B Online Appendix of "Multi-Type Information Acquisition in a Strategic Trading Model" by Ou-Yang, Wu, and Zhang

B.1 Existence of Pure-Strategy Equilibrium

In this section, we examine whether a pure strategy equilibrium exists under a given cost vector.

By Lemma 2, if $N_l \geq (<)\ 2R+1$, then $N_k^\dagger(N_l)$ increases (decreases) with N_l, i.e., $N_k^\dagger(N_l+1) \geq (<)\ N_k^\dagger(N_l)$. Consequently, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ may cross each other in four regions: 1) $N_1, N_2 < 2R+1$; 2) $N_1, N_2 \geq 2R+1$; 3) $N_1 < 2R+1$ and $N_2 \geq 2R+1$; and 4) $N_2 < 2R+1$ and $N_1 \geq 2R+1$. In the first two regions, where the slopes of $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ have the same signs, we say that $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ move in the same direction. In the latter two regions, where the slopes of $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ have opposite signs, we say that $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ move in the opposite direction.

Lemma B.1 *Suppose that $x_1^\dagger(x_2)$ and $x_2^\dagger(x_1)$ are monotonic discrete functions of x_2 and x_1 respectively, that x_i and $x_i^\dagger(x_j)$ are integers for $i,j=1,2$ and $i \neq j$, and that there are x_i^1 and x_i^2 with $x_i^1 < x_i^2$ such that $x_i^\dagger(x_j^\dagger(x_i^1)) \geq x_i^1$ and $x_i^\dagger(x_j^\dagger(x_i^2)) \leq x_i^2$. We obtain that*

1. *if $x_i^\dagger(x_j)$ increases (decreases) with x_j for $i,j=1,2$ and $i \neq j$, then there is one or more pairs of (x_1^*, x_2^*) such that $x_1^* = x_1^\dagger(x_2^*)$ and $x_2^* = x_2^\dagger(x_1^*)$, where $x_i^1 \leq x_i^* \leq x_i^2$.*
2. *if $x_i^\dagger(x_j)$ increases with x_j but $x_j^\dagger(x_i)$ decreases with x_i for $i,j=1,2$ and $i \neq j$, then there is one or no pairs of (x_1^*, x_2^*) such that $x_1^* = x_1^\dagger(x_2^*)$ and $x_2^* = x_2^\dagger(x_1^*)$, where $x_i^1 \leq x_i^* \leq x_i^2$.*

We illustrate the results of Lemma B.1 in Figure 8, in which the red stars represent $x_i^\dagger(x_j)$, and the black circles represent $x_j^\dagger(x_i)$. The condition of $x_i^\dagger(x_j^\dagger(x_i^1)) \geq x_i^1$ means that $x_j^\dagger(x_i)$ is above $x_i^\dagger(x_j)$ at $x_i = x_i^\dagger(x_j^\dagger(x_i^1))$, and the condition of $x_i^\dagger(x_j^\dagger(x_i^2)) \leq x_i^2$ means that $x_j^\dagger(x_i)$ is below $x_i^\dagger(x_j)$ at $x_i = x_i^\dagger(x_j^\dagger(x_i^2))$. These two conditions guarantee that $x_i^\dagger(x_j)$ and $x_j^\dagger(x_i)$ cross in the region of $x_i^1 \leq x_i \leq x_i^2$. Panel A in Figure 8 presents an example,

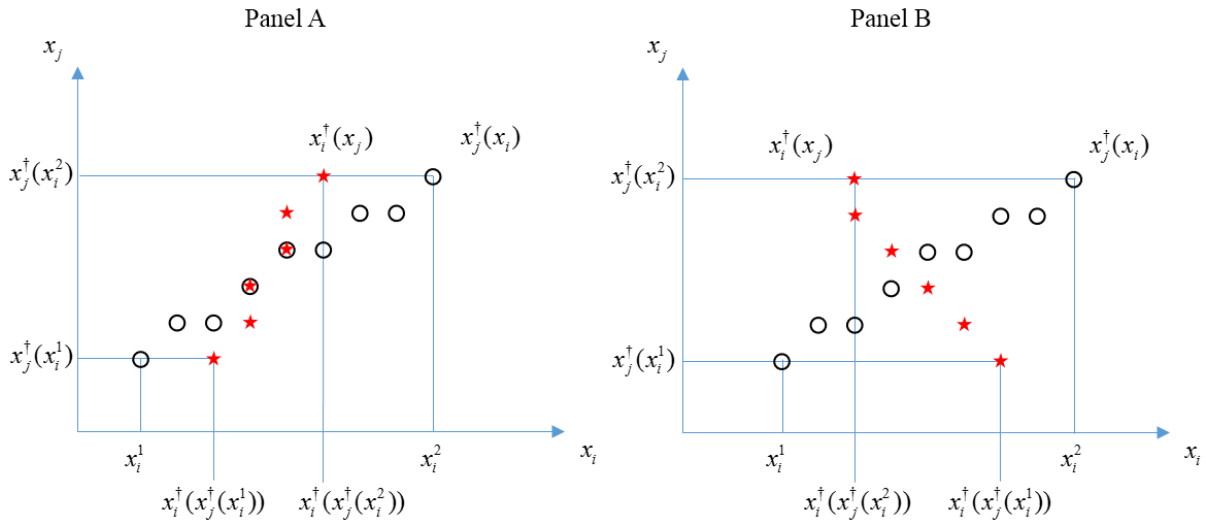

Figure 8. Illustrations of Lemma B.1. This figure illustrates when two monotonic discrete functions have an overlap. It shows the results of Lemma B.1. The red stars represent $x_i^\dagger(x_j)$, and the black circles represent $x_j^\dagger(x_i)$.

where $x_i^\dagger(x_j)$ and $x_j^\dagger(x_i)$ move in the same direction and there are two overlapping points, corresponding to two equilibria. Panel B in Figure 8 presents an example, where $x_i^\dagger(x_j)$ and $x_j^\dagger(x_i)$ move in the opposite directions and there is no overlapping point, indicating the nonexistence of a pure strategy equilibrium. In short, Point 1 in Lemma B.1 states that if $x_1^\dagger(x_2)$ and $x_2^\dagger(x_1)$ move in the same direction and cross, then there exists at least one equilibrium, and Point 2 in Lemma B.1 states that if $x_1^\dagger(x_2)$ and $x_2^\dagger(x_1)$ move in the opposite directions and cross, then there exists at most one equilibrium.

Using Lemma B.1, we investigate the existence of equilibrium when $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ cross in different regions. Because $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ are both non-monotonic, however, our problem is more complicated than the problem in Lemma B.1. Fortunately, we can show that $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ always cross only in a certain region, which greatly reduces the difficulty of our problem.

For any given c_k, let $\overline{N_k} = N_k^\dagger(0)$ and $\underline{N_k} = N_k^\dagger(2R+1)$ for $k = 1, 2$. Because $L(0) = L(+\infty) \leq L(N_l) \leq L(2R+1)$ for $N_l \geq 0$ by Equation (2), we have $\underline{N_k} \leq N_k^\dagger(N_l) \leq \overline{N_k}$. That is, $\underline{N_k}$ and $\overline{N_k}$ are the lower and upper bounds of $N_k^\dagger(N_l)$ for a given c_k, respectively. Based on the results in Lemma B.1, we obtain the following lemma that identifies the conditions under which $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ cross each other in each region and further

presents the existence of equilibrium in each scenario. Without loss of generality, the following lemma considers the case of $c_k \geq c_l$, which implies that $\overline{N_k} \leq \overline{N_l}$ and $\underline{N_k} \leq \underline{N_l}$.

Lemma B.2 *Suppose that $c_k \geq c_l$, the following results hold.*

1. *If $N_k^\dagger(\underline{N_l}) \geq 2R+1$, then $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross either in the region of $N_k, N_l \leq 2R+1$ or in the region of $N_k, N_l > 2R+1$, and therefore there exists at least one equilibrium.*
2. *If $N_k^\dagger(\underline{N_l}) < 2R+1$, then we have*
   (a) *if $N_l^\dagger(\underline{N_k}+1) \geq 2R+1$, then $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross in the region of $N_k \leq 2R+1$ and $N_l > 2R+1$, and therefore there exists at most one equilibrium, and*
   (b) *if $N_l^\dagger(\underline{N_k}+1) < 2R+1$, then $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross in the region of $N_k, N_l < 2R+1$, and therefore there exists at least one equilibrium.*

Figures 1 and 2 illustrate the results of Lemma B.2, in which green stars represent $N_k^\dagger(N_l)$ and blue circles represent $N_l^\dagger(N_k)$. The intuitions behind Lemma B.2 are as follows.

When $N_k^\dagger(\underline{N_l}) \geq 2R+1$, we can prove that $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross in the regions where $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ move in the same direction. Thus, we obtain Point 1 in Lemma B.2 according to Lemma B.1. Figure 2a presents an example in which $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross and have two overlapping points, corresponding to two equilibria.

When $N_k^\dagger(\underline{N_l}) < 2R+1$ and $N_l^\dagger(\underline{N_k}+1) \geq 2R+1$, we can prove that $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross in the region where $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ move in the opposite directions. Thus, we obtain Point 2a in Lemma B.2 according to Lemma B.1. Figure 1a presents an example where $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross each other and there is only one overlapping point, indicating a unique equilibrium. Figure 1b presents an example where $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross each other but there are no overlapping points, corresponding to the nonexistence of equilibrium.

When $N_k^\dagger(\underline{N_l}) < 2R+1$ and $N_l^\dagger(\underline{N_k}+1) < 2R+1$, we can prove that $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ cross in the region where $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ move in the same direction. Thus,

we obtain Point 2b in Lemma B.2 according to Lemma B.1. Figure 2b presents an example in which $N_k^\dagger(N_l)$ and $N_l^\dagger(N_k)$ have two overlapping points, corresponding to two equilibria.

Note that $\overline{N_k}$ and $\underline{N_k}$ are determined by c_k. Thus, by using Lemma B.2, we can determine whether an equilibrium exists for any given cost vector (c_1, c_2). Before that, we define π_H, π_M, and π_L as follows.

$$\begin{aligned}
\pi_H &\equiv \mathbb{E}[\pi_1(1,0)] = \mathbb{E}[\pi_2(0,1)],\\
\pi_M &\equiv \mathbb{E}[\pi_1(2R+1,0)] = \mathbb{E}[\pi_2(0,2R+1)],\\
\pi_L &\equiv \mathbb{E}[\pi_1(2R+1,2R+1)] = \mathbb{E}[\pi_2(2R+1,2R+1)].
\end{aligned}$$

If $0 < c_k \leq \pi_L$, then we have $\underline{N_k} \geq 2R+1$. If $\pi_L < c_k \leq \pi_M$, then we have $\underline{N_k} < 2R+1 \leq \overline{N_k}$. If $\pi_M < c_k \leq \pi_H$, then we have $1 \leq \overline{N_k} < 2R+1$. If $\pi_H < c_k$, then we have $\overline{N_k} = 0$. Using π_L, π_M, and π_H, we divide all possible cost vectors (c_1, c_2) into 11 sets as follows, and present the existence results in each set in Proposition B.1.

$$\begin{aligned}
\mathcal{S}_1 &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : c_1 > \pi_H \text{ or } c_2 > \pi_H\},\\
\mathcal{S}_2 &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : 0 < c_1, c_2 \leq \pi_L\},\\
\mathcal{S}_3 &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_M < c_1, c_2 \leq \pi_H\},\\
\mathcal{S}_{41} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_M < c_1 \leq \pi_H \text{ and } 0 < c_2 \leq \pi_L\},\\
\mathcal{S}_{42} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_M < c_2 \leq \pi_H \text{ and } 0 < c_1 \leq \pi_L\},\\
\mathcal{S}_{51} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_L < c_1 \leq \pi_M \text{ and } 0 < c_2 \leq \pi_L\},\\
\mathcal{S}_{52} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_L < c_2 \leq \pi_M \text{ and } 0 < c_1 \leq \pi_L\},\\
\mathcal{S}_{61} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_M < c_1 \leq \pi_H \text{ and } \pi_L < c_2 \leq \pi_M\},\\
\mathcal{S}_{62} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_M < c_2 \leq \pi_H \text{ and } \pi_L < c_1 \leq \pi_M\},\\
\mathcal{S}_{71} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_L < c_2 \leq c_1 \leq \pi_M\},\\
\mathcal{S}_{72} &\equiv \{(c_1,c_2) \in \mathbb{R}^2_{++} : \pi_L < c_1 < c_2 \leq \pi_M\}.
\end{aligned}$$

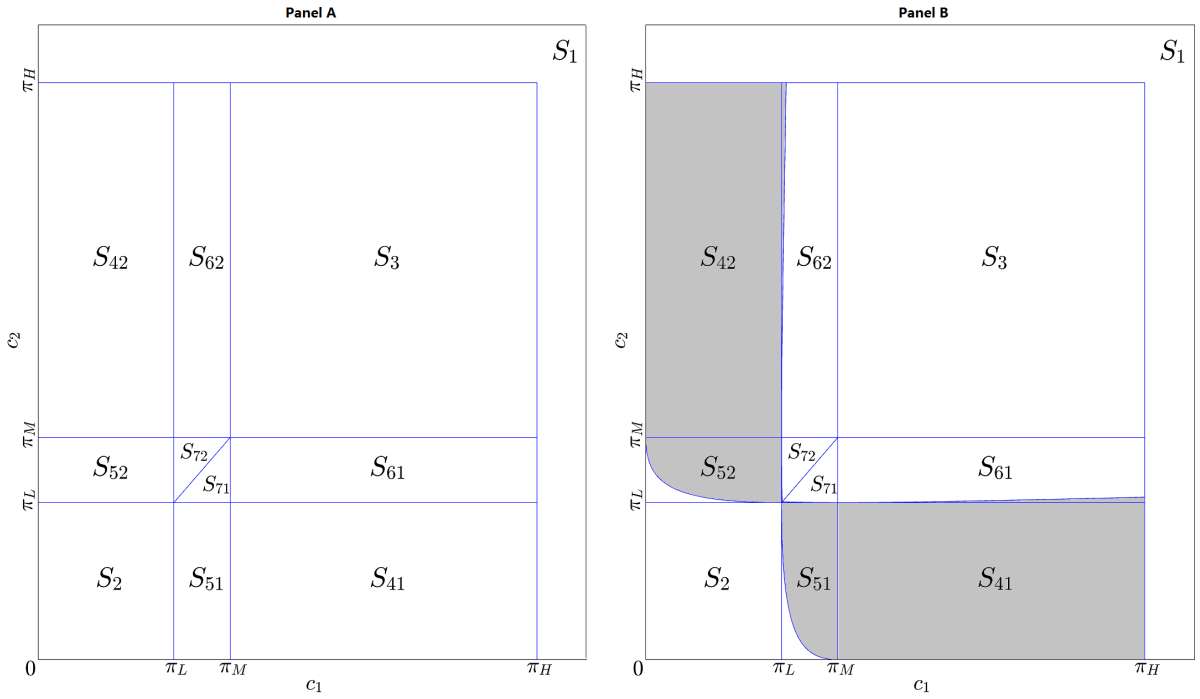

Figure 9. Sets and Existence of Equilibrium. Panel A demonstrates the locations of 11 sets that cover all possible cost vectors, and Panel B demonstrates the regions where there exists at most one equilibrium and the regions where there exists at least one equilibrium. The parameter values are $R = 1$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$.

