\chi\frac{\mu(Y+a)-\frac{\partial Y}{\partial p}(\mu p+a)}{(Y+a)^2}\right] + \frac{\alpha}{\chi'}\left[\chi\frac{\mu(Y+a)-\frac{\partial Y}{\partial p}(\mu p+a)}{(Y+a)^2}\right] = 0$$

which can be simplified to

$$\frac{1}{1-\beta}\left[\frac{\partial Y}{\partial p}\left(1-\beta\frac{\mu p+a}{Y+a}\right) + \beta\frac{\partial Y}{\partial \chi}\chi\frac{\mu}{Y+a}\right] + \alpha\frac{\mu(Y+a)-\frac{\partial Y}{\partial p}(\mu p+a)}{\mu p+a} = 0$$

$$\frac{\partial Y}{\partial p}\left[1-\alpha(1-\beta)-\beta\frac{\mu p+a}{Y+a}\right] + \frac{\partial Y}{\partial \chi}\frac{\beta\mu\chi}{Y+a} + \alpha\mu(1-\beta)\frac{Y+a}{\mu p+a} = 0$$

where $\frac{\partial Y}{\partial p} = 1 - p\phi^{-1} - \chi\frac{1+\phi}{2\phi}$ and $\frac{\partial Y}{\partial \chi} = -p\frac{1+\phi}{2\phi}$.

$$\left(1-\frac{p}{\phi}-\chi\frac{1+\phi}{2\phi}\right)\left[1-\alpha(1-\beta)-\beta\frac{\mu p+a}{Y+a}\right] - p\frac{1+\phi}{2\phi}\frac{\beta\mu\chi}{Y+a} + \alpha\mu(1-\beta)\frac{Y+a}{\mu p+a} = 0$$

$$\left(\phi-p-\frac{1+\phi}{2}\chi\right)\left[1-\alpha(1-\beta)-\beta\frac{\mu p+a}{Y+a}\right] - \frac{1+\phi}{2}\chi\frac{\beta\mu p}{Y+a} + \alpha\phi\mu(1-\beta)\frac{Y+a}{\mu p+a} = 0$$

$$p = \phi - \frac{1+\phi}{2}\chi\left(1+\frac{\beta\mu p}{(1-\alpha+\alpha\beta)(Y+a)-\beta(\mu p+a)}\right) + \frac{\alpha\phi\mu(1-\beta)(Y+a)^2/(\mu p+a)}{(1-\alpha+\alpha\beta)(Y+a)-\beta(\mu p+a)} \tag{A1}$$

Note that if $\alpha = 0$, then the problem comes back to our baseline case.

We first consider the steady state:

$$\left(\phi - p - \frac{1+\phi}{2}\chi\right)(1-\beta)(1-\alpha) - \frac{1+\phi}{2}\chi\frac{\beta\mu p}{\mu p + a} + \alpha\phi\mu(1-\beta) = 0$$

$$(\phi - p)(1-\beta)(1-\alpha) - \frac{1+\phi}{2}\chi\left[(1-\beta)(1-\alpha) + \beta\frac{\mu p}{\mu p + a}\right] + \alpha\phi\mu(1-\beta) = 0$$

$$\begin{aligned} p &= \phi - \frac{1+\phi}{2}\chi\left[1 + \frac{\beta}{(1-\beta)(1-\alpha)}\frac{\mu p}{\mu p + a}\right] + \frac{\alpha\mu}{1-\alpha}\phi \\ &= \frac{1-\alpha(1-\mu)}{1-\alpha}\phi - \frac{1+\phi}{2}\chi\left[1 + \frac{\beta}{(1-\beta)(1-\alpha)}\frac{\mu p}{\mu p + a}\right] \end{aligned}$$

Next we linearize (A1).

$$\begin{aligned} \frac{\partial p}{\partial \chi} =& -\frac{1+\phi}{2}\left(1 + \frac{\beta\mu p^*}{(1-\alpha)(1-\beta)(\mu p^* + a)}\right) \\ & - \frac{1+\phi}{2}\chi^*\frac{\beta\mu\frac{\partial p}{\partial \chi}(1-\alpha)(1-\beta)(\mu p^* + a) - \beta\mu p^*\left[(1-\alpha+\alpha\beta)\frac{\partial Y}{\partial \chi} - \beta\mu\frac{\partial p}{\partial \chi}\right]}{(1-\alpha)^2(1-\beta)^2(\mu p^* + a)^2} \\ & + \frac{\alpha\phi\mu(1-\beta)\frac{2(\mu p^*+a)^2\frac{\partial Y}{\partial \chi} - \mu\frac{\partial p}{\partial \chi}(\mu p^*+a)^2}{(\mu p^*+a)^2}}{(1-\alpha)(1-\beta)(\mu p^* + a)} - \frac{\alpha\phi\mu(1-\beta)(\mu p^* + a)\left[(1-\alpha+\alpha\beta)\frac{\partial Y}{\partial \chi} - \beta\mu\frac{\partial p}{\partial \chi}\right]}{(1-\alpha)^2(1-\beta)^2(\mu p^* + a)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial p}{\partial \chi} =& -\frac{1+\phi}{2}\left(1 + \frac{\beta\mu p^*}{(1-\alpha)(1-\beta)(\mu p^* + a)}\right) - \frac{1+\phi}{2}\chi^*\frac{\beta\mu}{(1-\alpha)(1-\beta)(\mu p^* + a)}\frac{\partial p}{\partial \chi} \\ & + \frac{1+\phi}{2}\chi^*\frac{\beta\mu p^*}{(1-\alpha)^2(1-\beta)^2(\mu p^* + a)^2}\left[(1-\alpha+\alpha\beta)\frac{\partial Y}{\partial \chi} - \beta\mu\frac{\partial p}{\partial \chi}\right] \\ & + \frac{\alpha\phi\mu}{(1-\alpha)(\mu p^* + a)}\left(2\frac{\partial Y}{\partial \chi} - \mu\frac{\partial p}{\partial \chi}\right) - \frac{\alpha\phi\mu\left[(1-\alpha+\alpha\beta)\frac{\partial Y}{\partial \chi} - \beta\mu\frac{\partial p}{\partial \chi}\right]}{(1-\alpha)^2(1-\beta)(\mu p^* + a)} \end{aligned}$$

