Online Appendix for
Experimenting under Stochastic Congestion

Shuangning Li, Ramesh Johari, Xu Kuang, Stefan Wager

Appendix A: Additional Proofs

A.1. Notation and preliminaries

For summations, if $a < b$, then we take the convention that $\sum_{i=a}^{b} x_i = 0$. We use zero-based matrix indexing. Let $\vec{e}_k$ be the vector with its k-th entry (zero-based indexing) 1 and other entries 0. Let $\vec{1}$ be the vector with all entries equal to 1. For a matrix A, let $\|A\|_{op}$ be the operator norm of matrix A.

Let $\vec{v} = (v_0, \ldots, v_K)$ be the initial distribution of the queue length process. Let

$$Q(p) = \begin{bmatrix} -\lambda_0(p) & \lambda_0(p) & & & \\ \mu & -\lambda_1(p) - \mu & \lambda_1(p) & & \\ & \mu & -\lambda_2(p) - \mu & \lambda_2(p) & \\ & & & \cdots & \\ & & & \mu & -\mu \end{bmatrix} \tag{1}$$

be the transition rate matrix (a.k.a. generator) of the continuous-time Markov chain corresponding to the queue length process. Throughout the section, we assume that $\lambda_k(p) > 0$ for any $p > 0$ and $k = 0, 1, \ldots, K-1$. Let $\pi(p)$ be the steady-state probabilities. Let $D_{\sqrt{\pi}}(p)$ be the diagonal matrix with entries $D_{\sqrt{\pi},i,i}(p) = \sqrt{\pi_i(p)}$. Let $\tilde{Q}(p) = D_{\sqrt{\pi}}(p) Q(p) D_{\sqrt{\pi}}^{-1}(p)$. By properties of the steady-state probabilities, we have that $\tilde{Q}(p)$ is a symmetric matrix. Let $\tilde{Q}(p) = \tilde{U}(p) D(p) \tilde{U}(p)^\top$ be the eigen-decomposition of $\tilde{Q}(p)$, where $\tilde{U}(p)$ is an orthogonal matrix and $D(p)$ is a diagonal matrix. With this, we can express $Q(p)$ as

$$Q(p) = D_{\sqrt{\pi}}^{-1}(p) Q(p) D_{\sqrt{\pi}}(p) = D_{\sqrt{\pi}}^{-1}(p) \tilde{U}(p) D(p) \tilde{U}(p)^\top D_{\sqrt{\pi}}(p) = U(p) D(p) U^{-1}(p), \tag{2}$$

where $U(p) = D_{\sqrt{\pi}}^{-1}(p) \tilde{U}(p)$.

The continuous-time Markov chain is clearly irreducible, and thus by Perron–Frobenius theorem, $Q(p)$ has a zero eigenvalue, and the rest of the eigenvalues are negative. Without loss of generality, we assume $D_{0,0}(p) = 0$. Let $Q^{\#}(p)$ be the group inverse of $Q(p)$, i.e.,

$$Q^{\#}(p) = U(p) D^{\#}(p) U^{-1}(p), \tag{3}$$

where $D_{0,0}^{\#}(p) = 0$ and $D_{k,k}^{\#}(p) = D_{k,k}^{-1}(p)$ for $k > 0$.[1]

Corresponding to the zero eigenvalue, the right eigenvector of $Q(p)$ is $\vec{1}$, and the left eigenvector is $\vec{\pi}(p)$. Thus we can decompose $U(p)$ into two parts: $U(p) = (\vec{1}, U_2(p))$, where $U_2(p)$ is a $(K+1) \times K$ matrix. We can do the same for $U^{-1}(p)$: $U^{-1}(p) = \left(\vec{\pi}(p), U_2^{-1}(p)\right)^\top$, where $U_2^{-1}(p)$ is a $(K+1) \times K$ matrix.

For a matrix M, we write $\mathrm{ME}(M) = \max_{i,j} \left|M_{ij}\right|$ as the maximum of the absolute values of the entries of M. For any $\alpha \geq \mathrm{ME}(Q(p))$, let $P(p, \alpha) = I + Q(p)/\alpha$. By construction, $P(p, \alpha)$ is a stochastic matrix with nonnegative entries and $P(p, \alpha)\vec{1} = \vec{1}$. By (2), we can also write

$$P(p, \alpha) = U(p) \left(I + \frac{1}{\alpha} D(p) \right) U^{-1}(p). \tag{4}$$

[1] Note another way of defining $Q^{\#}$ is by setting $Q^{\#}(p) = -(\vec{1}\vec{\pi}^\top - Q)^{-1}(I - \vec{1}\vec{\pi})$. The two ways are equivalent.

Again, by Perron–Frobenius theorem, we have that $D_{k,k}(p)/\alpha + 1 \geq 0$, and hence $\left|D_{k,k}(p)\right| \leq \alpha$.

For a fixed price p_0 and price perturbation ζ_0, let

$$\alpha_0(p_0,\zeta_0) = \sup_{p\in[p_0-\zeta_0,p_0+\zeta_0]} \mathrm{ME}(Q(p)) = \sup_{p\in[p_0-\zeta_0,p_0+\zeta_0]} \max_{i,j} \left|Q_{ij}(p)\right|. \tag{5}$$

Furthermore, let

$$\beta_0(p_0,\zeta_0) = \inf_{p\in[p_0-\zeta_0,p_0+\zeta_0]} \min_{k\geq 1} \left|D_{k,k}(p)\right|. \tag{6}$$

Note that $\beta_0(p_0,\zeta_0) > 0$ since $\left|D_{k,k}(p)\right| > 0$ for any $k \geq 1$ and $p > 0$, and eigenvalues are continuous functions of matrices. Further, by the analysis following (4), we note that $\alpha_0 \geq \beta_0$.

In the following sections, we use the notation $C(A,B,\ldots,)$ to denote constants that depend on $A,B,\ldots,$ but not on other parameters. The notation C may take different values at different places.

A.2. Lemmas

LEMMA 1. *Let $R_i(p,p_0,\zeta_0) = P^i(p,\alpha_0(p_0,\zeta_0)) - \vec{1}\vec{\pi}(p)^\top$, where α_0 is defined in (5) and $P(p,\alpha) = I + Q(p)/\alpha$. Then there exists a constant $C(K,p_0,\zeta_0)\in(0,1)$, such that for any $p\in[p_0-\zeta_0,p_0+\zeta_0]$ and any $i\in\mathbb{N}_+$,*

$$\|R_i(p,p_0,\zeta_0)\|_{op} \leq C^i(K,p_0,\zeta_0), \tag{7}$$

where $\|\|_{op}$ stands for the operator norm of a matrix.

Proof. We have shown in (4) that $P(p,\alpha_0(p_0,\zeta_0)) = U(p)\left(I + \frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)U^{-1}(p)$. Therefore,

$$P^i(p,\alpha_0(p_0,\zeta_0)) = U(p)\left(I + \frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)^i U^{-1}(p). \tag{8}$$

As discussed in Section A.1, $D_{0,0}(p)$ is zero, and all the rest diagonal entries $D_{k,k}(p)$ are negative. We also note from the discussion in Section A.1 that $U(p) = (\vec{1},U_2(p))$, where $U_2(p)$ is a $(K+1)\times K$ matrix and $U^{-1}(p) = \left(\vec{\pi}(p),U_2^{-1}(p)\right)^\top$, where $U_2^{-1}(p)$ is a $(K+1)\times K$ matrix. Therefore,

$$\begin{aligned} R_i(p,p_0,\zeta_0) &= P^i(p,\alpha_0(p_0,\zeta_0)) - \vec{1}\vec{\pi}^\top \\ &= U(p)\left(\left(I + \frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)^i - \vec{e}_0\vec{e}_0^\top\right)U^{-1}(p). \end{aligned} \tag{9}$$

Let $\Sigma = \left(\left(I + \frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)^i - \vec{e}_0\vec{e}_0^\top\right)$. The matrix Σ is clearly a diagonal matrix with $\Sigma_{0,0} = 0$ and

$$\Sigma_{k,k} = \left(1 + \frac{D_{k,k}(p)}{\alpha_0(p_0,\zeta_0)}\right)^i \leq \left(1 - \frac{\beta_0(p_0,\zeta_0)}{\alpha_0(p_0,\zeta_0)}\right)^i \tag{10}$$

for $k\geq 1$, where β_0 is defined in (6). Take $C(K,p_0,\zeta_0) = 1 - \beta_0(p_0,\zeta_0)/\alpha_0(p_0,\zeta_0)$. Note that we have shown that $C(K,p_0,\zeta_0)\in[0,1)$ in Section A.1. Therefore, we have that the absolute value of the eigenvalues of $R_i(p,p_0,\zeta_0)$ are all bounded by $C(K,p_0,\zeta_0)^i$. The result then follows from noting that the operator norm of a diagonalizable matrix is the same as its largest eigenvalue (in absolute value).

LEMMA 2. *There exists a constant $C(K,p_0,\zeta_0)\in(0,1)$ and $C_1(K,p_0,\zeta_0) > 0$, such that for any $p\in[p_0-\zeta_0,p_0+\zeta_0]$, and any $m\in\mathbb{N}_+$,*

$$\begin{aligned} &\left\|\sum_{i=0}^{m-1} P^i(p,\alpha_0(p_0,\zeta_0)) - m\vec{1}\vec{\pi}(p)^\top + \alpha_0(p_0,\zeta_0)Q^\#(p)\right\|_{op} \\ &\leq C_1(K,p_0,\zeta_0)C^m(K,p_0,\zeta_0). \end{aligned} \tag{11}$$

Here α_0 is defined in (5).

Proof. We note that by (4),

$$\begin{aligned}\sum_{i=0}^{m-1} P^i(p,\alpha_0(p_0,\zeta_0)) &= \sum_{i=0}^{m-1}\left(U(p)\left(I+\frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)U^{-1}(p)\right)^i \\ &= \sum_{i=0}^{m-1} U(p)\left(I+\frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)^i U^{-1}(p) \\ &= U(p)\left(\sum_{i=0}^{m-1}\left(I+\frac{1}{\alpha_0(p_0,\zeta_0)}D(p)\right)^i\right)U^{-1}(p).\end{aligned} \tag{12}$$

As discussed in Section A.1, $D_{0,0}(p)$ is zero, and all the rest diagonal entries are negative. Therefore,

$$\sum_{i=0}^{m-1}\left(1+\frac{1}{\alpha_0(p_0,\zeta_0)}D_{0,0}(p)\right)^i = \sum_{i=0}^{m-1} 1 = m. \tag{13}$$

At the same time, for $k \geq 1$,

$$\sum_{i=0}^{m-1}\left(1+\frac{1}{\alpha_0(p_0,\zeta_0)}D_{k,k}(p)\right)^i = -\frac{\alpha_0(p_0,\zeta_0)}{D_{k,k}(p)} + \frac{\alpha_0(p_0,\zeta_0)}{D_{k,k}(p)}\left(1+\frac{D_{k,k}(p)}{\alpha_0(p_0,\zeta_0)}\right)^m. \tag{14}$$

where

$$0 \leq \left(1+\frac{D_{k,k}(p)}{\alpha_0(p_0,\zeta_0)}\right)^m \leq \left(1-\frac{\beta_0(p_0,\zeta_0)}{\alpha_0(p_0,\zeta_0)}\right)^m. \tag{15}$$

Thus,

$$\begin{aligned}&\sum_{i=0}^{m-1} P^i(p,\alpha_0(p_0,\zeta_0)) \\ &= U(p)\begin{bmatrix} m & & & \\ & -\frac{\alpha_0(p_0,\zeta_0)}{D_{1,1}(p)} & & \\ & & \ldots & \\ & & & -\frac{\alpha_0(p_0,\zeta_0)}{D_{K,K}(p)}\end{bmatrix}U^{-1}(p) + U(p)\Sigma U^{-1}(p) \\ &= (\vec{1}, U_2(p))\begin{bmatrix} m & & & \\ & -\frac{\alpha_0(p_0,\zeta_0)}{D_{1,1}(p)} & & \\ & & \ldots & \\ & & & -\frac{\alpha_0(p_0,\zeta_0)}{D_{K,K}(p)}\end{bmatrix}\left(\vec{\pi}(p), U_2^{-1}(p)\right)^\top + U(p)\Sigma U^{-1}(p) \\ &= m\vec{1}\vec{\pi}(p)^\top - \alpha_0(p_0,\zeta_0)U(p)D^\#(p)U^{-1}(p) + U(p)\Sigma U^{-1}(p) \\ &= m\vec{1}\vec{\pi}(p)^\top - \alpha_0(p_0,\zeta_0)Q^\#(p) + U(p)\Sigma U^{-1}(p),\end{aligned} \tag{16}$$

where Σ is a diagonal matrix with $\Sigma_{k,k}=0$ and $\Sigma_{k,k} = \frac{\alpha_0(p_0,\zeta_0)}{D_{k,k}(p)}\left(1+\frac{D_{k,k}(p)}{\alpha_0(p_0,\zeta_0)}\right)^m$ for $k \geq 1$. Therefore,

$$\mathrm{ME}(\Sigma) \leq \frac{\alpha_0(p_0,\zeta_0)}{\beta_0(p_0,\zeta_0)}\left(1-\frac{\beta_0(p_0,\zeta_0)}{\alpha_0(p_0,\zeta_0)}\right)^m. \tag{17}$$

Take $C(K,p_0,\zeta_0) = 1-\beta_0(p_0,\zeta_0)/\alpha_0(p_0,\zeta_0)$ and $C_1(K,p_0,\zeta_0) = \alpha_0(p_0,\zeta_0)/\beta_0(p_0,\zeta_0)$. Note that we have shown that $C(K,p_0,\zeta_0) \in [0,1)$ in Section A.1. Therefore, we have that the absolute value of the eigenvalues of $U(p)\Sigma U^{-1}(p)$ are all bounded by $C_1(K,p_0,\zeta_0)C^m(K,p_0,\zeta_0)$. The result then follows from noting that the operator norm of a diagonalizable matrix is the same as its largest eigenvalue (in absolute value).

LEMMA 3 **(Number of eligible jumps)**. *Consider the continuous-time Markov chain with transition rate matrix defined in* (1)*. Let $\mathcal{J}$ be a set of eligible jumps, i.e., $\mathcal{J}$ is a set of (i,j) pairs such that $i,j \in \{0,\ldots,K\}$. Assume that*

$(k,k)\notin\mathcal{J}$. Here the pair (i,j) stands for a jump from state i to state j. Let $N_{\mathcal{J}}(T,p)$ be the number of eligible jumps in time $[0,T]$. Let $\tilde{Q}$ be a matrix such that $\tilde{Q}_{i,j}=Q_{i,j}$ if $(i,j)\in\mathcal{J}$ and otherwise $\tilde{Q}_{i,j}=0$.

There is a constant $C(K,p_0,\zeta_0,\mu)$ such that for any $p\in[p_0-\zeta_0,p_0+\zeta_0]$,

$$\left|\mathbb{E}\left[N_{\mathcal{J}}(T,p)\right]-T\vec{\pi}(p)^\top\tilde{Q}(p)\vec{1}\right|\le C(K,p_0,\zeta_0,\mu), \tag{18}$$

$$\left|\mathrm{Var}\left[N_{\mathcal{J}}(T,p)\right]-T\sigma^2_{\mathcal{J}}(p)\right|\le C(K,p_0,\zeta_0,\mu), \tag{19}$$

where

$$\sigma^2_{\mathcal{J}}(p)=\vec{\pi}(p)^\top\tilde{Q}(p)\vec{1}-2\pi(p)^\top\tilde{Q}(p)Q^\#(p)\tilde{Q}(p)\vec{1}. \tag{20}$$

Here $Q^\#$ is the group inverse of Q defined in (3).

Proof. We write $\alpha=\alpha_0(p_0,\zeta_0)$, where α_0 is defined in (5). Throughout this proof, we omit "(p)" and "(α)" for simplicity. We use constants $C_1,C_2,\dots$ that could depend on K,μ,p_0,ζ_0 but not on p.

We start with a uniformization step. The continuous-time Markov chain we considered can be described by a discrete-time Markov chain with transition matrix $P=I+Q/\alpha$ where jumps occur according to a Poisson process with intensity α. Let $M(T)$ be the total number of jumps of the discrete-time Markov chain. Note that among the jumps, there are some self jumps that go from state k to state k. We are interested in the eligible jumps in $\mathcal{J}$.

Note that $M(T)\sim\mathrm{Poisson}(\alpha T)$ and that $\mathrm{Var}\left[N_{\mathcal{J}}(T)\right]=\mathrm{Var}\left[\mathbb{E}\left[N_{\mathcal{J}}(T)\mid M(T)\right]\right]+\mathbb{E}\left[\mathrm{Var}\left[N_{\mathcal{J}}(T)\mid M(T)\right]\right]$. We will then study the distribution of $N_{\mathcal{J}}(T)$ conditioning on $M(T)=m$. To this end, let $\tilde{P}=\tilde{Q}/\alpha$. By Lemma 2, the first moment satisfies

$$\begin{aligned}\mathbb{E}\left[N_{\mathcal{J}}(T)\mid M(T)=m\right]&=\vec{\nu}^\top(I+P+P^2+\dots P^{m-1})\tilde{P}\vec{1}\\&=\vec{\nu}^\top\left(m\vec{1}\vec{\pi}^\top-\alpha Q^\#\right)\tilde{P}\vec{1}+\mathrm{error}_1\\&=m\vec{\pi}^\top\tilde{P}\vec{1}-\alpha\vec{\nu}^\top Q^\#\tilde{P}\vec{1}+\mathrm{error}_1,\end{aligned} \tag{21}$$

where $|\mathrm{error}_1|\le C_1C_2^m$ for some constants $C_1>0$ and $C_2\in(0,1)$. Therefore,

$$\begin{aligned}\mathbb{E}\left[N_{\mathcal{J}}(T)\right]=\mathbb{E}\left[\mathbb{E}\left[N_{\mathcal{J}}(T)\mid M(T)\right]\right]&=\mathbb{E}\left[M(T)\right]\vec{\pi}^\top\tilde{P}\vec{1}+\mathrm{error}_1'\\&=\vec{\pi}^\top\tilde{Q}\vec{1}T+\mathrm{error}_1',\end{aligned} \tag{22}$$

where $\left|\mathrm{error}_1'\right|\le C_1'$ for some $C_1'>0$.

