Appendix

Advertising by Recruiting Influencers

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Technical Appendix

Modeling Note: We begin this technical appendix with a brief note on deriving the value of C from an equilibrium between the firm and influencers. Our focus is on the scenario of "hiring" influencers when the profitability threshold is satisfied for both influencer groups, as described in Proposition 2. According to this proposition, the firm optimally selects a single influencer group.

We model a sequential game where the firm first sets the compensation, c, and influencers subsequently choose their effort levels, which in turn determine their compensation. Let group A influencers choose effort e_A and group B influencers choose e_B, which determine promotional effectiveness parameters $\alpha = a \times e_A$ and $\beta = b \times e_B$, where a and b are constants. The cost of effort for each influencer in group $G \in \{A, B\}$ is $e_G^2/2$, and C is proportional to effort, modeled as $C = e_A \times c$. Upon recruitment, an influencer's utility is $\psi_G \cdot c \cdot e_G - e_G^2/2$, leading each influencer in group G to optimally set $e_G = \psi_G \cdot c$, thereby determining α and β.

We now consider the case where group A influencers are chosen. For simplicity, assume $\gamma \to 0$. The firm's optimal profit from recruiting m_A influencers is:

$$E[\Pi] = \frac{(M_1 + \delta \cdot M_2) \cdot p \cdot \phi}{\bar{x} - \underline{x}} \times \left[1 - (1 - \psi_A + \psi_A \cdot e^{-a \cdot e_A^*})^{m_A}\right] - \psi_A \cdot c \cdot e_A^* - \kappa.$$

Substituting $e_A^* = \psi_A \cdot c$ into $E[\Pi]$, we get:

$$E[\Pi] = \frac{(M_1 + \delta \cdot M_2) \cdot p \cdot \phi}{\bar{x} - \underline{x}} \times \left[1 - (1 - \psi_A + \psi_A \cdot e^{-a \cdot \psi_A \cdot c})^{m_A}\right] - \psi_A^2 \cdot c^2 - \kappa.$$

Taking the derivative of $E[\Pi]$ with respect to c gives:

$$\frac{\partial E[\Pi]}{\partial c} = \frac{(M_1 + \delta M_2)\, p\, \phi}{\bar{x} - \underline{x}} \left[a\, m_A\, \psi_A^2\, e^{-a\psi_A c}(1 - \psi_A + \psi_A e^{-a\psi_A c})^{m_A - 1}\right] - 2\psi_A^2\, c.$$

Setting the derivative equal to zero, the optimal c satisfies:

$$\frac{(M_1 + \delta \cdot M_2) \cdot p \cdot \phi}{\bar{x} - \underline{x}} \left[a \cdot m_A \cdot \psi_A^2 e^{-a\psi_A c}(1 - \psi_A + \psi_A e^{-a\psi_A c})^{m_A - 1}\right] = 2\psi_A^2 \cdot c.$$

Finally, $C = e_A \times c$. A similar analysis holds for group B influencers, where C is optimally set based on the group yielding higher expected profit.

This demonstrates the firm's trade-off between higher compensation, which raises influencer effort and potential sales, and the associated cost. The equilibrium effort depends jointly on the firm's compensation decision and the characteristics of the influencer groups.

Proof of Lemma 1:

A non-loyal customer j ($x_j < x^M$) who follows s_j^A and s_j^B influencers makes a purchase with probability:

$$P\left(X_j > p - q_p - \phi \cdot (1 - \exp\{-\alpha \cdot S_j^A - \beta \cdot S_j^B\} | X_j < p - q_p,\ S_j^A = s_j^A,\ S_j^B = s_j^B\right)$$
$$= \frac{\phi \cdot (1 - \exp\{-\alpha \cdot s_j^A - \beta \cdot s_j^B\})}{x^M - \underline{x}}.$$

However, since s_j^A and s_j^B are unknown to the firm, it observes the following purchase probability for a non-loyal customer:

$$P(X_j > p - q_p - \phi \cdot (1 - \exp\{-\alpha \cdot S_j^A - \beta \cdot S_j^B\} | X_j < p - q_p) = \sum_{k_2^B=0}^{n_B} \sum_{k_2^A=0}^{n_A} \sum_{k_1^B=0}^{m_B} \sum_{k_1^A=0}^{m_A} \binom{m_A}{k_1^A} (\psi_A)^{k_1^A}$$
$$(1-\psi_A)^{m_A-k_1^A} \times \binom{n_A}{k_2^A} (\rho_A\psi_A)^{k_2^A} (1-\rho_A\psi_A)^{n_A-k_2^A} \times \binom{m_B}{k_1^B} (\psi_B)^{k_1^B} (1-\psi_B)^{m_B-k_1^B} \times \binom{n_B}{k_2^B}$$
$$(\rho_B\psi_B)^{k_2^B} (1-\rho_B\psi_B)^{n_B-k_2^B} \times \frac{\phi \cdot (1 - \exp\{-\alpha \cdot (k_1^A + k_2^A) - \beta \cdot (k_1^B + k_2^B)\})}{x^M - \underline{x}},$$

by binomial expansion and independence of followers. Multiplying this probability by the probability that a random customer is non-loyal, $\frac{x^M - \underline{x}}{\bar{x} - \underline{x}}$, gives the probability v^S that a random customer converts from a non-buyer to a buyer in period 1. The value of v^S can be simplified as follows:

$$v^S = \frac{x^M - \underline{x}}{\bar{x} - \underline{x}} \times P(X_j > p - q_p - \phi \cdot (1 - \exp\{-\alpha \cdot S_j^A - \beta \cdot S_j^B\} | X_j < p - q_p)$$
$$= \frac{\phi}{\bar{x} - \underline{x}} \left[1 - \left(1 - \psi_A + \psi_A e^{-\alpha}\right)^{m_A} \times \left(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha}\right)^{n_A} \times \left(1 - \psi_B + \psi_B e^{-\beta}\right)^{m_B} \right.$$
$$\left. \times \left(1 - \rho_B\psi_B + \rho_B\psi_B e^{-\beta}\right)^{n_B} \right].$$

The total potential revenue is $M_1 \cdot p$ and thus, $E[\Pi^S]$ can be written as follows:

$$E[\Pi^S] = \frac{M_1 \times p \times \phi}{\bar{x} - \underline{x}} \left[1 - \left(1 - \psi_A + \psi_A e^{-\alpha}\right)^{m_A} \times \left(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha}\right)^{n_A} \times \left(1 - \psi_B + \psi_B e^{-\beta}\right)^{m_B} \right.$$
$$\left. \times \left(1 - \rho_B\psi_B + \rho_B\psi_B e^{-\beta}\right)^{n_B} \right].$$

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Proof of Proposition 1:

(a) We consider:

$$E[\Pi] = E[\Pi^L] + E[\Pi^S] + \delta E[\Pi^R] - \Gamma.$$

We have previously expressed

$$E[\Pi^L] = M_1 \times p \times \frac{\bar{x} - x^M}{\bar{x} - \underline{x}}, \text{ and}$$
$$\Gamma = m_A \times (\psi_A \times C + \kappa) + n_A \times \kappa$$

and from Lemma 1,

$$E[\Pi^S] = \frac{M_1 \times p \times \phi}{\bar{x} - \underline{x}} \left[1 - \left(1 - \psi_A + \psi_A e^{-\alpha}\right)^{m_A} \times \left(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha}\right)^{n_A} \right].$$

Next, the purchase probability in period 1 is:

$$v = v^L + v^S = \frac{\bar{x} - x^M}{\bar{x} - \underline{x}} + \frac{\phi}{\bar{x} - \underline{x}} \times \Big(1 - (1 - \psi_A + \psi_A e^{-\alpha})^{m_A} \times (1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha})^{n_A}\Big).$$