Proposition B.1 *1. If $(c_1, c_2) \in \mathcal{S}_1$, then there exists a unique equilibrium.*

2. If $(c_1, c_2) \in \mathcal{S}_2 \cup \mathcal{S}_3$, then there exists at least one equilibrium.

3. If $(c_1, c_2) \in \mathcal{S}_{41} \cup \mathcal{S}_{42}$, then there exists at most one equilibrium.

4. For $(c_1, c_2) \in \mathcal{S}_{51}$, if $\mathbb{E}[\pi_1(2R+1, \underline{N_2})] \geq c_1$, then there is at least one equilibrium; otherwise, there exists at most one equilibrium. For $(c_1, c_2) \in \mathcal{S}_{52}$, if $\mathbb{E}[\pi_2(\underline{N_1}, 2R+1)] \geq c_2$, then there is at least one equilibrium; otherwise, there is at most one equilibrium.

5. For $(c_1, c_2) \in \mathcal{S}_{61} \cup \mathcal{S}_{71}$, if $\mathbb{E}[\pi_2(\underline{N_1}+1, 2R+1)] \geq c_2$, then there is at most one equilibrium; otherwise, there is at least one equilibrium. For $(c_1, c_2) \in \mathcal{S}_{62} \cup \mathcal{S}_{72}$, if $\mathbb{E}[\pi_1(2R+1, \underline{N_2}+1)] \geq c_1$, then there is at most one equilibrium; otherwise, there is at least one equilibrium.

We illustrate the results of Proposition B.1 in Figure 9. Panel A in Figure 9 presents the locations of those sets. Panel B in Figure 9 illustrates the results of Proposition B.1 in the case of $R = 1$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$. The light grey regions represent the regions

where there exists at most one equilibrium, and the white regions represent the regions where there exists at least one equilibrium.

The intuitions behind Proposition B.1 are as follows. For a simple example, if $(c_1, c_2) \in \mathcal{S}_2$, then we have $\underline{N_1}, \underline{N_2} \geq 2R+1$. Note that $N_k^\dagger(N_l) \geq \underline{N_k}$ for $N_l \geq 0$. Thus, $N_1^\dagger(\underline{N_2}) \geq 2R+1$ always holds for $(c_1, c_2) \in \mathcal{S}_2$ and there exists at least one equilibrium according to Point 1 of Lemma B.2. If $(c_1, c_2) \in \mathcal{S}_{51}$, then we have $\underline{N_2} \geq 2R+1$. Thus, $N_2^\dagger(\underline{N_1}+1) \geq 2R+1$ always holds. If $\mathbb{E}[\pi_1(2R+1, \underline{N_2})] \geq c_1$, then we have $N_1^\dagger(\underline{N_2}) \geq 2R+1$, and thus there exists at least one equilibrium according to Point 1 of Lemma B.2. If $\mathbb{E}[\pi_1(2R+1, \underline{N_2})] < c_1$, then we have $N_1^\dagger(\underline{N_2}) < 2R+1$, and thus there exists at most one equilibrium according to Point 2a of Lemma B.2. The other results in Proposition B.1 can be proved in a similar way.

B.2 Mixed Strategy Equilibrium

This section considers mixed-strategy equilibrium. We start with the case where there exists only one type of information that can be acquired as in Kyle (1985) and Subrahmanyam and Titman (1999).

Note that there are an infinite number of traders who can acquire private information. We assume that N traders actually choose to acquire information with positive probability, and other traders quit the market. For simplicity, we consider only symmetric equilibria; that is, all traders who acquire information choose the same probability. Note that the number of traders who obtain information is uncertain. For simplicity, we assume that the market makers can observe the number of informed traders; that is, the market makers do not need to guess it.

If all other informed traders acquire information with probability p, then the expected profit of the kth informed trader is given by

$$\mathbb{E}[\pi(p_k, p, N)] = p_k \left(\sum_{n=0}^{N-1} \{ C_{N-1}^n p^n (1-p)^{N-n-1} \mathbb{E}[\pi(n+1)] \} - c \right), \tag{B.70}$$

where $C_{N-1}^n = \frac{(N-1)!}{n!(N-n-1)!}$. Obviously, if $\sum_{n=0}^{N-1} \{ C_{N-1}^n p^n (1-p)^{N-n-1} \mathbb{E}[\pi(n+1)] \} > c$, then

the optimal p_k is unity; if $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{n}(1-p)^{N-n-1}\mathbb{E}[\pi(n+1)]\}<c$, then the optimal p_k is zero; if $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{n}(1-p)^{N-n-1}\mathbb{E}[\pi(n+1)]\}=c$, then the value of p_k makes no difference for the kth informed trader. Therefore, if all informed traders choose a p such that $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{n}(1-p)^{N-n-1}\mathbb{E}[\pi(n+1)]\}=c$, then no informed trader has an incentive to deviate.

An increase in p implies that the probability of having a larger number of informed traders increases and the probability of having a smaller number of informed traders decreases. We can prove rigorously that $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{n}(1-p)^{N-n-1}\mathbb{E}[\pi(n+1)]\}$ decreases with p.

Notice that when $p=1$, $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{n}(1-p)^{N-n-1}\mathbb{E}[\pi(n+1)]\}-c=\mathbb{E}[\pi(N)]-c$. Therefore, if $\mathbb{E}[\pi(N)]<c$, then there exists a unique p^* such that $\sum_{n=0}^{N-1}\{C_{N-1}^{n}p^{*n}(1-p^*)^{N-n-1}\mathbb{E}[\pi(n+1)]\}=c$ holds. That is, for any N satisfying $\mathbb{E}[\pi(N)]<c$, there exists a unique mixed equilibrium.

Next, we move to the case where there are two types of private information that can be acquired. We still consider symmetric equilibria. That is, in an equilibrium, all informed traders of type k acquire private information with the same probability, denoted as p_k, for $k=1,2$.

Assume that there are N_1 informed traders who acquire type 1 private information and N_2 informed traders who acquire type 2 private information. If all other informed traders in group 1 acquire type 1 private information with probability p_1 and all informed traders in group 2 acquire type 2 private information with probability p_2, then the expected profit of the kth informed trader in group 1, who acquires type 1 private information with probability $p_{1,k}$, is given by

$$\begin{aligned}&\mathbb{E}[\pi_1(p_{1,k},p_1,N_1,p_2,N_2)]\\&=p_{1,k}\left(\sum_{n_1=0}^{N_1-1}\left\{C_{N_1-1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1-1}\times\sum_{n_2=0}^{N_2}C_{N_2}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2}\mathbb{E}[\pi_1(n_1+1,n_2)]\right\}-c_1\right)\end{aligned}\tag{B.71}$$

If the following inequality holds, then the optimal $p_{1,k}$ is unity.

$$\sum_{n_1=0}^{N_1-1}\left\{C_{N_1-1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1-1}\times\sum_{n_2=0}^{N_2}C_{N_2}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2}\mathbb{E}[\pi_1(n_1+1,n_2)]\right\}>c_1,$$

If the following inequality holds, then the optimal $p_{1,k}$ is zero.

$$\sum_{n_1=0}^{N_1-1}\left\{C_{N_1-1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1-1}\times\sum_{n_2=0}^{N_2}C_{N_2}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2}\mathbb{E}[\pi_1(n_1+1,n_2)]\right\}<c_1,$$

If the following equality holds, then the value of $p_{1,k}$ makes no difference for the kth informed trader in group 1.

$$\sum_{n_1=0}^{N_1-1}\left\{C_{N_1-1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1-1}\times\sum_{n_2=0}^{N_2}C_{N_2}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2}\mathbb{E}[\pi_1(n_1+1,n_2)]\right\}=c_1,$$

Therefore, if all N_1 informed traders in group 1 choose p_1 and all N_2 informed traders in group 2 choose p_2 such that the two following equalities hold, then no informed traders have an incentive to deviate.

$$\sum_{n_1=0}^{N_1-1}\left\{C_{N_1-1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1-1}\times\sum_{n_2=0}^{N_2}C_{N_2}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2}\mathbb{E}[\pi_1(n_1+1,n_2)]\right\}=c_1, \tag{B.72}$$

$$\sum_{n_2=0}^{N_2-1}\left\{C_{N_2-1}^{n_2}p_2^{n_2}(1-p_2)^{N_2-n_2-1}\times\sum_{n_1=0}^{N_1}C_{N_1}^{n_1}p_1^{n_1}(1-p_1)^{N_1-n_1}\mathbb{E}[\pi_2(n_1,n_2+1)]\right\}=c_2. \tag{B.73}$$

For given p_2, N_1, N_2, there exists a unique p_1 such that Equation (B.72) holds and for given p_1, N_1, N_2, there is a unique p_2 such that Equation (B.73) holds. For given N_1, N_2 such that $\mathbb{E}[\pi_1(N_1,0)]<c_1$ and $\mathbb{E}[\pi_2(0,N_2)]<c_2$ hold, then there exist p_1^* and p_2^* such that both Equations (B.72) and (B.73) hold. That is, a mixed strategy equilibrium exists.

B.3 Stock Payoff Contains More Than Two Fundamentals

When the stock payoff contains more than two fundamentals, our main conclusions remain intact. A major additional finding is that the number of multiple equilibria increases rapidly with the number of fundamentals contained in the stock payoff.

Suppose that the stock payoff is affected by M fundamentals, and thus there are M groups of informed traders correspondingly. Each group of informed traders is informed of a certain fundamental. According to Definition 1, an equilibrium in the information market is determined by M inequalities. Further, according to Lemma A.1, to verify whether there exists a cost vector $\vec{c} = (c_1, \cdots, c_M)$ such that $\vec{N}^1, \cdots, \vec{N}^J$ are all equilibria under $\vec{c}$, we need to examine $M \times J \times (J-1)$ inequalities. The number of inequalities increases rapidly with M. As space is limited, this section presents our main findings in a narrative manner.

Learning substitutability can still lead to multiplicity of equilibria. Similar to the case when $M = 2$, if $(N_1, N_2, \cdots, N_M)$ is an equilibrium under $(c_1, \cdots, c_M)$ and $N_1, N_2 \leq 2R$, then $(N_1+1, N_2-1, N_3, \cdots, N_M)$, $(N_1+2, N_2-2, N_3, \cdots, N_M)$, $(N_1+3, N_2-3, N_3, \cdots, N_M)$, $\cdots$ may also be equilibria under $(c_1, \cdots, c_M)$. Furthermore, when $M > 2$, the form of multiple equilibria derived from substitutability can be more complicated and diverse. Take $M = 3$ as an example. Suppose that (N_1, N_2, N_3) is an equilibrium under (c_1, c_2, c_3) and $N_1, N_2, N_3 \leq 2R$. Then, information allocations such as (N_1+1, N_2-1, N_3), (N_1+1, N_2, N_3-1), (N_1, N_2+1, N_3-1), (N_1-1, N_2+1, N_3), (N_1-1, N_2, N_3+1), (N_1, N_2-1, N_3+1) may also be equilibria under (c_1, c_2, c_3). In addition, (N_1+2, N_2-1, N_3-1), (N_1-2, N_2+1, N_3+1), (N_1-1, N_2+2, N_3-1), (N_1+1, N_2-2, N_3+1), (N_1-1, N_2-1, N_3+2), (N_1+1, N_2+1, N_3-2) may also be equilibria under (c_1, c_2, c_3). According to the above pattern, it is intuitive that the number of multiple equilibria due to substitutability increases rapidly with M. Further, the above results also hold for the case of $c_1 = \cdots = c_M$. Therefore, the number of symmetry breaking equilibria also increases rapidly with M.

Learning complementarity can still lead to multiplicity of equilibria. Similar to Proposition 1, if $(N_1, N_2, N_3, \cdots, N_M)$ is an equilibrium under $(c_1, \cdots, c_M)$ and $N_1, N_2 \geq 2R+1$,

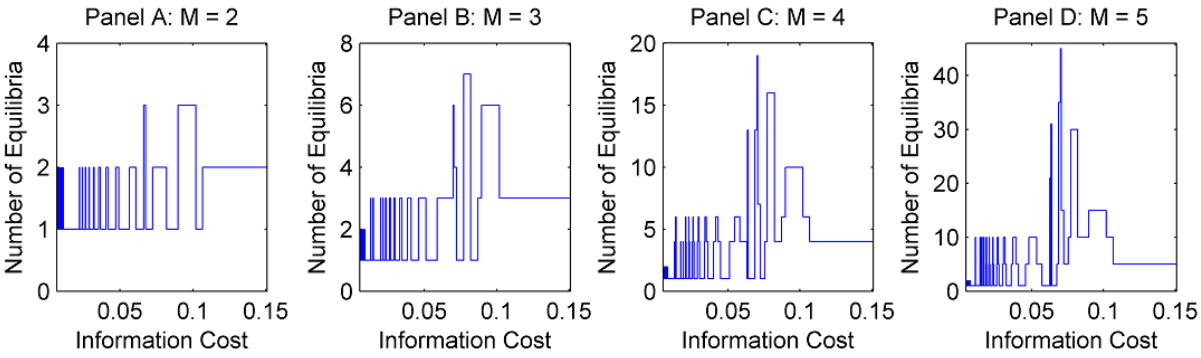

Figure 10. Number of Multiple Equilibria vs M. This figure illustrates how the number of multiple equilibria in the information market varies with the number of fundamentals in the payoff, M. The parameter values are $R = 10$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$.

then $(N_1+1, N_2+1, N_3, \cdots, N_M)$ may also be an equilibrium under $(c_1, \cdots, c_M)$. Recall from Proposition 1 that when $M = 2$, there are up to two equilibria due to complementarity under a given cost vector. When $M > 2$, however, there can be more than two equilibria derived from complementarity. The intuition is that as M becomes larger, there are more types of informed traders who provide liquidity mutually; that is, the complementarity effect becomes stronger. When the effect is strong enough, there can be more than two equilibria. For example, if M is sufficiently large, and $N_1, N_2 > N_3, \cdots, N_M \geq 2R+1$, then $(N_1, N_2, N_3, \cdots, N_M)$, $(N_1+1, N_2+1, N_3+1, \cdots, N_M+1)$, $(N_1+2, N_2+2, N_3+1, \cdots, N_M+1)$, $(N_1+3, N_2+3, N_3+1, \cdots, N_M+1)$, $(N_1+4, N_2+4, N_3+1, \cdots, N_M+1)$, $\cdots$ may all be equilibria under the same cost vector.