where $\frac{\partial Y}{\partial \chi} = Y_2\frac{\partial p}{\partial \chi} + Y_1$ with $Y_2 = -p + 1 - 2p(1-\phi)/2\phi - \chi\left(1 + (1-\phi)/2\phi\right)$, $Y_1 = -p\left(1 + (1-\phi)/2\phi\right)$, define

$$\Xi = \frac{1+\phi}{2}\chi^*\frac{\beta\mu}{(1-\alpha)(1-\beta)(\mu p^* + a)} + \frac{\alpha\phi\mu^2}{(1-\alpha)(\mu p^* + a)}$$

$$\Pi = \frac{1+\phi}{2}\chi^*\frac{\beta\mu p^*}{(1-\alpha)^2(1-\beta)^2(\mu p^* + a)^2} - \frac{\alpha\phi\mu}{(1-\alpha)^2(1-\beta)(\mu p^* + a)}$$

$$\Upsilon = \frac{2\alpha\phi\mu}{(1-\alpha)(\mu p^* + a)}$$

Then the above equation can be simplified to:

$$\frac{\partial p}{\partial \chi} = -\frac{1+\phi}{2}\left(1+\frac{\beta\mu p^*}{(1-\alpha)(1-\beta)(\mu p^*+a)}\right) - \Xi\frac{\partial p}{\partial \chi}$$
$$+\Pi\left[(1-\alpha+\alpha\beta)\left(Y_2\frac{\partial p}{\partial \chi}+Y_1\right)-\beta\mu\frac{\partial p}{\partial \chi}\right]+\Upsilon\left(Y_2\frac{\partial p}{\partial \chi}+Y_1\right)$$

which gives:

$$\frac{\partial p}{\partial \chi} = \frac{\mathcal{A}}{\mathcal{B}}$$

where

$$\mathcal{A} = -\frac{1+\phi}{2}\left(1+\frac{\beta\mu p^*}{(1-\alpha)(1-\beta)(\mu p^*+a)}\right) - \left[\Pi(1-\alpha+\alpha\beta)+\Upsilon\right]Y_1$$

$$\mathcal{B} = 1+\Xi-\left[\Pi(1-\alpha+\alpha\beta)+\Upsilon\right]Y_2+\Pi\beta\mu$$

B General Characterization

B.1 Non-Zero Minimum Productivity Level

We have so far used the uniform distribution $[0,1]$ and production function $y_i = z_i k_i$ to make sense of the key mechanism of the paper. Now we generalize the analysis by assuming $z_{\min} > 0$ and $y_i = Az_ik_i$, where A is an aggregate productivity shock. In this scenario, the market-clearing condition for the lemon asset (6) may have two solutions. To distinguish the two solutions, we assume that the net demand of lemon assets (which is characterized in Lemma 6) is mazimized at z^*,and $z^{**} = z^{**}(z^*;\phi)$ is an implicit function of z^* which is defined in equation (7).Then we have $\lim_{z^*\to z_{\min}} z^{**}(z^*;\phi) = z_{\max}$. Equation (6) can be rewritten as net demand and net supply of lemon assets, which is summarized in the following lemma.

Lemma 1. *If the market exists, there always exist multiple equilibria to $\underline{z}$, which satisfies*

$$\mathcal{D}(\underline{z}) = \chi, \tag{A2}$$

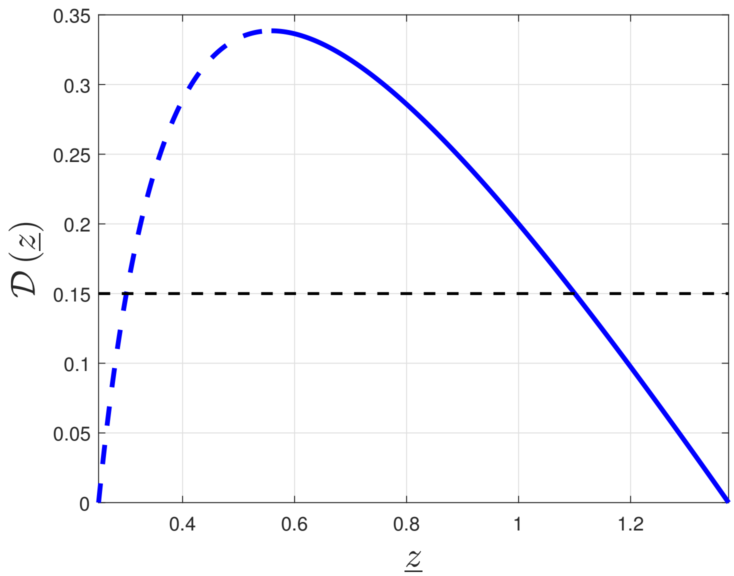
where the net demand of lemon assets is given by

$$\mathcal{D}(\underline{z}) \equiv \frac{\phi}{1-\phi}\left[1-F(\bar{z})\right]-F(\underline{z}), \tag{A3}$$

and $\bar{z}$ is an implicit function of $\underline{z}$ from equation (7).

