Online Appendices to
"The Economics of Waste Valorization"

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This appendix is organized into five sections. Section A presents a detailed comparison between waste valorization, remanufacturing, and coproduction strategies. We also provide a summary of case studies on waste valorization across industries, in which sales data reveal varying degrees of demand correlation among the involved products. Section B introduces and summarizes the notations used in the main text, alongside data employed in our numerical analysis. Section C provides the analysis and proofs related to the model outlined in the Benchmark section. Section D focuses on the analysis and proofs for the model incorporating waste valorization and demand correlation. Section E delivers essential analysis and proofs for the models discussed in the Extensions section. Finally, Section F offers the analysis and proofs for the model of waste supply chain.

Appendix A: Strategy Comparisons, Practical Examples, and Empirical Analysis

A.1. Comparison of Different Waste Reduction Strategies

Table A.1 Comparison among Waste Valorization, Remanufacturing, and Coproduction

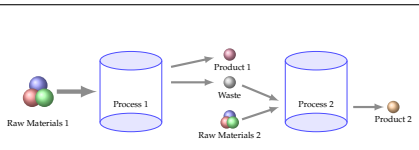
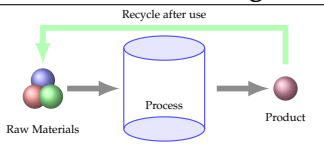
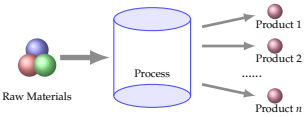
Aspect	Waste Valorization	Remanufacturing	Coproduction
Illustration			
Process	2 processes	1 process	1 process
Input	materials + waste	materials + used products	materials
Output	two products	same product	different quality
Implications	two interdependencies	cannibalization	vertical differentiation
Examples	Fuwa, Kencoco	automobile (1940s)	semiconductor (1980s)

Table A.1 facilitates a comprehensive comparison of waste valorization, remanufacturing, and coproduction. As illustrated in the table, the initial point of distinction lies in the production process. Waste valorization encompasses two complete processes, contrasting with remanufacturing and coproduction, each characterized by a singular process. Regarding production input and output, all three operations, waste valorization, remanufacturing, and coproduction, share virgin raw materials as input. However, waste materials become viable input exclusively for waste valorization, while remanufacturing may incorporate used products, typically in the form of components. Remanufacturing yields a single product output, whereas waste valorization and coproduction generate two or more products. It is noteworthy that the output products of waste valorization can vary significantly, although they may exhibit demand correlation in end markets. Conversely, coproduction's output is confined to vertically differentiated products, differing in

product quality. In essence, the output for remanufacturing and coproduction primarily involves substitution or partial substitution. The pivotal distinction emerges in the waste reduction considerations. Waste valorization encompasses dual perspectives, targeting both disposal/waste and source reduction. On the other hand, remanufacturing and coproduction solely focus on source reduction. Consequently, the managerial implications diverge based on these nuanced distinctions.

Specifically, waste valorization primarily focuses on pre-consumer (or in-production) waste, whereas upcycling and downcycling are more concerned with post-consumer waste. Pre-consumer waste exhibits distinct characteristics compared to post-consumer waste: 1. Minimal contamination, although this may incur non-negligible maintenance costs. 2. Reliable supply in terms of both quantity and quality. Most importantly, upcycling (or downcycling, respectively) focuses on reusing waste materials to create products of higher (or lower, respectively) value or quality than the original. In other words, upcycled or downcycled products typically exhibit an upward or downward substitute demand relationship with the original product. In contrast, waste valorization involves an unpredictable demand relationship between waste-generating and waste-valorizing products, which can be independent, complementary, or substitute in their market demand. Thus, when considering upcycling and downcycling, it is essential to integrate factors such as refurbishing processes, supply risks, and value/quality substitution (cannibalization effects) into the model. While refurbishing processes and supply risks may not be directly relevant to waste valorization, the demand correlation observed in waste valorization shares notable similarities with upcycling and downcycling. For instance, a recent study by Long and Gui (2024) examines how upcycling, combined with a quick response strategy, helps mitigate deadstock (i.e., waste) generation in the fashion industry. Although their focus is on in-production waste within the context of upcycling, the use of deadstock fabric for higher-value products may lead to a cannibalization effect, impacting the original demand.

A.2. Waste Valorization Cases Across Industries

Table A.2 Summary of Waste Valorization Across Industries

	Product A	Product B	Waste	Cor.*	Firm(s)
Intra-firm	coffee	tumbler	coffee grounds	+	Starbucks
	ice cream	Cremissimo	ice cream	-	Unilever
	bedlinen	Tanum	leftover fabric	+/-	IKEA
	shoes and clothes	dumbbell	manufacturing scrap	+	Nike
Inter-firm	hummus	Fabanaise	aquafaba	Ind.	Ithaca & Sir Kensington
	beer	Marmite	brewer yeast	+	Heineken & Unilever
	clothes	clothes	leftover fabric	+/-	Jeplan & Patagonia

Note: "Cor." denotes demand correlation, where "+" indicates positive correlation, "–" indicates negative correlation, and "+/–" indicates that either relationship may exist. The correlations are derived from our empirical analysis below.

Waste valorization in a single firm:



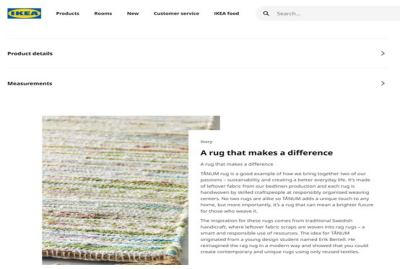
(a) Coffee Ground Made Tumbler



(b) Promotion in Store



(c) Cremissimo Ice Cream



(d) Tanum Rug



(e) Nike Grind Projects



(f) Nike Grind Dumbbell

Figure A.1 Waste Valorization in a Single Firm

Starbucks: `www.starbucks.com.cn/en/about/news/good-good-lifestyles-towards-a-better-planet/`

Unilever: `www.unilever.com/news/news-search/2021/why-our-fight-on-food-waste-is-a-chain-reaction/`

Ikea: `www.ikea.is/en/products/rugs-mats/rugs/small-rugs-and-runners/tanum-rug-flatwoven-art-30212675`

Nike: `www.nikegrind.com/about/`

1. **Starbucks coffee grounds valorization.** Starbucks' "Good Good" movement, launched in China on April 22, 2021, where coffee grounds are repurposed into tumblers sold in store. Customers purchasing select coffee products and pay an extra 68 yuan to receive a tumbler (retail price 119 yuan), a promotion targeting coffee drinkers with a sustainable, branded product. This bundling suggests strong complementarity, as coffee consumption likely drives tumbler demand, consistent with trends in sustainable consumer goods (Starbucks Sustainability Report, 2023). This initiative involves the creation of tumblers and notebooks, partially made from recycled coffee grounds collected at Starbucks stores. The grounds are processed and blended with polypropylene to produce an industrial-grade material, containing over 30% coffee grounds. This material can potentially be used to manufacture a wide range of lifestyle products. According to Starbucks' annual report, coffee ground waste represents a significant challenge in their daily operations. While Starbucks successfully composts a large portion of the coffee grounds, the lack of local composting infrastructure remains a major barrier. As a result, over 50 kilotons of coffee grounds are still sent to landfills annually (Starbucks Report, 2020). By repurposing coffee grounds into tumblers, notebooks, and straws, Starbucks has found a more economically viable and sustainable solution.

2. **Unilever ice cream valorization.** Unilever's Cremissimo, where ice cream waste from production line changeovers is remelted and blended into a new flavor. Since its launch, Cremissimo

has sold over 1.2 million tubs, repurposing approximately 160 metric tons of waste annually, and ranks among Unilever's top five flavors. Its success indicates a substitution effect, as consumers shift preferences within the brand's ice cream portfolio. As the world's largest ice cream manufacturer, Unilever has pioneered innovative solutions to reduce waste in its production facilities. During the changeover of production lines, a significant amount of ice cream is typically wasted. However, its German Cremissimo brand has found a way to repurpose this waste by re-melting it and reintegrating it into the production process. The resulting flavor, aptly named "Chocolate Hero," became an immediate best-seller and is now one of the brand's top five flavors. Unilever has sold over 1.2 million tubs of "Chocolate Hero," effectively repurposing 160 metric tons (over 352,739 pounds) of ice cream waste annually. (Food Dive, 2021 `https://www.fooddive.com/news/how-unilever-is-turning-sustainability-into-opportunity/595399/`) At Unilever, over 60% of production food waste is repurposed through methods such as anaerobic digestion, composting, land application, and energy recovery. However, more than 620 tonnes of food waste are still discarded in 2023 (Unilever Food Waste Report, 2023).

3. **Ikea leftover fabric valorization.** Tanum rug is made of leftover fabric from Ikea bedlinen production and each rug is handwoven by skilled craftspeople at responsibly organised weaving centers. The inspiration for these rugs comes from traditional Swedish handicraft, where leftover fabric scraps are woven into rag rugs – a smart and responsible use of resources. The idea for Tanum originated from a young design student named Erik Bertell. He reimagined the rag rug in a modern way and showed that you could create contemporary and unique rugs using only reused textiles.
4. **Nike grind project.** Nike Grind materials primarily consist of manufacturing scraps, alongside a blend of recycled and unsold shoes. Many of these shoes are collected through Nike's Reuse-A-Shoe program, which has been running since the early 1990s and receives approximately 1.5 million pairs annually. The program accepts worn-out shoes from any brand, donated by consumers, often at Nike retail locations. Once collected, the shoes are sent to one of two recycling facilities in the United States or Belgium, where they are processed into Nike Grind materials. From each shoe, three types of material are extracted: rubber from the outsole, foam from the midsole, and fabric from the upper. These materials—rubber, foam, fiber, leather, and textile blends—are then ground into granules, which can be reformed into various products. Nike Grind materials are subsequently incorporated into a range of Nike products, including footwear, apparel, and the yarn used to make them.

Waste valorization in a suppl chain:

1. **Aquafaba valorization.** Fabanaise is the first egg-free, vegan mayonnaise made from aquafaba—the liquid leftover from cooking chickpeas. Sir Kensington's discovered that this otherwise discarded cooking water possessed remarkable egg-like properties, leading them to partner with a hummus company to rescue it. The result is a versatile vegan spread that retains the tangy bite and smooth texture of traditional mayonnaise. But sourcing aquafaba was no easy feat. Although aquafaba is routinely discarded when chickpeas are strained, it was not available for purchase. With the help of Elly



(a) Hummus (b) Aquafaba Made Fabanaise (c) Heineken Beer

(d) Yeast Made Marmite (e) Pre-consumer Fabric Waste (f) PG Maded Fertilizer

Figure A.2 Waste Valorization in a Supply Chain

Fabanaise: `food52.com/blog/16841-aquafaba-returns-this-time-in-creamy-jarred-fabanaise?srsltid=AfmBOorg951D-_pjf1TyBm6n7fU-NOOfBn-X85H9kBRM4nEA0wAbBQ1p`

Marmite: `www.marmite.co.uk/journey.html`

Patagonia: `www.patagonia.com/our-footprint/jeplan.html`

PG Application: `www.fertilizer.org/wp-content/uploads/2023/01/2020_IFA_Phosphogypsum_Leadership_Innovation_Partnership-A4.pdf`

Truesdell, a local forager for Whole Foods Market, Sir Kensington's connected with Chris Kirby at Ithaca Hummus. Through this partnership, they've rescued between 50,000 and 100,000 pounds of aquafaba annually. According to Scott, "More hours of work went into Fabanaise than any other of our products because there was no existing supply chain." Ithaca Hummus had been pouring tens of thousands of gallons of aquafaba down the drain, so they had to figure out how to transform it into a viable product. And they succeeded. As reported by the New York Times, Sir Kensington's plans to purchase 20,000 gallons of aquafaba from Ithaca Hummus this year for their Fabanaise. While Scott did not disclose the exact price paid to Ithaca Hummus, he emphasized that quality and safety are the top priorities: "The price is secondary to the value we place on securing a stable supply. If we weren't paying a premium for it, we couldn't guarantee that."

2. **Brewing yeast valorization.** Marmite is a British savoury food spread based on yeast extract, invented by the German scientist Justus von Liebig. It is made from byproducts of beer brewing (lees) and is produced by the British company Unilever. Marmiteis a vegan source of B vitamins, including supplemental vitamin B12. The product that was to become Marmite was invented during the late 19th

century when the German scientist Justus von Liebig discovered that brewer's yeast could be concentrated, bottled and eaten. In 1902, the Marmite Food Extract Company was formed in Burton upon Trent, Staffordshire, England with Marmite as its main product and Burton as the site of the first factory. The byproduct yeast needed for the paste was supplied by Bass Brewery. By 1907, the product had become successful enough to warrant construction of a second factory at Camberwell Green in London. The beer brewing process causes fermentation; brewer's yeast multiplies and creates a large surplus. For a long time, brewers were stumped just what to do with it all, other than throw it out.

3. **Fabric valorization.** Only 15% of textiles are recycled, 19% are incinerated, and a staggering 66% end up in landfills, where decomposing garments release methane, further exacerbating greenhouse gas emissions. JEPLAN, a Japan-based recycling company, is addressing this issue on a global scale. By utilizing both pre-consumer textile waste (fabric scraps and yarn remnants from the manufacturing process) and post-consumer textile waste collected through their take-back program, BRING, JEPLAN chemically recycles polyethylene terephthalate (PET), commonly known as polyester, into new clothing. This innovative process diverts waste from landfills, reduces reliance on virgin petroleum for raw materials, and advances the chemical recycling sector with essential technology and infrastructure. According to Patagonia, more than 48,000 pounds of pre-consumer JEPLAN material will be incorporated into the body fabric of their Fall 2024 Better Sweater products. Traditionally, brand-supplier relationships have placed the majority of responsibility on the supplier. However, Patagonia's collaboration with JEPLAN marks a significant shift toward a more integrated, partnership-driven approach within the supply chain.

4. **Phosphogypsum (PG) valorization.** Phosphogypsum, a hazardous waste byproduct of the wet-acid process used in phosphoric acid production, presents significant environmental and economic challenges. During this process, sulfuric acid reacts with phosphate rock, generating a slurry of phosphoric acid and calcium sulfate (phosphogypsum). The phosphoric acid is extracted through filtration, leaving phosphogypsum as a residual waste. Although emerging valorization strategies explore its applications in construction materials, soil amendments, and fertilizers (IFA, 2022), its management remains costly due to stringent environmental and safety regulations, such as those mandated by the Resource Conservation and Recovery Act (RCRA). This case underscores a critical reality: waste valorization is not inherently profitable. Its economic feasibility hinges on factors such as waste composition, market demand, and regulatory constraints.

Building on the above examples, we investigate a broader range of waste types within MSW and present a categorization of their economic values in Table A.3.

Table A.3 MSW Categories and Waste Economic Value

Waste Type	Proportion in MSW	Example	Waste Economic Value
food	$30-40\%$	Starbucks, Unilever	$\eta \in \{0,1\}$
paper	$15-20\%$	NA	
plastic	$10-20\%$	NA	
textile	$5-10\%$	Nike Grind, Patagonia	$\eta \in \{0,1\}$
metal	$5-10\%$	NA	
e-waste	$2-5\%$	NA	
hazardous	$<5/\%$	phosphogypsum (PG) fertilizer	$\eta = 0$

Data Source: EPA (2018)

A.3. Demand Correlation Analysis with Sales Data

In the main text, we analyze demand correlations using sales data from Ito Yokado spanning October 2024 to May 2025, as shown in the following figure.

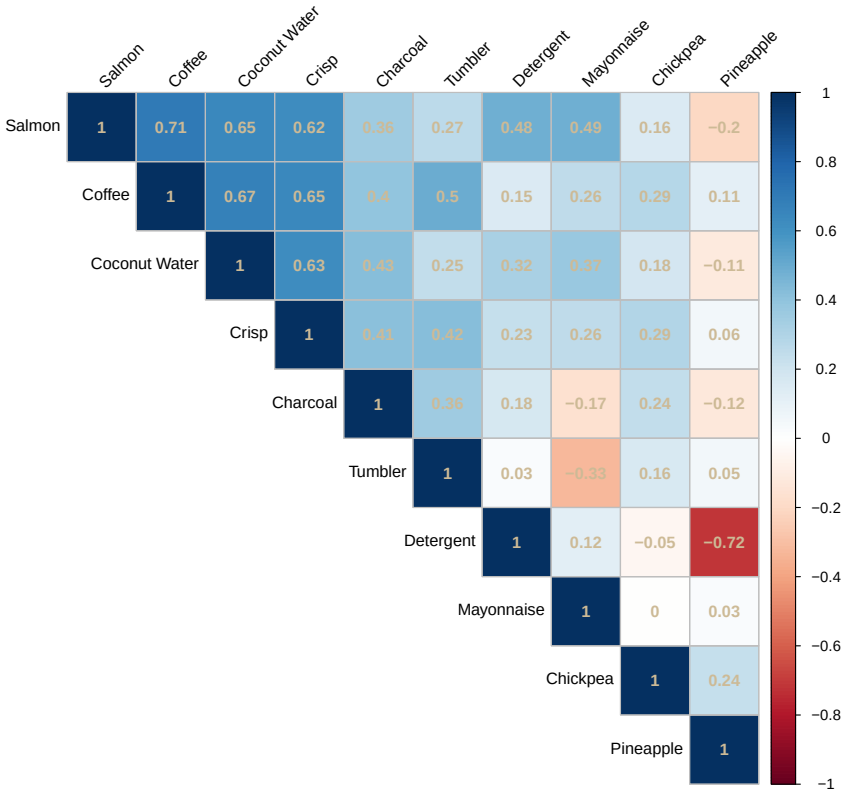


Figure A.3 Pearson Correlation Analysis of Sales Among Selected Grocery Products (Ito Yokado Data)

To uncover latent demand correlations, we further analyzed weekly, category-level sales data from Nielsen spanning January to December 2018. As illustrated in Figure A.4, we identified a strong positive correlation between beer and salad dressings (a proxy for Marmite), with a Pearson coefficient (PC) of 0.75. Similarly, a significant positive correlation (PC = 0.61) was observed between snacks (e.g., crisps) and seafood (e.g., salmon), closely aligning with patterns in our alternative dataset. In addition, we found a negative correlation between detergent and fruit (PC = –0.43), again consistent with our alternative data. A non-significant negative correlation was also observed between vegetables (including chickpeas) and mayonnaise. Furthermore, the results reveal a positive correlation between soft drinks (e.g., coconut water) and charcoal (PC = 0.42), and between coffee and glassware (e.g., tumblers) (PC = 0.56). Overall, the two datasets collectively underscore the presence and consistency of meaningful demand correlations among our focal product categories.

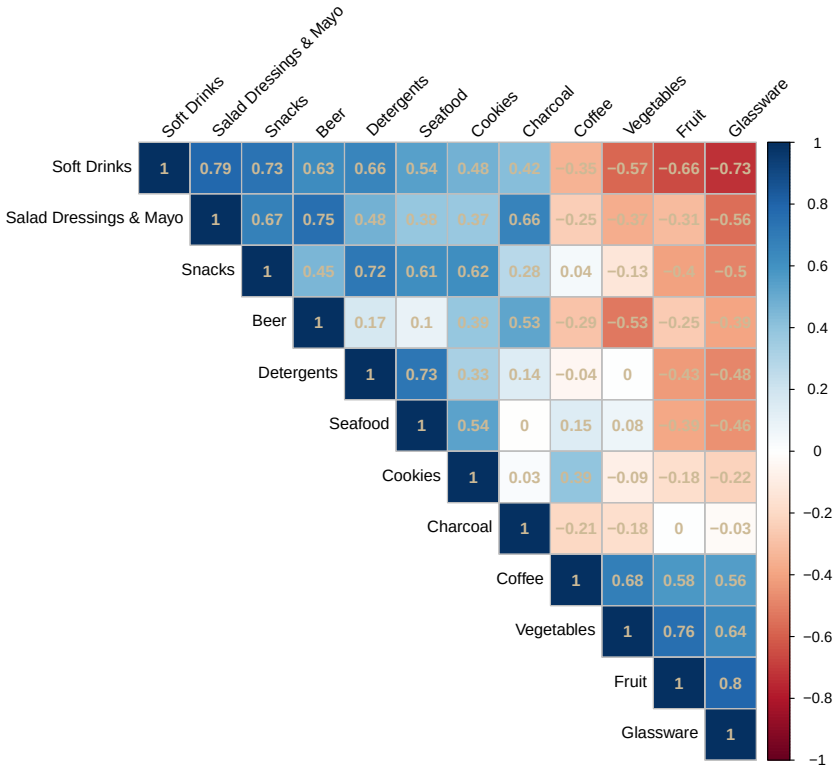


Figure A.4 Pearson Correlation Analysis of Selected Grocery Products (Nielsen Data)

Appendix B: Notations and Numerical Data

Table B.1 Primary Notations Throughout the Paper

Notation	Definition
c_a	The unit procurement cost for product A
c_Φ	The unit processing cost for product A
c_X	The unit production cost for product A, $c_X = c_a + c_\Phi$
c_η	The unit economic value of waste for per unit production of A, $c_\eta = (1-\eta)c_d - \eta c_s$, $\eta \in \{0,1\}$
c'_η	The unit economic value of waste for per unit production of B
$Q_j^{\mathcal{X}}$	The quantity for product j in model $\mathcal{X}$, $\mathcal{X} \in \{\mathcal{B}, \mathcal{BC}, \mathcal{W}, \mathcal{WC}, \mathcal{D}\}$, $j \in \{a,b\}$
$p_j^{\mathcal{X}}$	The market-clearing price for product j in model $\mathcal{X}$, $\mathcal{X} \in \{\mathcal{B}, \mathcal{BC}, \mathcal{W}, \mathcal{WC}, \mathcal{D}\}$, $j \in \{a,b\}$
M_j	The market size for product j, $j \in \{a,b\}$
E_j	The price-quantity sensitivity for product j, $j \in \{a,b\}$
ξ_j	The price-demand shock for product j, $j \in \{a,b\}$
$\hat{\xi}_j$	The realization of market uncertainty for product j, $j \in \{a,b\}$
c_{bm}	The unit procurement cost of raw materials for product B
c_{bw}	The unit procurement cost of waste for product B
c_Y	The unit processing cost for product B
γ	The waste valorization rate, $\gamma > 0$
ρ	The demand correlation coefficient, $\rho \in (-1,1)$
μ	The mean value of ϕ_j, $\mathbb{E}[\phi_j] = \mu$
θ	The probability of being in a good market state, $\theta \in (0,1)$
σ^2	The variance of ϕ_j, $Var[\phi_j] = \sigma^2$ and $\Delta = \sqrt{\mu^2 + \sigma^2} > 0$
$\pi_{II}^{\mathcal{X}}$	The expected profit for the manufacturer in stage II of model $\mathcal{X}$, $\mathcal{X} \in \{\mathcal{B}, \mathcal{BC}, \mathcal{W}, \mathcal{WC}\}$
$\pi^{\mathcal{X}}$	The total expected profit for the manufacturer in model $\mathcal{X}$, $\mathcal{X} \in \{\mathcal{B}, \mathcal{BC}, \mathcal{W}, \mathcal{WC}\}$
π_M^D	The expected profit for the manufacturer in model $\mathcal{D}$
π_S^D	The expected profit for the supplier in model $\mathcal{D}$
`DE`	The `disposal-reduction effect`, $\frac{\mathbb{1}_{\{\eta=0\}} \cdot c_\eta \mathbb{E}[\min(Q_b,\, \gamma Q_a)]}{\gamma}$
`RE`	The `resource-reduction effect`, $c_{bw}\mathbb{E}\left[\min\left(Q_b,\, \gamma Q_a\right)\right]$
`AE`	The `aggregate effect`, $\left(\mathbb{1}_{\{\eta=0\}} \cdot c_\eta/\gamma + c_{bw}\right) \mathbb{E}\left[\min(Q_b,\, \gamma Q_a)\right]$
`EE`	The `environment effect`, $\mathbb{E}\left[\alpha_1 \left(\frac{Q_b^{\mathcal{X}}}{\gamma} \cdot \mathbb{1}_{\{\mathcal{X} \in \{\mathcal{W}, \mathcal{WC}\}\}} - Q_a^{\mathcal{X}}\right)^+ - \alpha_2 \left(Q_a^{\mathcal{X}} - \frac{Q_b^{\mathcal{X}}}{\gamma} \cdot \mathbb{1}_{\{\mathcal{X} \in \{\mathcal{W}, \mathcal{WC}\}\}}\right)^+\right]$

Table B.2 Data for Numerical Examples

Parameters	Base Value	Range
c_X	0.20	0.20–0.35
c_Y	0.15	0.15–0.30
c_{bw}	0.20	0.20–0.35
c_{bm}	0.20	0.20–0.35
c_η	0.15	-0.10–0.30
γ	–	0.00–4.00
ρ	–	-1.00–1.00
Δ	–	$0.00–\frac{1-\max\{c_{bw},\, c_{bm}\}-c_\mathrm{Y}}{\lvert\rho\rvert}$

Appendix C: Benchmark Models

C.1. Uncorrelated Demands (Model $\mathcal{B}$)

The following analysis demonstrate that the manufacturer's overall profit maximization coincides with the optimal decision made for each product individually. Without waste valorization, the manufacturer has to incur procurement cost (i.e., virgin raw materials) for product B and disposal cost (or salvage value) for product A. That is, the total cost of product B includes c_Y and c_{bm} per unit; the net cost of product A includes $c_X = c_\Phi + c_a$ and c_η per unit. The manufacturer's expected profit, $\pi_{II}^{\mathcal{B}}(Q_b)$, in stage 2 is as follows:

$$\pi_{II}^{\mathcal{B}}(Q_b) = \mathbb{E}\left[p_b Q_b\right] - (c_Y + c_{bm})Q_b - c_\eta Q_a. \tag{C.1}$$

The manufacturer's total expected profit, $\pi^{\mathcal{B}}(Q_a)$, is as follows:

$$\pi^{\mathcal{B}}(Q_a) = \mathbb{E}\left[p_a Q_a\right] - c_X Q_a + \pi_{II}^{\mathcal{B}}(Q_a). \tag{C.2}$$

Note that incorporating the waste cost (or salve value) $c_\eta Q_a$ into the decision of $\pi^{\mathcal{B}}(Q_a)$ yields identical results, indicating that the decisions regarding Q_a and Q_b are independent. In the absence of both waste valorization and demand correlation, the optimal decision for the manufacturer, as proposed, is as follows.

LEMMA C.1. *When $\gamma = 0$ and $\rho = 0$, the optimal quantity decisions for products A and B are $Q_a^{\mathcal{B}} = \frac{1-c_X-c_\eta}{2}$ and $Q_b^{\mathcal{B}} = \frac{1-c_Y-c_{bm}}{2}$. The manufacturer's total expected profit is $\pi^{\mathcal{B}} = \frac{(1-c_X-c_\eta)^2+(1-c_Y-c_{bm})^2}{4}$.*

It is evident that, as shown in Lemma C.1, the decisions regarding $\{Q_a, Q_b\}$ are independent, although this assumption may not hold in two significant scenarios: 1) when there is correlation among demands, and 2) when there is interdependence in production due to the implementation of waste valorization.