The second moment involves more analyses. We write $N_{\mathcal{J}}(T)=\sum_{i=1}^{M(T)}X_i$, where $X_i=1$ if the i-th jump in the discrete-time Markov chain is an eligible jump. Then

$$\begin{aligned}\mathbb{E}\left[N^2_{\mathcal{J}}(T)\mid M(T)=m\right]&=\mathbb{E}\left[\left(\sum_{i=1}^m X_i\right)^2\mid M(T)=m\right]\\&=\mathbb{E}\left[\sum_{i=1}^m X_i+2\sum_{j=1}^{m-1}\sum_{i=1}^{m-j}X_iX_{i+j}\mid M(T)=m\right]\\&=\sum_{i=1}^m\vec{\nu}^\top P^{i-1}\tilde{P}\vec{1}+2\sum_{j=1}^{m-1}\sum_{i=1}^{m-j}\vec{\nu}^\top P^{i-1}\tilde{P}P^{j-1}\tilde{P}\vec{1}.\end{aligned} \tag{23}$$

For the first summation, we can again apply Lemma 2 and get

$$\begin{aligned}\sum_{i=1}^m\vec{\nu}^\top P^{i-1}\tilde{P}\vec{1}&=\vec{\nu}^\top\left(m\vec{1}\vec{\pi}^\top-\alpha Q^\#\right)\tilde{P}\vec{1}+\mathrm{error}_1\\&=m\vec{\pi}^\top\tilde{P}\vec{1}-\alpha\vec{\nu}^\top Q^\#\tilde{P}\vec{1}+\mathrm{error}_1,\end{aligned} \tag{24}$$

where $|\text{error}_1| \leq C_1 C_2^m$ for some constants $C_1 > 0$ and $C_2 \in (0,1)$. For the second summation, we have that

$$\begin{aligned} 2\sum_{j=1}^{m-1}\sum_{i=1}^{m-j} \vec{v}^\top P^{i-1}\tilde{P}P^{j-1}\tilde{P}\vec{1} &= 2\sum_{j=1}^{m-1}\vec{v}^\top\left(\sum_{i=1}^{m-j}P^{i-1}\right)\tilde{P}P^{j-1}\tilde{P}\vec{1} \\ &= 2\sum_{j=1}^{m-1}\vec{v}^\top\left((m-j)\vec{1}\vec{\pi}^\top - \alpha Q^{\#} + E_j\right)\tilde{P}P^{j-1}\tilde{P}\vec{1}, \end{aligned} \tag{25}$$

where $\|E_j\|_{op} \leq C_1 C_2^{m-j}$ for some constants $C_1 > 0$ and $C_2 \in (0,1)$. We further note that

$$2\sum_{j=1}^{m-1}\vec{v}^\top E_j\tilde{P}P^{j-1}\tilde{P}\vec{1} \leq \sum_{j=1}^{m-1} C_3(1-C_2)^{m-j} \leq C_4, \tag{26}$$

for some constants $C_3, C_4 > 0$. Plugging this back into (25), we get that

$$\begin{aligned} &2\sum_{j=1}^{m-1}\sum_{i=1}^{m-j}\vec{v}^\top P^{i-1}\tilde{P}P^{j-1}\tilde{P}\vec{1} \\ &\quad= 2\sum_{j=1}^{m-1}\vec{v}^\top\left((m-j)\vec{1}\vec{\pi}^\top - \alpha Q^{\#} + E_j\right)\tilde{P}P^{j-1}\tilde{P}\vec{1} \\ &\quad= 2\vec{\pi}^\top\tilde{P}\left(\sum_{j=1}^{m-1}(m-j)P^{j-1}\right)\tilde{P}\vec{1} - 2\vec{v}^\top\alpha Q^{\#}\tilde{P}\left(\sum_{j=1}^{m-1}P^{j-1}\right)\tilde{P}\vec{1} + \text{error}_2, \end{aligned} \tag{27}$$

where $|\text{error}_2| \leq C_4$. For the first summation in (27), note that

$$\begin{aligned} \sum_{j=1}^{m-1}(m-j)P^{j-1} &= \sum_{i=1}^{m-1}\sum_{j=1}^{i}P^{j-1} = \sum_{i=1}^{m-1}\left(i\vec{1}\vec{\pi}^\top - \alpha Q^{\#} + E_i\right) \\ &= \sum_{i=1}^{m-1}\left(i\vec{1}\vec{\pi}^\top - \alpha Q^{\#}\right) + E_0 \\ &= \frac{m(m-1)}{2}\vec{1}\vec{\pi}^\top - (m-1)\alpha Q^{\#} + E_0, \end{aligned} \tag{28}$$

where, again by Lemma 2, $\|E_i\|_{op} \leq C_1 C_2^i$ for some constants $C_1 > 0$ and $C_2 \in (0,1)$ and $\|E_0\|_{op} \leq C_5$ for some constant $C_5 > 0$. This then implies that the first summation in (27) satisfies

$$\begin{aligned} &2\vec{\pi}^\top\tilde{P}\left(\sum_{j=1}^{m-1}(m-j)P^{j-1}\right)\tilde{P}\vec{1} \\ &\quad= 2\vec{\pi}^\top\tilde{P}\left(\frac{m(m-1)}{2}\vec{1}\vec{\pi}^\top - (m-1)\alpha Q^{\#}\right)\tilde{P}\vec{1} + \text{error}_3 \\ &\quad= m(m-1)\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 - 2\alpha(m-1)\vec{\pi}^\top\tilde{P}Q^{\#}\tilde{P}\vec{1} + \text{error}_3, \end{aligned} \tag{29}$$

where $|\text{error}_3| \leq C_6$ for some constant $C_6 > 0$. For the second summation in (27), we can again apply Lemma 2 and get that

$$\begin{aligned} 2\vec{v}^\top\alpha Q^{\#}\tilde{P}\left(\sum_{j=1}^{m-1}P^{j-1}\right)\tilde{P}\vec{1} &= 2\vec{v}^\top\alpha Q^{\#}\tilde{P}\left(\sum_{j=1}^{m-1}P^{j-1}\right)\tilde{P}\vec{1} \\ &= 2\vec{v}^\top\alpha Q^{\#}\tilde{P}\left((m-1)\vec{1}\vec{\pi}^\top - \alpha Q^{\#}\right)\tilde{P}\vec{1} + \text{error}_4 \\ &= 2\alpha(m-1)\vec{v}^\top Q^{\#}\tilde{P}\vec{1}\left(\vec{\pi}^\top\tilde{P}\vec{1}\right) - 2\alpha^2\vec{v}^\top Q^{\#}\tilde{P}Q^{\#}\tilde{P}\vec{1} + \text{error}_4, \end{aligned} \tag{30}$$

where $|\text{error}_4| \le C_7 C_2^m$ for some constant $C_7 > 0$ and $C_2 \in (0,1)$. Finally, combining (27), (29) and (30), we have that the second summation in (23) satisfies

$$2\sum_{j=1}^{m-1}\sum_{i=1}^{m-j} \vec{v}^\top P^{i-1}\tilde{P}P^{j-1}\tilde{P}\vec{1} = m(m-1)\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 - 2\alpha m\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1} \\ - 2\alpha m\vec{v}^\top Q^\#\tilde{P}\vec{1}\left(\vec{\pi}^\top\tilde{P}\vec{1}\right) + \text{error}_5, \tag{31}$$

where $|\text{error}_5| \le C_8$ for some constant $C_8 > 0$. Now, together with (24) and (23), the above implies that

$$\begin{aligned}\mathbb{E}&\left[N_{\mathcal{J}}^2(T) \mid M(T)=m\right] \\ &= m\vec{\pi}^\top\tilde{P}\vec{1} + m(m-1)\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 - 2\alpha m\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1} \\ &\quad - 2\alpha m\vec{v}^\top Q^\#\tilde{P}\vec{1}\left(\vec{\pi}^\top\tilde{P}\vec{1}\right) + \text{error}_6,\end{aligned} \tag{32}$$

where $|\text{error}_6| \le C_9$ for some constant $C_9 > 0$.

We are ready to study the variance of $N_{\mathcal{J}}(T)$ conditional on $M(T)$.

$$\begin{aligned}\text{Var}&\left[N_{\mathcal{J}}(T) \mid M(T)=m\right] \\ &= \mathbb{E}\left[N_{\mathcal{J}}^2(T) \mid M(T)=m\right] - \left(\mathbb{E}\left[N_{\mathcal{J}}(T) \mid M(T)=m\right]\right)^2 \\ &= m\vec{\pi}^\top\tilde{P}\vec{1} + m(m-1)\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 - 2\alpha m\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1} \\ &\quad - 2m\alpha\vec{v}^\top Q^\#\tilde{P}\vec{1}\left(\vec{\pi}^\top\tilde{P}\vec{1}\right) - \left(m\vec{\pi}^\top\tilde{P}\vec{1} - \alpha\vec{v}^\top Q^\#\tilde{P}\vec{1}\right)^2 + \text{error}_7 \\ &= m\left(-\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 + \left(\vec{\pi}^\top\tilde{P}\vec{1}\right) - 2\alpha\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1}\right) + \text{error}_8,\end{aligned} \tag{33}$$

where $|\text{error}_7| \le C_{10}$ and $|\text{error}_8| \le C_{10}$ for some constant $C_{10} > 0$.

Finally, note that

$$\begin{aligned}\text{Var}\left[N_{\mathcal{J}}(T)\right] &= \text{Var}\left[\mathbb{E}\left[N_{\mathcal{J}}(T) \mid M(T)\right]\right] + \mathbb{E}\left[\text{Var}\left[N_{\mathcal{J}}(T) \mid M(T)\right]\right] \\ &= \left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 \text{Var}\left[M(T)\right] + \text{error}_9 \\ &\quad + \left(-\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 + \left(\vec{\pi}^\top\tilde{P}\vec{1}\right) - 2\alpha\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1}\right)\mathbb{E}\left[M(T)\right] \\ &= \left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 \alpha T + \text{error}_9 \\ &\quad + \left(-\left(\vec{\pi}^\top\tilde{P}\vec{1}\right)^2 + \left(\vec{\pi}^\top\tilde{P}\vec{1}\right) - 2\alpha\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1}\right)\alpha T \\ &= \left(\vec{\pi}^\top\tilde{P}\vec{1} - 2\alpha\vec{\pi}^\top\tilde{P}Q^\#\tilde{P}\vec{1}\right)\alpha T + \text{error}_9, \\ &= \left(\vec{\pi}^\top\tilde{Q}\vec{1} - 2\vec{\pi}^\top\tilde{Q}Q^\#\tilde{Q}\vec{1}\right)T + \text{error}_9,\end{aligned} \tag{34}$$

where $|\text{error}_9| \le C_{11}$ and $|\text{error}_9| \le C_{11}$ for some constant $C_{11} > 0$.

LEMMA 4 (**Number of departures**). *Consider the continuous-time Markov chain with transition rate matrix defined in (1). Treating it as a queue length process, let $N_{\text{dep}}(T,p)$ be the number of departures in time $[0,T]$. There is a constant $C(K,p_0,\zeta_0)$ such that for any $p \in [p_0-\zeta_0, p_0+\zeta_0]$,*

$$\left|\mathbb{E}\left[N_{\text{dep}}(T,p)\right] - T\mu(1-\pi_0(p))\right| \le C(K,p_0,\zeta_0,\mu), \tag{35}$$

$$\left|\text{Var}\left[N_{\text{dep}}(T,p)\right] - T\sigma_{\text{dep}}^2(p)\right| \le C(K,p_0,\zeta_0,\mu), \tag{36}$$

where

$$\sigma^2_{\text{dep}}(p) = \mu(1-\pi_0(p)) + 2\mu^2 \sum_{i=0}^{K-1} \pi_{i+1}(p) Q^{\#}_{i0}(p). \tag{37}$$

Here $Q^{\#}$ is the group inverse of Q defined in (3).

Proof. Throughout this proof, we omit "(p)" and "(α)" for simplicity. We use constants $C_1, C_2, \dots$ that could depend on K, μ, p_0, ζ_0 but not on p. We will use Lemma 3 to show the results. We note that the eligible jumps are $\mathcal{J} = \left\{(i,j) \in \{1,\dots,K\}^2 : j = i-1\right\}$. Therefore, in this setting,

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & \dots & & \\ \mu & 0 & & & \\ & \mu & 0 & & \\ & & & \dots & \\ & & & \mu & 0 \end{bmatrix}. \tag{38}$$

We start with the expectation. Note that $\tilde{Q}\vec{1} = (0, \mu, \dots, \mu)^\top$. Therefore,

$$\vec{\pi}^\top \tilde{Q}\vec{1} = \mu \left(\sum_{i=1}^{K} \pi_i \right) = \mu(1-\pi_0). \tag{39}$$

For the variance, we note that since $Q^{\#}\vec{1} = \vec{0}$,

$$Q^{\#}\tilde{Q}\vec{1} = \mu Q^{\#} \begin{bmatrix} 0 \\ 1 \\ \dots \\ 1 \end{bmatrix} = \mu Q^{\#}\vec{1} - \mu Q^{\#} \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} = -\mu \begin{bmatrix} Q^{\#}_{0,0} \\ Q^{\#}_{1,0} \\ \dots \\ Q^{\#}_{K,0} \end{bmatrix}. \tag{40}$$

Note also that $\vec{\pi}^\top \tilde{Q} = \mu(\pi_1, \dots, \pi_K, 0)$. Therefore, $\vec{\pi}^\top \tilde{Q} Q^{\#} \tilde{Q}\vec{1} = -\mu^2 \sum_{i=0}^{K-1} \pi_{i+1} Q^{\#}_{i,0}$. Hence,

$$\vec{\pi}^\top \tilde{Q}\vec{1} - 2\vec{\pi}^\top \tilde{Q} Q^{\#} \tilde{Q}\vec{1} = (1-\pi_0)\mu + 2\mu^2 \sum_{i=0}^{K-1} \pi_{i+1} Q^{\#}_{i,0}. \tag{41}$$

LEMMA 5 **(Uniform integrability).** *Set the price at p throughout the experiment. Let $N_{\text{arr}}(T,p)$ be the total number of arrivals in time $[0,T]$. Under Assumptions 1-3, there is a constant $C(K,p) > 0$ such that*

$$\mathbb{E}\left[(N_{\text{arr}}(T,p) - \mu(1-\pi_0(p))T)^4 \right] \le C(K,p)T^2. \tag{42}$$

Proof. We prove the lemma in a more general case. In particular, we will show that under the conditions of Lemma 3, there is a constant $C(K, p_0, \zeta_0, \mu)$ such that for any $p \in [p_0 - \zeta_0, p_0 + \zeta_0]$,

$$\mathbb{E}\left[\left(N_{\mathcal{J}}(T,p) - \vec{\pi}(p)^\top \tilde{Q}(p)\vec{1}T \right)^4 \right] \le C(K, p_0, \zeta_0, \mu) T^2. \tag{43}$$

Recall that in Lemma 3, $\mathcal{J}$ is a set of eligible jumps, i.e., $\mathcal{J}$ is a set of (i,j) pairs such that $i, j \in \{0,\dots,K\}$, and $N_{\mathcal{J}}(T,p)$ is the number of eligible jumps in time $[0,T]$. We require that $(k,k) \notin \mathcal{J}$. We have also defined $\tilde{Q}$ to be a matrix such that $\tilde{Q}_{i,j} = Q_{i,j}$ if $(i,j) \in \mathcal{J}$ and otherwise $\tilde{Q}_{i,j} = 0$. In the context of Lemma 5, we simply take $\mathcal{J} = \left\{(i,j) \in \{1,\dots,K\}^2 : j = i+1\right\}$.

From this point on, we will work under the conditions of Lemma 3 and aim at proving (43).

Similar to the proof of Lemma 3, we make some notation simplification. We write $\alpha = \alpha_0(p_0, \zeta_0)$, where α_0 is defined in (5). Throughout this proof, we omit "(p)" and "(α)" for simplicity. We use constants $C_1, C_2, \dots$ that could depend on K, μ, p_0, ζ_0 but not on p.

Again, same as in the proof of Lemma 3, we start with a uniformization step. The continuous-time Markov chain we considered can be described by a discrete-time Markov chain with transition matrix $P = I + Q/\alpha$ where jumps occur according to a Poisson process with intensity αT. Let $M(T)$ be the total number of jumps of the discrete-time Markov chain. Note that among the jumps, there are some self jumps that go from state k to state k. We are interested in the eligible jumps in $\mathcal{J}$.

