Risk pricing under gain-loss asymmetry
Marianne Andries
ONLINE APPENDIX

Appendix A Additional Figures

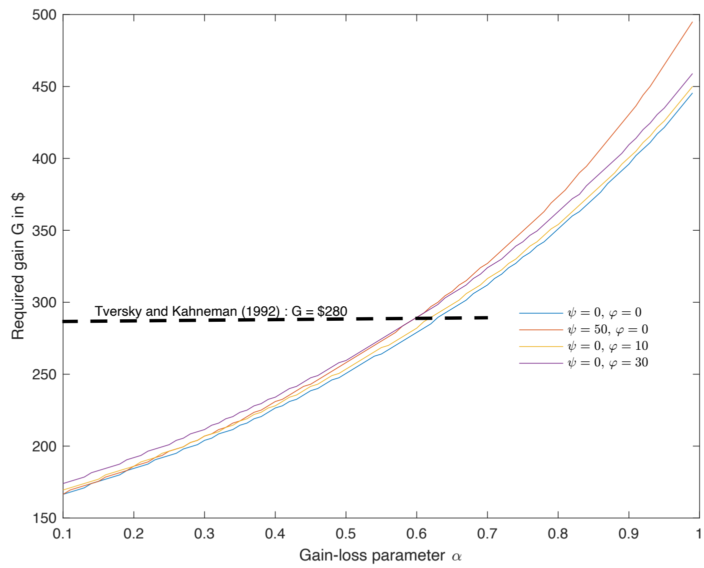

FIGURE A.1: **Roles of parameters α, ψ and φ for the gain G, required to compensate potential loss $L = \$150$.** THE SOLUTIONS ARE DERIVED FOR AN AGENT WITH WEALTH $W = \$100,000$.

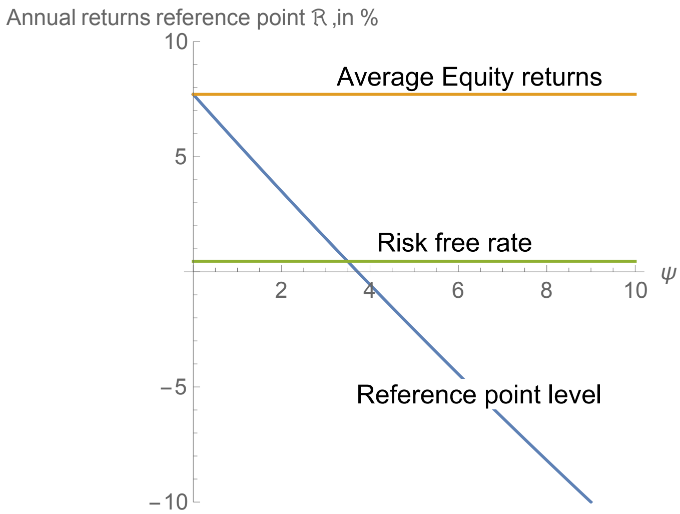

Figure A.2: **Reference point model.** The payoff $X \mid \mathcal{I}$ is lognormaly distributed, calibrated to match the annual US equity market log returns $\mathbb{E}(r_m) = 5.43\%$ and $\sigma(r_m) = 19.98\%$; with a risk-free rate $\mathbb{E}(r_f) = 0.46\%$ (from CRSP, 1930-2011, converted to real terms by Schorfheide et al. (2018)). A constant reference point $\overline{\mathcal{R}} = \mathbb{E}(R_m)$ is achieved for $\psi = 0$; the constant $\log \overline{\mathcal{R}} = \mathbb{E}(r_f)$ is achieved for $\psi = 3.8$.