PROOF OF LEMMA C.1:

In stage 2, according to the manufacturer's expected profit function of $\pi_{II}^{\mathcal{B}}$, it shows that $\pi_{II}^{\mathcal{B}}$ is concave in Q_b. Hence, based on the derivative of $\pi_{II}^{\mathcal{B}}$ in Equation (C.1), we have that $Q_b^{\mathcal{B}} = \frac{1-c_{bm}-c_Y}{2}$. In stage 1, according to the manufacturer's expected profit function of $\pi^{\mathcal{B}}$, it shows that $\pi^{\mathcal{B}}$ is concave in Q_a. We have that $Q_a^{\mathcal{B}} = \frac{1-c_\eta-c_X}{2}$ based on the derivative of $\pi^{\mathcal{B}}$ in Equation (C.2). Lemma C.1 follows. □

C.2. Correlated Demands (Model $\mathcal{BC}$)

PROOF OF LEMMA 1:

Given $\hat{\xi}_a$ as the realization of ξ_a, the manufacturer's expected profit in stage 2 is

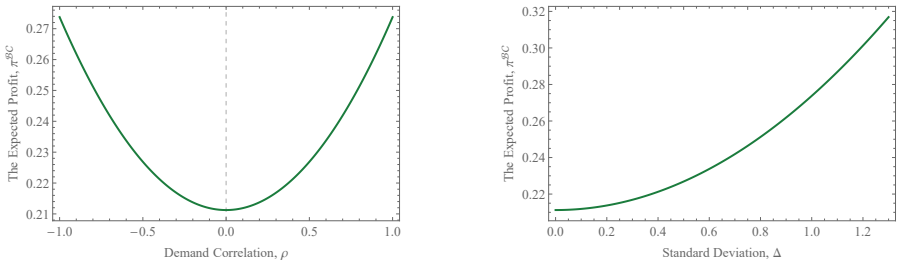
$$\pi_{II}^{\mathcal{BC}}(Q_b) = \mathbb{E}\left[p_b Q_b\right] - (c_Y + c_{bm})Q_b - c_\eta Q_a = Q_b\left(1 - Q_b + \rho\hat{\xi}_a\right) - (c_Y + c_{bm})Q_b - c_\eta Q_a. \tag{C.3}$$

Regardless of $\hat{\xi}_a = \{\Delta, -\Delta\}$, we have that

$$Q_b^{\mathcal{BC}} = \begin{cases} \frac{1-c_Y-c_{bm}+\rho\hat{\xi}_a}{2}, & \text{if } \rho > 0 \text{ and } \hat{\xi}_a > \frac{c_{bm}+c_Y-1}{\rho}, \text{ or } \rho < 0 \text{ and } \hat{\xi}_a < \frac{c_{bm}+c_Y-1}{\rho} \\ 0, & \text{otherwise} \end{cases}.$$

With regard to $\hat{\xi}_a = \{\Delta, -\Delta\}$, we have that

$$Q_b^{\mathcal{BC}} = \begin{cases} \frac{1-c_Y-c_{bm}+\rho\hat{\xi}_a}{2}, & \text{if } 0 < \Delta < \frac{1-c_{bm}-c_Y}{|\rho|} \\ 0, & \text{otherwise} \end{cases}.$$



(a) $\Delta = 0.5$ (b) $|\rho| = 0.5$

Figure C.1 The Manufacturer's Optimal Profits with $\gamma = 0$ and $\rho \neq 0$

The manufacturer's total expected profit, $\pi^{\mathcal{BC}}$, is as follows:

$$\pi^{\mathcal{BC}}(Q_a) = \mathbb{E}\left[p_a Q_a\right] - c_X Q_a + \pi_{II}^{\mathcal{BC}} = Q_a(1 - Q_a) - c_X Q_a + \pi_{II}^{\mathcal{BC}}. \tag{C.4}$$

In stage 2, according to the manufacturer's expected profit function of $\pi_{II}^{\mathcal{BC}}$, it shows that $\pi_{II}^{\mathcal{BC}}$ is concave in Q_b. Hence, based on the derivative of $\pi_{II}^{\mathcal{BC}}$, we have that $Q_b^{\mathcal{BC}} = \frac{1-c_{bm}-c_Y+\rho\hat{\xi}_a}{2}$. Note that, since $\hat{\xi}_a = \{-\Delta, \Delta\}$ and $\frac{1-c_{bm}-c_Y+\rho\hat{\xi}_a}{2} > 0$, we have that if $\rho \geq 0$ and $\hat{\xi}_a > \frac{c_{bm}+c_Y-1}{\rho}$, or $\rho < 0$ and $\hat{\xi}_a < \frac{c_{bm}+c_Y-1}{\rho}$, the optimal $Q_b^{\mathcal{BC}} = \frac{1-c_{bm}-c_Y+\rho\hat{\xi}_a}{2}$ only if $0 < \Delta|\rho| < 1 - c_{bm} - c_Y$; otherwise, $Q_b^{\mathcal{BC}} = 0$. In stage 1, according to the manufacturer's expected profit function of $\pi^{\mathcal{BC}}$, it shows that $\pi^{\mathcal{BC}}$ is concave in Q_a. We have that $Q_a^{\mathcal{BC}} = \frac{1-c_\eta-c_X}{2}$ based on the derivative of $\pi^{\mathcal{BC}}$.

Given the expected profit in Lemma 1, we have that $\frac{d\pi^{\mathcal{BC}}}{d\Delta} = \frac{\Delta\rho^2}{2} > 0$ and $\frac{d\pi^{\mathcal{BC}}}{d|\rho|} = \frac{\Delta^2|\rho|}{2} > 0$. We also illustrate the manufacturer's expected profit in Figure C.1. Lemma 1 follows. □

Appendix D: The Analysis of Waste Valorization

D.1. Waste Valorization Without Demand Correlation (Model $\mathcal{W}$)

PROOF OF PROPOSITION 1:

When $(\eta, c_s) \in C_I$, the manufacturer only utilizes the market-sourced waste, which the expect profit is given by $\pi_{II}^{\mathcal{W}} = \mathbb{E}[p_b Q_b] - Q_a c_\eta - (c_Y + c_{bw})Q_b$. In this case, we show that $Q_b = \frac{1-c_Y-c_{bw}}{2}$. Consequently, the optimal decision for product A is $Q_a = \frac{1-c_X-c_\eta}{2}$, which is independent of Q_b. When $(\eta, c_s) \in C_D$, we first summarize the optimal solutions in Table D.1.

Table D.1 Optimal Results for Proposition 1 with $(\eta, c_s) \in C_D$

Res.	$0 < \gamma < \hat{\gamma}$ $(Q_b^{\mathcal{W}} > \gamma Q_a^{\mathcal{W}})$	$\hat{\gamma} \le \gamma < \bar{\gamma}$ $(Q_b^{\mathcal{W}} = \gamma Q_a^{\mathcal{W}})$	$\bar{\gamma} \le \gamma$ $(Q_b^{\mathcal{W}} < \gamma Q_a^{\mathcal{W}})$
$Q_a^{\mathcal{W}}$	$\frac{1-c_X+\gamma c_{bw}}{2}$	$\frac{1-c_X+\gamma-\gamma c_Y}{2(1+\gamma^2)}$	$\frac{1-c_X-c_\eta}{2}$
$Q_b^{\mathcal{W}}$	$\frac{1-c_{bw}-c_Y}{2}$	$\frac{\gamma(1-c_X+\gamma-\gamma c_Y)}{2(1+\gamma^2)}$	$\frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}$
$\pi_{II}^{\mathcal{W}}$	$\frac{c_{bw}^2+2c_{bw}\left(\gamma+\gamma^2 c_{bw}-\gamma c_X+c_Y-1\right)+(c_Y-1)^2}{4}$	$\frac{\gamma(c_X+\gamma(c_Y-1)-1)\left(\gamma-\gamma^2-\gamma c_X+\left(\gamma^2+2\right)c_Y-2\right)}{4\left(\gamma^2+1\right)^2}$	$\frac{c_\eta^2+2\gamma c_\eta(\gamma(c_\eta+c_X-1)-c_Y+1)+\gamma^2(c_Y-1)^2}{4\gamma^2}$
$\pi^{\mathcal{W}}$	$\frac{\left(\gamma^2+1\right)c_{bw}^2+2c_{bw}(\gamma-\gamma c_X+c_Y-1)+c_X^2-2c_X+c_Y^2-2c_Y+2}{4}$	$\frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)}$	$\frac{\left(\gamma^2+1\right)c_\eta^2+2\gamma c_\eta(\gamma(c_X-1)-c_Y+1)+\gamma^2\left(c_X^2-2c_X+c_Y^2-2c_Y+2\right)}{4\gamma^2}$

In stage 2, the manufacturer's maximization problem is as follows:

$$\pi_{II}^{\mathcal{W}}(Q_b) = \begin{cases} \mathbb{E}[p_b Q_b] - c_\eta Q_a - (c_Y - c_\eta/\gamma)Q_b, & \text{if } Q_b \le \gamma Q_a \\ \mathbb{E}[p_b Q_b] - (c_Y + c_{bw})Q_b + \gamma c_{bw} Q_a, & \text{if } Q_b > \gamma Q_a \end{cases}. \tag{D.1}$$

It shows that $Q_b = \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}$ maximizes the first piece of Equation (D.1), and $Q_b = \frac{1-c_{bw}-c_Y}{2}$ maximizes the second piece of Equation (D.1). Hence, we distinguish three cases as follows:

- Case 1. $0 < \frac{1-c_{bw}-c_Y}{2} < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma} < \gamma Q_a$

  In Case 1, $Q_b = \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}$ maximizes Equation (D.1) only if $Q_a > \frac{c_\eta+\gamma-c_Y\gamma}{2\gamma^2}$.
- Case 2. $\frac{1-c_{bw}-c_Y}{2} < \gamma Q_a < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}$

  In Case 2, $Q_b = \gamma Q_a^B$ maximizes Equation (D.1) only if $\frac{1-c_{bw}-c_Y}{2\gamma} < Q_a < \frac{c_\eta+\gamma-c_Y\gamma}{2\gamma^2}$.
- Case 3. $\gamma Q_a < \frac{1-c_{bw}-c_Y}{2} < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}$

  In Case 3, $Q_b = \frac{1-c_{bw}-c_Y}{2}$ maximizes Equation (D.1) only if $0 < Q_a < \frac{1-c_{bw}-c_Y}{2\gamma}$.

Hence, the optimal production quantity for product B is

$$Q_b^{\mathcal{W}} = \begin{cases} \frac{1-c_{bw}-c_Y}{2}, & \text{if } 0 < Q_a < \frac{1-c_{bw}-c_Y}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y}{2\gamma} \le Q_a < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{otherwise} \end{cases}. \tag{D.2}$$

According to Equation (D.2), the manufacturer's total expected profit, $\pi^{\mathcal{W}}$, is as follows:

$$\pi^{\mathcal{W}}(Q_a) = \begin{cases} \frac{c_{bw}^2+2c_{bw}(c_Y+2\gamma Q_a-1)-4Q_a(c_X+Q_a-1)+(c_Y-2)c_Y+1}{4}, & \text{if } 0 < Q_a < \frac{1-c_{bw}-c_Y}{2\gamma} \\ -Q_a\left(c_X+\gamma(c_Y-1)+\gamma^2 Q_a+Q_a-1\right), & \text{if } \frac{1-c_{bw}-c_Y}{2\gamma} \le Q_a < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2} \\ \frac{c_\eta^2+\gamma^2(-4Q_a(c_\eta+c_X+Q_a-1)+(c_Y-2)c_Y+1)-2\gamma c_\eta(c_Y-1)}{4\gamma^2}, & \text{otherwise} \end{cases}. \tag{D.3}$$

We define $\mathcal{N} = -Q_a\left(c_X + \gamma(c_Y - 1) + \gamma^2 Q_a + Q_a - 1\right)$, Equation (D.3) can be reduced to

$$\pi^{\mathcal{W}}(Q_a) = \begin{cases} \mathcal{N} + \frac{(1-c_{bw}-c_Y-2\gamma Q_a)^2}{4} = \mathcal{N}_1, & \text{if } 0 < Q_a < \frac{1-c_{bw}-c_Y}{2\gamma} \\ \mathcal{N}, & \text{if } \frac{1-c_{bw}-c_Y}{2\gamma} \le Q_a < \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2} \\ \mathcal{N} + \frac{(c_\eta-\gamma(c_Y+2\gamma Q_a-1))^2}{4\gamma^2} = \mathcal{N}_2, & \text{otherwise} \end{cases}. \tag{D.4}$$

It shows that $\frac{d\mathcal{N}}{dQ_a}|\left\{Q_a = \left[\frac{1-c_{bw}-c_Y}{2\gamma} + \epsilon\right]\right\} = \frac{d\mathcal{N}_1}{dQ_a}|\left\{Q_a = \left[\frac{1-c_{bw}-c_Y}{2\gamma} - \epsilon\right]\right\}$, where ϵ is a sufficiently small and positive real number. Similarly, $\frac{d\mathcal{N}}{dQ_a}|\left\{Q_a = \left[\frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2} - \epsilon\right]\right\} = \frac{d\mathcal{N}_2}{dQ_a}|\left\{Q_a = \left[\frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2} + \epsilon\right]\right\}$. Additionally, $\mathcal{N}_1(Q_a = \frac{1-c_{bw}-c_Y}{2\gamma}) = \mathcal{N}(Q_a = \frac{1-c_{bw}-c_Y}{2\gamma})$ and $\mathcal{N}_2(Q_a = \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2}) = \mathcal{N}(Q_a = \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2})$. Hence, $\pi^{\mathcal{W}}$ is differentiable given $Q_a > 0$.

We can show that, $Q_a = \frac{1-c_X+\gamma c_{bw}}{2}$ maximizes the first piece of Equation (D.4), $Q_a = \frac{1-c_X+\gamma-\gamma c_Y}{2(1+\gamma^2)}$ maximizes the second piece of Equation (D.4), and $Q_a = \frac{1-c_X-c_\eta}{2}$ maximizes the third piece of Equation (D.4). Additionally, since $\mathcal{N}_1 \ge \mathcal{N}$ and $\mathcal{N}_2 \ge \mathcal{N}$, we have that $Q_a = \frac{1-c_X+\gamma c_{bw}}{2}$ maximizes $\pi^{\mathcal{W}}$ only if $\frac{1-c_X+\gamma c_{bw}}{2} < \frac{1-c_{bw}-c_Y}{2\gamma}$, which is equivalent to have that $0 < \gamma < \hat{\gamma}$. Also, $Q_a = \frac{1-c_X-c_\eta}{2}$ maximizes $\pi^{\mathcal{W}}$ only if $\frac{1-c_X-c_\eta}{2} > \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma^2}$, that is $\bar{\gamma} \le \gamma$. And, $Q_a = \frac{1-c_X+\gamma-\gamma c_Y}{2(1+\gamma^2)}$ maximizes $\pi^{\mathcal{W}}$ only if $\hat{\gamma} \le \gamma < \bar{\gamma}$. We present the optimal outcomes in Table D.1. Proposition 1 follows. □

PROOF OF COROLLARY 1:

According to our definition, $\texttt{DE} = \frac{\mathbb{1}_{\{\eta=0\}} c_\eta \mathbb{E}[\min\{Q_b, \gamma Q_a\}]}{\gamma}$ and $\texttt{RE} = c_{bw}\mathbb{E}[\min\{Q_b, \gamma Q_a\}]$. Hence, based on Proposition 1, we have that

$$\texttt{DE} = \frac{\mathbb{1}_{\{\eta=0\}} c_\eta}{\gamma} \times \begin{cases} \gamma\frac{1-c_X+\gamma c_{bw}}{2}, & \text{if } 0 < \gamma < \hat{\gamma} \\ \gamma\frac{1-c_X+\gamma-\gamma c_Y}{2(1+\gamma^2)}, & \text{if } \hat{\gamma} \le \gamma < \bar{\gamma} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{otherwise} \end{cases},$$

$$\texttt{RE} = c_{bw}\mathbb{1}_{\{(\eta,c_s)\in C_D\}} \times \begin{cases} \gamma\frac{1-c_X+\gamma c_{bw}}{2}, & \text{if } 0 < \gamma < \hat{\gamma} \\ \gamma\frac{1-c_X+\gamma-\gamma c_Y}{2(1+\gamma^2)}, & \text{if } \hat{\gamma} \le \gamma < \bar{\gamma} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{otherwise} \end{cases}.$$

Hence, when $\eta = 0$ and $\gamma < \hat{\gamma}$, both DE and RE increase with γ; when $\eta = 0$ and $\gamma > \bar{\gamma}$, both DE and RE decrease with γ. When $\eta = 0$ and $\hat{\gamma} < \gamma < \bar{\gamma}$, DE decreases with γ whereas RE increases with γ. When $(\eta, c_s) \in C_D$ and $\eta = 1$, RE increases with γ. When $c_\eta = 0$, RE is increasing in γ if $Q_b \ge \gamma Q_a$, and is independent of γ when $Q_b < \gamma Q_a$. It is easy to show that RE and $\pi^{\mathcal{W}}$ weakly increase with γ when $(\eta, c_s) \in C_D$ and $\eta = 1$, and $\pi^{\mathcal{W}}$ is independent of γ when $(\eta, c_s) \in C_I$.

Moreover, according to the results in Table D.1, we can derive that

$$\frac{d\pi^{\mathcal{W}}}{d\gamma}|_{\eta=0} = \begin{cases} \frac{c_{bw}(1-c_X+c_{bw}\gamma)}{2}, & \text{if } 0 < \gamma < \hat{\gamma} \\ \frac{(\gamma-\gamma c_X+c_Y-1)(c_X-1+\gamma(c_Y-1))}{2(\gamma^2+1)^2}, & \text{if } \hat{\gamma} \le \gamma < \bar{\gamma} \\ -\frac{c_\eta(c_\eta+\gamma-c_Y\gamma)}{2\gamma^3}, & \text{otherwise} \end{cases}.$$

Hence, the manufacturer's profit $\pi^{\mathcal{W}}$ increases with γ when $\gamma < \hat{\gamma}$, while it decreases with γ when $\gamma > \bar{\gamma}$. When $\hat{\gamma} < \gamma < \bar{\gamma}$, there exists a point at $\gamma = \frac{1-c_Y}{1-c_X}$ where $\frac{d\pi^{\mathcal{W}}}{d\gamma} > 0$ for $\gamma < \frac{1-c_Y}{1-c_X}$ and $\frac{d\pi^{\mathcal{W}}}{d\gamma} < 0$ for $\gamma > \frac{1-c_Y}{1-c_X}$. Therefore, $\gamma = \gamma^{\#} = \frac{1-c_Y}{1-c_X}$ is the point where the manufacturer can achieve the profit maximum. Corollary 1 follows. □

PROOF OF COROLLARY 2:

When the two product's demands are uncorrelated, the manufacturer's expected profit is $\pi^{\mathcal{B}} = \frac{(1-c_X-c_\eta)^2+(1-c_Y-c_{bm})^2}{4}$ without waste valorization. When $(\eta, c_s) \in C_I$, the manufacturer's expected profit is $\pi^{\mathcal{W}} = \frac{(1-c_X-c_\eta)^2+(1-c_Y-c_{bw})^2}{4}$. As a result, $\pi^{\mathcal{W}} > \pi^{\mathcal{B}}$ if $c_{bw} < c_{bm}$. When $(\eta, c_s) \in C_D$ and $\eta = 1$, we can show that $\pi^{\mathcal{W}} = \pi^{\mathcal{B}} = \frac{(1-c_X)^2+(1-c_Y)^2}{4}$ if $c_\eta = c_{bm} = 0$ and $\gamma > \bar{\gamma}$. According to the monotonicity in Corollary 1, it shows that $\pi^{\mathcal{W}} \geq \pi^{\mathcal{B}}$ when $\gamma > \gamma_0^{\#}$, where $\gamma_0^{\#}$ is given by the solution of equation $\pi^{\mathcal{W}} = \pi^{\mathcal{B}}$.

When $\eta = 0$, comparing with $\pi^{\mathcal{W}}$ in Table D.1, it shows that $\frac{(c_X+\gamma(c_Y-1)-1)^2}{4(\gamma^2+1)} > \frac{(1-c_X-c_\eta)^2+(1-c_Y-c_{bm})^2}{4}$ if and only if $\gamma_1^{\#} < \gamma < \gamma_2^{\#}$, where $\gamma_1^{\#} = -\frac{\frac{1}{2}\sqrt{4(c_X-1)^2(c_Y-1)^2-4\left(c_{bm}^2+2c_{bm}(c_Y-1)+(c_\eta+c_X-1)^2\right)\left(c_{bm}^2+2c_{bm}(c_Y-1)+c_\eta^2+2c_\eta(c_X-1)+(c_Y-1)^2\right)}-c_Xc_Y+c_X+c_Y-1}{c_{bm}^2+2c_{bm}(c_Y-1)+(c_\eta+c_X-1)^2}$ and $\gamma_2^{\#} = \frac{\frac{1}{2}\sqrt{4(c_X-1)^2(c_Y-1)^2-4\left(c_{bm}^2+2c_{bm}(c_Y-1)+(c_\eta+c_X-1)^2\right)\left(c_{bm}^2+2c_{bm}(c_Y-1)+c_\eta^2+2c_\eta(c_X-1)+(c_Y-1)^2\right)}+c_Xc_Y-c_X-c_Y+1}{c_{bm}^2+2c_{bm}(c_Y-1)+(c_\eta+c_X-1)^2}$. Moreover, when $\gamma = \gamma^{\#}$, $\pi^{\mathcal{W}} = \frac{(1-c_X)^2+(1-c_Y)^2}{4}$ which is always greater than $\frac{(1-c_X-c_\eta)^2+(1-c_Y-c_{bm})^2}{4}$.

Further, $\lim_{\gamma\to 0} \pi^{\mathcal{W}} = \frac{c_{bw}^2+2c_{bw}(c_Y-1)+c_X^2-2c_X+c_Y^2-2c_Y+2}{4}$ and $\lim_{\gamma\to\infty} \pi^{\mathcal{W}} = \frac{c_\eta^2+2c_\eta c_X-2c_\eta+c_X^2-2c_X+c_Y^2-2c_Y+2}{4}$. It shows that when $c_X < \frac{c_{bw}^2+2c_{bw}c_Y-2c_{bw}-c_{bm}^2-2c_{bm}c_Y+2c_{bm}-c_\eta^2+2c_\eta}{2c_\eta} = \hat{c}_X$, $\lim_{\gamma\to 0} \pi^{\mathcal{W}} > \pi^{\mathcal{B}}$; when $c_Y < \frac{2-c_{bm}}{2} = \hat{c}_Y$, $\lim_{\gamma\to\infty} \pi^{\mathcal{W}} > \pi^{\mathcal{B}}$. □

D.2. The Analysis of Demand Correlation (Model $\mathcal{WC}$)

PROOF OF LEMMA 2:

The manufacturer's decision problem in stage 2 can be described as follows:

$$\pi_{II}^{\mathcal{WC}}(Q_b) = \begin{cases} Q_b\left(1-Q_b+\rho\hat{\xi}_a\right) - Q_b\left(c_Y - c_\eta/\gamma\right) - Q_a c_\eta, & \text{if } 0 \leq Q_b \leq \gamma Q_a \\ Q_b\left(1-Q_b+\rho\hat{\xi}_a\right) + Q_a c_{bw}\gamma - Q_b\left(c_{bw}+c_Y\right), & \text{if } Q_b > \gamma Q_a \end{cases}. \tag{D.5}$$

Note that, Equation (D.5) reflects real costs in stage 2, that is the cost of waste disposal occurs in stage 2. In the case with $Q_b \leq \gamma Q_a$, the real cost of disposal for product A is $Q_a c_\eta - Q_b c_\eta/\gamma$. In fact, we can show the case which product A's disposal cost (or salvage value) $Q_a c_\eta$ was counted in stage 1 does not affect the optimal results. Given any realization of $-\infty < \hat{\xi}_a < \infty$ (i.e., regardless of Δ), we have following lemma to characterize Q_b. Again, when $(\eta, c_s) \in C_I$, the results in Model $\mathcal{WC}$ are the same as those in Model $\mathcal{BC}$, which we omit it.

LEMMA D.1. *For any realization of $-\infty < \hat{\xi}_a < \infty$, the decision of Q_b is:*

i. if $0 < \rho < 1$ and $\hat{\xi}_a < \frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho}$, or $-1 < \rho < 0$ and $\hat{\xi}_a \geq \frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho}$,

$$Q_b^{\mathcal{WC}} = 0;$$

ii. if $0 < \rho < 1$ and $\frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho} \leq \hat{\xi}_a < \frac{c_{bw}+c_Y-1}{\rho}$, or $-1 < \rho < 0$ and $\frac{c_{bw}+c_Y-1}{\rho} \leq \hat{\xi}_a < \frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho}$,

$$Q_b^{\mathcal{WC}} = \begin{cases} \gamma Q_a, & \text{if } Q_a < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\rho\hat{\xi}_a}{2\gamma^2} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\rho\hat{\xi}_a}{2\gamma}, & \text{otherwise} \end{cases};$$

iii. if $0<\rho<1$ *and* $\hat{\xi}_a \geq \frac{c_{bw}+c_Y-1}{\rho}$, *or* $-1<\rho<0$ *and* $\hat{\xi}_a < \frac{c_{bw}+c_Y-1}{\rho}$,

$$Q_b^{\mathcal{WC}} = \begin{cases} \frac{1-c_{bw}-c_Y+\rho\hat{\xi}_a}{2}, & \text{if } Q_a < \frac{1-c_{bw}-c_Y+\rho\hat{\xi}_a}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y+\rho\hat{\xi}_a}{2\gamma} \leq Q_a < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\rho\hat{\xi}_a}{2\gamma^2} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\rho\hat{\xi}_a}{2\gamma}, & \text{otherwise} \end{cases}.$$

The proof of Lemma D.1 is similar to that for Proposition 1. Lemma D.1 reveals that $Q_b^{\mathcal{WC}}=0$ occurs under the conditions where $\rho>0$ and $\hat{\xi}_a < \frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho}<0$, or $\rho<0$ and $\hat{\xi}_a \geq \frac{c_Y\gamma-\gamma-c_\eta}{\gamma\rho}>0$. This implies that the manufacturer's optimal decision is to refrain from producing product B if $|\rho|\hat{\xi}_a$ is negative and $|\rho\hat{\xi}_a|$ is sufficiently large. Similarly, it is optimal to have that $Q_b \leq \gamma Q_a$ if $|\rho|\hat{\xi}_a$ is negative and $|\rho\hat{\xi}_a|$ is moderate. Otherwise, the relationship between Q_b and γQ_a need to investigate further. Given the assumption of $\Delta|\rho|<1-c_Y-c_{bw}$, Lemma 2 follows. □

PROOF OF PROPOSITION 2:

According to Lemma 2, if $0 \leq \Delta < \min\left\{\frac{1-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\}$ (denoted as Region L_1), the manufacturer's maximization problem in stage 1 is,

$$\pi^{\mathcal{WC}}(Q_a) = \begin{cases} \mathcal{L}_1^{l_1}\ (Q_b>\gamma Q_a), & \text{if } 0 \leq Q_a \leq \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} \\ \mathcal{L}_2^{l_1}\ (Q_b>\gamma Q_a \text{ and } Q_b=\gamma Q_a, \textit{ Pooling Le}), & \text{if } \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} < Q_a \leq \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{L}_3^{l_1}\ (Q_b=\gamma Q_a), & \text{if } \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < Q_a \leq \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_4^{l_1}\ (Q_b=\gamma Q_a \text{ and } Q_b<\gamma Q_a, \textit{ Pooling Se}), & \text{if } \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < Q_a \leq \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_5^{l_1}\ (Q_b<\gamma Q_a), & \text{otherwise} \end{cases},$$

where

$$\mathcal{L}_1^{l_1} = \mathcal{L}_1^{L} = \frac{1}{4}\left(c_{bw}^2 + 2c_{bw}(c_Y+2\gamma Q_a-1) - 4Q_a(c_X+Q_a-1) + (c_Y-2)c_Y + \Delta^2\rho^2 + 1\right),$$

$$\begin{aligned}\mathcal{L}_2^{l_1} =& \mathcal{L}_2^{L} \\ =& \frac{1}{8}\left(c_{bw}^2 + 2c_{bw}(c_Y-\Delta|\rho|+2\gamma Q_a-1)\right) \\ &+ \frac{1}{2}\left(Q_a\left(\gamma - 2c_X - \left(\gamma^2+2\right)Q_a + 2\right)\right) \\ &+ \frac{1}{8}\left(c_Y^2 - 2c_Y(\Delta|\rho|+2\gamma Q_a+1) + \Delta^2\rho^2 + 2\Delta|\rho| - 4\gamma\Delta|\rho|Q_a + 1\right)\end{aligned}$$

$$\mathcal{L}_3^{l_1} = \mathcal{L}_3^{L} = -Q_a\left(c_X + \gamma(c_Y-1) + \gamma^2 Q_a + Q_a - 1\right),$$

$$\begin{aligned}\mathcal{L}_4^{l_1} =& \mathcal{L}_4^{L} \\ =& \frac{1}{8}\left(\frac{(c_\eta-\gamma(c_Y+\Delta|\rho|-1))^2}{\gamma^2} - 8Q_a(c_\eta+c_X) + 4Q_a(c_\eta-\gamma(c_Y-\Delta|\rho|+\gamma Q_a-1)) - 8(Q_a-1)Q_a\right),\end{aligned}$$

$$\mathcal{L}_5^{l_1} = \mathcal{L}_5^{L} = \frac{c_\eta^2 - 2\gamma c_\eta(c_Y+2\gamma Q_a-1) + \gamma^2\left(-4Q_a(c_X+Q_a-1) + (c_Y-2)c_Y + \Delta^2\rho^2+1\right)}{4\gamma^2}.$$

We distinguish relationships among $\mathcal{L}_i^{l_1}$ that is analogous to the proof for Proposition 1:

$$\mathcal{L}_5^{l_1} - \mathcal{L}_4^{l_1} = \frac{(c_\eta + \gamma(1 - c_Y + \Delta|\rho| - 2\gamma Q_a))^2}{8\gamma^2};$$

$$\mathcal{L}_4^{l_1} - \mathcal{L}_3^{l_1} = \frac{(c_\eta + \gamma(1 - c_Y - \Delta|\rho| - 2\gamma Q_a))^2}{8\gamma^2};$$

$$\mathcal{L}_2^{l_1} - \mathcal{L}_3^{l_1} = \frac{(c_{bw} + c_Y - \Delta|\rho| + 2\gamma Q_a - 1)^2}{8};$$

$$\mathcal{L}_1^{l_1} - \mathcal{L}_2^{l_1} = \frac{(c_{bw} + c_Y + \Delta|\rho| + 2\gamma Q_a - 1)^2}{8};$$

$$\mathcal{L}_i^{l_1} = \mathcal{M}_i^{l_1},\ \forall\, i$$

Here, $\mathcal{M}_i^{l_1}$ is the objective function when the term $c_\eta Q_a$ was counted at the first stage. We summarize this important observation in Table D.2. The remaining proofs are similar to those for Proposition 1, where we show that the interior solution for each piecewise segment is globally optimal only if it satisfies the condition within that piece. Define Region i ($i = 1, 2, ..., 5$), denoted as $\mathcal{R}_i^{l_1}$, as follows:

$\mathcal{R}_1^{l_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_X+\gamma c_{bw}}{2} < \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma}\right\}$,
$\mathcal{R}_2^{l_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} < \frac{2-\gamma\Delta|\rho|+\gamma+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\right\}$,
$\mathcal{R}_3^{l_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < \frac{\gamma-c_X-\gamma c_Y+1}{2\gamma^2+2} < \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta|\rho|}{2\gamma^2}\right\}$,
$\mathcal{R}_4^{l_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta|\rho|}{2\gamma^2} < \frac{2+\gamma\Delta|\rho|+\gamma-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2}\right\}$,
$\mathcal{R}_5^{l_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2} < \frac{1-c_\eta-c_X}{2}\right\}$, where $J \in \{X,\ Y,\ \eta,\ bw\}$.