Note that $M(T) \sim \text{Poisson}(\alpha T)$. We will then study the distribution of $N_{\mathcal{J}}(T)$ conditioning on $M(T) = m$. We define $\tilde{P} = \tilde{Q}/\alpha$. From now on, unless specified otherwise, we are conditioning on $M(T) = m$ and focus on the discrete-time Markov chain. We write $N_{\mathcal{J}}(T) = \sum_{i=1}^{M(T)} X_i$, where $X_i = 1$ if the i-th jump in the discrete-time Markov chain is an eligible jump. We are interested in the first, second, third and fourth moments of $N_{\mathcal{J}}(T)$. Expanding the parentheses, we get that

$$\begin{aligned}
N_{\mathcal{J}}(T) &= \sum_{i=1}^{M(T)} X_i, \\
N_{\mathcal{J}}^2(T) &= \sum_{i=1}^{M(T)} X_i + 2 \sum_{1 \le i < j \le M(T)} X_i X_j, \\
N_{\mathcal{J}}^3(T) &= \sum_{i=1}^{M(T)} X_i + 6 \sum_{1 \le i < j \le M(T)} X_i X_j + 6 \sum_{1 \le i < j < k \le M(T)} X_i X_j X_k, \\
N_{\mathcal{J}}^4(T) &= \sum_{i=1}^{M(T)} X_i + 14 \sum_{1 \le i < j \le M(T)} X_i X_j + 36 \sum_{1 \le i < j < k \le M(T)} X_i X_j X_k \\
&\qquad + 24 \sum_{1 \le i < j < k < l \le M(T)} X_i X_j X_k X_l.
\end{aligned} \tag{44}$$

This then implies that

$$\begin{aligned}
\mathbb{E}\left[N_{\mathcal{J}}(T) \mid M(T) = m\right] &= A_1, \\
\mathbb{E}\left[N_{\mathcal{J}}^2(T) \mid M(T) = m\right] &= A_1 + 2A_2, \\
\mathbb{E}\left[N_{\mathcal{J}}^3(T) \mid M(T) = m\right] &= A_1 + 6A_2 + 6A_3, \\
\mathbb{E}\left[N_{\mathcal{J}}^4(T) \mid M(T) = m\right] &= A_1 + 14A_2 + 36A_3 + 24A_4,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
A_1 &= \sum_{i=1}^{m} \mathbb{E}\left[X_i\right], \qquad A_2 = \sum_{1 \le i < j \le m} \mathbb{E}\left[X_i X_j\right], \\
A_3 &= \sum_{1 \le i < j < k \le m} \mathbb{E}\left[X_i X_j X_k\right], \qquad A_4 = \sum_{1 \le i < j < k < l \le m} \mathbb{E}\left[X_i X_j X_k X_l\right].
\end{aligned} \tag{46}$$

We will then study $A_1, A_2, \ldots A_4$. We have showed in the proof of Lemma 3 that $A_1 = \mathbb{E}\left[N_{\mathcal{J}}(T) \mid M(T) = m\right] = m\vec{\pi}^\top \tilde{P}\vec{1} - \alpha\vec{v}^\top Q^{\#}\tilde{P}\vec{1} + \text{error}$, where $|\text{error}| \le C_0 C^m$ for some constants $C_0 > 0$ and $C \in (0,1)$. For A_2, note that for $i < j$, $X_i X_j$ is the indicator that the i-th jump is eligible and the j-th jump is eligible. Therefore, $\mathbb{E}\left[X_i X_j\right] = \vec{v}^\top P^{i-1}\tilde{P}P^{j-i-1}\tilde{P}\vec{1}$. Hence, we can write

$$A_2 = \sum_{1 \le i < j \le m} \mathbb{E}\left[X_i X_j\right] = \sum_{1 \le i < j \le m} \vec{v}^\top P^{i-1}\tilde{P}P^{j-i-1}\tilde{P}\vec{1} = \vec{v}^\top \sum_{i+j \le m} P^{i-1}\tilde{P}P^{j-1}\tilde{P}\vec{1}, \tag{47}$$

where the last equality follows from a change of variable. Similarly, we find

$$
\begin{aligned}
A_3 &= \sum_{1\le i<j<k\le m} \mathbb{E}\left[X_i X_j X_k\right] = \sum_{1\le i<j<k\le m} \vec{\nu}^\top P^{i-1}\tilde{P}P^{j-i-1}\tilde{P}P^{k-j-1}\tilde{P}\vec{1} \\
&= \vec{\nu}^\top \sum_{i+j+k\le m} P^{i-1}\tilde{P}P^{j-1}\tilde{P}P^{k-1}\tilde{P}\vec{1},
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
A_4 &= \sum_{1\le i<j<k<l\le m} \mathbb{E}\left[X_i X_j X_k X_l\right] \\
&= \sum_{1\le i<j<k<l\le m} \vec{\nu}^\top P^{i-1}\tilde{P}P^{j-i-1}\tilde{P}P^{k-j-1}\tilde{P}P^{l-k-1}\tilde{P}\vec{1} \\
&= \vec{\nu}^\top \sum_{i+j+k+l\le m} P^{i-1}\tilde{P}P^{j-1}\tilde{P}P^{k-1}\tilde{P}P^{l-1}\tilde{P}\vec{1}.
\end{aligned}
\tag{49}
$$

Now we focus on A_3. The results for A_2 and A_4 are very similar. The key term in A_3 is $\sum_{i+j+k\le m} P^{i-1}\tilde{P}P^{j-1}\tilde{P}P^{k-1}\tilde{P}$, which is a sum of the product of terms like $P^{i-1}\tilde{P}$. Let $R_i = P^i - \vec{1}\vec{\pi}^\top$. Then we can decompose each P^{i-1} into two terms R_{i-1} and $\vec{1}\vec{\pi}^\top$, and thus

$$
\begin{aligned}
&P^{i-1}\tilde{P}P^{j-1}\tilde{P}P^{k-1}\tilde{P} \\
&= \left(R_{i-1} + \vec{1}\vec{\pi}^\top\right)\tilde{P}\left(R_{j-1} + \vec{1}\vec{\pi}^\top\right)\tilde{P}\left(R_{k-1} + \vec{1}\vec{\pi}^\top\right)\tilde{P} \\
&= \left(\vec{1}\vec{\pi}^\top\tilde{P}\right)^3 + R_{i-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right)^2 + \left(\vec{1}\vec{\pi}^\top\tilde{P}\right)R_{j-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right) + \left(\vec{1}\vec{\pi}^\top\tilde{P}\right)^2 R_{k-1}\tilde{P} \\
&\quad + R_{i-1}\tilde{P}R_{j-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right) + R_{i-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right)R_{k-1}\tilde{P} + \left(\vec{1}\vec{\pi}^\top\tilde{P}\right)R_{j-1}\tilde{P}R_{k-1}\tilde{P} \\
&\quad + R_{i-1}\tilde{P}R_{j-1}\tilde{P}R_{k-1}\tilde{P}.
\end{aligned}
\tag{50}
$$

By Lemma 1, we have that $\|R_{i-1}\|_{op} \le C^{i-1}$ for some constant $C \in (0,1)$. Therefore,

$$
\begin{aligned}
&\left\|\sum_{i+j+k\le m} R_{i-1}\tilde{P}R_{j-1}\tilde{P}R_{k-1}\tilde{P}\right\|_{op} \\
&\le \sum_{i+j+k\le m} \|R_{i-1}\|_{op}\left\|\tilde{P}\right\|_{op}\left\|R_{j-1}\right\|_{op}\left\|\tilde{P}\right\|_{op}\|R_{k-1}\|_{op}\left\|\tilde{P}\right\|_{op} \\
&\le \sum_{i+j+k\le m} C^{i+j+k-3} \le C_1,
\end{aligned}
\tag{51}
$$

for some constant $C_1 > 0$. Now we study terms involve two R_{i-1}.

$$
\begin{aligned}
&\left\|\sum_{i+j+k\le m} R_{i-1}\tilde{P}R_{j-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right)\right\|_{op} \\
&\le \sum_{i+j+k\le m} \|R_{i-1}\|_{op}\left\|\tilde{P}\right\|_{op}\left\|R_{j-1}\right\|_{op}\left\|\tilde{P}\right\|_{op}\left\|\vec{1}\vec{\pi}^\top\right\|_{op}\left\|\tilde{P}\right\|_{op} \\
&\le \sum_{i+j+k\le m} C^{i+j-2} \le \sum_{i+j\le m} C^{i+j-2} m \le C_2 m,
\end{aligned}
\tag{52}
$$

for some constant $C_2 > 0$. The same inequalities hold for $R_{i-1}\tilde{P}\left(\vec{1}\vec{\pi}^\top\tilde{P}\right)R_{k-1}\tilde{P}$ and $\left(\vec{1}\vec{\pi}^\top\tilde{P}\right)R_{j-1}\tilde{P}R_{k-1}\tilde{P}$.

Then, we look at terms involve only one R_{i-1}. Note that

$$
\begin{aligned}
\sum_{i+j+k\le m} R_{i-1} &= \sum_{i=1}^{m-2}\binom{m-i}{2}R_{i-1} = \sum_{i=1}^{m-2}\frac{(m-i)(m-i-1)}{2}R_{i-1} \\
&= \frac{m^2}{2}\sum_{i=1}^{m-2} R_{i-1} + \sum_{i=1}^{m-2}\frac{-(2i+1)m - i(i+1)}{2}R_{i-1}.
\end{aligned}
\tag{53}
$$

For the first term, by Lemma 2, we have that $\sum_{i=1}^{m-2} R_{i-1} = -\alpha Q^{\#} + E$, where $\|E\|_{op} \le C_1 C^m$ for some constant $C_1 > 0$, $C \in (0,1)$. For the second term, we note that by Lemma 1, $\|R_{i-1}\|_{op} \le C^{i-1}$ for some constant $C \in (0,1)$. Therefore,

$$\left\| \sum_{i=1}^{m-2} \frac{-(2i+1)m - i(i+1)}{2} R_{i-1} \right\|_{op} \le \sum_{i=1}^{m-2} \left| \frac{-(2i+1)m - i(i+1)}{2} \right| C^{i-1} \le C_1 m, \tag{54}$$

for some constant $C_1 > 0$. Combining the results and plugging back into (53), we have that

$$\sum_{i+j+k \le m} R_{i-1} = -\frac{m^2}{2} \alpha Q^{\#} + E_1, \tag{55}$$

where $\|E_1\|_{op} \le C_1 m$ for some constant $C_1 > 0$. By symmetry, the same result holds for $\sum_{i+j+k \le m} R_{j-1}$ and $\sum_{i+j+k \le m} R_{k-1}$.

We have analyzed all terms in (50). Combining the results, we have

$$\begin{aligned}
&\sum_{i+j+k \le m} P^{i-1} \tilde{P} P^{j-1} \tilde{P} P^{k-1} \tilde{P} \\
&= \sum_{i+j+k \le m} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^3 + E_3 \\
&\quad - \frac{\alpha m^2}{2} \left[Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 Q^{\#} \tilde{P} \right] \\
&= \binom{m}{3} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^3 + E_3 \\
&\quad - \frac{\alpha m^2}{2} \left[Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 Q^{\#} \tilde{P} \right]
\end{aligned} \tag{56}$$

where $\|E_3\|_{op} \le C_4 m$ for some constant $C_4 > 0$.

We can conduct the same analysis for A_2 and A_4 and get

$$\begin{aligned}
&\sum_{i+j \le m} P^{i-1} \tilde{P} P^{j-1} \tilde{P} \\
&\quad = \binom{m}{2} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 - \alpha m \left[Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) Q^{\#} \tilde{P} \right] + E_2,
\end{aligned} \tag{57}$$

where $\|E\|_{op} \le C_2$ for some constant $C_2 > 0$, and

$$\begin{aligned}
&\sum_{i+j+k+l \le m} P^{i-1} \tilde{P} P^{j-1} \tilde{P} P^{k-1} \tilde{P} P^{l-1} \tilde{P} \\
&\quad = \binom{m}{4} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^4 - \frac{\alpha m^3}{6} \left[Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^3 + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 \right. \\
&\qquad \left. + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^2 Q^{\#} \tilde{P} \left(\vec{1} \vec{\pi}^\top \tilde{P} \right) + \left(\vec{1} \vec{\pi}^\top \tilde{P} \right)^3 Q^{\#} \tilde{P} \right] + E_4,
\end{aligned} \tag{58}$$

where $\|E_4\|_{op} \le C_4 m^2$ for some constant $C_1 > 0$. Here, details of the derivation have been omitted for brevity.

Therefore, by (47) - (49), we have the following formulae for $A_1, \ldots, A_4$.

$$\begin{aligned}
A_1 &= m\left(\vec{\pi}^\top \tilde{P}\vec{1}\right) - \alpha \vec{v}^\top Q^\# \tilde{P}\vec{1} + \text{error}_1,\\
A_2 &= \frac{m^2}{2}\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^2 - m\Bigg[\alpha\left(\vec{\pi}^\top \tilde{P} Q^\# \tilde{P}\vec{1}\right)\\
&\qquad\qquad + \alpha\left(\vec{v}^\top Q^\# \tilde{P}\vec{1}\right)\left(\vec{\pi}^\top \tilde{P}\vec{1}\right) + \frac{1}{2}\left(\vec{\pi}^\top \tilde{P}\right)^2\Bigg] + \text{error}_2,\\
A_3 &= \frac{m^3}{6}\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^3 - m^2\Bigg[\alpha\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)\left(\vec{\pi}^\top \tilde{P} Q^\# \tilde{P}\vec{1}\right)\\
&\qquad\qquad + \frac{1}{2}\alpha\left(\vec{v}^\top Q^\# \tilde{P}\vec{1}\right)\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^2 + \frac{1}{2}\left(\vec{\pi}^\top \tilde{P}\right)^3\Bigg] + \text{error}_3,\\
A_4 &= \frac{m^4}{24}\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^4 - m^3\Bigg[\frac{1}{2}\alpha\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^2\left(\vec{\pi}^\top \tilde{P} Q^\# \tilde{P}\vec{1}\right)\\
&\qquad\qquad + \frac{1}{6}\alpha\left(\vec{v}^\top Q^\# \tilde{P}\vec{1}\right)\left(\vec{\pi}^\top \tilde{P}\vec{1}\right)^3 + \frac{1}{4}\left(\vec{\pi}^\top \tilde{P}\right)^3\Bigg] + \text{error}_4,
\end{aligned} \tag{59}$$

where $|\text{error}_1| \le C_1 C^m$, $|\text{error}_2| \le C_2$, $|\text{error}_3| \le C_3 m$, $|\text{error}_4| \le C_4 m^2$, for some constants $C \in (0,1), C_1, C_2, C_3, C_4 > 0$. To simplify notation, let

$$\gamma = \left(\vec{\pi}^\top \tilde{P}\vec{1}\right), \qquad \theta = \left(\vec{\pi}^\top \tilde{P} Q^\# \tilde{P}\vec{1}\right), \qquad \delta = \left(\vec{v}^\top Q^\# \tilde{P}\vec{1}\right). \tag{60}$$

Then we can express $A_1, \ldots A_4$ as

$$\begin{aligned}
A_1 &= m\gamma - \alpha\delta + \text{error}_1,\\
A_2 &= \frac{m^2}{2}\gamma^2 - m\left(\alpha\theta + \alpha\gamma\delta + \frac{1}{2}\gamma^2\right) + \text{error}_2,\\
A_3 &= \frac{m^3}{6}\gamma^3 - m^2\left(\alpha\gamma\theta + \frac{1}{2}\alpha\gamma^2\delta + \frac{\gamma^3}{2}\right) + \text{error}_3,\\
A_4 &= \frac{m^4}{24}\gamma^4 - m^3\left(\frac{1}{2}\alpha\gamma^2\theta + \frac{1}{6}\alpha\gamma^3\delta + \frac{\gamma^4}{4}\right) + \text{error}_4,
\end{aligned} \tag{61}$$

where $|\text{error}_1| \le C_1 C^m$, $|\text{error}_2| \le C_2$, $|\text{error}_3| \le C_3 m$, $|\text{error}_4| \le C_4 m^2$, for some constants $C \in (0,1), C_1, C_2, C_3, C_4 > 0$.