In addition, learning complementarity and learning substitutability may play a role at the same time in the existence of multiple equilibria. Take $M = 4$ as an example. Suppose that (N_1, N_2, N_3, N_4) with $N_1, N_2 < 2R$ and $N_3, N_4 \geq 2R+1$ is an equilibrium under (c_1, c_2, c_3, c_4). Then, $(N_1-1, N_2+1, N_3+1, N_4+1)$, $(N_1+1, N_2-1, N_3+1, N_4+1)$, $(N_1-2, N_2+2, N_3+1, N_4+1)$, $(N_1+2, N_2-2, N_3+1, N_4+1)$, $\cdots$ may also be equilibria under (c_1, c_2, c_3, c_4).

Figure 10 illustrates that the maximum number of equilibria increases with the number of fundamentals, M. For example, when $M = 2$, the maximum number of equilibria is 3, but when $M = 5$, the maximum number is 45.

B.4 Market Participation and Information Acquisition

We consider an extension in which traders decide whether or not to participate in the financial market and, if so, how much information to obtain. As in the baseline model, the numbers of each type of informed traders become common knowledge. Hence, when choosing their information precision, informed traders know the population sizes of both types of informed traders.

The m_kth informed trader in group k, denoted as (k, m_k), observes a private signal, s_{k,m_k}, about θ_k:

$$s_{k,m_k} = \theta_k + \epsilon_{k,m_k}.$$

The information precision of the (k, m_k) informed trader is defined as $\rho_{k,m_k} = \frac{\sigma^2}{\mathbf{Var}(\epsilon_{k,m_k})}$. Consistent with the benchmark model, we define $R_{k,m_k} = \rho_{k,m_k}^{-1}$.

We follow Xiong and Yang (2021) and assume the information cost takes the form of $c_i(\rho) := a_i + b\rho^\tau$, with $a_i, b \geq 0$ and $\tau > 1$.[16] Here, a_i is a fixed entry fee for acquiring the i-th fundamental. To keep the model tractable, we focus on symmetric equilibria in which informed traders of the same type choose the same level of information precision.

The timeline of the model goes as follows. First, traders choose whether they would participate or not. Second, given (N_1, N_2) each informed trader chooses their level of information precision. Lastly, traders engage in trading in the financial market. To facilitate exposition, we provide a sketch of how to solve the model and leave the detail to the end of the subsection. Similar to the main text, we first compute the financial market equilibrium. From their trading profit, we compute each trader's choice of information precision. The following lemma shows that there is a unique $(R_1^*(N_1, N_2), R_2^*(N_1, N_2))$ for any given (N_1, N_2).

Lemma B.3 *Given (N_1, N_2), there exists a unique $(R_1^*(N_1, N_2), R_2^*(N_1, N_2))$ under the symmetric equilibrium.*

[16] Note that Xiong and Yang (2021) set the parameter τ to 1. Our model is also well-posed when $\tau = 1$. For ease of exposition, we explicitly assume $\tau > 1$, which makes the information cost function strictly convex in precision, ensuring that the first-order condition always yields an interior solution (see the discussion after Equation B.76).

The proof of the lemma can be found at the end of this subsection. To determine equilibrium (N_1, N_2), each trader compares their expected profit minus the variable information cost with the participation cost.

Lemma B.4 *The expected profit of the k-th type of informed traders minus the variable information cost, decreases as N_k increases.*

The proof of the lemma can be found at the end of this subsection. Given the above lemma, the equilibrium condition for information allocation (N_1, N_2) is defined as follows.

Definition 2 *The information allocation $\vec{N} = (N_1, N_2)$ is an equilibrium if and only if the following conditions hold for $k = 1, 2$*

$$\begin{aligned} \mathbb{E}[\pi_k(\vec{N})] - \frac{b}{R_k^*(\vec{N})^\tau} \geq a_k > \mathbb{E}[\pi_k(\vec{N} + \vec{T}_k)] - \frac{b}{R_k^*(\vec{N} + \vec{T}_k)^\tau}, \quad & \text{if } N_k > 0, \\ a_k > \mathbb{E}[\pi_k(\vec{N} + \vec{T}_k)] - \frac{b}{R_k^*(\vec{N} + \vec{T}_k)^\tau}, \quad & \text{if } N_k = 0. \end{aligned}$$

Given the above definition, we use numerical analysis to search for equilibrium. Figure 11 shows the number of equilibria in the information market when the parameter values are $b = 100$, $\tau = 1.5$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$. As we can see, multiple equilibria can arise. Under the above parameter specification, there can be up to 19 different equilibrium information allocations (N_1, N_2) that constitute an equilibrium. As we further increase b from 100 to 1000, traders acquire noisier signals, i.e, R increases, in which case, the substitutability region expands. Consistent with the main text, the number of equilibria can be even larger. Figure 12 shows that the number of equilibria can be as many as 80 when the parameter values are $b = 1000$, $\tau = 1.5$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$. Overall, our result on equilibrium multiplicity remains robust when information precision is endogenized.

Proof of Lemma B.3. We solve for the optimal information precision chosen by the informed traders. Because the informed trader's trading strategy is assumed to be linear

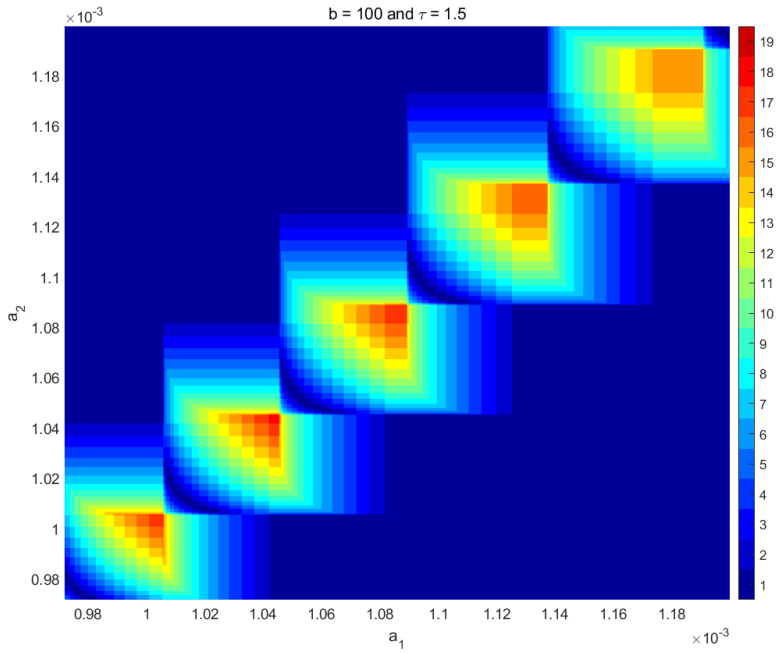

Figure 11. Multiplicity under endogenous market participation and information acquisition. This figure illustrates how the number of multiple equilibria in the information market when both the number of informed traders and information acquisition are endogenous. The parameter values are $b = 100$, $\tau = 1.5$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$.

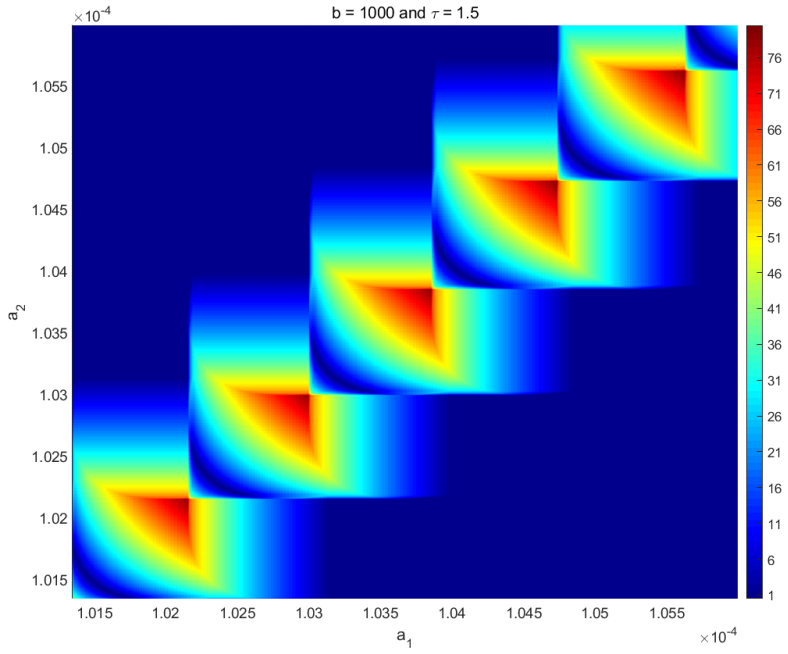

Figure 12. Multiplicity under endogenous market participation and information acquisition. This figure illustrates how the number of multiple equilibria in the information market varies when both the number of informed traders and information acquisition are endogenous. The parameter values are $b = 1000$, $\tau = 1.5$, $\sigma^2 = 1$, and $\sigma_u^2 = 1$.

in his private signal, the profit of informed trader (k, m_k) is given by

$$\pi_{k,m_k} = x_{k,m_k}(v - P) = x_{k,m_k}\left[v - \lambda\left(\sum_{l=1}^{2}\sum_{n_l=1}^{N_l}\beta_{l,n_l}s_{l,n_l} + u\right)\right].$$

The expected profit of informed trader (k, m_k) conditional on s_{k,m_k} is given by

$$\mathbb{E}(\pi_{k,m_k}|s_{k,m_k}) = x_{k,m_k}\left(\mathbb{E}(\theta_k|s_{k,m_k}) - \lambda\left[x_{k,m_k} + \sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\mathbb{E}(\theta_k|s_{k,m_k})\right]\right).$$

Taking the derivative of $\mathbb{E}(\pi_{k,m_k}|s_{k,m_k})$ with respect to x_{k,m_k}, we obtain the first-order condition for the optimization problem of informed trader (k, m_k):

$$\mathbb{E}(\theta_k|s_{k,m_k}) - \lambda\left[x_{k,m_k} + \sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\mathbb{E}(\theta_k|s_{k,m_k})\right] - \lambda x_{k,m_k} = 0.$$

It follows immediately that

$$\begin{aligned}
2\lambda x_{k,m_k} &= \mathbb{E}(\theta_k|s_{k,m_k}) - \lambda\sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\mathbb{E}(\theta_k|s_{k,m_k}),\\
&= \left(1 - \lambda\sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\right)\mathbb{E}(\theta_k|s_{k,m_k}),\\
&= \left(1 - \lambda\sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\right)\frac{1}{1+R_{k,m_k}}s_{k,m_k}.
\end{aligned}$$

Then, we have

$$\beta_{k,m_k} = \frac{1}{2\lambda}\left(1 - \lambda\sum_{\substack{n_k=1\\ n_k\neq m_k}}^{N_k}\beta_{k,n_k}\right)\frac{1}{1+R_{k,m_k}} \tag{B.74}$$

Next, we compute the unconditional expected profit informed trader (k, m_k). Note

that $x_{k,m_k} = \beta_{k,m_k} s_{k,m_k}$. Then, we have

$$\begin{aligned}
\mathbb{E}(\pi_{k,m_k}) &= \mathbb{E}\left\{\beta_{k,m_k} s_{k,m_k}\left[v - \lambda\left(\sum_{l=1}^{2}\sum_{n_l=1}^{N_l}\beta_{l,n_l}s_{l,n_l} + u\right)\right]\right\}. \\
&= \beta_{k,m_k}\left(\sigma^2 - \lambda\sum_{n_k\neq m_k}\beta_{k,n_k}\sigma^2 - \lambda\beta_{k,m_k}\sigma^2(1+R_{k,m_k})\right). \\
&= \frac{\sigma^2}{4\lambda}\left(1-\lambda\sum_{n_k\neq m_k}\beta_{k,n_k}\right)^2\frac{1}{1+R_{k,m_k}} \qquad \text{(B.75)}
\end{aligned}$$

The information cost takes the form $a+\frac{b}{R^{\tau}_{k,m_k}}$. We assume that the informed trader of type (k, m_k) chooses his information precision while taking the market maker and other traders' β and λ as given, as his choice is unobservable to them. Then, the optimal R_{k,m_k} is determined by the following first order condition.

$$\frac{\sigma^2}{4\lambda}\left(1-\lambda\sum_{n_k\neq m_k}\beta_{k,n_k}\right)^2\frac{R^{\tau+1}_{k,m_k}}{(1+R_{k,m_k})^2} = b\tau \qquad \text{(B.76)}$$

Given that $\tau > 1$, the left-hand side (LHS) of the above equation equals zero when $R_{k,m_k} = 0$, and the LHS approaches infinity as R_{k,m_k} tends to infinity. Therefore, a positive solution for R_{k,m_k} exists for this equation.

We consider a symmetric equilibrium in which the informed traders in group k choose the same level of information precision. Then, for any m_k, we have $R_k = R_{k,m_k}$ and $\beta_k = \beta_{k,m_k}$. Note that for a given k, if R_{k,m_k} is the same for all m_k, then we have

$$\beta_k = \frac{1}{\lambda(2R_k + 1 + N_k)}. \qquad \text{(B.77)}$$

The first order condition (B.76) can be further simplified as follows.

$$\frac{\sigma^2}{4\lambda}\left(1-\frac{N_k-1}{2R_k+1+N_k}\right)^2\frac{R^{\tau+1}_k}{(1+R_k)^2} = b\tau$$

Algebraic manipulation leads to the following equation.

$$\frac{\sigma^2 R_k^{\tau+1}}{\lambda\left(2R_k+1+N_k\right)^2}=b\tau \tag{B.78}$$

Note that λ is bounded. Given that $\tau > 1$, the left-hand side (LHS) of the above equation equals zero when $R_k = 0$, and the LHS approaches infinity as R_k tends to infinity. Therefore, a positive solution for R_k exists for this equation.

The optimal equilibrium information precision given N_1 and N_2 is determined by Equation (B.78). Note that the derivative of $\frac{R_k}{2R_k+1+N_k}$ with respect to R_k is always positive for $\tau \geq 1$. Therefore, given N_k, when liquidity worsens (i.e., $\lambda \uparrow$), R_k increases, meaning that information precision declines.