Appendix B Calibration parameters

	Parameters	Calibration
Preferences	α	0.625
	β	$0.999^{\frac{1}{12}}$
Consumption	μ_c	0.0015
	ϕ_c	1
	Σ_c	$\begin{bmatrix} 0.0072 & 0 \end{bmatrix}$
	A	0.975
	Σ_X	$\begin{bmatrix} 0 & 0.038 \times 0.0072 \end{bmatrix}$
Dividends	μ_d	0.0015
	ϕ_d	2.5
	Σ_d	$\begin{bmatrix} 2.6 \times 0.0072 & 0 \end{bmatrix}$
	$\widetilde{\Sigma}_d$	5.96×0.0072

Appendix C Proofs

Appendix C.1 Proof of Lemma 1 and Lemma 2

Equation (3) makes explicit $u(V_{t+1})$ as an increasing function of V_{t+1}, via both the direct terms in V_{t+1} and via the reference point term $\mathcal{R}_t(V_{t+1})$, so $\mu_t(V_{t+1})$ is increasing in V_{t+1}, and so is V_t.

Further

$$f(x, y) = \left[(1-\beta)\, x^{1-\rho} + \beta g(y)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

is concave if g is concave. By Cauchy-Schwarz inequality, $g\left(Y\right)=\mathbb{E}\left(k\left(Y\right)^{1-\varphi}\right)^{\frac{1}{1-\varphi}}$ is concave if k is concave.

Let's now show $u\left(V_{t+1}\right)$ displays both first-order and second-order stochastic dominance.

Rewrite $u\left(.\right)$ as

$$u\left(X,R\right)=\begin{cases}X^{1-\alpha}R^{\alpha} & \text{if } X\geq R\\ X & \text{if } X\leq R\end{cases},$$

a function of two independent variables X and R. Then, u is a well behave non-decreasing and (concave) utility function of X. Besides u is increasing in R. Let's consider F_1 and F_2 two cumulative distributions for payoffs X, such that F_1 first order stochastic dominates (second order dominates) F_2. Let's fix $R=\bar{R}$. Then, we know $\mathbb{E}(u(X,\bar{R})\mid F_1)\geq\mathbb{E}(u(X,\bar{R})\mid F_2)$. This inequality is valid for all specification of $\bar{R}$, and in particular for $\bar{R}=\mathbb{E}\left(X^{1-\psi}\mid\mathcal{I}\right)^{\frac{1}{1-\psi}}$, $\forall\mathcal{I}$. Further, because $u(X,R)$ is increasing in R point by point, $\mathbb{E}(u(X,\tilde{R})\mid F_1)\geq\mathbb{E}(u(X,\bar{R})\mid F_1)$ for any $\tilde{R}\geq\bar{R}$, and in particular for $\log\tilde{R}=\left(\mathbb{E}\left(X^{1-\psi}\mid F_1\right)\right)^{\frac{1}{1-\psi}}$ which is greater than $\log\bar{R}=\left(\mathbb{E}\left(X^{1-\psi}\mid F_2\right)\right)^{\frac{1}{1-\psi}}$ (since the power utility function with coefficient $1-\psi$ with $\psi\geq 0$ is a non-decreasing (and concave) utility function). We thus find $u(X\mid F_1)\geq u(X\mid F_2)$ under both fist order and second order stochastic dominance.

□

Appendix C.2 Proof of Lemma 3

As standard, small caps denote log levels in the proofs that follow. To simplify the notations, $\mathcal{R}_t\left(V_{t+1}\right)$ is replaced by $\mathcal{R}_t$.