Table D.2 Variables and Definitions

Variables	Definition	Variables	Definition
Notations		Notations	
$\mathcal{L}_1^{l_1}$	$\mathcal{L}_2^{l_1} + \frac{(c_{bw}+c_Y+\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}$	$\mathcal{M}_1^{l_1}$	$\mathcal{M}_2^{l_1} + \frac{(c_{bw}+c_Y+\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}$
$\mathcal{L}_2^{l_1}$	$\mathcal{L}_3^{l_1} + \frac{(c_{bw}+c_Y-\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}$	$\mathcal{M}_2^{l_1}$	$\mathcal{M}_3^{l_1} + \frac{(c_{bw}+c_Y-\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}$
$\mathcal{L}_3^{l_1}$	$-Q_a\left(c_X + \gamma(c_Y - 1) + \gamma^2 Q_a + Q_a - 1\right)$	$\mathcal{M}_3^{l_1}$	$-Q_a\left(c_X + \gamma(c_Y - 1) + \gamma^2 Q_a + Q_a - 1\right)$
$\mathcal{L}_4^{l_1}$	$\mathcal{L}_3^{l_1} + \frac{(c_\eta+\gamma(1-c_Y-\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}$	$\mathcal{M}_4^{l_1}$	$\mathcal{M}_3^{l_1} + \frac{(c_\eta+\gamma(1-c_Y-\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}$
$\mathcal{L}_5^{l_1}$	$\mathcal{L}_4^{l_1} + \frac{(c_\eta+\gamma(1-c_Y+\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}$	$\mathcal{M}_5^{l_1}$	$\mathcal{M}_4^{l_1} + \frac{(c_\eta+\gamma(1-c_Y+\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}$

Similarly, if $\min\left\{\frac{1-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{1-c_{bw}-c_Y}{|\rho|}$ (denoted as Region L_2),

$$\pi^{\mathcal{WC}}(Q_a) = \begin{cases} \mathcal{L}_1^{l_2} = \mathcal{L}_1^H\ (Q_b > \gamma Q_a), & \text{if } 0 \le Q_a \le \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} \\ \mathcal{L}_2^{l_2} = \mathcal{L}_2^H\ (Q_b > \gamma Q_a \text{ and } Q_b = \gamma Q_a,\ \textit{Pooling Le}), & \text{if } \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_3^{l_2} = \mathcal{L}_3^H\ (Q_b > \gamma Q_a \text{ and } Q_b < \gamma Q_a, \textit{Pooling LS}), & \text{if } \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < Q_a \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{L}_4^{l_2} = \mathcal{L}_4^H\ (Q_b = \gamma Q_a \text{ and } Q_b < \gamma Q_a,\ \textit{Pooling Se}), & \text{if } \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_5^{l_2} = \mathcal{L}_5^H\ (Q_b < \gamma Q_a), & \text{otherwise} \end{cases},$$

where $\mathcal{L}_1^{l_2} = \mathcal{L}_1^{l_1}$, $\mathcal{L}_2^{l_2} = \mathcal{L}_2^{l_1}$, $\mathcal{L}_4^{l_2} = \mathcal{L}_4^{l_1}$, $\mathcal{L}_5^{l_2} = \mathcal{L}_5^{l_1}$, and
$\mathcal{L}_3^{l_2} = \frac{\gamma^2\left(c_{bw}^2+2c_{bw}(c_Y-\Delta|\rho|+2\gamma Q_a-1)-8Q_a(c_x+Q_a-1)+2(c_Y-2)c_Y+2\Delta^2\rho^2+2\right)+c_\eta^2-2\gamma c_\eta(c_Y+\Delta|\rho|+2\gamma Q_a-1)}{8\gamma^2}$.

We also have that $\mathcal{L}_2^{l_2} + \frac{(c_\eta+\gamma(1-c_Y-\Delta|\rho|-2\gamma Q_a))^2}{8\gamma^2} = \mathcal{L}_3^{l_2}$, $\mathcal{L}_2^{l_2} + \frac{(c_{bw}+c_Y+\Delta|\rho|+2\gamma Q_a-1)^2}{8} = \mathcal{L}_1^{l_2}$, $\mathcal{L}_4^{l_2} + \frac{(c_{bw}+c_Y-\Delta|\rho|+2\gamma Q_a-1)^2}{8} = \mathcal{L}_3^{l_2}$, and $\mathcal{L}_4^{l_2} + \frac{(c_\eta+\gamma(1-c_Y+\Delta|\rho|-2\gamma Q_a))^2}{8\gamma^2} = \mathcal{L}_5^{l_2}$.

Define Region i $(i = 1,2,...,5)$, denoted as $\mathcal{R}_i^{l_2}$, as follows:

$\mathcal{R}_1^{l_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_X+\gamma c_{bw}}{2} < \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma}\right\}$,

$\mathcal{R}_2^{l_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} < \frac{2-\gamma\Delta|\rho|+\gamma+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4} \le \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_3^{l_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4} < \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\right\}$,

$\mathcal{R}_4^{l_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < \frac{2+\gamma\Delta|\rho|+\gamma-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_5^{l_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2} < \frac{1-c_\eta-c_X}{2}\right\}$, where $J \in \{X,\ Y,\ \eta,\ bw\}$.

In the main body of the paper, $\mathcal{R}_1^{LH} = \mathcal{R}_1^{l_1} \cup \mathcal{R}_1^{l_2}$, $\mathcal{R}_2^{LH} = \mathcal{R}_2^{l_1} \cup \mathcal{R}_2^{l_2}$, $\mathcal{R}_3^{L} = \mathcal{R}_3^{l_1}$, $\mathcal{R}_3^{H} = \mathcal{R}_3^{l_2}$, $\mathcal{R}_4^{LH} = \mathcal{R}_4^{l_1} \cup \mathcal{R}_4^{l_2}$, and $\mathcal{R}_5^{LH} = \mathcal{R}_5^{l_1} \cup \mathcal{R}_5^{l_2}$. We further provide the analysis for cases with $\min\left\{\frac{1-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta$ as follows.

If $\min\left\{\frac{1-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}$ (denoted as Region M_1),

$$\pi^{\mathcal{WC}}(Q_a) = \begin{cases} \mathcal{L}_1^{m_1}\ (Q_b > \gamma Q_a \text{ and } Q_b = \gamma Q_a,\ \textit{Pooling Le}), & \text{if } 0 \le Q_a \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{L}_2^{m_1}\ (Q_b = \gamma Q_a), & \text{if } \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_3^{m_1}\ (Q_b = \gamma Q_a \text{ and } Q_b < \gamma Q_a,\ \textit{Pooling Se}), & \text{if } \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_4^{m_1}\ (Q_b < \gamma Q_a), & \text{otherwise} \end{cases},$$

where $\mathcal{L}_2^{m_1} = \mathcal{L}_3^{l_1}$, $\mathcal{L}_1^{m_1} = \mathcal{L}_2^{l_1}$, $\mathcal{L}_3^{m_1} = \mathcal{L}_4^{l_1}$, and $\mathcal{L}_4^{m_1} = \mathcal{L}_5^{l_1}$. Hence, $\mathcal{L}_1^{m_1} = \mathcal{L}_2^{m_1} + \frac{(c_{bw}+c_Y-\Delta|\rho|+2\gamma Q_a-1)^2}{8}$, $\mathcal{L}_3^{m_1} = \mathcal{L}_2^{m_1} + \frac{(c_\eta+\gamma(1-c_Y-\Delta|\rho|-2\gamma Q_a))^2}{8\gamma^2}$, and $\mathcal{L}_4^{m_1} = \mathcal{L}_3^{m_1} + \frac{(c_\eta+\gamma(1-c_Y+\Delta|\rho|-2\gamma Q_a))^2}{8\gamma^2}$. Define Region i $(i = 1,2,...,5)$, denoted as $\mathcal{R}_i^{m_1}$, as follows:

$\mathcal{R}_1^{m_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{2-\gamma\Delta|\rho|+\gamma+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\right\}$,

$\mathcal{R}_2^{m_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < \frac{1+\gamma-c_X-\gamma c_Y}{2(1+\gamma^2)} \le \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_3^{m_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < \frac{2+\gamma+\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_4^{m_1}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2} < \frac{1-c_\eta-c_X}{2}\right\}$.

If $\max\left\{\frac{1-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{c_\eta+\gamma-\gamma c_Y}{\gamma|\rho|}$ (denoted as Region M_2),

$$\pi^{\mathcal{WC}}(Q_a) = \begin{cases} \mathcal{L}_1^{m_2}\ (Q_b > \gamma Q_a \text{ and } Q_b = \gamma Q_a,\ \textit{Pooling Le}), & \text{if } 0 \le Q_a \le \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_2^{m_2}\ (Q_b > \gamma Q_a \text{ and } Q_b < \gamma Q_a, \textit{Pooling LS}), & \text{if } \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < Q_a \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{L}_3^{m_2}\ (Q_b = \gamma Q_a \text{ and } Q_b < \gamma Q_a,\ \textit{Pooling Se}), & \text{if } \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_4^{m_2}\ (Q_b < \gamma Q_a), & \text{otherwise} \end{cases},$$

where $\mathcal{L}_2^{m_1} = \mathcal{L}_3^{l_2}$, $\mathcal{L}_1^{m_1} = \mathcal{L}_2^{l_2}$, $\mathcal{L}_3^{m_1} = \mathcal{L}_4^{l_2}$, and $\mathcal{L}_4^{m_1} = \mathcal{L}_5^{l_2}$. Define Region i $(i = 1,2,...,5)$, denoted as $\mathcal{R}_i^{m_2}$, as follows:

$\mathcal{R}_1^{m_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{2+\gamma-\gamma\Delta|\rho|+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_2^{m_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-c_Y\gamma-\gamma\Delta|\rho|}{2\gamma^2} < \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4} \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\right\}$,

$\mathcal{R}_3^{m_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < \frac{2+\gamma+\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4} < \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_4^{m_2}$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J)\ |\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta|\rho|}{2\gamma^2} < \frac{1-c_\eta-c_X}{2}\right\}$.

If $\Delta > \frac{c_\eta+\gamma-\gamma c_Y}{\gamma|\rho|}$ (denoted as Region H)

$$\pi^{\mathcal{WC}}(Q_a) = \begin{cases} \mathcal{L}_1^h \ (Q_b > \gamma Q_a \text{ and } Q_b = 0), & \text{if } 0 \le Q_a \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{L}_2^h \ (Q_b = \gamma Q_a \text{ and } Q_b = 0), & \text{if } \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < Q_a \le \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} \\ \mathcal{L}_3^h \ (Q_b < \gamma Q_a \text{ and } Q_b = 0), & \text{if } \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} < Q_a \end{cases},$$

where $\mathcal{L}_2^h = -\frac{1}{2}Q_a\left(c_\eta + 2c_X + \gamma(c_Y + \Delta|\rho| - 1) + \left(\gamma^2+2\right)Q_a - 2\right)$, $\mathcal{L}_1^h = \mathcal{L}_2^h + \frac{1}{8}(c_{bw} + c_Y - \Delta|\rho| + 2\gamma Q_a - 1)^2$, $\mathcal{L}_3^h = \mathcal{L}_2^h + \frac{(c_\eta+\gamma(-c_Y+\Delta\rho-2\gamma Q_a+1))^2}{8\gamma^2}$. Define Region i $(i = 1, 2, ..., 5)$, denoted as $\mathcal{R}_i^h$, as follows:

$\mathcal{R}_1^h$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J) \mid \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4} \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\right\}$,

$\mathcal{R}_2^h$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J) \mid \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} < \frac{2+\gamma+\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4} \le \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2}\right\}$,

$\mathcal{R}_3^h$: $\left\{(\gamma,\ \rho,\ \Delta,\ c_J) \mid \frac{c_\eta+\gamma-c_Y\gamma+\gamma\Delta|\rho|}{2\gamma^2} < \frac{1-c_\eta-c_X}{2}\right\}$.

When $\Delta > \frac{c_\eta+\gamma-\gamma c_Y}{\gamma|\rho|}$, Q_b might be zero. Hence, for certain piecewise, it shows that $\pi^{\mathcal{WC}}(Q_b = 0)$ can combine with $\pi^{\mathcal{WC}}(Q_b > \gamma Q_a)$, $\pi^{\mathcal{WC}}(Q_b < \gamma Q_a)$, or $\pi^{\mathcal{WC}}(Q_b = \gamma Q_a)$. We summarize all of the outcomes in Tables D.3, D.4, D.5, D.6, and D.7.

Table D.3 Optimal Results in Region L_1

Parameters	Results in Region L_1
$Q_a^{\mathcal{WC}}$	$\begin{cases} \frac{1-c_X+\gamma c_{bw}}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{l_1} \\ \frac{2+\gamma-\gamma\Delta\|\rho\|+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{l_1} \\ \frac{1+\gamma-c_X-\gamma c_Y}{2(1+\gamma^2)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{l_1} \\ \frac{2+\gamma+\gamma\Delta\|\rho\|-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{l_1} \\ \frac{1-c_\eta-c_X}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{l_1} \end{cases}$
$Q_b^{\mathcal{WC}}$	$Q_b^{\mathcal{WC}}\|_{\hat{\xi}_a\rho\ge 0} = \begin{cases} \frac{1-c_{bw}-c_Y+\Delta\|\rho\|}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{l_1} \bigcup \mathcal{R}_2^{l_1} \\ \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{l_1} \bigcup \mathcal{R}_4^{l_1} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\|\rho\|}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{l_1} \end{cases}$, $Q_b^{\mathcal{WC}}\|_{\hat{\xi}_a\rho<0} = \begin{cases} \frac{1-c_{bw}-c_Y-\Delta\|\rho\|}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{l_1} \\ \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{l_1} \bigcup \mathcal{R}_3^{l_1} \\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta\|\rho\|}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{l_1} \bigcup \mathcal{R}_5^{l_1} \end{cases}$
$\mathbb{E}\left[Q_b^{\mathcal{WC}}\right]$	$\begin{cases} \frac{1-c_{bw}-c_Y}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{l_1} \\ \frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta\|\rho\|}{2\left(\gamma^2+2\right)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{l_1} \\ \frac{\gamma(1+\gamma-c_X-\gamma c_Y)}{2(1+\gamma^2)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{l_1} \\ \frac{c_\eta+\gamma-c_Y\gamma+\gamma^2(1-c_X)+\gamma^3(1-c_Y)-\gamma\Delta\|\rho\|}{2\gamma(2+\gamma^2)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{l_1} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{l_1} \end{cases}$
$\pi_{II}^{\mathcal{WC}}$	$\pi^{\mathcal{WC}} - ((c_X)Q_a + Q_a(1 - Q_a))$
$\pi^{\mathcal{WC}}$	$\begin{cases} \frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)} + \frac{(c_{bw}+c_Y-\Delta\|\rho\|+2\gamma Q_a-1)^2}{8} + \frac{(c_{bw}+c_Y+\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{l_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)} + \frac{(c_{bw}+c_Y-\Delta\|\rho\|+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{l_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{l_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)} + \frac{(c_\eta+\gamma(1-c_Y-\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{l_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4\left(\gamma^2+1\right)} + \frac{(c_\eta+\gamma(1-c_Y-\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2} + \frac{(c_\eta+\gamma(1-c_Y+\Delta\|\rho\|-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{l_1} \end{cases}$

Table D.4 Optimal Results in Region L_2

Parameters	Results in Region L_2
$Q_a^{\mathcal{WC}}$	$\begin{cases} \frac{1-c_X+\gamma c_{bw}}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{l_2} \\ \frac{2+\gamma-\gamma\Delta\lvert\rho\rvert+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{l_2} \\ \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{l_2} \\ \frac{2+\gamma+\gamma\Delta\lvert\rho\rvert-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{l_2} \\ \frac{1-c_\eta-c_X}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_5^{l_2} \end{cases}$
$Q_b^{\mathcal{WC}}$	$Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho\geq 0}=\begin{cases} \frac{1-c_{bw}-c_Y+\Delta\lvert\rho\rvert}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{l_2}\cup\mathcal{R}_2^{l_2}\cup\mathcal{R}_3^{l_2} \\ \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{l_2} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_5^{l_2} \end{cases}$, $Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho<0}=\begin{cases} \frac{1-c_{bw}-c_Y-\Delta\lvert\rho\rvert}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{l_2} \\ \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{l_2} \\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{l_2}\cup\mathcal{R}_4^{l_2}\cup\mathcal{R}_5^{l_2} \end{cases}$
$\mathbb{E}\left[Q_b^{\mathcal{WC}}\right]$	$\begin{cases} \frac{1-c_{bw}-c_Y}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{l_2} \\ \frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta\lvert\rho\rvert}{2(\gamma^2+2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{l_2} \\ \frac{c_\eta-\gamma(c_{bw}+2c_Y-2)}{4\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{l_2} \\ \frac{c_\eta+\gamma-c_Y\gamma+\gamma^2(1-c_X)+\gamma^3(1-c_Y)-\gamma\Delta\lvert\rho\rvert}{2\gamma(2+\gamma^2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{l_2} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_5^{l_2} \end{cases}$
$\pi_{II}^{\mathcal{WC}}$	$\pi^{\mathcal{WC}}-((c_X)Q_a+Q_a(1-Q_a))$
$\pi^{\mathcal{WC}}$	$\begin{cases} \mathcal{L}_2^{l_2}+\frac{(c_{bw}+c_Y+\Delta\lvert\rho\rvert+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{l_2} \\ \mathcal{L}_3^{l_2}-\frac{(c_\eta+\gamma(1-c_Y-\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{l_2} \\ \frac{\gamma^2\left(\left(\gamma^2+2\right)c_{bw}^2+4c_{bw}(\gamma-\gamma c_X+c_Y-\kappa-1)+4\left(c_X^2-2c_X+c_Y^2-2c_Y+\kappa^2+2\right)\right)-2\gamma c_\eta\left(2\gamma+\gamma^2 cr-2\gamma c_X+2c_Y+2\kappa-2\right)+\left(\gamma^2+2\right)c_\eta^2}{16\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{l_2} \\ \mathcal{L}_3^{l_2}-\frac{(c_{bw}+c_Y-\Delta\lvert\rho\rvert+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{l_2} \\ \mathcal{L}_4^{l_2}+\frac{(c_\eta+\gamma(1-c_Y+\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_5^{l_2} \end{cases}$

Table D.5 Optimal Results in Region M_1

Parameters	Results in Region M_1
$Q_a^{\mathcal{WC}}$	$\begin{cases} \frac{2+\gamma-\gamma\Delta\lvert\rho\rvert+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_1} \\ \frac{1+\gamma-c_X-\gamma c_Y}{2(1+\gamma^2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_1} \\ \frac{2+\gamma+\gamma\Delta\lvert\rho\rvert-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_1} \\ \frac{1-c_\eta-c_X}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_1} \end{cases}$
$Q_b^{\mathcal{WC}}$	$Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho\geq 0}=\begin{cases} \frac{1-c_{bw}-c_Y+\Delta\lvert\rho\rvert}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_1} \\ \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_1}\cup\mathcal{R}_3^{m_1} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_1} \end{cases}$, $Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho<0}=\begin{cases} \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_1}\cup\mathcal{R}_2^{m_1} \\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_1}\cup\mathcal{R}_4^{m_1} \end{cases}$
$\mathbb{E}\left[Q_b^{\mathcal{WC}}\right]$	$\begin{cases} \frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta\lvert\rho\rvert}{2(\gamma^2+2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_1} \\ \frac{\gamma(1+\gamma-c_X-\gamma c_Y)}{2(1+\gamma^2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_1} \\ \frac{c_\eta+\gamma-c_Y\gamma+\gamma^2(1-c_X)+\gamma^3(1-c_Y)-\gamma\Delta\lvert\rho\rvert}{2\gamma(2+\gamma^2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_1} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_1} \end{cases}$
$\pi_{II}^{\mathcal{WC}}$	$\pi^{\mathcal{WC}}-((c_X)Q_a+Q_a(1-Q_a))$
$\pi^{\mathcal{WC}}$	$\begin{cases} \frac{(c_X+\gamma(c_Y-1)-1)^2}{4(\gamma^2+1)}+\frac{(c_{bw}+c_Y-\Delta\lvert\rho\rvert+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4(\gamma^2+1)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4(\gamma^2+1)}+\frac{(c_\eta+\gamma(1-c_Y-\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_1} \\ \frac{(c_X+\gamma(c_Y-1)-1)^2}{4(\gamma^2+1)}+\frac{(c_\eta+\gamma(1-c_Y-\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}+\frac{(c_\eta+\gamma(1-c_Y+\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_1} \end{cases}$

Table D.6 Optimal Results in Region M_2

Parameters	Results in Region M_2
$Q_a^{\mathcal{WC}}$	$\begin{cases} \frac{2+\gamma-\gamma\Delta\lvert\rho\rvert+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_2} \\ \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_2} \\ \frac{2+\gamma+\gamma\Delta\lvert\rho\rvert-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_2} \\ \frac{1-c_\eta-c_X}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_2} \end{cases}$
$Q_b^{\mathcal{WC}}$	$Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho\geq 0}=\begin{cases} \frac{1-c_{bw}-c_Y+\Delta\lvert\rho\rvert}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_2}\bigcup\mathcal{R}_2^{m_2} \\ \gamma Q_a^{\mathcal{WC}^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_2} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_2} \end{cases}$, $Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho<0}=\begin{cases} \gamma Q_a^{\mathcal{WC}}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_2} \\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_2}\bigcup\mathcal{R}_3^{m_2}\bigcup\mathcal{R}_4^{m_2} \end{cases}$
$\mathbb{E}\left[Q_b^{\mathcal{WC}}\right]$	$\begin{cases} \frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta\lvert\rho\rvert}{2(\gamma^2+2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_2} \\ \frac{c_\eta-\gamma(c_{bw}+2c_Y-2)}{4\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_2} \\ \frac{c_\eta+\gamma-c_Y\gamma+\gamma^2(1-c_X)+\gamma^3(1-c_Y)-\gamma\Delta\lvert\rho\rvert}{2\gamma(2+\gamma^2)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_2} \\ \frac{c_\eta+\gamma-\gamma c_Y}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_2} \end{cases}$
$\pi_{II}^{\mathcal{WC}}$	$\pi^{\mathcal{WC}}-((c_X)Q_a+Q_a(1-Q_a))$
$\pi^{\mathcal{WC}}$	$\begin{cases} \mathcal{L}_3^{l_2}-\frac{(c_\eta+\gamma(1-c_Y-\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{m_2} \\ \frac{\gamma^2\left(\left(\gamma^2+2\right)c_{bw}^2+4c_{bw}(\gamma-\gamma c_X+c_Y-\kappa-1)+4\left(c_X^2-2c_X+c_Y^2-2c_Y+\kappa^2+2\right)\right)-2\gamma c_\eta\left(2\gamma+\gamma^2 cr-2\gamma c_X+2c_Y+2\kappa-2\right)+\left(\gamma^2+2\right)c_\eta^2}{16\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{m_2} \\ \mathcal{L}_3^{l_2}-\frac{(c_{bw}+c_Y-\Delta\lvert\rho\rvert+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{m_2} \\ \mathcal{L}_4^{l_2}+\frac{(c_\eta+\gamma(1-c_Y+\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_4^{m_2} \end{cases}$

Table D.7 Optimal Results in Region H

Parameters	Results in Region H
$Q_a^{\mathcal{WC}}$	$\begin{cases} \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{h} \\ \frac{2+\gamma+\gamma\Delta\lvert\rho\rvert-c_\eta-2c_X-\gamma c_Y}{2\gamma^2+4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{h} \\ \frac{1-c_\eta-c_X}{2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{h} \end{cases}$
$Q_b^{\mathcal{WC}}$	$Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho\geq 0}=\begin{cases} \frac{1-c_{bw}-c_Y+\Delta\lvert\rho\rvert}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{h} \\ \gamma Q_a^{\mathcal{WC}^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{h} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\lvert\rho\rvert}{2\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{h} \end{cases}$, $Q_b^{\mathcal{WC}}\lvert_{\hat{\xi}_a\rho<0}=0.$
$\mathbb{E}\left[Q_b^{\mathcal{WC}}\right]$	$\begin{cases} \frac{1-c_{bw}-c_Y+\Delta\lvert\rho\rvert}{4}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{h} \\ -\frac{\gamma(-\gamma\Delta\lvert\rho\rvert-\gamma+c_\eta+2c_X+\gamma c_Y-2)}{4\left(\gamma^2+2\right)}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{h} \\ \frac{c_\eta+\gamma-\gamma c_Y+\gamma\Delta\lvert\rho\rvert}{4\gamma}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{h} \end{cases}$
$\pi_{II}^{\mathcal{WC}}$	$\pi^{\mathcal{WC}}-((c_X)Q_a+Q_a(1-Q_a))$
$\pi^{\mathcal{WC}}$	$\begin{cases} \frac{(\gamma\Delta\lvert\rho\rvert+\gamma-c_\eta-2c_X-\gamma c_Y+2)^2}{8\gamma^2+16}+\frac{(c_{bw}+c_Y-\Delta\lvert\rho\rvert+2\gamma Q_a-1)^2}{8}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_1^{h} \\ \frac{(\gamma\Delta\lvert\rho\rvert+\gamma-c_\eta-2c_X-\gamma c_Y+2)^2}{8\gamma^2+16}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_2^{h} \\ \frac{(\gamma\Delta\lvert\rho\rvert+\gamma-c_\eta-2c_X-\gamma c_Y+2)^2}{8\gamma^2+16}+\frac{(c_\eta+\gamma(1-c_Y+\Delta\lvert\rho\rvert-2\gamma Q_a))^2}{8\gamma^2}, & \text{if } (\gamma,\rho,\Delta,c_J)\in\mathcal{R}_3^{h} \end{cases}$

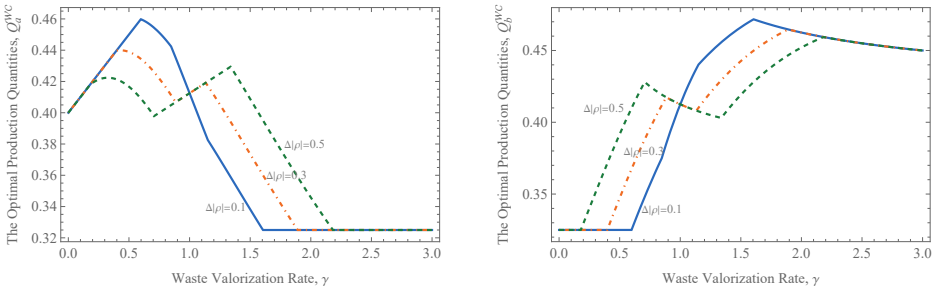
□

PROOF OF COROLLARY 3:

Define $c_{X1}=\frac{2\gamma+\gamma^2c_{bw}+2c_{bw}-\gamma c_\eta+2c_Y-2\Delta\lvert\rho\rvert-2}{2\gamma}$, $c_{X2}=\frac{2\gamma^2-2\gamma+\gamma^3c_{bw}+\gamma c_{bw}-\gamma^2c_\eta-c_\eta+2\gamma c_Y}{2\gamma^2}$, and $c_{X3}=\frac{2\gamma^2+2\gamma\Delta\lvert\rho\rvert-2\gamma+\gamma^3c_{bw}-\gamma^2c_\eta-2c_w+2\gamma c_Y}{2\gamma^2}$. According to the results in Tables D.3 and D.4, it shows that $\frac{2+\gamma-\gamma\Delta\lvert\rho\rvert+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}<\frac{1-c_X+\gamma c_{bw}}{2}$ and $\frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta\lvert\rho\rvert}{2(\gamma^2+2)}>\frac{1-c_{bw}-c_Y}{2}$. Moreover,

$\frac{2+\gamma-\gamma\Delta|\rho|+\gamma c_{bw}-2c_X-\gamma c_Y}{2\gamma^2+4}$ decreases with $\Delta|\rho|$ and $\frac{1-c_{bw}-c_Y+\gamma(1-c_X)+\gamma^2(1-c_Y)+\Delta|\rho|}{2(\gamma^2+2)}$ increases with $\Delta|\rho|$. Hence, with *Pooling Le*, $Q_a^{\mathcal{WC}} < \frac{1-c_X+\gamma c_{bw}}{2}$ and $\mathbb{E}[Q_b^{\mathcal{WC}}] > \frac{1-c_{bw}-c_Y}{2}$. Likewise, we can prove the results with both *Pooling LS* and *Pooling Se*. Corollary 3 follows.