Because $P = \mathbb{E}(v|X+u) = \lambda(X+u)$, λ is given by

$$\lambda=\frac{\operatorname{Cov}(v,X+u)}{\operatorname{Var}(X+u)}=\frac{\sum_{l=1}^{2}\operatorname{Cov}(\theta_l,X+u)}{\operatorname{Var}(X+u)}=\frac{\sum_{l=1}^{2}N_l\beta_l\sigma^2}{\sum_{l=1}^{2}N_l^2\beta_l^2\sigma^2+\sum_{l=1}^{2}N_l\beta_l^2\sigma^2\rho_l^{-1}+\sigma_u^2}.$$

Substituting β_k given by Equation (B.77) into the above equation, we have

$$\lambda^2=\frac{\sigma^2}{\sigma_u^2}\sum_{l=1}^{2}\frac{N_l(\rho_l^{-1}+1)}{(2\rho_l^{-1}+1+N_l)^2}=\frac{\sigma^2}{\sigma_u^2}\sum_{l=1}^{2}L_l.$$

where

$$L_l=\frac{N_l(\rho_l^{-1}+1)}{(2\rho_l^{-1}+1+N_l)^2}.$$

We are now ready to prove the uniqueness in the level of information precision given (N_1, N_2). We argue by contradiction. Given N_1 and N_2, suppose there exist two equilibria (R_1^*, R_2^*) and $(R_1^\dagger, R_2^\dagger)$.

Using the expression for λ, we can rewrite Equation (B.78) in the following form.

$$\frac{1}{L_1+L_2}\frac{N_k^2(R_k+1)^2}{(2R_k+1+N_k)^4}=\frac{b^2\tau^2N_k^2(R_k+1)^2}{\sigma^2\sigma_u^2R_k^{2\tau+2}}.$$

The above expression is equivalent to the following one.

$$\frac{L_k^2}{L_1+L_2} = \frac{b^2\tau^2 N_k^2 (R_k+1)^2}{\sigma^2\sigma_u^2 R_k^{2\tau+2}} \tag{B.79}$$

Assume $L^\dagger > L^*$. Let $L_1^\dagger = \alpha_1 L_1^*$ and $L_2^\dagger = \alpha_2 L_2^*$. Let $\alpha_1 > \alpha_2$. Then, we have $\alpha_1 > 1$.

$$\frac{(L_1^\dagger)^2}{L_1^\dagger + L_2^\dagger} = \frac{(\alpha_1 L_1^*)^2}{\alpha_1 L_1^* + \alpha_2 L_2^*} > \frac{(\alpha_1 L_1^*)^2}{\alpha_1 L_1^* + \alpha_1 L_2^*} = \frac{\alpha_1 (L_1^*)^2}{L_1^* + L_2^*} > \frac{(L_1^*)^2}{L_1^* + L_2^*}.$$

Given $L^\dagger > L^*$, we have $R_k^\dagger > R_k^*$ for any k. Moreover, the derivative of $\frac{(R_k+1)^2}{R_k^{2\tau+2}}$ with respect to R_k is negative. Thus, if the information precision moves from (R_1^*, R_2^*) to $(R_1^\dagger, R_2^\dagger)$, the left-hand side of the Equation (B.79) increases, while the right-hand side decreases, contradicted to the equality for both (R_1^*, R_2^*) and $(R_1^\dagger, R_2^\dagger)$. Hence, there is a unique (R_1^*, R_2^*) that solves (B.78).

Proof of Lemma B.4. We first show that R_1 increases as a function of N_1, holding N_2 as constant. Taking the logarithm of both sides of the Equation (B.78) is equivalent to the following formula:

$$2\ln(\sigma) + (\tau+1)\ln(R_k) - 2\ln(2R_k + 1 + N_k) - \ln(\lambda) = \ln(b) + \ln(\tau). \tag{B.80}$$

Take the total differential of both sides of the above formula for $k = 1$ and $k = 2$ separately.

$$\left(\frac{\tau+1}{R_1} - \frac{4}{2R_1+1+N_1}\right) \mathrm{d}R_1 - \frac{2}{2R_1+1+N_1}\mathrm{d}N_1 - \frac{\mathrm{d}\lambda}{\lambda} = 0, \tag{B.81}$$

$$\left(\frac{\tau+1}{R_2} - \frac{4}{2R_2+1+N_2}\right) \mathrm{d}R_2 - \frac{\mathrm{d}\lambda}{\lambda} = 0. \tag{B.82}$$

Plugging $\frac{\mathrm{d}\lambda}{\lambda}$ from (B.81) into (B.82), $\mathrm{d}R_2$ can be expressed in terms of $\mathrm{d}R_1$ and $\mathrm{d}N_1$.

$$\mathrm{d}R_2 = \frac{\left(\frac{\tau+1}{R_1} - \frac{4}{2R_1+1+N_1}\right)}{\left(\frac{\tau+1}{R_2} - \frac{4}{2R_2+1+N_2}\right)}\mathrm{d}R_1 - \frac{\left(\frac{2}{2R_1+1+N_1}\right)}{\left(\frac{\tau+1}{R_2} - \frac{4}{2R_2+1+N_2}\right)}\mathrm{d}N_1 \tag{B.83}$$

Note that

$$\mathrm{d}\ln\lambda = \frac{1}{2(L_1+L_2)}\left(\frac{\partial L_1}{\partial N_1}\mathrm{d}N_1 + \frac{\partial L_1}{\partial R_1}\mathrm{d}R_1 + \frac{\partial L_2}{\partial R_2}\mathrm{d}R_2\right). \tag{B.84}$$

Plugging $\mathrm{d}R_2$ Equation (B.83) into $\mathrm{d}\ln\lambda$, and then plugging $\mathrm{d}\ln\lambda$ in to Equation (B.81). We have the following equation.

$$\Gamma_{R_1}\mathrm{d}R_1 = \Gamma_{N_1}\mathrm{d}N_1, \tag{B.85}$$

where

$$\Gamma_{R_1} = \left(\frac{\tau+1}{R_1} - \frac{4}{2R_1+1+N_1}\right)\left[1 - \frac{\frac{\partial L_2}{\partial R_2}}{2(L_1+L_2)\left(\frac{\tau+1}{R_2} - \frac{4}{2R_2+1+N_2}\right)}\right] - \frac{1}{2(L_1+L_2)}\frac{\partial L_1}{\partial R_1},$$

$$\Gamma_{N_1} = \left(\frac{2}{2R_1+1+N_1}\right)\left[1 - \frac{\frac{\partial L_2}{\partial R_2}}{2(L_1+L_2)\left(\frac{\tau+1}{R_2} - \frac{4}{2R_2+1+N_2}\right)}\right] + \frac{1}{2(L_1+L_2)}\frac{\partial L_1}{\partial N_1}.$$

Since Γ_{R_1} and Γ_{N_1} are both greater than zero, (B.85) implies that R_1 increases as a function of N_1.

We are now ready to show that the expected profit of the k-th type of informed traders minus the variable information cost, decreases as N_k increases.

We define $\mathbb{E}(\hat{\pi}_k) = \mathbb{E}(\pi_k) - \frac{b}{R_k^\tau}$. From (B.75), we have

$$\mathbb{E}(\hat{\pi}_k) = \frac{\sigma^2(R_k+1)}{\lambda(N_k+2R_k+1)^2} - \frac{b}{R_k^\tau}. \tag{B.86}$$

Note that

$$\ln\left(\mathbb{E}(\pi_k)\right) = 2\ln(\sigma) + \ln(R_k+1) - \ln(\lambda) - 2\ln(N_k+2R_k+1). \tag{B.87}$$

Then, we have

$$\frac{\mathrm{d}\mathbb{E}(\pi_k)}{\mathbb{E}(\pi_k)} = \left(\frac{1}{R_k+1} - \frac{4}{N_k+2R_k+1}\right)\mathrm{d}R_k - \left(\frac{2}{N_k+2R_k+1}\right)\mathrm{d}N_k - \frac{d\lambda}{\lambda} \tag{B.88}$$

Substituting (B.81) into Formula (B.88), we obtain the following formula.

$$\frac{\mathrm{d}\mathbb{E}(\pi_k)}{\mathbb{E}(\pi_k)} = \left(\frac{1}{R_k+1} - \frac{\tau+1}{R_k}\right)\mathrm{d}R_k \tag{B.89}$$

Now we consider $\mathrm{d}\mathbb{E}(\hat{\pi}_k)$. From the above equation we have

$$\mathrm{d}\mathbb{E}(\pi_k) - \mathrm{d}\left(\frac{b}{R_k^\tau}\right) = \mathbb{E}(\pi_k)\left(\frac{1}{R_k+1} - \frac{\tau+1}{R_k}\right)\mathrm{d}R_k + \frac{b\tau}{R_k^{\tau+1}}\mathrm{d}R_k \tag{B.90}$$

$$= \left[\frac{\sigma^2(R_k+1)}{\lambda(N_k+2R_k+1)^2}\left(\frac{1}{R_k+1} - \frac{\tau+1}{R_k}\right) + \frac{b\tau}{R_k^{\tau+1}}\right]\mathrm{d}R_k \tag{B.91}$$

We aim to show that the expression inside the brackets is negative. Plugging (B.78) into it leads to

$$\frac{b\tau}{R_k^{\tau+1}}\left(1 - \tau - \frac{\tau+1}{R_k}\right). \tag{B.92}$$

Given $\tau > 1$, the above equation is negative. Therefore, an increase in N_k leads to a decrease in $\mathbb{E}(\hat{\pi}_k)$.

B.5 The Possibility of A Trader Switching From One Type of Information to the Other

In the case where a trader can switch from one type of information to the other, the multiplicity due to substitutability in learning still exists, but the number of equilibria will be sharply reduced, and the multiplicity due to complementarity in learning still holds.

A trader has private information about one fundamental only, but can switch from one fundamental to the other. In this case, the equilibrium conditions add the following two inequalities on the basis of Equation (4).

$$\mathbb{E}[\pi_1(\vec{N}^*)] - c_1 \geq \mathbb{E}[\pi_2(\vec{N}^* + \vec{S}_2)] - c_2, \tag{B.93}$$

$$\mathbb{E}[\pi_2(\vec{N}^*)] - c_2 \geq \mathbb{E}[\pi_1(\vec{N}^* + \vec{S}_1)] - c_1. \tag{B.94}$$

where $\vec{S}_1 = (1, -1)$ and $\vec{S}_2 = (-1, 1)$. Inequality (B.93) ensures that the trader who acquires type 1 information has no incentive to switch to type 2, and Inequality (B.94) ensures that the trader who acquires type 2 information has no incentive to switch to type 1.

First, the multiplicity due to complementarity in learning still holds. For $N_1, N_2 \geq 2R + 2$, under a given cost vector (c_1, c_2), if (N_1, N_2) is an equilibrium under Definition 1, then we can show that (N_1, N_2) is also an equilibrium under the new equilibrium conditions; that is, it satisfies Inequalities (B.93) and (B.94) as well. Our proof is as follows. Because (N_1, N_2) is an equilibrium under Definition 1, we have $\mathbb{E}[\pi_1(N_1, N_2)] \geq c_1$ and $\mathbb{E}[\pi_2(N_1, N_2 + 1)] < c_2$. In addition, because the market liquidity is better under $(N_1, N_2{+}1)$ than under $(N_1{-}1, N_2{+}1)$, we have $\mathbb{E}[\pi_2(N_1{-}1, N_2{+}1)] < \mathbb{E}[\pi_2(N_1, N_2{+}1)] < c_2$. Thus, we have $\mathbb{E}[\pi_1(N_1, N_2)] - c_1 \geq 0 > \mathbb{E}[\pi_2(N_1 - 1, N_2 + 1)] - c_2$; that is, Inequality (B.93) holds. Similarly, we can show that Inequality (B.94) also holds.

Second, the multiplicity due to substitutability in learning still exists, but the number of equilibria will be sharply reduced. Suppose that both (N_1, N_2) and $(N_1 - 1, N_2 + 1)$ are equilibria under Definition 1. Unless $\mathbb{E}[\pi_1(N_1, N_2)] - c_1 = \mathbb{E}[\pi_2(N_1 - 1, N_2 + 1)] - c_2$ and $\mathbb{E}[\pi_2(N_1, N_2)] - c_2 = \mathbb{E}[\pi_1(N_1 - 1, N_2 + 1)] - c_1$, one of the two is not an equilibrium under the new equilibrium conditions. For instance, if $\mathbb{E}[\pi_1(N_1, N_2)] - c_1 < \mathbb{E}[\pi_2(N_1 - 1, N_2 + 1)] - c_2$, then a type 1 informed trader will switch to type 2, and thus (N_1, N_2) is no longer an equilibrium under the new equilibrium conditions. Therefore, under the new equilibrium conditions, the multiplicity due to substitutability in learning only exists in a smaller parameter interval, and the number of equilibria will be greatly reduced.

Overall, considering the case where a trader can switch from one type of information to the other does not substantially undermine the main results of our paper.

B.6 Learning Both Types of Information

One might also wonder what the nature of the equilibrium is in the case where a trader has access to two types of information. In fact, in this relaxed case, our results stay unchanged.

Because we assume that informed traders are risk neutral, the expected profit of an informed trader who obtains two types of information equals the sum of the expected profits of two traders, with one trader obtaining type 1 information and the other obtaining type 2 information. Thus, in this relaxed case, the equilibrium conditions are exactly the same as the ones in Definition 1, so all conclusions are unaffected.

B.7 Comparative Statics in the Strategic Complementarity Region

In the strategic complementarity region, the existence of multiplicity leads to a comparative statics result that is opposite to the prediction of the continuous solution. Normally, a decrease in information cost leads to an increase in the number of informed traders and both liquidity and stock price informativeness improve in the complementarity region. This is consistent with the prediction of the continuous solution. However, due to multiplicity, stock liquidity and stock price informativeness can decline as costs decrease.

Consider the case illustrated in Figure 13. In region A, there are two equilibria (N_1, N_2) and $(N_1 + d, N_2 + 1)$, and in region B, there are two equilibria (N_1, N_2) and $(N_1 + d + 1, N_2 + 1)$. If the market moves from $(N_1 + d, N_2 + 1)$ to (N_1, N_2) due to a decrease in c_1, then both stock liquidity and stock price informativeness decrease.

Table 3 presents a numerical example. In this setup, c_2 is held constant at 98.3392×10^{-4}, while c_1 is reduced from 1.0186×10^{-4} to 1.0162×10^{-4}. Prior to the reduction in c_1, two equilibria, (400, 30) and $(401, 31)$, exist. Following the reduction, the equilibria are (400, 30) and $(402, 31)$. When the market shifts from equilibrium $(401, 31)$ to equilibrium $(400, 30)$, both liquidity and SPI decrease. This result contrasts with the continuous solution, where both liquidity and stock price informativeness increase.