Equation (3) is re-written as:

$$\log u\left(V_{t+1}\right)=\left(1-\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}\right)v_{t+1}+\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}\frac{\log\mathbb{E}_t\exp\left(\left(1-\psi\right)v_{t+1}\right)}{1-\psi}\ ,$$

so

$$\frac{u\left(V_{t+1}\left(C_{t+1}+\delta R_{t+1}\right)\right)-u\left(V_{t+1}\left(C_{t+1}\right)\right)}{u\left(V_{t+1}\right)}=\begin{cases}\left(1-\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}\right)\frac{dV_{t+1}/dC_{t+1}}{V_{t+1}}\delta R_{t+1}\\ +\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}\frac{\mathbb{E}_t\left(\frac{dV_{t+1}/dC_{t+1}}{V_{t+1}}\delta R_{t+1}\exp((1-\psi)v_{t+1})\right)}{\mathbb{E}_t\exp((1-\psi)v_{t+1})}\end{cases}.$$

The first order condition of the optimum consumption path is:

$$\mathbb{E}_t\left[R_{t+1}S_{t,t+1}\right]=\frac{V_t\left(C_{t+1}+\delta R_{t+1}\right)-V_t\left(C_{t+1}\right)}{V_t\left(C_t+\delta\right)-V_t\left(C_t\right)}=1\ .$$

$$\begin{aligned}\mathbb{E}_t\left[R_{t+1}S_{t,t+1}\right]=&\ \beta\left(\mu_t\left(V_{t+1}\right)\right)^{\varphi-\rho}\mathbb{E}_t\left(u\left(V_{t+1}\right)^{1-\varphi}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}V_{t+1}^{\rho-1}\left(1-\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}\right)R_{t+1}\right)\\ &+\beta\left(\mu_t\left(V_{t+1}\right)\right)^{\varphi-\rho}\mathbb{E}_t\left(\alpha\mathbf{1}_{V_{t+1}\geq\mathcal{R}_t}u\left(V_{t+1}\right)^{1-\varphi}\right)\frac{\mathbb{E}_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}V_{t+1}^{\rho-1}R_{t+1}V_{t+1}^{1-\psi}\right)}{\mathbb{E}_t\left(V_{t+1}^{1-\psi}\right)}\ ,\end{aligned}$$

and

$$S_{t,t+1} = \beta \left(\frac{V_{t+1}}{\mu_t(V_{t+1})}\right)^{\rho-\varphi} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{u(V_{t+1})}{V_{t+1}}\right)^{1-\varphi} \times$$
$$\left(\left(1-\alpha \mathbf{1}_{V_{t+1}\geq \mathcal{R}_t}\right) + \alpha \frac{\mathbb{E}_t\left(\mathbf{1}_{V_{t+1}\geq \mathcal{R}_t} u(V_{t+1})^{1-\varphi}\right)}{u(V_{t+1})^{1-\varphi}} \frac{V_{t+1}^{1-\psi}}{\mathbb{E}_t\left(V_{t+1}^{1-\psi}\right)}\right) .$$

$\square$

Appendix C.3 Proof of Lemma 4

As standard, small caps denote log levels in the proofs that follow.

Consider the model with unit intertemporal elasticity of substitution ($\rho = 1$) and constant volatility of consumption:

$$v_t = (1-\beta) c_t + \frac{\beta}{1-\varphi} \log \mathbb{E}_t \left[\exp(1-\varphi) \log u(V_{t+1})\right]$$
$$\log u(V_{t+1}) = v_{t+1} - \alpha \max\left(0, v_{t+1} - \frac{\log \mathbb{E}_t \exp((1-\psi) v_{t+1})}{1-\psi}\right) ,$$

and

$$\log C_{t+1} - \log C_t = \mu_c + \phi_c X_t + \Sigma_c W_{t+1}$$
$$X_{t+1} = A X_t + \Sigma_X W_{t+1} .$$

Assume (and verify) $v_t - c_t = \mu_v + \phi_v X_t$, then:

$$v_{t+1} - \frac{\log \mathbb{E}_t \exp((1-\psi) v_{t+1})}{1-\psi} = (\Sigma_c + \phi_v \Sigma_X) W_{t+1} - \frac{1}{2}(1-\psi)|\Sigma_c + \phi_v \Sigma_X|^2 .$$