Further, under *Pooling Le*, $Q_a^{\mathcal{WC}}$ ($\mathbb{E}[Q_b^{\mathcal{WC}}]$, resp.) decreases (increases, resp.) with both Δ and $|\rho|$, whereas under *Pooling Se*, $Q_a^{\mathcal{WC}}$ ($\mathbb{E}[Q_b^{\mathcal{WC}}]$, resp.) increases (decreases, resp.) with both Δ and $|\rho|$. These opposite comparative statics reflect how correlation and volatility reshape the balance between waste generation and utilization across demand states. This observation leads to an important finding: the "pool-to-center" effect becomes more pronounced as $\Delta|\rho|$ increases. Intuitively, stronger correlation or greater demand uncertainty intensifies the mismatch between waste generation and utilization across states, which in turn strengthens the incentive for pooling policies to pull both overproduction and underproduction toward economically meaningful center values. We illustrate this effect numerically in Figure D.1.



(a) Optimal quantity for product A (b) Optimal quantity for product B

Figure D.1 Optimal Decisions with Pooling Policies

Note: The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$ and $c_Y = c_\eta = 0.15$.

□

PROOF OF COROLLARY 4:

Based on Proposition 2, we can have the following equations:

$$\frac{\gamma c_{bw} - c_\eta - 2c_X + 2}{4} = \frac{\gamma - c_X - \gamma c_Y + 1}{2\gamma^2 + 2}, \tag{D.6}$$

and

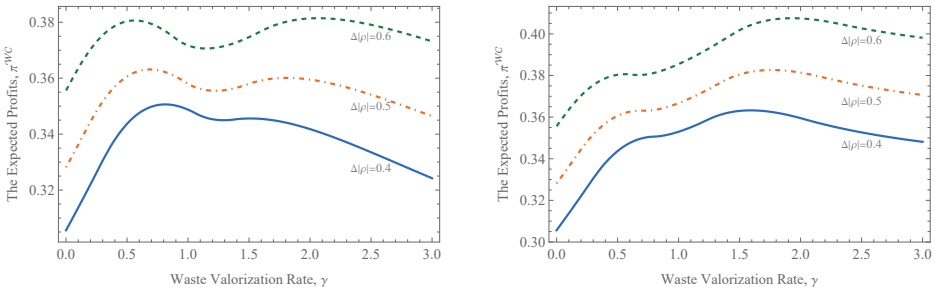
$$\frac{c_\eta - \gamma(c_{bw} + 2c_Y - 2)}{4\gamma} = \frac{\gamma(\gamma - c_X - \gamma c_Y + 1)}{2\gamma^2 + 2}. \tag{D.7}$$

By solving Equations (D.6) and (D.7), we can obtain a unique $\gamma^p = \frac{c_\eta + 2c_X - \sqrt[3]{\sqrt{\mathcal{P}^2+\mathcal{J}^3}+\mathcal{P}} + \frac{\mathcal{J}}{\sqrt[3]{\sqrt{\mathcal{P}^2+\mathcal{J}^3}+\mathcal{P}}} - 2}{3c_{bw}}$, where $\mathcal{P} = 9c_{bw}(c_Y - 1)(c_\eta + 2c_X - 2) - 9c_{bw}^2(c_\eta - c_X + 1) - (c_\eta + 2c_X - 2)^3$ and $\mathcal{J} = 3c_{bw}(c_{bw} + 2c_Y - 2) - (c_\eta + 2c_X - 2)^2$. Moreover, when $\gamma < \gamma^p$, $Q_a^{\mathcal{WC}}$ is lower under *Pooling Le* compared to the case without demand correlation; conversely, when $\gamma > \gamma^p$, $Q_b^{\mathcal{WC}}$ is lower under *Pooling Se*. Consequently, for $\gamma < \gamma^p$, we have $\mathbb{E}[\min\{Q_b, \gamma Q_a\}] = \gamma Q_a^{\mathcal{WC}}$; for $\gamma > \gamma^p$, it holds that $\mathbb{E}[\min\{Q_b, \gamma Q_a\}] = \mathbb{E}[Q_b^{\mathcal{WC}}]$; for $\gamma = \gamma^p$, both expressions yield $\mathbb{E}[\min\{Q_b, \gamma Q_a\}] = \gamma Q_a^{\mathcal{WC}} = \mathbb{E}[Q_b^{\mathcal{WC}}]$. Notably, at $\gamma = \gamma^p$, the optimal production quantities for both products under demand correlation coincide with those in the absence of demand correlation. Therefore, DE and RE are diminished in the presence of demand correlation. Corollary 4 follows. □

PROOF OF COROLLARY 5:

According to Proposition 2, we can have the expected optimal profit for the manufacturer as shown in Tables D.3 and Table D.4. It's easy to show that $\pi^{\mathcal{WC}}$ increases with γ when $Q_b^{\mathcal{WC}} > \gamma Q_a^{\mathcal{WC}}$; and decreases with γ when $Q_b^{\mathcal{WC}} < \gamma Q_a^{\mathcal{WC}}$. In the case of $Q_b^{\mathcal{WC}} = \gamma Q_a^{\mathcal{WC}}$, it shows that there is only one extreme point of $\gamma = \frac{1-c_Y}{1-c_X} \in \mathcal{R}_3^{l_1}$, $\pi^{\mathcal{WC}}$ increases with γ if $\gamma < \frac{1-c_Y}{1-c_X}$, and $\pi^{\mathcal{WC}}$ decreases with γ if $\gamma > \frac{1-c_Y}{1-c_X}$. Likewise, within *Pooling Le*, it shows that there is only one extreme point of $\gamma = \frac{1+c_{bw}-c_Y-\Delta|\rho|}{1-c_X} \in \mathcal{R}_2^{l_1} \bigcup \mathcal{R}_2^{l_2}$, $\pi^{\mathcal{WC}}$ increases with γ if $\gamma < \frac{1+c_{bw}-c_Y-\Delta|\rho|}{1-c_X}$, and $\pi^{\mathcal{WC}}$ decreases with γ if $\gamma > \frac{1+c_{bw}-c_Y-\Delta|\rho|}{1-c_X}$. Within *Pooling LS*, it shows that there is only one extreme point of $\gamma_{x_1} \in \mathcal{R}_3^{l_2}$, where $\gamma_{x_1} = \text{Root}\left[x^4 c_{bw}^2 + x^3(2c_{bw} - c_{bw}c_\eta - 2c_{bw}c_X) + x(2c_\eta c_Y + 2c_\eta\Delta|\rho| - 2c_\eta) - 2c_\eta^2,\ 2\right]$, which $\pi^{\mathcal{WC}}$ decreases with γ if $\gamma < \gamma_{x_1}$, and $\pi^{\mathcal{WC}}$ increases with γ if $\gamma > \gamma_{x_1}$. Within *Pooling Se*, it shows that there is only one extreme point of $\gamma_{x_2} \in \mathcal{R}_4^{l_1} \bigcup \mathcal{R}_4^{l_2}$, where $\gamma_{x_2} = \text{Root}[x^5(c_\eta\Delta|\rho| - c_X c_Y + c_X\Delta|\rho| + c_X + c_Y - \Delta|\rho| - 1) + x^4\left(-c_\eta^2 - 2c_\eta c_X + 2c_\eta - 2c_X^2 + 4c_X + c_Y^2 - 2c_X\Delta|\rho| - 2c_Y + \Delta^2\rho^2 + 2\Delta|\rho| - 1\right) + x^3(3c_\eta c_Y + c_\eta\Delta|\rho| - 3c_\eta + 2c_X c_Y - 2c_X\Delta|\rho| - 2c_X - 2c_Y + 2\Delta|\rho| + 2) - 2x^2c_\eta^2 + x(2c_\eta c_Y + 2c_\eta\Delta|\rho| - 2c_\eta) - 2c_\eta^2,\ 3]$, which $\pi^{\mathcal{WC}}$ increases with γ if $\gamma < \gamma_{x_2}$, and $\pi^{\mathcal{WC}}$ decreases with γ if $\gamma > \gamma_{x_2}$. Hence, $\pi^{\mathcal{WC}}$ has at most two local maxima. Corollary 5 follows.

Moreover, as shown in Figure D.2, the optimal profit $\pi^{\mathcal{WC}}$ increases with $\Delta|\rho|$ across all cases. In the presence of valorization, higher $\Delta|\rho|$ strengthens pooling incentives, which mitigates the mismatch between waste generation and utilization across demand states. As a result, correlation and volatility not only improve information utilization, but also amplify the efficiency gains from production interdependence induced by waste valorization.



(a) $c_d = 0.35$ $(\eta = 0)$ (b) $c_d = 0.15$ $(\eta = 0)$

Figure D.2 Manufacturer's Optimal Profits

Note: The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$ and $c_Y = 0.15$.

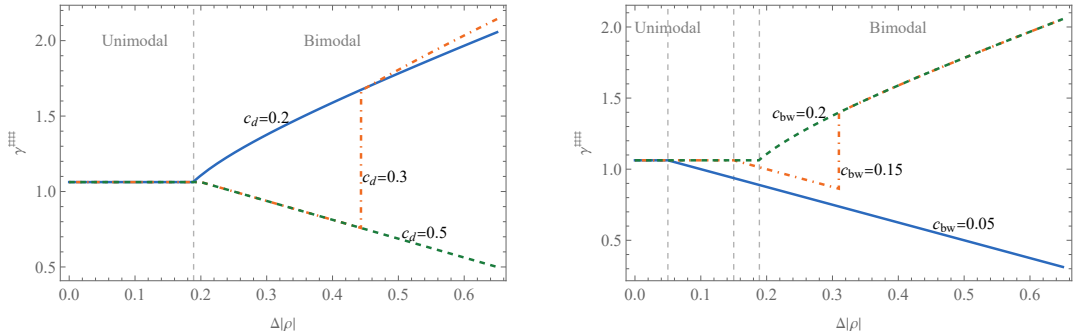
□

PROOF OF COROLLARY 6:

According to Corollary 5, the results indicate that when $\mathcal{T} < \Delta|\rho| < \frac{c_\eta + c_{bw}\gamma}{2\gamma}$ and $c_Y > \frac{(\gamma^2+1)c_\eta - \gamma(2\gamma+\gamma^2 c_{bw}+c_{bw}-2\gamma c_X-2)}{2\gamma}$, or when $\Delta|\rho| > \frac{c_\eta + c_{bw}\gamma}{2\gamma}$ and $c_Y < \frac{(\gamma^2+1)c_\eta - \gamma(2\gamma+\gamma^2 c_{bw}+c_{bw}-2\gamma c_X-2)}{2\gamma}$, the manufacturer achieves higher profit with *Pooling Le* than with *Pooling Se*, and vice versa. Moreover, $\gamma = \frac{1+c_{bw}-c_Y-\Delta|\rho|}{1-c_X}$ (i.e., the optimal with *Pooling Le*) that is independent of c_η; while $\gamma = \gamma_{x_2}$ (i.e., the optimal

with *Pooling Se*) is independent of c_{bw}. According to Table D.3, the manufacturer's profit under *Pooling Le* is independent of c_η and decreases with c_{bw}. Also, the manufacturer's profit under *Pooling Se* is independent of c_{bw} and decreases with c_η. Corollary 6 follows.

We also illustrate the optimal waste valorization rate, $\gamma^{\#\#}$, that achieves the maximum profit in the $(\Delta|\rho|, \gamma^{\#\#})$ plane.



(a) $c_{bw} = 0.20$ (b) $c_\eta = 0.20$

Figure D.3 Impacts of ρ and Δ on $\gamma^{\#\#}$ ($c_X = 0.20$, $c_Y = 0.15$)

We conduct a sub-optimality loss analysis by quantifying the profit deviation incurred when demand correlation is neglected and the recommended pooling-based production quantities are not adopted. Based on empirical observations, the correlation coefficient between products A (pineapple) and B (detergent) is $\rho = -0.72$. We further assume the following parameter values: $c_X = c_{bw} = 0.20$, $c_Y = 0.15$, $c_\eta = 0.10$, and $\Delta = 0.72$. As illustrated in Figure D.4, the LoP exhibits a non-monotonic pattern with respect to γ, initially decreasing then increasing. The minimum loss is approximately 10%, with the maximum value exceeding 25%. Notably, the *Pooling Le* and *Pooling Se* policies yield profit improvements of over 15% relative to the suboptimal decisions.

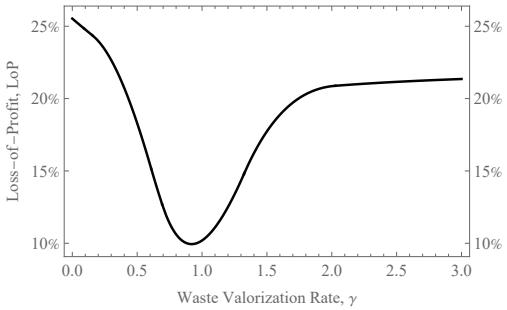


Figure D.4 Sub-optimality Loss Analysis

□

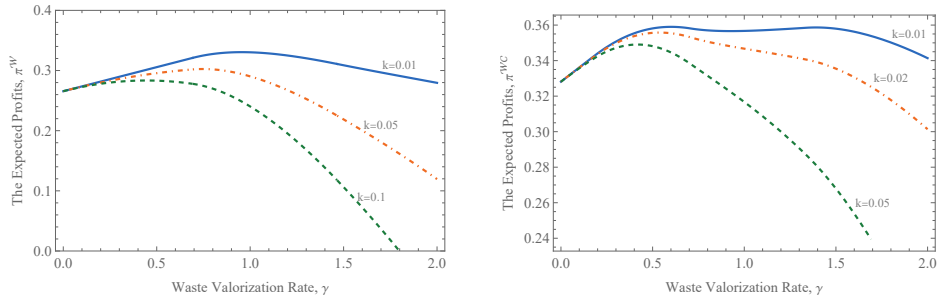
PROOF OF COROLLARY 7: Environmental Effect (EE)

Given the optimal production quantity $Q_a^{\mathcal{B}} = Q_a^{\mathcal{BC}} = \frac{1-c_X-c_\eta}{2}$ in the benchmark, a comparison with the optimal results in Proposition 2 and Corollary 7 follows. □

D.3. Discussion on an Endogenous γ Setting

To further clarify the relevance of our results to an endogenous-technology setting, we now provide a numerical illustration that incorporates a simple convex investment cost, $k\gamma^2$, into the firm's profit function. This exercise is not intended to introduce a new model, but rather to illustrate how our main insights carry over when γ is chosen optimally.

The numerical results show a clear and intuitive pattern as shown in Figure D.5. When demands are uncorrelated ($\rho = 0$), the profit function remains inverse U-shaped in γ, and introducing a convex cost simply shifts the optimal efficiency level without altering the unimodal structure. In contrast, when demands are correlated, the profit function becomes sensitive to the investment cost parameter k: for low k, the M-shaped pattern persists, whereas higher values of k compress the high-efficiency region and may restore a single interior optimum. Importantly, these changes are driven by the same mismatch mechanism between waste generation and utilization identified in our previous analysis.



(a) Optimal Profit $\pi^{\mathcal{W}}$ (b) Optimal Profit $\pi^{\mathcal{WC}}$

Figure D.5 Optimal Profits Under a Convex Cost Function $k\gamma 2$

Note: The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$, $c_Y = c_\eta = 0.15$, and $\Delta|\rho| = 0.5$.

Taken together, this numerical illustration reinforces that our analysis of profit shapes in γ is economically meaningful even when γ is endogenized. While we do not explicitly model technology investment decisions in the main analysis, the regions characterized in the paper directly inform optimal technology choices in such extended settings.

Appendix E: The Analysis for Extensions

E.1. Waste Generation from Both Productions

In this section, we extend our analysis to the case where both production processes generate waste. Specifically, we consider the case where the production of product B (regardless of whether it uses raw or waste materials) generates waste. We denote the economic value of this waste by c'_η, where $c'_\eta > 0$ reflects a unit disposal cost, and $c'_\eta < 0$ indicates a unit positive economic value. All other model assumptions remain consistent with the baseline setting. Our focus lies in comparing scenarios with and without waste valorization, and the insights derived naturally extend to the context of correlated demand.

In the absence of waste valorization, the optimal production quantities are $Q_a = \frac{1-c_X-c_\eta}{2}$ and $Q_b = \frac{1-c_Y-c_{bm}-c'_\eta}{2}$, indicating that the production decisions are mutually independent. However, this independence no longer holds in the presence of waste valorization, as demonstrated in the following proposition.

PROPOSITION E.1. *When both productions generate waste under uncorrelated demand, the optimal production quantity for product j with waste valorization is*

$$\left(Q_a^{\mathcal{W}},\, Q_b^{\mathcal{W}}\right) = \begin{cases} \left(\frac{1-c_X+\gamma c_{bw}}{2},\, \frac{1-c_{bw}-c_Y-c'_\eta}{2}\right), & \text{if } (\gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_1^b \\ \left(\frac{1-\gamma(c'_\eta+c_Y-1)-c_X}{2(\gamma^2+1)},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } (\gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_2^b \\ \left(\frac{1-c_\eta-c_X}{2},\, \frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma}\right), & \text{if } (\gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_3^b \\ \left(\frac{1-c_\eta-c_X}{2},\, 1\right), & \text{if } (\gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_4^b \\ \left(\frac{1}{\gamma},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } (\gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_5^b \end{cases}.$$

where $J \in \{X,\, Y,\, \eta,\, bw,\, \eta'\}$, $\mathcal{R}_i^b$ $(i = 1, 2, ..., 5)$ *is shown in the Appendix.*

Proposition E.1 illustrates the implications of additional waste generation from the production of product B. When $c'_\eta > 0$, representing a disposal cost, the optimal production quantity of product B decreases compared to the benchmark case in Proposition 1. Due to the interdependence between productions, the optimal quantity of product A may also decline. Conversely, when $c'_\eta < 0$, reflecting a positive economic value associated with waste generation, the optimal quantities of both products can increase relative to Proposition 1. However, this production increment may stall or even disrupt the interdependence between the two production processes when $c'_\eta < 0$. In particular, under the condition $Q_b \le \gamma Q_a$, the manufacturer opts to overproduce product B in order to exploit both RE and DE benefits. Yet, Proposition E.1 indicates that such overproduction is constrained when $\frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma} > 1$, primarily owing to the limited potential market size. As a result, the manufacturer either preserves the production interdependence by setting $Q_a = \frac{1}{\gamma}$, or abandons this linkage altogether, opting instead for the monopoly production quantity.

This outcome arises when waste valorization is driven primarily by two factors: the cost-saving potential from reducing disposal fees associated with product A, and the salvage value obtained from the waste generated by product B itself. When the economic gains from waste treatment outweigh the revenue from product B's market sale, manufacturers may adopt waste valorization strategies that are decoupled from

traditional profit motives. A practical manifestation of this phenomenon is the use of waste-derived materials in packaging rather than in standalone products. This approach enables firms to integrate waste valorization in a manner that avoids market oversupply and maintains product price stability.

PROOF OF PROPOSITION E.1:

Since the production of B also generates waste, we denote the economic value of waste per unit of B as c'_η. Analogous to earlier definitions, $c'_\eta > 0$ reflects a unit disposal cost, whereas $c'_\eta < 0$ indicates a unit positive economic value derived from the waste. Our analysis centers on the case where $(\eta, c_s) \in C_D$, as $(\eta, \gamma) \in C_I$ would eliminate production interdependence between A and B. Indeed, when $c'_\eta > 0$, our previous results remain valid, as the positive value of c'_η can be interpreted as an additional processing cost. Accordingly, we redefine the effective unit cost as $c'_Y = c_Y + c'_\eta$. Under this formulation, and in the absence of waste valorization, the optimal production quantity of B without demand correlation is given by $Q_b^{\mathcal{B}} = \frac{1-c_{bm}-c'_Y}{2}$; with demand correlation, the optimal quantity becomes $Q_b^{\mathcal{BC}} = \frac{1-c_{bm}-c'_Y+\rho\hat{\xi}_a}{2}$. However, our previous results might change when $c'_\eta < 0$. With waste valorization, the manufacturer's expected profit function in the second stage is given by

$$\pi_{II}^{\mathcal{W}}(Q_b; Q_a) = \begin{cases} \mathbb{E}[p_b Q_b] - Q_a c_\eta - Q_b\left(c_Y - c_\eta/\gamma - c'_\eta\right), & \text{if } Q_b \le \gamma Q_a \\ \mathbb{E}[p_b Q_b] - Q_b\left(c_{bw} + c_Y - c'_\eta\right) + \gamma Q_a c_{bw}, & \text{if } Q_b > \gamma Q_a \end{cases}. \tag{E.1}$$

The constraint $0 \le Q_b \le 1$ is imposed to ensure a non-negative expected sales price for product B, as $\mathbb{E}[p_b] = 1 - Q_b$. We identify the optimal Q_b by considering two cases:

1. Case 1. $\gamma < \frac{c_\eta}{1+c'_\eta+c_Y}$.
$$Q_b^{\mathcal{W}} = \begin{cases} \frac{1-c_{bw}-c_Y-c'_\eta}{2}, & \text{if } Q_a < \frac{1-c_{bw}-c_Y-c'_\eta}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y-c'_\eta}{2\gamma} \le Q_a < \frac{1}{\gamma} \\ 1, & \text{otherwise} \end{cases},$$
2. Case 2. $\max\left\{\frac{c_\eta}{1+c'_\eta+c_Y}, -\frac{c_{\eta|\eta=1}}{c_{bw}}\right\} \le \gamma$.
$$Q_b^{\mathcal{W}} = \begin{cases} \frac{1-c_{bw}-c_Y-c'_\eta}{2}, & \text{if } Q_a < \frac{1-c_{bw}-c_Y-c'_\eta}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y-c'_\eta}{2\gamma} \le Q_a < \frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma^2} \\ \frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma}, & \text{otherwise} \end{cases},$$

When γ is relatively low (i.e., $\gamma < \frac{c_\eta}{1+c'_\eta+c_Y}$), the manufacturer tends to generate more waste for valorization, potentially pushing Q_b to its upper bound, i.e., $Q_b^{\mathcal{W}} = 1$. In contrast, when γ is relatively high (i.e., $\max\left\{\frac{c_\eta}{1+c'_\eta+c_Y}, -\frac{c_{\eta|\eta=1}}{c_{bw}}\right\} \le \gamma$), our earlier analytical structure largely holds, where Q_b may exceed, equal, or fall short of γQ_a. Based on the optimal determination of Q_b, we now derive the optimal production quantities for both products. To facilitate this, we define the following regions, denoted as $\mathcal{R}_i^b$:

$\mathcal{R}_1^b$: $\left\{(\gamma,\, \rho,\, \Delta,\, c_J) \mid -\frac{c_\eta}{c_{bw}} < \gamma \text{ and } \frac{1-c_X+\gamma c_{bw}}{2} < \frac{1-c_{bw}-c'_\eta-c_Y}{2\gamma}\right\}$,

$\mathcal{R}_2^b$: $\left\{(\gamma,\, \rho,\, \Delta,\, c_J) \mid -\frac{c_\eta}{c_{bw}} < \gamma \text{ and } \frac{1-c_{bw}-c'_\eta-c_Y}{2\gamma} < \frac{1-\gamma(c'_\eta+c_Y-1)-c_X}{2(\gamma^2+1)} < \min\left(\frac{1}{\gamma}, \frac{\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$,

$\mathcal{R}_3^b$: $\left\{(\gamma,\, \rho,\, \Delta,\, c_J) \mid \max\left\{\frac{c_\eta}{1+c'_\eta+c_Y}, -\frac{c_\eta}{c_{bw}}\right\} \le \gamma \text{ and } \frac{1-c_\eta-c_X}{2} > \frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma^2}\right\}$,

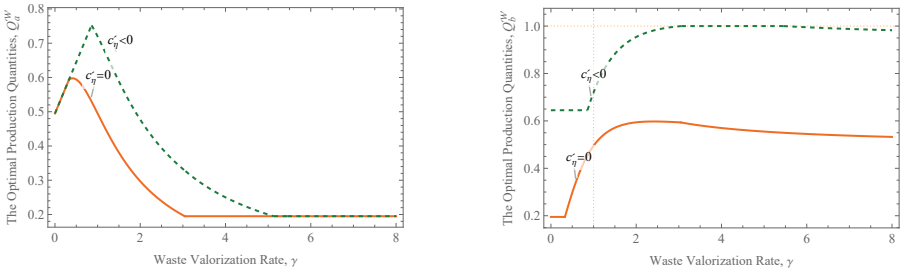
$\mathcal{R}_4^b$: $\left\{(\gamma,\, \rho,\, \Delta,\, c_J) \mid -\frac{c_\eta}{c_{bw}} < \gamma < \frac{c_\eta}{1+c'_\eta+c_Y} \text{ and } \frac{1-c_\eta-c_X}{2} > \frac{1}{\gamma}\right\}$,

$\mathcal{R}_5^b$: $\left\{(\gamma,\,\rho,\,\Delta,\,c_J)\mid -\frac{c_\eta}{c_{bw}}<\gamma<\frac{c_\eta}{1+c'_\eta+c_Y}\text{ and }\frac{1-\gamma(c'_\eta+c_Y-1)-c_X}{2(\gamma^2+1)}>\frac{1}{\gamma}>\frac{1-c_\eta-c_X}{2}\right\}$, where $J\in\{X,Y,bw,\eta,\eta'\}$

The optimal production quantity for product j is

$$\left(Q_a^{\mathcal{W}},\,Q_b^{\mathcal{W}}\right)=\begin{cases}\left(\frac{1-c_X+\gamma c_{bw}}{2},\,\frac{1-c_{bw}-c_Y-c'_\eta}{2}\right), & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_1^b\\ \left(\frac{1-\gamma(c'_\eta+c_Y-1)-c_X}{2(\gamma^2+1)},\,\gamma Q_a^{\mathcal{W}}\right), & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_2^b\\ \left(\frac{1-c_\eta-c_X}{2},\,\frac{c_\eta-(c_Y+c'_\eta-1)\gamma}{2\gamma}\right), & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_3^b\\ \left(\frac{1-c_\eta-c_X}{2},\,1\right), & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_4^b\\ \left(\frac{1}{\gamma},\,\gamma Q_a^{\mathcal{W}}\right), & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_5^b\end{cases}.$$

Therefore, Proposition E.1 follows. We further illustrate the optimal quantity decisions for $Q_a^{\mathcal{W}}$ and $Q_b^{\mathcal{W}}$ with $c'_\eta<0$ in the following figures.



(a) Optimal $Q_a^{\mathcal{W}}$ (b) Optimal $Q_b^{\mathcal{W}}$

Figure E.1 The Manufacturer's Optimal Quantity Decisions

□

We further derive the optimal production quantities under correlated demand in Proposition E.2. The underlying logic parallels that in Proposition E.1, wherein the economic value derived from waste in product B may reduce its production interdependence with product A. Nonetheless, the core insights of our previously proposed pooling policies remain largely valid. We also compare the manufacturer's optimal profit under both uncorrelated and correlated demand scenarios (see Figure E.2). The findings confirm the robustness of Corollaries 1 and 5: while the expected profit remains an inverse U-shaped pattern in the uncorrelated case, it can become M-shaped under demand correlation.