Proof. Note that the capital market clear condition is given by:

Figure A.1. $\mathcal{D}\left(\underline{z}\right)=\chi$



$$\frac{\phi}{1-\phi}[1-F(\bar{z})]=F(\underline{z})+\chi$$

Then we immediately get the equation (A3). Then from the asset quality condition:

$$\rho=\frac{F(\underline{z})}{F(\underline{z})+\chi},$$

and that $\bar{z}=\underline{z}/\rho$, we can write $\bar{z}$ as a function of $\underline{z}$. □

Comments: If there were no adverse selection, i.e., $\chi=0$, and thus $\bar{z}=\underline{z}$. In that case, equation (A3) can be further written as

$$\mathcal{D}^{\star}\left(\underline{z}\right)=\frac{\phi}{1-\phi}-\left(1+\frac{\phi}{1-\phi}\right)F\left(\underline{z}\right)>\mathcal{D}\left(\underline{z}\right),$$

and the cutoff in equilibrium is determined by $\mathcal{D}^{\star}\left(\underline{z}\right)=0$. That is, $\underline{z}=z^{**}\left(\phi\right)=F^{-1}\left(\phi\right)$.

We consider *Pareto-dominant equilibrium,* and thus $\underline{z}\in\left(z^{*},z^{**}\right)\subset\left(z_{\min},z_{\max}\right)$, where

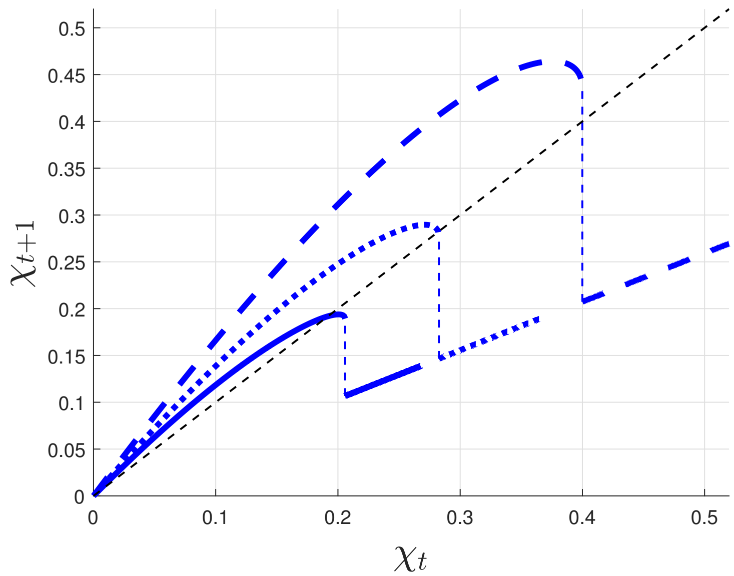
$$z^{*}=\arg\max_{z\in\left(z_{\min},z^{**}\right)}\mathcal{D}\left(z\right)=\mathcal{D}^{-1}\left(\overline{\chi}\right).$$

We can prove that $\underline{z}$ strictly increases with λ.

Proposition 1. *The market exists iff $\chi<\overline{\chi}$, where $\bar{\chi}$ is determined by*

$$\overline{\chi}=\max_{z\in\left(z_{\min},z^{**}\right)}\mathcal{D}\left(z\right)=\mathcal{D}\left(z^{*}\right),\tag{A4}$$

Figure A.2. Law of motion on χ_t



Notes: Law of motion on χ_t under different levels of ϕ.

with $\overline{\chi}$ increasing with ϕ. Then the asset quality and asset price are respectively given by

$$\rho = \begin{cases} \frac{F(\underline{z})}{F(\underline{z})+\chi}, & \text{if } \chi \leq \bar{\chi} \\ 0, & \text{if } \chi > \bar{\chi} \end{cases},$$

$$p = \begin{cases} A\underline{z}, & \text{if } \chi \leq \bar{\chi} \\ 0, & \text{if } \chi > \bar{\chi} \end{cases}.$$

Proposition 2. *The law of motion of χ_t is characterized as below.*

$$\chi_{t+1} = \Theta_t(\chi_t) = \begin{cases} \Theta_t^+(\chi_t), & \text{if } \chi_t \leq \overline{\chi}_t \\ \Theta_t^-(\chi_t), & \text{if } \chi_t > \overline{\chi}_t \end{cases},$$

where $\overline{\chi}$ is defined in equation (A4), and $\left\{\Theta_t^+(\chi_t), \Theta_t^-(\chi_t)\right\}$ are defined as

$$\Theta^+(\chi) = \frac{\mu p + (1-\delta)}{AZ - p\chi + (1-\delta)}\chi,$$

$$\Theta^-(\chi) = \frac{1-\delta}{A\mathbb{E}(z) + (1-\delta)}\chi.$$

Paradox of credit expansion: In the static environment, a higher ϕ always strengthen the risk capacity. The bright side of the market as insurance increases. The dark side is adverse selection and cream skimming. With a credit expansion, the high-productivity firms can borrow more to produce more, which in turn makes it more profitable for the lemon firms to transform capital to lemon, and the asset quality tends to decrease.