Write $(\Sigma_c + \phi_v \Sigma_X) W_{t+1} = \Sigma_v W_{t+1} = |\Sigma_v| w_{t+1}$, then

$$(\mu_v + \phi_v X_t) \times \frac{1-\varphi}{\beta} =$$
$$\log \mathbb{E}_t \exp\left[(1-\varphi)\left(\mu_v + \phi_v X_{t+1} - \alpha |\Sigma_v| \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|}\left(w_{t+1} - \frac{1}{2}(1-\psi)|\Sigma_v|\right) + \mu_c + \phi_c X_t + \Sigma_c W_{t+1}\right)\right] .$$

Grouping the terms in X_t:

$$\phi_v = \beta \phi_c (I - \beta A)^{-1} ,$$

Grouping the constant terms:

$$\frac{1-\beta}{\beta}\mu_v = \mu_c + \frac{1}{2}(1-\varphi)|\Sigma_v|^2 +$$
$$\frac{1}{1-\varphi} \log\left[\begin{cases} \Phi\left(-\left((1-\varphi) - \frac{1}{2}(1-\psi)\right)|\Sigma_v|\right) + \\ \Phi\left(\left((1-\varphi)(1-\alpha) - \frac{1}{2}(1-\psi)\right)|\Sigma_v|\right) \exp\left(-\frac{\alpha(1-\varphi)((1-\varphi)(2-\alpha)-(1-\psi))|\Sigma_v|^2}{2}\right) \end{cases}\right] .$$

$\square$

Appendix C.4 Proof of Proposition 1, and Corollaries 1, 2, 3, 4, and 5

$$S_{t,t+1} = \beta \left(\frac{V_{t+1}}{\mu_t (V_{t+1})} \right)^{1-\varphi} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\frac{u(V_{t+1})}{V_{t+1}} \right)^{1-\varphi} \times$$
$$\left(\left(1 - \alpha \mathbf{1}_{V_{t+1} \geq \mathcal{R}_t}\right) + \alpha \frac{\mathbb{E}_t \left(\mathbf{1}_{V_{t+1} \geq \mathcal{R}_t} u(V_{t+1})^{1-\varphi} \right)}{u(V_{t+1})^{1-\varphi}} \frac{V_{t+1}^{1-\psi}}{\mathbb{E}_t \left(V_{t+1}^{1-\psi} \right)} \right).$$

Taking the logs,

$$\begin{aligned} s_{t,t+1} = {} & \log \beta - \phi_c X_t - (1-\varphi) \frac{1-\beta}{\beta} \mu_v - \varphi \mu_c + (1-\varphi) \left(1 - \alpha \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|}\right) |\Sigma_v| w_{t+1} - \Sigma_c W_{t+1} \\ & + \frac{1}{2} (1-\psi)(1-\varphi) \alpha |\Sigma_v|^2 \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \\ & + \log \left[\begin{cases} \left(1 - \alpha \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|}\right) + \\ \alpha \exp \left(\begin{cases} \left[(1-\psi) - (1-\varphi) \left(1 - \alpha \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|}\right) \right] |\Sigma_v| w_{t+1} \\ -\alpha (1-\varphi) \frac{1}{2} (1-\psi) |\Sigma_v|^2 \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \\ + \frac{1}{2} \left((1-\varphi) \left((1-\varphi)(1-\alpha)^2 + \alpha (1-\psi) \right) - (1-\psi)^2 \right) |\Sigma_v|^2 \end{cases} \right) \times \\ \Phi \left(\left((1-\varphi)(1-\alpha) - \frac{1}{2}(1-\psi) \right) |\Sigma_v| \right) \end{cases} \right]. \end{aligned}$$