Plugging the above into (45), we have that

$$\begin{aligned}
\mathbb{E}\left[N_{\mathcal{J}} \mid M(T) = m\right] &= m\gamma - \alpha\delta + \text{error}_1,\\
\mathbb{E}\left[N_{\mathcal{J}}^2 \mid M(T) = m\right] &= m^2\gamma^2 - m\left(2\alpha\theta + 2\alpha\gamma\delta + \gamma^2 - \gamma\right) + \text{error}_2,\\
\mathbb{E}\left[N_{\mathcal{J}}^3 \mid M(T) = m\right] &= m^3\gamma^3 - m^2\left(6\alpha\gamma\theta + 3\alpha\gamma^2\delta + 3\gamma^3 - 3\gamma^2\right) + \text{error}_3,\\
\mathbb{E}\left[N_{\mathcal{J}}^4 \mid M(T) = m\right] &= m^4\gamma^4 - m^3\left(12\alpha\gamma^2\theta + 4\alpha\gamma^3\delta + 6\gamma^4 - 6\gamma^3\right) + \text{error}_4,
\end{aligned} \tag{62}$$

where $|\text{error}_1| \le C_1 C^m$, $|\text{error}_2| \le C_2$, $|\text{error}_3| \le C_3 m$, $|\text{error}_4| \le C_4 m^2$, for some constants $C \in (0,1), C_1, C_2, C_3, C_4 > 0$. Then note that since $M(T) \sim \text{Poisson}(\alpha T)$, we have that

$$\begin{aligned}
\mathbb{E}\left[M(T)\right] &= \alpha T,\\
\mathbb{E}\left[M(T)^2\right] &= (\alpha T)^2 + \alpha T,\\
\mathbb{E}\left[M(T)^3\right] &= (\alpha T)^3 + 3(\alpha T)^2 + \alpha T,\\
\mathbb{E}\left[M(T)^3\right] &= (\alpha T)^4 + 6(\alpha T)^3 + 7(\alpha T)^2 + \alpha T.
\end{aligned} \tag{63}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[N_{\mathcal{J}}\right] &= (\alpha T)\,\gamma - \alpha\delta + \text{error}_1,\\
\mathbb{E}\left[N_{\mathcal{J}}^2\right] &= (\alpha T)^2\,\gamma^2 - (\alpha T)\,(2\alpha\theta + 2\alpha\gamma\delta - \gamma) + \text{error}_2,\\
\mathbb{E}\left[N_{\mathcal{J}}^3\right] &= (\alpha T)^3\,\gamma^3 - (\alpha T)^2\left(6\alpha\gamma\theta + 3\alpha\gamma^2\delta - 3\gamma^2\right) + \text{error}_3,\\
\mathbb{E}\left[N_{\mathcal{J}}^4\right] &= (\alpha T)^4\,\gamma^4 - (\alpha T)^3\left(12\alpha\gamma^2\theta + 4\alpha\gamma^3\delta - 6\gamma^3\right) + \text{error}_4,
\end{aligned}
\tag{64}
$$

where $|\text{error}_1| \le C_1 C^T$, $|\text{error}_2| \le C_2$, $|\text{error}_3| \le C_3 T$, $|\text{error}_4| \le C_4 T^2$, for some constants $C \in (0,1), C_1, C_2, C_3, C_4 > 0$. Finally, we note that

$$
\begin{aligned}
\mathbb{E}\left[\left(N_{\mathcal{J}} - \mathbb{E}\left[N_{\mathcal{J}}\right]\right)^4\right] &= \mathbb{E}\left[N_{\mathcal{J}}^4\right] - 4\mathbb{E}\left[N_{\mathcal{J}}^3\right]\mathbb{E}\left[N_{\mathcal{J}}\right] + 6\mathbb{E}\left[N_{\mathcal{J}}^2\right]\left(\mathbb{E}\left[N_{\mathcal{J}}\right]\right)^2 - 3\mathbb{E}\left[N_{\mathcal{J}}\right]^4\\
&= \cancel{\alpha^4 T^4 \gamma^4} + \alpha^3 T^3\left(\cancel{-12\alpha\gamma^2\theta} - \cancel{4\alpha\gamma^3\delta} + \cancel{6\gamma^3}\right)\\
&\qquad - \cancel{4\alpha^4 T^4 \gamma^4} + \alpha^3 T^3\left(\cancel{4\alpha\gamma^3\delta} + \cancel{24\alpha\gamma^2\theta} + \cancel{12\alpha\gamma^3\delta} - \cancel{12\gamma^3}\right)\\
&\qquad + \cancel{6\alpha^4 T^4 \gamma^4} + \alpha^3 T^3\left(\cancel{-12\alpha\gamma^3\delta} - \cancel{12\alpha\gamma^2\theta} - \cancel{12\alpha\gamma^3\delta} + \cancel{6\gamma^3}\right)\\
&\qquad - \cancel{3\alpha^4 T^3 \gamma^4} + \alpha^3 T^3(\cancel{12\alpha\gamma^3\delta}) + \text{error}\\
&= \text{error},
\end{aligned}
\tag{65}
$$

where $|\text{error}| \le CT^2$ for some constant $C > 0$.

LEMMA 6 **(Triangular array version of Anscombe's theorem).** *Let $X_1(\zeta), X_2(\zeta), \ldots$ be a sequence of i.i.d. function of ζ. Let $\mu(\zeta) = \mathbb{E}\left[X_1(\zeta)\right]$, $\sigma^2(\zeta) = \operatorname{Var}\left[X_1(\zeta)\right]$, and $m(\zeta) = \mathbb{E}\left[(X_1(\zeta) - \mu(\zeta))^4\right]$. Assume that μ, σ^2 and m are continuous functions of ζ and that $m(\zeta) \le C^2$ uniformly over ζ for some constant $C > 0$. Let N_n be a sequence of random variables such that $N_n/n \xrightarrow{p} 1$. Let ζ_n be a sequence of numbers such that $\zeta_n \to 0$. Let*

$$
S_n(\zeta) = X_1(\zeta) + X_2(\zeta) + \cdots + X_n(\zeta). \tag{66}
$$

Then as $n \to \infty$,

$$
\sqrt{N_n}\left(\frac{S_{N_n}(\zeta_n)}{N_n} - \mu(\zeta_n)\right) \Rightarrow \mathcal{N}(0, \sigma^2(0)). \tag{67}
$$

Proof. *For notation simplicity, let $Y_i(\zeta) = X_i(\zeta) - \mu(\zeta)$ and $\tilde{S}_n(\zeta) = Y_1(\zeta) + Y_2(\zeta) + \cdots + Y_n(\zeta)$. We start with showing convergence of $S_n(\zeta_n)$. This can be shown easily with the Lindeberg-Feller theorem (Durrett 2019). In particular, we have that $\sum_{i=1}^n \mathbb{E}\left[Y_i(\zeta_n)^2\right]/n = \sigma^2(\zeta_n) \to \sigma^2(0)$, and that*

$$
\sum_{i=1}^n \mathbb{E}\left[\frac{Y_i(\zeta_n)^2}{n}; \frac{Y_i(\zeta_n)^2}{n} > \epsilon\right] \le \sum_{i=1}^n \mathbb{E}\left[Y_i(\zeta_n)^4\right]/n^2 \to 0. \tag{68}
$$

Therefore, the Linderberg conditions are satisfied and thus

$$
\sqrt{n}\left(\frac{S_n(\zeta_n)}{n} - \mu(\zeta_n)\right) \Rightarrow \mathcal{N}(0, \sigma^2(0)). \tag{69}
$$

We will then show that $S_n(\zeta_n) - n\mu(\zeta_n)$ is close to $S_{N_n}(\zeta_{N_n}) - N_n\mu(\zeta_n)$, i.e., $\tilde{S}_n(\zeta_n)$ is close to $\tilde{S}_{N_n}(\zeta_n)$. For any $\epsilon, \delta > 0$, by Kolmogorov's inequality (Durrett 2019),

$$
\begin{aligned}
\mathbb{P}\left[\left|\tilde{S}_{N_n}(\zeta_n) - \tilde{S}_n(\zeta_n)\right| > \epsilon\sqrt{n}\right] &\le \mathbb{P}\left[|N_n - n| > \delta n\right] + 2\mathbb{P}\left[\max_{1\le m\le\lceil\delta n\rceil}\left|\tilde{S}_m(\zeta_n)\right| \ge \epsilon\sqrt{n}\right]\\
&\le \mathbb{P}\left[|N_n - n| > \delta n\right] + 2\operatorname{Var}\left[\tilde{S}_{\lceil\delta n\rceil}(\zeta_n)\right]/(n\epsilon^2)\\
&\le \mathbb{P}\left[|N_n - n| > \delta n\right] + 2C\lceil\delta n\rceil/(n\epsilon^2).
\end{aligned}
\tag{70}
$$

Since $N_n/n \xrightarrow{p} 0$, we have that

$$\limsup_{n\to\infty} \mathbb{P}\left[\left|\tilde{S}_{N_n}(\zeta_n) - \tilde{S}_n(\zeta_n)\right| > \epsilon\sqrt{n}\right] \le \limsup_{n\to\infty} \mathbb{P}\left[|N_n - n| > 2\delta n\right] + \frac{2C\delta}{\epsilon} = \frac{2C\delta}{\epsilon}. \tag{71}$$

Finally, since the choice of δ is arbitrary, we have that

$$\limsup_{n\to\infty} \mathbb{P}\left[\left|\tilde{S}_{N_n}(\zeta_n) - \tilde{S}_n(\zeta_n)\right| > \epsilon\sqrt{n}\right] = 0 \tag{72}$$

and thus $\tilde{S}_{N_n}(\zeta_n) = \tilde{S}_n(\zeta_n) + o_p(\sqrt{n})$. Therefore,

$$\sqrt{N_n}\left(\frac{S_{N_n}(\zeta_n)}{N_n} - \mu(\zeta_n)\right) \Rightarrow \mathcal{N}(0, \sigma^2(0)). \tag{73}$$

LEMMA 7. *If a matrix A satisfies the following two conditions:*

1. $Q(p)A = I - \vec{1}\vec{\pi}(p)^\top$,
2. $\vec{\pi}(p)^\top A = \vec{0}^\top$,

then $A = Q^\#(p)$.

Proof. We will omit the dependence on p throughout the proof. Recall that in Section A.1, we define $\tilde{Q} = D_{\sqrt{\pi}} Q D_{\sqrt{\pi}}^{-1}$, which is a symmetric matrix with eigen-decomposition $\tilde{Q} = \tilde{U}D\tilde{U}^\top$. Here $\tilde{U}$ is an orthogonal matrix. Let $\tilde{A} = D_{\sqrt{\pi}} A D_{\sqrt{\pi}}^{-1}$. We have that

$$\tilde{Q}\tilde{A} = D_{\sqrt{\pi}} Q D_{\sqrt{\pi}}^{-1} D_{\sqrt{\pi}} A D_{\sqrt{\pi}}^{-1} = D_{\sqrt{\pi}} Q A D_{\sqrt{\pi}}^{-1} = D_{\sqrt{\pi}} (I - \vec{1}\vec{\pi}^\top) D_{\sqrt{\pi}}^{-1} = I - \tilde{u}_1\tilde{u}_1^\top, \tag{74}$$

where $\tilde{u}_1 = (\sqrt{p_0}, \ldots, \sqrt{p_K})^\top$. We also note that $\tilde{u}_1$ corresponds to the first eigenvector of $\tilde{Q}$. Now we write $\tilde{U} = (\tilde{u}_1, \tilde{U}_2)$. Since $\tilde{Q} = \tilde{U}D\tilde{U}^\top$, we have that $\tilde{U}D\tilde{U}^\top\tilde{A} = I - \tilde{u}_1\tilde{u}_1^\top$, which then implies that

$$D\tilde{U}^\top\tilde{A} = \tilde{U}^\top - \tilde{U}^\top\tilde{u}_1\tilde{u}_1^\top = \tilde{U}^\top - (1, 0, \ldots, 0)^\top\tilde{u}_1 = \begin{bmatrix} \tilde{u}_1^\top \\ \tilde{U}_2^\top \end{bmatrix} - \begin{bmatrix} \tilde{u}_1^\top \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{U}_2^\top \end{bmatrix}. \tag{75}$$

Then since $D_{0,0} = 0$, we further have that $\tilde{U}_2^\top\tilde{A} = D_2^{-1}\tilde{U}_2$, where $D_2 = D_{1:K,1:K}$.

Now, Condition 2 implies that $\tilde{u}_1^\top\tilde{A} = \vec{1}^\top D_{\sqrt{\pi}} D_{\sqrt{\pi}} A D_{\sqrt{\pi}}^{-1} = \pi^\top A D_{\sqrt{\pi}}^{-1} = \vec{0}^\top$. Thus,

$$\tilde{U}^\top\tilde{A} = \begin{bmatrix} \tilde{u}_1^\top \\ \tilde{U}_2^\top \end{bmatrix} A = \begin{bmatrix} \vec{0}^\top \\ D_2^{-1}\tilde{U}_2^\top \end{bmatrix} = D^\#\tilde{U}^\top. \tag{76}$$

This further implies that

$$\tilde{A} = \tilde{U}D^\#\tilde{U}^\top, \tag{77}$$

and hence

$$A = D_{\sqrt{\pi}}^{-1} A D_{\sqrt{\pi}} = D_{\sqrt{\pi}}^{-1}\tilde{U}D^\#\tilde{U}^\top D_{\sqrt{\pi}} = UD^\#U^{-1} = Q^\#. \tag{78}$$

LEMMA 8. *Consider the continuous-time Markov chain with transition rate matrix defined in (1). Let $N_k(t)$ be the number of jumps from state k to state $k+1$ in the time interval $[0,t]$. Let $T_k(t)$ be the amount of time in state k in the time interval $[0,t]$. Then,*

$$\sup_{t\in[0,T]} |N_k(t) - t\lambda_k\pi_k| = o_p(T). \tag{79}$$

$$\sup_{t\in[0,T]} |T_k(t) - t\pi_k| = o_p(T). \tag{80}$$

Proof. Let τ_i be the time of the i-th jump from state k to state $k+1$. We can then divide the time interval into subintervals of $[0,\tau_1]$, $[\tau_1,\tau_2],\dots$. The Markov chain behaves independently within each subinterval. Therefore, we can apply Kolmogorov's inequality and get the desired result.

LEMMA 9. *Consider the continuous-time Markov chain with transition rate matrix defined in (1). For two sets $\mathcal{J}_1$ and $\mathcal{J}_2$, we have that*

$$\mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}(m)\right] = m\vec{\pi}^\top \tilde{P}_{\mathcal{J}_1}\vec{1} + O(1), \tag{81}$$

$$\mathbb{E}\left[\tilde{T}_{\mathcal{J}_1}(m)\right] = \mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}(m)\right]/\alpha, \tag{82}$$

$$\begin{aligned}\mathrm{Cov}\left[\tilde{N}_{\mathcal{J}_1}(m), \tilde{N}_{\mathcal{J}_2}(m)\right] = m\bigg(&\vec{\pi}^\top \tilde{P}_{\mathcal{J}_1\cap\mathcal{J}_2}\vec{1} - (\vec{\pi}^\top \tilde{P}_{\mathcal{J}_1}\vec{1})(\vec{\pi}^\top \tilde{P}_{\mathcal{J}_2}\vec{1}) \\ &- \alpha\vec{\pi}^\top \tilde{P}_{\mathcal{J}_1}Q^\#\tilde{P}_{\mathcal{J}_2}\vec{1} - \alpha\vec{\pi}^\top \tilde{P}_{\mathcal{J}_2}Q^\#\tilde{P}_{\mathcal{J}_1}\vec{1}\bigg) + O(1),\end{aligned} \tag{83}$$

$$\mathrm{Cov}\left[\tilde{T}_{\mathcal{J}_1}(m), \tilde{T}_{\mathcal{J}_2}(m)\right] = \mathrm{Cov}\left[\tilde{N}_{\mathcal{J}_1}(m), \tilde{N}_{\mathcal{J}_2}(m)\right]/\alpha^2 + \mathbb{E}\left[\tilde{N}_{\mathcal{J}_1\cap\mathcal{J}_2}(m)\right]/\alpha^2, \tag{84}$$

$$\mathrm{Cov}\left[\tilde{T}_{\mathcal{J}_1}(m), \tilde{N}_{\mathcal{J}_2}(m)\right] = \mathrm{Cov}\left[\tilde{N}_{\mathcal{J}_1}(m), \tilde{N}_{\mathcal{J}_2}(m)\right]/\alpha. \tag{85}$$

Here, $\tilde{N}_{\mathcal{J}}(m)$, $\tilde{T}_{\mathcal{J}}(m)$, and $\tilde{P}_{\mathcal{J}}$ are defined in Definition 1.

Proof. We focus on $\tilde{N}_{\mathcal{J}}(m)$ first and will move on to study $\tilde{T}_{\mathcal{J}}(m)$ later. For simplicity, we drop the dependency "(m)".

We write $\tilde{N}_{\mathcal{J}_1} = \sum_{i=1}^m X_i$, where $X_i = 1$ if the i-th jump in the discrete-time Markov chain is an eligible jump in $\mathcal{J}_1$. Similarly, we write $\tilde{N}_{\mathcal{J}_2} = \sum_{i=1}^m Y_i$, where $Y_i = 1$ if the i-th jump in the discrete-time Markov chain is an eligible jump in $\mathcal{J}_2$.

We are interested in the first and second moments of $\tilde{N}_{\mathcal{J}}$. Expanding the parentheses, we get that

$$\begin{aligned}\tilde{N}_{\mathcal{J}_1} &= \sum_{i=1}^m X_i, \\ \tilde{N}_{\mathcal{J}_1}^2 &= \sum_{i=1}^m X_i + 2\sum_{1\le i<j\le m} X_iX_j, \\ \tilde{N}_{\mathcal{J}_1}\tilde{N}_{\mathcal{J}_2} &= \sum_{i=1}^m X_iY_i + \sum_{1\le i<j\le m} X_iY_j + \sum_{1\le i<j\le m} Y_iX_j.\end{aligned} \tag{86}$$

This then implies that

$$\begin{aligned}\mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}\right] &= A_1, \\ \mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}^2\right] &= A_1 + 2A_2, \\ \mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}N_{\mathcal{J}_2}\right] &= B_1 + B_2 + B_2',\end{aligned} \tag{87}$$

where

$$\begin{aligned}&A_1 = \sum_{i=1}^m \mathbb{E}\left[X_i\right], \qquad A_2 = \sum_{1\le i<j\le m} \mathbb{E}\left[X_iX_j\right], \\ &B_1 = \sum_{i=1}^m \mathbb{E}\left[X_iY_i\right], \qquad B_2 = \sum_{1\le i<j\le m} \mathbb{E}\left[X_iY_j\right], \qquad B_2' = \sum_{1\le i<j\le m} \mathbb{E}\left[Y_iX_j\right].\end{aligned} \tag{88}$$

Furthermore, we have that

$$\mathrm{Cov}\left[\tilde{N}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right] = B_1 + B_2 + B_2' - A_1A_1', \tag{89}$$

where $A_1' = \sum_{i=1}^m \mathbb{E}[Y_i]$.