PROPOSITION E.2. *When both productions generate waste under correlated demand, the optimal production quantity for product A is*

$$Q_a^{\mathcal{WC}}=\begin{cases}\frac{1-c_X+\gamma c_{bw}}{2}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_1^{dd1}\cup\mathcal{R}_1^{dd2}\cup\mathcal{R}_1^{dd3}\cup\mathcal{R}_1^{dd4}\cup\mathcal{R}_1^{dd5}\\ \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_2^{dd1}\cup\mathcal{R}_2^{dd2}\cup\mathcal{R}_2^{dd3}\cup\mathcal{R}_2^{dd4}\cup\mathcal{R}_2^{dd5}\\ \frac{1-\gamma(c'_\eta+c_Y-1)-c_X}{2(\gamma^2+1)}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_3^{dd1}\cup\mathcal{R}_3^{dd2}\cup\mathcal{R}_3^{dd4}\\ \frac{2-c_\eta-2c_X+\gamma c_{bw}}{4}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_3^{dd3}\cup\mathcal{R}_3^{dd5}\\ \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_4^{dd2}\cup\mathcal{R}_4^{dd3}\cup\mathcal{R}_4^{dd4}\cup\mathcal{R}_4^{dd5}\\ \frac{1-c_\eta-c_X}{2}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_5^{dd4}\cup\mathcal{R}_5^{dd5}\cup\mathcal{R}_4^{dd1}\cup\mathcal{R}_5^{dd2}\cup\mathcal{R}_5^{dd3}\\ \frac{1}{\gamma}, & \text{if }(\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_5^{dd1}\cup\mathcal{R}_6^{dd2}\cup\mathcal{R}_6^{dd3}\end{cases}.$$

PROOF OF PROPOSITION E.2:

With demand correlation, we first identify the optimal Q_b in two cases.

1. Case 1. $\gamma < \frac{c_\eta}{1+c_Y+c'_\eta-|\rho|\hat{\xi}_a}$.

$$Q_b^{\mathcal{WC}} = \begin{cases} \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2}, & \text{if } Q_a < \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2\gamma} \le Q_a < \frac{1}{\gamma} \\ 1, & \text{otherwise} \end{cases}$$

2. Case 2. $\gamma > \frac{c_\eta}{1+c_Y+c'_\eta-|\rho|\hat{\xi}_a}$

$$Q_b^{\mathcal{WC}} = \begin{cases} \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2}, & \text{if } Q_a < \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2\gamma} \\ \gamma Q_a, & \text{if } \frac{1-c_{bw}-c_Y-c'_\eta+|\rho|\hat{\xi}_a}{2\gamma} \le Q_a < \frac{c_\eta+\gamma-\gamma c_Y-\gamma c'_\eta+\gamma|\rho|\hat{\xi}_a}{2\gamma^2} \\ \frac{c_\eta+\gamma-\gamma c_Y-\gamma c'_\eta+\gamma|\rho|\hat{\xi}_a}{2\gamma}, & \text{otherwise} \end{cases}$$

We derive the optimal solution for Q_a across five cases.

1. Case 1. $\gamma < \frac{c_\eta}{1+c_Y+c'_\eta+\Delta|\rho|}$.

   In case 1, we have five solutions for Q_a. Define the following regions of $\mathcal{R}_i^{dd1}$.

   $\mathcal{R}_1^{dd1}$: $\left\{\frac{1}{2}(\gamma c_{bw} - c_X + 1) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y-\Delta\rho+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+1}{2}$;

   $\mathcal{R}_2^{dd1}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma} < \frac{\gamma(c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1)-2c_X+2}{2(\gamma^2+2)} < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}$;

   $\mathcal{R}_3^{dd1}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma} < -\frac{\gamma(c'_\eta+c_Y-1)+c_X-1}{2(\gamma^2+1)} < \min\left(1, \frac{1}{\gamma}\right)\right\}$ with $Q_a = \frac{1-c_X+\gamma(1-c_Y-c'_\eta)}{2(1+\gamma^2)}$;

   $\mathcal{R}_4^{dd1}$: $\left\{\frac{1}{2}(-c_\eta - c_X + 1) > \frac{1}{\gamma}\right\}$ with $Q_a = \frac{1-c_\eta-c_X}{2}$;

   $\mathcal{R}_5^{dd1}$: $\left\{\frac{\gamma(c'_\eta+c_Y-1)+c_X-1}{2(\gamma^2+1)} > \frac{1}{\gamma} > \frac{1}{2}(-c_\eta - c_X + 1)\right\}$ with $Q_a = \frac{1}{\gamma}$;

2. Case 2. $\frac{c_\eta}{1+c_Y+c'_\eta+\Delta|\rho|} < \gamma < \frac{c_\eta}{1+c_Y+c'_\eta-\Delta|\rho|}$ and $\Delta < \min\left\{\frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\}$.

   In case 2, we have six solutions for Q_a. Define the following regions of $\mathcal{R}_i^{dd2}$.

   $\mathcal{R}_1^{dd2}$: $\left\{\frac{1}{2}(\gamma c_{bw} - c_X + 1) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+1}{2}$;

   $\mathcal{R}_2^{dd2}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma} < \frac{\gamma(c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1)-2c_X+2}{2(\gamma^2+2)} < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}$;

   $\mathcal{R}_3^{dd2}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma} < -\frac{\gamma(c'_\eta+c_Y-1)+c_X-1}{2(\gamma^2+1)} < \min\left(1, \frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{1-c_X+\gamma(1-c_Y-c'_\eta)}{2(1+\gamma^2)}$;

   $\mathcal{R}_4^{dd2}$: $\left\{\frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} < \min\left(1, \frac{1}{\gamma}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4}$;

   $\mathcal{R}_5^{dd2}$: $\left\{\frac{1}{2}(-c_\eta - c_X + 1) > \frac{1}{\gamma}\right\}$ with $Q_a = \frac{1-c_\eta-c_X}{2}$;

   $\mathcal{R}_6^{dd2}$: $\left\{\frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} > \frac{1}{\gamma} > \frac{1}{2}(-c_\eta - c_X + 1)\right\}$ with $Q_a = \frac{1}{\gamma}$;

3. Case 3. $\frac{c_\eta}{1+c_Y+c'_\eta+\Delta|\rho|} < \gamma < \frac{c_\eta}{1+c_Y+c'_\eta-\Delta|\rho|}$ and $\min\left\{\frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}$.
   In case 3, we have six solutions for Q_a. Define the following regions of $\mathcal{R}_i^{dd3}$.
   $\mathcal{R}_1^{dd3}$: $\left\{\frac{1}{2}(\gamma c_{bw}-c_X+1) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+1}{2}$;
   $\mathcal{R}_2^{dd3}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma} < \frac{\gamma(c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1)-2c_X+2}{2(\gamma^2+2)} < \min\left(1, \frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}$;
   $\mathcal{R}_3^{dd3}$: $\left\{\frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{4}(\gamma c_{bw}-c_\eta-2c_X+2) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{2-c_\eta-2c_X+\gamma c_{bw}}{4}$;
   $\mathcal{R}_4^{dd3}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma} < \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} < \min\left(1, \frac{1}{\gamma}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4}$;
   $\mathcal{R}_5^{dd3}$: $\left\{\frac{1}{2}(-c_\eta-c_X+1) > \frac{1}{\gamma}\right\}$ with $Q_a = \frac{1-c_\eta-c_X}{2}$;
   $\mathcal{R}_6^{dd3}$: $\left\{\frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} > \frac{1}{\gamma} > \frac{1}{2}(-c_\eta-c_X+1)\right\}$ with $Q_a = \frac{1}{\gamma}$;
4. Case 4. $\gamma > \frac{c_\eta}{1+c_Y+c'_\eta-\Delta|\rho|}$ and $\Delta < \min\left\{\frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\}$.
   In case 4, we have five solutions for Q_a. Define the following regions of $\mathcal{R}_i^{dd4}$.
   $\mathcal{R}_1^{dd4}$: $\left\{\frac{1}{2}(\gamma c_{bw}-c_X+1) < \min\left(\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma}, 1\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+1}{2}$;
   $\mathcal{R}_2^{dd4}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma} < \frac{\gamma(c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1)-2c_X+2}{2(\gamma^2+2)} < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}$;
   $\mathcal{R}_3^{dd4}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma} < -\frac{\gamma(c'_\eta+c_Y-1)+c_X-1}{2(\gamma^2+1)} < \min\left(1, \frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{1-c_X+\gamma(1-c_Y-c'_\eta)}{2(1+\gamma^2)}$;
   $\mathcal{R}_4^{dd4}$: $\left\{\frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} < \min\left(1, \frac{\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4}$;
   $\mathcal{R}_5^{dd4}$: $\left\{\frac{\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{2}(-c_\eta-c_X+1)\right\}$ with $Q_a = \frac{1-c_\eta-c_X}{2}$;
5. Case 5. $\gamma > \frac{c_\eta}{1+c_Y+c'_\eta-\Delta|\rho|}$ and $\min\left\{\frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{1-c_{bw}-c_Y-c'_\eta}{|\rho|}$.
   In case 5, we have five solutions for Q_a. Define the following regions of $\mathcal{R}_i^{dd5}$.
   $\mathcal{R}_1^{dd5}$: $\left\{\frac{1}{2}(\gamma c_{bw}-c_X+1) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+1}{2}$;
   $\mathcal{R}_2^{dd5}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1}{2\gamma} < \frac{\gamma(c_{bw}-c'_\eta-c_Y-\Delta|\rho|+1)-2c_X+2}{2(\gamma^2+2)} < \min\left(1, \frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-c'_\eta-\Delta|\rho|+1)-2c_X+2}{4+2\gamma^2}$;
   $\mathcal{R}_3^{dd5}$: $\left\{\frac{-\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{4}(\gamma c_{bw}-c_\eta-2c_X+2) < \min\left(1, \frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma}\right)\right\}$ with $Q_a = \frac{2-c_\eta-2c_X+\gamma c_{bw}}{4}$;

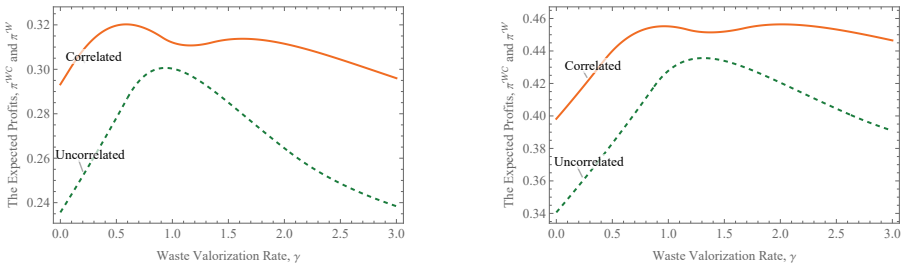
$\mathcal{R}_4^{dd5}$: $\left\{\frac{-c_{bw}-c'_\eta-c_Y+\Delta|\rho|+1}{2\gamma} < \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4} < \min\left(1, \frac{\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|+\gamma-c_\eta-\gamma c'_\eta-2c_X-\gamma c_Y+2}{2\gamma^2+4}$;

$\mathcal{R}_5^{dd5}$: $\left\{\frac{\gamma\Delta|\rho|+\gamma+c_\eta-\gamma c'_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{2}(-c_\eta - c_X + 1)\right\}$ with $Q_a = \frac{1-c_\eta-c_X}{2}$;

Combining the solutions above, we identify some corner solutions with are absent in our base model. In our base model, $Q_b > \gamma Q_a$, $Q_b = \gamma Q_a$, $Q_b < \gamma Q_a$, *Pooling Le*, *Pooling Se*, and *Pooling LS* occur with five interior solutions. Under the case with waste generation from both productions, we list the additional solutions that $Q_b = 1$ or $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+c'_\eta-3+\Delta|\rho|)}{4\gamma}$ as follows.

- $Q_a = \frac{1-c_\eta-c_X}{2}$ and $Q_b = 1$ for $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_4^{dd1}$;
- $Q_a = \frac{1}{\gamma}$ and $Q_b = 1$ for $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_5^{dd1}$;
- $Q_a = \frac{1-c_\eta-c_X}{2}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+c'_\eta-3+\Delta|\rho|)}{4\gamma}$ for $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_5^{dd2}$ or $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_5^{dd3}$;
- $Q_a = \frac{1}{\gamma}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+c'_\eta-3+\Delta|\rho|)}{4\gamma}$ for $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_6^{dd2}$ or $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_6^{dd3}$.

Hence, we have the optimal decisions for the manufacturer. We further illustrate the optimal profit for the manufacturer with correlated demand in the following figures.



(a) Economic Value, $c'_\eta = 0.1$ (b) Economic Value, $c'_\eta = -0.2$

Figure E.2 The Manufacturer's Expected Profits

□

E.2. Asymmetric Markets

In this extension, we generalize the model to account for two products sold in asymmetric markets. Specifically, we consider an inverse demand function of the form $p_j = M_j - E_j Q_j + \xi_j$, where both M_j and E_j are positive constants. This relaxes the normalization adopted in the base model, where $M_j = 1$ and $E_j = 1$. To ensure non-negative prices, we impose the constraint $0 < Q_j < \frac{M_j}{E_j}$. For economic viability, we further assume $-M_a < c_\eta < M_a - c_X$ and $0 < c_Y < M_b - c_{bw}$. The following proposition characterizes the optimal production decisions under viable waste valorization.

PROPOSITION E.3. *When the two products are sold in asymmetric markets under uncorrelated demand, the optimal production quantity for product j with waste valorization is*

$$\left(Q_a^{\mathcal{W}},\, Q_b^{\mathcal{W}}\right) = \begin{cases} \left(\frac{M_a - c_X + c_{bw}\gamma}{2E_a},\, \frac{M_b - c_Y - c_{bw}}{2E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_1^{AM} \\ \left(\frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_2^{AM} \\ \left(\frac{M_a - c_X - c_\eta}{2E_a},\, \frac{c_\eta + (M_b - c_Y)\gamma}{2E_b\gamma}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_3^{AM} \\ \left(\frac{M_a - c_X - c_\eta}{2E_a},\, \frac{M_b}{E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_4^{AM} \\ \left(\frac{M_b}{\gamma E_b},\, \frac{M_b}{E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_5^{AM} \\ \left(\frac{M_a}{E_a},\, \frac{M_b - c_Y - c_{bw}}{2E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_6^{AM} \\ \left(\frac{M_a}{E_a},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_7^{AM} \end{cases}.$$

where $j \in \{a, b\}$ and $J \in \{X,\, \mathrm{Y},\, \eta,\, bw\}$, $\mathcal{R}_i^{AM}$ $(i = 1, 2, ..., 7)$ is shown in the Appendix.

PROOF OF PROPOSITION E.3:

Likewise, when $(\eta, \gamma) \in C_I$, the two products are produced independently, with the optimal quantities are given by $Q_a = \frac{M_a - c_X - c_\eta}{2E_a}$ and $Q_b = \frac{M_b - c_Y - c_{bw}}{2E_b}$. Accordingly, the following analysis focuses on the case with $(\eta, c_s) \in C_D$. We begin by identifying the optimal Q_b in two cases.

1. Case 1. $\gamma < \frac{c_\eta}{M_b + c_Y}$.

$$Q_b^{\mathcal{W}} = \begin{cases} \frac{M_b - c_{bw} - c_Y}{2E_b}, & \text{if } Q_a < \frac{M_b - c_{bw} - c_Y}{2\gamma E_b} \\ \gamma Q_a, & \text{if } \frac{M_b - c_{bw} - c_Y}{2E_b\gamma} \le Q_a < \frac{M_b}{\gamma E_b} \\ \frac{M_b}{E_b}, & \text{otherwise} \end{cases},$$

2. Case 2. $\max\left\{\frac{c_\eta}{M_b + c_Y}, -\frac{c_{\eta|\eta=1}}{c_{bw}}\right\} < \gamma$.

$$Q_b^{\mathcal{W}} = \begin{cases} \frac{M_b - c_{bw} - c_Y}{2E_b}, & \text{if } Q_a < \frac{M_b - c_{bw} - c_Y}{2\gamma E_b} \\ \gamma Q_a, & \text{if } \frac{M_b - c_{bw} - c_Y}{2E_b\gamma} \le Q_a < \frac{c_\eta + M_b\gamma - c_Y\gamma}{2\gamma^2 E_b} \\ \frac{c_\eta + M_b\gamma - c_Y\gamma}{2\gamma E_b}, & \text{otherwise} \end{cases},$$

Based on the optimal Q_b, we derive the optimal production quantities for both products. Define the following regions, denoted as $\mathcal{R}_i^{AM}$:

$\mathcal{R}_1^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid -\frac{c_\eta}{c_{bw}} < \gamma \text{ and } \frac{M_a - c_X + c_{bw}\gamma}{2E_a} < \min\left\{\frac{M_a}{E_a}, \frac{M_b - c_Y - c_{bw}}{2E_b\gamma}\right\}\right\}$,

$\mathcal{R}_2^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid -\frac{c_\eta}{c_{bw}} < \gamma \text{ and } \frac{M_b - c_Y - c_{bw}}{2E_b\gamma} < \frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)} < \min\left\{\frac{M_a}{E_a}, \frac{M_b}{\gamma E_b}, \frac{c_\eta + (M_b - c_Y)\gamma}{2E_b\gamma^2}\right\}\right\}$,

$\mathcal{R}_3^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid \max\left\{\frac{c_\eta}{M_b + c_Y}, -\frac{c_\eta}{c_{bw}}\right\} < \gamma \text{ and } \frac{M_a - c_X - c_\eta}{2E_a} > \frac{c_\eta + (M_b - c_Y)\gamma}{2E_b\gamma^2}\right\}$,

$\mathcal{R}_4^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid \gamma < \frac{c_\eta}{M_b + c_Y} \text{ and } \frac{M_a - c_X - c_\eta}{2E_a} > \frac{M_b}{\gamma E_b}\right\}$,

$\mathcal{R}_5^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid \gamma < \frac{c_\eta}{M_b + c_Y} \text{ and } \frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)} > \frac{M_b}{\gamma E_b} > \frac{M_a - c_X - c_\eta}{2E_a}\right\}$,

$\mathcal{R}_6^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid \max\left\{\frac{c_\eta}{M_b + c_Y}, -\frac{c_\eta}{c_{bw}}\right\} < \gamma \text{ and } \frac{M_a}{E_a} < \min\left\{\frac{M_b - c_Y - c_{bw}}{2E_b\gamma}, \frac{M_a - c_X + c_{bw}\gamma}{2E_a}\right\}\right\}$,

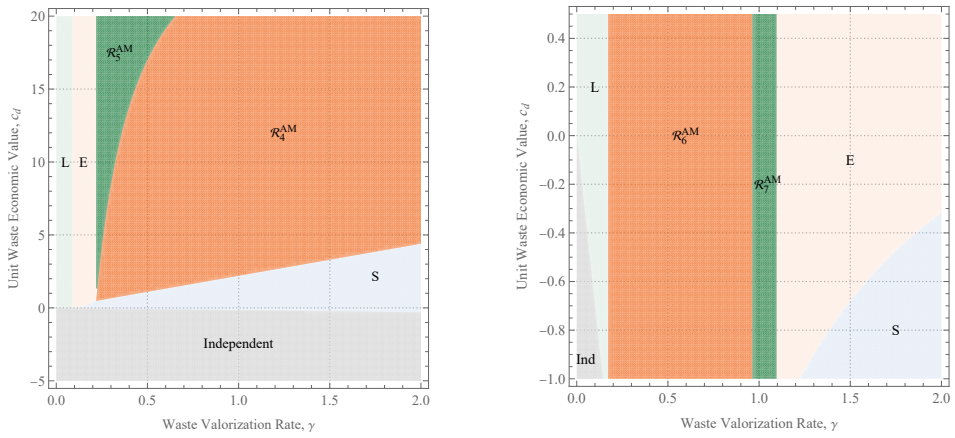
$\mathcal{R}_7^{AM}$: $\left\{\left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \mid \max\left\{\frac{c_\eta}{M_b + c_Y}, -\frac{c_\eta}{c_{bw}}\right\} < \gamma \text{ and } \frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)} > \frac{M_a}{E_a} \text{ and } \frac{M_b - c_Y - c_{bw}}{2E_b\gamma} < \frac{M_a}{E_a} < \frac{c_\eta + (M_b - c_Y)\gamma}{2E_b\gamma^2}\right\}$,

where $j \in \{a, b\}$ and $J \in \{X, \mathrm{Y}, bw, \eta\}$.

The optimal production quantity for product j is

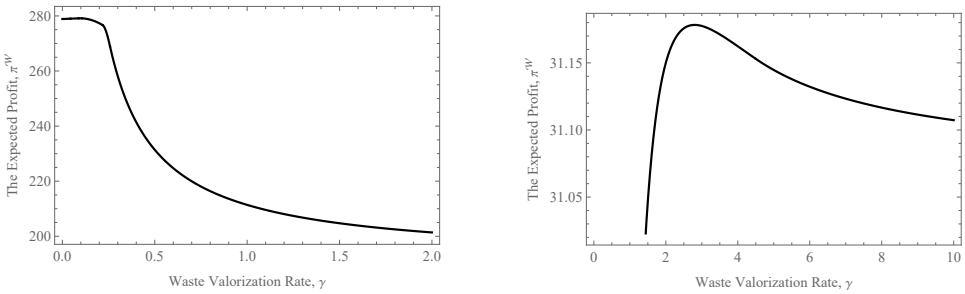
$$\left(Q_a^{\mathcal{W}},\, Q_b^{\mathcal{W}}\right) = \begin{cases} \left(\frac{M_a - c_X + c_{bw}\gamma}{2E_a},\, \frac{M_b - c_Y - c_{bw}}{2E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_1^{AM} \\ \left(\frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_2^{AM} \\ \left(\frac{M_a - c_X - c_\eta}{2E_a},\, \frac{c_\eta + (M_b - c_Y)\gamma}{2E_b\gamma}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_3^{AM} \\ \left(\frac{M_a - c_X - c_\eta}{2E_a},\, \frac{M_b}{E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_4^{AM} \\ \left(\frac{M_b}{\gamma E_b},\, \frac{M_b}{E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_5^{AM} \\ \left(\frac{M_a}{E_a},\, \frac{M_b - c_Y - c_{bw}}{2E_b}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_6^{AM} \\ \left(\frac{M_a}{E_a},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } \left(M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J\right) \in \mathcal{R}_7^{AM} \end{cases}.$$

Therefore, Proposition E.3 follows. We illustrate the optimal policies across different regions in the following figure. Notably, market size and price sensitivity exert opposing effects. We illustrate the findings under asymmetric market sizes, while analogous results hold in scenarios with asymmetric price sensitivities. The main difference between Figures E.3 and 2 lies in the shift from the previously maintained relationship $Q_b = \gamma Q_a$ to new formulations: either $Q_a = \frac{M_a}{E_a}$ when $M_a \ll M_b$, or $Q_b = \frac{M_b}{E_b}$ when $M_a \gg M_b$.



(a) $M_a \gg M_b$ (b) $M_a \ll M_b$

Figure E.3 Optimal Policy with Asymmetric Markets Under Uncorrelated Demand



(a) $M_a \gg M_b$ (b) $M_a \ll M_b$

Figure E.4 Optimal Profit with Asymmetric Markets Under Uncorrelated Demand

We further investigate the manufacturer's profit under asymmetric markets with uncorrelated demand, as depicted in Figure E.4. The results reveal that, despite significant market heterogeneity, the profit function retains its unimodal structure under uncorrelated demand (i.e., $\eta = 0$). Moreover, the profit maximum continues to align with the condition $Q_b = \gamma Q_a$. Notably, market asymmetry can shift the profit-maximizing γ either upward or downward, relative to the benchmark case of symmetric markets. □

When two product's demands are correlated under asymmetric markets, we have the proposition as follows.

PROPOSITION E.4. *When the two products are sold in asymmetric markets under correlated demand, the optimal production quantity for product A with waste valorization is*

$$Q_a^{\mathcal{WC}} = \begin{cases} \frac{M_a - c_X + c_{bw}\gamma}{2E_a}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{a1} \cup \mathcal{R}_1^{a2} \cup \mathcal{R}_1^{a3} \cup \mathcal{R}_1^{a4} \cup \mathcal{R}_1^{a5} \\ \frac{\gamma(c_{bw} - c_Y - \Delta|\rho| + M_b) - 2c_X + 2M_a}{4E_a + 2\gamma^2 E_b}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{a1} \cup \mathcal{R}_2^{a2} \cup \mathcal{R}_2^{a3} \cup \mathcal{R}_2^{a4} \cup \mathcal{R}_2^{a5} \\ \frac{M_a - c_X + (M_b - c_Y)\gamma}{2(E_a + E_b\gamma^2)}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{a1} \cup \mathcal{R}_3^{a2} \cup \mathcal{R}_3^{a4} \\ \frac{\gamma\Delta|\rho| - c_\eta - 2c_X - \gamma c_Y + 2M_a + \gamma M_b}{4E_a + 2\gamma^2 E_b}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{a2} \cup \mathcal{R}_4^{a3} \cup \mathcal{R}_4^{a4} \cup \mathcal{R}_4^{a5} \\ \frac{\gamma c_{bw} - c_\eta - 2c_X + 2M_a}{4E_a}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{a3} \cup \mathcal{R}_3^{a5} \\ \frac{M_a - c_X - c_\eta}{2E_a}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{a4} \cup \mathcal{R}_5^{a5} \cup \mathcal{R}_4^{a1} \cup \mathcal{R}_5^{a2} \cup \mathcal{R}_5^{a3} \\ \frac{M_b}{\gamma E_b}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{a1} \cup \mathcal{R}_6^{a2} \cup \mathcal{R}_6^{a3} \\ \frac{M_a}{E_a}, & if\ (M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_6^{a4} \cup \mathcal{R}_7^{a4} \cup \mathcal{R}_8^{a4} \cup \mathcal{R}_9^{a4} \cup \mathcal{R}_6^{a5} \cup \mathcal{R}_7^{a5} \cup \mathcal{R}_8^{a5} \cup \mathcal{R}_9^{a5} \end{cases}.$$

PROOF OF PROPOSITION E.4:

With demand correlation, we identify the optimal Q_b in two cases.

1. Case 1. $\gamma < \frac{c_\eta}{M_b + c_Y - |\rho|\hat{\xi}_a}$.

$$Q_b^{\mathcal{WC}} = \begin{cases} \frac{M_b - c_{bw} - c_Y + |\rho|\hat{\xi}_a}{2E_b}, & \text{if } Q_a < \frac{M_b - c_{bw} - c_Y + |\rho|\hat{\xi}_a}{2\gamma E_b} \\ \gamma Q_a, & \text{if } \frac{M_b - c_{bw} - c_Y + |\rho|\hat{\xi}_a}{2\gamma E_b} \le Q_a < \frac{M_b}{\gamma E_b} \\ \frac{M_b}{E_b}, & \text{otherwise} \end{cases}$$

2. Case 2. $\gamma > \frac{c_\eta}{M_b + c_Y - |\rho|\hat{\xi}_a}$

$$Q_b^{\mathcal{WC}} = \begin{cases} \frac{M_b - c_{bw} - c_Y + |\rho|\hat{\xi}_a}{2E_b}, & \text{if } Q_a < \frac{M_b - c_{bw} - c_Y + |\rho|\hat{\xi}_a}{2\gamma E_b} \\ \gamma Q_a, & \text{if } \frac{M_b - c_{bw} - c_Y + \rho\hat{\xi}_a}{2\gamma E_b} \le Q_a < \frac{c_\eta + \gamma M_b - \gamma c_Y + \gamma|\rho|\hat{\xi}_a}{2E_b\gamma^2} \\ \frac{c_\eta + \gamma M_b - \gamma c_Y + \gamma|\rho|\hat{\xi}_a}{2E_b\gamma}, & \text{otherwise} \end{cases}$$

We derive the optimal solution for Q_a across five cases.

1. Case 1. $\gamma < \frac{c_\eta}{M_b + c_Y + \Delta|\rho|}$.

   In case 1, we have five solutions for Q_a. Define the following regions of $\mathcal{R}_i^{a1}$.