The Euler Equation is $\log \mathbb{E}_t \left(\exp \left(ee_{t,t+1} (r_{t+1}) \right) \right) = 0$, with:

$$\begin{aligned} ee_{t,t+1}(r_{t+1}) = {} & r_{t+1} + \log \beta + (1-\varphi) \left(-\frac{1-\beta}{\beta} \mu_v - \phi_c X_t + \phi_v \Sigma_X W_{t+1} \right) - \varphi \left(\mu_c + \phi_c X_t + \Sigma_c W_{t+1} \right) \\ & - (1-\varphi) \alpha |\Sigma_v| \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \left(w_{t+1} - \frac{1}{2} (1-\psi) |\Sigma_v| \right) \\ & + \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \log \left((1-\alpha) + \alpha \frac{\mathbb{E}_t \left(\exp \left(r_{t+1} - \Sigma_c W_{t+1} + (1-\psi) \Sigma_v W_{t+1} \right) \right)}{\exp \left(r_{t+1} - \Sigma_c W_{t+1} + \frac{1}{2} (1-\psi)^2 |\Sigma_v|^2 \right)} \right). \end{aligned}$$

Risk Free Rate

$$rf_t = -\log \mathbb{E}_t \left(\exp ee_{t,t+1}(0) \right),$$

and

$$\begin{aligned} rf_t = {} & -\log \beta + \mu_c + \phi_c X_t \\ & + \log \mathbb{E}_t \exp \left[(1-\varphi) \left(\Sigma_v W_{t+1} - \alpha |\Sigma_v| \mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \left(w_{t+1} - \frac{1}{2} (1-\psi) |\Sigma_v| \right) \right) \right] \\ & - \log \mathbb{E}_t \exp \left[\begin{cases} (1-\varphi) \left(\phi_v \Sigma_X + \frac{\varphi}{\varphi - 1} \Sigma_c \right) W_{t+1} \\ -\mathbf{1}_{w_{t+1} \geq \frac{1}{2}(1-\psi)|\Sigma_v|} \left[\begin{cases} \alpha (1-\varphi) |\Sigma_v| \left(w_{t+1} - \frac{1}{2}(1-\psi)|\Sigma_v| \right) - \\ \log \left((1-\alpha) + \alpha \exp \left(\Sigma_c W_{t+1} + \frac{1}{2} |\Sigma_c|^2 - (1-\psi) \Sigma_c \Sigma_v' \right) \right) \end{cases} \right] \end{cases} \right], \end{aligned}$$

$$
\begin{aligned}
rf_t &= -\log\beta + \mu_c + \phi_c X_t - \frac{1}{2}\left|\Sigma_c\right|^2 + (1-\varphi)(1-\alpha)\Sigma_v\Sigma_c' \\
&+ \log\left[\begin{cases}\Phi\left(-\left((1-\varphi)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|\right)\exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\Sigma_v\right|^2}{2}\right)+ \\ \Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\alpha(1-\varphi)(1-\psi)\left|\Sigma_v\right|^2}{2}\right)\end{cases}\right] \\
&- \log\left[\begin{cases}\Phi\left(-\left((1-\varphi)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|+\frac{\Sigma_c\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\Sigma_v\right|^2}{2}\right)\exp\left(-\alpha(1-\varphi)\Sigma_v\Sigma_c'\right)+ \\ (1-\alpha)\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|-\frac{\Sigma_c\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\left(\frac{\alpha(1-\varphi)(1-\psi)\left|\Sigma_v\right|^2}{2}\right)+ \\ \alpha\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|\right)\exp\left(\left((1-\varphi)(1-\alpha)-(1-\psi)\right)\Sigma_v\Sigma_c'\right)\end{cases}\right].
\end{aligned}
$$

Expected Excess Returns: For returns $\log R_{t+1} = \left(\bar{r}_t - \frac{1}{2}\left|\Sigma_R\right|^2 - \frac{1}{2}\left|\widetilde{\Sigma_R}\right|^2\right) + \Sigma_R W_{t+1} + \widetilde{\Sigma_R}\widetilde{W_{t+1}}$,

$$
\bar{r}_t(\Sigma_R) = -\log\mathbb{E}_t\left(\exp ee_{t,t+1}\left(\Sigma_R W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right)\right).
$$