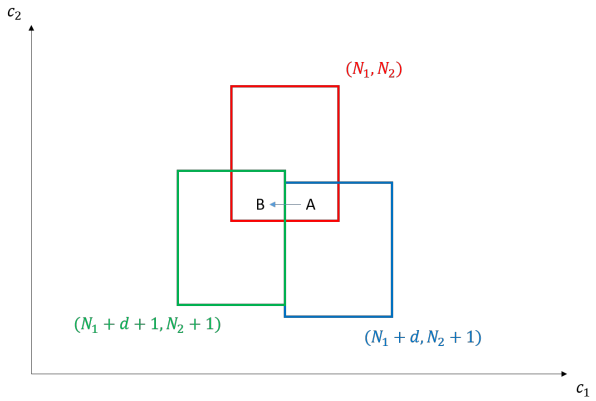

Figure 13. This figure illustrates the comparative statics of changes in information costs. Each box represents a cost region that supports a specific equilibrium. For example, the region inside the red box supports (N_1, N_2). The overlapping areas between boxes indicate the existence of multiple equilibria. When c_2 remains constant and c_1 decreases, the equilibrium moves from point A to point B, transitioning from a unique-equilibrium region to a multiple-equilibrium region.

B.8 Comparative Statics under the Continuous Solution

B.8.1 Comparative Statics in the Information Costs

This section studies the impact of changes in acquisition costs on information allocation, stock liquidity, and stock price informativeness under the continuous solution. The following proposition provides a summary of the main results.

Proposition B.2 *As c_i increases, under the continuous solution, the following conclusions hold.*

1. *n_i^* decreases.*
2. *If $n_i^* < 2R+1$, then n_j^* and liquidity increases; otherwise, n_j^* and liquidity decreases.*
3. *stock price informativeness decreases.*

The economic intuition is as follows. An increase in c_i leads to a decline in n_i. Since liquidity dynamics are primarily driven by changes in n_i, liquidity increases when $n_i^* < 2R + 1$ and decreases otherwise. The change in n_i affects n_j through its impact on liquidity. When $n_i^* < 2R + 1$, improved liquidity causes n_j to rise; in contrast, when $n_i^* \geq 2R+1$, deteriorating liquidity results in a decrease in n_j. Stock price informativeness is also determined by n_i. As n_i decreases, the informativeness of stock prices declines.

Table 3. Comparative Statics in Information Cost

This table presents a numerical example to illustrate that, under the equilibrium, when c_1 decreases, liquidity and stock price informativeness may either increase or decrease in the strategic complementarity region. In contrast, under the continuous solution, both liquidity and stock price informativeness increase. The parameter values are $\sigma^2 = 1$, $\sigma_u^2 = 1$, and $R = 5$.

	Equilibrium		Continuous Solution	
$\vec{c}(10^{-4})$	(1.0186, 98.3392)	(1.0162, 98.3392)	(1.0186, 98.3392)	(1.0162, 98.3392)
Allocation	(400, 30); (401, 31)	(400, 30); (402, 31)	(401.73, 31.004)	(402.23, 31.006)
Liquidity	2.8714; 2.8914	2.8714; 2.8918	2.89175	2.89199
SPI	0.8525; 0.85569	0.8525; 0.85573	0.85573	0.85576

Proof of Proposition B.2. On Information Allocation. Under the information cost vector (c_1, c_2), the continuous solution is given by (n_1, n_2). Without loss of generality, we consider the case where c_1 increases by $\mathrm{d}c_1$. The new continuous solution is denoted as $(n_1 + \mathrm{d}n_1, n_2 + \mathrm{d}n_2)$. Letting $G_k = \left[\frac{(1+R)}{(n_k+2R+1)^2}\right]^2$, by Equation (3), we have

$$[\mathbb{E}(\pi_k)]^2 \propto \frac{G_k}{L} = \frac{G_k}{L_1 + L_2}. \tag{B.95}$$

To ease the exposition, we define the following notations:

$$G_k' = \frac{\partial G_k}{\partial n_k}, \quad (\ln G_k)' = \frac{1}{G_k}\frac{\partial G_k}{\partial n_k}, \quad L_k' = \frac{\partial L_k}{\partial n_k}. \tag{B.96}$$

The condition (5) is equivalent to

$$(\ln G_1)'\mathrm{d}n_1 = \frac{1}{L}\sum_{l=1}^{2} L_l'\mathrm{d}n_l + 2\frac{\mathrm{d}c_1}{c_1}, \tag{B.97}$$

$$(\ln G_2)'\mathrm{d}n_2 = \frac{1}{L}\sum_{l=1}^{2} L_l'\mathrm{d}n_l. \tag{B.98}$$

Subtracting both sides of Equation (B.97) from Equation (B.98) yields

$$\mathrm{d}n_2 = \frac{(\ln G_1)'}{(\ln G_2)'}\mathrm{d}n_1 - \frac{2}{(\ln G_2)'}\frac{\mathrm{d}c_1}{c_1}. \tag{B.99}$$

Plugging Equation (B.99) into Equation (B.97), we have

$$(\ln G_1)' \left[1 - \frac{1}{L}\sum_{l=1}^{2} \frac{L_l'}{(\ln G_l)'}\right] \mathrm{d}n_1 = 2\frac{\mathrm{d}c_1}{c_1}\left[1 - \frac{1}{L}\frac{L_2'}{(\ln G_2)'}\right]. \tag{B.100}$$

From the above equation, we have

$$\frac{\mathrm{d}n_1}{\mathrm{d}c_1/c_1} = \frac{2}{(\ln G_1)'} \frac{L_1 + \left[L_2 - \frac{L_2'}{(\ln G_2)'}\right]}{\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right]}. \tag{B.101}$$

Substituting Equation (B.101) into Equation (B.99), we have

$$\frac{\mathrm{d}n_2}{\mathrm{d}c_1/c_1} = \frac{2}{(\ln G_2)'} \frac{\frac{L_1'}{(\ln G_1)'}}{\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right]}. \tag{B.102}$$

Notice that

$$\frac{1}{(\ln G_l)'} = -\frac{(2R+1+n_l)}{4} < 0, \tag{B.103}$$

$$L_l' = \frac{(R+1)(2R+1-n_l)}{(n_l+2R+1)^3}, \tag{B.104}$$

$$\frac{L_l'}{(\ln G_l)'} = \frac{(R+1)(n_l-2R-1)}{4(2R+1+n_l)^2}, \tag{B.105}$$

$$L_l - \frac{L_l'}{(\ln G_l)'} = \frac{(R+1)(2R+1+3n_l)}{4(2R+1+n_l)^2} > 0. \tag{B.106}$$

By Equations (B.101) and (B.106), we conclude that

$$\frac{\mathrm{d}n_1}{\mathrm{d}c_1/c_1} < 0. \tag{B.107}$$

By Equation (B.102), the sign of $\frac{\mathrm{d}n_2}{\mathrm{d}c_1/c_1}$ is the same as the sign of L_1'. Thus, by Equations (B.102) and (B.104), we have

$$\frac{\mathrm{d}n_2}{\mathrm{d}c_1/c_1} \begin{cases} > 0 & n_1 < 2R+1, \\ < 0 & n_1 > 2R+1. \end{cases} \tag{B.108}$$

On Liquidity. Recall that $\lambda^2 \propto L := L_1 + L_2$. Then, we have $\mathrm{d}L = L_1' \mathrm{d}n_1 + L_2' \mathrm{d}n_2$. Substituting Equations (B.101) and (B.102) into $\mathrm{d}L$, we have

$$\mathrm{d}L = \frac{2L_1'}{(\ln G_1)'} \frac{L - \frac{L_2'}{(\ln G_2)'}}{L - \sum_{l=1}^{2} \frac{L_l'}{(\ln G_l)'}} \frac{\mathrm{d}c_1}{c_1} + \frac{2L_2'}{(\ln G_2)'} \frac{\frac{L_1'}{(\ln G_1)'}}{L - \sum_{l=1}^{2} \frac{L_l'}{(\ln G_l)'}} \frac{\mathrm{d}c_1}{c_1} \tag{B.109}$$

By Equation (B.106), we have $L - \sum_{l=1}^{2} \frac{L_l'}{(\ln G_l)'} > 0$. Thus, we have

$$\mathrm{d}L \propto \frac{2L_1'}{(\ln G_1)'} \left[L - \frac{L_2'}{(\ln G_2)'} \right] \frac{\mathrm{d}c_1}{c_1} + \frac{2L_1'}{(\ln G_1)'} \frac{L_2'}{(\ln G_2)'} \frac{\mathrm{d}c_1}{c_1} \tag{B.110}$$

$$= \frac{2L_1'}{(\ln G_1)'} L \frac{\mathrm{d}c_1}{c_1} \tag{B.111}$$

By Equation (B.105), we have

$$\frac{\mathrm{d}L}{\mathrm{d}c_1/c_1} \propto \frac{L_1'}{(\ln G_1)'} \begin{cases} < 0 & n_1 < 2R + 1, \\ > 0 & n_1 > 2R + 1. \end{cases} \tag{B.112}$$

On Stock Price Informativeness. For any l, let

$$I_l = \frac{\mathrm{Cov}(\theta_l, P)}{\sigma^2} = \lambda n_l \beta_l = \frac{n_l}{2R + 1 + n_l}. \tag{B.113}$$

The stock price informativeness is $I \equiv [\mathrm{Corr}(v, P)]^2$, and $\lambda = \frac{\mathrm{Cov}(v, X+u)}{\mathrm{Var}(X+u)}$. Then, we have

$$\mathrm{Var}(P) = \lambda^2 \mathrm{Var}(X + u) = \lambda \mathrm{Cov}(v, X + u) = \mathrm{Cov}(v, P). \tag{B.114}$$

Therefore, we have

$$I \equiv \frac{\mathrm{Cov}(v, P)^2}{\mathrm{Var}(v)\mathrm{Var}(P)} = \frac{\mathrm{Cov}(v, P)}{\mathrm{Var}(v)} = \frac{I_1 + I_2}{2}. \tag{B.115}$$

Letting $I_l' \equiv \frac{\partial I_l}{\partial n_l}$, we have $I_l' > 0, \forall l$ and

$$\mathrm{d}I \propto I_1' \mathrm{d}n_1 + I_2' \mathrm{d}n_2. \tag{B.116}$$

Substituting Equations (B.101) and (B.102) into the above equation, and note that $L - \sum_{l=1}^{2} \frac{L_l'}{(\ln G_l)'} > 0$ from Equation (B.106), we have

$$\frac{\mathrm{d}I}{\mathrm{d}c_1/c_1} \propto \frac{2I_1'}{(\ln G_1)'}\left(L_1 + \left[L_2 - \frac{L_2'}{(\ln G_2)'}\right]\right) + \frac{L_1'}{(\ln G_1)'}\frac{2I_2'}{(\ln G_2)'} \tag{B.117}$$

$$\propto \frac{2I_1'L_1}{(\ln G_1)'} + \frac{2I_1'}{(\ln G_1)'}\left[L_2 - \frac{L_2'}{(\ln G_2)'}\right] + \frac{L_1'}{(\ln G_1)'}\frac{2I_2'}{(\ln G_2)'}. \tag{B.118}$$

Note that $(\ln G_1)' < 0$. Based on the above equation, we have

$$-\frac{\mathrm{d}I}{\mathrm{d}c_1/c_1} \propto I_1'L_1 + I_1'\left[L_2 - \frac{L_2'}{(\ln G_2)'}\right] + L_1'\frac{I_2'}{(\ln G_2)'}. \tag{B.119}$$

Therefore, by Equation (B.119), if $n_1 > 2R+1$, then $L_1' < 0$ and thus $\frac{\mathrm{d}I}{\mathrm{d}c_1/c_1} < 0$. On the other hand, if $n_1 < 2R+1$, then $L_1' > 0$ and we have

$$-\frac{\mathrm{d}I}{\mathrm{d}c_1/c_1} \propto \frac{L_1I_1'}{L_1'} + \frac{I_1'}{L_1'}\left[L_2 - \frac{L_2'}{(\ln G_2)'}\right] + \frac{I_2'}{(\ln G_2)'}, \tag{B.120}$$

$$\propto \frac{n_1(2R+1)}{(2R+1)^2 - n_1^2} - \frac{(2R+1)}{(n_2+2R+1)} + \frac{(2R+1)(2R+1+n_1)}{(R+1)(2R+1-n_1)}\left[\frac{(3n_2+2R+1)(R+1)}{(n_2+2R+1)^2}\right]. \tag{B.121}$$

When $n_1 = 0$, the above expression attains its minimum. Substituting $n_1 = 0$, it is easy to verify that the expression is positive. Thus, we conclude that $\frac{\mathrm{d}I}{\mathrm{d}c_1}$ is always negative.

B.8.2 Comparative Statics in the Information Precision

This section discusses the impact of changes in information precision on information allocation, stock liquidity, and stock price informativeness under the continuous solution. The following proposition provides a summary of the main results.

Proposition B.3 *As R increases (i.e., the precision of the signal decreases), under the continuous solution, the following conclusion holds.*

1. *If $n_i^*, n_j^* < 2R+3$, then liquidity increases;*
2. *If $n_i^*, n_j^* > 2R+3$, then liquidity decreases.*
3. *stock price informativeness decreases.*

Proof of Proposition B.3. In this proof, we assume that the signal precisions of the two types of informed traders may differ. Accordingly, R_1 and R_2 are defined as the inverses of their respective signal precisions. We consider a simple case where R_1 changes while R_2 remains constant.