We will then study A_1, A_2, B_1, B_2, B_2'. The analysis of A_1' is essentially the same as that of A_1. We have showed in the proof of Lemma 3 that $A_1 = m\vec{\pi}^\top \tilde{P}\vec{1} - \alpha\vec{\nu}^\top Q^\# \tilde{P}_{\mathcal{J}_1}\vec{1} + \text{error}$, where $|\text{error}| \le C_0 C^m$ for some constants $C_0 > 0$ and $C \in (0,1)$. For A_2, note that for $i < j$, $X_i X_j$ is the indicator that the i-th jump is eligible and the j-th jump is eligible. Therefore, $\mathbb{E}\left[X_i X_j\right] = \vec{\nu}^\top P^{i-1}\tilde{P}_{\mathcal{J}_1} P^{j-i-1}\tilde{P}_{\mathcal{J}_1}\vec{1}$. Hence, we can write

$$A_2 = \sum_{1\le i<j\le m} \vec{\nu}^\top P^{i-1}\tilde{P}_{\mathcal{J}_1} P^{j-i-1}\tilde{P}_{\mathcal{J}_1}\vec{1} = \vec{\nu}^\top \sum_{i+j\le m} P^{i-1}\tilde{P}_{\mathcal{J}_1} P^{j-1}\tilde{P}_{\mathcal{J}_1}\vec{1}, \tag{90}$$

where the last equality follows from a change of variable. Similarly, we have that $B_1 = m\vec{\pi}^\top\tilde{P}\vec{1} - \alpha\vec{\nu}^\top Q^\#\tilde{P}_{\mathcal{J}_1\cap\mathcal{J}_2}\vec{1} + \text{error}_1$, where $|\text{error}_1| \le C_0 C^m$ for some constants $C_0 > 0$ and $C \in (0,1)$. Furthermore,

$$\begin{aligned} B_2 &= \sum_{1\le i<j\le m} \vec{\nu}^\top P^{i-1}\tilde{P}_{\mathcal{J}_1} P^{j-i-1}\tilde{P}_{\mathcal{J}_2}\vec{1} = \vec{\nu}^\top \sum_{i+j\le m} P^{i-1}\tilde{P}_{\mathcal{J}_1} P^{j-1}\tilde{P}_{\mathcal{J}_2}\vec{1}, \\ B_2' &= \sum_{1\le i<j\le m} \vec{\nu}^\top P^{i-1}\tilde{P}_{\mathcal{J}_2} P^{j-i-1}\tilde{P}_{\mathcal{J}_1}\vec{1} = \vec{\nu}^\top \sum_{i+j\le m} P^{i-1}\tilde{P}_{\mathcal{J}_2} P^{j-1}\tilde{P}_{\mathcal{J}_1}\vec{1}. \end{aligned} \tag{91}$$

Now we focus on B_2'. The results for A_2 and B_2' are very similar. The key term in A_2 is $\sum_{i+j\le m} P^{i-1}\tilde{P}_{\mathcal{J}_1}P^{j-1}\tilde{P}_{\mathcal{J}_2}$, which is a sum of the product of terms like $P^{i-1}\tilde{P}_{\mathcal{J}}$. Let $R_i = P^i - \vec{1}\vec{\pi}^\top$. Then we can decompose each P^{i-1} into two terms R_{i-1} and $\vec{1}\vec{\pi}^\top$, and thus

$$\begin{aligned} & P^{i-1}\tilde{P}_{\mathcal{J}_1}P^{j-1}\tilde{P}_{\mathcal{J}_2} \\ &= \left(R_{i-1} + \vec{1}\vec{\pi}^\top\right)\tilde{P}_{\mathcal{J}_1}\left(R_{j-1} + \vec{1}\vec{\pi}^\top\right)\tilde{P}_{\mathcal{J}_2} \\ &= \left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\right)\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\right) + R_{i-1}\tilde{P}_{\mathcal{J}_1}\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\right) \\ &\qquad + \left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\right)R_{j-1}\tilde{P}_{\mathcal{J}_2} + R_{i-1}\tilde{P}_{\mathcal{J}_1}R_{j-1}\tilde{P}_{\mathcal{J}_2}. \end{aligned} \tag{92}$$

By Lemma 1, we have that $\|R_{i-1}\|_{op} \le C^{i-1}$ for some constant $C \in (0,1)$. Therefore,

$$\begin{aligned} \left\|\sum_{i+j\le m} R_{i-1}\tilde{P}_{\mathcal{J}_1}R_{j-1}\tilde{P}_{\mathcal{J}_2}\right\| &\le \sum_{i+j\le m} \|R_{i-1}\|_{op}\left\|\tilde{P}_{\mathcal{J}_1}\right\|_{op}\left\|R_{j-1}\right\|_{op}\left\|\tilde{P}_{\mathcal{J}_2}\right\|_{op} \\ &\le \sum_{i+j\le m} C^{i+j-2} \le C_1, \end{aligned} \tag{93}$$

for some constant $C_1 > 0$.

Then, we look at terms that involve only one R_{i-1}. Note that

$$\sum_{i+j\le m} R_{i-1} = \sum_{i=1}^{m-1}(m-i)R_{i-1} = m\sum_{i=1}^{m-1}R_{i-1} - \sum_{i=1}^{m-1} iR_{i-1}. \tag{94}$$

For the first term, by Lemma 2, we have that $\sum_{i=1}^{m-1} R_{i-1} = -\alpha Q^\# + E$, where $\|E\|_{op} \le C_1 C^m$ for some constant $C_1 > 0$, $C \in (0,1)$. For the second term, we note that by Lemma 1, $\|R_{i-1}\|_{op} \le C^{i-1}$ for some constant $C \in (0,1)$. Therefore,

$$\left\|\sum_{i=1}^{m-1} iR_{i-1}\right\|_{op} \le \sum_{i=1}^{m-1} iC^{i-1} \le C_1, \tag{95}$$

for some constant $C_1 > 0$. Combining the results and plugging back into (94), we have that

$$\sum_{i+j\le m} R_{i-1} = -m\alpha Q^\# + E_1, \tag{96}$$

where $\|E_1\|_{op} \le C_1$ for some constant $C_1 > 0$.

We have analyzed all terms in (92). Combining the results, we have

$$\begin{aligned}
&\sum_{i+j\le m} P^{i-1}\tilde{P}P^{j-1}\tilde{P} \\
&\quad = \sum_{i+j\le m}\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\right)\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\right) - m\alpha\left(Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2} + \vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_2}\right) + E_2 \\
&\quad = \binom{m}{2}\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\right)\left(\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\right) - m\alpha\left(Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2} + \vec{1}\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_2}\right) + E_2
\end{aligned} \tag{97}$$

where $\|E_2\|_{op} \le C_4$ for some constant $C_4 > 0$.

Therefore, by (90)-(91), we have the following formulae for A_1, A_2, B_1, B_2, B_2'.

$$\begin{aligned}
A_1 &= m\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right) - \alpha\vec{v}^\top Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1} + \text{error}_1, \\
A_2 &= \frac{m^2}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right)^2 - m\Bigg[\alpha\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\right) \\
&\qquad + \alpha\left(\vec{v}^\top Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right) + \frac{1}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\right)^2\Bigg] + \text{error}_2, \\
B_1 &= m\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1\cap\mathcal{J}_2}\vec{1}\right) - \alpha\vec{v}^\top Q^{\#}\tilde{P}_{\mathcal{J}_1\cap\mathcal{J}_2}\vec{1} + \text{error}_3, \\
B_2 &= \frac{m^2}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}\right) - m\Bigg[\alpha\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_2}\vec{1}\right) \\
&\qquad + \alpha\left(\vec{v}^\top Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}\right) + \frac{1}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}\right)\Bigg] + \text{error}_4, \\
B_2' &= \frac{m^2}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}\right) - m\Bigg[\alpha\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_2}\vec{1}\right) \\
&\qquad + \alpha\left(\vec{v}^\top Q^{\#}\tilde{P}_{\mathcal{J}_2}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right) + \frac{1}{2}\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}\right)\left(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1}\right)\Bigg] + \text{error}_5\,.
\end{aligned} \tag{98}$$

where $|\text{error}_1| \le C_1 C^m$, $|\text{error}_2| \le C_2$, $|\text{error}_3| \le C_3 C^m$, $|\text{error}_4| \le C_4$, and $|\text{error}_5| \le C_5$, for some constants $C \in (0,1), C_1, C_2, C_3, C_4, C_5 > 0$.

Plugging things back into (87) and (89), we have that

$$\mathbb{E}\left[\tilde{N}_{\mathcal{J}_1}\right] = m\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1} + O(1), \tag{99}$$

$$\begin{aligned}
\text{Cov}\left[\tilde{N}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right] = m\Big(&\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1\cap\mathcal{J}_2}\vec{1} - (\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}\vec{1})(\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}\vec{1}) \\
&- \alpha\vec{\pi}^\top\tilde{P}_{\mathcal{J}_1}Q^{\#}\tilde{P}_{\mathcal{J}_2}\vec{1} - \alpha\vec{\pi}^\top\tilde{P}_{\mathcal{J}_2}Q^{\#}\tilde{P}_{\mathcal{J}_1}\vec{1}\Big) + O(1).
\end{aligned} \tag{100}$$

Finally, for $\tilde{T}_{\mathcal{J}}$, we note that we can write $\tilde{T}_{\mathcal{J}} = \sum_{i=1}^{\tilde{N}_{\mathcal{J}}} H_i$, where $H_i \sim \text{Exp}(\alpha)$ independently and H_i's are independent of $\tilde{N}_{\mathcal{J}}$. Therefore,

$$\mathbb{E}\left[\tilde{T}_{\mathcal{J}}\right] = \mathbb{E}\left[\tilde{N}_{\mathcal{J}}\right]/\alpha, \tag{101}$$

$$\begin{aligned}
&\text{Cov}\left[\tilde{T}_{\mathcal{J}_1}, \tilde{T}_{\mathcal{J}_2}\right] \\
&\quad = \text{Cov}\left[\mathbb{E}\left[\tilde{T}_{\mathcal{J}_1} \mid \tilde{N}_{\mathcal{J}_1}\right], \mathbb{E}\left[\tilde{T}_{\mathcal{J}_2} \mid \tilde{N}_{\mathcal{J}_2}\right]\right] + \mathbb{E}\left[\text{Cov}\left[\tilde{T}_{\mathcal{J}_1}, \tilde{T}_{\mathcal{J}_2} \mid \tilde{N}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right]\right] \\
&\quad = \text{Cov}\left[\tilde{N}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right]/\alpha^2 + \mathbb{E}\left[\tilde{N}_{\mathcal{J}_1\cap\mathcal{J}_2}\right]/\alpha^2,
\end{aligned} \tag{102}$$

and

$$\mathrm{Cov}\left[\tilde{T}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right] = \mathrm{Cov}\left[\tilde{N}_{\mathcal{J}_1}, \tilde{N}_{\mathcal{J}_2}\right]/\alpha. \tag{103}$$

LEMMA 10 **(Uniform Integrability Cont.)**. *Consider the continuous-time Markov chain with transition rate matrix Q defined in (1). There exists a constant C such that*

$$\mathbb{E}\left[\left(\tilde{N}_{\mathcal{J}}(m) - \vec{\pi}^\top \tilde{P}_{\mathcal{J}}\vec{1}m\right)^4\right] \le Cm^2, \tag{104}$$

$$\mathbb{E}\left[\left(\tilde{T}_{\mathcal{J}}(m) - \vec{\pi}^\top \tilde{P}_{\mathcal{J}}\vec{1}m/\alpha\right)^4\right] \le Cm^2, \tag{105}$$

where $\tilde{N}_{\mathcal{J}}(m)$, $\tilde{T}_{\mathcal{J}}(m)$ and $\tilde{P}_{\mathcal{J}}$ are defined in Definition 1.

Proof. The results for $\tilde{N}_{\mathcal{J}}(m)$ are included in the proof of Lemma 5. For $\tilde{T}_{\mathcal{J}}(m)$, we note that we can write

$$\tilde{T}_{\mathcal{J}}(m) = \sum_{i=1}^{\tilde{N}_{\mathcal{J}}(m)} H_i \tag{106}$$

where $H_i \sim \mathrm{Exp}(\alpha)$ independently and H_i's are independent of $\tilde{N}_{\mathcal{J}}(m)$. Therefore,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}_{\mathcal{J}}(m) - \vec{\pi}^\top \tilde{P}_{\mathcal{J}}\vec{1}m/\alpha\right)^4\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\tilde{T}_{\mathcal{J}}(m) - \vec{\pi}^\top \tilde{P}_{\mathcal{J}}\vec{1}m/\alpha\right)^4 \mid \tilde{N}_{\mathcal{J}}(m)\right]\right] \\ &= \mathbb{E}\left[\left(\tilde{N}_{\mathcal{J}}(m) - \vec{\pi}^\top \tilde{P}_{\mathcal{J}}\vec{1}m\right)^4\right]/\alpha^4.\end{aligned} \tag{107}$$

LEMMA 11. *Let $X_n \sim \mathrm{Binom}(n, p_n)$ be a sequence of Binomial random variables. Assume that $p_n \to c$ for some constant $c \in (0,1)$. Then*

$$\frac{\sqrt{n}}{\sqrt{p_n(1-p_n)}}\left(\frac{X_n}{n} - p_n\right) \Rightarrow \mathcal{N}(0,1). \tag{108}$$

Proof. We prove the lemma using the Lindeberg-Feller theorem (Durrett 2019). We start by rewriting $X_n = Y_{n,1} + Y_{n,2} + \ldots Y_{n,n}$, where $Y_{n,i}$'s are i.i.d. $\mathrm{Bern}(p_n)$ random variables. Let $Z_{n,i} = (Y_{n,i} - p_n)/\sqrt{np_n(1-p_n)}$. Then, $\mathbb{E}\left[Z_{n,i}\right] = 1$ and $\mathrm{Var}\left[Z_{n,i}\right] = 1/n$. For any $\epsilon > 0$,

$$\mathbb{E}\left[\left|Z_{n,i}\right|^2; \left|Z_{n,i}\right| > \epsilon\right] \le \mathbb{P}\left[\left|Z_{n,i}\right| > \epsilon\right] \le \frac{\mathbb{E}\left[\left|Z_{n,i}^4\right|\right]}{\epsilon^4} \le \frac{1}{n^2 p_n^2 (1-p_n)^2 \epsilon^4}. \tag{109}$$

Thus,

$$\sum_{i=1}^n \mathbb{E}\left[\left|X_{n,i}\right|^2; \left|X_{n,i}\right| > \epsilon\right] \le \frac{n}{n^2 p_n^2 (1-p_n)^2 \epsilon^4}, \tag{110}$$

which goes to 0 as $n \to \infty$. By Lindeberg-Feller theorem,

$$\frac{\sqrt{n}}{\sqrt{p_n(1-p_n)}}\left(\frac{X_n}{n} - p_n\right) = \sum_{i=1}^n Z_{n,i} \Rightarrow \mathcal{N}(0,1). \tag{111}$$

LEMMA 12. *Under the conditions of Theorem 7, if we define $N_k(T, \zeta_T) = N_{k,+}(T, \zeta_T) + N_{k,-}(T, \zeta_T)$, then*

$$N_k(T, \zeta_T)/T \xrightarrow{p} \pi_k(p)\lambda_k(p). \tag{112}$$

Proof. We start with noting that the arrival rate when the length of the queue is k is $\tilde{\lambda}_k(\zeta) = (\lambda_k(p+\zeta) + \lambda_k(p-\zeta))/2$. In other words, the arrival rate can be written as a function of ζ. If we treat ζ as the new price and $\tilde{\lambda}$ as the new

λ function, we can still apply results in Section A.1 and Lemma 3. By Lemma 3, if we define the eligible jumps to be jumps from state k to state $k+1$, then we have that

$$N_k(T,\zeta_T)/T = \tilde{\pi}_k(\zeta_T)\tilde{\lambda}_k(\zeta_T) + o_p(1), \tag{113}$$

where $\tilde{\pi}$ is the steady-state probabilities corresponding to the arrival rate $\tilde{\lambda}$. Since $\tilde{\pi}_k(\zeta_T) \to \tilde{\pi}_k(0) = \pi_k(p)$ and $\tilde{\lambda}_k(\zeta_T) \to \tilde{\lambda}_k(0) = \lambda_k(p)$, we have that

$$N_k(T,\zeta_T)/T = \pi_k(p)\lambda_k(p) + o_p(1). \tag{114}$$

LEMMA 13. *Under the conditions of Theorem 7, if we define $\pi_i(p,\zeta_T)$ to be the steady-state probability of state i when the state dependent arrival rate is set to be $\tilde{\lambda}(p,\zeta_T) = (\lambda_i(p-\zeta_T) + \lambda_i(p+\zeta_T))/2$, then for any k,*

$$|\pi_k(p,\zeta_T) - \pi_k(p,0)| \le C\zeta_T^2 \tag{115}$$

for some constant C depending on K, B_0, B_1, B_2. Note that $\pi_k(p,0)$ corresponds to our previous notation of $\pi_k(p)$.