Thus $E[\Pi^L + \Pi^S] = M_1 \times p \times v$. In the absence of social learning ($\gamma \to 0$),

$$E[\Pi^R] = M_2 \times p \times v.$$

Therefore,

$$E[\Pi] = (M_1 + \delta \times M_2) \times p \times v - \Gamma.$$

We take partial derivatives of $E[\Pi]$ with respect to m_A and n_A:

$$\begin{aligned}
\frac{\partial E[\Pi]}{\partial m_A} &= -\frac{(M_1 + \delta \times M_2) \cdot p \cdot \phi}{\bar{x} - \underline{x}} \times \ln(1 - \psi_A + \psi_A e^{-\alpha}) \times (1 - \psi_A + \psi_A e^{-\alpha})^{m_A} \times (1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha})^{n_A} \\
&\quad - (\psi_A C + \kappa), \\
\frac{\partial E[\Pi]}{\partial n_A} &= -\frac{(M_1 + \delta \times M_2) \cdot p \times \phi}{\bar{x} - \underline{x}} \times \ln(1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha}) \times (1 - \psi_A + \psi_A e^{-\alpha})^{m_A} \\
&\quad \times (1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha})^{n_A} - \kappa.
\end{aligned}$$

Let us establish the threshold, X, which will prove useful when evaluating the profitability of each campaign:

$$X = \frac{(M_1 + \delta \times M_2) \times p \times \phi}{\bar{x} - \underline{x}},$$

Under the positivity constraints $m_A \geq 0$, $m_B \geq 0$, $n_A \geq 0$, and $n_B \geq 0$, the Kuhn-Tucker equalities are:

$$m_A \times \frac{\partial E[\Pi]}{\partial m_A} = 0 \text{ and } n_A \times \frac{\partial E[\Pi]}{\partial n_A} = 0,$$

and the inequalities are:

$$\frac{\partial E[\Pi]}{\partial m_A} \leq 0 \text{ and } \frac{\partial E[\Pi]}{\partial n_A} \leq 0.$$

Using the shorthand notation of Definition 1, these become:

$$\begin{aligned}
(1 - \psi_A + \psi_A e^{-\alpha})^{m_A} \times (1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha})^{n_A} &\leq X/h(A), \\
(1 - \psi_A + \psi_A e^{-\alpha})^{m_A} \times (1 - \rho_A \psi_A + \rho_A \psi_A e^{-\alpha})^{n_A} &\leq X/g(A).
\end{aligned}$$

Since the left-hand side is always below 1, an inequality can only bind if its RHS is ≤ 1, which requires $h(A) > X$ or $g(A) > X$. Hence $m_A^* = n_A^* = 0$ if $X > \max\{h(A), g(A)\}$.

When $X < \max\{h(A), g(A)\}$, only one KKT inequality can bind, so only one of $\left\{\frac{\partial E[\Pi]}{\partial m_A}, \frac{\partial E[\Pi]}{\partial n_A}\right\}$, can be zero. Combined with the equalities, only one of $\{m_A, n_A\}$ can be strictly positive. The binding inequality is the one with the smaller RHS, i.e. $\min\left\{\frac{1}{h(A)}, \frac{1}{g(A)}\right\}$. inequality.

(b) If $h(A) = \max\{h(A), g(A), X\}$, then $m_A^* > 0$ and $n_A^* = 0$. From the binding inequality, $(1 - \psi_A + \psi_A e^{-\alpha})^{m_A^*} = X/h(A)$, or equivalently,

$$m_A^* = \frac{\ln(X) - \ln(h(A))}{\ln(1 - \psi_A + \psi_A e^{-\alpha})} = m_A^X.$$

Similarly,

$$n_A^X = \frac{\ln(X) - \ln(g(A))}{\ln(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})}.$$

(c) We now study the behavior of $g(A) - h(A)$ w.r.t α, including its monotonicity. We first look for the roots of $g(A) - h(A) = 0$. One root is $\alpha = 0$ because: $h(A) = \dfrac{-\ln(1 - \psi_A + \psi_A e^{-0})}{\psi_A C + \kappa} = 0$ and $g(A) = \dfrac{-\ln(1 - \rho_A\psi_A + \rho_A\psi_A e^{-0})}{\kappa} = 0$. To investigate the existence of other roots, we expand the expression:

$$g(A) - h(A) = \frac{-\ln(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})}{\kappa} - \frac{-\ln(1 - \psi_A + \psi_A e^{-\alpha})}{\psi_A C + \kappa}.$$

The first derivative with respect to α is:

$$\frac{\rho_A\psi_A e^{-\alpha}}{\kappa(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})} - \frac{\psi_A e^{-\alpha}}{(\psi_A C + \kappa)(1 - \psi_A + \psi_A e^{-\alpha})},$$

This derivative is positive if and only if:

$$\frac{\rho_A\psi_A e^{-\alpha}}{\kappa(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})} > \frac{\psi_A e^{-\alpha}}{(\psi_A C + \kappa)(1 - \psi_A + \psi_A e^{-\alpha})},$$

which simplifies to:

$$e^{-\alpha} > \frac{\kappa(1 - \rho_A) - \psi_A C\rho_A(1 - \psi_A)}{\psi_A^2 C\rho_A}.$$

Let us define:

$$\alpha^C = -\ln\left(\frac{\kappa(1 - \rho_A) - \psi_A C\rho_A(1 - \psi_A)}{\psi_A^2 C\rho_A}\right)$$

assuming $0 < \kappa(1 - \rho_A) - \psi_A(1 - \psi_A)C\rho_A < \psi_A^2 C\rho_A$ to ensure that α^C is well defined. We can distinguish the following cases:

(1) If $\dfrac{\kappa}{\psi_A C} < \dfrac{\rho_A}{1 - \rho_A} < \dfrac{\kappa}{\psi_A C(1 - \psi_A)}$, or equivalently if $\dfrac{\kappa}{\kappa + \psi_A C} < \rho_A < \dfrac{\kappa}{\kappa + \psi_A C(1 - \psi_A)}$, then there exists α^T such that $\alpha^T > \alpha^C$ and: $g(A) > h(A)$ if $\alpha < \alpha^T$, $g(A) < h(A)$ if $\alpha > \alpha^T$, and $g(A) = h(A)$ if $\alpha = \alpha^T$.

(2) If $\dfrac{\rho_A}{1 - \rho_A} \geq \dfrac{\kappa}{\psi_A C(1 - \psi_A)}$, or equivalently if $\rho_A \geq \dfrac{\kappa}{\kappa + \psi_A C(1 - \psi_A)}$, (the partial derivative is always positive), then $g(A) > h(A)$.

(3) If $\dfrac{\rho_A}{1 - \rho_A} \leq \dfrac{\kappa}{\psi_A C}$, or equivalently if $\rho_A \leq \dfrac{\kappa}{\kappa + \psi_A C}$, (the partial derivative is always negative),then $g(A) < h(A)$.

(d) We analyze $g(A) - h(A)$ as a function of ρ_A,

$$g(A) - h(A) = \frac{-\ln(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})}{\kappa} - \frac{-\ln(1 - \psi_A + \psi_A e^{-\alpha})}{\psi_A C + \kappa}.$$

The partial derivative with respect to ρ_A is: $\frac{\psi_A(1 - e^{-\alpha})}{\kappa(1 - \rho_A\psi_A + \rho_A\psi_A e^{-\alpha})}$. Since the derivative is positive, $g(A) - h(A)$ is strictly increasing in ρ_A. Moreover: when $\rho_A = 0$, we have $g(A) - h(A) < 0$ and when $\rho_A = 1$, we have that $g(A) - h(A) > 0$. Let ρ_A^T denote the unique positive root of $g(A) - h(A) = 0$.