On Information Allocation. Without loss of generality, we consider the case where R_1 increases by $\mathrm{d}R_1$. The new continuous solution is denoted as $(n_1 + \mathrm{d}n_1, n_2 + \mathrm{d}n_2)$. The condition (5) is equivalent to

$$(\ln G_1)'\mathrm{d}n_1 + \frac{\partial \ln G_1}{\partial R_1}\mathrm{d}R_1 = \frac{1}{L}\left(\frac{\partial L_1}{\partial R_1}\mathrm{d}R_1 + \sum_{l=1}^{2} L_l'\mathrm{d}n_l\right), \tag{B.122}$$

$$(\ln G_2)'\mathrm{d}n_2 = \frac{1}{L}\left(\frac{\partial L_1}{\partial R_1}\mathrm{d}R_1 + \sum_{l=1}^{2} L_l'\mathrm{d}n_l\right). \tag{B.123}$$

Subtracting both sides of the above equations yields

$$\mathrm{d}n_2 = \frac{(\ln G_1)'}{(\ln G_2)'}\mathrm{d}n_1 + \frac{\partial \ln G_1/\partial R_1}{(\ln G_2)'}\mathrm{d}R_1. \tag{B.124}$$

Plugging Equation (B.124) into Equation (B.122), we have

$$(\ln G_1)'\left[1 - \frac{1}{L}\sum_{l=1}^{2}\frac{L_l'}{(\ln G_l)'}\right]\mathrm{d}n_1 = \left[\frac{1}{L}\frac{\partial L_1}{\partial R_1} - \frac{\partial \ln G_1}{\partial R_1} + \frac{\partial \ln G_1}{\partial R_1}\frac{1}{L}\sum_{l=2}^{2}\frac{L_l'}{(\ln G_l)'}\right]\mathrm{d}R_1.$$

From the above equation, we have

$$\frac{\mathrm{d}n_1}{\mathrm{d}R_1} = \frac{\left(\frac{\partial L_1}{\partial R_1} - \frac{\partial \ln G_1}{\partial R_1}L_1\right) - \frac{\partial \ln G_1}{\partial R_1}\sum_{l=2}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right]}{(\ln G_1)'\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right]}. \tag{B.125}$$

Substituting Equation (B.125) into Equation (B.124), we have

$$\frac{\mathrm{d}n_2}{\mathrm{d}R_1} = \frac{1}{(\ln G_2)'}\frac{\frac{\partial L_1}{\partial R_1} - \frac{L_1'}{(\ln G_1)'}\frac{\partial \ln G_1}{\partial R_1}}{\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right]}. \tag{B.126}$$

Thus, from (B.125) we have

$$\frac{\mathrm{d}n_1}{\mathrm{d}R_1}\begin{cases} > 0 & n_1 > 2R_1 + 3, \\ < 0 & n_1 < 2R_1 + 3, \end{cases} \tag{B.127}$$

and from (B.126) we have

$$\frac{\mathrm{d}n_2}{\mathrm{d}R_1}\begin{cases} < 0 & n_1 > 2R_1 + 3, \\ > 0 & n_1 < 2R_1 + 3. \end{cases} \tag{B.128}$$

On Market Liquidity. Recall that $\lambda^2 \propto L := \sum_{l=1}^{2} L_l$. Then, plugging Equations (B.125) and (B.126) into $\mathrm{d}L$, we have

$$\frac{\mathrm{d}L}{\mathrm{d}R_1} = \frac{\partial L_1}{\partial R_1} + L_1'\frac{\mathrm{d}n_1}{\mathrm{d}R_1} + L_2'\frac{\mathrm{d}n_2}{\mathrm{d}R_1}, \tag{B.129}$$

$$\propto \frac{\partial L_1}{\partial R_1}\sum_{l=1}^{2}\frac{L_l'}{(\ln G_l)'} - \frac{L_l'}{(\ln G_l)'}\frac{\partial \ln G_1}{\partial R_1}\sum_{l=1}^{2} L_l + \frac{\partial L_1}{\partial R_1}\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right], \tag{B.130}$$

$$= \left[\frac{\partial L_1}{\partial R_1} - \frac{L_l'}{(\ln G_l)'}\frac{\partial \ln G_1}{\partial R_1}\right]\sum_{l=1}^{2} L_l, \tag{B.131}$$

$$\propto (n_1 - 2R_1 - 3). \tag{B.132}$$

Then, we have

$$\frac{\mathrm{d}L}{\mathrm{d}R_1}\begin{cases} > 0 & n_1 > 2R_1 + 3, \\ < 0 & n_1 < 2R_1 + 3. \end{cases} \tag{B.133}$$

Since in our case n_1 and n_2 are both less than $2R + 1$, when $R_1 = R_2 = R$ and R increases, L decreases, implying that liquidity rises.

On Stock Price Informativeness. Note that

$$\frac{\mathrm{d}I}{\mathrm{d}R_1} \propto \frac{\partial I_1}{\partial R_1} + I_1'\frac{\mathrm{d}n_1}{\mathrm{d}R_1} + I_2'\frac{\mathrm{d}n_2}{\mathrm{d}R_1}. \tag{B.134}$$

Substituting Equations (B.125) and (B.126) into the above equation, we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}R_1} \propto & \frac{\partial I_1}{\partial R_1}\sum_{l=1}^{2}\left[L_l - \frac{L_l'}{(\ln G_l)'}\right] + I_1'\frac{\left(\frac{\partial L_1}{\partial R_1} - L_1\frac{\partial \ln G_1}{\partial R_1}\right) - \frac{\partial \ln G_1}{\partial R_1}\left[L_2 - \frac{L_2'}{(\ln G_2)'}\right]}{(\ln G_1)'} \\ & + \left(\frac{\partial L_1}{\partial R_1} - \frac{\partial \ln G_1}{\partial R_1}\frac{L_1'}{(\ln G_1)'}\right)\frac{I_2'}{(\ln G_2)'}.\end{aligned} \tag{B.135}$$

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}R_1} \propto & -\frac{n_1(R_1+5/4)}{(2R_1+1+n_1)^3} - \frac{(2R_1+3)}{2(R_1+1)(2R_1+1+n_1)}\left[L_2 - \frac{L_2'}{(\ln G_2)'}\right] \\ & - \frac{(2R_1+3-n_1)}{2(2R_1+1+n_1)^2}\frac{I_2'}{(\ln G_2)'}.\end{aligned} \tag{B.136}$$

If $2R_1+3-n_1<0$ then $\frac{\mathrm{d}I}{\mathrm{d}R_1}<0$.

If $2R_1+3-n_1>0$ then we have

$$\begin{aligned}-\frac{\mathrm{d}I}{\mathrm{d}R_1} \propto & \frac{2n_1(R_1+5/4)}{(2R_1+1+n_1)(2R_1+3-n_1)} - \frac{(2R_2+1)}{(n_2+2R_2+1)} \\ & + \frac{(2R_1+3)(2R_1+1+n_1)}{(R_1+1)(2R_1+3-n_1)}\left[\frac{(3n_2+2R_2+1)(R_2+1)}{(n_2+2R_2+1)^2}\right].\end{aligned} \tag{B.137}$$

Consider the right hand side of the above equation. When $n_1=0$, the expression attains its minimum. Therefore, we focus on the case where $n_1=0$. That is,

$$-\frac{\mathrm{d}I}{\mathrm{d}R_1} \propto \frac{(2R_1+1)}{(R_1+1)}\left[\frac{(3n_2+2R_2+1)(R_2+1)}{(n_2+2R_2+1)^2}\right] - \frac{(2R_2+1)}{(n_2+2R_2+1)} \tag{B.138}$$

Given $R_1=R_2$, substituting it into the above equation, it is straightforward to see that the right hand side is always positive. Then, we have $\frac{\mathrm{d}I}{\mathrm{d}R_1}<0$. It is obviously to see $\frac{\mathrm{d}I}{\mathrm{d}R_2}<0$. And, if $\mathrm{d}R=\mathrm{d}R_1=\mathrm{d}R_2$, we have $\frac{\mathrm{d}I}{\mathrm{d}R}<0$.

B.9 Omitted Proofs from Section B.1

B.9.1 Proof of Lemma B.1

Without loss of generality, we let $i=1$ and $j=2$.

Point 1: Suppose that $x_2^\dagger(x_1)$ increases with x_1 and $x_1^\dagger(x_2)$ increases with x_2. If $x_1^\dagger(x_2^\dagger(x_1^1))=x_1^1$, then $(x_1^1, x_2^\dagger(x_1^1))$ is an equilibrium. If $x_1^\dagger(x_2^\dagger(x_1^2))=x_1^2$, then $(x_1^2, x_2^\dagger(x_1^2))$ is an equilibrium. Then, we consider the case in which $x_1^\dagger(x_2^\dagger(x_1^1))>x_1^1$ and $x_1^\dagger(x_2^\dagger(x_1^2))<$

x_1^2 with $x_1^1 < x_1^2$.

Because $x_2^\dagger(x_1)$ increases with x_1, $x_1^\dagger(x_2^\dagger(x_1))$ also increases with x_1. Let $g(x_1) = x_1^\dagger(x_2^\dagger(x_1)) - x_1$. Taking x_1^1 as a starting point, we increase x_1 until we find an $\hat{x}_1$ such that $g(\hat{x}_1) \geq 0$ and $g(\hat{x}_1 + 1) \leq 0$. We first check $x_1 = x_1^1 + 1$. If $g(x_1^1 + 1) \leq 0$, then we have $\hat{x}_1 = x_1^1$; otherwise, we check $x_1 = x_1^1 + 2$, and so on. Because $x_1^\dagger(x_2^\dagger(x_1^2)) < x_1^2$, there must exist an $\hat{x}_1$ with $x_1^1 \leq \hat{x}_1 < x_1^2$ such that $g(\hat{x}_1) \geq 0$ and $g(\hat{x}_1 + 1) \leq 0$.

If $g(\hat{x}_1) = 0$, then $(\hat{x}_1, x_2^\dagger(\hat{x}_1))$ is an equilibrium. If $g(\hat{x}_1 + 1) = 0$, then $(\hat{x}_1 + 1, x_2^\dagger(\hat{x}_1 + 1))$ is an equilibrium. If $g(\hat{x}_1) > 0$, then we have $x_1^\dagger(x_2^\dagger(\hat{x}_1)) \geq \hat{x}_1 + 1$. Thus, because $x_1^\dagger(x_2^\dagger(x_1))$ increases with x_1, $x_1^\dagger(x_2^\dagger(\hat{x}_1 + 1)) < \hat{x}_1 + 1$ will not hold. Therefore, $g(\hat{x}_1) > 0$ and $g(\hat{x}_1 + 1) < 0$ cannot hold simultaneously. Thus, at least one of $(\hat{x}_1, x_2^\dagger(\hat{x}_1))$ and $(\hat{x}_1 + 1, x_2^\dagger(\hat{x}_1 + 1))$ must be an equilibrium.

Point 2: If $x_1^\dagger(x_2^\dagger(x_1^1)) = x_1^1$, then $(x_1^1, x_2^\dagger(x_1^1))$ is an equilibrium. If $x_1^\dagger(x_2^\dagger(x_1^2)) = x_1^2$, then $(x_1^2, x_2^\dagger(x_1^2))$ is an equilibrium. Then, we consider $x_1^\dagger(x_2^\dagger(x_1^1)) > x_1^1$ and $x_1^\dagger(x_2^\dagger(x_1^2)) < x_1^2$ with $x_1^1 < x_1^2$.

Because $x_2^\dagger(x_1)$ decreases with x_1, $x_1^\dagger(x_2^\dagger(x_1))$ also decreases with x_1. Thus, if $x_1^\dagger(x_2^\dagger(\dot{x}_1)) = \dot{x}_1$, then we have $x_1^\dagger(x_2^\dagger(x_1)) \leq \dot{x}_1$ for $x_1 > \dot{x}_1$ and $x_1^\dagger(x_2^\dagger(x_1)) \geq \dot{x}_1$ for $x_1 < \dot{x}_1$. That is, $(\dot{x}_1, x_2^\dagger(\dot{x}_1))$ is the unique equilibrium.

Taking x_1^1 as a starting point, we increase x_1 until we find an $\hat{x}_1$ such that $x_1^\dagger(x_2^\dagger(\hat{x}_1)) \geq \hat{x}_1$ and $x_1^\dagger(x_2^\dagger(\hat{x}_1 + 1)) \leq \hat{x}_1 + 1$. If $x_1^\dagger(x_2^\dagger(\hat{x}_1)) = \hat{x}_1$, then $(\hat{x}_1, x_2^\dagger(\hat{x}_1))$ is the unique equilibrium. If $x_1^\dagger(x_2^\dagger(\hat{x}_1 + 1)) = \hat{x}_1 + 1$, then $(\hat{x}_1 + 1, x_2^\dagger(\hat{x}_1 + 1))$ is the unique equilibrium. Because $x_1^\dagger(x_2^\dagger(x_1))$ decreases with x_1, $x_1^\dagger(x_2^\dagger(\hat{x}_1)) > \hat{x}_1$ and $x_1^\dagger(x_2^\dagger(\hat{x}_1 + 1)) < \hat{x}_1 + 1$ may hold simultaneously. Thus, at most one of $(\hat{x}_1, x_2^\dagger(\hat{x}_1))$ and $(\hat{x}_1 + 1, x_2^\dagger(\hat{x}_1 + 1))$ is an equilibrium, and a pure strategy equilibrium may not even exist.

B.9.2 Proof of Lemma B.2

We first prove two useful lemmas.

Lemma B.5 *1. If $N_l \leq 2R$ and $N_k^\dagger(N_l) \leq 2R + 1$, then we have*

(a) $N_k^\dagger(N_l) - N_k^\dagger(N_l + 1) \leq 2$, and

(b) *if* $R \geq 7$ *and* $N_l \leq 1$, *then there exist some* $c_k's$ *such that* $N_k^\dagger(N_l+1) = N_k^\dagger(N_l) - 2$; *otherwise,* $N_k^\dagger(N_l) - N_k^\dagger(N_l+1) \leq 1$.

2. *If* $N_l \leq 2R+1$ *and* $N_k^\dagger(N_l) \leq 2R+1$, *then we have* $N_k^\dagger(N_l-1) - N_k^\dagger(N_l) \leq 2$.
3. *If* $N_k^\dagger(N_l) \leq 2R+1$ *and* $N_l \geq 2R+1$, *then we have* $N_k^\dagger(N_l+1) - N_k^\dagger(N_l) \leq 1$.
4. *If* $N_k^\dagger(2R+1) \leq 2R-1$, *then we have*
   (a) $N_k^\dagger(2R-x) - N_k^\dagger(2R+1) \leq 1$ *for* $x = 0, 1, 2$, *and*
   (b) *if* $R \geq 2$, *then* $N_k^\dagger(2R-3) - N_k^\dagger(2R+1) \leq 1$.
5. *If* $N_l \leq 2R+1$ *and* $N_k^\dagger(N_l) \leq 2R$, *then we have* $N_k^\dagger(N_l-1) - N_k^\dagger(N_l) \leq 2$.