   $\mathcal{R}_1^{a1}$: $\left\{ \frac{\gamma c_{bw} - c_X + M_a}{2E_a} < \min\left(\frac{-c_{bw} - c_Y - \Delta|\rho| + M_b}{2\gamma E_b}, \frac{M_a}{E_a} \right) \right\}$ with $Q_a = \frac{\gamma c_{bw} - c_X + M_a}{2E_a}$;

   $\mathcal{R}_2^{a1}$: $\left\{ \frac{-c_{bw} - c_Y - \Delta|\rho| + M_b}{2\gamma E_b} < \frac{\gamma(c_{bw} - c_Y - \Delta|\rho| + M_b) - 2c_X + 2M_a}{4E_a + 2\gamma^2 M_b} < \min\left(\frac{-c_{bw} - c_Y + \Delta|\rho| + M_b}{2\gamma E_b}, \frac{M_a}{E_a} \right) \right\}$ with $Q_a = \frac{\gamma(c_{bw} - c_Y - \Delta|\rho| + M_b) - 2c_X + 2M_a}{4E_a + 2\gamma^2 E_b}$;

   $\mathcal{R}_3^{a1}$: $\left\{ \frac{-c_{bw} - c_Y + \Delta|\rho| + M_b}{2\gamma E_b} < \frac{-c_X + \gamma(M_b - c_Y) + M_a}{2(E_a + \gamma^2 E_b)} < \min\left(\frac{M_b}{\gamma E_b}, \frac{M_a}{E_a} \right) \right\}$ with $Q_a = \frac{-c_X + \gamma(M_b - c_Y) + M_a}{2(E_a + \gamma^2 E_b)}$;

   $\mathcal{R}_4^{a1}$: $\left\{ -\frac{c_\eta + c_X - M_a}{2E_a} > \frac{M_b}{\gamma E_b} \right\}$ with $Q_a = \frac{M_a - c_\eta - c_X}{2E_a}$;

   $\mathcal{R}_5^{a1}$: $\left\{ \frac{-c_X + \gamma(M_b - c_Y) + M_a}{2(E_a + \gamma^2 E_b)} > \frac{M_b}{\gamma E_b} > -\frac{c_\eta + c_X - M_a}{2E_a} \right\}$ with $Q_a = \frac{M_b}{\gamma E_b}$;

2. Case 2. $\frac{c_\eta}{M_b+c_Y+\Delta|\rho|} < \gamma < \frac{c_\eta}{M_b+c_Y-\Delta|\rho|}$ and $\Delta < \min\left\{\frac{M_b-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\}$.
In case 2, we have six solutions for Q_a. Define the following regions of $\mathcal{R}_i^{a2}$.
$\mathcal{R}_1^{a2}$: $\left\{\frac{\gamma c_{bw}-c_X+M_a}{2E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+M_a}{2E_a}$;
$\mathcal{R}_2^{a2}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b} < \min\left(\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_3^{a2}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{-c_X+\gamma(M_b-c_Y)+M_a}{2(E_a+\gamma^2E_b)} < \min\left(\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{-c_X+\gamma(M_b-c_Y)+M_a}{2(E_a+\gamma^2E_b)}$;
$\mathcal{R}_4^{a2}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b} < \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b} < \min\left(\frac{M_b}{\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_5^{a2}$: $\left\{-\frac{c_\eta+c_X-M_a}{2E_a} > \frac{M_b}{\gamma E_b}\right\}$ with $Q_a = -\frac{c_\eta+c_X-M_a}{2E_a}$;
$\mathcal{R}_6^{a2}$: $\left\{\frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b} > \frac{M_b}{\gamma E_b} > -\frac{c_\eta+c_X-M_a}{2E_a}\right\}$ with $Q_a = \frac{M_b}{\gamma E_b}$;
3. Case 3. $\frac{c_\eta}{M_b+c_Y+\Delta|\rho|} < \gamma < \frac{c_\eta}{M_b+c_Y-\Delta|\rho|}$ and $\min\left\{\frac{M_b-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{M_b-c_{bw}-c_Y}{|\rho|}$.
In case 3, we have six solutions for Q_a. Define the following regions of $\mathcal{R}_i^{a3}$.
$\mathcal{R}_1^{a3}$: $\left\{\frac{\gamma c_{bw}-c_X+M_a}{2E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+M_a}{2E_a}$;
$\mathcal{R}_2^{a3}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b} < \min\left(\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_3^{a3}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b} < -\frac{-\gamma c_{bw}+c_\eta+2c_X-2M_a}{4E_a} < \min\left(\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = -\frac{-\gamma c_{bw}+c_\eta+2c_X-2M_a}{4E_a}$;
$\mathcal{R}_4^{a3}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b} < \min\left(\frac{M_b}{\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_5^{a3}$: $\left\{-\frac{c_\eta+c_X-M_a}{2E_a} > \frac{M_b}{\gamma E_b}\right\}$ with $Q_a = -\frac{c_\eta+c_X-M_a}{2E_a}$;
$\mathcal{R}_6^{a3}$: $\left\{\frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b} > \frac{M_b}{\gamma E_b} > -\frac{c_\eta+c_X-M_a}{2E_a}\right\}$ with $Q_a = \frac{M_b}{\gamma E_b}$;
4. Case 4. $\gamma > \frac{c_\eta}{M_b+c_Y-\Delta|\rho|}$ and $\Delta < \min\left\{\frac{M_b-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\}$.
In case 4, we have nine solutions for Q_a. Define the following regions of $\mathcal{R}_i^{a4}$.
$\mathcal{R}_1^{a4}$: $\left\{\frac{\gamma c_{bw}-c_X+M_a}{2E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+M_a}{2E_a}$;
$\mathcal{R}_2^{a4}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b} < \min\left(\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_3^{a4}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{-c_X+\gamma(M_b-c_Y)+M_a}{2(E_a+\gamma^2E_b)} < \min\left(\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{-c_X+\gamma(M_b-c_Y)+M_a}{2(E_a+\gamma^2E_b)}$;
$\mathcal{R}_4^{a4}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2E_b} < \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b} < \min\left(\frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2E_b}$;
$\mathcal{R}_5^{a4}$: $\left\{\frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2E_b} < -\frac{c_\eta+c_X-M_a}{2E_a}\right\}$ with $Q_a = -\frac{c_\eta+c_X-M_a}{2E_a}$;
$\mathcal{R}_6^{a4}$: $\left\{\frac{M_a}{E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{\gamma c_{bw}-c_X+M_a}{2E_a}\right)\right\}$ with $Q_a = \frac{M_a}{E_a}$;
$\mathcal{R}_7^{a4}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{M_a}{E_a} < \frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b}$ and $\frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2E_b} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;

$\mathcal{R}_8^{a4}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{M_a}{E_a} < \frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b} \text{ and } \frac{-c_X+\gamma(M_b-c_Y)+M_a}{2(E_a+\gamma^2 E_b)} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;
$\mathcal{R}_9^{a4}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b} < \frac{M_a}{E_a} < \frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2 E_b} \text{ and } \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2 E_b} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;

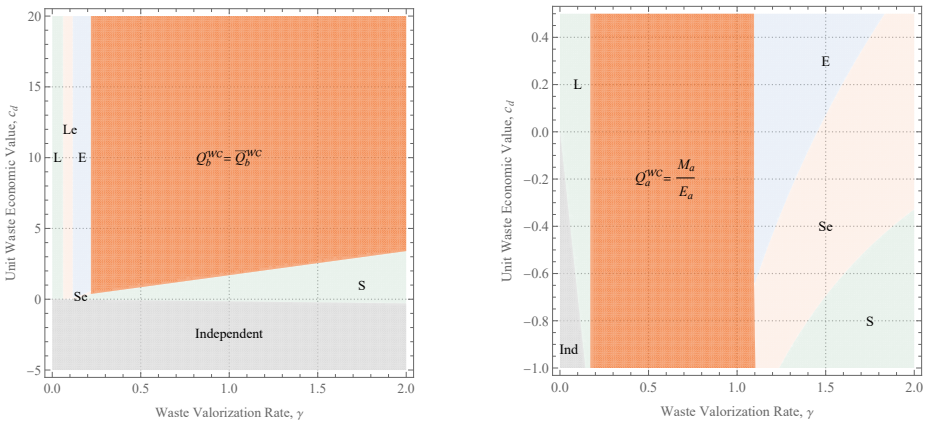
5. Case 5. $\gamma > \frac{c_\eta}{M_b+c_Y-\Delta|\rho|}$ and $\min\left\{\frac{M_b-c_{bw}-c_Y}{|\rho|}, \frac{c_\eta+c_{bw}\gamma}{2\gamma|\rho|}\right\} < \Delta < \frac{M_b-c_{bw}-c_Y}{|\rho|}$.
In case 4, we have nine solutions for Q_a. Define the following regions of $\mathcal{R}_i^{a5}$.
$\mathcal{R}_1^{a5}$: $\left\{\frac{\gamma c_{bw}-c_X+M_a}{2E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma c_{bw}-c_X+M_a}{2E_a}$;
$\mathcal{R}_2^{a5}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2 E_b} < \min\left(\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2 E_b}$;
$\mathcal{R}_3^{a5}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b} < -\frac{-\gamma c_{bw}+c_\eta+2c_X-2M_a}{4E_a} < \min\left(\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = -\frac{-\gamma c_{bw}+c_\eta+2c_X-2M_a}{4E_a}$;
$\mathcal{R}_4^{a5}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2 E_b} < \min\left(\frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2 E_b}, \frac{M_a}{E_a}\right)\right\}$ with $Q_a = \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2 E_b}$;
$\mathcal{R}_5^{a5}$: $\left\{\frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2 E_b} < -\frac{c_\eta+c_X-M_a}{2E_a}\right\}$ with $Q_a = -\frac{c_\eta+c_X-M_a}{2E_a}$;
$\mathcal{R}_6^{a5}$: $\left\{\frac{M_a}{E_a} < \min\left(\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b}, \frac{\gamma c_{bw}-c_X+M_a}{2E_a}\right)\right\}$ with $Q_a = \frac{M_a}{E_a}$;
$\mathcal{R}_7^{a5}$: $\left\{\frac{-c_{bw}-c_Y-\Delta|\rho|+M_b}{2\gamma E_b} < \frac{M_a}{E_a} < \frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b} \text{ and } \frac{\gamma(c_{bw}-c_Y-\Delta|\rho|+M_b)-2c_X+2M_a}{4E_a+2\gamma^2 E_b} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;
$\mathcal{R}_8^{a5}$: $\left\{\frac{c_\eta+\gamma(-c_Y-\Delta|\rho|+M_b)}{2\gamma^2 E_b} < \frac{M_a}{E_a} < \frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} \text{ and } -\frac{-\gamma c_{bw}+c_\eta+2c_X-2M_a}{4E_a} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;
$\mathcal{R}_9^{a5}$: $\left\{\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{2\gamma E_b} < \frac{M_a}{E_a} < \frac{c_\eta+\gamma(-c_Y+\Delta|\rho|+M_b)}{2\gamma^2 E_b} \text{ and } \frac{\gamma\Delta|\rho|-c_\eta-2c_X-\gamma c_Y+2M_a+\gamma M_b}{4E_a+2\gamma^2 E_b} > \frac{M_a}{E_a}\right\}$ with $Q_a = \frac{M_a}{E_a}$;

Combining the solutions above, we identify some corner solutions which are absent in our base model. In our base model, we have five interior solutions including $Q_b > \gamma Q_a$, $Q_b = \gamma Q_a$, $Q_b < \gamma Q_a$, *Pooling Le*, *Pooling Se*, and *Pooling LS*. Under asymmetric markets and correlated demand, we list the additional solutions as follows.

- $Q_a = \frac{M_a-c_\eta-c_X}{2E_a}$ and $Q_b = \frac{M_b}{E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{a1}$;
- $Q_a = \frac{M_b}{\gamma E_b}$ and $Q_b = \frac{M_b}{E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{a1}$;
- $Q_a = \frac{M_a-c_\eta-c_X}{2E_a}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+\Delta|\rho|-3M_b)}{4\gamma E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{a2}$ or $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{a3}$;
- $Q_a = \frac{M_b}{\gamma E_b}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+\Delta|\rho|-3M_b)}{4\gamma E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_6^{a2}$ or $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_6^{a3}$;
- $Q_a = \frac{M_a}{E_a}$ and $\mathbb{E}[Q_b] = -\frac{c_{bw}+c_Y-M_b}{2E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_6^{a4}$ or $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_6^{a5}$;
- $Q_a = \frac{M_a}{E_a}$ and $\mathbb{E}[Q_b] = \frac{1}{4}\left(\frac{-c_{bw}-c_Y+\Delta|\rho|+M_b}{E_b} + \frac{2\gamma M_a}{E_a}\right)$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_7^{a4}$ or $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_7^{a5}$
- $Q_a = \frac{M_a}{E_a}$ and $Q_b = \frac{\gamma M_a}{E_a}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_8^{a4}$;
- $Q_a = \frac{M_a}{E_a}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_{bw}+2c_Y-2M_b)}{4\gamma E_b}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_8^{a5}$;
- $Q_a = \frac{M_a}{E_a}$ and $\mathbb{E}[Q_b] = \frac{c_\eta-\gamma(c_Y+\Delta|\rho|-M_b)}{4\gamma E_b} + \frac{\gamma M_a}{2E_a}$ for $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_9^{a4}$ or $(M_j,\ E_j,\ \gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_9^{a5}$.

Hence, we have the optimal decisions for the manufacturer. To better illustrate the policy regions, we define that

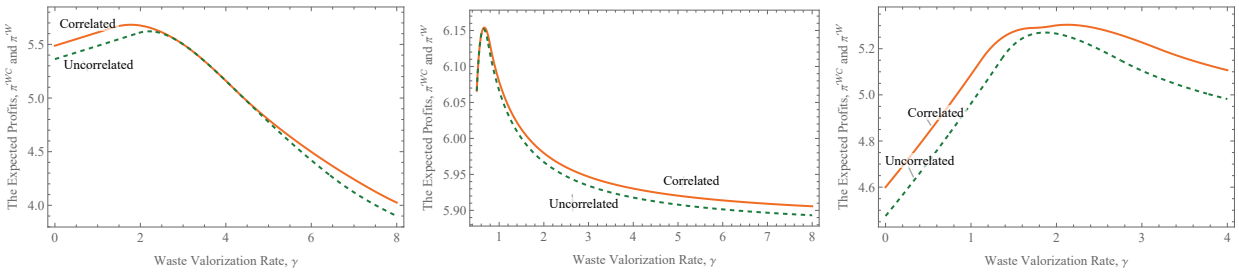
$$\bar{Q}_b^{\mathcal{WC}} = \begin{cases} \frac{M_b}{E_b}, & \text{if } (M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_4^{a1} \cup \mathcal{R}_5^{a1} \\ \frac{c_\eta - \gamma(c_\Upsilon + \Delta|\rho| - 3M_b)}{4\gamma E_b}, & \text{if } (M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_5^{a2} \cup \mathcal{R}_5^{a3} \\ \frac{c_\eta - \gamma(c_\Upsilon + \Delta|\rho| - 3M_b)}{4\gamma E_b}, & \text{if } (M_j,\, E_j,\, \gamma,\, \rho,\, \Delta,\, c_J) \in \mathcal{R}_6^{a2} \cup \mathcal{R}_6^{a3} \end{cases}.$$



(a) $M_a \gg M_b$ (b) $M_a \ll M_b$

Figure E.5 Optimal Policy with Asymmetric Markets and Correlated Demand

As illustrated in Figure E.5, similar patterns arise under asymmetric markets, following the same rationale as in the uncorrelated demand case. Two corner solutions emerge: either $Q_a^{\mathcal{WC}} = \frac{M_a}{E_a}$ or $Q_b = \bar{Q}_b^{\mathcal{WC}}$. We further explore the implications for the manufacturer's profit in the subsequent figure.



(a) $M_a \gg M_b$ (b) $M_a \ll M_b$ (c) $M_a \sim M_b$

Figure E.6 Optimal Profit with Asymmetric Markets and Correlated Demand

As illustrated in Figure E.6, market asymmetry can significantly dampen production interdependence and thereby reduce profits. This constrained interdependence leads to two notable effects. First, as depicted in Figures 6(a) and 6(a), the profit gains under correlated demand diminish in the presence of substantial market heterogeneity, relative to the case with symmetric market structures. This is primarily because boundary production solutions restrict overproduction and hinder the effectiveness of pooling strategies. Second, the constrained interdependence tends to produce an inverse U-shaped profit pattern with respect

to γ, even when demand is correlated. However, as shown in Figure 6(c), when market heterogeneity is reduced, pooling policies become more effective, and the manufacturer's profit once again exhibits an M-shaped pattern, consistent with the findings in Corollary 5. □

E.3. Asymmetric Discrete Distribution

A relatively general assumption regarding the asymmetric distribution of ϕ_j can be characterized as follows:

$$\phi_j = \begin{cases} \Delta_1, & \text{with probability of } \theta \\ -\Delta_2, & \text{with probability of } 1-\theta \end{cases}, \quad j \in \{a, b\}. \tag{E.2}$$

Essentially, this formulation is equivalent to considering either $\Delta_1 \neq \Delta_2$ (with $\Delta_i > 0$ for $i = \{1, 2\}$) or $\theta \neq 1/2$. We confine our analysis to the cases where $\theta \neq 1/2$ and $\Delta_1 = \Delta_2 = \Delta$. Hence, we have that $\mathbb{E}(\phi_j) = (2\theta - 1)\Delta := \mu$ and $Var(\phi_j) = 4\theta(1-\theta)\Delta^2 := \sigma^2$, which derives $\theta = \frac{\mu^2+\sigma^2+\mu\sqrt{\mu^2+\sigma^2}}{2(\mu^2+\sigma^2)}$ and $\Delta = \sqrt{\mu^2+\sigma^2}$. Our analysis shows that the principle of pooling applies even with asymmetric distributions, as illustrated in the $(\gamma, |\rho|)$ plane in Figure E.7. The results highlight that pooling policies are especially significant when μ is small or σ is large.

COROLLARY E.1. *When the two products' demands are correlated with asymmetric discrete distribution,*

a). $Q_a^{\mathcal{WC}}$ decreases with $|\rho|$ under Pooling Le, while it increases with $|\rho|$ under Pooling Se;

b). If $\theta \in (\theta_1,\ \theta_2)$, $\mathbb{E}[Q_b^{\mathcal{WC}}]$ increases with $|\rho|$ under Pooling Le, while it decreases with $|\rho|$ under Pooling Se, where $\theta_1 = \frac{3\gamma^2+1}{4\gamma^2} - \frac{1}{4}\sqrt{\frac{\gamma^4+6\gamma^2+1}{\gamma^4}}$ and $\theta_2 = \frac{\gamma^2-1}{4\gamma^2} + \frac{1}{4}\sqrt{\frac{\gamma^4+6\gamma^2+1}{\gamma^4}}$;

c). The manufacturer's expected profit exhibits a bimodal distribution with respect to γ, attaining maximum profit when Pooling Le or Pooling Se occurs.

Corollary E.1 presents a finding akin to that of Corollary 3, demonstrating that the pooling policy effectively aligns the production of both products and exhibits a "pool-to-center" pattern influenced by demand correlation. Notably, $Q_a^{\mathcal{WC}}$ shows a monotonic relationship with $|\rho|$ under both *Pooling Le* and *Pooling Se*; however, $Q_b^{\mathcal{WC}}$ may not. This discrepancy arises because the mean value μ is affected by θ: when θ is either very small or very large, the production of product B becomes more deterministic. Nevertheless, Corollary E.1(c) shows that an asymmetric discrete distribution does not affect the bimodal nature identified with our earlier result in Corollary 5. This finding further indicates that the "go-to-extreme" pattern remains robust in the presence of demand correlation.

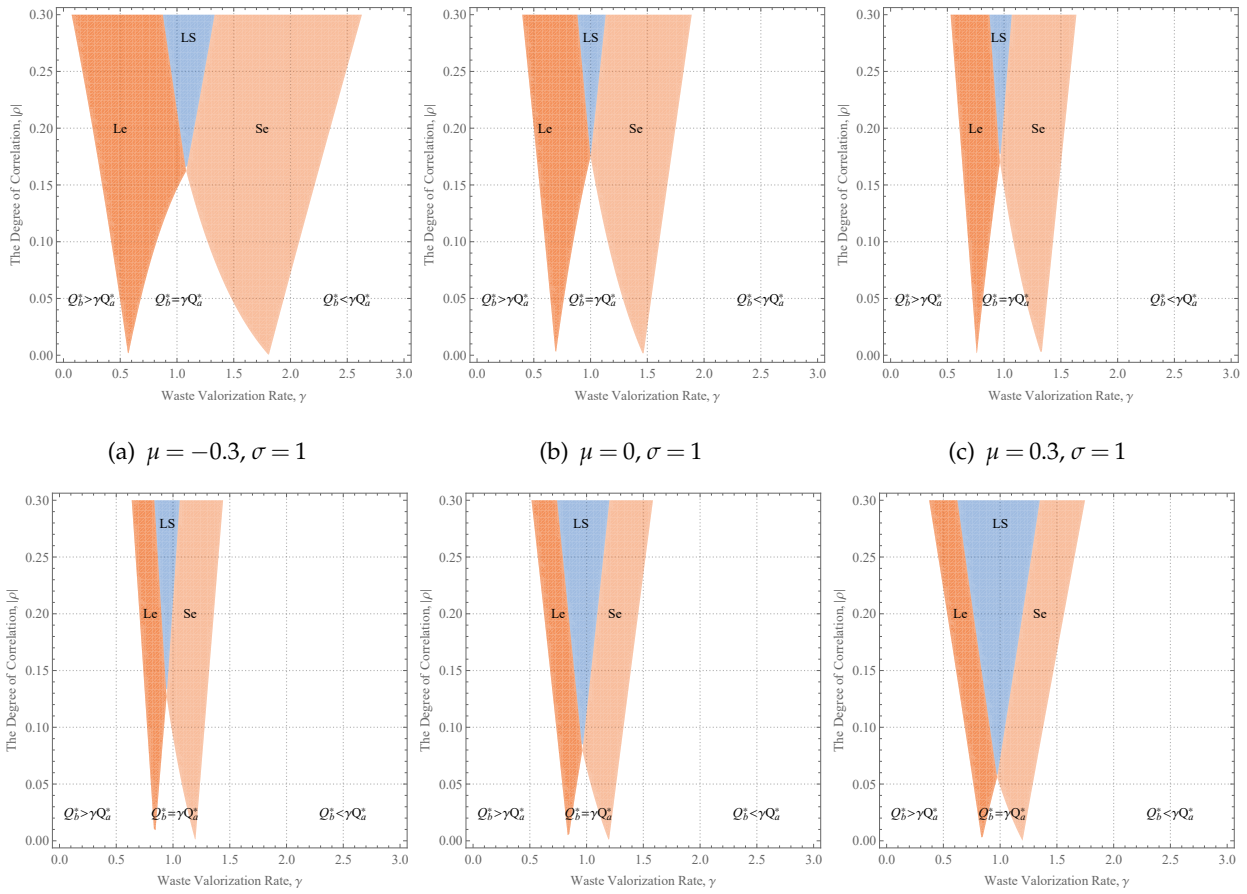
PROOF OF COROLLARY E.1:

When the two products' demands are uncorrelated, our result in Proposition 1 qualitatively holds. The optimal production quantity for products j is

$$\left(Q_a^{\mathcal{W}},\ Q_b^{\mathcal{W}}\right) = \begin{cases} \left(\frac{1-c_X+\gamma c_{bw}+\mu}{2}, \frac{1+\mu-c_{bw}-c_Y}{2}\right), & \text{if } 0 < \gamma < \hat{\gamma}_A \\ \left(\frac{1-c_X+(1+\mu)\gamma-\gamma c_Y+\mu}{2(1+\gamma^2)},\ \gamma Q_a^{\mathcal{W}}\right), & \text{if } \hat{\gamma} \leq \gamma < \bar{\gamma}_A\,, \\ \left(\frac{1-c_X-c_\eta+\mu}{2},\ \frac{c_\eta+\gamma(1+\mu)-\gamma c_Y}{2\gamma}\right), & \text{otherwise} \end{cases}$$

where $\hat{\gamma}_A = \frac{c_X-(1+\mu)+\sqrt{(1-c_X+\mu)^2+4c_{bw}(1+\mu-c_{bw}-c_Y)}}{2c_{bw}}$ and $\bar{\gamma}_A = \frac{1+\mu-c_Y+\sqrt{(1+\mu-c_Y)^2+4c_\eta(1+\mu-c_\eta-c_X)}}{2(1+\mu-c_\eta-c_X)}$. In this case, the optimal decision only account for the mean value of μ, because of the uncorrelated demands for

both products. Likewise, we can have an inverse U-shaped pattern for the manufacturer's profit due to the trade-off between DE and RE. When $Q_b^{\mathcal{W}} = \gamma Q_a^{\mathcal{W}}$, the manufacturer can attain the profit maximum with $\gamma^{\#} = \frac{1-c_Y}{1-c_X}$, which is consistent with our findings in Corollary 1.



(a) $\mu = -0.3, \sigma = 1$ (b) $\mu = 0, \sigma = 1$ (c) $\mu = 0.3, \sigma = 1$

(d) $\mu = 1, \sigma = 1$ (e) $\mu = 1, \sigma = 2$ (f) $\mu = 1, \sigma = 3$

Figure E.7 Optimal Policy Under Asymmetric Distribution

Note: 'Le' denotes *Pooling Le*, 'Se' denotes *Pooling Se*, and 'LS' denotes *Pooling LS*. The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$, $c_Y = c_\eta = 0.15$.

When the two products' demands are correlated, parallel with Proposition 2, the $Q_a^{\mathcal{WC}}$ with asymmetric distribution is given by following:

$$Q_a^{\mathcal{WC}} = \begin{cases} \frac{1+\mu-c_X+\gamma c_{bw}}{2}\ (Q_b^{\mathcal{WC}} > \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A1}^{LH} \\ \frac{\gamma(\theta(1-c_{bw}-\Delta|\rho|+\mu)-c_Y\theta+c_Y+\Delta|\rho|-\mu-1)+c_X-\mu-1}{2\gamma^2(\theta-1)-2}\ (\textit{Pooling Le}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A2}^{LH} \\ \frac{1+\mu-c_X+\gamma(1-c_Y+\mu-\Delta|\rho|+2\theta\Delta|\rho|)}{2(1+\gamma^2)}\ (Q_b^{\mathcal{WC}} = \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{L} \\ \frac{\gamma c_{bw}\theta+c_\eta(\theta-1)-c_X+\mu+1}{2}\ (\textit{Pooling LS}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{H} \\ \frac{\gamma\theta\Delta|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1}{2\gamma^2\theta+2}\ (\textit{Pooling Se}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A4}^{LH} \\ \frac{1+\mu-c_\eta-c_X}{2}\ (Q_b^{\mathcal{WC}} < \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A5}^{LH} \end{cases},$$

where $J \in \{X,\ \mathrm{Y},\ \eta,\ bw\}$, $\mathcal{R}_{Ai}^{LH}$, $\mathcal{R}_{A3}^{L}$, and $\mathcal{R}_{A3}^{H}$ $(i = 1,2,4,5)$ are shown later.