(e) We analyze $g(A) - h(A)$ as a function of ψ_A. For $\psi_A = 0$, we have $g(A) = h(A) = 0$. Now consider $\psi_A \to 0^+$. The derivative of $g(A)$ with respect to ψ_A is: $\frac{\rho_A(1-e^{-\alpha})}{\kappa(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})}$. Evaluating at $\psi_A = 0$ yields: $\frac{\rho_A(1-e^{-\alpha})}{\kappa}$. The partial derivative of $h(A)$ with respect to ψ_A is $\frac{(\psi_A C+\kappa)\times\frac{1-e^{-\alpha}}{1-\psi_A+\psi_A e^{-\alpha}}+C(\ln(1-\psi_A+\psi_A e^{-\alpha}))}{(\psi_A C+\kappa)^2}$. At $\psi_A = 0$, this simplifies to: $\frac{1-e^{-\alpha}}{\kappa}$. Since: $\frac{\rho_A(1-e^{-\alpha})}{\kappa} \leq \frac{(1-e^{-\alpha})}{\kappa}$, we conclude that $g(A) < h(A)$ for $\psi_A \to 0^+$ and $\rho_A < 1$. This establishes the existence of a region of $\psi_A \in [0, \psi_A^L]$ such that $h(A) > g(A)$ if $\rho_A < 1$.

(f) We analyze $g(A) - h(A)$ for $\psi_A = 1$: $g(A) = \frac{-\ln(1-\rho_A+\rho_A e^{-\alpha})}{\kappa}$ and $h(A) = \frac{\alpha}{C+\kappa}$. To establish a region where $g(A) < h(A)$, we need: $C < \frac{\alpha\kappa}{-\ln(1-\rho_A+\rho_A e^{-\alpha})} - \kappa$. This inequality defines the region $\psi_A \in [\psi_A^H, 1]$ where $g(A) < h(A)$.

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Proof of Corollary 1:

Based on Proposition 1, $n_A^X = \dfrac{\ln(X) - \ln(g(A))}{\ln(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})}$. The derivative of n_A^X with respect to ρ_A can be expressed as follows:

$$\frac{\partial n_A^X}{\partial \rho_A} = \frac{\psi_A(1-e^{-\alpha})}{1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha}} \times \left(1 + \ln\left(\frac{X}{g(A)}\right)\right),$$

such that $\frac{\psi_A(1-e^{-\alpha})}{1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha}}$ is strictly positive, and $g(A) > X$ is a necessary condition for gifting to be profitable for this influencer group. Since $g(A)$ is strictly increasing in ρ_A, then the value of $\left(1 + \ln\left(\frac{X}{g(A)}\right)\right)$ is strictly decreasing in ρ_A and can have both positive and negative values. This implies the existence of a threshold ρ_A^X, such that the derivative $\frac{\partial n_A^X}{\partial \rho_A}$ is positive for $\rho_A < \rho_A^X$ and negative for $\rho_A > \rho_A^X$. Therefore, the value of n_A^X exhibits an increasing then decreasing relationship with respect to ρ_A.

If we consider a linear relationship between ρ_A and q, we can use a similar argument to prove the existence of a threshold q^X, such that the derivative $\frac{\partial n_A^X}{\partial q}$ is positive for $q < q^X$ and negative for $q > q^X$.

Proof of Proposition 2:

(a) Using a similar proof to that of Proposition 1, we can write:

$$E[\Pi] = (M_1 + \delta M_2)\, p\, (1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A} \times (1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B} - \Big[(\psi_A C+\kappa)m_A + (\psi_B C+\kappa)m_B + \kappa n_A + \kappa n_B\Big].$$

We then take partial derivatives of $E[\Pi]$ with respect to m_A, m_B, n_A, and n_B:

$$\frac{\partial E[\Pi]}{\partial m_A} = -\frac{(M_1+\delta M_2)\, p\, \phi}{\bar{x}-\underline{x}} \ln(1-\psi_A+\psi_A e^{-\alpha})\,(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B} \times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B} - (\psi_A C+\kappa),$$

$$\frac{\partial E[\Pi]}{\partial m_B} = -\frac{(M_1+\delta M_2)\, p\, \phi}{\bar{x}-\underline{x}} \ln(1-\psi_B+\psi_B e^{-\beta})\,(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B} \times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B} - (\psi_B C+\kappa),$$

$$\frac{\partial E[\Pi]}{\partial n_A} = -\frac{(M_1+\delta M_2)\, p\, \phi}{\bar{x}-\underline{x}} \ln(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})\,(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}$$

$$\times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}-\kappa,$$

$$\frac{\partial E[\Pi]}{\partial n_B}=-\frac{(M_1+\delta M_2)\,p\,\phi}{\bar{x}-\underline{x}}\ln(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})\,(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}$$

$$\times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}-\kappa.$$

Under the constraints $m_A, m_B, n_A, n_B \geq 0$, the Kuhn–Tucker equalities are:

$$m_A\frac{\partial E[\Pi]}{\partial m_A}=0,\quad m_B\frac{\partial E[\Pi]}{\partial m_B}=0,\quad n_A\frac{\partial E[\Pi]}{\partial n_A}=0,\quad n_B\frac{\partial E[\Pi]}{\partial n_B}=0.$$

The inequalities are:

$$\frac{\partial E[\Pi]}{\partial m_A}\leq 0,\qquad \frac{\partial E[\Pi]}{\partial m_B}\leq 0,\qquad \frac{\partial E[\Pi]}{\partial n_A}\leq 0,\qquad \frac{\partial E[\Pi]}{\partial n_B}\leq 0,$$

which expand to:

$$(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\leq\frac{X}{h(A)},$$

$$(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\leq\frac{X}{h(B)},$$

$$(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\leq\frac{X}{g(A)},$$

$$(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\leq\frac{X}{g(B)}.$$

A standard argument (as in Proposition 1) shows that at most one inequality can bind. If $X=\max\{h(A),h(B),g(A),g(B),X\}$, then none binds and

$$m_A^*=m_B^*=n_A^*=n_B^*=0.$$

Otherwise the binding constraint is the one with the smallest right-hand side, that is:

$$\min\left\{\tfrac{1}{h(A)},\tfrac{1}{h(B)},\tfrac{1}{g(A)},\tfrac{1}{g(B)}\right\},$$

which identifies the optimal partnership type.

(b) The KKT system yields:

$$m_A^X=\frac{\ln X-\ln h(A)}{\ln(1-\psi_A+\psi_A e^{-\alpha})},\qquad m_B^X=\frac{\ln X-\ln h(B)}{\ln(1-\psi_B+\psi_B e^{-\beta})},$$

$$n_A^X=\frac{\ln X-\ln g(A)}{\ln(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})},\qquad n_B^X=\frac{\ln X-\ln g(B)}{\ln(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})}.$$

(c) As $\bar{x}-\underline{x}$ increases, $X=\frac{\bar{x}-\underline{x}}{M_1+\delta M_2}$ increases while $h(A)$, $h(B)$, $g(A)$, and $g(B)$ are constant. Thus the optimal campaign type depends only on whether it exceeds the profitability threshold X, not on heterogeneity. Since $\ln(1-\psi_A+\psi_A e^{-\alpha})<0$ is constant in $\bar{x}-\underline{x}$, m_A^X decreases as heterogeneity increases. The same holds for m_B^X, n_A^X, and n_B^X.

(d) The partial derivative of m_A^X with respect to α is:

$$\frac{\partial m_A^X}{\partial \alpha} = \left[\left(\frac{\psi_A e^{-\alpha}}{1-\psi_A+\psi_A e^{-\alpha}}\right)\left(\ln(X)-\ln(h(A))-\frac{\ln(1-\psi_A+\psi_A e^{-\alpha})}{h(A)\cdot(\psi_A C+\kappa)}\right)\right] \div \big(\ln(1-\psi_A+\psi_A e^{-\alpha})\big)^2$$

$$= \left[\left(\frac{\psi_A e^{-\alpha}}{1-\psi_A+\psi_A e^{-\alpha}}\right) \div \big(\ln(1-\psi_A+\psi_A e^{-\alpha})\big)^2\right]\big(\ln(X)-\ln(h(A))+1\big),$$

and has the same sign as $\ln(X)-\ln(h(A))+\ln(e)$. Focusing on the region where $h(A)>X$, we note that $\frac{\partial m_A^X}{\partial \alpha}=0$ when $h(A)=X$. As α increases, $h(A)$ increases and $\frac{\partial m_A^X}{\partial \alpha}$ decreases. It becomes zero when $h(A)=X\cdot e$, which we denote as α^n. For $\alpha>\alpha^n$, as α increases, the derivative becomes strictly negative. Hence, if there exists an α such that $h(A)=X\cdot e$, then m_A^X increases monotonically for $\alpha<\alpha^n$ and decreases monotonically for $\alpha>\alpha^n$. Otherwise, m_A^X is strictly increasing in α. We summarize the findings in the following two cases:

I. If $-\dfrac{\ln(1-\psi_A)}{\psi_A C+\kappa}\le X\cdot e$, then $\dfrac{\partial m_A^X}{\partial \alpha}>0$ for all values of α. This implies that m_A^X is monotonically increasing in α.

