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Learning to cover: online learning and optimization with irreversible decisions toward target coverage
Electronic Companion

EC.1. Appendix to Section 3

EC.1.1. Proof of Proposition 1

We first prove that $\widehat{\theta}_t$ is a consistent estimator.

LEMMA EC.1. *For every $\varepsilon_0, \varepsilon_1, \gamma_1, \gamma_2 > 0$, there exists a such that, for all $N_{t-1} \geq a$,*

$$\mathbb{P}\left(\|\widehat{\theta}_t - \theta_0\|_2 \geq \frac{\varepsilon_0}{(N_{t-1})^{1/2-\gamma_1}}\right) \leq \frac{\varepsilon_1}{(N_{t-1})^{1/2-\gamma_2}} \tag{EC.1}$$

Proof of Lemma EC.1 First, we show that $\widehat{\theta}_t$ from Equation (4) is the (conditional) maximum likelihood estimator based on independent, but not necessarily identically distributed, observations. While the facility success indicators S_{it} may in general depend on prior facility openings, conditioning on the features $\boldsymbol{x}_i$ and $y_{it} = 1$ fixes the success probability $\eta_i = f_{\theta_0}(\boldsymbol{x}_i)$ and the set of facilities we choose. Concretely, define $X := \{x_i\}_{i=1}^M$, $Y_t := \{y_{it}\}_{i=1}^M$ and $Z_t := \{z_{it}\}_{i=1}^M$. We define the history up to period t as:

$$\mathcal{H}_s := \sigma(X, \{(Y_r, Z_r)\}_{r \leq s})$$

where $\sigma(\cdot)$ is the sigma algebra induced by the randomness in the sampling of Y_t and the success of Z_t in our setup. By the chain rule, we have:

$$p_\theta(Z_{1:t-1}, Y_{1:t-1} \mid X) = \prod_{s=1}^{t-1} p_\theta(Z_s \mid \mathcal{H}_{s-1}, Y_s) p(Y_s \mid \mathcal{H}_{s-1})$$
$$p(Y_{1:t-1} \mid X) = \prod_{s=1}^{t-1} p(Y_s \mid \mathcal{H}_{s-1})$$

We obtain:

$$\begin{aligned}
p_\theta(Z_{1:t-1} \mid X, Y_{1:t-1}) &= \frac{p_\theta(Z_{1:t-1}, Y_{1:t-1} \mid X)}{p(Y_{1:t-1} \mid X)} \\
&= \frac{\prod_{s=1}^{t-1} p_\theta(Z_s \mid \mathcal{H}_{s-1}, Y_s) p(Y_s \mid \mathcal{H}_{s-1})}{\prod_{s=1}^{t-1} p(Y_s \mid \mathcal{H}_{s-1})} \\
&= \prod_{s=1}^{t-1} p_\theta(Z_s \mid \mathcal{H}_{s-1}, Y_s) \\
&= \prod_{s=1}^{t-1} \prod_{i: y_{is}=1} (f_\theta(\boldsymbol{x}_i))^{z_{is}} (1 - f_\theta(\boldsymbol{x}_i))^{1-z_{is}}
\end{aligned}$$

where the last equality follows from the independence of z_{is} across different facilities given y_{is}. This shows that, conditioned on observed X and Y, Equation (4) is a (conditional) likelihood based on independent (z_{is}) but not necessarily identical distributions (due to the different $f_\theta(\boldsymbol{x}_i)$), and $\hat{\theta}_t$ is the corresponding maximum likelihood estimator (MLE).

Hoadley (1971) defines the following regularity conditions for MLE estimators based on independent but not necessarily identical distributions:

(a) Θ is a closed set.

(b) $f_\theta(\boldsymbol{x})$ is an upper semi-continuous function of θ uniformly in $\boldsymbol{x} \in \mathcal{X}$.

(c) The log-likelihood $\log p_\theta(Z_{1:t-1} \mid X, Y_{1:t-1})$ is a measurable function of $Z_{1:t-1}$ and is twice continuously differentiable with respect to θ.

(d) The log-likelihood $\log p_\theta(Z_{1:t-1} \mid X, Y_{1:t-1})$ is uniformly upper bounded by an integrable function $g(X, Y_{1:t-1})$ with respect to X and $Y_{1:t-1}$.

(e) The derivative of the log-likelihood $\frac{\partial \log p_\theta(Z_{1:t-1}|X,Y_{1:t-1})}{\partial\theta}$ has finite third moments.

(f) The expected log-likelihood is identifiable: $R_t(\theta \mid \mathcal{J}_t) = R_t(\theta_0 \mid \mathcal{J}_t) \iff \theta = \theta_0 \qquad \forall t$

(g) The Fisher information is positive definite at $\theta = \theta_0$:

$$\mathcal{I}_t(\theta_0 \mid \mathcal{J}_t) := \frac{1}{|\mathcal{J}_t|} \sum_{i \in \mathcal{J}_t} \frac{1}{f_{\theta_0}(\boldsymbol{x}_i)(1 - f_{\theta_0}(\boldsymbol{x}))} \frac{\partial f_{\theta_0}(\boldsymbol{x}_i)}{\partial\theta} \frac{\partial f_{\theta_0}(\boldsymbol{x}_i)}{\partial\theta^T} \succ 0 \qquad \forall t$$

Condition (a) is implied by the first point of Statistical Condition 5. Conditions (b), (c) and (d) are implied by the second point in Statistical Condition 5. Condition (e) directly follows because all moments of the Bernoulli distribution are finite and $f_\theta(\boldsymbol{x}_i)$ is uniformly bounded. Conditions (f) and (g) are satisfied by the last two statements in Statistical Condition 5.

Therefore, per the central limit theorem for MLE estimation, $\sqrt{N_{t-1}}\left(\widehat{\theta}_t - \theta_0\right) \xrightarrow{d} N(0, \boldsymbol{\Sigma})$ for some covariance matrix $\boldsymbol{\Sigma}$. Furthermore, under Statistical Condition 5, a multivariate Berry–Esseen bound holds (e.g. Raič 2019), so there exist constants c, C such that for all ε, we have:

$$\mathbb{P}\left(\|\widehat{\theta}_t - \theta_0\|_2 \geq \varepsilon\right) \leq \mathbb{P}(\|W\|_2 \geq c\varepsilon\sqrt{N_{t-1}}) + \frac{C}{(N_{t-1})^{1/2}}$$

where $W \sim N(0, \sum_t I_t(\theta_0 \mid \mathcal{J}_t))$ follows an ℓ-dimensional normal distribution. Then, for any $\gamma_1, \gamma_2 > 0$, we have that:

$$\mathbb{P}\left(\|\widehat{\theta}_t - \theta_0\|_2 \geq \frac{\varepsilon}{(N_{t-1})^{1/2-\gamma_1}}\right) \leq \mathbb{P}(\|W\| \geq c\varepsilon N_{t-1}^{\gamma_1}) + \frac{C}{(N_{t-1})^{1/2}} \leq \frac{\varepsilon_1}{(N_{t-1})^{1/2-\gamma_2}}$$

where the last inequality is true for all N_{t-1} sufficiently large. □

Next, we bound the difference in accuracy between event probabilities conditioning on the classifier $h_t(\boldsymbol{x}_i) = 1$ versus the optimal classifier $h^*(\boldsymbol{x}_i) = 1$, as a function of the functions $f_{\widehat{\theta}_t}(\boldsymbol{x})$ and $f_\theta(\boldsymbol{x})$, of the concentration of $f_{\theta_0}(\boldsymbol{x})$ around τ, and of the mass of exploration facilities.

Lemma EC.2. *Let E be any probability event defined over the set of facilities $\mathcal{F}$, so that $\mathbb{P}(E) = \frac{1}{|\mathcal{F}|}\sum_i \mathbf{1}\{E\}$. For any $\varepsilon > 0$, the following holds:*

$$|\mathbb{P}(E \mid h_t(\boldsymbol{x}_i)=1) - \mathbb{P}(E \mid h^*(\boldsymbol{x}_i)=1)| \leq 2\cdot \frac{\mathbb{P}\left(\left|f_{\widehat{\theta}_t}(\boldsymbol{x}_i) - f_{\theta_0}(\boldsymbol{x}_i)\right| > \varepsilon\right) + \mathbb{P}\left(|f_{\theta_0}(\boldsymbol{x}_i) - \tau| < \varepsilon\right) + \mathbb{P}(i \in \mathcal{E}_t)}{\mathbb{P}(h^*(\boldsymbol{x}_i)=1)}$$

Proof of Lemma EC.2 The result is derived as follows:

$$\begin{aligned}
&|\mathbb{P}(E \mid h_t(\boldsymbol{x}_i)=1) - \mathbb{P}(E \mid h^*(\boldsymbol{x}_i)=1)| \\
=&\left|\frac{\sum_{i\in\mathcal{F}}\mathbb{P}(E)\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\}} - \frac{\sum_{i\in\mathcal{F}}\mathbb{P}(E)\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}}\right| \\
\leq&\left|\sum_{i\in\mathcal{F}}\mathbb{P}(E)\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\}\left(\frac{1}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\}} - \frac{1}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}}\right)\right| \\
&+\left|\frac{\sum_{i\in\mathcal{F}}\mathbb{P}(E)\left(\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\} - \mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}\right)}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}}\right| \\
\leq&2\cdot\frac{\sum_{i\in\mathcal{F}}\mathbb{P}(E)\left|\left(\mathbf{1}\{h_t(\boldsymbol{x}_i)=1\} - \mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}\right)\right|}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
\leq&2\cdot\frac{\sum_{i\in\mathcal{F}}\left|\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\} - \mathbf{1}\{h_t(\boldsymbol{x}_i)=1\}\right|}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
=&2\cdot\frac{\sum_{i\in\mathcal{F}}\mathbf{1}\{f_{\widehat{\theta}_t}(\boldsymbol{x}_i) > \tau, f_{\theta_0}(\boldsymbol{x}_i) < \tau\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{f_{\widehat{\theta}_t}(\boldsymbol{x}_i) < \tau, f_{\theta_0}(\boldsymbol{x}_i) > \tau\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{i\in\mathcal{E}_t\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
\leq&2\cdot\frac{\sum_{i\in\mathcal{F}}\mathbf{1}\{f_{\widehat{\theta}_t}(\boldsymbol{x}_i) > \tau, f_{\theta_0}(\boldsymbol{x}_i) < \tau-\varepsilon\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{f_{\widehat{\theta}_t}(\boldsymbol{x}_i) < \tau, f_{\theta_0}(\boldsymbol{x}_i) > \tau+\varepsilon\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{|f_{\theta_0}(\boldsymbol{x}_i) - \tau| < \varepsilon\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
&+2\cdot\frac{\sum_{i\in\mathcal{F}}\mathbf{1}\{i\in\mathcal{E}_t\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
\leq&2\cdot\frac{\sum_{i\in\mathcal{F}}\mathbf{1}\{|f_{\widehat{\theta}_t}(\boldsymbol{x}_i) - f_{\theta_0}(\boldsymbol{x}_i)| > \varepsilon\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{|f_{\theta_0}(\boldsymbol{x}_i) - \tau| < \varepsilon\} + \sum_{i\in\mathcal{F}}\mathbf{1}\{i\in\mathcal{E}_t\}}{\sum_{i\in\mathcal{F}}\mathbf{1}\{h^*(\boldsymbol{x}_i)=1\}} \\
=&2\cdot\frac{\mathbb{P}\left(\left|f_{\widehat{\theta}_t}(\boldsymbol{x}_i) - f_{\theta_0}(\boldsymbol{x}_i)\right| > \varepsilon\right) + \mathbb{P}\left(|f_{\theta_0}(\boldsymbol{x}_i) - \tau| < \varepsilon\right) + \mathbb{P}(i\in\mathcal{E}_t)}{\mathbb{P}(h^*(\boldsymbol{x}_i)=1)}.
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Proposition 1 Fix $\gamma > 0$. Since $N_t = \mathcal{O}(m)$ for all t, so we write $N_t \leq \Psi m$ for some constant Ψ and for $m \geq m_A$. Fix $\widehat{\varepsilon} > 0$ and define $p^* = \mathbb{P}(h^*(\boldsymbol{x}_i)=1)$. Per Statistical Condition 1, we can define m_B such that $\frac{m}{M} \leq \frac{\widehat{\varepsilon}}{9\Psi(N_{t-1})^{1/2-\gamma}}$ for all $m \geq m_B$. This implies that:

$$\mathbb{P}(y_{i\tau}=1 \text{ for } \tau \leq t-1) \leq \frac{\Psi m}{M} \leq \frac{\widehat{\varepsilon}}{9(N_{t-1})^{1/2-\gamma}} \tag{EC.2}$$

Now define $\varepsilon_0 > 0$ such that $\frac{C\varepsilon_0^\alpha}{9^\alpha} = \frac{\widehat{\varepsilon}}{9}$. We proceed to prove the main result.

$$\begin{aligned}
1-\mu_t =&\mathbb{P}(S_{it}=1 \mid y_{i1}=\cdots=y_{i,t-1}=0\ ;\ h_t(\boldsymbol{x}_i)=1) \\
=&\frac{\mathbb{P}(S_{it}=1 \mid h_t(\boldsymbol{x}_i)=1) - \mathbb{P}(y_{is}=1 \text{ for } s\leq t-1 \mid h_t(\boldsymbol{x}_i)=1)\mathbb{P}(S_{it}=1 \mid h_t(\boldsymbol{x}_i)=1, y_{is}=1 \text{ for } s\leq t-1)}{\mathbb{P}(y_{i1}=\cdots=y_{i,t-1}=0 \mid h_t(\boldsymbol{x}_i)=1)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{P}(S_{it}=1 \mid h_t(\boldsymbol{x}_i)=1) - \mathbb{P}(y_{is}=1 \text{ for } s\leq t-1 \mid h_t(\boldsymbol{x}_i)=1)\\
&\geq \mathbb{P}(S_{it}=1 \mid h^*(\boldsymbol{x}_i)=1) - \mathbb{P}(y_{is}=1 \text{ for } s\leq t-1 \mid h^*(\boldsymbol{x}_i)=1)\\
&\quad -4\cdot\frac{\mathbb{P}\left(\left|f_{\widehat{\theta}_t}(\boldsymbol{x}_i)-f_{\theta_0}(\boldsymbol{x}_i)\right| > \frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)}{\mathbb{P}(h^*(\boldsymbol{x}_i)=1)} -4\cdot\frac{\mathbb{P}\left(\left|f_{\theta_0}(\boldsymbol{x}_i)-\tau\right| < \frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)}{\mathbb{P}(h^*(\boldsymbol{x}_i)=1)} -4\cdot\frac{\mathbb{P}(i\in\mathcal{E}_t)}{\mathbb{P}(h^*(\boldsymbol{x}_i)=1)}\\
&\geq \mathbb{P}(S_{it}=1 \mid h^*(\boldsymbol{x}_i)=1) - \frac{\widehat{\varepsilon}}{9p^*(N_{t-1})^{1/2-\gamma}}\\
&\quad -4\cdot\frac{\mathbb{P}\left(\left|f_{\widehat{\theta}_t}(\boldsymbol{x}_i)-f_{\theta_0}(\boldsymbol{x}_i)\right| > \frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)}{p^*} -4\cdot\frac{\mathbb{P}\left(\left|f_{\theta_0}(\boldsymbol{x}_i)-\tau\right| < \frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)}{p^*} -4\cdot\frac{K}{p^*(N_{t-1})^{1/2}},
\end{aligned}
$$

where the equality comes from the law of total probabilities, the first inequality is obtained by lower bounding probabilities by 1, and the second inequality is derived from applying EC.2 twice. The third inequality comes from Assumption 2 which states that $p^*\geq\xi>0$ and Statistical Condition 3 which bounds the probability of i in $\mathcal{E}_t$. Now from Statistical Condition 4, we know that $\mathbb{P}\left(|f_\theta(\boldsymbol{x}_i)-\tau|\leq\frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)\leq C\left(\frac{\varepsilon_0}{9(N_{t-1})^{1/2-\gamma}}\right)^\alpha=\frac{\widehat{\varepsilon}}{9(N_{t-1})^{\alpha\cdot(1/2-\gamma)}}$. And from Statistical Condition 5, we know that f is twice continuously differentiable in θ and that Θ is compact, so f is Lipschitz continuous in $\theta\in\Theta$. Let L be the Lipschitz constant. We obtain:

$$
\begin{aligned}
1-\mu_t \geq& \mathbb{P}(S_{it}=1 \mid h^*(\boldsymbol{x}_i)=1) - \frac{\widehat{\varepsilon}}{9p^*(N_{t-1})^{1/2-\gamma}}\\
&-\frac{4}{p^*}\mathbb{P}\left(\|\widehat{\theta}_t-\theta\|_2>\frac{\varepsilon_0}{9L(N_{t-1})^{1/2-\gamma}}\right)-\frac{4\widehat{\varepsilon}}{9p^*(N_{t-1})^{\alpha\cdot(1/2-\gamma)}}-\frac{4K}{p^*(N_{t-1})^{1/2}}
\end{aligned}
$$

By Lemma EC.1, define m_C such that for all $m\geq m_C$, N_{t-1} will be sufficiently large for the following to hold:

$$
\mathbb{P}\left(\|\widehat{\theta}_t-\theta\|_2>\frac{\varepsilon_0}{9L(N_{t-1})^{1/2-\gamma}}\right)\leq\frac{\widehat{\varepsilon}}{9(N_{t-1})^{1/2-\gamma}},\ \forall t=1,\cdots,T \tag{EC.3}
$$

Now fix $m_0=\max\{m_A,m_B,m_C\}$ and consider any $m\geq m_0$. We obtain:

$$
\begin{aligned}
1-\mu_t \geq& \mathbb{P}(S_{it}=1 \mid h^*(\boldsymbol{x}_i)=1) - \frac{\widehat{\varepsilon}}{9p^*(N_{t-1})^{1/2-\gamma}}-\frac{4\widehat{\varepsilon}}{9p^*(N_{t-1})^{1/2-\gamma}}-\frac{4\widehat{\varepsilon}}{9p^*(N_{t-1})^{\alpha\cdot(1/2-\gamma)}}-\frac{4K}{p^*(N_{t-1})^{1/2}}\\
\geq& 1-\mu^*-\frac{\widehat{\varepsilon}+4K}{p^*(N_{t-1})^{\min\{\alpha,1\}\times(1/2-\gamma)}}.
\end{aligned}
$$

Therefore, we can define $\overline{\varepsilon}=\frac{2(\widehat{\varepsilon}+4K)}{p^*}$ such that:

$$
1-\mu_t\geq 1-\mu^*-\frac{\overline{\varepsilon}}{(N_{t-1}+1)^{\min\{\alpha,1\}\times(1/2-\gamma)}}.
$$

This completes the proof. □

EC.1.2. Proof of Corollary 1

Define $\widetilde{\mu}_t$ such that $\widetilde{S}_{it} \sim \text{Bern}(1-\widetilde{\mu}_t)$. Per Proposition 1, $\widetilde{\mu}_t \geq \mu_t$, irregardless of the possible dependence of μ_t on the previous set of selected facilities. Since $B_t \mid \mu_t \sim \text{Bino}\,(A_t, 1-\mu_t)$ and $\widetilde{B}_t \sim \text{Bino}\,(A_t, 1-\widetilde{\mu}_t)$, this implies:

$$\mathbb{P}\left[\sum_{t=1}^{T} \widetilde{B}_t \geq m\right] \leq \mathbb{P}\left[\sum_{t=1}^{T} B_t \geq m \mid \mu_t\right]$$

Therefore, by the law of total expectation, taking the expectation over μ_t gives:

$$\mathbb{P}\left[\sum_{t=1}^{T} \widetilde{B}_t \geq m\right] \leq \mathbb{P}\left[\sum_{t=1}^{T} B_t \geq m\right]$$

EC.1.3. Proof of Corollary 2

Recall that $B_t = \sum_{i:\, y_{it}=1} S_{it}$. Under Assumption 1 the indices with $y_{it}=1$ are drawn *with* replacement from the set $\mathcal{J}_t$. Therefore, Assumption 3 implies that every draw from $\mathcal{J}_t$ has the same conditional success probability

$$p_t = \frac{1}{|\mathcal{J}_t|}\sum_{i \in \mathcal{J}_t} \eta_i = \mathbb{P}(h_t(\boldsymbol{x}_i) = S_{it} \mid i \in \mathcal{J}_t) = 1 - \frac{\varepsilon p}{(N_{t-1}+1)^r} - \varepsilon\,(1-p),$$

so that $S_{it} \mid y_{it} = 1 \sim \text{Bernoulli}(p_t)$ for all $i \in \mathcal{J}_t$. The model specification $S_{it} \sim \text{Bernoulli}(\eta_i)$ further assumes independence across facilities, i.e. $S_{it} \perp\!\!\!\perp S_{jt}$ for $i \neq j$. Because sampling is with replacement, the selection indicators $\{y_{it}\}_{i\in\mathcal{J}_t}$ are independent conditional on $i \in \mathcal{J}_t$ as well; hence, conditional on $\mathcal{J}_t$, the random variables $\{S_{it} : y_{it} = 1\}$ are i.i.d. Bernoulli(p_t). Therefore

$$B_t \sim \text{Binomial}\big(A_t, p_t\big),$$

which completes the proof. □

EC.2. Proofs from Section 4

EC.2.1. Proof of Proposition 2

The fully-learned benchmark and the no-learning baseline exhibit similar formulations, with differences in the error rates. We make use of the following lemma, which proves a general result on the asymptotic number of trials required for a binomial distribution to satisfy the chance constraint.

Lemma EC.3. *Consider the following problem and denote by $n^*(m)$ its optimal solution:*

$$\min \quad \{A \;:\; \mathbb{P}\left[\text{Bino}\,(n,q) \geq m\right] \geq 1-\delta\}$$

The following holds:

$$n^* = \frac{m}{q} - \frac{\Phi^{-1}(\delta)\sqrt{1-q}}{q}\sqrt{m} + \mathcal{O}(1)$$

EC.2.1.1. Proof of Lemma EC.3. Let us define $S_n = \mathrm{Bino}\,(n,q)$.