Given the optimal production quantity $Q_a^{\mathcal{WC}}$ as follows:

$$Q_a^{\mathcal{WC}} = \begin{cases} \frac{1+\mu-c_X+\gamma c_{bw}}{2}\ (Q_b^{\mathcal{WC}} > \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A1}^{LH} \\ \frac{\gamma(\theta(1-c_{bw}-\Delta|\rho|+\mu)-c_Y\theta+c_Y+\Delta|\rho|-\mu-1)+c_X-\mu-1}{2\gamma^2(\theta-1)-2}\ (\textit{Pooling Le}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A2}^{LH} \\ \frac{1+\mu-c_X+\gamma(1-c_Y+\mu-\Delta|\rho|+2\theta\Delta|\rho|)}{2(1+\gamma^2)}\ (Q_b^{\mathcal{WC}} = \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{L} \\ \frac{\gamma c_{bw}\theta+c_\eta(\theta-1)-c_X+\mu+1}{2}\ (\textit{Pooling LS}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{H} \\ \frac{\gamma\theta\Delta|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1}{2\gamma^2\theta+2}\ (\textit{Pooling Se}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A4}^{LH} \\ \frac{1+\mu-c_\eta-c_X}{2}\ (Q_b^{\mathcal{WC}} < \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A5}^{LH} \end{cases},$$

and

$$\mathbb{E}[Q_b^{\mathcal{WC}}] = \begin{cases} \frac{1-c_{bw}-c_Y+2\Delta\theta|\rho|-\Delta|\rho|+\mu}{2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A1}^{LH} \\ \frac{\theta(c_{bw}+c_Y-\Delta|\rho|-\mu-1)-\gamma(\theta-1)(c_X-\mu-1)+\gamma^2(\theta-1)(1-c_Y+2\Delta\theta|\rho|-\Delta|\rho|+\mu)}{2\gamma^2(\theta-1)-2}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A2}^{LH} \\ \frac{\gamma(1-c_X+\gamma(-c_Y+2\Delta\theta|\rho|-\Delta|\rho|+\mu+1)+\mu)}{2(\gamma^2+1)}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{L} \\ \frac{c_\eta-\gamma(c_{bw}\theta+c_Y-2\Delta\theta|\rho|+\Delta|\rho|-\mu-1)-c_\eta\theta}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A3}^{H} \\ \frac{\gamma\theta(\gamma\Delta\theta|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1)}{2\gamma^2\theta+2} - \frac{(\theta-1)(c_\eta+\gamma(1-c_Y-\Delta|\rho|+\mu))}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A4}^{LH} \\ \frac{c_\eta+\gamma(1-c_Y+2\Delta\theta|\rho|-\Delta|\rho|+\mu)}{2\gamma}, & \text{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_{A5}^{LH} \end{cases},$$

where $\mathcal{R}_{A1}^{LH}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{\gamma c_{bw}-c_X+\mu+1}{2} < \frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu}{2\gamma}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{\gamma c_{bw}-c_X+\mu+1}{2} < \frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu}{2\gamma}\}$,
$\mathcal{R}_{A2}^{LH}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{-c_{bw}-c_Y-\Delta|\rho|+\mu+1}{2\gamma} < \frac{\gamma(\theta(-c_{bw}-\Delta|\rho|+\mu+1)+c_Y(-\theta)+c_Y+\Delta|\rho|-\mu-1)+c_X-\mu-1}{2\gamma^2(\theta-1)-2} < \frac{-c_{bw}-c_Y+\Delta|\rho|+\mu+1}{2\gamma}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{-c_{bw}-c_Y-\Delta|\rho|+\mu+1}{2\gamma} < \frac{\gamma(\theta(-c_Y-\Delta|\rho|+\mu+1)+c_Y(-\theta)+c_Y+\Delta|\rho|-\mu-1)+c_X-\mu-1}{2\gamma^2(\theta-1)-2} < \frac{-\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}\}$,
$\mathcal{R}_{A3}^{L}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{-c_{bw}-c_Y+\Delta|\rho|+\mu+1}{2\gamma} < \frac{-c_X+\gamma(-c_Y+2\Delta\theta|\rho|-\Delta\rho+\mu+1)+\mu+1}{2(\gamma^2+1)} < \frac{-\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}\}$,
$\mathcal{R}_{A3}^{H}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{-\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{2}(\gamma c_{bw}\theta + c_w(\theta-1) - c_X + \mu + 1) < \frac{-c_{bw}-c_Y+\Delta|\rho|+\mu+1}{2\gamma}\}$,
$\mathcal{R}_{A4}^{LH}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{-\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < \frac{\gamma\Delta\theta|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1}{2\gamma^2\theta+2} < \frac{\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}$,or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{-c_{bw}-c_Y+\Delta|\rho|+\mu+1}{2\gamma} < \frac{\gamma\Delta\theta|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1}{2\gamma^2\theta+2} < \frac{\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}\}$,
$\mathcal{R}_{A5}^{LH}$: $\{(\gamma,\ \rho,\ \Delta,\ c_J)\,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{2}(-c_\eta - c_X + \mu + 1)$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{\gamma\Delta|\rho|+\gamma\mu+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < \frac{1}{2}(-c_\eta - c_X + \mu + 1)\}$.
Hence, it is straight forward to have the monotonicity for either $Q_a^{\mathcal{WC}}$ or $\mathbb{E}[Q_b^{\mathcal{WC}}]$. Corollary E.1 follows. □

PROOF: Alternative Correlation Formulation with $\xi_b = \rho\xi_a + (1-\rho)\phi_b$.

When demands are uncorrelated ($\rho = 0$), $\mathbb{E}[\xi_b] = 0$ under both formulations. When $\rho \neq 0$, $\mathbb{E}[\xi_b] = \rho\hat{\xi}_a$, which is a constant from the perspective of the decision variable Q_b. As a result, the optimal decisions and comparative statics are largely unaffected.

Next, we explicitly verify that our key results continue to hold under the alternative specification $\xi_b = \rho\xi_a + (1-\rho)\phi_b$, allowing for asymmetric discrete distributions as in Section 5.3.1. These additional analyses confirm that our conclusions regarding correlated demands are robust to the choice of correlation formulation.

When the two products' demands are uncorrelated, our result qualitatively holds. The optimal production quantity for products j is

$$\left(Q_a^{\mathcal{W}},\, Q_b^{\mathcal{W}}\right) = \begin{cases} \left(\frac{1-c_X+\gamma c_{bw}+\mu}{2}, \frac{1+\mu-c_{bw}-c_Y}{2}\right), & \text{if } 0<\gamma<\hat{\gamma}_A \\ \left(\frac{1-c_X+(1+\mu)\gamma-\gamma c_Y+\mu}{2(1+\gamma^2)},\, \gamma Q_a^{\mathcal{W}}\right), & \text{if } \hat{\gamma}\le\gamma<\bar{\gamma}_A\,, \\ \left(\frac{1-c_X-c_\eta+\mu}{2},\, \frac{c_\eta+\gamma(1+\mu)-\gamma c_Y}{2\gamma}\right), & \text{otherwise} \end{cases}$$

where $\hat{\gamma}_A = \frac{c_X-(1+\mu)+\sqrt{(1-c_X+\mu)^2+4c_{bw}(1+\mu-c_{bw}-c_Y)}}{2c_{bw}}$ and $\bar{\gamma}_A = \frac{1+\mu-c_Y+\sqrt{(1+\mu-c_Y)^2+4c_\eta(1+\mu-c_\eta-c_X)}}{2(1+\mu-c_\eta-c_X)}$. In this case, the optimal decision only account for the mean value of μ, because of the uncorrelated demands for both products. Likewise, we can have the inverse U-shaped pattern for the manufacturer's profit due to the trade-off between DE and RE. When $Q_b^{\mathcal{W}} = \gamma Q_a^{\mathcal{W}}$, the manufacturer can attain the profit maximum with $\gamma^{\#} = \frac{1-c_Y}{1-c_X}$, which is consistent with our findings.

When the two products' demands are correlated, parallel with Proposition 2, the $Q_a^{\mathcal{WC}}$ with asymmetric distribution is given by following:

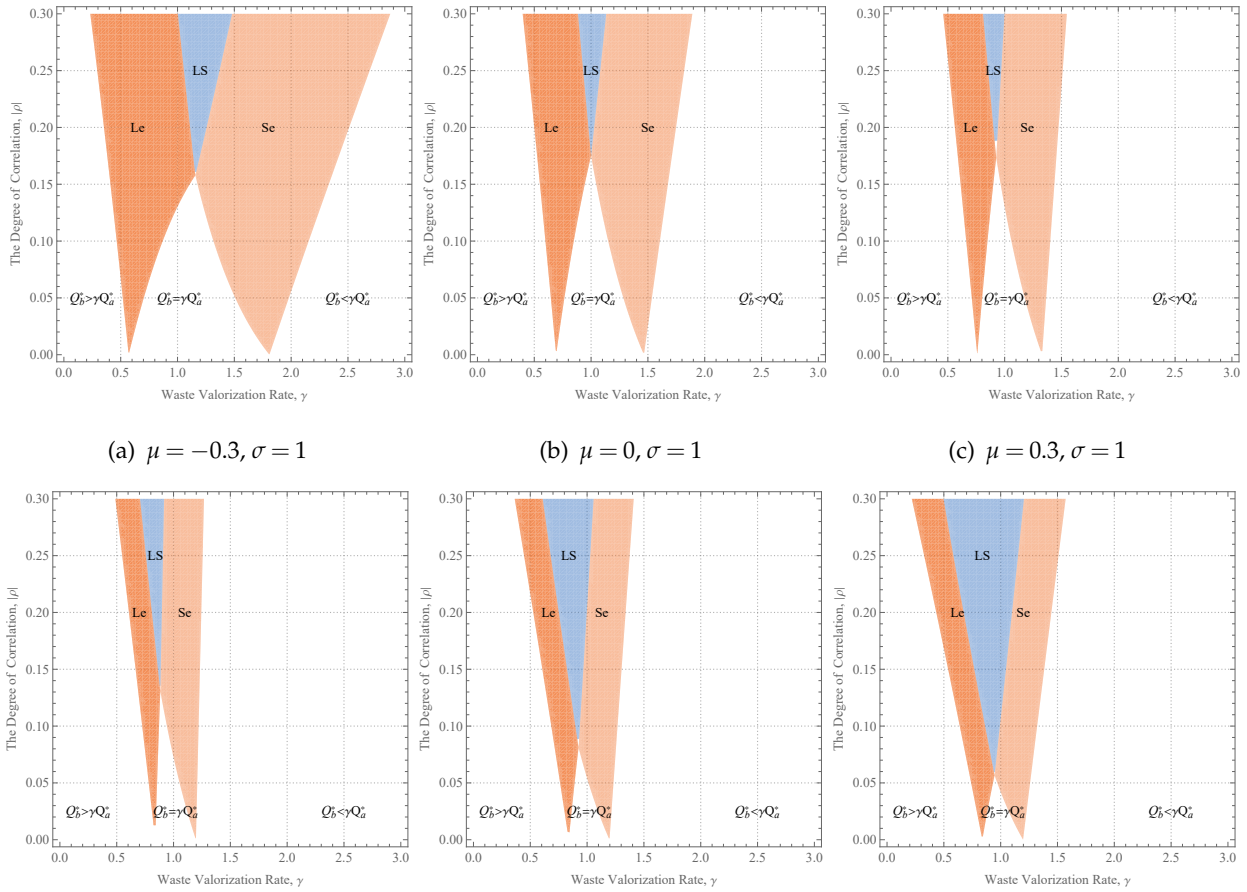
$$Q_a^{\mathcal{WC}} = \begin{cases} \frac{1+\mu-c_X+\gamma c_{bw}}{2} := Q_{a1}\ (Q_b^{\mathcal{WC}} > \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A1}^{LH} \\ \frac{c_X-\mu-1-\gamma(c_{bw}\theta+c_Y\theta-c_Y-(1-\theta)|\rho|(\Delta+\mu)-\theta\mu-\theta+\mu+1)}{2\gamma^2(\theta-1)-2} := Q_{a2}\ (\textit{Pooling Le}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A2}^{LH} \\ \frac{1+\mu-c_X+\gamma(1-c_Y+\mu(1-|\rho|)-\Delta|\rho|+2\theta\Delta|\rho|)}{2(1+\gamma^2)} := Q_{a3}\ (Q_b^{\mathcal{WC}} = \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A3}^{L} \\ \frac{\gamma c_{bw}\theta+c_\eta(\theta-1)-c_X+\mu+1}{2} := Q_{a4}\ (\textit{Pooling LS}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A3}^{H} \\ \frac{\gamma\Delta\theta|\rho|-\gamma\theta\mu|\rho|+\gamma\theta\mu+\gamma\theta+c_\eta(\theta-1)-c_X-\gamma c_Y\theta+\mu+1}{2\gamma^2\theta+2} := Q_{a5}\ (\textit{Pooling Se}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A4}^{LH} \\ \frac{1+\mu-c_\eta-c_X}{2} := Q_{a6}\ (Q_b^{\mathcal{WC}} < \gamma Q_a^{\mathcal{WC}}), & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A5}^{LH} \end{cases},$$

where $J\in\{X,\,Y,\,\eta,\,bw\}$, $\mathcal{R}_{Ai}^{LH}$, $\mathcal{R}_{A3}^{L}$, and $\mathcal{R}_{A3}^{H}$ $(i=1,2,4,5)$ are shown later.

Given the optimal production quantity above, $\mathbb{E}[Q_b^{\mathcal{WC}}]$ is shown as follows:

$$\mathbb{E}[Q_b^{\mathcal{WC}}] = \begin{cases} \frac{1-c_{bw}-c_Y-2\Delta\theta|\rho|+\Delta|\rho|-\mu|\rho|+\mu}{2}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A1}^{LH} \\ \frac{\theta(c_{bw}+c_Y-\Delta|\rho|+\mu(|\rho|-1)-1)-\gamma(\theta-1)(c_X-\mu-1)+\gamma^2(\theta-1)(-c_Y-|\rho|(-2\Delta\theta+\Delta+\mu)+\mu+1)}{2\gamma^2(\theta-1)-2}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A2}^{LH} \\ \frac{\gamma(1-c_X-\gamma(c_Y-2\Delta\theta|\rho|+\Delta|\rho|+\mu(|\rho|-1)-1)+\mu)}{2(\gamma^2+1)}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A3}^{L} \\ \frac{c_\eta-\gamma(c_{bw}\theta+c_Y-2\Delta\theta|\rho|+\Delta|\rho|+\mu|\rho|-\mu-1)-c_\eta\theta}{2\gamma}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A3}^{H} \\ \frac{\gamma\theta(c_\eta(\theta-1)-c_X+\gamma\theta(-c_Y+\Delta|\rho|-\mu|\rho|+\mu+1)+\mu+1)}{2\gamma^2\theta+2} - \frac{(\theta-1)(c_\eta-\gamma(c_Y+|\rho|(\Delta+\mu)-\mu-1))}{2\gamma}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A4}^{LH} \\ \frac{c_\eta-\gamma(c_Y-2\Delta\theta|\rho|+\Delta|\rho|+\mu(|\rho|-1)-1)}{2\gamma}, & \text{if } (\gamma,\,\rho,\,\Delta,\,c_J)\in\mathcal{R}_{A5}^{LH} \end{cases},$$

where $\mathcal{R}_{A1}^{LH}$: $\{(\gamma,\,\rho,\,\Delta,\,c_J)\,|0<\Delta<\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $Q_{a1} < \frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu(1-|\rho|)}{2\gamma}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $Q_{a1} < \frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu(1-|\rho|)}{2\gamma}\}$,
$\mathcal{R}_{A2}^{LH}$: $\{(\gamma,\,\rho,\,\Delta,\,c_J)\,|0<\Delta<\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu(1-|\rho|)}{2\gamma} < Q_{a2} < \frac{1-c_{bw}-c_Y+\Delta|\rho|+\mu(1-|\rho|)}{2\gamma}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{1-c_{bw}-c_Y-\Delta|\rho|+\mu(1-|\rho|)}{2\gamma} < Q_{a2} < \frac{\gamma-\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+c_\eta-\gamma c_Y}{2\gamma^2}\}$,



(a) $\mu = -0.3, \sigma = 1$ (b) $\mu = 0, \sigma = 1$ (c) $\mu = 0.3, \sigma = 1$

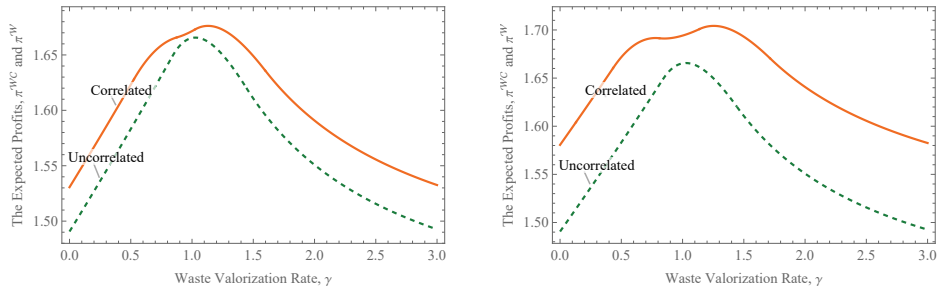
(d) $\mu = 1, \sigma = 1$ (e) $\mu = 1, \sigma = 2$ (f) $\mu = 1, \sigma = 3$

Figure E.8 Optimal Policy Under Asymmetric Distribution with $\xi_b = \rho\xi_a + (1-\rho)\phi_b$

Note: 'Le' denotes *Pooling Le*, 'Se' denotes *Pooling Se*, and 'LS' denotes *Pooling LS*. The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$, $c_Y = c_\eta = 0.15$.

$\mathcal{R}_{A3}^{L} : \{(\gamma, \rho, \Delta, c_J) \,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{1-c_{bw}-c_Y+\Delta|\rho|+\mu(1-|\rho|)}{2\gamma} < Q_{a3} < \frac{\gamma-\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+c_\eta-\gamma c_Y}{2\gamma^2}\}$,
$\mathcal{R}_{A3}^{H} : \{(\gamma, \rho, \Delta, c_J) \,|\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{\gamma-\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+c_\eta-\gamma c_Y}{2\gamma^2} < Q_{a4} < \frac{1-c_{bw}-c_Y+\Delta|\rho|+\mu(1-|\rho|)}{2\gamma}\}$,
$\mathcal{R}_{A4}^{LH} : \{(\gamma, \rho, \Delta, c_J) \,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{\gamma-\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+c_\eta-\gamma c_Y}{2\gamma^2} < Q_{a5} < \frac{\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{1-c_{bw}-c_Y+\Delta|\rho|+\mu(1-|\rho|)}{2\gamma} < Q_{a5} < \frac{\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}\}$,
$\mathcal{R}_{A5}^{LH} : \{(\gamma, \rho, \Delta, c_J) \,|0 < \Delta < \min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right)$ and $\frac{\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < Q_{a6}$, or $\min\left(\frac{1-c_{bw}-c_Y+\mu}{|\rho|}, \frac{\gamma c_{bw}+c_\eta}{2\gamma|\rho|}\right) < \Delta < \frac{1-c_{bw}-c_Y+\mu}{|\rho|}$ and $\frac{\gamma\Delta|\rho|+\gamma\mu(1-|\rho|)+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < Q_{a6}\}$.
Hence, it is straight forward to have the monotonicity for either $Q_a^{\mathcal{WC}}$ or $\mathbb{E}[Q_b^{\mathcal{WC}}]$.

According to the above analysis, we can conclude: (i) Figure E.8 shows that pooling policies persist under the alternative correlation formulation (i.e., a convex combination). Moreover, these pooling policies are



(a) $\mu = 1, \sigma = 2$ (b) $\mu = 1, \sigma = 3$

Figure E.9 Optimal Profits Under Asymmetric Distribution with $\xi_b = \rho\xi_a + (1-\rho)\phi_b$

Note: The parameter values in the numerical example are: $c_X = c_{bw} = 0.20$, $c_Y = 0.15$, $c_\eta = 0.35$, and $\rho = 0.2$.

influenced by the mean and variance in a manner analogous to that observed under our proposed correlation formulation (i.e., a linear combination), as illustrated in Figure E.7. (ii) As shown in Figure E.9, the profit distributions with respect to γ under the alternative formulation exhibit the same qualitative properties as those established in the main model analysis. Overall, we show that our results, including the proposed pooling policies (see Figure E.8), profit patterns (see Figure E.9), and key insights, largely hold under either correlation specification. □

E.4. Continuous Distribution

In this extension, we consider continuous distribution of ϕ, to explore whether our results hold valid. We assume that $\xi_j = \phi_j$ in the absence of demand correlation, where its pdf and CDF is given by $f(\cdot)$ and $F(\cdot)$, respectively. For model tractability, we also assume that $\mathbb{E}[\phi_j] = 0$ and the distribution support over $[-\Delta, \Delta]$. Similarly, in the presence of demand correlation, the random variable satisfy $\xi_a = \phi_a$ and $\xi_b = \rho\xi_a + \phi_b$, which give us $\mathbb{E}[\xi_a] = 0$ and $\mathbb{E}[\xi_b] = \rho\hat{\xi}_a$ when ξ_a is realized.

PROPOSITION E.5. *Under continuous distribution, comparing the optimal production quantities in models with and without demand correlation,*

a). Under Pooling Le, $Q_a^{\mathcal{WC}} < Q_a^{\mathcal{W}}$ and $\mathbb{E}[Q_b^{\mathcal{WC}}] > Q_b^{\mathcal{W}}$;

b). Under Pooling Se, $Q_a^{\mathcal{WC}} > Q_a^{\mathcal{W}}$ and $\mathbb{E}[Q_b^{\mathcal{WC}}] < Q_b^{\mathcal{W}}$.

PROOF OF PROPOSITION OF E.5:

We can show that Lemma 2 holds if $(\eta, c_s) \in C_D$. We have the following derivations. When $0 < \Delta < \frac{c_\eta + c_{bw}\gamma}{2\gamma\rho}$,

- If $Q_a < \frac{1-c_{bw}-c_Y-\Delta\rho}{2\gamma} \iff \frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho} < -\Delta$, $Q_b > \gamma Q_a$. Hence, $\pi^{\mathcal{WC}}(Q_a) = (1 - c_X - Q_a)Q_a + \int_{-\Delta}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$.
- If $\frac{-c_{bw}-c_Y-\Delta\rho+1}{2\gamma} < Q_a < \frac{-c_{bw}-c_Y+\Delta\rho+1}{2\gamma} \iff -\Delta < \frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho} < \Delta$. Hence, $\pi^{\mathcal{WC}}(Q_a) = (1 - c_X - Q_a)Q_a + \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \mathbf{L}_2 f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$

- If $\frac{-c_{bw}-c_Y+\Delta\rho+1}{2\gamma} < Q_a < \frac{-\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} \iff \frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho} < -\Delta < \Delta < \frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}$, $Q_b = \gamma Q_a$. Hence, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\Delta} \mathbf{L}_2 f(\xi)d\xi$.
- If $\frac{-\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < Q_a < \frac{\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} \iff -\Delta < \frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho} < \Delta < \frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}$. Hence, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} \mathbf{L}_3 f(\xi)d\xi + \int_{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}}^{\Delta} \mathbf{L}_2 f(\xi)d\xi$.
- If $Q_a > \frac{\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} \iff \Delta < \frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}$, $Q_a < \gamma Q_a$. Hence, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\Delta} \mathbf{L}_3 f(\xi)d\xi$.

$\mathbf{L}_1(Q_b > \gamma Q_a) = \frac{1}{4}\left(c_{bw}^2 + 2c_{bw}(c_Y - \xi\rho + 2\gamma Q_a - 1) + (-c_Y + \xi\rho + 1)^2\right)$, $\mathbf{L}_2(Q_b = \gamma Q_a) = \gamma(-Q_a)(c_Y - \xi\rho + \gamma Q_a - 1)$, and $\mathbf{L}_3(Q_b < \gamma Q_a) = \frac{c_\eta^2 - 2\gamma c_\eta(c_Y - \xi\rho + 2\gamma Q_a - 1) + \gamma^2(-c_Y+\xi\rho+1)^2}{4\gamma^2}$

Similarly, when $\Delta > \frac{c_\eta + c_{bw}\gamma}{2\gamma\rho}$,

- If $Q_a < \frac{1-c_{bw}-c_Y-\Delta\rho}{2\gamma}$, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$.
- If $\frac{1-c_{bw}-c_Y-\Delta\rho}{2\gamma} < Q_a < \frac{c_\eta-\gamma\Delta\rho+\gamma-\gamma c_Y}{2\gamma^2}$, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \mathbf{L}_2 f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$
- If $\frac{c_\eta-\gamma\Delta\rho+\gamma-\gamma c_Y}{2\gamma^2} < Q_a < \frac{1-c_{bw}-c_Y+\Delta\rho}{2\gamma}$, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} \mathbf{L}_3 f(\xi)d\xi + \int_{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \mathbf{L}_2 f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$.
- If $\frac{1-c_{bw}-c_Y+\Delta\rho}{2\gamma} < Q_a < \frac{\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}$, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} \mathbf{L}_3 f(\xi)d\xi + \int_{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}}^{\Delta} \mathbf{L}_2 f(\xi)d\xi$.
- If $Q_a > \frac{\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2}$, $\pi^{\mathcal{WC}}(Q_a) = (1-c_X-Q_a)Q_a + \int_{-\Delta}^{\Delta} \mathbf{L}_3 f(\xi)d\xi$.

We can prove that if Pooling Le or Pooling Se occurs (i.e., when optimal), our *pool-to-center* effect, that is smoothing production result holds for any symmetric distribution. However, bimodal profit distribution may or may not hold with respect to specific distribution. We can numerically show that Beta distribution can behave bimodal pattern.

Without demand correlation, we see that $Q_b > \gamma Q_a$ given $Q_a < \frac{1-c_{bw}-c_Y}{2\gamma}$. With demand correlation, we consider the range of $\frac{1-c_{bw}-c_Y-\Delta\rho}{2\gamma} < Q_a < \frac{1-c_{bw}-c_Y}{2\gamma}$. Then, the expected profit in model $\mathcal{W}$ is as follows:

$$\pi^{\mathcal{W}} = (1-Q_a-c_X)Q_a + \frac{c_{bw}^2 + 2c_{bw}(c_Y + 2\gamma Q_a - 1) + (1-c_Y)^2}{4}$$

The optimal Q_a satisfies the following:

$$\begin{aligned}
\frac{d\pi^{\mathcal{W}}}{dQ_a} =& 1 - c_X - 2Q_a + \gamma c_{bw} \\
=& 1 - c_X - 2Q_a + \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \gamma c_{bw} f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \gamma c_{bw} f(\xi)d\xi \\
=& 0
\end{aligned}$$

We see the derivation function is monotone decreasing in Q_a. Denote that the optimal solution as $Q_a^{\mathcal{W}}$.

The expected profit function in model $\mathcal{WC}$ is as follows:

$$\pi^{\mathcal{WC}}(Q_a) = (1 - c_X - Q_a)Q_a + \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \mathbf{L}_2 f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \mathbf{L}_1 f(\xi)d\xi$$

The optimal Q_a satisfies the following:

$$\begin{aligned}\frac{d\pi^{\mathcal{WC}}}{dQ_a} =& 1 - c_X - 2Q_a + \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \gamma(1 - c_Y - 2Q_a\gamma + \rho\xi)f(\xi)d\xi + \int_{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}}^{\Delta} \gamma c_{bw} f(\xi)d\xi \\ =& 0\end{aligned}$$

Similarly, we see the derivation is monotone decreasing in Q_a. Denote that the optimal solution as $Q_a^{\mathcal{WC}}$. We have that

$$\frac{d\pi^{\mathcal{W}}}{dQ_a} - \frac{d\pi^{\mathcal{WC}}}{dQ_a} = \int_{-\Delta}^{\frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}} \gamma(c_{bw} + c_Y + 2Q_a\gamma - 1 - \rho\xi)f(\xi)d\xi$$

Given $-\Delta < \xi < \frac{c_{bw}+c_Y+2\gamma Q_a-1}{\rho}$, we can show that $\frac{1-c_{bw}-c_Y+\rho\xi}{2\gamma} \leq Q_a$, which means that $\gamma(c_{bw} + c_Y + 2Q_a\gamma - 1 - \rho\xi) \geq 0$. Hence, $\frac{d\pi^{\mathcal{W}}}{dQ_a} - \frac{d\pi^{\mathcal{WC}}}{dQ_a} \geq 0$, and $Q_a^{\mathcal{WC}} \leq Q_a^{\mathcal{W}}$ is followed. In that sense, $\mathbb{E}[Q_b^{\mathcal{WC}}] \geq \mathbb{E}[Q_b^{\mathcal{W}}]$ is followed according to Lemma 2. In a similar vein, we can prove the case where $\frac{1-c_{bw}-c_Y}{2\gamma} < Q_a < \frac{-c_{bw}-c_Y+\Delta\rho+1}{2\gamma}$, so that pull to center is valid with respect to Pooling Le policy.

Next, we prove the case with Pooling Se. Consider the case where $\frac{-\gamma\Delta\rho+\gamma+c_\eta-\gamma c_Y}{2\gamma^2} < Q_a < \frac{\gamma+c_\eta-\gamma c_Y}{2\gamma^2}$. Without demand correlation,

$$\begin{aligned}\frac{d\pi^{\mathcal{W}}}{dQ_a} =& 1 - c_X + \gamma - c_Y\gamma - 2Q_a(1 + \gamma^2) \\ =& 1 - c_X - 2Q_a \\ &+ \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} \gamma(1 - c_Y - 2Q_a\gamma + \xi\rho)f(\xi)d\xi + \int_{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}}^{\Delta} \gamma(1 - c_Y - 2Q_a\gamma + \xi\rho)f(\xi)d\xi\end{aligned}$$

With demand correlation,

$$\begin{aligned}\frac{d\pi^{\mathcal{WC}}}{dQ_a} =& 1 - c_X - 2Q_a \\ &+ \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} -c_\eta f(\xi)d\xi + \int_{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}}^{\Delta} \gamma(1 - c_Y - 2Q_a\gamma + \xi\rho)f(\xi)d\xi\end{aligned}$$

We can prove that

$$\begin{aligned}\frac{d\pi^{\mathcal{WC}}}{dQ_a} - \frac{d\pi^{\mathcal{W}}}{dQ_a} =& \int_{-\Delta}^{\frac{-\gamma-c_\eta+\gamma c_Y+2\gamma^2 Q_a}{\gamma\rho}} -c_\eta - (\gamma(1 - c_Y - 2Q_a\gamma + \xi\rho))f(\xi)d\xi \\ \geq& 0\end{aligned}$$

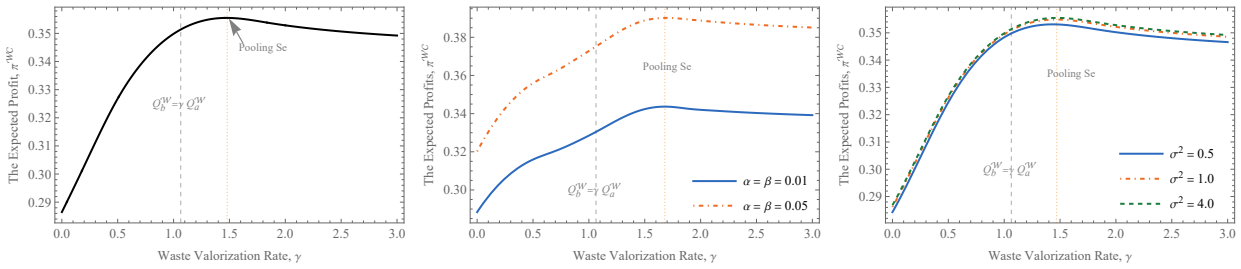
Hence, with Pooling Se, we have that $Q_a^{\mathcal{WC}} \geq Q_a^{\mathcal{W}}$. The proposition E.5 follows.