II. If $-\dfrac{\ln(1-\psi_A)}{\psi_A C+\kappa}> X\cdot e$, then $\dfrac{\partial m_A^X}{\partial \alpha}=0$ for some $\alpha=\alpha^n$, $\dfrac{\partial m_A^X}{\partial \alpha}>0$ for $\alpha<\alpha^n$, and $\dfrac{\partial m_A^X}{\partial \alpha}<0$ for $\alpha>\alpha^n$. This means that m_A^X first increases and then decreases in α.

We also study $-\dfrac{\ln(1-\psi_A)}{\psi_A C+\kappa}$ as a function of ψ_A for $0\le\psi_A<1$. Taking the first derivative with respect to ψ_A yields

$$\frac{\psi_A C+\kappa}{1-\psi_A}+C\ln(1-\psi_A),$$

which is the sum of a positive term and a negative term. For $\psi_A=0$, this derivative is strictly positive. Moreover, the positive term increases at a higher rate (derivative $=\frac{C+\kappa}{(1-\psi_A)^2}$) compared to the rate of decrease of the negative term in absolute terms (derivative $=\frac{C}{1-\psi_A}$). Thus,

$$\frac{\psi_A C+\kappa}{1-\psi_A}+C\ln(1-\psi_A)>0$$

for all ψ_A, indicating that $-\dfrac{\ln(1-\psi_A)}{\psi_A C+\kappa}$ is continuous and strictly increasing in ψ_A with range $[0,+\infty)$.

This implies that there exists a threshold on ψ_A, strictly between 0 and 1, such that case (I) holds below the threshold and case (II) holds above the threshold. A similar analysis applies to m_B^X, n_A^X, and n_B^X.

■

Proof of Corollary 2:

Proposition 2 shows that it is optimal to recruit at most one group of influencers through a single form of partnership. We now determine which group should be preferred under each form of partnership.

a. **Gifting:** Group A is more profitable than group B if and only if $g(A)>g(B)$. This implies:

$$\frac{-\ln(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})}{\kappa}<\frac{-\ln(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})}{\kappa}.$$

Since $\kappa>0$, this is equivalent to:

$$\rho_B\psi_B(1-e^{-\beta})<\rho_A\psi_A(1-e^{-\alpha}),$$

or:

$$\psi_A > \frac{\rho_B \psi_B (1-e^{-\beta})}{\rho_A (1-e^{-\alpha})}.$$

b. **Hiring:** Group A is more profitable than B if and only if $h(A) > h(B)$:

$$\frac{-\ln(1-\psi_B+\psi_B e^{-\beta})}{\psi_B C+\kappa} < \frac{-\ln(1-\psi_A+\psi_A e^{-\alpha})}{\psi_A C+\kappa}.$$

Since $\psi_B > \psi_A$, we have $\psi_B C+\kappa > \psi_A C+\kappa$, and multiplying both sides by the positive denominator $\psi_A C+\kappa$ yields:

$$-\frac{\ln(1-\psi_B+\psi_B e^{-\beta})}{\psi_A C+\kappa} < -\frac{\ln(1-\psi_A+\psi_A e^{-\alpha})}{\psi_A C+\kappa}.$$

This inequality holds if:

$$1-\psi_B+\psi_B e^{-\beta} > 1-\psi_A+\psi_A e^{-\alpha},$$

equivalently, $\psi_A > \tau_H$, where

$$\tau_H = \frac{\psi_B(1-e^{-\beta})}{1-e^{-\alpha}}.$$

Consequently, hiring group B is more profitable than hiring group A iff $h(B) > h(A)$:

$$\frac{-\ln(1-\psi_B+\psi_B e^{-\beta})}{\psi_B C+\kappa} > \frac{-\ln(1-\psi_A+\psi_A e^{-\alpha})}{\psi_A C+\kappa}. \tag{3}$$

Bounding the left-hand side from below and the right-hand side from above, (3) holds if:

$$\frac{-\ln(1-\psi_B+\psi_B e^{-\beta})}{C+\kappa} > \frac{-\ln(1-\psi_A+\psi_A e^{-\alpha})}{\kappa},$$

which can be rewritten as:

$$(1-\psi_B+\psi_B e^{-\beta})^{\kappa} < (1-\psi_A+\psi_A e^{-\alpha})^{C+\kappa},$$

equivalently:

$$\psi_A < \tau_L, \qquad \tau_L = \frac{1-\sqrt[C+\kappa]{(1-\psi_B+\psi_B e^{-\beta})^{\kappa}}}{1-e^{-\alpha}}.$$

Note that $\tau_L = \tau_H$ when $C=0$, and τ_L is decreasing in C, thus $\tau_L \le \tau_H$ always holds.

■

Proof of Proposition 3:

We can follow a similar proof to that of Proposition 1 to write the expression of $E[\Pi]$ under various conditions. For simplicity, we assume that $n_A = n_B = 0$ in what follows.

(a) If $\phi < x^M - x^S$, we can write:

$$\begin{aligned} E[\Pi] = {} & (1-P_{niche})\frac{M_1+\delta M_2}{\bar{x}-\underline{x}}\, p\,(1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B} \\ & + P_{niche} f_2(M_1+\delta M_2)\, p\,(1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B} \\ & - \big((\psi_A C+\kappa)m_A + (\psi_B C+\kappa)m_B\big). \end{aligned}$$

We now take partial derivatives:

$$\frac{\partial E[\Pi]}{\partial m_A} = (M_1+\delta M_2)p\phi\,(1-\psi_B+\psi_B e^{-\beta})^{m_B}\Bigg[\frac{-(1-P_{niche})}{\bar{x}-\underline{x}}\ln(1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha}) \times (1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha})^{m_A} - P_{niche}f_2\ln(1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha}) \times (1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}\Bigg] - (\psi_A C+\kappa),$$

$$\frac{\partial E[\Pi]}{\partial m_B} = (M_1+\delta M_2)p\phi\,(1-\psi_B+\psi_B e^{-\beta})^{m_B}\Bigg[\frac{-(1-P_{niche})}{\bar{x}-\underline{x}}\ln(1-\psi_B+\psi_B e^{-\beta}) \times (1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha})^{m_A} - P_{niche}f_2\ln(1-\psi_B+\psi_B e^{-\beta}) \times (1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}\Bigg] - (\psi_B C+\kappa).$$

Under positivity constraints $m_A, m_B \geq 0$, the Kuhn–Tucker equalities are

$$m_A\,\frac{\partial E[\Pi]}{\partial m_A}=0, \qquad m_B\,\frac{\partial E[\Pi]}{\partial m_B}=0.$$

and the inequalities:

$$\frac{\partial E[\Pi]}{\partial m_A}\leq 0, \qquad \frac{\partial E[\Pi]}{\partial m_B}\leq 0.$$

Replacing the derivatives, inequality $\frac{\partial E[\Pi]}{\partial m_A}\leq 0$ becomes:

$$\left(\frac{(1-P_{niche})}{\bar{x}-\underline{x}}\,h^{\mathcal{G}}(A)\,(1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha})^{m_A}+P_{niche}f_2\,h^{\mathcal{N}}(A)\,(1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}\right) \leq \frac{1}{(M_1+\delta M_2)p\phi\,(1-\psi_B+\psi_B e^{-\beta})^{m_B}}. \tag{4}$$