Claim 1: Denote $\overline{n} = \frac{m}{q} - \frac{\Phi^{-1}(\delta)\sqrt{1-q}}{q}\sqrt{m}$. The following holds:

$$\mathbb{P}(S_{\overline{n}} \le m-1) = \delta + \mathcal{O}\left(\frac{1}{\sqrt{m}}\right)$$

We write $S_n = X_1 + \cdots + X_n$ with $X_1, \cdots, X_n$ defined as independent Bernoulli variables with probability of success q. We have:

$$\begin{aligned}
&\mathbb{E}(X_i) = q \\
&\mathrm{Var}(X_i) = q(1-q) \\
&\mathbb{E}(|X_i - \mathbb{E}(X_i)|^3) = q(1-q)(q^2 + (1-q)^2)
\end{aligned}$$

By the Berry-Esseen theorem, we obtain for some constant C:

$$\left|\mathbb{P}\left(\frac{S_n - nq}{\sqrt{nq(1-q)}} \le a\right) - \Phi(a)\right| \le C \cdot \frac{q(1-q)(q^2+(1-q)^2)}{q^{3/2}(1-q)^{3/2}} \cdot \frac{1}{\sqrt{n}} = C \cdot \frac{q^2+(1-q)^2}{\sqrt{q(1-q)}} \cdot \frac{1}{\sqrt{n}}$$

Let us denote by $x(n) = \frac{m-1/2-nq}{\sqrt{nq(1-q)}}$. Note that

$$\mathbb{P}(S_n \le m-1) = \mathbb{P}(S_n \le m-1/2) = \mathbb{P}\left(\frac{S_n - nq}{\sqrt{nq(1-q)}} \le x(n)\right)$$

We derive, for a constant $c_1 = C \cdot \frac{(q^2+(1-q)^2)}{\sqrt{q(1-q)}}$:

$$|\mathbb{P}(S_n \le m-1) - \Phi(x(n))| \le \frac{c_1}{\sqrt{n}}$$

Using the definition of $x(n)$, we obtain:

$$\left|\mathbb{P}(S_{\overline{n}} \le m-1) - \Phi\left(\frac{m - 1/2 - \overline{n}q}{\sqrt{\overline{n}q(1-q)}}\right)\right| \le \frac{c_1}{\sqrt{\overline{n}}},$$

that is, by construction of $\overline{n}$:

$$\left|\mathbb{P}(S_{\overline{n}} \le m-1) - \Phi\left(\frac{\Phi^{-1}(\delta)\sqrt{1-q}\sqrt{m} - 1/2}{\sqrt{(1-q)} \cdot \sqrt{m - \Phi^{-1}(\delta)\sqrt{1-q}\sqrt{m}}}\right)\right| \le \frac{c_1 \cdot \sqrt{q}}{\sqrt{m - \Phi^{-1}(\delta)\sqrt{1-q}\sqrt{m}}}$$

Note in particular that:

$$\Phi(x(n)) = \delta + \mathcal{O}\left(\frac{1}{\sqrt{m}}\right).$$

Hence:

$$\mathbb{P}(S_{\overline{n}} \le m-1) = \delta + \mathcal{O}\left(\frac{1}{\sqrt{m}}\right)$$

This completes the proof of Claim 1. We then want to show that the optimal value of n is within $\mathcal{O}(1)$ of $\overline{n}$. To this end, we need to study the sensitivity of the chance constraint with respect to n, which requires to study the sensitivity of $x(n)$ with respect to n.

Claim 2: There exists a constant c_2 such that, for any constant $\Delta > 0$, we have:

$$x(n+\Delta) - x(n) \leq -c_2 \cdot \frac{\Delta}{\sqrt{m}}$$

We write:

$$x(n) = \frac{m-1/2}{\sqrt{q(1-q)}} \cdot \frac{1}{\sqrt{n}} - \frac{\sqrt{q}}{\sqrt{1-q}} \cdot \sqrt{n}$$

Taking the differential with respect to n, we derive:

$$x'(n) = -\frac{1}{2}\frac{1}{\sqrt{1-q}}\left(\frac{m-1/2}{\sqrt{q}} \cdot \frac{1}{n^{3/2}} + \sqrt{q} \cdot \frac{1}{\sqrt{n}}\right)$$

In particular, for $n = \Theta(m)$, we get $x'(n) = -\Theta\left(\frac{1}{\sqrt{m}}\right)$, hence there exists a constant c_2 such that $x'(n) \leq -c_2 \cdot \frac{1}{\sqrt{m}}$ for m large enough. By the mean value theorem, we write $x(n+\Delta) - x(n) = x'(\widetilde{n}) \cdot \Delta$ for some $\widetilde{n} \in [n, n+\Delta]$ hence $\widetilde{n} = \Theta(m)$, which completes the proof of Claim 2.

Claim 3: There exist a constant Δ such that $\overline{n} - \Delta \leq n^* \leq \overline{n} + \Delta$.

Let us denote by $\varphi = \Phi'$. We know that φ is continuous and $\varphi(\Phi^{-1}(\delta)) > 0$, so there exist constants $\eta > 0$ and $\underline{\varphi} > 0$ such that $\varphi(x) \geq \underline{\varphi}$ for all x satisfying $|x - \Phi^{-1}(\delta)| \leq \eta$. Recall that $\Phi(x(\overline{n})) = \delta + \mathcal{O}\left(\frac{1}{\sqrt{m}}\right)$ (Claim 1) and that $x(\overline{n}+\Delta) - x(\overline{n}) = \mathcal{O}\left(\frac{1}{\sqrt{m}}\right)$ with $x(\overline{n}+\Delta) - x(\overline{n}) < 0$ (Claim 2). Therefore, for sufficiently large m, we have $|x(\overline{n}) - \Phi^{-1}(\delta)| \leq \eta$ and $|x(\overline{n}+\Delta) - \Phi^{-1}(\delta)| \leq \eta$. By the mean value theorem and Claim 2, there exist $\widehat{x} \in [x(\overline{n}), x(\overline{n}+\Delta)]$ such that $\Phi(x(\overline{n}+\Delta)) - \Phi(x(\overline{n})) = \varphi(\widehat{x})(x(\overline{n}+\Delta) - x(\overline{n}))$. Therefore, it comes:

$$\Phi(x(\overline{n}+\Delta)) - \Phi(x(\overline{n})) \leq \underline{\varphi} \cdot (x(\overline{n}+\Delta) - x(\overline{n})) \leq -\underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}}$$

Similarly, by proceeding as in Claim 2 and above, we can prove that:

$$\Phi(x(\overline{n}-\Delta)) - \Phi(x(\overline{n})) \geq \underline{\varphi} \cdot (x(\overline{n}-\Delta) - x(\overline{n})) \geq \underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}}$$

Using Claim 1, we can now establish that, for a sufficiently large constant Δ, $\overline{n}+\Delta$ is feasible whereas $\overline{n}-\Delta$ is not. We have, for some constants c_3 and c_4 coming from Claim 1, using $n = \Theta(m)$:

$$\begin{aligned}
\mathbb{P}(S_{\overline{n}+\Delta} \leq m-1) &\leq \Phi(x(\overline{n}+\Delta)) + \frac{c_3}{\sqrt{m}} \\
&\leq \Phi(x(\overline{n})) - \underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}} + \frac{c_3}{\sqrt{m}} \\
&\leq \delta + \frac{c_4}{\sqrt{m}} - \underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}} + \frac{c_3}{\sqrt{m}} \\
&\leq \delta
\end{aligned}$$

for Δ sufficiently large. Similarly:

$$\begin{aligned}\mathbb{P}(S_{\overline{n}-\Delta} \leq m-1) &\geq \Phi(x(\overline{n}-\Delta)) - \frac{c_3}{\sqrt{m}} \\ &\geq \Phi(x(\overline{n})) + \underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}} - \frac{c_3}{\sqrt{m}} \\ &\geq \delta - \frac{c_4}{\sqrt{m}} + \underline{\varphi} \cdot c_2 \cdot \frac{\Delta}{\sqrt{m}} - \frac{c_3}{\sqrt{m}} \\ &> \delta\end{aligned}$$

This proves that $\mathbb{P}(S_{\overline{n}+\Delta} \geq m) \geq 1-\delta$ and $\mathbb{P}(S_{\overline{n}-\Delta} \geq m) < 1-\delta$. This proves that $n^* = \overline{n} + \mathcal{O}(1)$, and completes the proof of the lemma.

EC.2.1.2. Proof of Proposition 2 Per Lemma EC.3, we know that:

$$\widehat{m}(\varepsilon,p) = \frac{m}{1-\varepsilon(1-p)} - \frac{\Phi^{-1}(\delta)\sqrt{\varepsilon(1-p)}}{1-\varepsilon(1-p)}\sqrt{m} + \mathcal{O}(1) \tag{EC.4}$$

$$m^{\mathrm{NL}}(\varepsilon) = \frac{m}{1-\varepsilon} - \frac{\Phi^{-1}(\delta)\sqrt{\varepsilon}}{1-\varepsilon}\sqrt{m} + \mathcal{O}(1) \tag{EC.5}$$

Therefore:

$$m^{\mathrm{NL}}(\varepsilon) - \widehat{m}(\varepsilon,p) = \frac{m\varepsilon p}{(1-\varepsilon)(1-\varepsilon(1-p))} + \mathcal{O}\left(\sqrt{m}\right)$$

This proves that $\texttt{Reg}^{\mathrm{NL}}(\varepsilon,p) = \Theta(m)$ as $m \to \infty$.

EC.2.2. Proof of Theorem 1

Throughout this proof, we denote by A_t^* the optimal solution of the stochastic problem (Problem $(\mathcal{P}^*)$), by $A_t^\dagger$ the solution of Algorithm 1, and by A_t^{D} the solution of the deterministic approximation (Problem $(\mathcal{P}^D)$). We denote by B_t^*, $B_t^\dagger$, and B_t^{D} the corresponding numbers of facilities successfully opened *at* period t, and by N_t^*, $N_t^\dagger$, and N_t^{D} the corresponding number of facilities successfully opened *by* period t.

Lemma 1 elicits the solution A_t^{D}; Lemma 2 establishes the feasibility of solution $A_t^\dagger$ in view of the chance constraint in Problem $(\mathcal{P}^*)$, thus providing a constructive upper bound of $m^*(\varepsilon,p,T)$; and Lemma 3 provides a lower bound of $m^*(\varepsilon,p,T)$. Throughout the proof, we will focus on the case where $r \neq 1$. The case where $r = 1$ is treated identically.

EC.2.2.1. Proof of Lemma 1 Note that the constraint $N_T^D \geq m$ is binding at optimality, so the problem can be re-written as:

$$\min \left\{ \sum_{t=1}^{T} A_t^D \;\middle|\; \sum_{t=1}^{T} A_t^D \left(1 - \frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{t-1} A_s^D + 1\right)^r} - \varepsilon \cdot (1-p)\right) = m \right\}$$

Let $\lambda \in \mathbb{R}$ be the dual variable associated with the equality constraint. The Karush–Kuhn–Tucker (KKT) conditions yield the following vanishing gradient equation:

$$1+\lambda\left(1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\varepsilon\cdot(1-p)\right)+\lambda\sum_{\tau=t+1}^{T}\frac{\varepsilon\cdot p\cdot r\cdot A_\tau^D}{\left(\sum_{s=1}^{\tau-1}A_s^D+1\right)^{r+1}}=0,\ \forall t\in\mathcal{T}$$

Clearly, $\lambda\neq 0$. After re-arranging the terms, we obtain:

$$\frac{1}{\left(\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\sum_{\tau=t+1}^{T}\frac{r\cdot A_\tau^D}{\left(\sum_{s=1}^{\tau-1}A_s^D+1\right)^{r+1}}=\frac{1+\lambda}{p\lambda\varepsilon}-\frac{1-p}{p},\ \forall t\in\mathcal{T}$$

Note that this expression is independent of t. Therefore, for all $t=1,\cdots,T-1$, we have:

$$\frac{1}{\left(\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\sum_{\tau=t+1}^{T}\frac{r\cdot A_\tau^D}{\left(\sum_{s=1}^{\tau-1}A_s^D+1\right)^{r+1}}=\frac{1}{\left(\sum_{s=1}^{t}A_s^D+1\right)^r}-\sum_{\tau=t+2}^{T}\frac{r\cdot A_\tau^D}{\left(\sum_{s=1}^{\tau-1}A_s^D+1\right)^{r+1}}.$$

This system of equations gives rise to a simple recurrence relation for A_t^D:

$$A_{t+1}^D=\frac{\left(\sum_{s=1}^{t}A_s^D+1\right)^{r+1}}{r\left(\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\frac{1}{r}\left(\sum_{s=1}^{t}A_s^D+1\right),\ \forall t=1,\cdots,T-1.$$

Let us define $u_t=\sum_{s=1}^{t}A_s^D+1$. By expressing $A_t^D=u_t-u_{t-1}$, we can re-write the recursion as:

$$u_{t+1}-u_t=\frac{u_t^{r+1}}{ru_{t-1}^r}-\frac{u_t}{r},\ \forall t=1,\cdots,T-1,$$

i.e., as:

$$u_{t+1}+\left(\frac{1}{r}-1\right)u_t=\frac{u_t^{r+1}}{ru_{t-1}^r},\ \forall t=1,\cdots,T-1. \tag{EC.6}$$

We are interested in the behavior of u_t as $m\to\infty$. We perform a leading-order analysis and a leading-coefficient analysis, assuming that $u_t=\Theta(g(t)m^{f(t)})$. Note, first, that the boundary condition of $u_0=1=\Theta(1)$ implies that $g(0)=1$ and $f(0)=0$. Moreover, we prove that $u_T=\frac{m}{1-(1-p)\varepsilon}+o(m)=\Theta(m)$, so that $g(T)=c_0$ and $f(T)=1$. Indeed, consider the solution $\bar{A}_1=\frac{m^{1/(r+1)}}{1-\varepsilon}$ and $\bar{A}_2=\frac{m}{1-(1-p)\varepsilon}$. We have:

$$\begin{aligned}N_2^D=&\bar{A}_1(1-\varepsilon)+\bar{A}_2\left(1-\frac{\varepsilon\cdot p}{(\bar{A}_1+1)^r}-(1-p)\varepsilon\right)\\ \geq&m^{1/(r+1)}+m-\frac{m}{(\bar{A}_1+1)^r}\\ \geq&m^{1/(r+1)}+m-\frac{m}{m^{r/(r+1)}}\\ =&m\end{aligned}$$

Therefore, $\left(\frac{m^{1/(r+1)}}{1-\varepsilon},\frac{m}{1-(1-p)\varepsilon},0,\cdots,0\right)$ is a feasible solution of Problem $(\mathcal{P}^D)$ with $T\geq 2$, so

$$\frac{m}{1-(1-p)\varepsilon}\leq u_T\leq\frac{m^{1/(r+1)}}{1-\varepsilon}+\frac{m}{1-(1-p)\varepsilon}+1$$

This proves that $u_T = \frac{m}{1-(1-p)\varepsilon} + o(m) = \Theta(m)$, hence $g(T) = c_0$ and $f(T) = 1$.

Let us derive the expression of $f(t)$. From Equation (EC.6), we obtain the following recursion:

$$f(t+1) = (r+1)f(t) - rf(t-1),$$

i.e.:

$$f(t+1) - f(t) = r\left(f(t) - f(t-1)\right).$$

We derive:

$$f(t+1) - f(t) = Kr^t, \quad \text{where } K = f(1) - f(0) = f(1).$$

By summation and by telescoping the sum, this yields:

$$\sum_{s=0}^{T-1}(f(s+1) - f(s)) = f(T) - f(0) = \frac{K}{1-r}\left(1 - r^T\right).$$

From the boundary conditions, we have $f(T) - f(0) = 1$, so that:

$$f(t+1) - f(t) = r^t \cdot \frac{1-r}{1-r^T}$$

Again, by summation and by telescoping the sum, the following holds:

$$f(t) = \sum_{s=0}^{t-1}(f(s+1) - f(s)) = \frac{1-r}{1-r^T}\sum_{s=0}^{t-1} r^s = \alpha_T(1-r^t).$$

In conclusion, the leading-order analysis leads to:

$$u_t = \Theta\left(m^{\alpha_T(1-r^t)}\right) = \Theta\left(m^{\alpha_T\left(1-r^t\right)}\right).$$

We conduct a similar analysis for the leading coefficient of u_t. Equation (EC.6) yields

$$g(t+1) = \frac{g(t)^{1+r}}{r(g(t-1))^r}$$

Define $h(t) = \log g(t)$ and we have the following recurrent relation with $h(0) = h(T) = 0$:

$$h(t+1) = (r+1)h(t) - rh(t-1) - \log r$$

We proceed as above, with the added drift term. We derive:

$$h(t+1) - h(t) = r\left(h(t) - h(t-1)\right) - \log(r)$$

$$h(t+1) - h(t) = Kr^t - \log r \cdot \sum_{s=0}^{t-1} r^s = K'r^t - \frac{1}{1-r}\log r, \quad \text{where } K = h(1) - h(0) \text{ and } K' = K + \frac{\log r}{1-r}.$$

$$\sum_{s=0}^{T-1}(h(s+1) - h(s)) = \underbrace{h(T) - h(0)}_{=\log(c_0)} = K'\frac{(1-r^T)}{1-r} - \frac{T\log r}{1-r} \implies K' = \frac{T\log r}{1-r^T} + \frac{\log(c_0)(1-r)}{1-r^T}$$

$$h(t+1) - h(t) = \left(\frac{T\log r}{1-r^T} + \frac{\log(c_0)(1-r)}{1-r^T}\right)r^t - \frac{\log r}{1-r}$$

$$h(t) = \frac{-\log r}{1-r}\left(t - T\alpha_T(1-r^t)\right) + \log(c_0) \cdot \alpha_T(1-r^t).$$

Bringing the two results together, we obtain:

$$u_t = r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)} m^{\alpha_T\left(1-r^t\right)} + o\left(m^{\alpha_T\left(1-r^t\right)}\right)$$

For ease of the exposition, let us write $u_t = r^{-\frac{\overline{h}(t)}{1-r}} c_0^{f(t)} m^{f(t)} + o\left(m^{f(t)}\right)$. Note that:

$$\overline{h}(t) - r\overline{h}(t-1) = (t - T\alpha_T(1-r^t)) - r\left(t-1-T\alpha_T(1-r^{t-1})\right) = (1-r)t + r - \frac{T(1-r)}{1-r^T} \tag{EC.7}$$

$$f(t) - rf(t-1) = \alpha_T(1-r^t) - r\cdot\alpha_T(1-r^{t-1}) = \alpha_T\cdot(1-r) \tag{EC.8}$$

We conclude as follows:

$$\begin{aligned}
(1-(1-p)\varepsilon))\sum_{t=1}^T A_t^D &= m + \sum_{t=1}^T \frac{A_t^D\cdot\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1} A_s^D + 1\right)^r} \\
&= m + \sum_{t=1}^T \frac{\varepsilon\cdot p\cdot(u_t - u_{t-1})}{u_{t-1}^r} \\
&= m + \varepsilon\cdot p\cdot\sum_{t=1}^T r^{-\frac{\overline{h}(t)-r\overline{h}(t-1)}{1-r}}\cdot c_0^{f(t)-rf(t-1)}\cdot m^{f(t)-rf(t-1)} + o\left(m^{f(t)-rf(t-1)}\right) \\
&= m + \varepsilon\cdot p\cdot\left(\sum_{t=1}^T \frac{1}{r^t}\right)\cdot\left(\frac{1}{r}\right)^{\frac{r}{1-r}}\cdot r^{T\cdot\alpha_T}\cdot c_0^{\alpha_T\cdot(1-r)}\cdot m^{\alpha_T\cdot(1-r)} + o\left(m^{\alpha_T\cdot(1-r)}\right) \\
&= m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + o\left(m^{\alpha_T\cdot(1-r)}\right),
\end{aligned}$$

where the fourth equality follows from Equations (EC.7) and (EC.8). The expressions for A_t^D follow immediately from the expression of u_t and the fact that $A_t^D = u_t - u_{t-1}$. □

EC.2.2.2. Proof of Remark 1 Per Lemma 1, we have:

$$\sum_{t=1}^T r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot m^{\alpha_T\cdot(1-r^t)} = c_0\left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + o\left(m^{\alpha_T\cdot(1-r)}\right)\right)$$

$$\sum_{t=1}^T r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot m^{\alpha_T\cdot(1-r^t)} \geq c_0\left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}\right)$$

The derivative of the last term of the left-hand side with respect to m is equal to c_0, so the derivative of the left-hand side is at least equal to c_0. Therefore, for m large enough:

$$\sum_{t=1}^T r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot (m+\Delta(m))^{\alpha_T\cdot(1-r^t)} \geq c_0\left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m)\right)$$

$$\sum_{t=1}^T r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot \left(m - m^{\alpha_T\cdot(1-r)}\right)^{\alpha_T\cdot(1-r^t)} \leq c_0\left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}\right)$$

This implies that $\ell \leq m + \Delta(m)$ and $\ell \geq m - m^{\alpha_T\cdot(1-r)}$. □

EC.2.2.3. Proof of Lemma 2 We make use of the following lemma, which provides a lower bound on the expected number of facilities that get successfully opened.

LEMMA EC.4. *We have:*

$$\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] \geq \begin{cases} m + \sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m} - \varepsilon\cdot p\cdot \alpha_T\cdot\zeta_T\cdot(1-r)\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}m^{\alpha_T\cdot(1-r)-\frac{1}{2}} & \text{if } p\neq 1 \\ m - \Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}m^{1/2\cdot\alpha_T\cdot(1-r)} + \varepsilon\cdot p\cdot\alpha_T\cdot\zeta_T\cdot(1-r)\Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}m^{3/2\cdot\alpha_T\cdot(1-r)-1} & \text{if } p=1. \end{cases}$$

Proof: By definition of B_t and by construction of $A_t^\dagger$, we have:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] &= \frac{1}{c_0}\sum_{t=1}^{T} A_t^\dagger - \sum_{t=1}^{T}\frac{A_t^\dagger\cdot\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r} \\ &= m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m) - \sum_{t=1}^{T}\frac{A_t^\dagger\cdot\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\end{aligned}$$

We then bound the last term as follows:

$$\begin{aligned}\sum_{t=1}^{T}\frac{A_t^\dagger}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r} &\leq \sum_{t=1}^{T}\frac{r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot\ell^{\alpha_T(1-r^t)}}{\left(r^{-\frac{1}{1-r}\left(t-1-T\cdot\alpha_T\cdot(1-r^{t-1})\right)}\cdot c_0^{\alpha_T\cdot(1-r^{t-1})}\cdot\ell^{\alpha_T(1-r^{t-1})}\right)^r} \\ &= \left(\sum_{t=1}^{T} r^{-t}\right) r^{-\frac{r}{1-r}} r^{T\cdot\alpha_T} c_0^{\alpha_T\cdot(1-r)}\ell^{\alpha_T\cdot(1-r)} \\ &= \zeta_T\cdot\ell^{\alpha_T\cdot(1-r)} \\ &\leq \zeta_T\cdot(m+\Delta(m))^{\alpha_T\cdot(1-r)} \\ &\leq \zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \alpha_T\zeta_T(1-r)\Delta(m)m^{\alpha_T\cdot(1-r)-1}\end{aligned}$$

where the first inequality comes from bounding the sum by the term associated with $s=t-1$, the next equalities follows from algebraic manipulations, the next inequality follows from Remark 1, and the last inequality stems from the fact that $(1+a)^\delta \leq 1+\delta a$ for $\delta<1$.

By plugging this bound into the previous equation, we obtain:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{T} B_t\right] &\geq m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m) - \varepsilon\cdot p\cdot\left(\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \alpha_T\zeta_T(1-r)\Delta(m)m^{\alpha_T\cdot(1-r)-1}\right) \\ &= m + \Delta(m) - \varepsilon\cdot p\cdot\alpha_T\zeta_T(1-r)\Delta(m)m^{\alpha_T\cdot(1-r)-1}\end{aligned}$$

This completes the proof by plugging the values of $\Delta(m)$ into the bound. □

Proof of Lemma 2.