Proof. Let $k=1$ and $l=2$. If $N_1 = N_1^\dagger(N_2)$ and $N_1' = N_1^\dagger(N_2')$, then we have

$$\mathbb{E}[\pi_1(N_1, N_2)] \geq c_1 > \mathbb{E}[\pi_1(N_1+1, N_2)],$$
$$\mathbb{E}[\pi_1(N_1', N_2')] \geq c_1 > \mathbb{E}[\pi_1(N_1'+1, N_2')].$$

The above two inequalities suggest that

$$\mathbb{E}[\pi_1(N_1', N_2')] > \mathbb{E}[\pi_1(N_1+1, N_2)], \tag{B.139}$$
$$\mathbb{E}[\pi_1(N_1, N_2)] > \mathbb{E}[\pi_1(N_1'+1, N_2')]. \tag{B.140}$$

We define X and Y as follows. Inequalities (B.139) and (B.140) are equivalent to $X > 0$ and $Y > 0$, respectively.

$$X = -\left[\mathcal{L}_{(N_1', N_2')} C^4_{(N_1')} - \mathcal{L}_{(N_1+1, N_2)} C^4_{(N_1+1)}\right],$$
$$Y = -\left[\mathcal{L}_{(N_1, N_2)} C^4_{(N_1)} - \mathcal{L}_{(N_1'+1, N_2')} C^4_{(N_1'+1)}\right].$$

Point 1: Let $N_1 \leq 2R+1$, $N_2 \leq 2R$, $N_2' = N_2 + 1$, and $N_1' = N_1 - d$. If $d=1$, then $X > 0$ and $Y > 0$ always hold. If $d=3$, then $Y > 0$ never holds. If $d=2$, then $X > 0$ always holds but $Y > 0$ only holds under certain conditions. Specifically, if and only if $R \geq 7$, $N_2 \leq 1$ and $N_1 = 2R+1$, then $Y > 0$ holds; otherwise $Y < 0$.

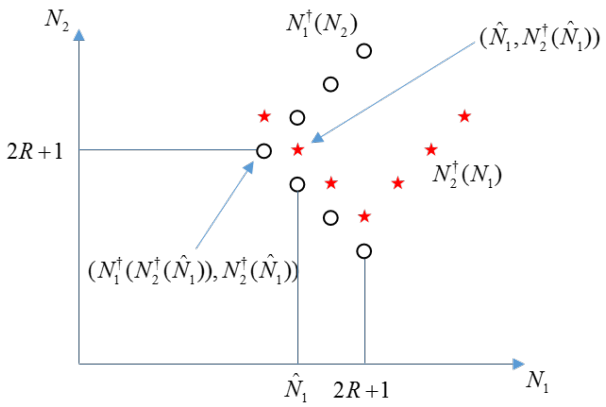

Figure 14. Intuition behind Lemma B.6. This figure illustrates the intuition behind Lemma B.6. The red stars represent $N_2^\dagger(N_1)$ and the black circles represent $N_1^\dagger(N_2)$.

Point 2: Let $N_1 \leq 2R+1$, $N_2 \leq 2R+1$, $N_2' = N_2 - 1$, and $N_1' = N_1 + d$. If $d = 3$, then $X > 0$ never holds.

Point 3: Let $N_1 \leq 2R+1$, $N_2 \geq 2R+1$, $N_2' = N_2 + 1$, and $N_1' = N_1 + d$. If $d \geq 2$, then $X > 0$ never holds.

Point 4: Let $N_2 = 2R+1$, $N_1 \leq 2R-1$, $N_2' = 2R - x$, and $N_1' = N_1 + d$. If $d \geq 2$ and $0 \leq x \leq 2$, then $X > 0$ never holds. If $R \geq 2$, $d \geq 2$ and $x = 3$, then $X > 0$ never holds.

Point 5: Let $N_2 \leq 2R+1$, $N_1 \leq 2R$, $N_2' = N_2 - 1$, and $N_1' = N_1 + d$. Note that X decreases with d. Let $d = 3$, and we find that $X < 0$. Therefore, we have $d \leq 2$.

Lemma B.6 *If there exists an* $\hat{N}_k$ *with* $\underline{N_k} + 1 \leq \hat{N}_k \leq 2R+1$ *such that* $N_l^\dagger(\hat{N}_k) \geq 2R$ *and* $N_k^\dagger(N_l^\dagger(\hat{N}_k)) < \hat{N}_k$*, then we have* $N_k^\dagger(N_l^\dagger(N_k)) < N_k$ *for* $N_k \geq \hat{N}_k$*.*

Proof. Let $k = 1$ and $l = 2$. $N_1^\dagger(N_2^\dagger(\hat{N}_1)) < \hat{N}_1$ means that point $(\hat{N}_1, N_2^\dagger(\hat{N}_1))$ is to the right of point $(N_1^\dagger(N_2^\dagger(\hat{N}_1)), N_2^\dagger(\hat{N}_1))$. Note that if $N_1^\dagger(N_2^\dagger(\hat{N}_1)) = \hat{N}_1$, then these two points overlap and thus $(\hat{N}_1, N_2^\dagger(\hat{N}_1))$ is an equilibrium. If point $(\hat{N}_1, N_2^\dagger(\hat{N}_1))$ is to the right of point $(N_1^\dagger(N_2^\dagger(\hat{N}_1)), N_2^\dagger(\hat{N}_1))$, then there is no pure strategy equilibrium at $N_1 = \hat{N}_1$. Lemma B.6 says that if there exists an $\hat{N}_k$ with $\underline{N_k} + 1 \leq \hat{N}_k \leq 2R+1$ such that $N_l^\dagger(\hat{N}_k) \geq 2R$ and there is no pure strategy equilibrium at $N_1 = \hat{N}_1$, then there is also no pure strategy equilibrium for $N_k \geq \hat{N}_k$. We plot Figure 14 to illustrate the intuition behind Lemma B.6.

Because there is no close-form solution to $N_k^\dagger(N_l)$, we consider all possible cases in all regions and prove that Lemma B.6 always holds.

For a given nonnegative real number x, we define y_1 and y_2 as follows.

$$E[\pi_1(x, y_1)] = c_1, \tag{B.141}$$

$$E[\pi_2(x, y_2)] = c_2. \tag{B.142}$$

Let $\hat{c}_1 = \frac{c_1}{[\sigma_u^2\sigma^2(R+1)]^{1/2}}$, $\hat{c}_2 = \frac{c_2}{[\sigma_u^2\sigma^2(R+1)]^{1/2}}$, and $C_{(N)} = N + 2R + 1$. Equation (3) implies

$$\frac{x}{C_{(x)}^2} + \frac{y_1}{C_{(y_1)}^2} = \frac{1}{\hat{c}_1^2 C_{(x)}^4}, \tag{B.143}$$

$$\frac{x}{C_{(x)}^2} + \frac{y_2}{C_{(y_2)}^2} = \frac{1}{\hat{c}_2^2 C_{(y_2)}^4}. \tag{B.144}$$

Let $f_1 = dy_1/dx$ and $f_2 = dy_2/dx$. Then, it follows

$$f_1 = -\frac{C_{(y_1)}^3}{2R+1-y_1}\left[\frac{2R+1-x}{C_{(x)}^3} + \frac{4}{\hat{c}_1^2 C_{(x)}^5}\right],$$
$$f_2 = -\frac{2R+1-x}{C_{(x)}^3}\left[\frac{2R+1-y_2}{C_{(y_2)}^3} + \frac{4}{\hat{c}_2^2 C_{(y_2)}^5}\right]^{-1}.$$

Substituting Equations (B.143) and (B.144) into f_1 and f_2 leads to

$$f_1 = -\frac{1}{2R+1-y_1}\left[\frac{C_{(y_1)}^3}{C_{(x)}^3}(3x + 2R + 1) + 4\frac{C_{(y_1)}}{C_{(x)}}y_1\right],$$
$$f_2 = -(2R+1-x)\left[\frac{C_{(x)}^3}{C_{(y_2)}^3}(3y_2 + 2R + 1) + 4\frac{C_{(x)}}{C_{(y_2)}}x\right]^{-1}.$$

For a given y_1, there is a unique $x(y_1)$ determined by Equation (B.141). Let $[x(y_1)]$ be the integer part of $x(y_1)$. For example, if $x(y_1) = 1.9$, then $[x(y_1)] = 1$. By comparing Inequality (6) and Equation (B.141), if y_1 is a nonnegative integer, then we have $N_1^\dagger(y_1) = [x(y_1)]$. Similarly, for a given x, there is a unique $y_2(x)$ determined by Equation (B.142). Let $[y_2(x)]$ be the integer part of $y_2(x)$. If x is a nonnegative integer, then $N_2^\dagger(x) = [y_2(x)]$.

If x is a nonnegative integer and $x = [x([y_2(x)])]$, then $(x, [y_2(x)])$ is an equilibrium.

Case 1: $1 \le x \le 2R+1$ and $0 \le y_1 \le 2R+1$.

In this region, we have $f_1 < 0$ and $f_2 < 0$. For $0 \le y_1 \le 2R+1$, $x(y_1)$ decreases monotonically with y_1, so $x(y_1)$ has an inverse function $y_1(x)$. We can prove that if $y_2(x) - y_1(x) = 2$, then we have $f_2 > f_1$ for $x \ge 1$, which implies that $y_2(x) - y_1(x)$ increases with x. We can also prove that f_1 decreases with y_1 for $y_1 \le 2R+1$ and f_2 decreases with y_2 for $x \le 2R+1$. Thus, if $y_2(x) - y_1(x) \le 2$, then we have $f_2 > f_1$ for $x \ge 1$. As a result, if there is an $\dot{x} \ge 1$, such that $y_2(\dot{x}) - y_1(\dot{x}) > 1$, then we have $y_2(x) - y_1(x) > 1$ for $x \ge \dot{x}$. If $y_2(x) - y_1(x) > 1$, then we have $[x([y_2(x)])] < x$.

For a given $\dot{x} \ge 1$, if $[x([y_2(\dot{x})] - 1)] < \dot{x}$ then we have $y_2(\dot{x}) - y_1(\dot{x}) > 1$. Thus, if there is a positive integer $\dot{x}$ such that $[x([y_2(\dot{x})] - 1)] < \dot{x}$, then we have $[x([y_2(x)])] < x$ for $x \ge \dot{x}$.

Case 2: $1 \le x \le 2R+1$ and $y_1 > 2R+1$.

In this region, we have $f_1 > 0$ and $f_2 < 0$. Because $y_2(x)$ decreases with x, we have $[y_2(x)]$ decreases with x. Because $x(y_1)$ increases monotonically with y_1, we have $x([y_2(x)])$ decreases with x. Thus, for a given positive integer $\dot{x}$, if $[x([y_2(\dot{x})])] < \dot{x}$, then we have $[x([y_2(x)])] < x$ for $x \ge \dot{x}$.

Case 3: $x > 2R+1$ and $0 \le y_1 \le 2R+1$.

In this region, we have $f_1 < 0$ and $f_2 > 0$. Because $y_2(x)$ increases with x, we have $[y_2(x)]$ increases with x. Because $x(y_1)$ decreases monotonically with y_1, we have $x([y_2(x)])$ decreases with x. Thus, for a given positive integer $\dot{x}$, if $[x([y_2(\dot{x})])] < \dot{x}$, then we have $[x([y_2(x)])] < x$ for $x \ge \dot{x}$.

Case 4: $x > 2R+1$ and $y_1 > 2R+1$.

In this region, we have $f_1 > 0$ and $f_2 > 0$. For $y_1 > 2R+1$, $x(y_1)$ increases monotonically with y_1, so $x(y_1)$ has an inverse function $y_1'(x)$. We can prove that if $y_1'(x) > y_2(x)$, then we have $f_1 - f_2 > 0$. Thus, if there is an $\dot{x} \ge 2R+1$ such that $y_1'(\dot{x}) > y_2(\dot{x})$, then we have $y_1'(x) > y_2(x)$ for $x \ge \dot{x}$. If $y_1'(x) > y_2(x)$, then we have $[x([y_2(x)])] < x$.

Because $x(y_1)$ increases with y_1, if $x([y_2(x)] + 1) < x$, then we have $y_1'(x) > y_2(x)$. Thus, if there is a positive integer $\dot{x} \ge 2R+1$ such that $[x([y_2(\dot{x})] + 1)] \le \dot{x}$, then we

have $[x([y_2(x)])] < x$ for $x \geq \dot{x}$.

Case 1: $\hat{N}_1 = 2R + 1$.

Subcase 1: $\underline{N_2} < 2R + 1$. Because $\underline{N_2} < 2R + 1$, we have $y_2(2R + 1) < 2R + 1$. Because $\underline{N_1} \leq 2R$ and $y_1'(x)$ increases with x, we have $y_1'(2R + 1) \geq 2R + 1$. Thus, we have $y_2(x) < y_1'(x)$ for $x \geq 2R + 1$. Using the results in Cases 3 and 4 in Preliminary Results, we have $N_1^\dagger(N_2^\dagger(N_1) + 1) < N_1$ for $N_1 \geq 2R + 1$.

Subcase 2: $\underline{N_2} \geq 2R + 1$ and $N_1^\dagger(\underline{N_2}) \leq 2R - 1$.

We have $N_1^\dagger(\underline{N_2} + 1) \leq 2R$ according to Point 3 of Lemma B.5. Therefore, we have $N_1^\dagger(N_2^\dagger(2R + 1) + 1) < 2R + 1$ and thus $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $N_1 \geq 2R + 1$ according to Case 4 in Preliminary Results.

Subcase 3: $\underline{N_2} \geq 2R + 1$ and $N_1^\dagger(\underline{N_2}) = 2R$.

By definition of $\underline{N_2}$, we have $c_2 > \mathbb{E}[\pi_2(2R + 1, \underline{N_2} + 1)]$. In addition, if $\underline{N_2} + d_2 = N_2^\dagger(2R + d_1)$, then we have $\mathbb{E}[\pi_2(2R + d_1, \underline{N_2} + d_2)] \geq c_2$. Thus, if $\underline{N_2} + d_2 = N_2^\dagger(2R + d_1)$, then we have $\mathbb{E}[\pi_2(2R + d_1, \underline{N_2} + d_2)] > \mathbb{E}[\pi_2(2R + 1, \underline{N_2} + 1)]$.

Because $N_1^\dagger(\underline{N_2}) = 2R$, we have $c_1 > \mathbb{E}[\pi_1(2R+1, \underline{N_2})]$. If $2R+d_1 = N_1^\dagger(\underline{N_2}+d_2)$, then we have $\mathbb{E}[\pi_1(2R+d_1, \underline{N_2}+d_2)] \geq c_1$. Thus, a necessary condition for $2R+d_1 = N_1^\dagger(\underline{N_2}+d_2)$ is that $\mathbb{E}[\pi_1(2R + d_1, \underline{N_2} + d_2)] > \mathbb{E}[\pi_1(2R + 1, \underline{N_2})]$.

For $d_1 = 2$, we can obtain that if $\mathbb{E}[\pi_2(2R+2, \underline{N_2}+d_2)] > \mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)]$, then we have $\mathbb{E}[\pi_1(2R+2, \underline{N_2} + d_2)] < \mathbb{E}[\pi_1(2R+1, \underline{N_2})]$. This implies that $N_1^\dagger(\underline{N_2} + d_2) < 2R+2$ or equivalently $N_1^\dagger(N_2^\dagger(2R + 2)) < 2R + 2$.