Similarly, $\frac{\partial E[\Pi]}{\partial m_B}\leq 0$ gives:

$$\left(\frac{(1-P_{niche})}{\bar{x}-\underline{x}}\,h(B)\,(1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha})^{m_A}+P_{niche}f_2\,h(B)\,(1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}\right) \leq \frac{1}{(M_1+\delta M_2)p\phi\,(1-\psi_B+\psi_B e^{-\beta})^{m_B}}. \tag{5}$$

If $h^{\mathcal{N}}(A)>h^{\mathcal{G}}(A)>h(B)$, inequality (4) binds, giving $m_A^*>0, m_B^*=0$, and m_A^* increases in f_2.
If $h(B)>h^{\mathcal{N}}(A)>h^{\mathcal{G}}(A)$, inequality (5) binds, giving $m_B^*>0, m_A^*=0$, and m_B^* increases in f_2.
If $h^{\mathcal{N}}(A)>h(B)>h^{\mathcal{G}}(A)$, both bind. Equating left-hand sides yields:

$$\frac{(1-P_{niche})}{P_{niche}(\bar{x}-\underline{x})}\left(\frac{h(B)-h^{\mathcal{G}}(A)}{h^{\mathcal{N}}(A)-h(B)}\right)\left(\frac{1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha}}{1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha}}\right)^{m_A}=f_2. \tag{6}$$

Since $\psi_A^{\mathcal{G}}<\psi_A^{\mathcal{N}}$, the ratio exceeds 1, so m_A^* increases with f_2.

Continuing from (5), we obtain:

$$\left[\frac{(1-P_{niche})}{\bar{x}-\underline{x}}\frac{h(B)}{f_2}\left(\frac{1-\psi_A^{\mathcal{G}}+\psi_A^{\mathcal{G}}e^{-\alpha}}{1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha}}\right)^{m_A}+P_{niche}\,h(B)\right] = \frac{1}{(M_1+\delta M_2)p\phi(1-\psi_B+\psi_B e^{-\beta})^{m_B}}\frac{1}{f_2(1-\psi_A^{\mathcal{N}}+\psi_A^{\mathcal{N}}e^{-\alpha})^{m_A}}. \tag{7}$$

Thus m_A increases with f_2 and m_B decreases with f_2.

(b) If $x^S > x^M$, the reasoning mirrors part (a):
- If $h^{\mathcal{N}}(A) > h^{\mathcal{G}}(A) > h(B)$: $m_A^* > 0, m_B^* = 0$ and m_A^* increases in f_1.
- If $h(B) > h^{\mathcal{N}}(A) > h^{\mathcal{G}}(A)$: $m_B^* > 0, m_A^* = 0$ and m_B^* increases in f_1.
- If $h^{\mathcal{N}}(A) > h(B) > h^{\mathcal{G}}(A)$: both positive, m_A^* increases and m_B^* decreases in f_1.

■

Proof of Proposition 4:

(a) The probability that a randomly chosen customer j makes a purchase in period 1 is:

$$\begin{aligned} v &= v^L + v^S \\ &= \frac{\bar{x}-x^M}{\bar{x}-\underline{x}} + \frac{\phi}{\bar{x}-\underline{x}}\Bigg[1-(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B} \\ &\qquad \times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\Bigg]. \end{aligned}$$

Expected revenue in the second period given $T=j$:

$$\begin{aligned} E[\Pi^R|T=j] = M_2\,p\Bigg[&\frac{\bar{x}-p+E[\widetilde{q}_u|T=j]}{\bar{x}-\underline{x}} + \frac{\phi}{\bar{x}-\underline{x}}\Bigg(1-(1-\psi_A+\psi_A e^{-\alpha})^{m_A} \\ &\times (1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\Bigg)\Bigg]. \end{aligned} \tag{8}$$

Expected second-period revenue:

$$\begin{aligned} E[\Pi^R] &= \sum_{j=0}^{M_1}\binom{M_1}{j}v^j(1-v)^{M_1-j}E[\Pi^R|T=j] \\ &= \sum_{j=0}^{M_1}\binom{M_1}{j}v^j(1-v)^{M_1-j}M_2\,p\Bigg[\frac{\phi}{\bar{x}-\underline{x}}\Big(1-(1-\psi_A+\psi_A e^{-\alpha})^{m_A} \\ &\qquad \times (1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}\Big) \\ &\qquad + \frac{\bar{x}-p}{\bar{x}-\underline{x}} + \frac{1}{\bar{x}-\underline{x}}\frac{q_p+qj\gamma}{1+j\gamma}\Bigg]. \end{aligned}$$

Derivative w.r.t. m_A:

$$\frac{\partial E[\Pi]}{\partial m_A} = -\frac{p\phi\ln(1-\psi_A+\psi_A e^{-\alpha})}{\bar{x}-\underline{x}}(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}$$

$$\times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}$$
$$\times \left[M_1+\delta M_2-\delta M_2\sum_{j=0}^{M_1}\binom{M_1}{j}\frac{v^j(1-v)^{M_1-j}}{v(1-v)}\frac{M_1v-j}{\bar{x}-\underline{x}}\frac{q_p+qj\gamma}{1+j\gamma}\right]$$
$$-(\psi_A C+\kappa).$$

Similarly, the other partial derivatives:

$$\frac{\partial E[\Pi]}{\partial m_B}=-\frac{p\phi\ln(1-\psi_B+\psi_B e^{-\beta})}{\bar{x}-\underline{x}}(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}$$
$$\times (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}$$
$$\times \left[M_1+\delta M_2-\delta M_2\sum_{j=0}^{M_1}\binom{M_1}{j}\frac{v^j(1-v)^{M_1-j}}{v(1-v)}\frac{M_1v-j}{\bar{x}-\underline{x}}\frac{q_p+qj\gamma}{1+j\gamma}\right]-(\psi_B C+\kappa),$$
$$\frac{\partial E[\Pi]}{\partial n_A}=-\frac{p\phi\ln(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})}{\bar{x}-\underline{x}}\cdots-\kappa,$$
$$\frac{\partial E[\Pi]}{\partial n_B}=-\frac{p\phi\ln(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})}{\bar{x}-\underline{x}}\cdots-\kappa.$$

Threshold Y:

$$Y=\frac{\bar{x}-\underline{x}}{p\phi}\Bigg/\left[M_1+\delta M_2-\delta M_2\sum_{j=0}^{M_1}\binom{M_1}{j}\frac{v^j(1-v)^{M_1-j}}{v(1-v)}\frac{M_1v-j}{\bar{x}-\underline{x}}\frac{q_p+qj\gamma}{1+j\gamma}\right].$$

KKT inequalities:

$$(1-\psi_A+\psi_A e^{-\alpha})^{m_A}(1-\psi_B+\psi_B e^{-\beta})^{m_B}(1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A}(1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B}$$
$$\le Y/h(A),\quad Y/h(B),\quad Y/g(A),\quad Y/g(B).$$

(b) If $h(A)=\max\{h(A),h(B),g(A),g(B),Y\}$ with $Y>0$, then $m_B^*=n_A^*=n_B^*=0$, $m_A^*>0$:

$$m_A^*=\frac{\ln(Y)-\ln(h(A))}{\ln(1-\psi_A+\psi_A e^{-\alpha})}.$$

Substitute $q=q_p$ in Y gives $Y=X$, so $m_A^*=m_A^X$ and $s^*=s^X$. Similar reasoning applies for $q>q_p$, $q<q_p$, and the other campaigns.

(c) Expected profit increases in q, q_p, and δ:

$$\frac{\partial V_q}{\partial q}=\delta\sum_{j=0}^{M_1}\binom{M_1}{j}v^j(1-v)^{M_1-j}M_2p\frac{j\gamma}{(\bar{x}-\underline{x})(1+j\gamma)}>0,$$
$$\frac{\partial V_{q_p}}{\partial q_p}=\frac{M_1p}{\bar{x}-\underline{x}}+\delta\sum_{j=0}^{M_1}\binom{M_1}{j}v^j(1-v)^{M_1-j}M_2p\frac{1}{(\bar{x}-\underline{x})(1+j\gamma)}>0,$$
$$\frac{\partial V_\delta}{\partial\delta}>0.$$

(d) Variation w.r.t. γ:

$$\frac{\partial V_\gamma}{\partial \gamma} = \delta \sum_{j=0}^{M_1} \binom{M_1}{j} v^j (1-v)^{M_1-j} \frac{M_2 p j (q-q_p)}{(\bar{x}-\underline{x})(1+j\gamma)^2}.$$

Sign depends on $q - q_p$; applies similarly for other campaigns.