Case 1: $p\neq 1$.

Note that $0\leq B_t^\dagger\leq A_t^\dagger$ for all $t=1,\cdots,T$. By Hoeffding's inequality, we have:

$$\mathbb{P}\left(N_T^\dagger < m\right) = \mathbb{P}\left(\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] - \sum_{t=1}^{T} B_t^\dagger < \mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] - m\right)$$

$$\leq \mathbb{P}\left(\left|\sum_{t=1}^{T} B_t^\dagger - \mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right]\right| > \left|\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] - m\right|\right)$$
$$\leq 2\exp\left(-2\frac{\left(\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] - m\right)^2}{\sum_{t=1}^{T} A_t^\dagger}\right)$$

By Lemma EC.4 and by construction of $A_t^\dagger$, this yields:

$$\mathbb{P}\left(N_T^\dagger < m\right) \leq 2\exp\left(-2\frac{\left(\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m} - \varepsilon\cdot p\alpha_T\zeta_T(1-r)\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}m^{\alpha_T\cdot(1-r)-\frac{1}{2}}\right)^2}{c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m}\right)}\right)$$

By taking the limit as m goes to infinity, we conclude:

$$\lim_{m\to\infty}\mathbb{P}\left(N_T^\dagger < m\right) \leq 2\exp\left(-2\left(\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\right)^2\frac{1}{c_0}\right) = \delta$$

Case 2: $p = 1$.

Since B_t follows a binomial distribution with A_t trials and failure probability $\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}$,

$$\mathrm{Var}(B_t^\dagger) = A_t^\dagger\left(1 - \frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}$$
$$\mathbb{E}(B_t^\dagger - E(B_t^\dagger))^3 = A_t^\dagger\left(1 - \frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\left(2\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r} - 1\right)$$

Moreover, since the variables $B_1, \cdots, B_T$ are independent, we have:

$$\begin{aligned}\mathrm{Var}\left(\sum_{t=1}^{T} B_t^\dagger\right) &= \sum_{t=1}^{T}\mathrm{Var}(B_t^\dagger)\\ &\leq \sum_{t=1}^{T} A_t^\dagger\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\\ &\leq \varepsilon\left(\zeta_T\cdot m^{\alpha_T\cdot(1-r)} - \alpha_T\zeta_T(1-r)\Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}m^{3/2\cdot\alpha_T\cdot(1-r)-1}\right)\\ &\leq 2\varepsilon\zeta_T\cdot m^{\alpha_T\cdot(1-r)},\end{aligned}$$

where the second inequality follows from the same procedure in the proof of Lemma EC.4 and the last inequality holds for m large enough. Similarly, we can also prove that:

$$\begin{aligned}\mathrm{Var}\left(\sum_{t=1}^{T} B_t^\dagger\right) &= \sum_{t=1}^{T}\mathrm{Var}(B_t^\dagger)\\ &\geq \frac{1}{2}\sum_{t=1}^{T} A_t^\dagger\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{t=1}^{T} \frac{\varepsilon r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot c_0^{\alpha_T\cdot(1-r^t)}\cdot \ell^{\alpha_T(1-r^t)}}{\left(\sum_{s=1}^{t-1} r^{-\frac{1}{1-r}(s-T\cdot\alpha_T\cdot(1-r^s))}\cdot c_0^{\alpha_T\cdot(1-r^s)}\cdot \ell^{\alpha_T(1-r^s)}+1\right)^r}\\
&\geq \frac{1}{2}\frac{\varepsilon c_0 \ell}{\left(\sum_{s=1}^{T-1} r^{-\frac{1}{1-r}(s-T\cdot\alpha_T\cdot(1-r^s))}\cdot c_0^{\alpha_T\cdot(1-r^s)}\cdot \ell^{\alpha_T(1-r^s)}+1\right)^r}\\
&= \frac{1}{2}\frac{\varepsilon c_0 \ell}{c_0^r\left(m+\varepsilon\cdot p\cdot \zeta_T\cdot m^{\alpha_T\cdot(1-r)}+\Delta(m)-\ell\right)^r}\\
&\geq \frac{1}{2}\frac{\varepsilon c_0\left(m-m^{\alpha_T(1-r)}\right)}{c_0^r\left(m+\varepsilon\cdot p\cdot \zeta_T\cdot m^{\alpha_T\cdot(1-r)}+\Delta(m)-(m-m^{\alpha_T(1-r)})\right)^r}\\
&= \Omega(m^{1-\alpha_T\cdot r(1-r)}),
\end{aligned}
$$

where the last inequality comes from Remark 1.

From the Berry-Esseen theorem, we obtain:

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{t=1}^{T} B_t^\dagger - \mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] \leq -k\sqrt{\sum_{t=1}^{T}\mathrm{Var}(B_t^\dagger)}\right)\\
&\leq \Phi(-k)+\mathcal{O}\left(\frac{\sum_{t=1}^{T} A_t^\dagger\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\left(\frac{2\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}-1\right)}{\left(\sum_{t=1}^{T} A_t^\dagger\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)^{3/2}}\right)\\
&= \Phi(-k)+\mathcal{O}\left(\left(\sum_{t=1}^{T} A_t^\dagger\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s^\dagger+1\right)^r}\right)^{-1/2}\right)\\
&= \Phi(-k)+\mathcal{O}\left(m^{-1/2(1-\alpha_T\cdot r(1-r))}\right),
\end{aligned}
$$

where the last equality stems from the lower bound on the variance term. For sufficiently large m, we get, using the upper bound on the variance term:

$$
\mathbb{P}\left(\sum_{t=1}^{T} B_t^\dagger - \mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] \leq -k\sqrt{2\varepsilon\zeta_T\cdot m^{\alpha_T\cdot(1-r)}}\right)\leq \Phi(-k) \tag{EC.9}
$$

Then, we have:

$$
\mathbb{P}\left(N_T^\dagger < m\right) = \mathbb{P}\left(\sum_{t=1}^{T} B_t^\dagger - \mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right] < m-\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right]\right)\leq \Phi\left(-\frac{\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right]-m}{\sqrt{2\varepsilon\zeta_T\cdot m^{\alpha_T\cdot(1-r)}}}\right)
$$

By Lemma EC.4, since $\mathbb{E}\left[\sum_{t=1}^{T} B_t^\dagger\right]\geq m+\Delta(m)-\varepsilon\cdot p\cdot\alpha_T\zeta_T(1-r)\Delta(m)m^{\alpha_T\cdot(1-r)-1}$ this yields:

$$
\mathbb{P}\left(N_T^\dagger < m\right)\leq \Phi\left(-\frac{-\Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}\cdot m^{1/2\cdot\alpha_T\cdot(1-r)}+\varepsilon\cdot p\cdot\alpha_T\cdot\zeta_T\cdot(1-r)\Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}m^{3/2\cdot\alpha_T\cdot(1-r)-1}}{\sqrt{2\varepsilon\zeta_T\cdot m^{\alpha_T\cdot(1-r)}}}\right)
$$

By taking the limit as m goes to infinity, we conclude: $\lim_{m\to\infty}\mathbb{P}\left(N_T^\dagger<m\right)\leq\delta$ □

EC.2.2.4. Proof of Lemma 3 Case 1: $p=1$.

Let k be such that $\frac{1}{2}\alpha_T\cdot(1-r)<k<\alpha_T\cdot(1-r)$. Let us define the following "good" event:

$$\mathcal{E}=\left\{\left|B_t-A_t\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}\right)\right|\leq m^k,\ \forall t\in\{1,\cdots,T\}\right\}$$

Lemma EC.5. *Assume that $p=1$. If $A_t\leq m+\mathcal{O}(m^{\alpha_T\cdot(1-r)})$ for all $t=1,\cdots,T$, then $\mathbb{P}(\mathcal{E}^c)=\mathcal{O}(\exp\left(-m^{2k-\alpha_T\cdot(1-r)}\right))$ as $m\to\infty$.*

Proof: By construction, $\mathbb{E}[B_t]=A_t\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}\right)\leq A_t$. From Bernstein's inequality, using the definition of $B_t=\sum_{i:y_{it}=1}S_{it}$:

$$\begin{aligned}\mathbb{P}(|B_t-\mathbb{E}[B_t]|\geq m^k)&\leq 2\exp\left(-\frac{1/2\cdot m^{2k}}{\sum_{i:y_{it}=1}\mathrm{Var}(S_{it})+1/3\cdot m^k}\right)\\&=2\exp\left(-\frac{1/2\cdot m^{2k}}{\mathrm{Var}(B_t)+1/3\cdot m^k}\right)\\&\leq 2\exp\left(-\frac{m^{2k}}{4\varepsilon\zeta_T m^{\alpha_T\cdot(1-r)}+2m^k/3}\right)\\&=\mathcal{O}\left(\exp\left(-m^{2k-\alpha_T\cdot(1-r)}\right)\right),\end{aligned}$$

where the equality follows from the independence of $B_1,\cdots,B_T$ and the second inequality uses the lower bound on the variance term from the proof of Lemma 2. By assumption, $2k-\alpha_T\cdot(1-r)>0$, so $\mathbb{P}\left(|B_t-\mathbb{E}[B_t]|\geq m^k\right)\to 0$ as $m\to\infty$. We obtain from the union bound:

$$\begin{aligned}\mathbb{P}(\mathcal{E}^c)&=\mathbb{P}\left(\exists t\in\{1,\cdots,T\}\ :\ \left|B_t-A_t\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}\right)\right|\geq m^k\right)\\&\leq\sum_{t=1}^{T}\mathbb{P}\left(\left|B_t-A_t\left(1-\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}\right)\right|\geq m^k\right)\\&=\mathcal{O}(\exp\left(-m^{2k-\alpha_T\cdot(1-r)}\right))\end{aligned}$$

This completes the proof of Lemma EC.5. □

Lemma EC.6. *Assume that $p=1$. Let $(B_1^*,\cdots,B_T^*)$ be a solution of Problem $(\mathcal{P}^*)$. If $\delta=o\left(m^{\alpha_T\cdot(1-r)-1}\right)$, then as $m\to\infty$:*

$$\sum_{\sigma=0}^{\infty}\sigma\mathbb{P}\left(\sum_{t=1}^{T}B_t^*=\sigma\mid\mathcal{E}\right)\geq m+o\left(m^{\alpha_T\cdot(1-r)}\right)$$

$$\sum_{\sigma=0}^{\infty}(\sigma-Tm^k)^{\alpha_T\cdot(1-r)}\mathbb{P}\left(\sum_{t=1}^{T}B_t^*=\sigma\mid\mathcal{E}\right)\geq m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)$$

Proof: Note that the optimal solution of Problem $(\mathcal{P}^*)$ must satisfy the constraint:

$$\mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m\right) \geq 1-\delta = 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right) \tag{EC.10}$$

From the law of total probabilities, we obtain, using Equation (EC.10) and Lemma EC.5:

$$\begin{aligned}
\mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m \mid \mathcal{E}\right) &= \frac{\mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m\right) - \mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m \mid \mathcal{E}^c\right)\mathbb{P}(\mathcal{E}^c)}{\mathbb{P}(\mathcal{E})} \\
&\geq 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right) - \mathcal{O}\left(\exp\left(-m^{2k-\alpha_T\cdot(1-r)}\right)\right) \\
&= 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)
\end{aligned}$$

Then, we have:

$$\sum_{\sigma=0}^{\infty} \sigma \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) \geq m\cdot\mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m \mid \mathcal{E}\right) \geq m + o\left(m^{\alpha_T\cdot(1-r)}\right)$$

Similarly:

$$\begin{aligned}
&\sum_{\sigma=0}^{\infty} (\sigma - Tm^k)^{\alpha_T\cdot(1-r)} \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) \\
&\geq (m - Tm^k)^{\alpha_T\cdot(1-r)} \mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m \mid \mathcal{E}\right) \\
&= m^{\alpha_T\cdot(1-r)}(1-Tm^{k-1})^{\alpha_T\cdot(1-r)}\left(1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)\right) \\
&= m^{\alpha_T\cdot(1-r)}\left(1 - T^{\alpha_T\cdot(1-r)} m^{(k-1)\alpha_T\cdot(1-r)} + o\left(m^{(k-1)\alpha_T\cdot(1-r)}\right)\right)\left(1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)\right) \\
&= m^{\alpha_T\cdot(1-r)} + o(m^{\alpha_T\cdot(1-r)}),
\end{aligned}$$

where the first equality results from Lemma EC.5, the second equality follows from Taylor expansion and the last equality follows from the assumption that $k < \alpha_T(1-r) \leq 1$. This completes the proof of Lemma EC.6. □

Now, let $(A_1^*, \cdots, A_T^*)$ and $(B_1^*, \cdots, B_T^*)$ denote the optimal solution of Problem $(\mathcal{P}^*)$. We obtain from Lemma EC.5:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} A_t^*\right] &= \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right]\mathbb{P}(\mathcal{E}) + \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}^c\right]\mathbb{P}(\mathcal{E}^c) \\
&\geq \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right]\left(1 - \mathcal{O}(\exp(-m^{2k-\alpha_T\cdot(1-r)}))\right) \\
&= \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right] + o(1)
\end{aligned}$$

Conditioned on $\mathcal{E}$, we have:

$$\sum_{t=1}^{T} B_t^* = \left|\sum_{t=1}^{T} \mathbb{E}(B_t^*) - \sum_{t=1}^{T}(B_t^* - \mathbb{E}(B_t^*))\right| \leq \sum_{t=1}^{T} \mathbb{E}(B_t^*) + \sum_{t=1}^{T} |B_t^* - \mathbb{E}(B_t^*)| \leq \sum_{t=1}^{T} \mathbb{E}(B_t^*) + Tm^k$$

Therefore, the following holds:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right] &= \sum_{\sigma=0}^{\infty} \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}, \sum_{t=1}^{T} B_t^* = \sigma\right] \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) \\
&= \sum_{\sigma=0}^{\infty} \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}, \sum_{t=1}^{T} B_t^* = \sigma, \sum_{t=1}^{T} A_t^* \left(1 - \frac{\varepsilon}{\left(\sum_{s=1}^{t-1} A_s^* + 1\right)^r}\right) \geq \sigma - Tm^k\right] \\
&\qquad \cdot \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right)
\end{aligned}$$

The first expression resembles the deterministic variant of the full problem, with a right-hand side target of $\sigma - Tm^k$. From Lemma 1, we can lower bound it to obtain:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right] \geq\; & c_0 \sum_{\sigma=0}^{\infty} \left(\sigma - Tm^k + \varepsilon \cdot \zeta_T \cdot \left(\sigma - Tm^k\right)^{\alpha_T \cdot (1-r)} + o\left(\left(\sigma - Tm^k\right)^{\alpha_T \cdot (1-r)}\right)\right) \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) \\
& + o(1) \\
\geq\; & c_0 \sum_{\sigma=0}^{\infty} \sigma \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) + c_0 \cdot \varepsilon \cdot \zeta_T \cdot \sum_{\sigma=0}^{\infty} (\sigma - Tm^k)^{\alpha_T \cdot (1-r)} \mathbb{P}\left(\sum_{t=1}^{T} B_t^* = \sigma \mid \mathcal{E}\right) \\
& + o\left(m^{\alpha_T \cdot (1-r)}\right) \quad \text{(because } k < \alpha_T(1-r)\text{)} \\
\geq\; & c_0 \left(m + \varepsilon \cdot \zeta_T \cdot m^{\alpha_T \cdot (1-r)} + o\left(m^{\alpha_T \cdot (1-r)}\right)\right) \quad \text{(Lemma EC.6)}
\end{aligned}$$

The proof is completed since $\mathbb{E}\left[\sum_{t=1}^{T} A_t^*\right] \geq \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}\right] + o(1)$.

Case 2: $p \neq 1$.

First, we could follow the same logic as in the proof for $p = 1$, and get:

$$m^*(\varepsilon, p, T) \geq c_0 \left(m + \varepsilon \cdot p \cdot \zeta_T \cdot m^{\alpha_T \cdot (1-r)} + o\left(m^{\alpha_T \cdot (1-r)}\right)\right)$$

This bound is tight if $\alpha_T(1-r) \geq 1/2$, since the upper-bounding solution admits a buffer of $\Theta(m^{\alpha_T(1-r)})$ in that case; however, if $\alpha_T(1-r) < 1/2$, the bound may no longer be tight since the the upper-bounding buffer then grows in $\Theta(\sqrt{m})$. To improve it, we assume by contradiction that:

$$m^*(\varepsilon, p, T) = \sum_{t=1}^{T} A_t^* < c_0 \left(m - \Phi^{-1}(\delta) \sqrt{2\varepsilon(1-p)m}\right)$$

Consider independent trials $S'_{it} \sim \text{Bern}\left(1 - \varepsilon \cdot (1-p)\right)$ and define $B'_t = \sum_{i \in \mathcal{A}_t} S'_{it}$. We have:

$$\mathbb{P}\left(\sum_{t=1}^{T} B_t^* \geq m\right) \leq \mathbb{P}\left(\sum_{t=1}^{T} B'_t \geq m\right)$$

Then by the Berry-Esseen Theorem, we have:

$$\mathbb{P}\left(\sum_{t=1}^{T} B'_t < m\right) = \mathbb{P}\left(\sum_{t=1}^{T} B'_t - \mathbb{E}\left[\sum_{t=1}^{T} B'_t\right] < m - \sum_{t=1}^{T} A_t^* (1 - \varepsilon(1-p))\right)$$

$$=\Phi\left(\frac{m-\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))}{\sqrt{\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))\varepsilon(1-p)}}\right)+\mathcal{O}\left(\frac{\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))\varepsilon(1-p)(2\varepsilon(1-p)-1)}{\left(\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))\varepsilon(1-p)\right)^{3/2}}\right)$$

$$=\Phi\left(\frac{m-\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))}{\sqrt{\sum_{t=1}^{T}A_t^*(1-\varepsilon(1-p))\varepsilon(1-p)}}\right)+o(1)$$

We use the contradiction assumption $\sum_{t=1}^{T}A_t^*<c_0\left(m-\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m}\right)$ to bound the numerator and the fact that $\sum_{t=1}^{T}A_t^*<2c_0m$ to bound the denominator. We obtain

$$\mathbb{P}\left(\sum_{t=1}^{T}B_t'<m\right)>\Phi\left(\frac{\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m}}{\sqrt{2\varepsilon(1-p)m}}\right)+o(1)=\delta+o(1)$$

Therefore, for sufficiently large m:

$$\mathbb{P}\left(\sum_{t=1}^{T}B_t^*\geq m\right)\leq\mathbb{P}\left(\sum_{t=1}^{T}B_t'\geq m\right)<1-\delta$$

This contradicts the fact that A_t^* is a feasible (and optimal) solution to Problem $(\mathcal{P}^*)$. Therefore, for sufficiently large m:

$$m^*(\varepsilon,p,T)=\sum_{t=1}A_t^*\geq c_0\left(m-\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m}\right)$$

Combining both bounds gives us:

$$m^*(\varepsilon,p,T)=\sum_{t=1}A_t^*\geq c_0\left(m+\max\{-\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m},\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}\}+o\left(m^{\alpha_T(1-r)}\right)\right)$$

This completes the proof. $\square$

EC.2.2.5. Proof of Theorem 1 The main result follows directly from the lemmas. Per Lemma 2, the solution given in Algorithm 1 is feasible in Problem $(\mathcal{P}^*)$. Per Equations (6) and (7), it achieves an upper bound of:

$$m^\dagger(\varepsilon,p,T)=\begin{cases}c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m}\right) & \text{if } p\neq 1\\ c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)\right) & \text{if } p=1\end{cases}$$

Moreover, since $\delta=o\left(m^{\alpha_T\cdot(1-r)-1}\right)$ by assumption, the optimal solution of Problem $(\mathcal{P}^*)$ admits the following lower bound, per Lemma 3:

$$m^*(\varepsilon,p,T)\geq\begin{cases}c_0\left(m+\max\{-\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m},\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}\}\right)+o\left(m^{\alpha_T(1-r)}\right) & \text{if } p\neq 1\\ c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)\right) & \text{if } p=1\end{cases}$$

Recall from Equation (EC.4) that

$$\widehat{m}(\varepsilon,p)=c_0m-c_0\cdot\Phi^{-1}(\delta)\sqrt{\varepsilon(1-p)m}+\mathcal{O}(1)$$

Therefore, if $p=1$ or if ($p\neq 1$ and $\alpha_T\cdot(1-r)>1/2$), then:

$$m^{\dagger}(\varepsilon,p,T)=c_0m+c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)$$
$$m^{*}(\varepsilon,p,T)=c_0m+c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)$$
$$\widehat{m}(\varepsilon,p)=c_0m+o\left(m^{\alpha_T\cdot(1-r)}\right)$$

This proves that:

$$\texttt{Reg}^{\dagger}(\varepsilon,p,T)=c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)$$
$$\texttt{Reg}^{*}(\varepsilon,p,T)=c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+o\left(m^{\alpha_T\cdot(1-r)}\right)$$

If $p\neq 1$ and $\alpha_T\cdot(1-r)<1/2$ (the case $\alpha_T\cdot(1-r)=1/2$ is treated similarly by adjusting the second-leading coefficient term of $m^{\dagger}(\varepsilon,p,T)$), we have:

$$m^{\dagger}(\varepsilon,p,T)=c_0m+c_0\cdot\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m}+o\left(\sqrt{m}\right)$$
$$m^{*}(\varepsilon,p,T)=c_0m-c_0\cdot\Phi^{-1}(\delta)\sqrt{2\varepsilon(1-p)m}+o\left(\sqrt{m}\right)$$
$$\widehat{m}(\varepsilon,p)=c_0m-c_0\cdot\Phi^{-1}(\delta)\sqrt{\varepsilon(1-p)m}+o(\sqrt{m})$$

This proves that:

$$\texttt{Reg}^{\dagger}(\varepsilon,p,T)=\Theta(\sqrt{m})$$
$$\texttt{Reg}^{*}(\varepsilon,p,T)=\Theta(\sqrt{m})$$

This completes the proof. □

EC.3. Details on the application to real-world datasets

We apply the online learning and optimization methodology to the four datasets (bank, default, occupancy, and online) from the University of California at Irvine (UCI) (2020), as described in Section 5.2. Table EC.1 reports descriptive statistics on the datasets.

We show the improvement in predictive performance as a function of the accrued sample size under the online learning environment from Section 3.1, thereby providing empirical evidence for the learning function in Assumption 3 from Proposition 1 and Corollary 1. For every dataset, we train a random forest classification model and follow the learning procedure outlined in Section 3.1 by iteratively randomly selecting 10 samples with positive predictions from the current classification model to serve as training samples for the next round (Assumption 1, with $\mathcal{E}=\emptyset$). We repeat the procedure 10 times for each dataset. Figure EC.1 plots the out-of-sample classification error as a function of the sample size, averaged across the ten instances. At each iteration, the out-of-sample classification error is calculated using all samples that have not been utilized for training.