Proof. We start with noting that

$$\pi_0(p,\zeta) = \frac{1}{1+\sum_{k=1}^{K}\prod_{i=1}^{k}\tilde{\lambda}_{i-1}(p,\zeta)/\mu}. \tag{116}$$

Therefore, it is straightforward to see that

$$\left|\frac{d}{d\zeta}\pi_0(p,\zeta)\right| \le C\max_k\left|\frac{d}{d\zeta}\tilde{\lambda}_k(p,\zeta)\right|, \tag{117}$$

for some constant C depending on K, B_0, B_1, B_2. We also note that

$$\pi_k(p,\zeta) = \pi_0(p,\zeta)\prod_{i=1}^{k}\frac{\tilde{\lambda}_{i-1}(p,\zeta)}{\mu}. \tag{118}$$

Therefore, for any $k \ge 0$,

$$\left|\frac{d}{d\zeta}\pi_k(p,\zeta)\right| \le C\max_k\left|\frac{d}{d\zeta}\tilde{\lambda}_k(p,\zeta)\right|, \tag{119}$$

for some constant C depending on K, B_0, B_1, B_2. Then we will turn our attention to $\frac{d}{d\zeta}\tilde{\lambda}_k(p,\zeta)$. Note that since $|\lambda_k''(p)| \le B_2$,

$$\left|\frac{d}{d\zeta}\tilde{\lambda}_k(p,\zeta)\right| = \frac{1}{2}\left|\lambda_k'(p+\zeta) - \lambda_k'(p-\zeta)\right| \le \zeta B_2. \tag{120}$$

This then implies that

$$\left|\frac{d}{d\zeta}\pi_k(p,\zeta)\right| \le C\max_k\left|\frac{d}{d\zeta}\tilde{\lambda}_k(p,\zeta)\right| \le C_1\zeta \tag{121}$$

for some constant C_1. Finally, note that the above result holds for any ζ, and thus

$$|\pi_k(p,\zeta_T) - \pi_k(p,0)| \le (C_1\zeta_T)\zeta_T \le C_1\zeta_T^2. \tag{122}$$

A.3. Proofs of propositions

A.3.1. Proof of Proposition 1

Regenerative switchback experiment We start with analyzing the regenerative switchback experiment. Let $M(T,\zeta_T)$ be the number of times the queue hits the state k_r. Let $W_i(\zeta_T)=1$ if the price changes to $p+\zeta_T$ at the i-th time of hitting k_r and let $W_i(\zeta_T)=0$ otherwise. Let $A_{i,+}(\zeta_T)$ ($A_{i,-}(\zeta_T)$) be the number of arrivals between the i-th and the $(i+1)$-th time of hitting k_r if the price is $p+\zeta_T$ ($p-\zeta_T$). In reality, we can only observe one of them; and thus, without loss of generality, we assume $A_{i,+}(\zeta_T)\perp\!\!\!\perp A_{i,-}(\zeta_T)$. Let $B_{i,+}(\zeta_T)$ ($B_{i,-}(\zeta_T)$) be the amount of time between the i-th and the $(i+1)$-th hitting of k_r if the price is $p+\zeta_T$ ($p-\zeta_T$). Again, without loss of generality, we assume $B_{i,+}(\zeta_T)\perp\!\!\!\perp B_{i,-}(\zeta_T)$.[2] Let $A_i(\zeta_T)=W_i(\zeta_T)A_{i,+}(\zeta_T)+(1-W_i(\zeta_T))A_{i,-}(\zeta_T)$ and $B_i(\zeta_T)=W_i(\zeta_T)B_{i,+}(\zeta_T)+(1-W_i(\zeta_T))B_{i,-}(\zeta_T)$ be the realized number of arrivals and hitting time.

Note that $A_{1,+}(\zeta_T),A_{2,+}(\zeta_T),\dots$ are i.i.d. random variables with bounded third moment (uniformly across ζ_T). The same hold for $A_{i,-}(\zeta_T)$, $B_{i,+}(\zeta_T)$ and $B_{i,-}(\zeta_T)$. By definition,

$$\sum_{i=0}^{M(T,\zeta_T)-1} B_i(\zeta_T)\le T\le \sum_{i=0}^{M(T,\zeta_T)} B_i(\zeta_T). \tag{123}$$

Thus by law of large numbers, $M(T,\zeta_T)/T\xrightarrow{p}1/\mathbb{E}\left[B_i(0)\right]$.

We can write similar inequalities for $N_+(T,\zeta_T)$, $T_+(T,\zeta_T)$, $N_-(T,\zeta_T)$ and $T_-(T,\zeta_T)$:

$$\begin{aligned}
\sum_{i=0}^{M(T,\zeta_T)-1} A_{i,+}(\zeta_T)W_i(\zeta_T)\le N_+(T,\zeta_T)&\le \sum_{i=0}^{M(T,\zeta_T)} A_{i,+}(\zeta_T)W_i(\zeta_T),\\
\sum_{i=0}^{M(T,\zeta_T)-1} B_{i,+}(\zeta_T)W_i(\zeta_T)\le T_+(T,\zeta_T)&\le \sum_{i=0}^{M(T,\zeta_T)} B_{i,+}(\zeta_T)W_i(\zeta_T),\\
\sum_{i=0}^{M(T,\zeta_T)-1} A_{i,-}(\zeta_T)(1-W_i(\zeta_T))\le N_-(T,\zeta_T)&\le \sum_{i=0}^{M(T,\zeta_T)} A_{i,-}(\zeta_T)(1-W_i(\zeta_T)),\\
\sum_{i=0}^{M(T,\zeta_T)-1} B_{i,-}(\zeta_T)(1-W_i(\zeta_T))\le T_-(T,\zeta_T)&\le \sum_{i=0}^{M(T,\zeta_T)} B_{i,-}(\zeta_T)(1-W_i(\zeta_T)).
\end{aligned} \tag{124}$$

By Lemma 6, we can show that

$$\sqrt{M(T,\zeta_T)}\begin{pmatrix}\frac{N_+(T,\zeta_T)}{M(T,\zeta_T)}-\mathbb{E}\left[A_{1,+}(\zeta_T)/2\right]\\ \frac{T_+(T,\zeta_T)}{M(T,\zeta_T)}-\mathbb{E}\left[B_{1,+}(\zeta_T)/2\right]\\ \frac{N_-(T,\zeta_T)}{M(T,\zeta_T)}-\mathbb{E}\left[A_{1,-}(\zeta_T)/2\right]\\ \frac{T_-(T,\zeta_T)}{M(T,\zeta_T)}-\mathbb{E}\left[B_{1,-}(\zeta_T)/2\right]\end{pmatrix}\Rightarrow\mathcal{N}(\vec{0},\Sigma), \tag{125}$$

as $T\to\infty$, where $\Sigma=\begin{pmatrix}\Sigma_0 & 0\\ 0 & \Sigma_0\end{pmatrix}$, and

$$\Sigma_0=\frac{1}{2}\begin{pmatrix}\mathrm{Var}\left[A_1(0)\right]+\frac{1}{2}\mathbb{E}\left[A_1(0)\right]^2 & \mathrm{Cov}\left[A_1(0),B_1(0)\right]+\frac{1}{2}\mathbb{E}\left[A_1(0)\right]\mathbb{E}\left[B_1(0)\right]\\ \mathrm{Cov}\left[A_1(0),B_1(0)\right]+\frac{1}{2}\mathbb{E}\left[A_1(0)\right]\mathbb{E}\left[B_1(0)\right] & \mathrm{Var}\left[B_1(0)\right]+\frac{1}{2}\mathbb{E}\left[B_1(0)\right]^2\end{pmatrix}. \tag{126}$$

Then by delta method (Lehmann and Casella 2006), the above suggests that

$$\sqrt{M(T,\zeta_T)}\begin{pmatrix}\frac{N_+(T,\zeta_T)}{T_+(T,\zeta_T)}-\frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]}\\ \frac{N_-(T,\zeta_T)}{T_-(T,\zeta_T)}-\frac{\mathbb{E}\left[A_{1,-}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,-}(\zeta_T)\right]}\end{pmatrix}\Rightarrow\mathcal{N}(\vec{0},\Sigma_1), \tag{127}$$

[2] For notation simplicity, here we set 0 to be the 0-th time of hitting k_r, even if we do not start from k_r.

as $T \to \infty$, where $\Sigma_1 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}$ and

$$\sigma_1^2 = 2\frac{\mathbb{E}\left[A_1(0)\right]^2}{\mathbb{E}\left[B_1(0)\right]^2}\left(\frac{\mathrm{Var}\left[A_1(0)\right]}{\mathbb{E}\left[A_1(0)\right]^2} - \frac{2\,\mathrm{Cov}\left[A_1(0), B_1(0)\right]}{\mathbb{E}\left[A_1(0)\right]\mathbb{E}\left[B_1(0)\right]} + \frac{\mathrm{Var}\left[B_1(0)\right]}{\mathbb{E}\left[B_1(0)\right]^2}\right). \tag{128}$$

This further implies that

$$\sqrt{T}\begin{pmatrix} \frac{N_+(T,\zeta_T)}{T_+(T,\zeta_T)} - \frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]} \\ \frac{N_-(T,\zeta_T)}{T_-(T,\zeta_T)} - \frac{\mathbb{E}\left[A_{1,-}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,-}(\zeta_T)\right]} \end{pmatrix} \Rightarrow \mathcal{N}(\vec{0}, \mathbb{E}\left[B_1(0)\right]^2 \Sigma_1), \tag{129}$$

as $T \to \infty$.

Interval switchback experiment For the interval switchback experiment, the analysis is very similar.

Let $\tau_1, \tau_2, \ldots, \tau_{n_t}$ represent the times at which the queue hits state k_r. Let $I_1 = [\tau_1, \tau_2)$, $I_2 = [\tau_2, \tau_3)$, $\ldots$, $I_{n_t-1} = [\tau_{n_t-1}, \tau_{n_t})$ be the time intervals between the hitting times. We call an interval a positive interval if the price stays at $p + \zeta_T$ throughout the time interval, a negative interval if the price stays at $p - \zeta_T$ throughout the time interval, and a mixed interval if the price changes during the interval. Let $M_+(T, \zeta_T)$ be the total number of positive intervals. Let $A_{i,+}(\zeta_T)$ be the number of arrivals in the i-th positive interval. Let $B_{i,+}(\zeta_T)$ be the amount of time in the i-th positive interval. We define $M_-(T, \zeta_T)$, $A_{i,-}(\zeta_T)$ and $B_{i,-}(\zeta_T)$ similarly for the negative intervals.

By Assumption 4, the total number of switches is upper bounded by $T/l_T = o(\sqrt{T})$, and thus the total number of mixed intervals is $o_p(\sqrt{T})$. Let T_+ be the amount of time the price is at $p + \zeta$. Therefore, we have that

$$\sum_{i=1}^{M_+(T,\zeta_T)} B_{i,+}(\zeta_T) = T_+ + o_p(\sqrt{T}). \tag{130}$$

Furthermore, because $T_+ = T/2 + o_p(1)$, we have that

$$\sum_{i=1}^{M_+(T,\zeta_T)} B_{i,+}(\zeta_T) = T/2 + o_p(T). \tag{131}$$

Note that $B_{1,+}(\zeta_T), B_{2,+}(\zeta_T), \ldots$ are i.i.d. random variables with bounded third moment (uniformly across ζ_T), and thus by law of large numbers, $M_+(T, \zeta_T)/T \xrightarrow{p} 1/(2\mathbb{E}\left[B_i(0)\right])$.

Similarly, we have that

$$\begin{aligned} \sum_{i=1}^{M_+(T,\zeta_T)} A_{i,+}(\zeta_T) &= N_+(T,\zeta_T) + o_p(\sqrt{T}), \\ \sum_{i=1}^{M_-(T,\zeta_T)} B_{i,-}(\zeta_T) &= T_- + o_p(\sqrt{T}), \\ \sum_{i=1}^{M_-(T,\zeta_T)} A_{i,-}(\zeta_T) &= N_-(T,\zeta_T) + o_p(\sqrt{T}). \end{aligned} \tag{132}$$

By Lemma 6, we can show that

$$\sqrt{M_+(T,\zeta_T)}\begin{pmatrix} \frac{N_+(T,\zeta_T)}{M_+(T,\zeta_T)} - \mathbb{E}\left[A_{1,+}(\zeta_T)/2\right] \\ \frac{T_+}{M_+(T,\zeta_T)} - \mathbb{E}\left[B_{1,+}(\zeta_T)/2\right] \end{pmatrix} \Rightarrow \mathcal{N}(\vec{0}, \Sigma_0/2), \tag{133}$$

as $T \to \infty$, where Σ_0 is defined in (126). Then by delta method (Lehmann and Casella 2006), the above suggests that

$$\sqrt{M_+(T,\zeta_T)}\left(\frac{N_+(T,\zeta_T)}{T_+} - \frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]}\right) \Rightarrow \mathcal{N}(0, \sigma_1^2/2), \tag{134}$$

as $T \to \infty$, where σ_1^2 is defined in (128). This further implies that

$$\sqrt{T}\left(\frac{N_+(T,\zeta_T)}{T_+} - \frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]}\right) \Rightarrow \mathcal{N}(0, \mathbb{E}\left[B_1(0)\right]^2 \sigma_1^2). \tag{135}$$

Note that we can conduct the exact same analysis for the price $p - \zeta_T$, and get

$$\sqrt{T}\left(\frac{N_-(T,\zeta_T)}{T_-} - \frac{\mathbb{E}\left[A_{1,-}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,-}(\zeta_T)\right]}\right) \Rightarrow \mathcal{N}(0, \mathbb{E}\left[B_1(0)\right]^2 \sigma_1^2). \tag{136}$$

Finally, note that in the proof, we indeed have $A_{i,+}(\zeta_T)$ and $B_{i,+}(\zeta_T)$ independent of $A_{i,-}(\zeta_T)$ and $B_{i,-}(\zeta_T)$, and thus a joint analysis of them gives

$$\sqrt{T}\begin{pmatrix} \frac{N_+(T,\zeta_T)}{T_+} - \frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]} \\ \frac{N_-(T,\zeta_T)}{T_-} - \frac{\mathbb{E}\left[A_{1,-}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,-}(\zeta_T)\right]} \end{pmatrix} \Rightarrow \mathcal{N}(\vec{0}, \mathbb{E}\left[B_1(0)\right]^2 \Sigma_1), \tag{137}$$

as $T \to \infty$, where $\Sigma_1 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix}$, and σ_1 is defined in (128).

Notation To make the notation consistent with the one in the statement of the proposition, we take $\tilde{\mu}(p + \zeta_T) = \frac{\mathbb{E}\left[A_{1,+}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,+}(\zeta_T)\right]}$, $\tilde{\mu}(p - \zeta_T) = \frac{\mathbb{E}\left[A_{1,-}(\zeta_T)\right]}{\mathbb{E}\left[B_{1,-}(\zeta_T)\right]}$, and $\tilde{\sigma}^2 = \mathbb{E}\left[B_1(0)\right]^2 \sigma_1^2$. Here, σ_1 is defined in (128).

A.3.2. Proof of Proposition 2 Let $N_{\text{arr}}(T)$ be the number of arrivals to the system, and let $M_{k_r}(T)$ be the number of times the queue hits the state k_r. Let A_i be the number of arrivals between the i-th and the $(i+1)$-th time of hitting k_r. Let B_i be the amount of time between the i-th and the $(i+1)$-th hitting of k_r.

Note that $A_1, A_2, \ldots$ are i.i.d. random variables with bounded third moment. The same hold for B_i. By definition,

$$\sum_{i=0}^{M_{k_r}(T)-1} B_i \leq T \leq \sum_{i=0}^{M_{k_r}(T)} B_i. \tag{138}$$

Thus by law of large numbers, $M_{k_r}(T)/T \xrightarrow{p} 1/\mathbb{E}\left[B_1\right]$.

We can write similar inequalities for $N_{\text{arr}}(T)$:

$$\sum_{i=0}^{M_{k_r}(T)-1} A_i \leq N_{\text{arr}}(T) \leq \sum_{i=0}^{M_{k_r}(T)} A_i, \tag{139}$$

By Anscombe's theorem (Anscombe 1952), we can show that, we can show that

$$\sqrt{M_{k_r}(T)}\begin{pmatrix} \frac{N_{\text{arr}}(T)}{M_{k_r}(T)} - \mathbb{E}\left[A_i\right] \\ \frac{T}{M_{k_r}(T)} - \mathbb{E}\left[B_1\right] \end{pmatrix} \Rightarrow \mathcal{N}(\vec{0}, \Sigma_2), \tag{140}$$

where

$$\Sigma_2 = \begin{pmatrix} \text{Var}\left[A_1\right] & \text{Cov}\left[A_1, B_1\right] \\ \text{Cov}\left[A_1, B_1\right] & \text{Var}\left[B_1\right] \end{pmatrix}. \tag{141}$$

Then by delta method (Lehmann and Casella 2006), the above suggests that

$$\sqrt{M_{k_r}(T)}\left(\frac{N_{\text{arr}}(T)}{T} - \frac{\mathbb{E}\left[A_1\right]}{\mathbb{E}\left[B_1\right]}\right) \Rightarrow \mathcal{N}(0, \sigma_2^2/2), \tag{142}$$

as $T \to \infty$, where

$$\sigma_2^2 = 2\frac{\mathbb{E}\left[A_1\right]^2}{\mathbb{E}\left[B_1\right]^2}\left(\frac{\text{Var}\left[A_1\right]}{\mathbb{E}\left[A_1\right]^2} - \frac{2\,\text{Cov}\left[A_1, B_1\right]}{\mathbb{E}\left[A_1\right]\mathbb{E}\left[B_1\right]} + \frac{\text{Var}\left[B_1\right]}{\mathbb{E}\left[B_1\right]^2}\right). \tag{143}$$

Furthermore, since $N_{\text{arr}}(T)/T \xrightarrow{p} 1/\mathbb{E}\left[B_1\right]$,

$$\sqrt{T}\left(\frac{N_{\text{arr}}(T)}{T} - \frac{\mathbb{E}\left[A_1\right]}{\mathbb{E}\left[B_1\right]}\right) \Rightarrow \mathcal{N}(0, \mathbb{E}\left[B_1\right]^2 \sigma_2^2/2), \tag{144}$$

as $T \to \infty$.