■

Proof of Proposition 5:

(a) Let $D_1 = \min\{T, Z_1\}$, and $D_2 \le Z_2$ denote the number of purchases in period 2. Given $D_1 = j$, the probability that a period-2 customer purchases is:

$$\begin{aligned} w_j = \frac{\phi}{\bar{x}-\underline{x}} \Big(& 1 - (1-\psi_A+\psi_A e^{-\alpha})^{m_A} (1-\rho_A\psi_A+\rho_A\psi_A e^{-\alpha})^{n_A} \\ & \times (1-\psi_B+\psi_B e^{-\beta})^{m_B} (1-\rho_B\psi_B+\rho_B\psi_B e^{-\beta})^{n_B} \Big) \\ & + \frac{\bar{x}-p}{\bar{x}-\underline{x}} + \frac{1}{\bar{x}-\underline{x}} \left(\frac{q_p}{1+j\gamma} + \frac{qj\gamma}{1+j\gamma} \right), \end{aligned}$$

where the products capture both customer types.

The expected profit is:

$$\begin{aligned} E[\Pi] = & \sum_{j=0}^{M_1} \binom{M_1}{j} v^j (1-v)^{M_1-j} \left[jp + \delta p \sum_{k=0}^{M_2} \binom{M_2}{k} w_j^k (1-w_j)^{M_2-k} \min(k, Z_2) \right] \\ & - \Gamma + \sum_{j>Z_1} (Z_1 - j)p + \delta p M_2 \frac{(Z_1-j)(q-q_p)\gamma}{(\bar{x}-\underline{x})(1+j\gamma)(1+Z_1\gamma)}. \end{aligned}$$

This reduces to the unconstrained case when $Z_1 \ge M_1$ and $Z_2 \ge M_2$. Hence, capacity reduces expected profit if $q > q_p$, completing part (a).

(b) For $q < q_p$, capacity increases profit if and only if

$$q_p - q > d_q := (\delta(M_2 - Z_2) + 1) \frac{(\bar{x}-\underline{x})(1+Z_1\gamma)(1+M_1\gamma)}{\delta M_2 \gamma}.$$

This follows from comparing expected profit under capacity limits versus unconstrained capacity and bounding the relevant terms.

(c) If $Z_1 \ge M_1$ and $Z_2 < M_2$, capacity cannot increase expected profit because the first-period term saturates ($D_1 = M_1$) and the second-period term is negative. Therefore, the inequality cannot hold.

■

To proceed we will need to introduce some auxiliary notation in the following definition.

Definition 5. We use the following shorthand notation.

$$\begin{aligned} \pi_R = & (1-a_S^R+a_S^R e^{-\alpha})^{m_A^R} (1-a_D^R+a_D^R e^{-\alpha})^{m_A^L} (1-\psi_A+\psi_A e^{-\alpha})^{m_A^N} (1-a_S^R\rho_A+a_S^R\rho_A e^{-\alpha})^{n_A^R} \\ & (1-a_D^R\rho_A+a_D^R\rho_A e^{-\alpha})^{n_A^L} (1-\psi_A\rho_A+\psi_A\rho_A e^{-\alpha})^{n_A^N} (1-b_S^R+b_S^R e^{-\beta})^{m_B^R} (1-b_D^R+b_D^R e^{-\beta})^{m_B^L} \\ & (1-\psi_B+\psi_B e^{-\beta})^{m_B^N} (1-b_S^R\rho_B+b_S^R\rho_B e^{-\beta})^{n_B^R} (1-b_D^R\rho_B+b_D^R\rho_B e^{-\beta})^{n_B^L} \end{aligned}$$

$$
\begin{aligned}
&(1-\psi_B\rho_B+\psi_B\rho_B e^{-\beta})^{n_B^N},\\
\pi_L =&(1-a_D^L+a_D^L e^{-\alpha})^{m_A^R}(1-a_S^L+a_S^L e^{-\alpha})^{m_A^L}(1-\psi_A+\psi_A e^{-\alpha})^{m_A^N}(1-a_D^L\rho_A+a_D^L\rho_A e^{-\alpha})^{n_A^R}\\
&(1-a_S^L\rho_A+a_S^L\rho_A e^{-\alpha})^{n_A^L}(1-\psi_A\rho_A+\psi_A\rho_A e^{-\alpha})^{n_A^N}(1-b_D^L+b_D^L e^{-\beta})^{m_B^R}(1-b_S^L+b_S^L e^{-\beta})^{m_B^L}\\
&(1-\psi_B+\psi_B e^{-\beta})^{m_B^N}(1-b_D^L\rho_B+b_D^L\rho_B e^{-\beta})^{n_B^R}(1-b_S^L\rho_B+b_S^L\rho_B e^{-\beta})^{n_B^L}\\
&(1-\psi_B\rho_B+\psi_B\rho_B e^{-\beta})^{n_B^N},\\
\pi_N =&(1-\psi_A+\psi_A e^{-\alpha})^{m_A^L+m_A^R+m_A^N}\cdot(1-\psi_B+\psi_B e^{-\beta})^{m_B^R+m_B^L+m_B^N}\cdot(1-\psi_A\rho_A+\psi_A\rho_A e^{-\alpha})^{n_A^R+n_A^L+n_A^N}\\
&(1-\psi_B\rho_B+\psi_B\rho_B e^{-\beta})^{n_B^R+n_B^L+n_B^N}.
\end{aligned}
$$

We will need the following two lemmas in the proof of Proposition 6.

Lemma 2. *Under no social learning ($\gamma\to 0$), if the three influencer classes exist, the optimal marketing campaign is to not recruit neutral influencers. That is, $m_A^N=m_B^N=n_A^N=n_B^N=0$.*

Proof. We show that recruiting an influencer of class N is never optimal.

Consider a campaign where hiring an additional influencer from group A and class N is more profitable than hiring one from class R or class L. The expected profit for any campaign is:

$$E[\Pi]=\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(1-P_R\pi_R-P_L\pi_L-P_N\pi_N\Big)-\Gamma.$$

If the additional influencer is from class N:

$$
\begin{aligned}
E[\Pi]_N=\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(&1-P_R\pi_R(1-\psi_A+\psi_A e^{-\alpha})-P_L\pi_L(1-\psi_A+\psi_A e^{-\alpha})\\
&-P_N\pi_N(1-\psi_A+\psi_A e^{-\alpha})\Big)-\Gamma-(\psi_A C+\kappa).
\end{aligned}
$$

If from class R:

$$
\begin{aligned}
E[\Pi]_R=\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(&1-P_R\pi_R(1-a_S^R+a_S^R e^{-\alpha})-P_L\pi_L(1-a_D^L+a_D^L e^{-\alpha})\\
&-P_N\pi_N(1-\psi_A+\psi_A e^{-\alpha})\Big)-\Gamma-(\psi_A C+\kappa).
\end{aligned}
$$

If from class L:

$$
\begin{aligned}
E[\Pi]_L=\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(&1-P_R\pi_R(1-a_D^R+a_D^R e^{-\alpha})-P_L\pi_L(1-a_S^L+a_S^L e^{-\alpha})\\
&-P_N\pi_N(1-\psi_A+\psi_A e^{-\alpha})\Big)-\Gamma-(\psi_A C+\kappa).
\end{aligned}
$$