Table EC.1 and Figure EC.1 also report the best fits of the form $\frac{\gamma}{n^r}+\beta$ (which corresponds to Assumption 3). These results suggest a clear downward trend as the sample size increases, as well as a concave dependency indicating diminishing returns as the sample size becomes increasingly large. The fitted curves yield a good first-order approximation of the learning error, so that the assumption that the learning error decay in $\mathcal{O}(1/n^r)$ characterizes fairly well the impact of additional data points on predictive performance—even in our online learning environment where data samples are not independent over time. These results complement our statistical learning results with an empirical justification of the general-purpose characterization of the machine learning error.

Table EC.1 Descriptive statistics of the 10 datasets and corresponding data-driven experimental setup.

Dataset	Positive class	# observations	# features	Positive rate	Best fit
Bank	521	4,521	16	11.5%	$\frac{0.30}{n^{0.097}}$
Default	6,636	30,000	23	22.1%	$\frac{1}{n^{0.682}}+0.18$
Occupancy	2,049	9,752	6	21.0%	$\frac{0.36}{n^{0.149}}$
Online	1,908	12,330	6	15.5%	$\frac{19}{n^{0.085}}$

EC.4. Proof of statements from Section 5.3

EC.4.1. Proof of Theorem 2

We define Problem $(\mathcal{Q}^*_T(m,N_0))$ as follows, with $N_0=\gamma m^\beta+\mathcal{O}(m^\beta)$:

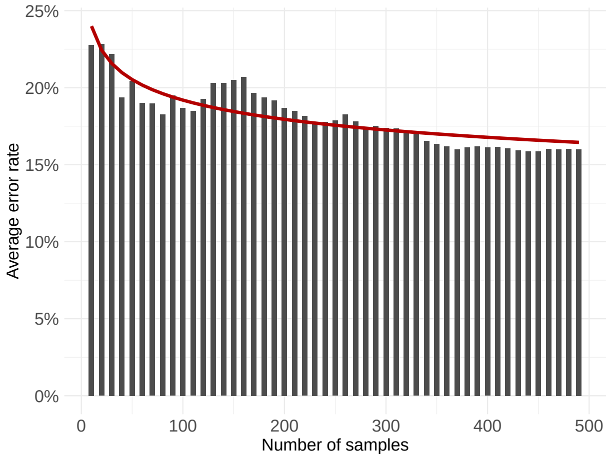
$$\begin{aligned}
\min \quad & \sum_{t=1}^T A_t && (\mathcal{Q}^*_T(m,N_0))\\
\text{s.t.} \quad & \mathbb{P}\left[\sum_{t=1}^T B_t \ge m\right] \ge 1-\delta \\
& B_t \sim \text{Bino}\left(A_t, 1-\frac{\varepsilon\cdot p}{\left(N_0\sum_{s=1}^{t-1}A_s+1\right)^r}-\varepsilon\cdot(1-p)\right),\ \forall t\in\mathcal{T}\\
& A_t, B_t \ge 0,\ \forall t\in\mathcal{T}
\end{aligned}$$

We adjust the algorithm to account for offline data as follows:

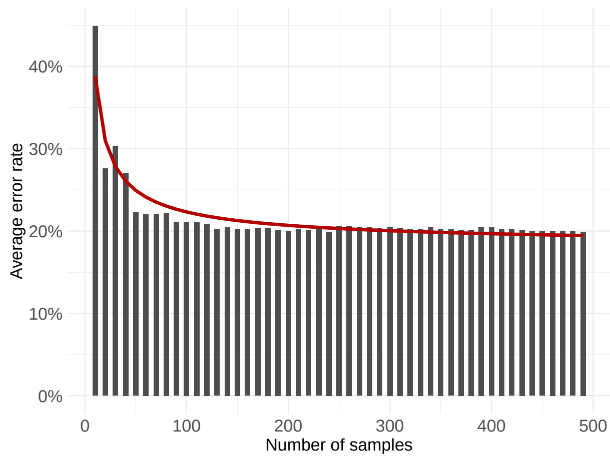
$$\mathcal{A}(T,m,\gamma,\beta):\quad A_t^\dagger = r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot\left(\frac{c_0}{\gamma}\right)^{\alpha_T\cdot(1-r^t)}\cdot\ell^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}, \tag{EC.11}$$

where ℓ is the unique positive solution to the following equation:

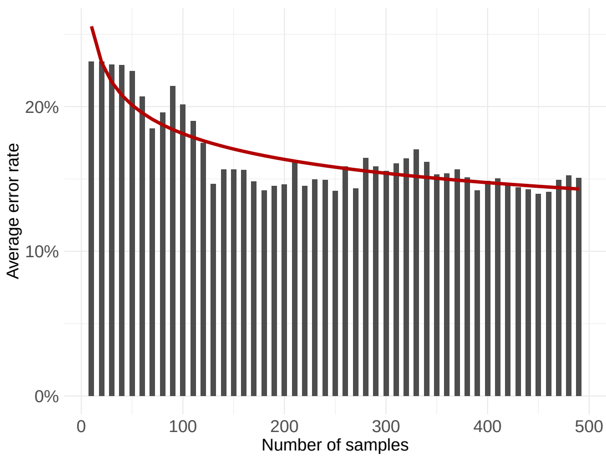
$$\mathcal{L}(T,m,\gamma,\beta):\quad \sum_{t=1}^T A_t^\dagger = c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot\gamma^{-\alpha_T(1-r)}\cdot m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)}+\Delta(m)\right),$$



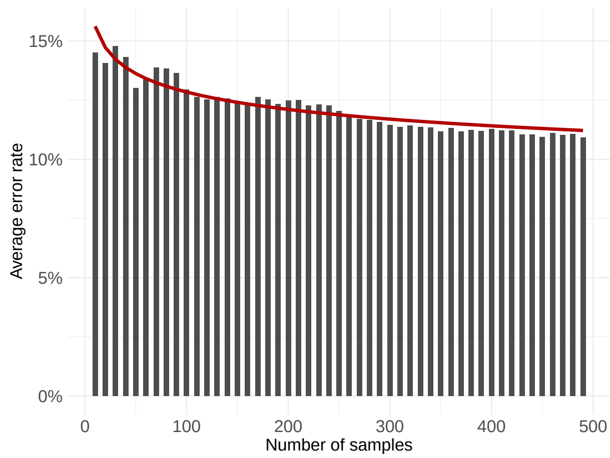
(a) 'Bank' dataset.



(b) 'Default' dataset.



(c) 'Occupancy' dataset.



(d) 'Online' dataset.

Figure EC.1 Average error of the classification models as a function of the accrued sample size. The red curve shows the best fit of the form $\frac{\gamma}{n^r}+\beta$.

and

$$\Delta(m)=\begin{cases}\sqrt{\frac{c_0}{2}\log\frac{2}{\delta}}\sqrt{m} & \text{if } p\neq 1,\\ -\Phi^{-1}(\delta)\sqrt{2\varepsilon\zeta_T}m^{\frac{1}{2}\cdot((1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r))} & \text{if } p=1.\end{cases}$$

The only non-trivial change from Theorem 1 comes from calculating the optimal solution of the its deterministic approximation, formulated as follows. The solution is given in Lemma EC.7, which is analogous to Lemma 1.

$$\min\left\{\sum_{t=1}^{T}A_t^D \;\middle|\; \sum_{t=1}^{T}A_t^D\left(1-\frac{\varepsilon\cdot p}{\left(N_0+\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\varepsilon\cdot(1-p)\right)=m\right\} \qquad (\mathcal{Q}_T^D(m,N_0))$$

Lemma EC.7. *Let $m^D(\varepsilon,T)$ be the optimal objective value of Problem $(\mathcal{Q}_T^D(m,N_0))$, and $A_1^D,\cdots,A_T^D$ be its solution. For any $T\geq 2$, we have as $m\to\infty$:*

$$A_t^D=r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot\left(\frac{c_0}{\gamma}\right)^{\alpha_T\cdot(1-r^t)}\cdot m^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}+o\left(m^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}\right),\ \forall t=1,\cdots,T$$

$$m^D(\varepsilon,T)=c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot\gamma^{-\alpha_T(1-r)}\cdot m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)}\right)+o\left(m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)}\right).$$

Proof of Lemma EC.7 Let $\lambda\in\mathbb{R}$ be the dual variable associated with the equality constraint. The KKT conditions yield the following vanishing gradient equation:

$$1+\lambda\left(1-\frac{\varepsilon\cdot p}{\left(N_0+\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\varepsilon\cdot(1-p)\right)+\lambda\sum_{\tau=t+1}^{T}\frac{r\cdot\varepsilon\cdot p\cdot A_\tau^D}{\left(N_0+\sum_{s=1}^{\tau-1}A_s^D+1\right)^{r+1}}=0,\ \forall t\in\mathcal{T}$$

Using the same logic as in the proof of Lemma 1, we can define $u_t=N_0+\sum_{s=1}^{t}A_s^D+1$ for $t=1,\cdots,T$, which gives rise to the same recursion:

$$u_{t+1}+\left(\frac{1}{r}-1\right)u_t=\frac{u_t^{r+1}}{ru_{t-1}^r},\ \forall t=1,\cdots,T-1.$$

The difference comes from the boundary condition. Let us write $u_t=\Theta(g(t)m^{f(t)})$ toward our leading-order and leading-coefficient analyses. Since $u_0=\gamma m^\beta+o(m^\beta)$, we now have $g(0)=\gamma$ and $f(0)=\beta$. Moreover, since the solution to Problem $(\mathcal{Q}_T^D(m,N_0))$ is upper bounded by that to Problem $(\mathcal{P}^D)$, we still have $g(T)=c_0$ and $f(T)=1$.

We perform the same leading-order analysis as in Lemma 1, with $f(T)-f(0)=1-\beta$ instead of $f(T)-f(0)=1$. Therefore:

$$\begin{aligned}
&f(T)-f(0)=1-\beta=\frac{K}{1-r}\left(1-r^T\right)\\
&f(t+1)-f(t)=Kr^t=(1-\beta)\cdot r^t\cdot\frac{1-r}{1-r^T}\\
&f(t)=\sum_{s=0}^{t-1}(f(s+1)-f(s))+\beta=(1-\beta)\cdot\frac{1-r}{1-r^T}\cdot\sum_{s=0}^{t-1}r^s+\beta=(1-\beta)\cdot\alpha_T(1-r^t)+\beta
\end{aligned}$$

In conclusion, the leading-order analysis leads to:

$$u_t=\Theta\left(m^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}\right).$$

We now proceed with the leading-coefficient analysis. Using $h(t)=\log g(t)$, we obtain the same recursion as in Lemma 1 except for the boundary conditions $h(T)=0$ and $h(0)=\log(\gamma)$:

$$h(t+1)=(r+1)h(t)-rh(t-1)-\log r$$

We obtain the following modified equations from the proof of Lemma 1:

$$\begin{aligned}
&\sum_{s=0}^{T-1}(h(s+1)-h(s))=\underbrace{h(T)-h(0)}_{=-\log(c_0)-\log(\gamma)}=K'\frac{(1-r^T)}{1-r}-\frac{T\log r}{1-r}\\
&\implies h(t)=\frac{-\log r}{1-r}\left(t-T\alpha_T(1-r^t)\right)+\log(c_0/\gamma)\cdot\alpha_T(1-r^t).
\end{aligned}$$

Bringing the two results together, we obtain:

$$u_t = r^{-\frac{1}{1-r}\left(t-T\cdot\alpha_T\cdot(1-r^t)\right)}\cdot\left(\frac{c_0}{\gamma}\right)^{\alpha_T\cdot(1-r^t)}\cdot m^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}+o\left(m^{(1-\beta)\alpha_T\left(1-r^t\right)+\beta}\right)$$

For ease of the exposition, let us write $u_t = r^{-\frac{\overline{h}(t)}{1-r}}\left(\frac{c_0}{\gamma}\right)^{f(t)} m^{\overline{f}(t)} + o\left(m^{\overline{f}(t)}\right)$. As earlier, $\overline{h}(t) - r\overline{h}(t-1) = (1-r)t + r - \frac{T(1-r)}{1-r^T}$ and $f(t) - rf(t-1) = \alpha_T\cdot(1-r)$. Moreover:

$$\overline{f}(t) - r\overline{f}(t-1) = (1-\beta)(f(t)-rf(t-1)) + \beta(1-r) = (1-\beta)\cdot\alpha_T\cdot(1-r) + \beta(1-r)$$

We conclude as follows:

$$\begin{aligned}
&(1-(1-p)\varepsilon))\sum_{t=1}^{T}A_t^D\\
&= m + \sum_{t=1}^{T}\frac{A_t^D\cdot\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s^D+1\right)^r}\\
&= m + \sum_{t=1}^{T}\frac{\varepsilon\cdot p(u_t-u_{t-1})}{u_{t-1}^r}\\
&= m + \varepsilon\cdot p\cdot\sum_{t=1}^{T} r^{-\frac{\overline{h}(t)-r\overline{h}(t-1)}{1-r}}\cdot\left(\frac{c_0}{\gamma}\right)^{f(t)-rf(t-1)}\cdot m^{\overline{f}(t)-r\overline{f}(t-1)} + o\left(m^{\overline{f}(t)-r\overline{f}(t-1)}\right)\\
&= m + \varepsilon\cdot p\cdot\left(\sum_{t=1}^{T}\frac{1}{r^t}\right)\cdot\left(\frac{1}{r}\right)^{\frac{r}{1-r}}\cdot r^{T\cdot\alpha_T}\cdot\left(\frac{c_0}{\gamma}\right)^{\alpha_T\cdot(1-r)}\cdot m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)} + o\left(m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)}\right)\\
&= m + \varepsilon\cdot p\cdot\zeta_T\cdot\gamma^{-\alpha_T\cdot(1-r)}\cdot m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)} + o\left(m^{(1-\beta)\cdot\alpha_T\cdot(1-r)+\beta(1-r)}\right)
\end{aligned}$$

This concludes the proof of Lemma EC.7. $\square$

EC.4.2. Proof of Theorem 3

We define Problem $(\mathcal{P}_{WS}^*)$ as follows:

$$\begin{aligned}
\min\quad & cA_0 + \sum_{t=1}^{T}A_t && (\mathcal{P}_{WS}^*)\\
\text{s.t.}\quad & \mathbb{P}\left[\sum_{t=1}^{T}B_t\geq m\right]\geq 1-\delta\\
& B_t\sim\text{Bino}\left(A_t, 1-\frac{\varepsilon\cdot p}{\left(A_0+\sum_{s=1}^{t-1}A_s+1\right)^r}-\varepsilon\cdot(1-p)\right),\ \forall t\in\mathcal{T}\\
& A_t, B_t\geq 0,\ \forall t\in\mathcal{T}
\end{aligned}$$

Again, the only non-trivial change from Theorem 1 comes from calculating the optimal solution of the its deterministic approximation, formulated as follows. The solution is given in Lemma EC.8.

$$\min\left\{cA_0^D+\sum_{t=1}^{T}A_t^D \;\middle|\; \sum_{t=1}^{T}A_t^D\left(1-\frac{\varepsilon\cdot p}{\left(A_0^D+\sum_{s=1}^{t-1}A_s^D+1\right)^r}-\varepsilon\cdot(1-p)\right)=m\right\} \qquad (\mathcal{P}_{WS}^D)$$

Lemma EC.8. *Let $m^D(\varepsilon, T)$ be the optimal objective value of Problem $(\mathcal{P}_{WS}^D)$, and $A_1^D, \cdots, A_T^D$ be its solution. Define $\gamma_\star = \left(\frac{\alpha_T(1-r)c_0\varepsilon \cdot p\zeta_T}{c}\right)^{\frac{1}{\alpha_T(1-r)+1}}$. For any $T \geq 2$, we have as $m \to \infty$:*

$$A_t^D = r^{-\frac{1}{1-r}\left(t - T\cdot\alpha_T\cdot(1-r^t)\right)} \cdot \left(\frac{c_0}{\gamma_\star}\right)^{\alpha_T\cdot(1-r^t)} \cdot m^{\alpha_{T+1}(1-r^{t+1})} + o\left(m^{\alpha_{T+1}(1-r^{t+1})}\right), \ \forall t = 1, \cdots, T$$
$$m^D(\varepsilon, T) = c_0 m + \left(c_0 \cdot \varepsilon \cdot p \cdot \zeta_T \cdot \gamma_\star^{-\alpha_T\cdot(1-r)} + c\cdot\gamma_\star\right) m^{\alpha_{T+1}\cdot(1-r)} + o\left(m^{\alpha_{T+1}\cdot(1-r)}\right).$$

Proof of Lemma EC.8 Let us define $\gamma_\star > 0$ and $\beta_\star \in [0,1)$ such that $A_0^D = \gamma_\star \cdot m^{\beta_\star} + o(m^{\beta_\star})$. Unlike in Lemma EC.7, $\gamma_\star$ and $\beta_\star$ are decision variables, reflecting the amount of offline data to be acquired. As in Lemma EC.7, we can define $u_t = A_0^D + \sum_{s=1}^t A_s^D + 1$ for $t = 1, \cdots, T$, and derive:

$$u_t = r^{-\frac{1}{1-r}\left(t - T\cdot\alpha_T\cdot(1-r^t)\right)} \cdot \left(\frac{c_0}{\gamma_\star}\right)^{\alpha_T\cdot(1-r^t)} \cdot m^{(1-\beta_\star)\alpha_T\left(1-r^t\right)+\beta_\star} + o\left(m^{(1-\beta_\star)\alpha_T\left(1-r^t\right)+\beta_\star}\right),$$

which achieves an objective of

$$c_0\left(m + \varepsilon\cdot p\cdot\zeta_T\cdot\gamma_\star^{-\alpha_T(1-r)}\cdot m^{(1-\beta_\star)\cdot\alpha_T\cdot(1-r)+\beta_\star(1-r)}\right) + o\left(m^{(1-\beta_\star)\cdot\alpha_T\cdot(1-r)+\beta_\star(1-r)}\right).$$

By construction, $A_0^D = \gamma_\star \cdot m^{\beta_\star} + o(m^{\beta_\star})$ and therefore the full objective value is:

$$c_0 m + \Theta\left(m^{(1-\beta_\star)\cdot\alpha_T\cdot(1-r)+\beta_\star(1-r)}\right) + \Theta\left(m^{\beta_\star}\right)$$

As $m \to \infty$, the minimum is achieved when:

$$(1-\beta_\star)\cdot\alpha_T\cdot(1-r) + \beta_\star(1-r) = \beta_\star$$

Solving this equation gives $\beta_\star = \frac{\alpha_T(1-r)}{r+\alpha_T(1-r)}$. Recalling the definition of $\alpha_T = \frac{1}{1-r^T}$, we obtain: $\beta_\star = \frac{1-r}{1-r^{T+1}} = (1-r)\alpha_{T+1}$. We get $A_0^D = \gamma_\star \cdot m^{\alpha_{T+1}\cdot(1-r)}$ and:

$$u_t = r^{-\frac{1}{1-r}\left(t - T\cdot\alpha_T\cdot(1-r^t)\right)} \cdot \left(\frac{c_0}{\gamma_\star}\right)^{\alpha_T\cdot(1-r^t)} \cdot m^{\alpha_{T+1}(1-r^{t+1})} + o\left(m^{\alpha_{T+1}(1-r^{t+1})}\right)$$

Proceeding as in the proof of Lemma EC.7, we derive:

$$(1-(1-p)\varepsilon))\sum_{t=1}^T A_t^D = m + \varepsilon\cdot p\cdot\zeta_T\cdot\gamma_\star^{-\alpha_T\cdot(1-r)}\cdot m^{\alpha_{T+1}\cdot(1-r)} + o\left(m^{\alpha_{T+1}\cdot(1-r)}\right)$$

The optimal objective value is therefore:

$$c_0 m + \left(c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot\gamma_\star^{-\alpha_T\cdot(1-r)} + c\cdot\gamma_\star\right) m^{\alpha_{T+1}\cdot(1-r)} + o\left(m^{\alpha_{T+1}\cdot(1-r)}\right)$$

Minimizing over $\gamma_\star$ yields the desired expression:

$$\gamma_\star = \left(\frac{\alpha_T(1-r)c_0\varepsilon\cdot p\zeta_T}{c}\right)^{\frac{1}{\alpha_T(1-r)+1}}$$

Then, we have:

$$c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot\gamma_\star^{-\alpha_T\cdot(1-r)}+c\cdot\gamma_\star=c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot\left(\frac{\alpha_T(1-r)c_0\varepsilon\cdot p\zeta_T}{c}\right)^{-\frac{\alpha_T\cdot(1-r)}{\alpha_T(1-r)+1}}+c\left(\frac{\alpha_T(1-r)c_0\varepsilon\cdot p\zeta_T}{c}\right)^{\frac{1}{\alpha_T(1-r)+1}}$$

$$=\left(c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot c^{\alpha_T(1-r)}\cdot\alpha_T(1-r)\right)^{\frac{1}{\alpha_T(1-r)+1}}\cdot\left(1+\frac{1}{\alpha_T(1-r)}\right)$$

Therefore, the optimal objective value can be written as:

$$c_0 m+\left(c_0\cdot\varepsilon\cdot p\cdot\zeta_T\cdot c^{\alpha_T(1-r)}\cdot\alpha_T(1-r)\right)^{\frac{1}{\alpha_T(1-r)+1}}\cdot\left(1+\frac{1}{\alpha_T(1-r)}\right)m^{\alpha_{T+1}\cdot(1-r)}+o\left(m^{\alpha_{T+1}\cdot(1-r)}\right)$$

This concludes the proof of Lemma $\mathcal{P}^D_{WS}$, hence of the theorem. □

EC.4.3. Adaptive policies, and proof of Theorem 4

Dynamic programming formulation. In this appendix, we first formalize the adaptive optimization problem via dynamic programming. The state variable includes the number of tentative facility openings at the beginning of each period t, that is, N_{t-1}, and the number successful facility openings, which we denote by $\overline{B}_{t-1}$. Mathematically, these variables are the cumulative sum of the variables in our model, that is $N_{t-1}=\sum_{s=1}^{t-1}A_s$ and $\overline{B}_{t-1}=\sum_{s=1}^{t-1}B_s$. The decision variable characterizes the number of tentative facility openings in the t^{th} period, that is A_t. The transition function specifies that $N_t=N_{t-1}+A_t$ and $\overline{B}_t=\overline{B}_{t-1}+W_t$, where W_t follows a binomial distribution with A_t trials and success probability $1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}-\varepsilon\cdot(1-p)$ (per Assumption 3). The Bellman equation is then given as follows where $J_t(N_{t-1},\overline{B}_{t-1})$ denotes the cost-to-go function. Note that the terminal condition is applied at time T rather than at time $T+1$ to reflect the chance constraint in the final-period decision.

$$\begin{aligned}
J_t(N_{t-1},\overline{B}_{t-1})&=\min_{A_t\geq 0}\quad\left\{A_t+\mathbb{E}J_{t+1}\left(N_{t-1}+A_t,\overline{B}_{t-1}+\text{Binom}\left(A_t,1-\frac{\varepsilon\cdot p}{(N_{t-1}+1)^r}-\varepsilon\cdot(1-p)\right)\right)\right\}\\
J_T(N_{T-1},\overline{B}_{T-1})&=\min_{A_T\geq 0}\quad N_{T-1}+A_T\\
&\qquad\text{s.t.}\quad\mathbb{P}\left[\text{Bino}\left(A_T,1-\frac{\varepsilon\cdot p}{(N_{T-1}+1)^r}-\varepsilon\cdot(1-p)\right)\geq m-\overline{B}_{T-1}\right]\geq 1-\delta
\end{aligned}$$

Unfortunately, the dynamic program would be very hard to solve analytically, in particular due to the interactions between the dynamics over the first $T-1$ periods and the final-period decision based on the chance constraint. We therefore resort to characterizing the solution of an adaptive re-solving algorithm, which applies the static problem iteratively over the planning horizon.