We present the numerical results with different distributions in the following table and figures.

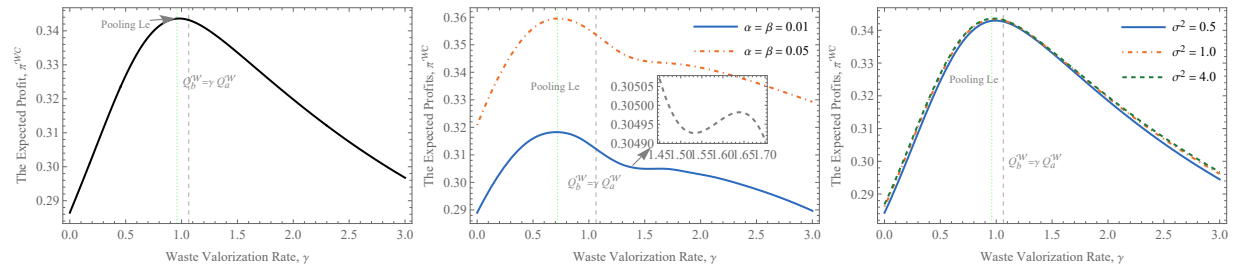
□

Table E.1 Optimal Profits Under Continuous Distributions

	Uncorrelated		Uniform		Beta (0.01)		Beta (0.05)		Normal (0.5)		Normal (1.0)		Normal (4.0)	
γ	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5	c_η=0.05	c_η=0.5
0.1	0.2737	0.2737	0.2946	0.2946	0.2965	0.2965	0.3305	0.3305	0.2919	0.2919	0.2939	0.2939	0.2945	0.2945
0.3	0.2905	0.2905	0.3113	0.3113	0.3086	0.3086	0.3461	0.3461	0.3086	0.3086	0.3106	0.3106	0.3112	0.3112
0.5	0.3081	0.3081	0.3268	0.3268	0.3159	0.3159	0.3560	0.3560	0.3246	0.3246	0.3262	0.3262	0.3268	0.3268
0.7	0.3265	0.3265	0.3386	0.3381	0.3204	**0.3183**	0.3621	**0.3595**	0.3370	0.3366	0.3382	0.3377	0.3386	0.3381
0.9	0.3383	0.3383	0.3469	**0.3432**	0.3256	0.3164	0.3691	0.3578	0.3455	**0.3424**	0.3465	**0.3430**	0.3469	**0.3432**
1.1	**0.3405**	**0.3405**	0.3521	0.3428	0.3316	0.3111	0.3766	0.3526	0.3505	0.3423	0.3517	0.3426	0.3521	0.3428
1.3	0.3380	0.3373	0.3548	0.3388	0.3379	0.3065	0.3840	0.3468	0.3527	0.3385	0.3543	0.3387	0.3548	0.3388
1.5	0.3357	0.3312	**0.3555**	0.3336	0.3424	**0.30495**	0.3888	0.3441	**0.3530**	0.3331	**0.3549**	0.3335	**0.3555**	0.3336
1.7	0.3340	0.3239	0.3547	0.3280	**0.3437**	0.3049	**0.3902**	0.3434	0.3521	0.3272	0.3540	0.3278	0.3547	0.3280
1.9	0.3326	0.3163	0.3534	0.3225	0.3425	0.3036	0.3893	0.3424	0.3508	0.3214	0.3528	0.3222	0.3534	0.3225
2.1	0.3315	0.3088	0.3523	0.3172	0.3416	0.3021	0.3882	0.3408	0.3497	0.3157	0.3517	0.3168	0.3523	0.3171
2.3	0.3306	0.3017	0.3514	0.3121	0.3409	0.3000	0.3873	0.3386	0.3488	0.3104	0.3508	0.3117	0.3514	0.3121
2.5	0.3300	0.2950	0.3507	0.3073	0.3403	0.2975	0.3866	0.3362	0.3480	0.3054	0.3500	0.3068	0.3506	0.3073
2.7	0.3292	0.2889	0.3500	0.3029	0.3398	0.2947	0.3859	0.3335	0.3474	0.3008	0.3494	0.3023	0.3500	0.3029
2.9	0.3287	0.2832	0.3495	0.2987	0.3394	0.2915	0.3854	0.3307	0.3468	0.2965	0.3488	0.2982	0.3494	0.2987



(a) Uniform Distribution (b) Beta Distribution (c) Normal Distribution



(d) Uniform Distribution (e) Beta Distribution (f) Normal Distribution

Note: $c_\eta = 0.05$ for (a), (b), (c), and $c_\eta = 0.5$ for (d), (e), (f). The parameter values in the numerical example are:

$c_X = c_{bw} = 0.20$, $c_Y = 0.15$, $\Delta|\rho| = 0.5$.

Figure E.10 Manufacture's Optimal Profits

E.5. Alternative Cost Structures

Consider the unit processing cost for raw materials (e.g., wood) as c_{Ym}, while the unit processing cost for waste (e.g., coconut shells) is $c_{Yw} \neq c_{Ym}$ due to differing production processes. Our model can be adapted

to this scenario with minor modifications without altering the optimal outcomes. Specifically, if we let $c_{bm} + c_{\Upsilon m} = c'_{bm} + c_{\Upsilon}$ and $c_{bw} + c_{\Upsilon w} = c'_{bw} + c_{\Upsilon}$, we can substitute c_{bm} and c_{bw} with c'_{bm} and c'_{bw} in our results, leading to identical conclusions as in our earlier analysis. While a comparison of different processing costs could yield additional insights, it would contribute marginally to the current study. Moreover, the model can be readily extended to accommodate a hybrid input setting, where both waste and raw materials are used, without altering the main insights. For instance, the manufacturer's objective function in such a case becomes $\pi_{II}^{\mathcal{W}} = \max_{Q_b} \{\mathbb{E}[p_b Q_b] - c_{\Upsilon} Q_b - (Q_a - \mathbb{1} \cdot Q_b/\gamma)^+ c_\eta - (Q_b - \mathbb{1} \cdot \gamma Q_a)^+ \min(c_{bm}, c_{bw})\}$. Also, during the investment phase, the manufacturer may consider the cost of waste valorization, much like other strategic initiatives such as product diversification and the implementation of new technology. However, our primary focus is on the operational decision-making process after the initial investment. The fixed investment cost does not alter the main results of this paper.

E.6. Substitution and Complementary Demands

Indeed, the degree of demand correlation significantly influences the optimal decisions of manufacturers and the related findings. More broadly, ρ can be interpreted as indicative of either a substitution or complementary relationship between the two products. For example, when $\hat{\xi}_a > 0$, a positive ρ signifies a complementary relationship, as the demand of product B tends to decrease as the price of product A increases. Conversely, a negative ρ indicates a substitution relationship, wherein the demand of product B tends to increase as the price of product A increases. In practice, the product portfolios at companies like Starbucks demonstrate complementary relationships, as seen with items like coffee and coffee tumblers, while Unilever's Cremissimo may serve as a substitute for other ice cream flavors available in the market. Our findings illuminate their production decisions, emphasizing not only the importance of sustainable operations through waste reduction but also how to effectively leverage waste valorization amidst demand correlation.

Appendix F: The Analysis for Waste Supply Chain

In this section, we explore a case where the supply chain is divided between two companies: the supplier (she), who produces product A and generates waste, and the manufacturer (he), who processes product B by valorizing that waste. The decision sequence remains consistent with Figure 1. We assume that the unit waste procurement cost, c_{bw}, is exogenous, as the supplier lacks pricing control over the waste, which is either undesired and requires disposal or is readily abundant in the market. If the supplier can fulfill the manufacturer's waste requirements, the manufacturer will exclusively procure from her. To facilitate this exclusivity, a small discount, ϵ, on the waste price (i.e., $c_{bw} - \epsilon$) incentivizes the manufacturer to prioritize waste procurement from this supplier.[14] If the supplier cannot fully meet the manufacturer's waste requirements, the manufacturer will source the remaining waste from alternative suppliers at the same unit cost of c_{bw}. All other assumptions remain consistent with those outlined in Section 3.[15] We use the superscript $\{\mathcal{D}\}$ to denote this decentralized waste supply chain model, and the subscripts $\{M, S\}$ to represent the manufacturer and supplier, respectively.

By employing backward induction, we derive $Q_b^{\mathcal{D}} = Q_b^{\mathcal{BC}}$. In the absence of a waste shortage, $Q_b^{\mathcal{D}}$ in this case is analogous to Lemma 1, where the manufacturer applies his monopolistic production quantity by considering demand correlation. Consequently, the supplier's expected profit function is as follows:

$$\pi_S^{\mathcal{D}}(Q_a) = \begin{cases} \mathcal{S}_1(Q_b > \gamma Q_a), & \text{if } 0 < Q_a \le \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} \\ \mathcal{S}_2(Q_b > \gamma Q_a \text{ and } Q_b < \gamma Q_a, \textit{Pooling LS}), & \text{if } \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma} < Q_a \le \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma} \\ \mathcal{S}_3(Q_b < \gamma Q_a), & \text{otherwise} \end{cases}. \tag{F.1}$$

In contrast to the centralized decision model, the supplier's decision problem reduces to assessing whether her available waste supply meets the manufacturer's requirement. This assessment yields three demand-contingent cases: (i) insufficient supply (i.e., $Q_b > \gamma Q_a$); (ii) sufficient supply (i.e., $Q_b < \gamma Q_a$); and (iii) sufficiency only in the low-demand state (i.e., $Q_b < \gamma Q_a$ when $\hat{\xi}_a = -\Delta$). Consequently, in Equation (F.1), it is clear that $\mathcal{S}_2$ shares an analogous objective function with $\mathcal{L}_3^H$ under *Pooling LS*, while $\mathcal{S}_1$ and $\mathcal{S}_3$ correspond to the insufficient- and sufficient-supply cases, respectively. We provide a comprehensive analysis of waste valorization with uncorrelated demand in Appendix F, which shows that the supplier can strategically adjust her production in response to the manufacturer's valorization decisions. The optimal $Q_b^{\mathcal{D}}$ may be greater than, equal to, or less than $\gamma Q_a^{\mathcal{D}}$, mirroring the outcomes presented in Proposition 1.

PROPOSITION F.1. *When the two products' demands are correlated with waste-from-market, the optimal production quantity for product A is:*

$$Q_a^{\mathcal{D}} = \begin{cases} \frac{1-c_X+\gamma c_{bw}}{2}\ (Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}), & \textit{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_1^{so} \\ \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma}\ (\textit{Boundary Le}), & \textit{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_2^{so} \\ \frac{\gamma c_{bw}-c_\eta-2c_X+2}{4}\ (\textit{Pooling LS}), & \textit{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_3^{so} \\ \frac{1-c_{bw}-c_Y+\Delta|\rho|}{2\gamma}\ (\textit{Boundary Se}), & \textit{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_4^{so} \\ \frac{1-c_\eta-c_X}{2}\ (Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}), & \textit{if } (\gamma,\ \rho,\ \Delta,\ c_J) \in \mathcal{R}_5^{so} \end{cases},$$

[14] In certain contexts, suppliers may implement quantity-based discount contracts for waste procurement. Also, the waste transaction price c'_{bw} may be lower than the manufacturer's procurement cost c_{bw} due to additional expenses such as transportation and hygiene. However, we maintain that $c'_{bw} = c_{bw}$ remains exogenous for the sake of consistency.

[15] The assumption $(\eta, c_s) \in C_D$ is preserved in the waste supply chain model, as any violation would disincentivize the supplier from transacting with the manufacturer.

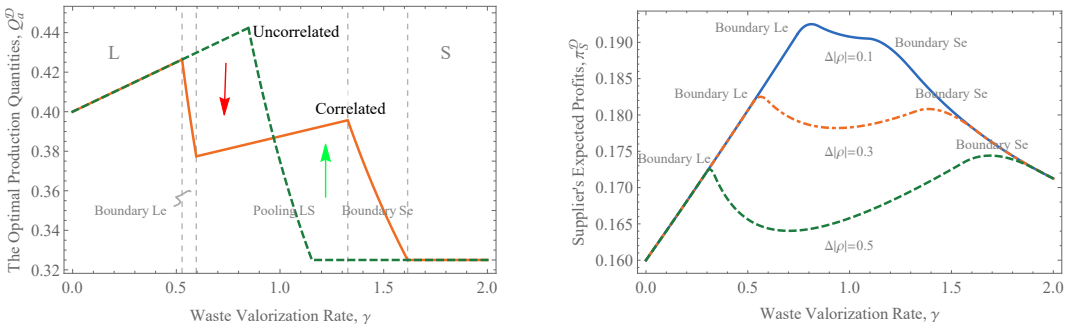
where $J \in \{X,\ Y,\ \eta,\ bw\}$, $\mathcal{R}_i^{so}$ $(i = 1,2,...,5)$ *is shown in the Appendix.*

In the presence of demand correlation, Proposition F.1 outlines the optimal policy within the waste supply chain framework. When $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_1^{so}$ (i.e., γ is small), the optimal $Q_a^{\mathcal{D}}$ takes the interior solution of $\mathcal{S}_1$ in Equation (F.1), resulting in $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$. When $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_2^{so}$, $Q_a^{\mathcal{D}}$ becomes the boundary solution of $\mathcal{S}_1$. Specifically, with $Q_a^{\mathcal{D}} = \frac{1-c_{bw}-c_Y-\Delta|\rho|}{2\gamma}$ and $Q_b^{\mathcal{D}} = \frac{1-c_{bw}-c_Y+\rho\hat{\xi}_a}{2}$, the condition $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$ holds only when $\rho\hat{\xi}_a < 0$; otherwise, $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$. We denote this boundary solution as *Boundary Le* (i.e., the policy results in equal probability for the 'larger than and equal to' pattern). When $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_3^{so}$ (i.e., γ is moderate), $Q_a^{\mathcal{D}}$ takes the interior solution of $\mathcal{S}_2$, akin to the outcome in *Pooling LS*. When $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_4^{so}$, $Q_a^{\mathcal{D}}$ reaches the boundary solution of $\mathcal{S}_2$, with $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$ occurring only when $\rho\hat{\xi}_a > 0$; otherwise, $Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}$. We denote this boundary solution as *Boundary Se* (i.e., the policy results in equal probability for the 'smaller than and equal to' pattern). Finally, when $(\gamma, \rho, \Delta, c_J) \in \mathcal{R}_5^{so}$ (i.e., γ is large), $Q_a^{\mathcal{D}}$ derives from the interior solution of $\mathcal{S}_3$, yielding $Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}$.

Interestingly, the optimal decisions for the two boundary solutions operate differently from the pooling mechanism discussed in Section 4.2. While the pooling policy integrates distinct qualitative segments into a single objective function that incorporates demand correlation, this integration does not occur in the waste supply chain equilibrium. For instance, under *Boundary Le*, the supplier anticipates the manufacturer's waste valorization decision and selects a production level so that all generated waste can be fully (or perfectly) utilized, thereby eliminating disposal costs. The case of *Boundary Se* exhibits similar characteristics. These boundary equilibria contrast sharply with the pooling strategies, revealing a unique aspect of waste supply chain configurations. Given the equilibrium, we are interested in whether the supplier's profit exhibits a pattern similar to previous results. Indeed, when $(\eta, \gamma) \in C_D$ and $\eta = 1$, the supplier's profit weakly increases with γ, aligning with the findings in Corollary 1. The following corollary presents the result for $\eta = 0$.

COROLLARY F.1. *When the two products' demands are correlated with waste-from-market and* $\eta = 0$*, the supplier's expected profit exhibits an M-shaped pattern with respect to* γ*, attaining maximum profit when Boundary Le or Boundary Se occurs.*

Corollary F.1 uncovers two key insights regarding the supplier's optimal decisions under demand correlation. First, as illustrated in Figure F.1(a), while the manufacturer's optimal production quantity remains at its monopolistic level, the supplier exhibits a "pool-to-center" behavior, similar to the result in Corollary 3. Essentially, akin to the pooling policy, the supplier reduces her optimal $Q_a^{\mathcal{D}}$ under the *Boundary Le* policy and increases $Q_a^{\mathcal{D}}$ under the *Boundary Se* policy compared to the case without correlation. The reasoning behind this is that when γ is relatively small, the supplier anticipates the manufacturer's production decisions by lowering $Q_a^{\mathcal{D}}$ to better manage the scenario where $\rho\hat{\xi}_a < 0$. In this case, with $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$, the production quantities of the two products align perfectly. Conversely, when γ is relatively large, the supplier increases $Q_a^{\mathcal{D}}$ to mitigate the loss of waste sales when $\rho\hat{\xi}_a > 0$. This smoothing of $Q_a^{\mathcal{D}}$ with respect to γ allows the supplier to handle situations where demand volatility and correlation are high. The concept of boundary equilibrium has been identified in supply chain literature (Wan and Wu 2017), particularly to



(a) Optimal $Q_a^{\mathcal{D}}$ ($\Delta|\rho| = 0.30$) (b) Optimal $\pi_S^{\mathcal{D}}$

Figure F.1 The Supplier's Production Quantity and Expected Profits

Note: The parameter values in the numerical example are: $c_X = 0.20$, $c_Y = c_\eta = 0.15$, $c_{bw} = 0.10$.

illustrate how suppliers can influence manufacturers to act in their favor. However, our findings introduce a novel aspect of boundary policy, showing how the "pool-to-center" feature emerges in the presence of demand correlation. Second, as depicted in Figure F.1(b), the supplier's profit exhibits an M-shaped pattern with respect to γ, with peaks occurring under *Boundary Le* or *Boundary Se*.

In practice, coordination between a single manufacturer and a single supplier, such as Sir Kensington and Ithaca Hummus, is essential, as their production decisions are interdependent and may require mutual adjustment. As noted by Scott, founder of Sir Kensington, the rising popularity of Fabanaise among the vegan community has created a pressing need to secure a larger supply of aquafaba (New York Times 2016). Our findings suggest that this shortage may stem not from capacity constraints, but from economically rational production choices by Ithaca Hummus. For waste utilizers facing such limitations, a viable solution is to diversify their sourcing base, exemplified by Unilever's Marmite, which collects brewer's yeast from across the United Kingdom (Marmite 2025).

PROOF OF PROPOSITION F.1:

Before proving Proposition F.1, we provide a comprehensive analysis for the waste supply chain model with $(\eta, c_s) \in C_D$ as follows.

Accordingly, the manufacturer's expected profit $\pi_M^{\mathcal{D}}(Q_b)$ is

$$\pi_M^{\mathcal{D}}(Q_b) = \mathbb{E}\left[p_b Q_b\right] - \left(c_{bw} + c_Y\right) Q_b. \tag{F.2}$$

The supplier's expected profit $\pi_S^{\mathcal{D}}(Q_a)$ is

$$\pi_S^{\mathcal{D}}\left(Q_a\right) = \begin{cases} \mathbb{E}\left[p_a Q_a\right] - c_X Q_a + c_{bw} Q_b - c_\eta \left(Q_a - Q_b/\gamma\right), & \text{if } Q_b \le \gamma Q_a \\ \mathbb{E}\left[p_a Q_a\right] - c_X Q_a + c_{bw}\gamma Q_a, & \text{if } Q_b > \gamma Q_a \end{cases}. \tag{F.3}$$

Equation (F.2) indicates that the waste supply from the market (i.e., encompassing multiple suppliers) is always sufficient. However, when consider a specific supplier, the waste supply may prove inadequate, as reflected in Equation (F.3). Likewise, the disposal cost depends on the production quantity Q_b; thus, the supplier must account for the manufacturer's waste valorization efforts along with the demand correlation in her production decision-making.

We first focus solely on the effect of waste valorization, assuming that the demands for products A and B are uncorrelated. As previously mentioned, with $\rho = 0$ and $\gamma \neq 0$, the manufacturer's optimal production quantity remains $Q_b^{\mathcal{D}} = \frac{1-c_{bw}-c_Y}{2}$, independent of γ. However, the supplier's production quantity may be influenced by the waste valorization process. The following proposition outlines the optimal production decisions for both the supplier and the manufacturer.

PROPOSITION F.2. *In the waste supply chain model with $\rho = 0$, the optimal production quantity for product j is*

$$\left(Q_a^{\mathcal{D}}, Q_b^{\mathcal{D}}\right) = \begin{cases} \left(\frac{1-c_X+\gamma c_{bw}}{2}, \frac{1-c_{bw}-c_Y}{2}\right), & \text{if } 0 < \gamma < \hat{\gamma} \\ \left(\frac{1-c_{bw}-c_Y}{2\gamma}, \gamma Q_a^{\mathcal{D}}\right), & \text{if } \hat{\gamma} \leq \gamma < \ddot{\gamma}, \\ \left(\frac{1-c_X-c_\eta}{2}, \frac{1-c_{bw}-c_Y}{2}\right), & \text{otherwise} \end{cases}$$

where $\ddot{\gamma} = \frac{c_{bw}+c_Y-1}{c_\eta+c_X-1}$.

Proposition F.2 outlines the relationship between the optimal production quantities of $Q_b^{\mathcal{D}}$ and $\gamma Q_a^{\mathcal{D}}$. When the waste valorization rate is relatively low (i.e., $0 < \gamma < \hat{\gamma}$), $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$; when the rate is high (i.e., $\gamma \geq \ddot{\gamma}$), the relationship reverses to $Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}$. For moderate valorization rates, $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$. These results align with those in Proposition 1, but Proposition F.2 introduces key differences. First, the production decision for product B depends solely on c_{bw} and c_Y in waste supply chain model. Second, when $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$, the production quantity of product A is determined only by the cost factors of product B (i.e., c_{bw} and c_Y), irrespective of A's own costs (i.e., c_η and c_X). This underscores the unique outcome where, under independent production decisions, waste suppliers may adjust their output purely in response to the impact of waste valorization.

Another significant insight emerges when comparing Proposition F.2 with Lemma 1. Specifically, when $c_{bw} = c_{bm}$, Proposition F.2 indicates that the supplier will overproduce if $\gamma < \ddot{\gamma}$; otherwise, she maintains product A's production at the monopolistic level. The rationale is that when $\gamma < \ddot{\gamma}$ and $Q_b^{\mathcal{D}} \geq \gamma Q_a^{\mathcal{D}}$, her waste supply is either insufficient or just enough to meet the manufacturer's requirements. In such cases, the supplier increases her production to better align with the manufacturer's needs. Once the waste supply matches the manufacturer's demand (i.e., $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$), the supplier reduces her production of product A as γ increases. When the waste supply exceeds the manufacturer's requirements (i.e., $Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}$), the supplier reverts to her monopolistic production quantities for product A, incurring some disposal costs for the leftover waste. This behavior is prevalent and observed in various practical scenarios.

According to our previous definition on `DE`, `RE`, and `AE`, we highlight some important findings in the corollary as follows.

COROLLARY F.2. *In the waste supply chain model with $\rho = 0$,*

a). `RE` either increases or remains constant with γ; while `DE` and `AE` first increase then decrease with γ.

b). The supplier's and total expected profits first increase then decrease with γ, while the manufacturer's expected profit is independent of γ;

The behaviors of DE, RE, and AE closely mirror those in Corollary 1. The maximum value of AE occurs at the threshold $\hat{\gamma}$, where both DE and RE also reach their peak. When $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$, some waste must be sourced externally (i.e., the supplier's waste is insufficient), but as γ increases, the supplier progressively satisfies the manufacturer's waste needs, leading to a rise in RE. Once $Q_b^{\mathcal{D}} \leq \gamma Q_a^{\mathcal{D}}$, RE becomes independent of γ since $Q_b^{\mathcal{D}}$ remains constant with γ and becomes a binding constraint. A similar explanation applies to DE. At $\gamma = \hat{\gamma}$, the supplier's waste generation exactly aligns with the manufacturer's requirements, maximizing AE. However, as γ continues to rise, generating excess waste, both AE and DE begin to decline.

The supplier achieves maximum profit when $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$, while the manufacturer's profit remains independent of γ. This leads to a similar unimodal pattern in the total profit of the waste chain, as observed in Corollary 1. Additionally, an optimal $\gamma^* = \frac{1-c_{bw}-c_Y}{1-c_X}$ emerges, depending on c_{bw}, c_Y, and c_X. When compared to $\gamma^{\#}$, this result highlights that the waste procurement cost for product B becomes a critical factor when waste material is exclusively sourced from the market.

In summary, when the two products' demands are uncorrelated, our results indicate that the supplier can adjust her production in response to the manufacturer's waste valorization. The optimal $Q_b^{\mathcal{D}}$ may be greater than, equal to, or less than $\gamma Q_a^{\mathcal{D}}$, mirroring the outcomes presented in Proposition 1. Specifically, when γ is relatively small (i.e., $\gamma < \hat{\gamma}$ and $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$), the optimal production quantities align with those in Proposition 1. This implies that the manufacturer's profit from product B (or the combined profit from both products) remains indifferent between generating waste materials in-house and sourcing them externally. The result also implies that the supplier's expected profit exhibits a unimodal pattern with respect to γ. When $Q_b^{\mathcal{D}} = \gamma Q_a^{\mathcal{D}}$ (i.e., $\gamma = \frac{1-c_{bw}-c_Y}{1-c_X}$), the supplier's profit achieve the maximum. Moreover, this unimodal profit pattern can be attributed to the trade-off between DE and RE, with AE attaining its maximum when the supplier's waste perfectly meets the manufacturer's requirements.

The remaining analysis of Proposition F.1 is similar to that in Proposition 1. Proposition F.1 follows. □

PROOF OF COROLLARY F.1:

According to Proposition F.1, we can have the expected profit for the supplier. It's easy to show that $\pi_S^{\mathcal{D}}$ increases with γ when $Q_b^{\mathcal{D}} > \gamma Q_a^{\mathcal{D}}$, since $\frac{d\pi_S^{\mathcal{D}}}{d\gamma} = \frac{c_{bw}(1-c_X+c_{bw}\gamma)}{2} > 0$; and decreases with γ when $Q_b^{\mathcal{D}} < \gamma Q_a^{\mathcal{D}}$, since $\frac{d\pi_S^{\mathcal{D}}}{d\gamma} = -\frac{c_\eta(1-c_{bw}-c_Y)}{2\gamma^2} < 0$. Within *Pooling LS*, it shows that there is only one extreme point of $\gamma_{y_1} \in \mathcal{R}_3^{so}$, where $\gamma_{y_1} = \text{Root}\left[x^3 c_{bw}^2 + x^2(2c_{bw} - c_{bw}c_\eta - 2c_{bw}c_X) + 2c_{bw}c_\eta + 2c_\eta c_Y + 2c_\eta\Delta|\rho| - 2c_\eta,\ 3\right]$, which $\pi_S^{\mathcal{D}}$ decreases with γ if $\gamma < \gamma_{y_1}$, and $\pi_S^{\mathcal{D}}$ increases with γ if $\gamma > \gamma_{y_1}$. Hence, γ_{y_1} attains the local minimum of $\pi_S^{\mathcal{D}}$ with respect to γ. Within *Boundary Le*, we have that there is only one extreme point of $\gamma = \frac{1-c_{bw}-c_Y-\Delta|\rho|}{1-c_X} \in \mathcal{R}_2^{so}$, $\pi_S^{\mathcal{D}}$ increases with γ if $\gamma < \frac{1-c_{bw}-c_Y-\Delta|\rho|}{1-c_X}$; and $\pi_S^{\mathcal{D}}$ decreases with γ if $\gamma > \frac{1-c_{bw}-c_Y-\Delta|\rho|}{1-c_X}$. Likewise, within *Boundary Se*, it shows that there is only one extreme point of $\gamma = \gamma_{y_2} = \frac{(c_{bw}+c_Y-\Delta|\rho|-1)^2}{c_{bw}(c_X-1)-c_\eta\Delta|\rho|+c_X(c_Y-\Delta|\rho|-1)-c_Y+\Delta|\rho|+1} \in \mathcal{R}_4^{so}$, $\pi_S^{\mathcal{D}}$ increases with γ if $\gamma < \gamma_{y_2}$; and $\pi_S^{\mathcal{D}}$ decreases with γ if $\gamma > \gamma_{y_2}$. Hence, $\pi_S^{\mathcal{D}}$ has at two most two local maxima. We further show that if $c_{bw} < \frac{2(1-\gamma)+\gamma(c_\eta+2c_X)-2c_Y}{2+\gamma^2}$, *Boundary Le* dominates *Boundary Se*. Corollary F.1 follows. □