B.2 Unbounded Distribution of Productivity

In the baseline model, we restrict our analysis on bounded uniform distribution. In this section, we extend our model to analyze the results when the productivity z follows a log normal distribution $LN(\nu,\sigma)$ with CDF $F(z)$ and support $(0, \infty)$. In this case, in equilibrium we still have two cutoffs for productivity, $\underline{z} = p$ and $\bar{z} = p/\rho$ such that firms with $z_i < \underline{z}$ will refer to lend out their capital stock ($l_i = k_i$and $b_i = 0$), while firms with $z_i > \bar{z}$ will choose to borrow from the capital market ($l_i = 0$ and $b_i = \frac{\phi}{1-\phi}k_i$). In equilibrium, the credit market clearing condition implies that:

$$\frac{\phi}{1-\phi}[1 - F(\bar{z})] = F(\underline{z}) + \chi,$$

and the two cutoffs has the relationship:

$$\frac{\phi}{1-\phi}\frac{1 - F(\bar{z})}{\bar{z}} = \frac{F(\underline{z})}{\underline{z}},$$

where $\bar{z} = \frac{\underline{z}}{\rho} = \underline{z}\frac{F(\underline{z})+\chi}{F(\underline{z})}$. In Figure A.3, we show the supply and demand functions that define the equilibrium on the credit market. In the Figure, the black solid line denotes the supply of credit $\frac{1-\phi}{\phi}\frac{F(\underline{z})}{\underline{z}}$, and the three green lines denotes the demand functions $\frac{1-F(\bar{z})}{\bar{z}}$ with different levels of χ. In this case, we find that there may be two equilibrium levels of $\underline{z}$ given any level of χ. As χ increases, the demand function moves downwards, the high level equilibrium of $\underline{z}$ goes down, and the low level one goes up. In this part of analysis, we focus on the higher level steady state since it is Pareto dominant. Note that when χ is high enough, $\chi > \bar{\chi}$, the supply and demand functions may not intersect with each other, and the credit market may collapse.

Given the credit market equilibrium outcome, we follow the same procedure to characterize the aggregate outcome. By independence of productivity draws, the aggregate output is $Y = ZK$ where

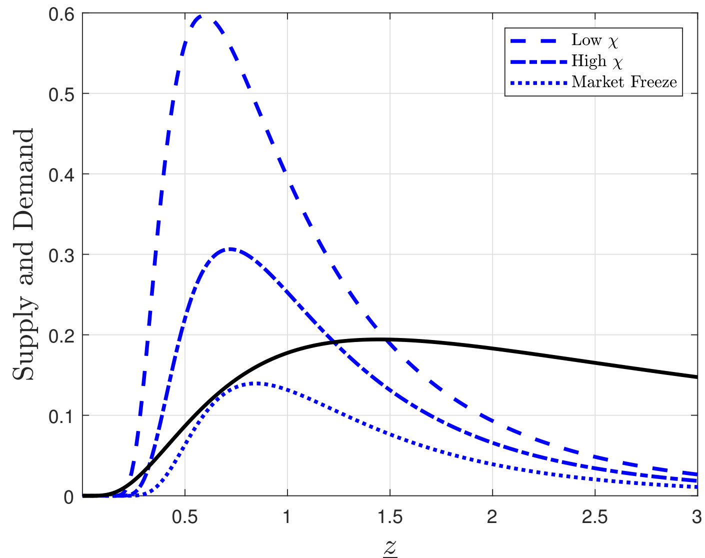
$$Z = (1 - F(\underline{z}))\mathbb{E}(z|z \geq \underline{z}) + F(\underline{z})\mathbb{E}(z|z \geq \bar{z}) = \int_{\underline{z}} z dF(z) + \frac{F(\underline{z})}{1 - F(\bar{z})}\int_{\bar{z}} z dF(z).$$

The dynamic optimization problems of the agents remain unchanged. From the above analysis, we can get the following results:

Proposition 3. *The credit market remains when $\chi_t < \bar{\chi}$, where $\bar{\chi}$is defined as:*

$$\bar{\chi} = \sup_{\chi}\left\{\min_{\underline{z}}\left[\frac{1-\phi}{\phi}\frac{F(\underline{z})}{\underline{z}} - \frac{1 - F(\bar{z}(\underline{z},\chi))}{\bar{z}(\underline{z},\chi)}\right] < 0\right\},$$

Figure A.3. Equilibrium on the Credit Market



and the law of motion of χ_t is characterized as below.

$$\chi_{t+1} = \Theta_t(\chi_t) = \begin{cases} \Theta_t^+(\chi_t), & \textit{if } \chi_t \leq \bar{\chi} \\ \Theta_t^-(\chi_t), & \textit{if } \chi_t > \bar{\chi} \end{cases},$$

where $\left\{\Theta_t^+(\chi_t), \Theta_t^-(\chi_t)\right\}$ *are defined as*

$$\begin{aligned} \Theta^+(\chi) &= \frac{\mu p + (1-\delta)}{Z - p\chi + (1-\delta)}\chi, \\ \Theta^-(\chi) &= \frac{1-\delta}{\mathbb{E}(z) + (1-\delta)}\chi, \end{aligned}$$

where:

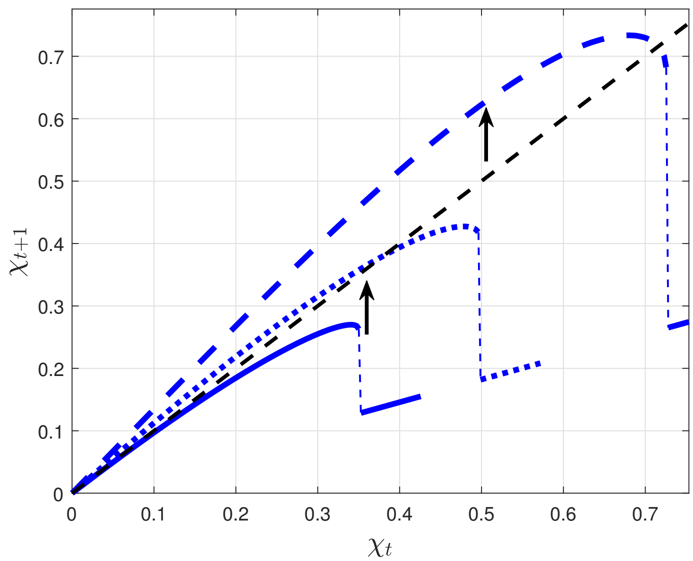
$$Z = (1 - F(\underline{z}))\mathbb{E}(z|z \geq \underline{z}) + F(\underline{z})\mathbb{E}(z|z \geq \bar{z}) = \int_{\underline{z}} z dF(z) + \frac{F(\underline{z})}{1-F(\bar{z})}\int_{\bar{z}} z dF(z).$$

In Figure A.4, we compare the law of motions of χ_t under different levels of ϕ. We find that in this case, as ϕ increases, the positive steady state of χ_t can turn from stable to unstable, which is consistent with the results in the baseline model and implies that determinant cycles may occur.

B.3 Labor Input in Production

In the baseline model, we assume that capital is the only input in production, and here we show a model extension with labor as an input. There are three types of agents: entrepreneurs, rent-seekers and workers. The first two types of agents owns productive and non-productive firms

Figure A.4. Law of motion of χ_t



respectively and make their own saving and production decisions, while workers just provide labor for production and are hand-to-mouth.

An entrepreneur i enters each period with initial productive asset stock k_i, draw an iid productivity $z_i \sim F$ and then decide whether to participate in the credit market to borrow or lend. The productive firms' production function is given by:

$$y_i = \mathcal{A}(z_i k_i)^{\alpha} n_i^{1-\alpha},$$

where $\mathcal{A} = A(ZK)^{1-\alpha}$denotes the aggregate productivity level, Z denotes the aggregate capital allocation efficiency which will be defined later, and K denotes the aggregate productive capital stock. Here we follow Romer (1986) to assume that there is a learning-by-doing effect in the productive sector, and the strength of such effect is determined by the *effective capital stock*, ZK. Assume that the wage rate is W, the intra-temporal optimal labor hiring problem of the productive firms is given by:

$$\pi_i = \max_{n_i} \mathcal{A}(z_i k_i)^{\alpha} n_i^{1-\alpha} - W n_i \Rightarrow \pi_i = R z_i k_i,$$

where $R = \alpha \left(\frac{1-\alpha}{W}\right)^{\frac{1}{\alpha}-1} \mathcal{A}^{\frac{1}{\alpha}}$. The credit market remains the same as in the baseline model, where the pooling capital price is given by p. If a productive firm with capital stock k_i borrows b_i from and lend l_i on the credit market, its net return will be $R z_i (k_i - l_i + \rho b_i) - p(b_i - l_i)$. In this case, it is obvious to show that the productive firm's decision to participate in the credit market still follows a threshold policy:

Lemma 2. *A productive firm's decision to participate in the credit market depends on two cutoffs* $(\underline{z}, \bar{z}) \equiv (p, p/\rho)$*, such that*

$$(b_i, \ell_i) = \begin{cases} (0, k_i), & \text{if } z_i < \underline{z} \\ (0, 0), & \text{if } z_i \in (\underline{z}, \overline{z}) \\ \left(\frac{\phi}{1-\phi} k_i, 0\right) & \text{if } z_i \geq \overline{z} \end{cases},$$

and the cutoff productivity level $\underline{z}$ *is determined by* $\underline{z} = p/R$.

By aggregating the decision of individual productive firms, we can solve for the equilibrium output and equilibrium capital price:

Proposition 4. *The aggregate output is given by:*

$$Y = \mathcal{A}(ZK)^{\alpha} N^{1-\alpha} = AZK,$$

where we normalize the aggregate labor supply to $N = 1$ *and the equilibrium capital allocation efficiency* Z *is defined as:*

$$Z = (1 - F(\underline{z}))\, \mathbb{E}[z|z \geq \underline{z}] + F(\underline{z})\, \mathbb{E}[z|z \geq \bar{z}],$$

when the credit market is functioning, and $Z = \mathbb{E}(z)$ *when the credit market freezes. In equilibrium, the capital return satisfies* $R = \frac{\alpha Y}{ZK} = \alpha A$ *and capital price equals* $p = \underline{z}R = \alpha A \underline{z}$.