Note that A_1 has the same distribution as $A_{1,+}(0)$ and $A_{1,-}(0)$ defined in proof of Proposition 1 (Section A.3.1). The same holds for B_1, $B_{1,+}(0)$ and $B_{1,-}(0)$. Therefore, $\sigma_2^2 = \sigma_1^2$, where σ_1 is defined in (128). Therefore, as $T \to \infty$,

$$\sqrt{T}\left(\frac{N_{\mathrm{arr}}(T,p)}{T} - \tilde{\mu}(p)\right) \Rightarrow \mathcal{N}(0, \tilde{\sigma}^2(p)/2). \tag{145}$$

A.3.3. Proof of Proposition 3 Note that the queue length is bounded above by K and thus the difference between the number of arrivals and the number of departures is always bounded above by K. It then follows directly from Lemma 4 and Proposition 6 that

$$\begin{aligned}
\sigma_{\mathrm{arr}}^2(p) &= (1-\pi_0(p))\mu + 2\mu^2 \sum_{i=0}^{K-1} \frac{\pi_{i+1}(p)\pi_0(p)}{\mu}\left(\sum_{j=1}^{i} \frac{S_j(p)}{\pi_j(p)} - \sum_{j=1}^{K} \frac{S_j^2(p)}{\pi_j(p)}\right) \\
&= (1-\pi_0(p))\mu + 2\mu\pi_0(p)\left(\sum_{i=0}^{K-1} \pi_{i+1}(p) \sum_{j=1}^{i} \frac{S_j(p)}{\pi_j(p)} - \sum_{i=0}^{K-1} \pi_{i+1}(p) \sum_{j=1}^{K} \frac{S_j^2(p)}{\pi_j(p)}\right) \\
&= (1-\pi_0(p))\mu + 2\mu\pi_0(p)\left(\sum_{j=1}^{K-1} \frac{S_j(p)}{\pi_j(p)} \sum_{i=j}^{K-1} \pi_{i+1}(p) - (1-\pi_0(p)) \sum_{j=1}^{K} \frac{S_j^2(p)}{\pi_j(p)}\right) \\
&= (1-\pi_0(p))\mu + 2\mu\pi_0(p)\left(\sum_{j=1}^{K-1} \frac{S_j(p)S_{j+1}(p)}{\pi_j(p)} - (1-\pi_0(p)) \sum_{j=1}^{K} \frac{S_j^2(p)}{\pi_j(p)}\right).
\end{aligned} \tag{146}$$

A.3.4. Proof of Proposition 4 The proof is very similar to that of Proposition 1. We only need to change the definition of $A_{i,+}$ and $A_{i,-}$. Here, we define $A_{i,+}(\zeta_T)$ ($A_{i,-}(\zeta_T)$) to be the amount of time in state 0 between the i-th and the $(i+1)$-th time of hitting k_r if the price is $p+\zeta_T$ ($p-\zeta_T$). For simplicity's sake, we omit the details.

A.3.5. Proof of Proposition 5 The proof is very similar to that of Proposition 2. We only need to change the definition of A_i. Here, we define A_i be the amount of time in state 0 between the i-th and the $(i+1)$-th time of hitting k_r.

A.3.6. Proof of Proposition 6 We will omit the dependence on p in this proof. We will make use of Lemma 7. Let A be a matrix with entries

$$A_{k,i} = \frac{\pi_i}{\mu}\left(-\sum_{j=1}^{\min(i,k)} \frac{1}{\pi_j} + \sum_{j=1}^{k} \frac{S_j}{\pi_j} + \sum_{j=1}^{i} \frac{S_j}{\pi_j} - \sum_{j=1}^{K} \frac{S_j^2}{\pi_j}\right). \tag{147}$$

Write $A = (a_0, \ldots, a_K)$. We will then show that $QA = I - \vec{1}\vec{\pi}^\top$ and $\vec{\pi}^\top A = \vec{0}$. We start with the second condition. For any $i \in \{0, \ldots, K\}$, note that since $\sum_{k=0}^{K} \pi_k = 1$,

$$\begin{aligned}
\vec{\pi}^\top a_i &= \frac{\pi_i}{\mu}\left(-\sum_{k=1}^{K} \sum_{j=1}^{\min(i,k)} \frac{\pi_k}{\pi_j} + \sum_{k=1}^{K} \sum_{j=1}^{k} \frac{\pi_k S_j}{\pi_j} + \sum_{j=1}^{i} \frac{S_j}{\pi_j} - \sum_{j=1}^{K} \frac{S_j^2}{\pi_j}\right) \\
&= \frac{\pi_i}{\mu}\left(-\sum_{j=1}^{i} \sum_{k=j}^{K} \frac{\pi_k}{\pi_j} + \sum_{j=1}^{K} \sum_{k=j}^{K} \frac{\pi_k S_j}{\pi_j} + \sum_{j=1}^{i} \frac{S_j}{\pi_j} - \sum_{j=1}^{K} \frac{S_j^2}{\pi_j}\right) \\
&= \frac{\pi_i}{\mu}\left(-\sum_{j=1}^{i} \frac{S_j}{\pi_j} + \sum_{j=1}^{K} \frac{S_j^2}{\pi_j} + \sum_{j=1}^{i} \frac{S_j}{\pi_j} - \sum_{j=1}^{K} \frac{S_j^2}{\pi_j}\right) = 0.
\end{aligned} \tag{148}$$

Therefore, $\vec{\pi}^\top A = \vec{0}$.

For the first condition, we need to verify that $QA = I - \vec{1}\vec{\pi}^\top$. Write $Q = (q_0, q_1, \ldots, q_K)^\top$. In particular, we will show that $q_k^\top a_i = -\pi_i$ if $k > i$ or if $k < i$, and that $q_k^\top a_i = 1 - \pi_i$ if $k = i$. Note that q_k has a very special structure: it has at most three entries non-zero and $q_k^\top \vec{1} = 0$. Therefore,

$$\begin{aligned} q_k^\top a_i &= Q_{k,k-1}A_{k-1,i} + Q_{k,k}A_{k,i} + Q_{k,k+1}A_{k+1,i} \\ &= -Q_{k,k-1}(A_{k,i} - A_{k-1,i}) + Q_{k,k+1}(A_{k+1,i} - A_{k,i}). \end{aligned} \tag{149}$$

Here for indices out of scope, we treat the corresponding entries of the matrix zero. Note that for $0 \le k \le K-1$,

$$A_{k+1,i} - A_{k,i} = \frac{\pi_i}{\mu}\left(-\frac{\mathbf{1}\{k+1 \le i\}}{\pi_{k+1}} + \frac{S_{k+1}}{\pi_{k+1}}\right). \tag{150}$$

Thus, for $1 \le k \le K-1$,

$$\begin{aligned} q_k^\top a_i &= -Q_{k,k-1}(A_{k,i} - A_{k-1,i}) + Q_{k,k+1}(A_{k+1,i} - A_{k,i}) \\ &= -\mu\frac{\pi_i}{\mu}\left(-\frac{\mathbf{1}\{k \le i\}}{\pi_k} + \frac{S_k}{\pi_k}\right) + \lambda_k\frac{\pi_i}{\mu}\left(-\frac{\mathbf{1}\{k+1 \le i\}}{\pi_{k+1}} + \frac{S_{k+1}}{\pi_{k+1}}\right) \\ &= \frac{\pi_i}{\pi_k}\left(\mathbf{1}\{k \le i\} - S_k - \mathbf{1}\{k \le i-1\} + S_{k+1}\right) \\ &= \frac{\pi_i}{\pi_k}\left(\mathbf{1}\{k = i\} - \pi_k\right) = \mathbf{1}\{k = i\} - \pi_i. \end{aligned} \tag{151}$$

Here we make use of the steady-state equation of $\pi_k\lambda_k = \pi_{k+1}\mu$. For the boundary cases, the results are similar. For $k = 0$,

$$\begin{aligned} q_0^\top a_i &= Q_{0,1}(A_{1,i} - A_{0,i}) = \lambda_0\frac{\pi_i}{\mu}\left(-\frac{\mathbf{1}\{i \ge 1\}}{\pi_1} + \frac{S_1}{\pi_1}\right) \\ &= \frac{\pi_i}{\pi_0}\left(-1 + \mathbf{1}\{i = 0\} + (1 - \pi_0)\right) = \mathbf{1}\{i = 0\} - \pi_i. \end{aligned} \tag{152}$$

For $k = K$,

$$\begin{aligned} q_0^\top a_i &= -Q_{K,K-1}(A_{K,i} - A_{K-1,i}) = -\mu\frac{\pi_i}{\mu}\left(-\frac{\mathbf{1}\{K \le i\}}{\pi_k} + \frac{S_K}{\pi_K}\right) \\ &= \frac{\pi_i}{\pi_K}\left(-\mathbf{1}\{i = K\} + \pi_K\right) = \mathbf{1}\{i = K\} - \pi_i. \end{aligned} \tag{153}$$

Combing the above three results, we get for any $k \in \{0, \ldots, K\}$,

$$q_k^\top a_i = \mathbf{1}\{k = i\} - \pi_i. \tag{154}$$

Hence, $QA = I - \vec{1}\vec{\pi}^\top$.

Finally, by Lemma 7, we have that $Q^\# = A$.

A.3.7. Proof of Propositions 7 and 8 Again, the proof closely resembles those of Propositions 1 and 2. Rather than focusing only on the total number of arrivals and the total amount of time when the price is either $p + \zeta_T$ or $p - \zeta_T$, we consider multiple quantities at the same time.

For Proposition 7, similar to the proof of Proposition 1, we define $A_{i,k,+}$ as the count of jumps from k to $k+1$ in the i-th positive interval. Similarly, we define $A_{i,k,-}$. We let $B_{i,k,+}$ represent the amount of time the queue length is k in the i-th positive interval. We define $B_{i,k,-}$ similarly. The subsequent analysis mirrors that of Proposition 1.

For Proposition 8, similar to the proof of Proposition 2, we define $A_{i,k}$ as the number of jumps from k to $k+1$ between the i-th and the $i+1$-th time of hitting k_r. We define $B_{i,k}$ as the amount of time in state k between the i-th and the $i+1$-th time of hitting k_r. The subsequent analysis mirrors that of Proposition 2.

For the sake of brevity, we omit further details.

A.3.8. Proof of Proposition 9 We are essentially dealing with a continuous-time Markov chain with transition rate matrix Q defined in (1). Take any $\alpha \geq \max_{i,j} \left|Q_{i,j}\right|$. We conduct a step of uniformization similar to the one in the proof of Lemma 3: The continuous-time Markov chain we considered can be described by a discrete-time Markov chain with transition matrix $P = I + Q/\alpha$ where jumps occur according to a Poisson process with intensity α. Let $M(T)$ be the total number of jumps of the discrete-time Markov chain. Note that among the jumps, there are some self jumps that go from state k to state k. From now on, when we refer to jumps, we are referring to jumps of the discrete-time Markov chain, and thus we include such self jumps.

One nice property of the uniformization step is that the times between the jumps are independent of the behavior of the discrete time Markov chain. However, because $M(T)$ is defined as the total number of jumps by time T, $M(T)$ itself is correlated with the times between the jumps, which complicates our analysis. To deal with this, we consider a process indexed by the total number of jumps instead: Define $\tilde{N}_k(m)$ to be the number of jumps from state k to state $k+1$ within the first m jumps and $\tilde{T}_k(m)$ to be the amount of time in state k within the first m jumps. Let $\tilde{T}(m)$ be the total amount of time spent till the m-th jump.

We will then aim to establish relationships between $\tilde{N}_k$, $\tilde{T}_k$ and N_k, T_k. Without loss of generality, we assume that αT is an integer. In particular, we note that $\tilde{N}_k(\alpha T) = N_k(\tilde{T}(\alpha T)) + O_p(1)$. Therefore, by Lemma 8, we have that

$$\begin{aligned} N_k(T) &= \tilde{N}_k(\alpha T) + (N_k(T) - N_k(\tilde{T}(\alpha T))) + O_p(1) \\ &= \tilde{N}_k(\alpha T) - \lambda_k \pi_k (\tilde{T}(\alpha T) - T) + o_p(|\tilde{T}(\alpha T) - T|) \\ &= \tilde{N}_k(\alpha T) - \lambda_k \pi_k (\tilde{T}(\alpha T) - T) + o_p(\sqrt{T}), \end{aligned} \tag{155}$$

and

$$\begin{aligned} T_k(T) &= \tilde{T}_k(\alpha T) + (T_k(T) - T_k(\tilde{T}(\alpha T))) + O_p(1) \\ &= \tilde{T}_k(\alpha T) - \pi_k (\tilde{T}(\alpha T) - T) + o_p(|\tilde{T}(\alpha T) - T|) \\ &= \tilde{T}_k(\alpha T) - \pi_k (\tilde{T}(\alpha T) - T) + o_p(\sqrt{T}), \end{aligned} \tag{156}$$

We then note that by Theorem 4.11 in (Asmussen 2003), $T_k(T) = \pi_k T + o_p(1)$, and thus

$$\frac{N_k(T)}{T_k(T)} - \lambda_k = \frac{N_k(T) - \lambda_k T_k(T)}{T_k(T)} = \frac{N_k(T) - \lambda_k T_k(T)}{\pi_k T}(1 + o_p(1)). \tag{157}$$

By (155) and (156), we have that

$$\frac{N_k(T)}{T_k(T)} = \lambda_k + \frac{1}{\pi_k T}\left(\tilde{N}_k(\alpha T) - \lambda_k \tilde{T}_k(\alpha T)\right)(1 + o_p(1)) + o_p(1/\sqrt{T}). \tag{158}$$

Then we turn our attention to the behavior of the Markov chain within a fixed $m = \alpha T$ total number of jumps. Firstly, we can easily establish a multivariate central limit theorem for $\tilde{N}_0(m), \ldots, \tilde{N}_{K-1}(m), \tilde{T}_0(m), \ldots, \tilde{T}_{K-1}(m)$: this is because the Markov chain we consider is a regenerative process. Secondly, we note that we have established a uniform integrability result in Lemma 10. Therefore, the limit of the mean, variance and covariance of $\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m)$ will corresponds to the limit in the central limit theorem.

It thus suffices to study the limit of the mean, variance and covariance of $\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m)$. Here, we study more general forms of "number of jumps" and "amount of time".

DEFINITION 1. Consider the continuous-time Markov chain with transition rate matrix Q defined in (1). With a step of uniformization, the continuous-time Markov chain can be described by a discrete-time Markov chain with transition matrix $P = I + Q/\alpha$ where jumps occur according to a Poisson process with intensity α.

Let $\mathcal{J}$ be a set of eligible jumps, i.e., $\mathcal{J}$ is a set of (i, j) pairs such that $i, j \in \{0, \ldots, K\}$. Here the pair (i, j) stands for a jump from state i to state j. Let $\tilde{N}_{\mathcal{J}}(m)$ be the number of eligible jumps within the first m jumps. Let $\tilde{T}_{\mathcal{J}}(m)$ be the cumulative amount of time between each eligible jump and its immediately preceding jump within the first m jumps. Let $\tilde{Q}_{\mathcal{J}}$ be a matrix such that $\tilde{Q}_{i,j} = Q_{i,j}$ if $(i, j) \in \mathcal{J}$ and otherwise $\tilde{Q}_{i,j} = 0$. Let $\tilde{P}_{\mathcal{J}}$ be a matrix such that $\tilde{P}_{i,j} = P_{i,j}$ if $(i, j) \in \mathcal{J}$ and otherwise $\tilde{P}_{i,j} = 0$.

Lemma 9 gives the mean, variance and covariance of the quantities $N_{\mathcal{J}}(m)$ and $T_{\mathcal{J}}(m)$. Using our notation of eligible jumps, $N_k = N_{\{(k,k+1)\}}$ and $T_k = T_{k\,\text{out}}$, where $k\,\text{out} = \{(k, k-1), (k, k), (k, k+1)\} \cap \{0, \ldots, K\} \times \{0, \ldots, K\}$. In particular, we note that for the mean,

$$\mathbb{E}\left[N_k(m) - \lambda_k T_k(m)\right] = O(1). \tag{159}$$

For the variance and covariance, Lemma 9 implies that for disjoint sets $\mathcal{J}_i$ and $\tilde{\mathcal{J}}_i$, if there exists real numbers w_i and $\tilde{w}_i$, such that $(\sum w_i \tilde{P}_{\mathcal{J}_i})\vec{1} = \vec{0}$ and $(\sum \tilde{w}_i \tilde{P}_{\tilde{\mathcal{J}}_i})\vec{1} = \vec{0}$, then

$$\operatorname{Var}\left[\sum_i w_i \tilde{N}_{\mathcal{J}_i}(m)\right] = m \sum_i w_i^2 \vec{\pi}^\top \tilde{P}_{\mathcal{J}_i}\vec{1} + O(1), \tag{160}$$

$$\operatorname{Cov}\left[\sum_i w_i \tilde{N}_{\mathcal{J}_i}(m), \sum_i \tilde{w}_i \tilde{N}_{\tilde{\mathcal{J}}_i}(m)\right] = O(1). \tag{161}$$

With this result, we can move on to study $\operatorname{Cov}\left[\tilde{N}_k(k) - \lambda_k \tilde{T}_k(m), \tilde{N}_l(m) - \lambda_l \tilde{T}_l(m)\right]$. Consider the case where $k \neq l$. By Lemma 9,

$$\begin{aligned}
&\operatorname{Cov}\left[\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m), \tilde{N}_l(m) - \lambda_l \tilde{T}_l(m)\right] \\
&= \frac{1}{\alpha^2}\operatorname{Cov}\left[\alpha \tilde{N}_{\{(k,k+1)\}}(m) - \lambda_k \tilde{N}_{k\,\text{out}}, \alpha \tilde{N}_{\{(l,l)\}}(m) - \lambda_l \tilde{N}_{l\,\text{out}}\right].
\end{aligned} \tag{162}$$

Clearly, $\tilde{P}_{k\,\text{out}}\vec{1} = \vec{e}_k$ and $\tilde{P}_{\{(k,k+1)\}}\vec{1} = (\lambda_k/\alpha)\vec{e}_k$. Therefore, $\left(\alpha \tilde{P}_{\{(k,k+1)\}} - \lambda_k \tilde{P}_{k\,\text{out}}\right)\vec{1} = \vec{0}$. By (161), we have that

$$\operatorname{Cov}\left[\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m), \tilde{N}_l(m) - \lambda_l \tilde{T}_l(m)\right] = O(1), \tag{163}$$

and thus

$$\lim_{m\to\infty} \frac{1}{m}\operatorname{Cov}\left[\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m), \tilde{N}_l(m) - \lambda_l \tilde{T}_l(m)\right] = 0. \tag{164}$$