Details on the adaptive algorithm. The adaptive algorithm is specified in Algorithm 3. At each time period $t=1,\cdots,T-1$, it solves Problem $\left(\mathcal{Q}^*_{T-t+1}\left(m-\sum_{s=1}^{t-1}B_s^{\text{AD}},\sum_{s=1}^{t-1}A_s^{\text{AD}}\right)\right)$. It then solves Problem $(\mathcal{S}^*\left(m-\sum_{s=1}^{T-1}B_s^{\text{AD}},\sum_{s=1}^{T-1}A_s^{\text{AD}}\right)$ in the last period, which is defined as follows:

$$\min\quad\left\{A_T:\mathbb{P}\left[\text{Bino}\left(A_T,1-\frac{\varepsilon\cdot p}{(N_0+1)^r}-\varepsilon\cdot(1-p)\right)\geq m\right]\geq 1-\delta,\ A_T\geq 0\right\}\qquad(\mathcal{S}^*(m,N_0))$$

Algorithm 3 Adaptive algorithm to solve Problem $(\mathcal{P}^*)$.

Repeat, for $t=1,\cdots,T-1$:

1. Optimize. Define $\beta_t=\alpha_T\left(1-r^{t-1}\right)$ and γ_t such that $\sum_{s=1}^{t-1} A_s^{\mathrm{AD}}=\gamma_t m^{\beta_t}+o\left(m^{\beta_t}\right)$. Solve Problem $\left(\mathcal{Q}_{T-t+1}^*\left(m-\sum_{s=1}^{t-1} B_s^{\mathrm{AD}}, \sum_{s=1}^{t-1} A_s^{\mathrm{AD}}\right)\right)$. Retrieve solution $A_1^{\dagger},\cdots,A_{T-t+1}^{\dagger}$.

2. Commit immediate decision. Implement the first-period decision: $A_t^{\mathrm{AD}}=A_1^{\dagger}$.

3. Observe realizations. Sample $B_t^{\mathrm{AD}}\sim \operatorname{Bino}\left(A_t^{\mathrm{AD}}, 1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1} A_s^{\mathrm{AD}}+1\right)^r}-\varepsilon\cdot(1-p)\right)$.

$A_T^{\mathrm{AD}} \leftarrow\left(\mathcal{S}^*\left(m-\sum_{s=1}^{T-1} B_s^{\mathrm{AD}}, \sum_{s=1}^{T-1} A_s^{\mathrm{AD}}\right)\right)$. Sample $B_T^{\mathrm{AD}}\sim \operatorname{Bino}\left(A_T^{\mathrm{AD}}, 1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{T-1} A_s^{\mathrm{AD}}+1\right)^r}-\varepsilon\cdot(1-p)\right)$.

At each iteration, the algorithm solves a variant of the problem with offline data, albeit with an increasing offline data sample over iterations reflected in $\sum_{s=1}^{t-1} A_s^{\mathrm{AD}}=\gamma_t m^{\beta_t}+o\left(m^{\beta_t}\right)$ and $\beta_t=\alpha_T\left(1-r^{t-1}\right)$. Therefore, at each iteration, we can invoke Theorem 2 to elicit the $A_1^{\dagger},\cdots,A_{T-t+1}^{\dagger}$ and the corresponding regret. Specifically, at each iteration, Problem $\left(\mathcal{Q}_{T-t+1}^*\left(m-\sum_{s=1}^{t-1} B_s, \sum_{s=1}^{t-1} A_s\right)\right)$ is solved via Equations $\mathcal{A}\left(T-t+1, m-\sum_{s=1}^{t-1} B_s^{\mathrm{AD}}, \gamma_t, \beta_t\right)$ and $\mathcal{L}\left(T-t+1, m-\sum_{s=1}^{t-1} B_s^{\mathrm{AD}}, \gamma_t, \beta_t\right)$, exploiting the fact that $\sum_{s=1}^{t-1} A_s=\Theta(\gamma_t m^{\beta_t})$. Note that γ_t and β_t change across iterations due to the increasing data sample, but the procedure to solve the problem remains identical across iterations, and also remains identical to the solution procedure used to solve the problem with offline data in Theorem 2.

Proof of Theorem 4. We prove by induction over $t=1,\cdots,T-1$ that

$$A_t^{\mathrm{AD}}=\Theta\left(m^{\alpha_T\left(1-r^t\right)}\right).$$

The result for $t=1$ follows directly from Theorem 1. Let us assume that it holds for $t-1\in\{1,\cdots,T-2\}$ and prove it for t. In period t, the decision-maker solves $\left(\mathcal{Q}_{T-t+1}^*\left(m-\sum_{s=1}^{t-1} B_s^{\mathrm{AD}}, \sum_{s=1}^{t-1} A_s^{\mathrm{AD}}\right)\right)$, where $\sum_{s=1}^{t-1} A_s^{\mathrm{AD}}=\Theta\left(m^{\alpha_T\left(1-r^{t-1}\right)}\right)$ per the induction hypothesis. Let us denote by $\beta_t=\alpha_T\left(1-r^{t-1}\right)$. Let us momentarily assume that $\delta= o\left(m^{(1-\beta_t)\cdot\alpha_{T-t+1}\cdot(1-r)+\beta_t(1-r)-1}\right)$. Then, per Theorem 2, we know that the solution of that problem, denoted by $A_\tau^{\dagger}$, satisfies $A_\tau^{\dagger}=\theta\left(m^{(1-\beta_t)\alpha_{T-t+1}(1-r^\tau)+\beta_t}\right)$ for $\tau=1,\cdots,T-t+1$. By construction, the adaptive policy follows the first-period solution in the problem solved at time t, so that $A_t^{\mathrm{AD}}=A_1^{\dagger}$. Therefore: $A_t^{\mathrm{AD}}=\Theta\left(m^{(1-\beta_t)\alpha_{T-t+1}(1-r)+\beta_t}\right)$. We conclude the induction as follows:

$$\begin{aligned}
(1-\beta_t)\alpha_{T-t+1}(1-r)+\beta_t &= \left(1-\alpha_T(1-r^{t-1})\right)\cdot\frac{1}{1-r^{T-t+1}}(1-r)+\alpha_T(1-r^{t-1})\\
&=\frac{1}{1-r^T}\left(\frac{(r^{t-1}-r^T)(1-r)}{1-r^{T-t+1}}+1-r^{t-1}\right)\\
&=\frac{1}{1-r^T}\left(r^{t-1}(1-r)+1-r^{t-1}\right)\\
&=\alpha_T(1-r^t)
\end{aligned}$$

Similarly, the leading-order term in the regret expression does not change over the iterations of the adaptive algorithm. Indeed, per Theorem 2, the algorithm achieves a regret of $\Theta\left(m^{(1-\beta_t)\cdot\alpha_{T-t+1}\cdot(1-r)+\beta_t(1-r)}\right)$ at each iteration, and

$$(1-\beta_t)\alpha_{T-t+1}(1-r)+\beta_t(1-r)=\frac{1}{1-r^T}\left(r^{t-1}(1-r)+(1-r^{t-1})(1-r)\right)=\alpha_T(1-r)$$

Note, also, that this property implies that $\delta = o\left(m^{(1-\beta_t)\cdot\alpha_{T-t+1}\cdot(1-r)+\beta_t(1-r)-1}\right)$, since by assumption $\delta = o\left(m^{\alpha_T(1-r)-1}\right)$, which ensures that the condition of Lemma EC.7 is satisfied. This concludes the proof that the algorithm maintains a regret of $\Theta\left(m^{\alpha_T\cdot(1-r)}\right)$ at each iteration.

EC.4.4. Semi-adaptive procedure with final-period adjustment

Recall that the semi-adaptive procedure implements $A_1^\dagger,\cdots,A_{T-1}^\dagger$ from Algorithm 1, and then solves Problem $\left(\mathcal{S}^*\left(m-\sum_{s=1}^{T-1}B\dagger_s,\sum_{s=1}^{T-1}A\dagger_s\right)\right)$ in period T. This is detailed in Algorithm 4.

Algorithm 4 Semi-adaptive algorithm to solve Problem $(\mathcal{P}^*)$.

Repeat, for $t=1,\cdots,T-1$:

1. **Optimize.** Solve Problem $(\mathcal{P}^*)$ using Algorithm 1 from the paper. Retrieve solution $A_1^\dagger,\cdots,A_T^\dagger$.
2. **Commit** $T-1$ **decisions.** Implement decisions $A_1^\dagger,\cdots,A_{T-1}^\dagger$.
3. **Observe realizations.** Sample $B_t\sim\text{Bino}\left(A_t^\dagger,1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{t-1}A_s+1\right)^r}-\varepsilon\cdot(1-p)\right)$, $t=1,\cdots,T-1$.

$A_T^\dagger\leftarrow\left(\mathcal{S}^*\left(m-\sum_{s=1}^{T-1}B_s^\dagger,\sum_{s=1}^{T-1}A_s^\dagger\right)\right)$. Sample $B_T^\dagger\sim\text{Bino}\left(A_T^\dagger,1-\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{T-1}A_s^\dagger+1\right)^r}-\varepsilon\cdot(1-p)\right)$.

EC.5. Proof of statements from Section 6

We will now focus on the case where the conditions does not hold. We will first prove the relationship between the solution of Algorithm $(\mathcal{P}_N^\dagger)$ and the optimal solution of Problem $(\mathcal{P}_N^*)$ for any value of p (Part 1 below). We then prove the relationship between the Problem $(\mathcal{P}_N^*)$ and the fully-learned benchmark when $p=1$ (Part 2).

– Part 1: Algorithm 2 yields a feasible solution to Problem $(\mathcal{P}_N^*)$ such that

$$m^\dagger(\varepsilon,p,T)\le m^*(\varepsilon,p,T)+\begin{cases}o\left(m^{\alpha_T\cdot(1-r)}\right) & \text{if } p=1 \text{ or } (p\neq 1 \text{ and } \alpha_T\cdot(1-r)>1/2)\\ \mathcal{O}(m^{1/2}) & \text{if } p\neq 1 \text{ and } \alpha_T\cdot(1-r)\le 1/2\end{cases}$$

– Part 2: regret of the optimal stochastic solution against the fully-learned solution when $p=1$:

$$m^*(\varepsilon,p,T)=\widehat{m}(\varepsilon,p)+\Theta\left(m^{g(r)}\right)$$

The restriction to $p=1$ for Part 2 stems from the fact that the characterization of the fully-learned solution is no longer tractable when $p<1$, which prevents us from bounding the solutions to the stochastic problem and the fully-learned problem. When $p=1$, the "fully-learned" information benchmark is formulated as a (deterministic) discrete optimization problem (Problem $(\widehat{\mathcal{P}}_N)$).

Throughout this proof, we denote by y^*_{it}, $y^\dagger_{it}$, and $\widehat{y}_{it}$ the solutions of the stochastic problem $(\mathcal{P}^*_N)$, our algorithm $(\mathcal{P}^\dagger_N)$, and the fully-learned problem $(\widehat{\mathcal{P}}_N)$, respectively. We denote by s^*_{jT}, $s^\dagger_{jT}$, and $\widehat{s}_{jT}$ the corresponding coverage variables. The optimal values are given as

$$m^*(\varepsilon,p,T)=\sum_{i=1}^{n}\sum_{t=1}^{T}y^*_{it} \qquad m^\dagger(\varepsilon,p,T)=\sum_{i=1}^{n}\sum_{t=1}^{T}y^\dagger_{it} \qquad \widehat{m}(\varepsilon,p)=\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{y}_{it}$$

We define $\pi_j(\boldsymbol{y})$ as the probability that customer $j=1,\sum_{j=1}^{q}$ is covered by a facility at the end of the horizon under the decision vector $\boldsymbol{y}=\{y_{it}\mid i=1,\cdots,n,\ t=1,\cdots,T\}$:

$$\pi_j(\boldsymbol{y})=\left(1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}+1\right)^r}+\varepsilon\cdot(1-p)\right)^{\sum_{i\in\mathcal{F}_j}y_{i\tau}}\right)$$

We use $\mathcal{N}(\boldsymbol{y})$ to denote the set of facilities that are tentatively opened during the planning horizon, and $\mathcal{M}(\boldsymbol{y})$ to denote the set of customers with at least one adjacent facility that has been tentatively opened under the decision vector $\boldsymbol{y}=\{y_{it}\mid i=1,\cdots,n,\ t=1,\cdots,T\}$. Formally:

$$\begin{aligned}
&\mathcal{N}(\boldsymbol{y})=\{i\in\{1,\cdots,n\}\mid \exists t\in\{1,\cdots,T\}:y_{it}=1\}\\
&\mathcal{M}(\boldsymbol{y})=\{j\in\{1,\cdots,q\}\mid \pi_j(\boldsymbol{y})>0\}=\{j\in\{1,\cdots,q\}\mid \mathcal{F}_j\cap\mathcal{N}(\boldsymbol{y})\neq\emptyset\}
\end{aligned}$$

We assume that $\mathcal{F}^{\text{CTL}}(d_\star)=\emptyset$ without loss of generality. Consider the graph $\mathcal{G}'$ where all central nodes $\mathcal{F}^{\text{CTL}}(d_\star)\cup\mathcal{C}^{\text{CTL}}(d_\star)$ and all connecting edges are removed. Let the optimal values of the stochastic problem, our algorithm, and the fully-learned problem in $\mathcal{G}'$ as $m^*_0(\varepsilon,T)$, $m^\dagger_0(\varepsilon,T)$, and $\widehat{m}_0(T)$ respectively. We can construct solutions to the original problems (with $\mathcal{G}$) by adding all central facilities, so that

$$\begin{aligned}
&m^*(\varepsilon,p,T)\le m^*_0(\varepsilon,T)\le m^*(\varepsilon,p,T)+|\mathcal{F}^{\text{CTL}}(d_\star)|\\
&m^\dagger(\varepsilon,p,T)\le m^\dagger_0(\varepsilon,T)\le m^\dagger(\varepsilon,p,T)+|\mathcal{F}^{\text{CTL}}(d_\star)|\\
&\widehat{m}(\varepsilon,T)\le \widehat{m}_0(\varepsilon,T)\le \widehat{m}(\varepsilon,T)+|\mathcal{F}^{\text{CTL}}(d_\star)|
\end{aligned}$$

Once we prove the theorem for any graph such that $\mathcal{F}^{\text{CTL}}(d_\star)=\emptyset$, we get $m^*_0(\varepsilon,T)=\widehat{m}_0(T)+\Theta\left(m^{g(r)}\right)$ and $m^\dagger_0(\varepsilon,T)=m^*_0(\varepsilon,T)+o\left(m^{g(r)}\right)$ or $m^\dagger_0(\varepsilon,T)=m^*_0(\varepsilon,T)+\Theta\left(m^{1/2}\right)$. Since the minimum order of terms is strictly larger than k, this implies per Assumption 4 that $m^*(\varepsilon,p,T)=\widehat{m}(\varepsilon,p)+\Theta\left(m^{g(r)}\right)$ and $m^\dagger(\varepsilon,p,T)=m^*(\varepsilon,p,T)+o\left(m^{g(r)}\right)$ or $m^\dagger(\varepsilon,p,T)=m^*(\varepsilon,p,T)+\Theta\left(m^{1/2}\right)$.

We make use of several lemmas. The logic of the proof is summarized in Figure EC.2. To prevent circular reasoning, we prove the lemmas in ascending order to derive a proof of Theorem 5.

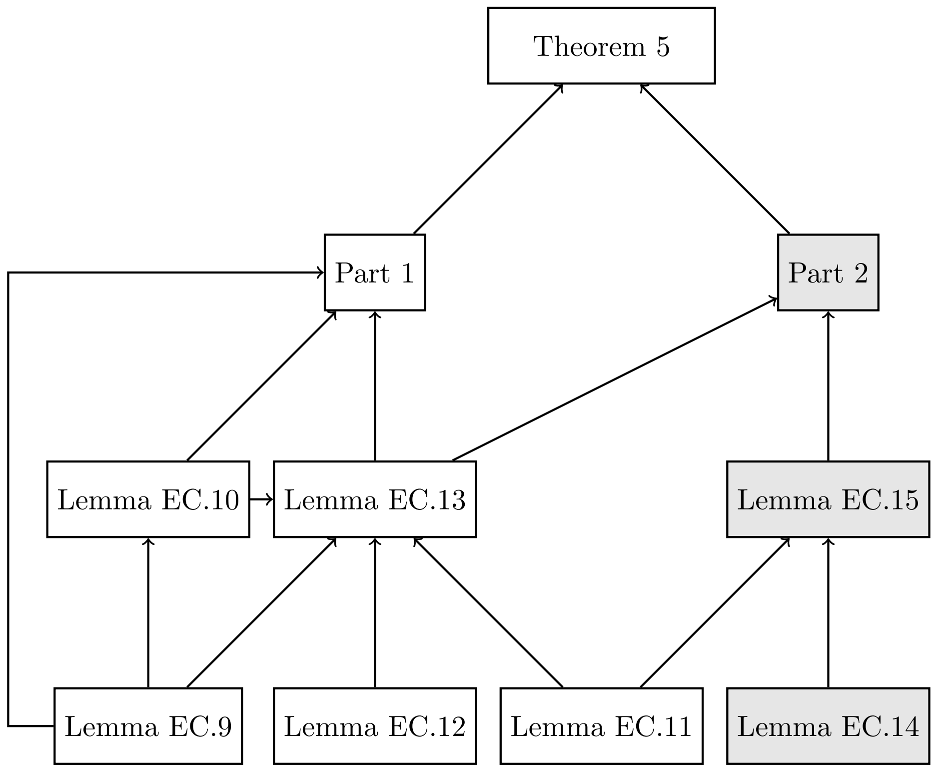


Figure EC.2 Logic of the proof of Theorem 5. Arrows indicate dependencies between results. Grayed boxes indicate results obtained for $p=1$, white boxes indicate results obtained for any value of $p\leq 1$.

EC.5.1. Statement and proof of Lemma EC.9

Lemma EC.9 shows that the number of facilities that need to be opened is on the order of m, both under the algorithm and in the optimal solution to the stochastic problem. The proof uses the degree assumption to identify sufficient feasibility conditions based on the *number* of facilities that are tentatively opened, which enables to leverage the technical results from Section 4.

Lemma EC.9. *For sufficiently large m, we have:*

$$\frac{m}{d_\star} \leq |\mathcal{N}(\boldsymbol{y}^\dagger)| \leq d_\star c_0 \left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m) + 3\Delta_G(m)\right)$$
$$\frac{m(1-\delta)}{d_\star} \leq \mathbb{E}[|\mathcal{N}(\boldsymbol{y}^*)|] \leq d_\star c_0 \left(m + \varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m)\right)$$

Proof: Let us prove the lower bounds and upper bounds separately.

Lower bounds. The lower bounds stem from the degree assumption: each facility covers up to $d_\star$ customers and m customers need to be covered, so at least $m/d_\star$ facilities are required. Formally, $\pi_j(\boldsymbol{y})=0$ if and only if $y_{it}=0$ for all $t=1,\cdots,T$ and for all $i\in\mathcal{F}_j$:

$$|\mathcal{M}(\boldsymbol{y}^\dagger)| \geq \sum_{j\in\mathcal{M}(\boldsymbol{y}^\dagger)} \pi_j(\boldsymbol{y}^\dagger) = \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^\dagger) \geq m + \Delta_G(m) \geq m$$

Moreover:

$$\begin{aligned}|\mathcal{M}(\boldsymbol{y}^\dagger)| &= \sum_{j=1}^{q} \mathbf{1}(\mathcal{F}_j \cap \mathcal{N}(\boldsymbol{y}^\dagger) \neq \emptyset) \\ &\leq \sum_{j=1}^{q}\sum_{i=1}^{n} \mathbf{1}(i \in \mathcal{F}_j \cap \mathcal{N}(\boldsymbol{y}^\dagger)) \\ &= \sum_{i=1}^{n}\left(\mathbf{1}\left(i \in \mathcal{N}(\boldsymbol{y}^\dagger)\right) \times \sum_{j=1}^{q} \mathbf{1}(i \in \mathcal{F}_j)\right) \\ &\leq d_\star |\mathcal{N}(\boldsymbol{y}^\dagger)|,\end{aligned}$$

due to the assumption that the degree is bounded by $d_\star$. We obtain that $m \leq d_\star|\mathcal{N}(\boldsymbol{y})|$.

Similarly, for the stochastic solution, we know that with probability $1-\delta$ the solution satisfies $\sum_{j=1}^{q} s^*_{jT} \geq m$. Conditioned on this event, we use the result above to write:

$$\mathbb{E}\left[|\mathcal{N}(\boldsymbol{y}^*)| \,|\, \sum_{j=1}^{q} s^*_{jT} \geq m\right] \geq \frac{m}{d_\star}$$

We conclude:

$$\mathbb{E}[|\mathcal{N}(\boldsymbol{y}^*)|] \geq \frac{m(1-\delta)}{d_\star}$$

Upper bounds. Let us start with Problem $(\mathcal{P}^*_N)$. Lemma 2 provides an algorithm that tentatively opens $d_\star c_0 \left(m + \varepsilon \cdot p \cdot \zeta_T \cdot m^{\alpha_T \cdot (1-r)} + \Delta(m)\right)$ facilities over T periods, so that at least $d_\star m$ facilities are actually opened with probability at least $1-\delta$. Per our degree assumption, each customer is connected to at most $d_\star$ facilities. Using the same argument as above, any set of $d_\star m$ facilities is connected to at least m customers. This means that the solution satisfies the chance constraint, and defines a feasible solution to problem $(\mathcal{P}^*_N)$. We have derived a solution that satisfies:

$$\begin{aligned}&\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it} = d_\star c_0 \left(m + \varepsilon \cdot p \cdot \zeta_T \cdot m^{\alpha_T \cdot (1-r)} + \Delta(m)\right) \\ &\mathbb{P}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} z_{it} \geq d_\star m\right] \geq 1-\delta \\ &\mathbb{P}\left[\sum_{j=1}^{q} s_{jT} \geq m\right] \geq 1-\delta\end{aligned}$$

Turning to Algorithm 2, define $q = m + \Delta_G(m)$. Using Lemma EC.4, there exists an algorithm such that tentatively opening the following number of facilities

$$\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it} = d_\star c_0 \left(q + \varepsilon \cdot p \cdot \zeta_T \cdot q^{\alpha_T \cdot (1-r)} + \Delta(q)\right)$$

would result in the number of opened facilities in expectation such that:

$$\mathbb{E}\left[\sum_{i=1}^{n}\sum_{t=1}^{T} z_{it}\right] \geq d_\star q,$$

for m sufficiently large. Again, since each customer is connected to at most $d_\star$ facilities, any set of $d_\star q$ facilities is connected to at least q customers. Thus, the expected number of connected customers must satisfy:

$$\mathbb{E}\left[\sum_{j=1}^{q} s_{jT} \geq m\right] \geq \frac{1}{d_\star}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{t=1}^{T} z_{it}\right] \geq q$$

Therefore, opening $d_\star c_0\left(q+\varepsilon\cdot p\cdot\zeta_T\cdot q^{\alpha_T\cdot(1-r)}+\Delta(q)\right)$ defines a feasible solution for Problem $(\mathcal{P}_N^\dagger)$, hence an upper bound to $|\mathcal{N}(\boldsymbol{y}^\dagger)|$. Note that the function within the brackets is concave in q and its derivative is less than 3 for q sufficiently large. Therefore, we obtain:

$$\begin{aligned}
|\mathcal{N}(\boldsymbol{y}^\dagger)| &\leq d_\star c_0\left(q+\varepsilon\cdot p\cdot\zeta_T\cdot q^{\alpha_T\cdot(1-r)}+\Delta(q)\right)\\
&\leq d_\star c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+\Delta(m)+3(q-m)\right)\\
&= d_\star c_0\left(m+\varepsilon\cdot p\cdot\zeta_T\cdot m^{\alpha_T\cdot(1-r)}+\Delta(m)+3\Delta_G(m)\right)
\end{aligned}$$

This completes the proof of Lemma EC.9. □

Statement and proof of Lemma EC.10.