Assuming hiring class N is more profitable than class R, we get:

$$
\begin{aligned}
&\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(1-P_R\pi_R(1-\psi_A+\psi_A e^{-\alpha})-P_L\pi_L(1-\psi_A+\psi_A e^{-\alpha})\\
&\quad-P_N\pi_N(1-\psi_A+\psi_A e^{-\alpha})\Big)-\Gamma-(\psi_A C+\kappa)\\
&>\frac{(M_1+\delta M_2)p\phi}{\bar{x}-\underline{x}}\Big(1-P_R\pi_R(1-a_S^R+a_S^R e^{-\alpha})-P_L\pi_L(1-a_D^L+a_D^L e^{-\alpha})\\
&\quad-P_N\pi_N(1-\psi_A+\psi_A e^{-\alpha})\Big)-\Gamma-(\psi_A C+\kappa).
\end{aligned}
$$

Simplifying leads to:

$$P_R\pi_R(a_S^R-\psi_A) < P_L\pi_L(\psi_A-a_D^L).$$

Using $a_S^R P_R + a_D^L P_L + \psi_A P_N = \psi_A$ and $P_R + P_L + P_N = 1$, we find:

$$a_S^R = \frac{\psi_A(P_R+P_L)-a_D^L P_L}{P_R} \implies \pi_R < \pi_L.$$

Similarly, hiring class N over class L would require $\pi_L < \pi_R$, a contradiction. Hence, $m_A^N = m_B^N = n_A^N = n_B^N = 0$. ∎

LEMMA 3. *Under no social learning ($\gamma \to 0$), if only one biased influencer class exists, the optimal marketing strategy is to recruit only neutral influencers. Specifically, if*

$$(m_A^R + m_B^R + n_A^R + n_B^R)\cdot(m_A^L + m_B^L + n_A^L + n_B^L) = 0,$$

then the optimal solution is

$$m_A^R = m_B^R = n_A^R = n_B^R = m_A^L = m_B^L = n_A^L = n_B^L = 0.$$

Proof. Consider hiring group A influencers when $m_A^L = 0$. Using the same approach as Lemma 2, we compare hiring class R versus class N:

$$\begin{aligned}&(1-a_S^R+a_S^R e^{-\alpha})^{m_A^R}(1-a_D^R+a_D^R e^{-\alpha})^{m_A^L}(1-\psi_A+\psi_A e^{-\alpha})^{m_A^N}\\ &> (1-a_D^L+a_D^L e^{-\alpha})^{m_A^R}(1-a_S^L+a_S^L e^{-\alpha})^{m_A^L}(1-\psi_A+\psi_A e^{-\alpha})^{m_A^N}.\end{aligned}$$

With $m_A^L = 0$, this reduces to:

$$(1-a_S^R+a_S^R e^{-\alpha})^{m_A^R} > (1-a_D^L+a_D^L e^{-\alpha})^{m_A^R},$$

which cannot hold because $0 < a_D^L < a_S^R < 1$ and $1-e^{-\alpha} > 0$.

Hence, the optimal strategy is $m_A^N \geq 0$ and $m_A^R = 0$ if $m_A^L = 0$. This extends similarly to all other classes, showing it is never optimal to recruit biased influencers when the other biased class is absent. ∎

Proof of Proposition 6:

Based on Lemma 2, the optimal marketing campaign does not include neutral influencers as $\gamma \to 0$. Recruiting influencers from each biased class is more profitable than relying solely on neutral influencers. Conversely, Lemma 3 shows that if only one biased class exists, recruiting neutral influencers is more profitable. Hence, in the absence of social learning, the optimal strategy involves recruiting influencers from each biased class while excluding neutral influencers, unless only one biased class is available.

Next, consider the conditions for profitability. For a given marketing campaign, the probability v of a first-period purchase is:

$$v = \frac{\bar{x}-x^M}{\bar{x}-\underline{x}} + \frac{\phi}{\bar{x}-\underline{x}}\Big(P_R(1-\pi_R)+P_L(1-\pi_L)+P_N(1-\pi_N)\Big).$$

The expected profit is:

$$E[\Pi] = E[\Pi^L] + E[\Pi^S] + \delta E[\Pi^R] - \Gamma$$

$$= M_1 pv \;+\; \delta \sum_{j=0}^{M_1} \binom{M_1}{j} v^j (1-v)^{M_1-j} M_2 p$$
$$\times \left[\frac{\phi}{\bar{x}-\underline{x}} \big(P_R(1-\pi_R) + P_L(1-\pi_L) + P_N(1-\pi_N) \big) \;+\; \frac{\bar{x}-p}{\bar{x}-\underline{x}} \;+\; \frac{q_p + qj\gamma}{(\bar{x}-\underline{x})(1+j\gamma)} \right] - \Gamma.$$

Define

$$W = \sum_{j=0}^{M_1} \binom{M_1}{j} \frac{v^j (1-v)^{M_1-j}}{v(1-v)} \frac{M_1 v - j}{\bar{x}-\underline{x}} \frac{q_p + qj\gamma}{1+j\gamma}.$$

The partial derivative of $E[\Pi]$ with respect to m_A^R is:

$$\frac{\partial E[\Pi]}{\partial m_A^R} = -\frac{p\phi}{\bar{x}-\underline{x}} \Bigg[P_R \pi_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) \;+\; P_L \pi_L \ln(1 - a_D^L + a_D^L e^{-\alpha})$$
$$+ P_N \pi_N \ln(1 - \psi_A + \psi_A e^{-\alpha}) \Bigg] \Big[M_1 + \delta M_2 (1 - W) \Big] - (\psi_A C + \kappa).$$

Recruiting group A influencers from class R is **unprofitable** if

$$W > \frac{M_1 + \delta M_2}{\delta M_2} \implies \frac{\partial E[\Pi]}{\partial m_A^R} < 0 \text{ for all } m_A^R.$$

If $W < \frac{M_1+\delta M_2}{\delta M_2}$, recruiting is profitable when all influencer counts are zero, i.e.,

$$m_A^N = m_B^N = n_A^N = n_B^N = m_A^R = m_B^R = n_A^R = n_B^R = m_A^L = m_B^L = n_A^L = n_B^L = 0,$$

if

$$- \frac{P_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) + P_L \ln(1 - a_D^L + a_D^L e^{-\alpha}) + P_N \ln(1 - \psi_A + \psi_A e^{-\alpha})}{\psi_A C + \kappa}$$
$$> \frac{\bar{x} - \underline{x}}{(M_1 + \delta M_2 (1 - W)) p\phi}.$$

Equivalently, recruiting is profitable if

$$h^R(A) > Y_C, \quad \text{where } Y_C = \frac{\bar{x} - \underline{x}}{(M_1 + \delta M_2 (1 - W)) p\phi}.$$

Here, $h^R(A)$ represents the cost-effectiveness of hiring group A influencers from class R (Definition 4). A similar calculation applies to other partnership schemes.

Inspecting Y_C, W, and X, we observe:

- $Y_C = X > 0$ if $q = q_p$,
- $0 < Y_C < X$ if $q > q_p$,
- if $q < q_p$ and $W > \frac{M_1+\delta M_2}{\delta M_2}$, then $Y_C < 0$ and $\partial E[\Pi]/\partial m_A^R < 0$, making recruitment unprofitable,
- if $q < q_p$ and $W < \frac{M_1+\delta M_2}{\delta M_2}$, then $Y_C > X > 0$.