For $d_1 \geq 3$, because $\underline{N_2} \geq 2R + 1$, if $\mathbb{E}[\pi_2(2R + d_1, \underline{N_2} + d_2)] > \mathbb{E}[\pi_2(2R + 1, \underline{N_2} + 1)]$, we have $\mathbb{E}[\pi_1(2R + d_1, \underline{N_2} + d_2)] < \mathbb{E}[\pi_1(2R + 2, \underline{N_2})]$ according to Lemma A.3. Thus, we have $\mathbb{E}[\pi_1(2R+d_1, \underline{N_2}+d_2)] < \mathbb{E}[\pi_1(2R+1, \underline{N_2})]$. This implies that $N_1^\dagger(\underline{N_2}+d_2) < 2R+d_1$ or equivalently $N_1^\dagger(N_2^\dagger(2R + d_1)) < 2R + d_1$.

Combining the results of the above three subcases, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $N_1 \geq \hat{N}_1$.

Case 2: $\underline{N_1} + 1 \leq \hat{N}_1 \leq 2R$ and $N_2^\dagger(\hat{N}_1) = 2R$

Subcase 1: $\hat{N}_1 = \underline{N_1} + 1$.

If $R < 7$, then we have $N_2^\dagger(\underline{N_1} + 2) \geq 2R - 1$ and $0 \leq N_1^\dagger(2R - 2) \leq \underline{N_1} + 1$ according

to Points 1 and 4 of Lemma B.5. Thus, we have $N_1^\dagger(N_2^\dagger(\underline{N_1}+2)-1) < \underline{N_1}+2$.

If $R \geq 7$, then we have $N_2^\dagger(\underline{N_1}+2) \geq 2R-2$ and $N_1^\dagger(2R-3) \leq \underline{N_1}+1$ according to Points 1 and 4 of Lemma B.5. Thus, we have $N_1^\dagger(N_2^\dagger(\underline{N_1}+2)-1) < \underline{N_1}+2$.

To sum up, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $\underline{N_1}+1 \leq N_1 \leq 2R+1$ according to Cases 1 and 2 in Preliminary Results.

Subcase 2: $\hat{N}_1 \geq \underline{N_1}+2$. We have $N_1^\dagger(N_2^\dagger(\hat{N}_1)-1) = N_1^\dagger(2R-1) \leq \underline{N_1}+1$ according to Point 4 of Lemma B.5. Therefore, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $\underline{N_1}+1 \leq N_1 \leq 2R+1$ according to Cases 1 and 2 in Preliminary Results.

Case 3: $\underline{N_1}+1 \leq \hat{N}_1 \leq 2R$ and $N_2^\dagger(\hat{N}_1) = 2R+1$

Subcase 1: $\hat{N}_1 = \underline{N_1}+1$. Consider two cases.

If $R < 7$, then we have $N_2^\dagger(\underline{N_1}+2) \geq 2R$ and $0 \leq N_1^\dagger(2R-1) \leq \underline{N_1}+1$ according to Points 1 and 4 of Lemma B.5. Thus, we have $N_1^\dagger(N_2^\dagger(\underline{N_1}+2)-1) < \underline{N_1}+2$.

If $R \geq 7$, then we have $N_2^\dagger(\underline{N_1}+2) \geq 2R-1$ and $N_1^\dagger(2R-2) \leq \underline{N_1}+1$ according to Points 1 and 4 of Lemma B.5. Thus, we have $N_1^\dagger(N_2^\dagger(\underline{N_1}+2)-1) < \underline{N_1}+2$.

To sum up, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $2R+1 \geq N_1 \geq \underline{N_1}+1$ according to Cases 1 and 2 in Preliminary Results.

Subcase 2: $\hat{N}_1 \geq \underline{N_1}+2$. We have $N_1^\dagger(N_2^\dagger(\hat{N}_1)-1) = N_1^\dagger(2R) \leq \underline{N_1}+1$ according to Point 4 of Lemma B.5. Therefore, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ for $\underline{N_1}+1 \leq N_1 \leq 2R+1$ according to Cases 1 and 2 in Preliminary Results.

Case 4: $\underline{N_1}+1 \leq \hat{N}_1 \leq 2R$ and $N_2^\dagger(\hat{N}_1) \geq 2R+2$

Because $N_2^\dagger(N_1)$ decreases with N_1 for $N_1 < 2R+1$ and $N_1^\dagger(N_2)$ increases with N_2 for $N_2 \geq 2R+1$, we have $N_1^\dagger(N_2^\dagger(\hat{N}_1)-1) < \hat{N}_1$. Thus, we have $N_1^\dagger(N_2^\dagger(N_1)) < N_1$ according to Cases 1 and 2.

Returning to the proof of Lemma B.2, we let $k=1$ and $l=2$.

Point 1: Conditions: $N_1^\dagger(\underline{N_2}) \geq 2R+1$.

If $N_1^\dagger(\underline{N_2}) = 2R+1$, then $(2R+1, N_1^\dagger(\underline{N_2}))$ is an equilibrium. Thus, we consider $N_1^\dagger(\underline{N_2}) > 2R+1$.

First, we prove that if $c_1 \geq c_2$ and $N_1^\dagger(\underline{N_2}) > 2R+1$, then we have $\underline{N_2} > 2R+1$. Because $N_1^\dagger(\underline{N_2}) > 2R+1$, we have $\mathbb{E}[\pi_1(2R+2, \underline{N_2})] \geq c_1$ by Inequality (6). By definition

of $\underline{N_2}$, we have $c_2 > \mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)]$. Because $c_1 \geq c_2$, we have $\mathbb{E}[\pi_1(2R+2, \underline{N_2})] > \mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)]$. It is straightforward to check that $\underline{N_2} > 2R+1$ is a necessary condition for $\mathbb{E}[\pi_1(2R+2, \underline{N_2})] > \mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)]$ to hold.

Let $N_1^1 = 2R+1$. Because $N_1^\dagger(\underline{N_2}) > 2R+1$, we have $N_1^\dagger(N_2^\dagger(N_1^1)) > N_1^1$.

Let $N_1^2 = N_1^\dagger(0)$. Note that $N_1^\dagger(N_2) \leq N_1^\dagger(0)$ for $N_2 \geq 0$. Thus, we have $N_1^\dagger(N_2^\dagger(N_1^2)) \leq N_1^2$.

Thus, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ cross in the region of $N_1, N_2 \geq 2R+1$. In addition, it can be checked that $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ do not cross in other regions. For example, because $\underline{N_2} > 2R+1$, there is no pure strategy equilibrium in the region of $N_2 \leq 2R+1$. Because $N_1^\dagger(N_2^\dagger(2R+1)) > 2R+1$, $N_2^\dagger(N_1)$ decreases with N_1 for $N_1 < 2R+1$, and $N_1^\dagger(N_2)$ increases with N_2 for $N_2 \geq 2R+1$, there is no pure strategy equilibrium in the region of $N_1 < 2R+1$ and $N_2 \geq 2R+1$. Using Lemma B.1, we obtain Point 1 in Lemma B.2.

Point 2a: Conditions: $N_1^\dagger(\underline{N_2}) < 2R+1$ and $N_2^\dagger(\underline{N_1}+1) \geq 2R+1$.

Note that if $N_2^\dagger(\underline{N_1}+1) \geq 2R+1$, then we have $N_2^\dagger(\underline{N_1}) \geq 2R+1$.

Let $N_1^1 = \underline{N_1} \equiv N_1^\dagger(2R+1)$. Because $N_2^\dagger(N_1^1) > 2R+1$ and $N_1^\dagger(N_2)$ increases with N_2 for $N_2 \geq 2R+1$, we have $N_1^\dagger(N_2^\dagger(N_1^1)) \geq N_1^\dagger(2R+1) \equiv N_1^1$.

Let $N_1^2 = 2R+1$. Because $N_1^\dagger(\underline{N_2}) < 2R+1$, we have $N_1^\dagger(N_2^\dagger(N_1^2)) < N_1^2$.

Thus, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ cross in the region of $N_1 \leq 2R+1$ and $N_2 \geq 2R+1$. In addition, because $N_1^\dagger(\underline{N_2}) < 2R+1$, there exists an $\hat{N}_1$ with $\underline{N_1}+1 \leq \hat{N}_1 \leq 2R+1$ such that $N_1^\dagger(N_2^\dagger(\hat{N}_1)) < \hat{N}_1$. If $\hat{N}_1 = \underline{N_1}+1$, then because $N_2^\dagger(\underline{N_1}+1) \geq 2R+1$, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ do not cross in other regions according to Lemma B.6. If $\hat{N}_1 > \underline{N_1}+1$, then we have $N_2^\dagger(\hat{N}_1 - 1) \geq 2R+2$, and thus $N_2^\dagger(\hat{N}_1) \geq 2R$ according to Point 2 of Lemma B.5. Therefore, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ do not cross in other regions according to Lemma B.6.

Thus, using Lemma B.1, we obtain Point 2a in Lemma B.2.

Point 2b: Conditions: $N_1^\dagger(\underline{N_2}) < 2R+1$ and $N_2^\dagger(\underline{N_1}+1) < 2R+1$.

Subcase 1: $N_2^\dagger(\underline{N_1}+1) < 2R$

If $N_2^\dagger(\underline{N_1}+1) < 2R$, then we have $N_2^\dagger(\underline{N_1}) \leq 2R+1$ according to Points 1 and 2 of Lemma B.5.

Let $N_1^1 = \underline{N_1} \equiv N_1^\dagger(2R+1)$. Because $N_1^\dagger(N_2)$ decreases with N_2 for $N_2 \leq 2R+1$ and

$N_2^\dagger(\underline{N_1}) \leq 2R+1$, we have $N_1^\dagger(N_2^\dagger(N_1^1)) \geq N_1^1$.

Let $N_1^2 = 2R+1$. Because $N_1^\dagger(\underline{N_2}) < 2R+1$, we have $N_1^\dagger(N_2^\dagger(N_1^2)) < N_1^2$.

Thus, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ cross in the region of $N_1, N_2 \leq 2R+1$ according to Lemma B.1. In addition, because $N_1^\dagger(\underline{N_2}) < 2R+1$, $N_1^\dagger(N_2)$ and $N_2^\dagger(N_1)$ do not cross in other regions according to Lemma B.6. Using Lemma B.1, we obtain Point 2b in Lemma B.2.

Subcase 2: $N_2^\dagger(\underline{N_1}+1) = 2R$

If $N_2^\dagger(\underline{N_1}+1) = 2R$, we have $N_2^\dagger(\underline{N_1}) \leq 2R+2$ according to Points 1 and 2 of Lemma B.5.

If $N_2^\dagger(\underline{N_1}) = 2R+1$, then the scenario is the same as Subcase 1. There exists one or more equilibria according to Lemma B.1.

If $N_2^\dagger(\underline{N_1}) = 2R+2$, then we consider $N_1^\dagger(2R+2)$ and $N_1^\dagger(2R)$. If $N_1^\dagger(2R+2) = \underline{N_1}$ and $N_1^\dagger(2R) = \underline{N_1}$, then $(\underline{N_1}, 2R+2)$ is the unique equilibrium. If $N_1^\dagger(2R+2) = \underline{N_1}$ and $N_1^\dagger(2R) = \underline{N_1}+1$, then both $(\underline{N_1}, 2R+2)$ and $(\underline{N_1}+1, 2R)$ are equilibria. It can be checked that if $\underline{N_1} \leq 2R+1$ and $N_1^\dagger(2R+2) = \underline{N_1}+1$, then we have $N_1^\dagger(2R) = \underline{N_1}+1$, and thus $(\underline{N_1}+1, 2R)$ is an equilibrium. Therefore, there exists at least one equilibrium.

B.9.3 Proof of Proposition B.1

The results in the case of $c_1 \geq c_2$ and those in the case of $c_1 < c_2$ are symmetrical, so we only need to consider the former case.

If $(c_1, c_2) \in \mathcal{S}_1$ and $c_1 > \mathbb{E}[\pi_1(1,0)]$, then we have $N_1 = 0$ and $N_2 = \overline{N_2}$ in equilibrium. If $(c_1, c_2) \in \mathcal{S}_1$ and $c_2 > \mathbb{E}[\pi_2(0,1)]$, then we have $N_2 = 0$ and $N_1 = \overline{N_1}$ in equilibrium.

If $(c_1, c_2) \in \mathcal{S}_2$, then we have $\underline{N_1}, \underline{N_2} \geq 2R+1$. Thus, there is one or more equilibria according to Point 1 of Lemma B.2.

If $(c_1, c_2) \in \mathcal{S}_3$ $\overline{N_1}, \overline{N_2} < 2R+1$. Thus, there is one or more equilibria according to Point 2b of Lemma B.2.

If $(c_1, c_2) \in \mathcal{S}_{41}$, we have $\overline{N_1} < 2R+1$ and $\underline{N_2} \geq 2R+1$. Thus, there is at most one equilibrium according to Point 2a of Lemma B.2.

If $(c_1, c_2) \in \mathcal{S}_{51}$, then we have $\underline{N_1} < 2R+1 \leq \overline{N_1}$ and $\underline{N_2} \geq 2R+1$. According to Points 1 and 2a of Lemma B.2, if $\mathbb{E}[\pi_1(2R+1, \underline{N_2})] \geq c_1$, then there is at least one

equilibrium; otherwise, there is at most one equilibrium.

If $(c_1, c_2) \in \mathcal{S}_{61}$, then we have $\overline{N_1} < 2R+1$ and $\underline{N_2} < 2R+1 \leq \overline{N_2}$. According to Points 2a and 2b of Lemma B.2, if $\mathbb{E}[\pi_2(\underline{N_1}+1, 2R+1)] \geq c_2$, then there can be at most one equilibrium; otherwise, there is at least one equilibrium.

If $(c_1, c_2) \in \mathcal{S}_{71}$, then we have $\underline{N_1} < 2R+1 \leq \overline{N_1}$ and $\underline{N_2} < 2R+1 \leq \overline{N_2}$. By definition of $\underline{N_2}$, we have $c_2 > \mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)]$. Given $\underline{N_2} < 2R+1$, it is straightforward to check that $\mathbb{E}[\pi_2(2R+1, \underline{N_2}+1)] > \mathbb{E}[\pi_1(2R+2, \underline{N_2})]$. Thus, because $c_1 \geq c_2$, we have $c_1 > \mathbb{E}[\pi_1(2R+2, \underline{N_2})]$ or equivalently $N_1^\dagger(\underline{N_2}) \leq 2R+1$. According to Points 2a and 2b of Lemma B.2, if $\mathbb{E}[\pi_2(\underline{N_1}+1, 2R+1)] \geq c_2$, then there can be at most one equilibrium; otherwise, there is at least one equilibrium.