Lemma EC.10. *Consider $\boldsymbol{y}=\{y_{it}\mid i=1,\cdots,n,\ t=1,\cdots,T\}$ such that $|\mathcal{N}(\boldsymbol{y})|\leq Km$ for some fixed constant K. Define the event:*

$$\mathcal{E}(\boldsymbol{y})=\left\{\left|\sum_{j=1}^{q} s_{jT}-\pi_j(\boldsymbol{y})\right|\leq\Delta_G(m)\right\} \tag{EC.12}$$

Then, $\mathbb{P}(\mathcal{E}(\boldsymbol{y}^\dagger))\geq 1-\delta$ and $\mathbb{P}(\mathcal{E}(\boldsymbol{y}^*))\geq 1-\delta$.

Lemma EC.10 proves that the "good" events $\mathcal{E}(\boldsymbol{y}^\dagger)$ and $\mathcal{E}(\boldsymbol{y}^*)$ occur with high probability under the stochastic and deterministic solutions. This shows that the total number of customers that are covered is likely to be close to its expected value within a factor of $\Delta_G(m)$.

The key technical difficulty is that whether a customer is covered is not independent from whether another customer is covered, due to the underlying graph structure. This prevents the application of standard concentration inequalities. Instead, we use the concentration inequality in dependency graphs from Janisch and Lehéricy (2024), based on the fractional chromatic number.

Definition EC.1. Let $\mathcal{G}=(\mathcal{N},\mathcal{A})$ define an undirected graph. A coloring of $\mathcal{G}$ labels each vertex by one color each such that no two adjacent vertices have the same color. The chromatic number $\chi(\mathcal{G})$ is defined as the smallest number of colors defining a coloring of $\mathcal{G}$.

Definition EC.2. Let $\mathcal{G}=(\mathcal{N},\mathcal{A})$ define an undirected graph. A b-fold coloring of $\mathcal{G}$ labels each vertex with up to b colors, such that no two adjacent vertices have any color in common. Its b-fold chromatic number $\chi_b(\mathcal{G})$ is defined as the smallest number of colors defining a b-fold coloring of $\mathcal{G}$. The fractional chromatic number, denoted by $\chi^*(\mathcal{G})$, is given by: $\chi^*(\mathcal{G})=\lim\limits_{b\to\infty}\frac{\chi_b(\mathcal{G})}{b}$

The following result extends the Berry-Esseen in the case of dependency graphs.

THEOREM EC.1 **(Janisch and Lehéricy (2024))**. *Let $X_1, \cdots, X_n$ be random variables in $[0, c_i]$. Let $\mathcal{G}$ denote the dependency graph of $X_1, \cdots, X_n$ and $\chi^*(\mathcal{D})$ denote its fractional chromatic number. Then we have:*

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i - \mathbb{E}\left[\sum_{i=1}^n X_i\right]\right| \geq s\sqrt{\mathrm{Var}\left(\sum_{i=1}^n X_i\right)}\right) = 2\Phi(-s) + \mathcal{O}\left(\frac{\max_i c_i \cdot \chi^*(\mathcal{G})^2 \sum_{i=1}^n \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^3\right]}{\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{3/2}}\right)$$

Let us prove Lemma EC.10 with $\boldsymbol{y} = \boldsymbol{y}^\dagger$. The arguments proceed in the same way for $\boldsymbol{y} = \boldsymbol{y}^*$. We define $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ as the dependency graph of the random variables s_{jT}, conditioned on the facilities tentatively opened under decision $\boldsymbol{y}$. Each node represents a customer. An edge connects two customers j and k if and only if s_{jT} and s_{kT} are not independent, i.e., if and only if a facility that can cover both has been tentatively opened: $(j,k) \in \mathcal{A} \iff \mathcal{F}_j \cap \mathcal{F}_k \cap \mathcal{N}(\boldsymbol{y}) \neq \emptyset$

By assumption, the degree of any node in the customer-facility graph is less than or equal to $d_\star$. Therefore, any customer can share an adjacent facility with up to $d_\star(d_\star - 1)$ other customers. By Brooks' Theorem (Brooks 1941), the chromatic number of $\mathcal{D}$ satisfies $\chi(\mathcal{D}) \leq (d_\star)^2 - d_\star + 1 \leq (d_\star)^2$. This implies that its fractional chromatic number satisfies $\chi^*(\mathcal{D}) \leq \chi(\mathcal{D}) \leq (d_\star)^2$.

We use this result to characterize $\left|\sum_{j=1}^q s_{jT} - \pi_j(\boldsymbol{y}^\dagger)\right|$. By definition, each variable s_{jT} is bounded by 1. We also need to bound $\mathrm{Var}\left(\sum_{j=1}^q s_{jT}^\dagger\right)$. By definition of $\chi^*(\mathcal{D})$, $s_{jT}^\dagger$ is correlated to at most $\chi^*(\mathcal{D})$ other variables $s_{iT}^\dagger$. Using the inequality $\mathrm{Var}(\sum_{i=1}^n X_i) \leq n \sum_{i=1}^n \mathrm{Var}(X_i)$, we obtain

$$\mathrm{Var}\left(\sum_{j=1}^q s_{jT}^\dagger\right) \leq \chi^*(\mathcal{D}) \sum_{j=1}^q \mathrm{Var}(s_{jT}^\dagger)$$

Now we separate the cases where $p = 1$ and $p \neq 1$.

<u>Case 1: $p \neq 1$.</u>

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j=1}^q s_{jT}^\dagger\right) \\
\leq &\chi^*(\mathcal{D}) \sum_{j=1}^q \mathrm{Var}(s_{jT}^\dagger) \\
\leq &d_\star^2 \sum_{j \in \mathcal{M}(\boldsymbol{y}^\dagger)} \mathrm{Var}(s_{jT}^\dagger) \\
\leq &d_\star^2 \sum_{j \in \mathcal{M}(\boldsymbol{y}^\dagger)} \prod_{\tau=1}^T \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y_{is}^\dagger + 1\right)^r} + \varepsilon \cdot (1-p)\right)^{\sum_{i \in \mathcal{F}_j} y_{i\tau}^\dagger} \\
= &d_\star^2 \sum_{j \in \mathcal{M}(\boldsymbol{y}^\dagger)} \prod_{\tau : \sum_{i \in \mathcal{F}_j} y_{i\tau}^\dagger > 0} \prod_{i : i \in \mathcal{F}_j, y_{i\tau}^\dagger > 0} \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y_{is}^\dagger + 1\right)^r} + \varepsilon \cdot (1-p)\right)^{y_{i\tau}^\dagger}
\end{aligned}$$

$$
\begin{aligned}
&\leq d_\star^2 \sum_{j\in\mathcal{M}(\boldsymbol{y}^\dagger)} \prod_{\tau:\sum_{i\in\mathcal{F}_j} y^\dagger_{i\tau}>0} \sum_{i:i\in\mathcal{F}_j, y^\dagger_{i\tau}>0} \left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y^\dagger_{is}+1\right)^r} + \varepsilon\cdot(1-p)\right) \frac{y^\dagger_{i\tau}}{|i: y^\dagger_{i\tau}>0|} \\
&\leq d_\star^2 \sum_{j\in\mathcal{M}(\boldsymbol{y}^\dagger)} \left(\prod_{\tau:\sum_{i\in\mathcal{F}_j} y^\dagger_{i\tau}>0} \left(\sum_{i:i\in\mathcal{F}_j, y^\dagger_{i\tau}>0} \left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y^\dagger_{is}+1\right)^r} + \varepsilon\cdot(1-p)\right) \frac{y^\dagger_{i\tau}}{|i: y^\dagger_{i\tau}>0|}\right)\right)^{1/|\tau:\sum_{i\in\mathcal{F}_j} y^\dagger_{i\tau}>0|} \\
&\leq d_\star^2 \sum_{j\in\mathcal{M}(\boldsymbol{y}^\dagger)} \sum_{\tau:\sum_{i\in\mathcal{F}_j} y^\dagger_{i\tau}>0} \sum_{i:i\in\mathcal{F}_j, y^\dagger_{i\tau}>0} \left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y^\dagger_{is}+1\right)^r} + \varepsilon\cdot(1-p)\right) \frac{y^\dagger_{i\tau}}{|\tau:\sum_{i\in\mathcal{F}_j} y^\dagger_{i\tau}>0||i: y^\dagger_{i\tau}>0|} \\
&\leq d_\star^3 \sum_{i\in\mathcal{N}(\boldsymbol{y}^\dagger)} \sum_{\tau=1}^T \left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y^\dagger_{is}+1\right)^r} + \varepsilon\cdot(1-p)\right) y^\dagger_{i\tau} \\
&\leq \varepsilon d_\star^4 c_0 (m + \varepsilon\cdot p\cdot \zeta_T\cdot m^{\alpha_T\cdot(1-r)} + \Delta(m) + 3\Delta_G(m)) \\
&\leq 2\varepsilon d_\star^4 c_0 m
\end{aligned}
$$

where in the fourth and sixth inequality, we are using the well-known AM-GM inequality twice. The third last inequality comes from the fact that each i,τ with $y^\dagger_{i\tau}>0$ has been counted at most $d_\star$ times in the triple sum; the second-to-last inequality comes from Lemma EC.9; and the last inequality holds for sufficiently large m. Similarly, since $p\neq 1$, the variance of each item satisfies $\mathrm{Var}(s^\dagger_{jT}) > (1-(1-p)\varepsilon)((1-p)\varepsilon) > 0$, and therefore $\mathrm{Var}\left(\sum_{j=1}^q s^\dagger_{jT}\right) = \Theta(m)$. This implies:

$$
\mathcal{O}\left(\frac{\max_i c_i\cdot \chi^*(\mathcal{G})^2 \sum_{i=1}^n \mathbb{E}\left[(X_i-\mathbb{E}[X_i])^3\right]}{\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{3/2}}\right) = \mathcal{O}\left(\frac{1}{\left(\mathrm{Var}\left(\sum_{i=1}^n X_i\right)\right)^{1/2}}\right) = o(1)
$$

From Janisch and Lehéricy (2024), we have for sufficiently large m:

$$
\mathbb{P}\left(\left|\sum_{j=1}^q s_{jT} - \pi_j(\boldsymbol{y}^\dagger)\right| \geq s\sqrt{2\varepsilon d_\star^4 c_0 m}\right) \leq 2\Phi(-s)
$$

By construction of $\Delta_G(m)$, we obtain:

$$
\mathbb{P}\left(\left|\sum_{j=1}^q s_{jT} - \pi_j(\boldsymbol{y}^\dagger)\right| > \Delta_G(m)\right) = \mathbb{P}\left(\left|\sum_{j=1}^q s_{jT} - \pi_j(\boldsymbol{y}^\dagger)\right| > -\Phi^{-1}\left(\frac{\delta}{2}\right)\sqrt{2\varepsilon d_\star^4 c_0 m}\right) \leq \delta
$$

Case 2: $p=1$.

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{j=1}^q s^\dagger_{jT}\right) \leq & d_\star^3 \sum_{i\in\mathcal{N}(\boldsymbol{y}^\dagger)} \sum_{\tau=1}^T \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y^\dagger_{is}+1\right)^r}\right) y^\dagger_{i\tau} \\
\leq & d_\star^4 \varepsilon \left(\zeta_T\cdot m^{\alpha_T\cdot(1-r)} + o\left(m^{\alpha_T\cdot(1-r)}\right)\right)
\end{aligned}
$$

where the first inequality follows the same logic as above, and the last inequality can be derived following the same procedure as Lemma 2. As in the proof of Lemma 2, we also have $\mathrm{Var}\left(\sum_{j=1}^{q} s_{jT}^{\dagger}\right) = \Omega(m^{1-\alpha_T\cdot r(1-r)})$. From Janisch and Lehéricy (2024), we have for sufficiently large m:

$$\mathbb{P}\left(\left|\sum_{j=1}^{q} s_{jT} - \pi_j(\boldsymbol{y}^{\dagger})\right| \geq s\sqrt{2\varepsilon d_\star^4 \zeta_T \cdot m^{\alpha_T\cdot(1-r)}}\right) \leq 2\Phi(-s)$$

By construction of $\Delta_G(m)$, we obtain

$$\mathbb{P}\left(\left|\sum_{j=1}^{q} s_{jT} - \pi_j(\boldsymbol{y}^{\dagger})\right| > \Delta_G(m)\right) = \mathbb{P}\left(\left|\sum_{j=1}^{q} s_{jT} - \pi_j(\boldsymbol{y}^{\dagger})\right| > -\Phi^{-1}\left(\frac{\delta}{2}\right)\sqrt{2\varepsilon d_\star^4 \zeta_T \cdot m^{\alpha_T\cdot(1-r)}}\right) \leq \delta$$

This completes the proof. □

EC.5.2. Statement and proof of Lemmas EC.11, EC.12 and EC.13

Lemma EC.11 provides a sensitivity result providing lower and upper bounds for how much the solution of Problem $(\mathcal{P}_N^{\dagger})$ changes with the coverage target. Then, Lemma EC.12 indicates the (conditional) feasibility of the solution of the deterministic problem, Lemma EC.13 bounds the difference between the deterministic solution of Algorithm 2 and the optimal solution to the stochastic problem $(\mathcal{P}^*)$, proving Part 1 of the theorem.

Let us define $\boldsymbol{y}^{\dagger}(m-\Delta)$ as the solution to Problem $(\mathcal{P}_N^{\dagger})$ with the constraint that $m-\Delta$ facilities must open, so the solution to Problem $(\mathcal{P}_N^{\dagger})$ can be written as $\boldsymbol{y}^{\dagger}(m+\Delta_G(m))$:

$$\boldsymbol{y}^{\dagger}(m-\Delta) \in \arg\min\left\{\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it} \;\middle|\; \sum_{j=1}^{q}\pi_j(\boldsymbol{y}) \geq m-\Delta;\; \boldsymbol{y}\in\mathbb{Z}^{N\times T}\right\}$$

Lemma EC.11. *The following holds, as $m,\Delta\to\infty$:*

$$\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(m) - d_\star c_0\left(\Delta+\mathcal{O}\left(\Delta^{g(r)}\right)\right) \leq \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(m-\Delta) \leq \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(m) - \frac{c_0\Delta}{d_\star}$$

Proof: Lemma EC.4 provides an algorithm that tentatively opens $d_\star c_0\left(\Delta+\mathcal{O}\left(\Delta^{g(r)}\right)\right)$ facilities and that opens at least $d_\star\Delta$ facilities on average, for sufficiently large Δ. Again, due to our degree assumption, this implies that the solution will serve at least Δ customers.

Let us consider the solution $\boldsymbol{y}^{\dagger}(m-\Delta)$ and add $d_\star c_0\left(\Delta+\mathcal{O}\left(\Delta^{g(r)}\right)\right)$ more facilities that connect to customers separate from those that are served in the current solution $\boldsymbol{y}^{\dagger}(m-\Delta)$ (those exist per the assumption of Theorem 5). This new solution serves Δ more customers than the solution $\boldsymbol{y}^{\dagger}(m-\Delta)$ in expectation. By proceeding as in the proof of Lemma EC.9, we can prove that it is a feasible solution to Problem $(\mathcal{P}_N^{\dagger})$ with a coverage target of m. We get the following bound:

$$\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(m) \leq \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(m-\Delta) + d_\star c_0\left(\Delta+\mathcal{O}\left(\Delta^{g(r)}\right)\right)$$

Similarly, let us consider the solution $\boldsymbol{y}^\dagger(m)$ with a coverage target of m, and subtract $\frac{c_0\Delta}{d_\star}$ facilities. By definition of c_0, every facility has at most $\frac{1}{c_0}$ chance of succeeding, and therefore, on average at most $\frac{\Delta}{d_\star}$ facilities can succeed. Per our degree assumption, each facility is connected to at most $d_\star$ customers, so $\frac{\Delta}{d_\star}$ facilities are connected to at most Δ customers. Therefore, the new solution covers at least equal as many customers as $\boldsymbol{y}^\dagger(m)$ minus Δ customers. The new solution is feasible for Problem $(\mathcal{P}_N^\dagger)$ with a coverage target of $m-\Delta$. We obtain:

$$\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m-\Delta) \leq \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m) - \frac{c_0\Delta}{d_\star}$$

This completes the proof of the lemma. □

Lemma EC.12 shows that the optimal stochastic solution satisfies the coverage constraint with high probability conditionally to the "good" event $\mathcal{E}(\boldsymbol{y}^*)$ (defined in Equation (EC.12)).

LEMMA EC.12. *The following holds:*

$$\mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m \mid \mathcal{E}(\boldsymbol{y}^*)\right) \geq 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)$$

Proof: Note that the optimal solution of Problem $(\mathcal{P}_N^*)$ must satisfy the constraint:

$$\mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m\right) \geq 1-\delta = 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right) \tag{EC.13}$$

Therefore, we have that:

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m \mid \mathcal{E}(\boldsymbol{y}^*)\right) &= \frac{\mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m\right) - \mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \mid \mathcal{E}^c(\boldsymbol{y}^*)\right)\mathbb{P}(\mathcal{E}^c(\boldsymbol{y}^*))}{\mathbb{P}(\mathcal{E}(\boldsymbol{y}^*))} \\
&\geq 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right) - \delta \quad \text{(Equation (EC.13) and Lemma EC.10)} \\
&= 1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)
\end{aligned}$$

This completes the proof of Lemma EC.12. □

Then, Lemma EC.13 bounds the suboptimality of the deterministic approximation solved in Algorithm 2 with a sub-linear term against the stochastic solution.

LEMMA EC.13. *The following holds:*

$$m^*(\varepsilon,p,T) \geq m^\dagger(\varepsilon,T) - \begin{cases} \mathcal{O}\left(m^{1/2}\right) & \text{if } p \neq 1 \\ o\left(m^{\alpha_T\cdot(1-r)}\right) & \text{if } p = 1 \end{cases}$$

Proof: We proceed by conditioning on the "good" event $\mathcal{E}(\boldsymbol{y}^*)$, as follows:

$$m^*(\varepsilon,p,T) = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^* \mid \mathcal{E}(\boldsymbol{y}^*)\right]\mathbb{P}(\mathcal{E}(\boldsymbol{y}^*)) + \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^* \mid \mathcal{E}^c(\boldsymbol{y}^*)\right]\mathbb{P}(\mathcal{E}^c(\boldsymbol{y}^*))$$

$$\geq \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^* \mid \mathcal{E}(\boldsymbol{y}^*)\right](1-\delta) \quad \text{(Lemma EC.10)}$$
$$= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^* \mid \mathcal{E}(\boldsymbol{y}^*)\right] + o(m^{\alpha_T\cdot(1-r)}) \text{ (Lemma EC.9)}$$

Conditioned on the good event, we have:

$$\sum_{j=1}^{q} s_{jT}^* \leq \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^*) + \left|\sum_{j=1}^{q}(s_{jT}^* - \pi_j(\boldsymbol{y}^*))\right| \leq \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^*) + \Delta_G(m)$$

By further conditioning on the value of s_{jT}^* and defining $A_t^* = \sum_{i=1}^{n} y_{it}^*$, we obtain:

$$\begin{aligned}
m^*(\varepsilon,p,T) &\geq \sum_{\sigma=0}^{\infty} \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}(\boldsymbol{y}^*), \sum_{j=1}^{q} s_{jT}^* = \sigma\right] \mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* = \sigma \mid \mathcal{E}(\boldsymbol{y}^*)\right) + o(m^{\alpha_T\cdot(1-r)})\\
&\geq \sum_{\sigma=m}^{\infty} \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}(\boldsymbol{y}^*), \sum_{j=1}^{q} s_{jT}^* = \sigma\right] \mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* = \sigma \mid \mathcal{E}(\boldsymbol{y}^*)\right) + o(m^{\alpha_T\cdot(1-r)})\\
&\geq \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}(\boldsymbol{y}^*), \sum_{j=1}^{q} s_{jT}^* = m\right] \mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m \mid \mathcal{E}(\boldsymbol{y}^*)\right) + o(m^{\alpha_T\cdot(1-r)})\\
&= \mathbb{E}\left[\sum_{t=1}^{T} A_t^* \mid \mathcal{E}(\boldsymbol{y}^*), \sum_{j=1}^{q} s_{jT}^* = m, \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^*) \geq m - \Delta_G(m)\right] \mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^* \geq m \mid \mathcal{E}(\boldsymbol{y}^*)\right) + o(m^{\alpha_T\cdot(1-r)})
\end{aligned}$$

The first term defines a feasible solution to Problem $(\mathcal{P}_N^\dagger)$ with a right-hand side of $m - \Delta_G(m)$. Per Lemma EC.11, the following holds, defining again $q = m + \Delta_G(m)$:

$$\begin{aligned}
\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\,(m - \Delta_G(m)) &\geq \begin{cases} \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m) - \Theta(m^{1/2}) & \text{if } p \neq 1 \\ \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m) - \Theta(m^{1/2\cdot\alpha_T\cdot(1-r)}) & \text{if } p = 1 \end{cases}\\
&= \begin{cases} \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - \Theta(m^{1/2}) & \text{if } p \neq 1 \\ \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - \Theta(m^{1/2\cdot\alpha_T\cdot(1-r)}) & \text{if } p = 1 \end{cases}
\end{aligned}$$

In fact, $\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q)$ is exactly the objective value achieved with Algorithm 2, that is, $\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}$. Combining this observation with Lemma EC.12, we obtain:

$$m^*(\varepsilon,p,T) \geq \begin{cases} \left(\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger - \Theta(m^{1/2})\right) \times \left(1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)\right) + o(m^{\alpha_T\cdot(1-r)}) & \text{if } p \neq 1 \\ \left(\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger - \Theta(m^{1/2\cdot\alpha_T\cdot(1-r)})\right) \times \left(1 - o\left(m^{\alpha_T\cdot(1-r)-1}\right)\right) + o(m^{\alpha_T\cdot(1-r)}) & \text{if } p = 1 \end{cases}$$

Therefore, we conclude:

$$m^*(\varepsilon,p,T) \geq m^\dagger(\varepsilon,T) - \begin{cases} \mathcal{O}\left(m^{\max\{\alpha_T\cdot(1-r),1/2\}}\right) & \text{if } p \neq 1 \\ o\left(m^{\alpha_T\cdot(1-r)}\right) & \text{if } p = 1 \end{cases}$$

This completes the proof. □

Statement and proof of Lemmas EC.14 and EC.15

In this section, we assume that $p=1$ (see Figure EC.2). We first provide in Lemma EC.14 lower and upper bounds for the sum of the probabilities of customer coverage under the optimal solution of the fully-learned problem (Problem ($\widehat{\mathcal{P}}_N$)), which will imply it is "close" to the right-hand side target in Problem ($\mathcal{P}_N^{\dagger}$). Combining this result with Lemma EC.11 , Lemma EC.15 relates the solution of Algorithm 2 (Problem ($\mathcal{P}_N^{\dagger}$)) to the fully-learned solution (Problem ($\widehat{\mathcal{P}}_N$)). In particular, the bounds on both directions are important to derive exact regret expressions.