From the above analysis, we find that when labor is used in production and there is a scale effect in final production, we can still rewrite the final output as a linear function of the aggregate effective capital stock, which is analogous to our baseline model except for that not all output is taken by the productive firms. Since we assume that the workers are hand-to-mouth, and that the intertemporal optimization problems are the same for entrepreneurs and rent-seekers as in the baseline model, we can get the following result about the equilibrium dynamics:

Proposition 5. *Aggregate capital evolves according to:*

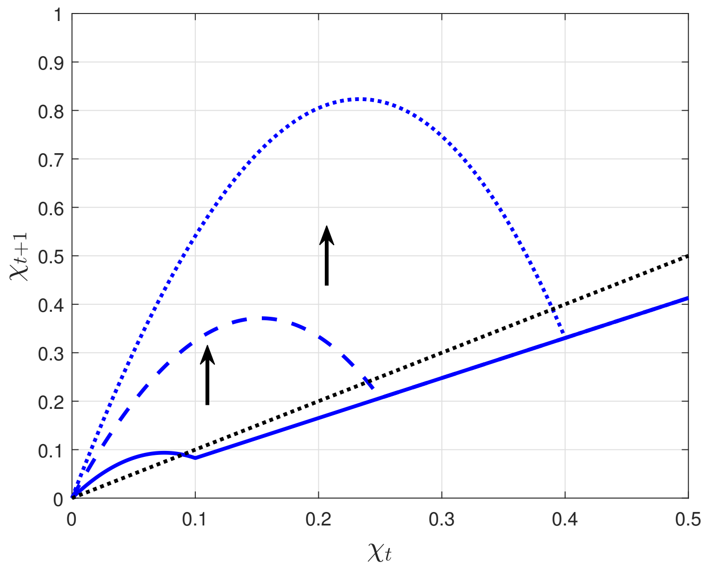
$$K'(\chi) = \beta\left[\alpha AZ(\chi) - p(\chi)\chi + 1 - \delta\right] K,$$

$$X'(\chi) = \beta\left[\mu p(\chi) + 1 - \delta\right] X,$$

and the equilibrium dynamics of the ratio between lemon asset and productive asset $\chi = \frac{X}{K}$ *is given by:*

$$\chi' = \frac{\mu p(\chi) + 1 - \delta}{\alpha AZ(\chi) - p(\chi)\chi + 1 - \delta}\chi. \tag{A5}$$

Figure A.5. Law of motion of χ_t



To characterize the equilibrium, we assume that $z \sim U[0,1]$ as in the baseline model. In this case, the credit market clearing condition is given by:

$$\frac{\phi}{1-\phi}\frac{1-F(\bar{z})}{\bar{z}} = \frac{F(\underline{z})}{\underline{z}} = 1 \Rightarrow \bar{z} = \phi.$$

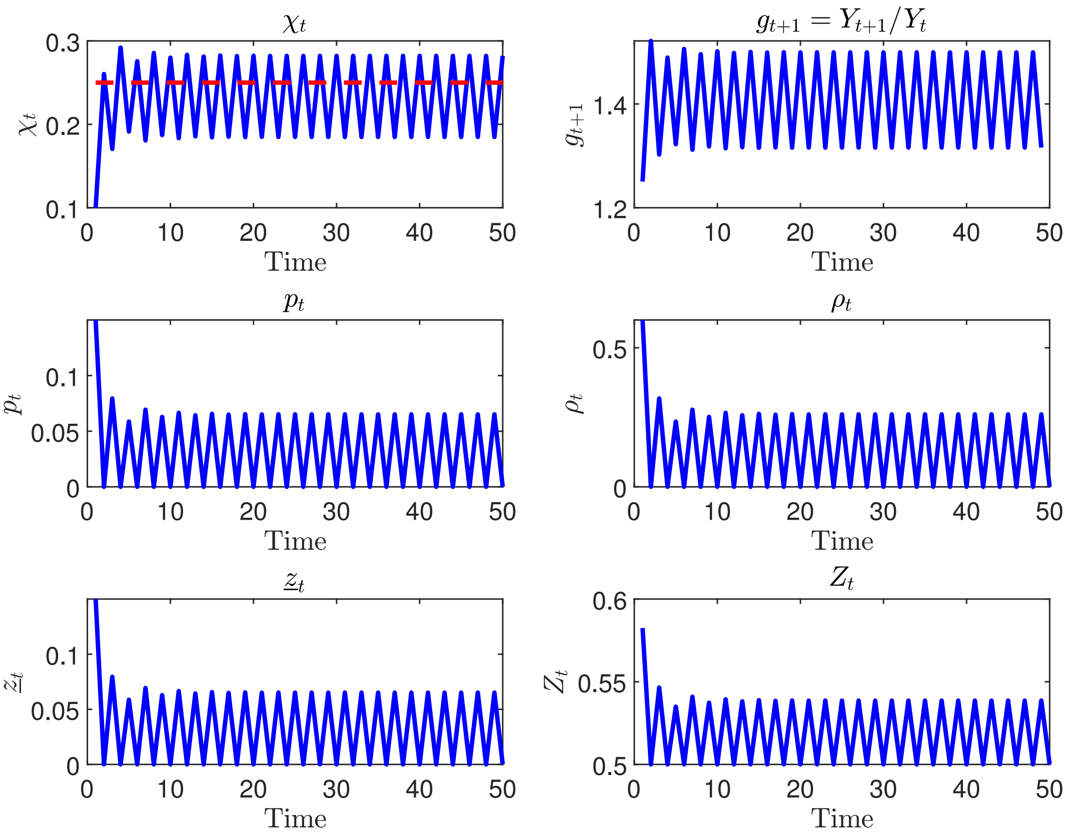
Since $\rho = \frac{\underline{z}}{\bar{z}} = \frac{\underline{z}}{\underline{z}+\chi}$ we can directly see that in equilibrium $\underline{z} = \phi - \chi$ such that capital price satisfies $p = \alpha A(\phi - \chi)$. For the credit market to remain functioning, we need $p > 0$ so that $\chi < \phi$. In the meantime, under uniform distribution, the equilibrium capital allocation efficiency is given by:

$$Z = \frac{1 + (1+\chi)\max\{\phi - \chi, 0\}}{2}.$$

In Figure A.5, we show the law-of-motion of χ_t with different levels of ϕ. Similar to our baseline mode, as ϕ increases, the positive steady state of χ may turn from stable to unstable, and potential cycles can occur.

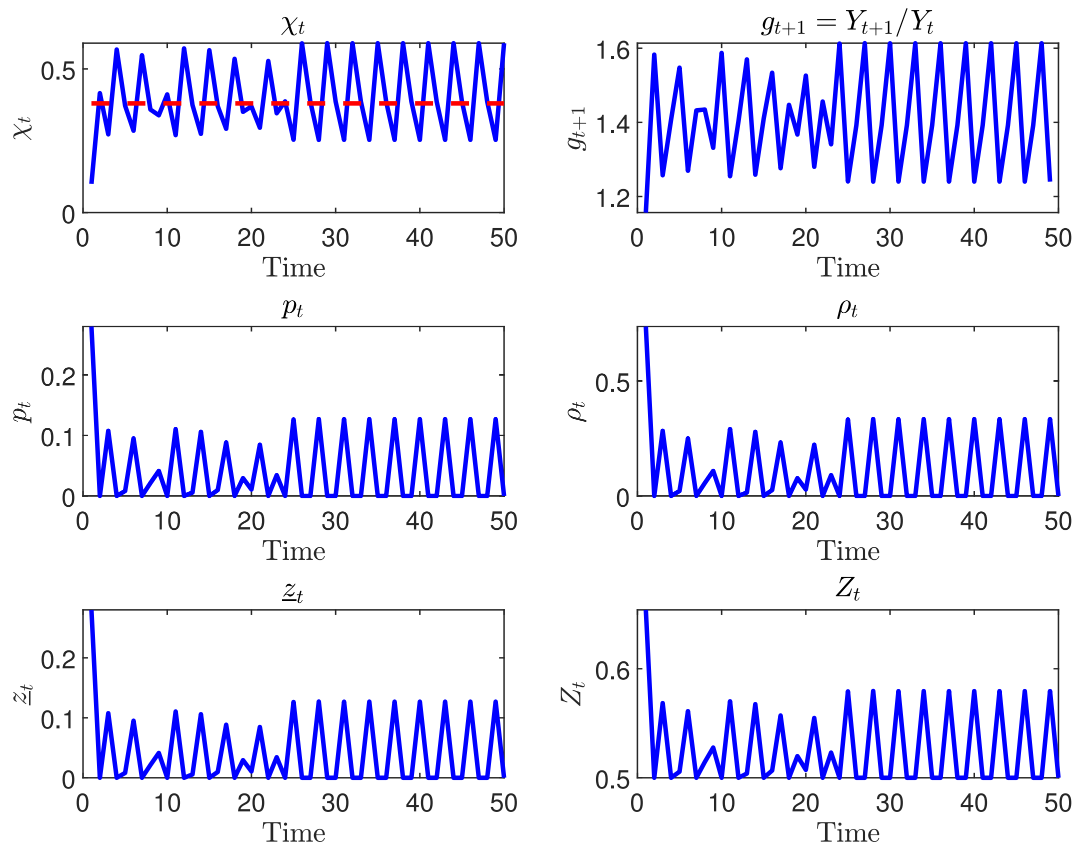
C Transition Paths with Cycles

Figure A.6. Transition dynamics of a 2-cycles



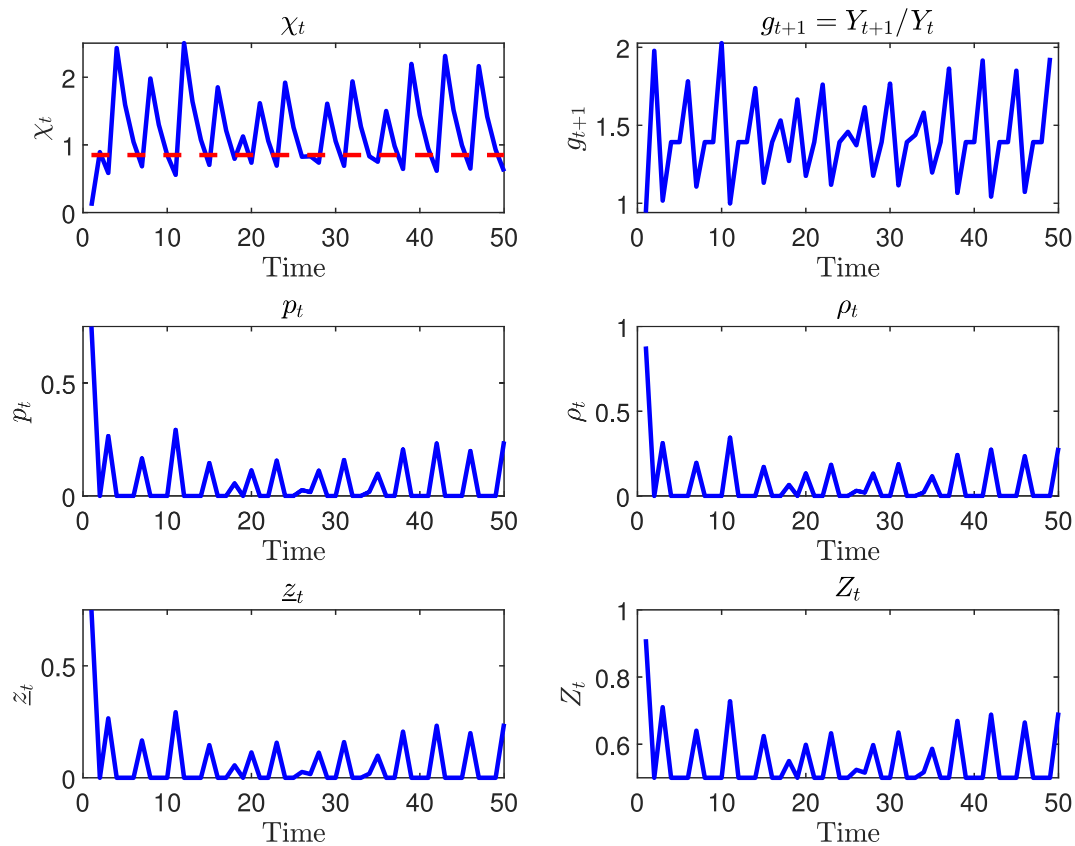
Notes: Parameters used are $\beta = 0.96$, $\mu = 20$, $\delta = 0.05$ and $\phi = 0.25$.

Figure A.7. Transition path of a 3-cycle