For the variance, again by Lemma 9,

$$\begin{aligned}
&\operatorname{Var}\left[\tilde{N}_k(m) - \lambda_k \tilde{T}_k(m)\right] \\
&\quad = \frac{1}{\alpha^2}\left(\operatorname{Var}\left[\alpha \tilde{N}_k(m) - \lambda_k \tilde{N}_{k\,\text{out}}(m)\right] + \lambda_k^2 \mathbb{E}\left[\tilde{N}_k(m)\right]\right) \\
&\quad = \frac{1}{\alpha^2}\left(\operatorname{Var}\left[\alpha \tilde{N}_{\{(k,k+1)\}}(m) - \lambda_k \tilde{N}_{k\,\text{out}}(m)\right] + \lambda_k^2 \mathbb{E}\left[\tilde{N}_k(m)\right]\right).
\end{aligned} \tag{165}$$

For $1 \leq k \leq K-1$, by (160),

$$\begin{aligned}
&\operatorname{Var}\left[\alpha \tilde{N}_{\{(k,k+1)\}}(m) - \lambda_k \tilde{N}_{k\,\text{out}}(m)\right] + \lambda_k^2 \mathbb{E}\left[\tilde{N}_{k\,\text{out}}(m)\right] \\
&\quad = m\left(\lambda_k^2 \vec{\pi}^\top \tilde{P}_{\{(k,k-1)\}}\vec{1} + \lambda_k^2 \vec{\pi}^\top \tilde{P}_{\{(k,k)\}}\vec{1} + (\alpha - \lambda_k)^2 \vec{\pi}^\top \tilde{P}_{\{(k,k+1)\}}\vec{1} + \lambda_k^2 \vec{\pi}^\top \tilde{P}_{k\,\text{out}}\vec{1}\right) + O_p(1) \\
&\quad = m\pi_k\left(\lambda_k^2 \mu/\alpha + \lambda_k^2(1 - (\mu + \lambda_k)/\alpha) + (\alpha - \lambda_k)^2 \lambda_k/\alpha + \lambda_k^2\right) + O_p(1) \\
&\quad = m\pi_k \lambda_k \alpha + O_p(1).
\end{aligned} \tag{166}$$

For $k = 1$, by (160),

$$\begin{aligned}
&\mathrm{Var}\left[\alpha\tilde{N}_{\{(k,k+1)\}}(m) - \lambda_k\tilde{N}_{k\,\mathrm{out}}(m)\right] + \lambda_k^2\mathbb{E}\left[\tilde{N}_{k\,\mathrm{out}}(m)\right] \\
&= m\left(\lambda_k^2\vec{\pi}^\top\tilde{P}_{\{(k,k)\}}\vec{1} + (\alpha-\lambda_k)^2\vec{\pi}^\top\tilde{P}_{\{(k,k+1)\}}\vec{1} + \lambda_k^2\vec{\pi}^\top\tilde{P}_{k\,\mathrm{out}}\vec{1}\right) + O_p(1) \\
&= m\pi_k\left(\lambda_k^2(1-\lambda_k/\alpha) + (\alpha-\lambda_k)^2\lambda_k/\alpha + \lambda_k^2\right) + O_p(1) \\
&= m\pi_k\lambda_k\alpha + O_p(1).
\end{aligned} \tag{167}$$

Therefore,

$$\mathrm{Var}\left[\tilde{N}_k(m) - \lambda_k\tilde{T}_k(m)\right] = \frac{1}{\alpha}m\pi_k\lambda_k\alpha + O(1), \tag{168}$$

and hence

$$\mathrm{Var}\left[\tilde{N}_k(\alpha T) - \lambda_k\tilde{T}_k(\alpha T)\right] = T\pi_k\lambda_k + O(1). \tag{169}$$

Combining with our discussion before Definition 1, we have that

$$\sqrt{T}(\mathrm{diff}_0, \mathrm{diff}_1, \ldots, \mathrm{diff}_{K-1}) \Rightarrow \mathcal{N}\left(\vec{0}, \Sigma_{\mathrm{diff}}\right), \tag{170}$$

where $\mathrm{diff}_k = \tilde{N}_k(\alpha T) - \lambda_k\tilde{T}_k(\alpha T)$, $\Sigma_{\mathrm{diff},k,k} = \pi_k\lambda_k$ and $\Sigma_{\mathrm{diff},k,l} = 0$ for $k \neq l$.

Finally, together with (158), we have that

$$\sqrt{T\zeta_T^2}(\mathrm{err}_0, \mathrm{err}_1, \ldots, \mathrm{err}_{K-1}) \Rightarrow \mathcal{N}\left(\vec{0}, \Sigma_{\mathrm{ratio}}(p)\right), \tag{171}$$

where $\mathrm{err}_k = \frac{N_k(T,p)}{T_k(T,p)} - \lambda_k(p)$, $\Sigma_{\mathrm{ratio},k,k}(p) = \lambda_k(p)/\pi_k(p)$ and $\Sigma_{\mathrm{ratio},k,l} = 0$ for $k \neq l$.

A.3.9. Proof of Proposition 10 Let $N_k(T,\zeta_T) = N_{k,+}(T,\zeta_T) + N_{k,-}(T,\zeta_T)$. For notation simplicity, we sometimes ignore the (T,ζ_T) and the subscript WDE in the proof.

We start with noting that by Lemma 12, $N_k/T \xrightarrow{p} \pi_k(p)\lambda_k(p)$. Then note that conditioning on N_k,

$$N_{k,+} \sim \mathrm{Binomial}\left(N_k, \frac{\lambda_k(p+\zeta_T)}{\lambda_k(p+\zeta_T) + \lambda_k(p+\zeta_T)}\right). \tag{172}$$

By Lemma 11 and Anscombe's theorem (Anscombe 1952),

$$\sqrt{N_k}\frac{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)}{\sqrt{\lambda_k(p+\zeta_T)\lambda_k(p-\zeta_T)}}\left(\frac{N_{k,+}}{N_k} - \frac{\lambda_k(p+\zeta_T)}{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)}\right) \Rightarrow \mathcal{N}(0,1), \tag{173}$$

as $T \to \infty$. Thus,

$$\sqrt{N_k}\frac{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)}{2\sqrt{\lambda_k(p+\zeta_T)\lambda_k(p-\zeta_T)}}\left(\frac{N_{k,+} - N_{k,-}}{N_k} - \frac{\lambda_k(p+\zeta_T) - \lambda_k(p-\zeta_T)}{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)}\right) \Rightarrow \mathcal{N}(0,1), \tag{174}$$

as $T \to \infty$. Since $\lambda_k(p+\zeta_T) \to \lambda_k(p)$, $\lambda_k(p-\zeta_T) \to \lambda_k(p)$, and $N_k/T \xrightarrow{p} \pi_k(p)\lambda_k(p)$, by Slutsky's theorem,

$$\sqrt{T}\left(\frac{N_{k,+} - N_{k,-}}{N_k} - \frac{\lambda_k(p+\zeta_T) - \lambda_k(p-\zeta_T)}{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)}\right) \Rightarrow \mathcal{N}\left(0, \frac{1}{\pi_k(p)\lambda_k(p)}\right). \tag{175}$$

Since the second derivative of λ_k is bounded above by a constant, we have that $\lambda_k(p+\zeta_T) - \lambda_k(p-\zeta_T) = 2\zeta_T\lambda_k'(p) + O\left(\zeta_T^2\right)$, and $\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T) = 2\lambda_k(p) + O\left(\zeta_T^2\right)$. Hence,

$$\frac{\lambda_k(p+\zeta_T) - \lambda_k(p-\zeta_T)}{\lambda_k(p+\zeta_T) + \lambda_k(p-\zeta_T)} = \zeta_T\frac{\lambda_k'(p)}{\lambda_k(p)} + O\left(\zeta_T^2\right). \tag{176}$$

Since $\sqrt{T}\zeta_T^2 \to 0$, we have that

$$\sqrt{T}\left(\frac{N_{k,+} - N_{k,-}}{N_k} - \zeta_T\frac{\lambda_k'(p)}{\lambda_k(p)}\right) \Rightarrow \mathcal{N}\left(0, \frac{1}{\pi_k(p)\lambda_k(p)}\right), \tag{177}$$

i.e.,

$$\sqrt{T\zeta_T^2}\left(\hat{\Delta}_k - \frac{\lambda_k'(p)}{\lambda_k(p)}\right) \Rightarrow \mathcal{N}\left(0, \frac{1}{\pi_k(p)\lambda_k(p)}\right). \tag{178}$$

For the joint distribution, the result is a consequence of the following fact: Conditioning on $(N_0, N_1, \ldots, N_{K-1})$, the random variables $N_{0,+}, \ldots, N_{K-1,+}$ are independent.

Appendix B: Additional Simulation Results and Details

B.1. Variance estimators

In Figure 1, we examine the accuracy of the proposed variance estimators discussed in Section 3.1. The variance estimators are generally quite accurate. For the model-free estimator, the idle-time-based estimator, and the WDE estimator in user-level randomization, the estimated variances closely track the true variances. For the WDE estimator in interval switchback experiments, the variance estimator can occasionally underestimate the true variance.

The simulation settings follow those in Section 3.2. We take $\zeta_T = 0.05$, hold the target price at $p = 1$, and assume that $\lambda_k(p) = \lambda_k(1)(2 - p)$.

B.2. Comparing the WDE estimator with the DQ estimator

Another estimator that addresses interference arising from Markovian systems using data from user-level randomization is the DQ estimator from Farias et al. (2022). To illustrate the performance of the WDE estimator in user-level randomization, we present a simulation-based comparison between the proposed WDE estimator and the DQ estimator.

As shown in Figure 2, the proposed WDE estimator achieves lower RMSE than both the DQ estimator and the difference-in-means estimator in most settings. We believe the efficiency gain arises from directly using the structural properties of the queueing system.

The simulation settings follow those in Section 3.2. We take $\zeta_T = 0.05$, hold the target price at $p = 1$, and assume that $\lambda_k(p) = \lambda_k(1)(2 - p)$. We also note that our analysis is conducted in a continuous-time Markov chain setting, while the DQ estimator was developed for discrete-time systems. To enable comparison, we discretize the trajectories when applying the DQ estimator, using a discretization interval of 0.1.

The DQ estimator was implemented with LSTD, with a single regularization hyperparameter α as described in Section G.1.2 (Implementation details, point 3) of Farias et al. (2022). We estimate the state–action value function Q and add an L_2 regularization term. In short, we solve for a fixed point of the regularized least-squares problem

$$Q = \arg\min_{Q'} \|Q' - r - PQ + \lambda\|_2^2 + \alpha \|Q'\|_2^2,$$

where $Q \in \mathbb{R}^{2(N+1)}$ stores the estimated $Q(s, a)$ values and $P \in \mathbb{R}^{2(N+1)\times 2(N+1)}$ is the state–action transition matrix. We construct plug-in estimates of r, P, and λ using sample means in each state. We set the hyperparameter to $\alpha = 0.1$.

B.3. Non-stationary simulator in Section 5

The departure rate $\mu = 2$. In week w, on day d, at time t, the state-dependent arrival rate is given by

$$\lambda_{w,d,t,k}(p) = \frac{4(2-p)}{1+k} a_w b_{d,t}, \tag{179}$$

where $b_{d,t}$ is the arrival rate depicted in Figure 9 and $a_1 = 0.9, a_2 = 1, a_3 = 1.1, a_4 = 1.2$.

B.4. Figure 8

We study the settings in Section 3.2. In Figure 8, we plot the mean squared error (scaled by $T\zeta_T^2$) of the estimators. We take $\zeta_T = 0.05$. We hold the target price at $p = 1$. In the first row of Figure 8, we assume that $\lambda_k(p) = \lambda_k(1)(2 - p)$; in the second row of Figure 8, we assume that $\lambda_k(p) = \lambda_k(1)((2 - p) + (1 - p)^2)$.

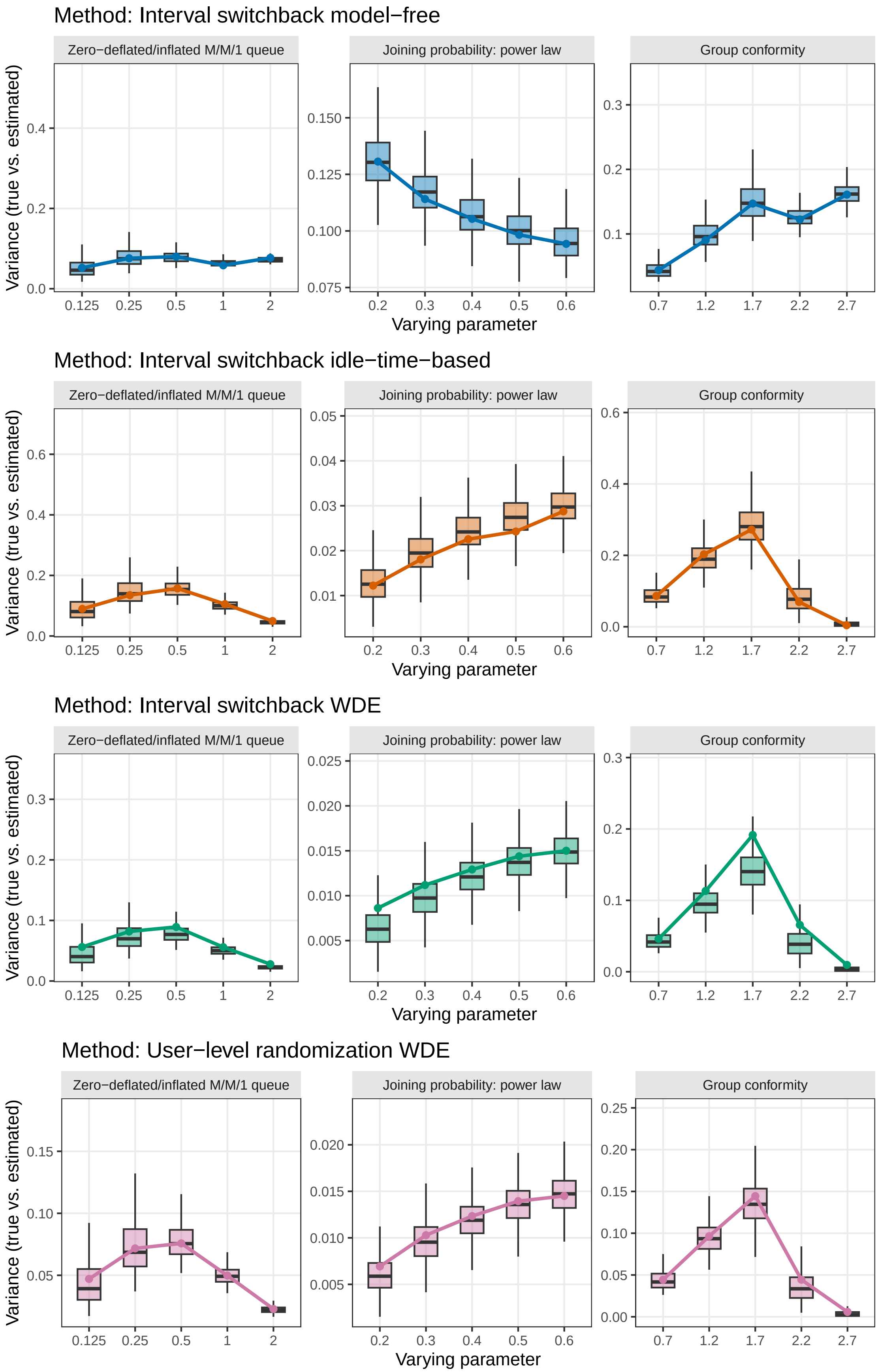

Figure 1 **Comparison between the true variance (solid lines) and the estimated variance (boxplots) across the four estimators and experimental settings.**

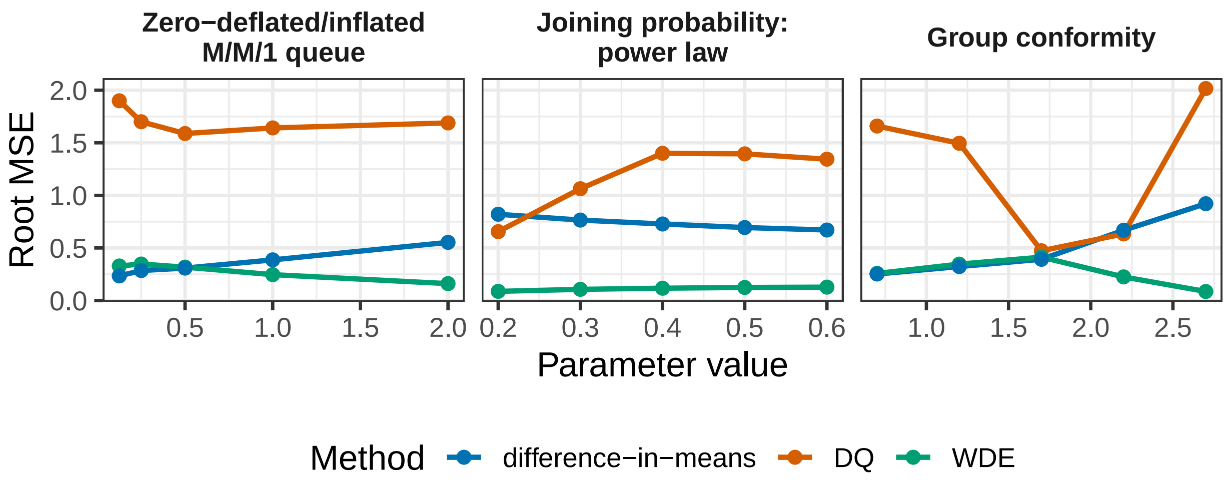

Figure 2 **Root mean squared error (RMSE) comparison of the estimators user-level randomization. We compare the proposed WDE estimator, the DQ estimator, and the difference-in-means estimator.**