LEMMA EC.14. *Consider an optimal solution $\widehat{\boldsymbol{y}}$ to Problem ($\widehat{\mathcal{P}}_N$). We have:*

$$\sum_{j=1}^{q}\pi_j(\widehat{\boldsymbol{y}}) \geq m - \sum_{\tau=1}^{T}\sum_{i=1}^{n}\left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}+\right)\widehat{y}_{i\tau}\cdot d_{\star}$$

$$\sum_{j=1}^{q}\pi_j(\widehat{\boldsymbol{y}}) \leq m - \sum_{\tau=1}^{T}\sum_{i=1}^{n}\left(\frac{\varepsilon}{\left(\sum_{s=1}^{t-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right)\widehat{y}_{i\tau}$$

Proof: Under solution $\widehat{y}_{it}$, every facility is opened at most once throughout the planning horizon (since $p=1$ and we know the success indicators). This implies that: $\sum_{t=1}^{T}\widehat{y}_{it}\leq 1,\ \forall i=1,\cdots,n$. Therefore, for the facilities $i\in\mathcal{N}(\widehat{\boldsymbol{y}})$, we can define the time $t(i)$ as the unique time period where facility i was opened, i.e., $\widehat{y}_{i,t(i)}=1$ and $\widehat{y}_{i\tau}=0$ for all $\tau\neq t(i)$. By convention, $t(i)=0$ and $\widehat{y}_{i,t(i)}=0$ if $i\notin\mathcal{N}(\widehat{\boldsymbol{y}})$. The lower bound is obtained as follows:

$$\begin{aligned}
\sum_{j=1}^{q}\pi_j(\widehat{\boldsymbol{y}}) &= \sum_{j=1}^{q}\left[1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right)^{\sum_{i\in\mathcal{F}_j}\widehat{y}_{i\tau}}\right]\\
&= \sum_{j=1}^{q}\left[1-\prod_{i\in\mathcal{F}_j}\left(\frac{\varepsilon}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right)^{\widehat{y}_{i,t(i)}}\right]\\
&= \sum_{j=1}^{q}\left[1-\prod_{i\in\mathcal{F}_j\cap\mathcal{N}(\widehat{\boldsymbol{y}})}\frac{\varepsilon}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right]\\
&= \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left[1-\prod_{i\in\mathcal{F}_j\cap\mathcal{N}(\widehat{\boldsymbol{y}})}\frac{\varepsilon}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right]\\
&\geq \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left[1-\sum_{i\in\mathcal{F}_j\cap\mathcal{N}(\widehat{\boldsymbol{y}})}\frac{\varepsilon}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right]\\
&= \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left[1-\sum_{i\in\mathcal{F}_j}\frac{\varepsilon\cdot\widehat{y}_{i,t(i)}}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}\right]\\
&= |\mathcal{M}(\widehat{\boldsymbol{y}})| - \sum_{i=1}^{n}\sum_{j\in\mathcal{C}_i\cap\mathcal{M}(\widehat{\boldsymbol{y}})}\frac{\varepsilon\cdot\widehat{y}_{i,t(i)}}{\left(\sum_{s=1}^{t(i)-1}\sum_{i=1}^{n}\widehat{y}_{is}+1\right)^r}y
\end{aligned}$$

$$\geq m - \sum_{i=1}^{n} \frac{\varepsilon \cdot \widehat{y}_{i,t(i)} \cdot d_\star}{\left(\sum_{s=1}^{t(i)-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}$$
$$= m - \sum_{\tau=1}^{T} \sum_{i=1}^{n} \frac{\varepsilon \cdot \widehat{y}_{i\tau} \cdot d_\star}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r},$$

where the first inequality stems from the fact that $ab \leq a+b$ for any $a,b \in [0,1]$, and the second inequality stems from the feasibility of the solution $\widehat{\boldsymbol{y}}$ (which implies that $|\mathcal{M}(\widehat{\boldsymbol{y}})| \geq m$) and from the fact that $|\mathcal{C}_i| \leq d_\star$ per our degree assumption.

We denote by $\mathcal{M}_t(\widehat{\boldsymbol{y}})$ the set of customers first tentatively covered in period t:

$$\mathcal{M}_t(\widehat{\boldsymbol{y}}) = \{j \in \{1,\cdots,q\} : \exists i \in \mathcal{F}_j : y_{i1} = \cdots = y_{i,t-1} = 0,\ y_{it} = 1,\ y_{k\tau} = 0,\ \forall \tau \leq t, \forall k \in \mathcal{F}_j \setminus \{i\}\}$$

Since $\widehat{\boldsymbol{y}}$ is the optimal solution to the fully-learned problem, every opened facility must serve at least one customer that is not served by any other facility (otherwise, this facility can be removed, contradicting the optimality of the solution). Thus, if $\widehat{y}_{it(i)} = 1$, we can denote such customer by $\widehat{j}(i)$. Then, we have:

$$\sum_{j=1}^{q} \pi_j(\widehat{\boldsymbol{y}}) = \sum_{j=1}^{q} \left[1 - \prod_{\tau=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}\right)^{\sum_{i\in\mathcal{F}_j} \widehat{y}_{i\tau}}\right]$$
$$= \sum_{t=1}^{T} \sum_{j\in\mathcal{M}_t(\widehat{\boldsymbol{y}})} \left[1 - \prod_{\tau=t}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}\right)^{\sum_{i\in\mathcal{F}_j} \widehat{y}_{i\tau}}\right]$$
$$\leq \left(\sum_{t=1}^{T} \sum_{j\in\mathcal{M}_t(\widehat{\boldsymbol{y}})} 1\right) - \left(\sum_{t=1}^{T} \sum_{j\in\mathcal{M}_t(\widehat{\boldsymbol{y}}), \exists k\ s.t.\ j=\widehat{j}(k)} \prod_{\tau=t}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}\right)^{\sum_{i\in\mathcal{F}_j} \widehat{y}_{i\tau}}\right]$$
$$\overset{(*)}{=} m - \left(\sum_{t=1}^{T} \sum_{j\in\mathcal{M}_t(\widehat{\boldsymbol{y}}), \exists k\ s.t.\ j=\widehat{j}(k)} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{t-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}\right) \widehat{y}_{kt}\right)$$
$$= m - \left(\sum_{i=1}^{n} \frac{\varepsilon \cdot \widehat{y}_{i,t(i)}}{\left(\sum_{s=1}^{t(i)-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}\right)$$
$$= m - \sum_{\tau=1}^{T} \sum_{i=1}^{n} \frac{\varepsilon \cdot \widehat{y}_{i\tau}}{\left(\sum_{s=1}^{t-1} \sum_{i=1}^{n} \widehat{y}_{is} + 1\right)^r}$$

The starred equality comes from two observations. First, the coverage constraint is binding (i.e., $|\mathcal{M}(\widehat{\boldsymbol{y}})| = m$) due to the optimality of the fully-learned solution. Second, for $j \in \mathcal{M}_t(\widehat{\boldsymbol{y}})$ (i.e., customer j is first tentatively covered at time t) such that there exists k such that $j = \widehat{j}(k)$ (i.e., customer j is only covered by facility k), the following holds: (i) $\widehat{y}_{kt} = 1$; (ii) $\widehat{y}_{it} = 0,\ \forall i \in \mathcal{F}_{\widehat{j}(k)} \setminus \{k\}$; and (iii) $\widehat{y}_{i\tau} = 0,\ \forall i \in \mathcal{F}_{\widehat{j}(k)},\ \forall \tau \neq t$. This concludes the proof of the lemma. □

We can now turn to Lemma EC.15, which proves that the deterministic approximation solved in Algorithm 2 achieves sub-linear regret against the fully-learned optimum.

LEMMA EC.15. *The following holds if $p=1$, as $m\to\infty$: $m^\dagger(\varepsilon,p,T)=\widehat{m}(\varepsilon,p)+\Theta\left(m^{g(r)}\right)$.*

Proof: Let $\boldsymbol{y}^{\text{L-OPT}}$ denote an optimal solution to Problem ($\widehat{\mathcal{P}}_N$). Recall that each facility is tentatively opened at most once in that solution, i.e., $\sum_{t=1}^T y_{it}^{\text{L-OPT}}\in\{0,1\}$ for all $i=1,\cdots,n$. Moreover, the success of each facility is time-invariant, i.e., $S_{it}=S_i$ for all $t=1,\cdots,T$. Therefore, any facility can be opened in any time period in the fully-learned solution. Out of all optimal solutions to Problem ($\widehat{\mathcal{P}}_N$), we select one that solves the following problem:

$$\begin{aligned}\min_{\boldsymbol{y}\in\{0,1\}^{T\times N}}\quad & \sum_{i=1}^n\sum_{\tau=1}^T\frac{\varepsilon\cdot y_{it}}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n y_{is}+1\right)^r}\\ \text{s.t.}\quad & \sum_{t=1}^T y_{it}=\sum_{t=1}^T y_{it}^{\text{L-OPT}},\ \forall i=1,\cdots,n\end{aligned}$$

By construction, the solution is feasible in Problem ($\widehat{\mathcal{P}}_N$) because of time invariance. Also by construction, the solution achieves the same cost as $\boldsymbol{y}^{\text{L-OPT}}$. It therefore remains an optimal solution to Problem ($\widehat{\mathcal{P}}_N$). We denote it by $\widehat{\boldsymbol{y}}$.

We operate the change of variables $A_t^{\text{L-OPT}}=\sum_{i=1}^n y_{it}^{\text{L-OPT}}$ and $\widehat{A}_t=\sum_{i=1}^n\widehat{y}_{it}$. Then, $\widehat{\boldsymbol{A}}$ is the optimal solution to the following problem:

$$\begin{aligned}\max_{\mathbf{A}\in\{0,1\}^T}\quad & \sum_{t=1}^T A_t-\sum_{\tau=1}^T\frac{\varepsilon\cdot A_t}{\left(\sum_{s=1}^{\tau-1}A_s+1\right)^r}\\ \text{s.t.}\quad & \sum_{t=1}^T A_t=\sum_{t=1}^T A_{it}^{\text{L-OPT}},\ \forall i=1,\cdots,n\end{aligned}$$

This problem now has an equivalent format to Problem ($\mathcal{P}^D$) in Section 4. Following the same procedure as Lemma 1, we can derive a solution such that $\sum_{t=1}^T A_t=m+K(T,\varepsilon,p)\cdot m^{g(r)}+o\left(m^{g(r)}\right)$ and $\sum_{t=1}^T A_t-\sum_{\tau=1}^T\left(\frac{\varepsilon}{(\sum_{s=1}^{\tau-1}A_s+1)^r}\right)A_t\geq\sum_{t=1}^T A_{it}^{\text{L-OPT}}\geq m$, so that

$$\sum_{\tau=1}^T\frac{\varepsilon\cdot\widehat{A}_t}{\left(\sum_{s=1}^{\tau-1}\widehat{A}_s+1\right)^r}=\sum_{i=1}^n\sum_{\tau=1}^T\frac{\varepsilon\cdot\widehat{y}_{it}}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^n\widehat{y}_{is}+1\right)^r}=K(T,\varepsilon,p)\cdot m^{g(r)}+o\left(m^{g(r)}\right)$$

for some constant $K(T,\varepsilon,p)>0$ that is absolutely bounded above by a constant $K_0<\infty$.

By Lemma EC.14, we derive:

$$m-d_\star K(T,\varepsilon,p)m^{g(r)}+o\left(m^{g(r)}\right)\leq\sum_{j=1}^q\pi_j(\widehat{\boldsymbol{y}})\leq m-K(T,\varepsilon,p)m^{g(r)}+o\left(m^{g(r)}\right)$$

Therefore, the fully-learned solution $\widehat{\boldsymbol{y}}$ is feasible in Problem $(\mathcal{P}_N^\dagger)$ with right-hand side $m - d_\star K(T,\varepsilon,p)\cdot m^{\alpha_T\cdot(1-r)} + o\left(m^{\alpha_T\cdot(1-r)}\right)$, that is:

$$\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\left(m - d_\star K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)\right) \le \widehat{m}(\varepsilon,p)$$

Let us again denote $q = m + \Delta_G(m)$. Lemma EC.11 implies:

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\left(m - d_\star K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)\right)\\
&\ge \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m) - d_\star^2 K(T,\varepsilon,p) m^{g(r)} + o\left(m^{g(r)}\right)\\
&\ge \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - d_\star\left((q-m) + \mathcal{O}\left((q-m)^{g(r)}\right)\right) - d_\star^2 K(T,\varepsilon,p) m^{g(r)} + o\left(m^{g(r)}\right)\\
&= \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - d_\star^2 K'(T,\varepsilon,p) m^{g(r)} + o\left(m^{g(r)}\right)
\end{aligned}$$

with

$$K'(T,\varepsilon,p) = \begin{cases} K(T,\varepsilon,p) & \text{if } p=1 \text{ or } (p\neq 1 \text{ and } \alpha_T\cdot(1-r) > 1/2)\\ K'(T,\varepsilon,p)+1 & \text{if } p\neq 1 \text{ and } \alpha_T\cdot(1-r)\le 1/2\end{cases}$$

Combining these observations, we obtain:

$$\widehat{m}(\varepsilon,p) \ge \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - \mathrm{d}_\star^2 K'(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)$$

Moreover, the fully-learned solution $\widehat{\boldsymbol{y}}$ is optimal in Problem $(\mathcal{P}_N^\dagger)$ with right-hand side $\sum_{j=1}^{q}\pi_j(\widehat{\boldsymbol{y}})$. Therefore, we have:

$$\widehat{m}(\varepsilon,p) \le \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\left(\sum_{j=1}^{q}\pi_j(\widehat{\boldsymbol{y}})\right) \le \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\left(m - K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)\right)$$

Again, using Lemma EC.11, we have:

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger\left(m - K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{\alpha_T\cdot(1-r)}\right)\right)\\
&\le \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(m) - \frac{1}{d_\star} K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)\\
&\le \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - \frac{1}{d_\star}\left((q-m) + \mathcal{O}\left((q-m)^{g(r)}\right)\right) - \frac{1}{d_\star} K(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)\\
&= \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^\dagger(q) - \frac{1}{d_\star} K'(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)
\end{aligned}$$

Combining these observations, we obtain:

$$\widehat{m}(\varepsilon,p) \leq \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\dagger}(q) - \frac{1}{d_\star}K'(T,\varepsilon,p)\cdot m^{g(r)} + o\left(m^{g(r)}\right)$$

We conclude that:

$$m^{\dagger}(\varepsilon,p,T) = \widehat{m}(\varepsilon,p) + \Theta\left(m^{g(r)}\right)$$

This completes the proof of Lemma EC.15. □

Proof of Theorem 5

Part 1. We prove that Algorithm 2 yields a feasible solution to $(\mathcal{P}_N^*)$. By definition of Problem $(\mathcal{P}_N^{\dagger})$:

$$\sum_{j=1}^{q} \pi_j(\boldsymbol{y}^{\dagger}) \geq m + \Delta_G(m)$$

Conditioned on $\mathcal{E}(\boldsymbol{y}^{\dagger})$, we have:

$$\sum_{j=1}^{q} s_{jT}^{\dagger} \geq \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^{\dagger}) - \left|\sum_{j=1}^{q}(s_{jT}^{\dagger} - \pi_j(\boldsymbol{y}^{\dagger})\right| \geq \sum_{j=1}^{q} \pi_j(\boldsymbol{y}^{\dagger}) - \Delta_G(m) \geq m$$

This implies from Lemma EC.10 that $\mathbb{P}\left(\sum_{j=1}^{q} s_{jT}^{\dagger} \geq m\right) \geq \mathbb{P}(\mathcal{E}(\boldsymbol{y}^{\dagger})) \geq 1-\delta$. Therefore, the decision vector $\boldsymbol{y}^{\dagger}$ defines a feasible solution to Problem $(\mathcal{P}_N^*)$. From Lemma EC.13, we also have

$$m^{\dagger}(\varepsilon,p,T) \leq m^*(\varepsilon,p,T) + \begin{cases} o\left(m^{\alpha_T\cdot(1-r)}\right) & \text{if } p=1 \text{ or } (p\neq 1 \text{ and } \alpha_T\cdot(1-r) > 1/2) \\ \mathcal{O}(m^{1/2}) & \text{if } p \neq 1 \text{ and } \alpha_T\cdot(1-r) \leq 1/2 \end{cases}$$

Part 2. This is obtained from Lemmas EC.13 and EC.15, when $p=1$:

$$m^{\dagger}(\varepsilon,p,T) \leq m^*(\varepsilon,p,T) + o\left(m^{\alpha_T\cdot(1-r)}\right)$$
$$m^{\dagger}(\varepsilon,p,T) = \widehat{m}(\varepsilon,p) + \Theta(m^{g(r)}) = \widehat{m}(\varepsilon,p) + \Theta(m^{\alpha_T(1-r)})$$

This proves Part 2 of the theorem: $m^*(\varepsilon,p,T) = \widehat{m}(\varepsilon,p) + \Theta(m^{\alpha_T(1-r)}) = \widehat{m}(\varepsilon,p) + \Theta(m^{g(r)})$. □

EC.5.3. Proof of Proposition 3

As in Lemma EC.11, let us define $\boldsymbol{y}^{\mathrm{NL}}(m-\Delta)$ as the solution to Problem $(\mathcal{P}_N^{NL})$ with the constraint that $m-\Delta$ facilities must open with probability $1-\delta$:

$$\begin{aligned}
\boldsymbol{y}^{NL}(m-\Delta) \in \arg\min \quad & \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it} \\
\text{s.t.} \quad & \mathbb{P}\left[\sum_{j=1}^{q} s_{jT} \geq m-\Delta\right] \geq 1-\delta \\
& s_{jT} \sim \mathrm{Ber}\left(1 - \prod_{\tau=1}^{T} \varepsilon^{\sum_{i\in\mathcal{F}_j} y_{i\tau}}\right), \quad \forall j \in \mathcal{C} \\
& y_{it} \in \{0,1\},\ \forall i \in \mathcal{F},\ \forall t \in \mathcal{T}
\end{aligned}$$

By the same argument as in Lemma EC.11, we have:

$$\sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{\mathrm{NL}}(m-\Delta) \le \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it}^{NL}(m) - \frac{c_0\Delta}{d_\star}$$

Recall that $\sum_{t=1}^{T}\widehat{y}_{it} \le 1,\ \forall i = 1,\cdots,n$ and that $|\mathcal{M}(\widehat{\boldsymbol{y}})| = m$. The expected customer coverage of the fully-learned solution in the no-learning problem satisfies:

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{q} s_{jT}\right] &= \sum_{j=1}^{q}\left(1-\prod_{\tau=1}^{T}\varepsilon^{\sum_{i\in\mathcal{F}_j}\widehat{y}_{i\tau}}\right) \\
&= \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left(1-\prod_{\tau=1}^{T}\varepsilon^{\sum_{i\in\mathcal{F}_j}\widehat{y}_{i\tau}}\right) \\
&= \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left(1-\varepsilon^{\sum_{i\in\mathcal{F}_j}\sum_{\tau=1}^{T}\widehat{y}_{i\tau}}\right) \\
&\le \sum_{j\in\mathcal{M}(\widehat{\boldsymbol{y}})}\left(1-\varepsilon^{d_\star}\right) \\
&= m - m\varepsilon^{d_\star}
\end{aligned}$$

By the same argument as Lemma EC.15,

$$\widehat{m}(\varepsilon,p) \le \sum_{t=1}^{T}\sum_{i=1}^{n}\boldsymbol{y}^{\mathrm{NL}}(m-m\varepsilon^{d_\star}) \le \sum_{t=1}^{T}\sum_{i=1}^{n}\boldsymbol{y}^{\mathrm{NL}}(m) - \frac{c_0 m\varepsilon^{d_\star}}{d_\star} = m^{\mathrm{NL}}(\varepsilon,T) - \frac{c_0 m\varepsilon^{d_\star}}{d_\star}.$$

This proves that $m^{\mathrm{NL}}(\varepsilon,T) \ge \widehat{m}(\varepsilon,p) + \Omega(m)$. □

EC.6. Computational Algorithm to Solve Problem $(\mathcal{P}_N^\dagger)$

Recall that the deterministic approximation in Algorithm 2 involves a mixed-integer non-convex optimization problem (Problem $(\mathcal{P}_N^\dagger)$), given as follows:

$$\begin{aligned}
\min\quad & \sum_{t=1}^{T}\sum_{i=1}^{n} y_{it} \\
& \sum_{j=1}^{q}\left(1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n} y_{is}+1\right)^r}+\varepsilon\cdot(1-p)\right)^{\sum_{i\in\mathcal{F}_j} y_{i\tau}}\right) \ge m+\Delta_G(m), \\
& y_{it}\in\{0,1\},\ \forall i\in\mathcal{F},\ \forall t\in\mathcal{T}
\end{aligned}$$

We propose in this appendix a computational algorithm to solve it efficiently. The algorithm proceeds by opening facilities by increasing order of expected incremental coverage: it ignores the least impactful facilities, initially targets moderately impactful ones for learning-based exploration, and then opens the most impactful ones for optimization-based exploitation. We show that the solution is optimal in star graphs and provides a principled heuristic in general bipartite graphs.

EC.6.1. Optimal algorithm in star graphs.

Proposition EC.1 characterizes the algorithm's solution.