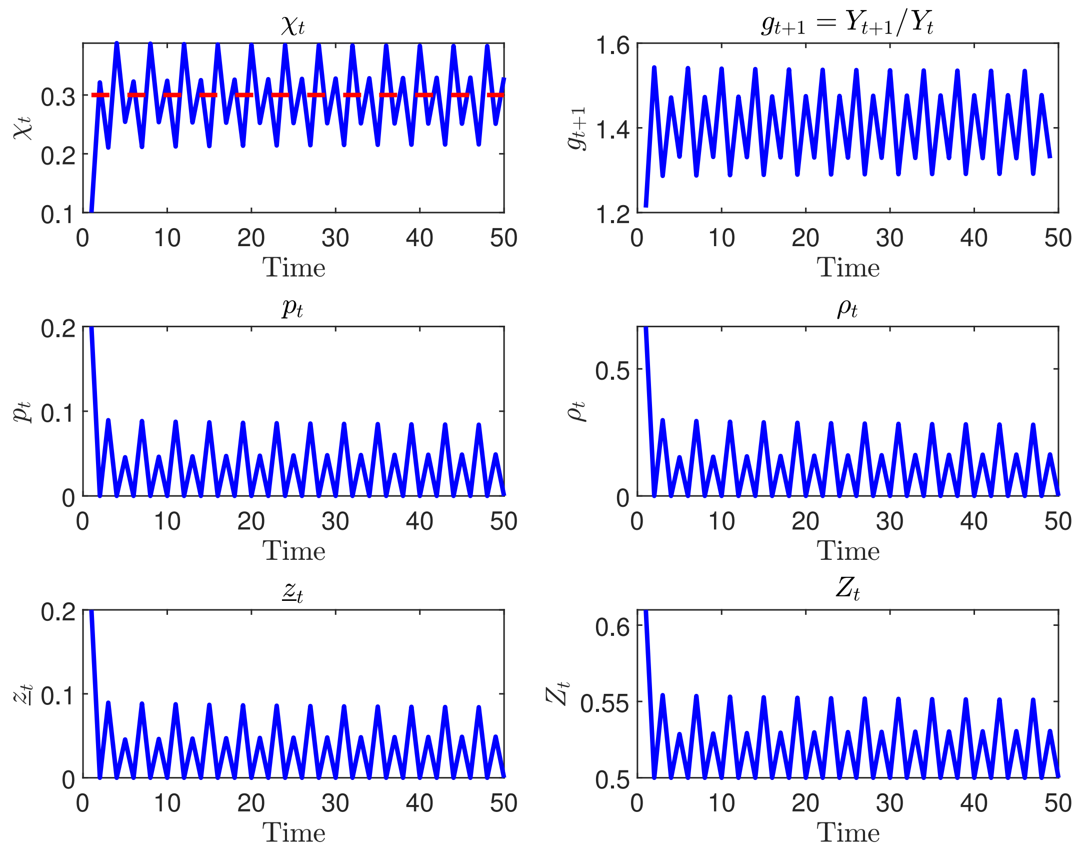
Notes: Parameters used are $\beta = 0.96$, $\mu = 20$, $\delta = 0.05$ and $\phi = 0.38$.

Figure A.8. Transition dynamics under chaos



Notes: Parameters used are $\beta = 0.96$, $\mu = 20$, $\delta = 0.05$ and $\phi = 0.85$.

Figure A.9. Transition dynamics in a 4-cycle



Notes: Parameters used are $\beta = 0.96$, $\mu = 20$, $\delta = 0.05$ and $\phi = 0.3$.

References

Romer, Paul M, "Increasing returns and long-run growth," *Journal of Political economy*, 1986, *94* (5), 1002–1037.