The same reasoning applies to all other partnership schemes. ∎

Proof of Proposition 7

(a) We adjust the partial derivatives of $E[\Pi]$ with respect to m_A^R and m_A^L to account for review propensity, similar to the proof of Proposition 6. Define the probability that a customer purchases in the first period and writes a review as

$$w = \frac{\bar{x} - x^M}{\bar{x} - \underline{x}}\Big(P_R\eta_R + P_L\eta_L + P_N\eta_N\Big) + \frac{\phi}{\bar{x} - \underline{x}}\Big(P_R\eta_R(1-\pi_R) + P_L\eta_L(1-\pi_L) + P_N\eta_N(1-\pi_N)\Big).$$

The expected profit is

$$\begin{aligned}
E[\Pi] &= E[\Pi^L] + E[\Pi^S] + \delta E[\Pi^R] - \Gamma \\
&= M_1 p\left(\frac{\bar{x} - x^M}{\bar{x} - \underline{x}} + \frac{\phi}{\bar{x} - \underline{x}}(P_R(1-\pi_R) + P_L(1-\pi_L) + P_N(1-\pi_N))\right) \\
&\quad + \delta M_2 p \sum_{j=0}^{M_1} \binom{M_1}{j} w^j (1-w)^{M_1-j}\left[\frac{\phi}{\bar{x} - \underline{x}}(P_R(1-\pi_R) + P_L(1-\pi_L) + P_N(1-\pi_N))\right. \\
&\quad \left. + \frac{\bar{x} - p}{\bar{x} - \underline{x}} + \frac{q_p + qj\gamma}{(\bar{x} - \underline{x})(1+j\gamma)}\right] - \Gamma.
\end{aligned}$$

The partial derivatives with respect to influencer counts are:

$$\begin{aligned}
\frac{\partial E[\Pi]}{\partial m_A^R} &= -\frac{p\phi(M_1 + \delta M_2)}{\bar{x} - \underline{x}}\Big(P_R\pi_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) + P_L\pi_L \ln(1 - a_D^L + a_D^L e^{-\alpha}) \\
&\quad + P_N\pi_N \ln(1 - \psi_A + \psi_A e^{-\alpha})\Big) \\
&\quad + \frac{M_2\delta p}{(\bar{x} - \underline{x})w(1-w)} \sum_{j=0}^{M_1} \binom{M_1}{j} w^j (1-w)^{M_1-j} \frac{\phi(M_1 w - j)}{\bar{x} - \underline{x}} \\
&\quad \times \Big(P_R\eta_R\pi_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) + P_L\eta_L\pi_L \ln(1 - a_D^L + a_D^L e^{-\alpha}) \\
&\quad + P_N\eta_N\pi_N \ln(1 - \psi_A + \psi_A e^{-\alpha})\Big)\frac{q_p + qj\gamma}{1 + j\gamma} - (\psi_A C + \kappa), \\
\frac{\partial E[\Pi]}{\partial m_A^L} &= -\frac{p\phi(M_1 + \delta M_2)}{\bar{x} - \underline{x}}\Big(P_R\pi_R \ln(1 - a_D^R + a_D^R e^{-\alpha}) + P_L\pi_L \ln(1 - a_S^L + a_S^L e^{-\alpha}) \\
&\quad + P_N\pi_N \ln(1 - \psi_A + \psi_A e^{-\alpha})\Big) \\
&\quad + \frac{M_2\delta p}{(\bar{x} - \underline{x})w(1-w)} \sum_{j=0}^{M_1} \binom{M_1}{j} w^j (1-w)^{M_1-j} \frac{\phi(M_1 w - j)}{\bar{x} - \underline{x}} \\
&\quad \times \Big(P_R\eta_R\pi_R \ln(1 - a_D^R + a_D^R e^{-\alpha}) + P_L\eta_L\pi_L \ln(1 - a_S^L + a_S^L e^{-\alpha}) \\
&\quad + P_N\eta_N\pi_N \ln(1 - \psi_A + \psi_A e^{-\alpha})\Big)\frac{q_p + qj\gamma}{1 + j\gamma} - (\psi_A C + \kappa).
\end{aligned}$$

Define the weighted sum

$$W_R = \sum_{j=0}^{M_1} \binom{M_1}{j} \frac{w^j (1-w)^{M_1-j}}{w(1-w)} \frac{M_1 w - j}{\bar{x} - \underline{x}} \frac{q_p + qj\gamma}{1+j\gamma}.$$

Case analysis:

- $q \geq q_p$: $W_R \leq 0$. Lower bounds of partial derivatives:

$$\frac{\partial E[\Pi]}{\partial m_A^R} \geq -\frac{p\phi(M_1+\delta M_2)}{\bar{x}-\underline{x}} \Big(P_R \pi_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) + \cdots \Big) - (\psi_A C + \kappa),$$
$$\frac{\partial E[\Pi]}{\partial m_A^L} \geq -\frac{p\phi(M_1+\delta M_2)}{\bar{x}-\underline{x}} \Big(P_R \pi_R \ln(1 - a_D^R + a_D^R e^{-\alpha}) + \cdots \Big) - (\psi_A C + \kappa).$$

- $q < q_p$: $W_R > 0$. Partial derivatives lower bounds:

$$\frac{\partial E[\Pi]}{\partial m_A^R} \geq -\frac{p\phi}{\bar{x}-\underline{x}} \Big(M_1 + \delta M_2 (1 - W_R) \Big) \Big(P_R \pi_R \ln(1 - a_S^R + a_S^R e^{-\alpha}) + \cdots \Big) - (\psi_A C + \kappa),$$
$$\frac{\partial E[\Pi]}{\partial m_A^L} \geq -\frac{p\phi}{\bar{x}-\underline{x}} \Big(M_1 + \delta M_2 (1 - W_R) \Big) \Big(P_R \pi_R \ln(1 - a_D^R + a_D^R e^{-\alpha}) + \cdots \Big) - (\psi_A C + \kappa).$$

The profitability threshold is

$$Y_R = \frac{\bar{x} - \underline{x}}{(M_1 + \delta M_2 (1 - W_R)) p\phi},$$

and hiring group A influencers from class R is profitable if $h^R(A) > Y_R > 0$, and from class L if $h^L(A) > Y_R > 0$.

(b) When the profitability conditions are met for both biased classes, equality of partial derivatives occurs when

$$P_R \pi_R \ln \frac{1 - a_S^R + a_S^R e^{-\alpha}}{1 - a_D^R + a_D^R e^{-\alpha}} \cdot U_R + P_L \pi_L \ln \frac{1 - a_D^L + a_D^L e^{-\alpha}}{1 - a_S^L + a_S^L e^{-\alpha}} \cdot U_L = 0,$$

where

$$U_R = -\frac{p\phi(M_1+\delta M_2)}{\bar{x}-\underline{x}} + \frac{\delta \eta_R M_2 p\phi}{(\bar{x}-\underline{x})^2 w(1-w)} \sum_{j=0}^{M_1} \binom{M_1}{j} w^j (1-w)^{M_1-j} (M_1 w - j) \frac{q_p + qj\gamma}{1+j\gamma},$$
$$U_L = -\frac{p\phi(M_1+\delta M_2)}{\bar{x}-\underline{x}} + \frac{\delta \eta_L M_2 p\phi}{(\bar{x}-\underline{x})^2 w(1-w)} \sum_{j=0}^{M_1} \binom{M_1}{j} w^j (1-w)^{M_1-j} (M_1 w - j) \frac{q_p + qj\gamma}{1+j\gamma}.$$

This leads to the logarithmic condition for (m_A^R, m_A^L):

$$m_A^R \ln \frac{1 - a_S^R + a_S^R e^{-\alpha}}{1 - a_D^L + a_D^L e^{-\alpha}} + m_A^L \ln \frac{1 - a_D^R + a_D^R e^{-\alpha}}{1 - a_S^L + a_S^L e^{-\alpha}} = \ln \frac{U_L}{U_R} + \ln \frac{P_L(\ln(1 - a_S^L + a_S^L e^{-\alpha}) - \ln(1 - a_D^L + a_D^L e^{-\alpha}))}{P_R(\ln(1 - a_S^R + a_S^R e^{-\alpha}) - \ln(1 - a_D^R + a_D^R e^{-\alpha}))}. \tag{9}$$

(c) The sign of $\ln(U_L/U_R)$ depends on q relative to q_p, which in turn determines how η_R and η_L affect m_A^R and m_A^L:

- $q > q_p$: Increasing η_R (or decreasing η_L) increases m_A^R and/or decreases m_A^L.
- $q < q_p$: Increasing η_R (or decreasing η_L) decreases m_A^R and/or increases m_A^L.
- $q = q_p$: m_A^R and m_A^L are unaffected by η_R or η_L.

By symmetry, the same applies to η_N with respect to neutral influencers.

■