PROPOSITION EC.1. *Let $y^\dagger_{it}$ be the solution of Algorithm 2 and $\pi_t(\boldsymbol{y}^\dagger) = \frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n} y^\dagger_{is}+1\right)^r} + \varepsilon \cdot (1-p)$. Consider $i_1 \neq i_2$ such that $y^\dagger_{i_1,t} = y^\dagger_{i_2,\tau} = 1$ and $y^\dagger_{i_1,\tau} = y^\dagger_{i_2,t} = 0$, with $t < \tau$. Then:*

$$\sum_{j\in\mathcal{C}_{i_1}\setminus\mathcal{C}_{i_2}}^{q}\prod_{\tau=1}^{T}\pi_t(\boldsymbol{y}^\dagger)^{\sum_{i\in\mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=t\}} \leq \sum_{j\in\mathcal{C}_{i_2}\setminus\mathcal{C}_{i_1}}^{q}\prod_{\tau=1}^{T}\pi_t(\boldsymbol{y}^\dagger)^{\sum_{i\in\mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=\tau\}}$$

Similarly, if $y^\dagger_{i_2,\tau} = 1$, $y^\dagger_{i_2,t} = 0$, $\forall t \neq \tau$ and $y^\dagger_{i_1,t} = 0$, $\forall t = 1, \cdots, T$ then:

$$\sum_{j\in\mathcal{C}_{i_1}\setminus\mathcal{C}_{i_2}}^{q}\prod_{\tau=1}^{T}\pi_t(\boldsymbol{y}^\dagger)^{\sum_{i\in\mathcal{F}_j} y^\dagger_{iv}} \leq \sum_{j\in\mathcal{C}_{i_2}\setminus\mathcal{C}_{i_1}}^{q}\prod_{\tau=1}^{T}\pi_t(\boldsymbol{y}^\dagger)^{\sum_{i\in\mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=\tau\}}$$

This result sheds light on the exploration-exploitation trade-off to balance the optimization and learning objectives. Specifically, the decision-maker disregards facilities with low marginal expected coverage, to maximize the impact of opened facilities. Then, they prioritize facilities with moderate coverage early on to enable exploration but still contribute toward the coverage target if successful. The facilities with the largest incremental coverage are opened later on for exploitation purposes once uncertainty is mitigated. Corollary EC.1 characterizes the exact solution in star graphs, when every customer is connected to exactly one facility. This star structure eliminates interdependencies across facilities, leading to a threshold-based solution: a facility gets selected if and only if its degree is larger than n_0; and selected facilities then get selected by non-decreasing degree order.

COROLLARY EC.1. *Assume that every customer is connected to one facility, i.e., $|\mathcal{F}_j| = 1$, $\forall j \in \mathcal{C}$, and that every facility is opened at most once, i.e., $\sum_{t=1}^{T} y^\dagger_{it} \leq 1$ $\forall i$. Without loss of generality, arrange the set of facilities in increasing degree order (i.e. $\deg(i_1) \leq \deg(i_2)$ for all $i_1 \leq i_2$). There exist $0 \leq n_0 \leq n_1 \leq \cdots \leq n_T = n$ such that the optimal solution $y^\dagger_{it}$ is of the form:*

$$y_{it} = \begin{cases} 1 & n_{t-1} < i \leq n_t \\ 0 & \textit{otherwise} \end{cases}$$

In star graphs, Problem $(\mathcal{P}^\dagger_N)$ therefore involves finding the largest possible threshold n_0 such that there exist subsequent thresholds $n_0 \leq n_1 \leq \cdots \leq n_T = n$ satisfying

$$\sum_{i=n_0+1}^{n}\deg(i) - \sum_{t=1}^{T}\sum_{i=n_{t-1}+1}^{n_t}\left(\frac{\varepsilon\deg(i)}{(n_t - n_0 + 1)^r} + \varepsilon\cdot(1-p)\right) \geq m + \Delta_G(m)$$

For any value of n_0, we therefore solve the following subproblem:

$$\min \quad \left\{\sum_{t=1}^{T}\sum_{i=n_{t-1}+1}^{n_t}\left(\frac{\varepsilon\deg(i)}{(n_t - n_0 + 1)^r} + \varepsilon\cdot(1-p)\right) \;\middle|\; n_0 \leq n_1 \leq \cdots \leq n_T = n\right\}. \qquad (\mathcal{P}^\dagger_0)$$

Problem $(\mathcal{P}_0^\dagger)$ can be cast as a shortest path problem over $(n-n_0+1)(T-1)+2$ nodes. The network representation, shown in Figure EC.3, includes one source and a sink, and one node for each value of $n_t \in \{n_0,\cdots,n\}$ for each period $t\in\{1,\cdots,T-1\}$. For $t\le T-2$, each node n_t is connected to the nodes $n_{t+1}\ge n_t$, with an arc cost of $\sum_{i=n_{t-1}+1}^{n_t}\left(\frac{\varepsilon\deg(i)}{(n_t-n_0+1)^r}+\varepsilon\cdot(1-p)\right)$. Finally, each node n_{T-1} is connected to the sink, with an arc cost of $\sum_{i=n_{T-1}+1}^{n}\left(\frac{\varepsilon\deg(i)}{(n-n_0+1)^r}+\varepsilon\cdot(1-p)\right)$. This shortest path problem can be solved in polynomial time in n, hence in polynomial time in m.

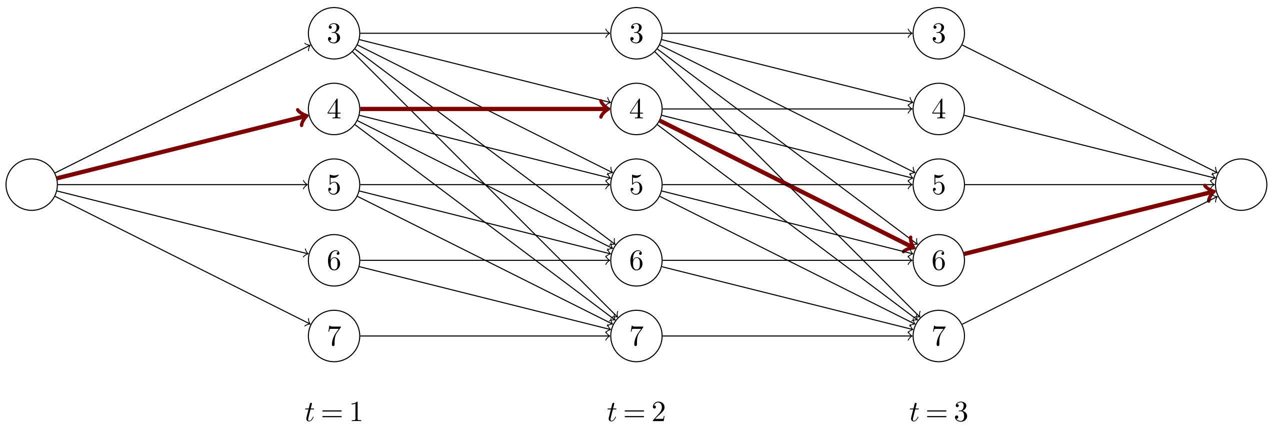


Figure EC.3 Shortest-path representation of Problem $(\mathcal{P}_0^\dagger)$, with $n=7$, $n_0=3$ and $T=4$. The path highlighted in red corresponds to $n_1=4$, $n_2=4$, $n_3=6$, and $n_4=7$.

If the optimum of Problem $(\mathcal{P}_0^\dagger)$ is less than $\frac{1}{\varepsilon}\left(\sum_{i=n_0+1}^{n}\deg(i)-m-\Delta_G(m)\right)$, then the solution to Problem $(\mathcal{P}_0^\dagger)$ is feasible for Problem $(\mathcal{P}_N^\dagger)$. Otherwise, n_0 needs to be increased to guarantee feasibility. We proceed by binary search over n_0 (Algorithm 5). Combined with the polynomial-time shortest path subproblem, this algorithm solves Problem $(\mathcal{P}_N^\dagger)$ in star graphs to optimality in polynomial time in n, hence in polynomial time in m (Proposition EC.2).

Algorithm 5 Algorithm to solve Problem $(\mathcal{P}_N^\dagger)$ in star graph $\mathcal{G}$: $\texttt{STAR}(\mathcal{G})$.

1: **Initialization:** $\underline{n}_0=0, \overline{n}_0=n, n_0=\left\lfloor\frac{\overline{n}_0+\underline{n}_0}{2}\right\rfloor$.

2: Iterate between Steps 1-3, until $\overline{n}_0-\underline{n}_0\le 1$.

Step 1. Solve Problem $(\mathcal{P}_0^\dagger)$ as a shortest path problem; retrieve solution $n_1^*(n_0),\cdots,n_{T-1}^*(n_0)$.

Step 2. If $\displaystyle\sum_{i=n_0+1}^{n}\deg(i)-\sum_{t=1}^{T}\sum_{i=n_{t-1}^*(n_0)+1}^{n_t}\left(\frac{\varepsilon\deg(i)}{(n_t^*(n_0)-n_0+1)^r+\varepsilon\cdot(1-p)}\right)\ge m+\Delta_G(m)$, update $\overline{n}_0=n_0$; otherwise, update $\underline{n}_0=n_0+1$.

Step 3. Update $n_0=\left\lfloor\frac{\overline{n}_0+\underline{n}_0}{2}\right\rfloor$.

3: **Output:** Set $n_0^*\leftarrow\overline{n}_0$; return $n-n_0^*$ as the optimal objective value of Problem $(\mathcal{P}_N^\dagger)$; use n_0^*, $n_1^*(n_0^*),\cdots,n_{T-1}^*(n_0^*)$ to reconstruct an optimal solution of Problem $(\mathcal{P}_N^\dagger)$ as given in Corollary EC.1.

Proposition EC.2. *Algorithm 5 returns the optimal solution of Problem* ($\mathcal{P}_N^\dagger$) *in polynomial time in* n *(hence, in polynomial time in* m*) if* $\mathcal{G}$ *is a star graph (i.e., if* $|\mathcal{F}_j| \leq 1,\ \forall j \in \mathcal{C}$*).*

EC.6.2. A heuristic in general graphs.

We leverage the exact algorithm in star graphs to devise a principled heuristic for Problem ($\mathcal{P}_N^\dagger$) in general bipartite graphs. This approach decomposes any bipartite graph $\mathcal{G}$ with maximum degree $d_\star$ into a weighted star graph $\mathcal{G}_w$ by duplicating every customer $j \in \mathcal{C}$ into $|\mathcal{F}_j|$ artificial customers—once per connected facility, as shown in Figure EC.4. By design, each artificial customer is connected to exactly one facility, so the decomposition recovers a star graph structure at the expense of ignoring interdependencies across facility-customer pairs. Thus, we weigh each connection in the star graph by $1/d_\star \leq w \leq 1$, so that each successful facility contributes a fraction w per customer toward total coverage. The resulting problem is formulated as Problem ($\mathcal{P}_N^\dagger$), except that the degree function is modified as $\deg(i) = w|\mathcal{C}_i|$.

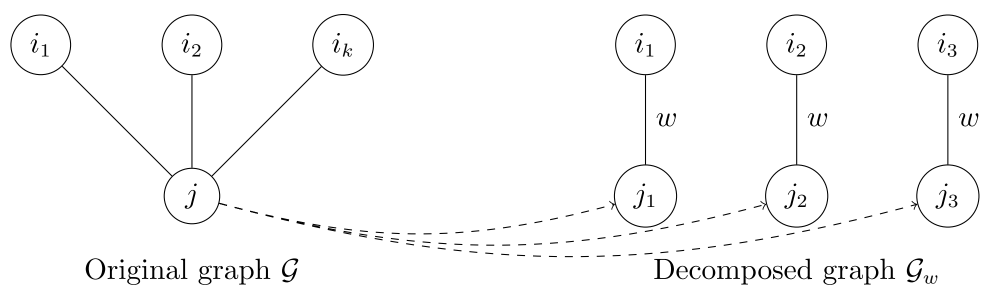


Figure EC.4 Decomposition of a general graph to a star graph.

At one extreme, a weight of $w = 1$ provides an aggressive approximation, in which the formulation allows each customer $j \in \mathcal{C}$ to be covered $|\mathcal{F}_j|$ times. Vice versa, a weight of $w = 1/d_\star$ provides a conservative approximation in which each customer $j \in \mathcal{C}$ can be covered at most once.

Corollary EC.2. *Let* $m^S(w)$ *be the optimum of Problem* ($\mathcal{P}_N^\dagger$) *in the star graph* $\mathcal{G}_w$ *with weights* $deg(i) = w|\mathcal{C}_i|$*, and* $m^\dagger$ *the one of Problem* ($\mathcal{P}_N^\dagger$)*. Then* $m^S(1) \leq m^\dagger \leq m^S(1/d_\star)$*.*

In between, the weight provides a degree of freedom to control the error resulting from the star graph approximation. We select the weight w via one-dimensional line search between $1/d_\star$ and 1:

$$\max \quad \{\mathtt{STAR}(\mathcal{G}_w) \ \mid \ 1/d_\star \leq w \leq 1\}. \tag{$\mathcal{P}_w^\dagger$}$$

Finally, we prune the resulting graph to further improve the solution. Let w^* denote the optimal weight from Problem ($\mathcal{P}_w^\dagger$) and let $\boldsymbol{y}^\dagger(w^*)$ denote the corresponding optimal solution. We perform a greedy search to prune binary decisions $y_{it}^\dagger(w^*)$, by removing facilities one at a time as long as the coverage constraint in Problem ($\mathcal{P}_N^\dagger$) is satisfied. The full algorithm is detailed in Algorithm 6.

Algorithm 6 Algorithm to solve Problem $(\mathcal{P}_N^\dagger)$ in a general graph $\mathcal{G}$.

Solve Problem $(\mathcal{P}_w^\dagger)$ by line search; retrieve optimal solution w^* and solution $\boldsymbol{y}^* \leftarrow \boldsymbol{y}^\dagger(w^*)$.

Repeat Steps 1 -2, until breaking condition is met.

Step 1. Select $i_0 \in \mathcal{F}$ and $t_0 \in \mathcal{T}$ such that $y^*_{i_0,t_0} = 1$. Define candidate solution

$$\widetilde{y}^\dagger_{it} = \begin{cases} y^*_{it} & \text{for all } (i,t) \in \mathcal{F} \times \mathcal{T} \setminus \{(i_0,t_0)\} \\ 0 & \text{if } i = i_0 \text{ and } t = t_0 \end{cases}$$

Step 2. Update $y^*_{i_0,t_0} \leftarrow 0$ if the following condition is met. Otherwise, break.

$$\sum_{j=1}^{q} \left(1 - \prod_{\tau=1}^{T} \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} \widetilde{y}^\dagger_{is} + 1\right)^r} + \varepsilon \cdot (1-p) \right)^{\sum_{i \in \mathcal{F}_j} \widetilde{y}^\dagger_{i\tau}} \right) \geq m + \Delta_G(m)$$

Output: $\boldsymbol{y}^*$

EC.6.3. Proof of Proposition EC.1

For notational ease, we focus on the case where $p = 1$, but all arguments trivially to the case with $p \in [0,1]$. The proof proceeds by swapping decisions $y^\dagger_{i_1,t} \to y^\dagger_{i_1,\tau}$ and $y^\dagger_{i_2,\tau} \to y^\dagger_{i_2,t}$. This swap does not change the coverage of any customer other than those in the neighborhood of the two facilities. It also leaves the objective function of Problem $(\mathcal{P}_N^\dagger)$ unchanged. Denote the new decisions as $\widetilde{\boldsymbol{y}}^\dagger$. Given that $y^\dagger_{it}$ is the optimal solution to Problem $(\mathcal{P}_N^\dagger)$, it is without loss of generality to assume that the left-hand side of the constraint is at least as large under $y^\dagger_{it}$ as under $\widetilde{y}^\dagger_{it}$, i.e.:

$$\sum_{j \in \mathcal{C}_{i_1} \cup \mathcal{C}_{i_2}}^{q} \left(1 - \prod_{\tau=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right)^{\sum_{i \in \mathcal{F}_j} y^\dagger_{iv}} \right) \geq \sum_{j \in \mathcal{C}_{i_1} \cup \mathcal{C}_{i_2}}^{q} \left(1 - \prod_{v=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{v-1} \sum_{i=1}^{n} \widetilde{y}^\dagger_{is} + 1\right)^r} \right)^{\sum_{i \in \mathcal{F}_j} \widetilde{y}^\dagger_{iv}} \right)$$

We can simplify this expression by noting that $\sum_{i \in \mathcal{F}_j} \widetilde{y}^\dagger_{iv} = \sum_{i \in \mathcal{F}_j} y^\dagger_{iv}$ for all $j \in \mathcal{C} \setminus (\mathcal{C}_{i_1} \cup \mathcal{C}_{i_2})$ and for all $v \in \mathcal{T}$:

$$\begin{aligned} &\sum_{j \in \mathcal{C}_{i_1} \setminus \mathcal{C}_{i_2}}^{q} \prod_{\tau=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right)^{\sum_{i \in \mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=t\}} \left(\left(\frac{\varepsilon}{\left(\sum_{s=1}^{t-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right) - \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right) \right) \\ &+ \sum_{j \in \mathcal{C}_{i_2} \setminus \mathcal{C}_{i_1}}^{q} \prod_{\tau=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right)^{\sum_{i \in \mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=\tau\}} \left(\left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right) - \left(\frac{\varepsilon}{\left(\sum_{s=1}^{t-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right) \right) \\ &\leq 0 \end{aligned}$$

Further simplification results in:

$$\sum_{j \in \mathcal{C}_{i_1} \setminus \mathcal{C}_{i_2}}^{q} \prod_{\tau=1}^{T} \left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y^\dagger_{is} + 1\right)^r} \right)^{\sum_{i \in \mathcal{F}_j} y^\dagger_{iv} - \mathbf{1}\{v=t\}}$$

$$-\sum_{j\in\mathcal{C}_{i_2}\setminus\mathcal{C}_{i_1}}^{q}\prod_{\tau=1}^{T}\left(\frac{\varepsilon}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}^{\dagger}+1\right)^{r}}\right)^{\sum_{i\in\mathcal{F}_j}y_{iv}^{\dagger}-\mathbf{1}\{v=\tau\}}\leq 0$$

We use a similar switching argument to show that those facilities that do not get selected have a smaller expected incremental coverage than those that do get selected. □

EC.6.4. Proof of Corollary EC.1

Consider two facilities $i_1\neq i_2$ such that $y_{i_1,t}^{\dagger}=y_{i_2,\tau}^{\dagger}=1$, with $t<\tau$. In the case of a star graph, we know that $\mathcal{C}_{i_1}\cap\mathcal{C}_{i_2}=\emptyset$ and that $|\mathcal{F}_j|=1$ for all $j\in\mathcal{C}$. Denote by N_{t-1} the number of samples available at time $t=1,\cdots,T$. Then, the condition from Proposition EC.1 reduces to:

$$\sum_{j\in\mathcal{C}_{i_1}}\left(\frac{\varepsilon\cdot p}{(N_{t-1}+1)^{r}}+\varepsilon\cdot(1-p)\right)\leq\sum_{j\in\mathcal{C}_{i_2}}\left(\frac{\varepsilon\cdot p}{(N_{t-1}+1)^{r}}+\varepsilon\cdot(1-p)\right),$$

i.e.:

$$\left(\frac{\varepsilon\cdot p}{(N_{t-1}+1)^{r}}+\varepsilon\cdot(1-p)\right)\deg(i_1)\leq\left(\frac{\varepsilon\cdot p}{(N_{t-1}+1)^{r}}+\varepsilon\cdot(1-p)\right)\deg(i_1)$$

Since $t<\tau$, we know that $N_{t-1}\leq N_{\tau-1}$, which implies that $\deg(i_1)\leq\deg(i_2)$. This proves that facilities are tentatively opened by increasing degree order. We proceed similarly to show that those facilities that do not get selected have a smaller degree than those that do get selected. □

EC.6.5. Proof of Corollary EC.2

With $\deg(i)=|\mathcal{C}_i|$, we are effectively solving the same problem as formulated as Problem $(\mathcal{P}_N^{\dagger})$, except that the left-hand side is written as:

$$\sum_{j=1}^{q}\sum_{i\in\mathcal{F}_j}\left(1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}+1\right)^{r}}+\varepsilon\cdot(1-p)\right)^{y_{i\tau}}\right)$$

Since $1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}+1\right)^{r}}+\varepsilon\cdot(1-p)\right)^{y_{i\tau}}<1$, this is larger than or equal to

$$\sum_{j=1}^{q}1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}+1\right)^{r}}+\varepsilon\cdot(1-p)\right)^{\sum_{i\in\mathcal{F}_j}y_{i\tau}}$$

Therefore, any solution that is feasible for the original problem is feasible for the modified one, with the same objective value. This implies that $m^{S}(1)\leq m^{\dagger}$.

Similarly, for $\deg(i)=w|\mathcal{C}_i|$ ans $w=1/d_{\star}$, we effectively solve the same problem as formulated as Problem $(\mathcal{P}_N^{\dagger})$, except the left side is written as:

$$\sum_{j=1}^{q}\sum_{i\in\mathcal{F}_j}\frac{1}{d_{\star}}\left(1-\prod_{\tau=1}^{T}\left(\frac{\varepsilon\cdot p}{\left(\sum_{s=1}^{\tau-1}\sum_{i=1}^{n}y_{is}+1\right)^{r}}+\varepsilon\cdot(1-p)\right)^{y_{i\tau}}\right)$$

$$\leq \sum_{j=1}^{q} \sum_{i \in \mathcal{F}_j} \frac{1}{|\mathcal{F}_j|} \left(1 - \prod_{\tau=1}^{T} \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y_{is} + 1\right)^r} + \varepsilon \cdot (1-p) \right)^{y_{i\tau}} \right)$$

$$= q - \sum_{i \in \mathcal{F}_j} \frac{1}{|\mathcal{F}_j|} \prod_{\tau=1}^{T} \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y_{is} + 1\right)^r} + \varepsilon \cdot (1-p) \right)^{y_{i\tau}}$$

$$\leq q - \prod_{\tau=1}^{T} \left(\frac{\varepsilon \cdot p}{\left(\sum_{s=1}^{\tau-1} \sum_{i=1}^{n} y_{is} + 1\right)^r} + \varepsilon \cdot (1-p) \right)^{\sum_{i \in \mathcal{F}_j} y_{i\tau}}$$

The last inequality follows from the fact that, for all $b > 0$ and $x_i > 0$ we have per the AM-GM inequality:

$$\frac{1}{k} \sum_{i=1}^{k} b^{x_i} \geq b^{\frac{1}{k} \sum_{i=1}^{k} x_i},$$

hence, for all $0 < b \leq 1$ and any $x_i > 0$,

$$\frac{1}{k} \sum_{i=1}^{k} b^{x_i} \geq b^{\sum_{i=1}^{k} x_i},$$

Thus, any feasible solution for $w = 1/d^\star$ is feasible for the original problem with the same objective value. Thus $m(1/d^\star) \geq m^\dagger$.