Because

$$
\begin{aligned}
& ee_{t,t+1}\left(\Sigma_R W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right) = \\
& \Sigma_R W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2 + \log\beta - \mu_c - \phi_c X_t \\
& -\log\mathbb{E}_t\exp\left[(1-\varphi)\left(\Sigma_v W_{t+1} - \alpha\left|\Sigma_v\right|\mathbf{1}_{w_{t+1}\geq\frac{1}{2}(1-\psi)\left|\Sigma_v\right|}\left(w_{t+1}-\frac{1}{2}(1-\psi)\left|\Sigma_v\right|\right)\right)\right] \\
& +\left((1-\varphi)\left(1-\alpha\mathbf{1}_{w_{t+1}\geq\frac{1}{2}(1-\psi)\left|\Sigma_v\right|}\right)\Sigma_v W_{t+1} - \Sigma_c\right)W_{t+1} \\
& +\mathbf{1}_{w_{t+1}\geq\frac{1}{2}(1-\psi)\left|\Sigma_v\right|}\left(\begin{cases}\frac{1}{2}(1-\varphi)\alpha(1-\psi)\left|\Sigma_v\right|^2+ \\ \log\left(\begin{cases}(1-\alpha)+ \\ \alpha\exp\left(\begin{cases}(\Sigma_c-\Sigma_R)W_{t+1}+ \\ \frac{1}{2}\left|\Sigma_c-\Sigma_R\right|^2-(1-\psi)(\Sigma_c-\Sigma_R)\Sigma_v'\end{cases}\right)\end{cases}\right)\end{cases}\right),
\end{aligned}
$$

the solution for the excess returns is:

$$
\begin{aligned}
\bar{r}_t(\Sigma_R) - rf_t &= \left(\Sigma_c + (\varphi-1)(1-\alpha)\Sigma_v\right)\Sigma_R' \\
&- \log\left[\begin{cases}\Phi\left(\left(\frac{1}{2}(1-\psi)-(1-\varphi)\right)\left|\Sigma_v\right|+\frac{(\Sigma_c-\Sigma_R)\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\alpha(1-\varphi)\left(\frac{(1-\varphi)(2-\alpha)\left|\Sigma_v\right|^2}{2}-\Sigma_v(\Sigma_c-\Sigma_R)'\right) \\ +(1-\alpha)\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|-\frac{(\Sigma_c-\Sigma_R)\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\left(\frac{\alpha(1-\varphi)(1-\psi)\left|\Sigma_v\right|^2}{2}\right) \\ +\alpha\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|\right)\exp\left(\left((1-\varphi)(1-\alpha)-(1-\psi)\right)\Sigma_v(\Sigma_c-\Sigma_R)'\right)\end{cases}\right] \\
&+ \log\left[\begin{cases}\Phi\left(-\left((1-\varphi)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|+\frac{\Sigma_c\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\alpha(1-\varphi)\left(\frac{(1-\varphi)(2-\alpha)\left|\Sigma_v\right|^2}{2}-\Sigma_v\Sigma_c'\right) \\ +(1-\alpha)\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|-\frac{\Sigma_c\Sigma_v'}{\left|\Sigma_v\right|}\right)\exp\left(\frac{\alpha(1-\varphi)(1-\psi)\left|\Sigma_v\right|^2}{2}\right) \\ +\alpha\Phi\left(\left((1-\varphi)(1-\alpha)-\frac{1}{2}(1-\psi)\right)\left|\Sigma_v\right|\right)\exp\left(\left((1-\varphi)(1-\alpha)-(1-\psi)\right)\Sigma_v\Sigma_c'\right)\end{cases}\right].
\end{aligned}
$$

For $\Sigma_R\Sigma_v' \to \infty$,

if $((\varphi - 1)(1-\alpha)) > \psi - 1$

$$\bar{r}_t(\Sigma_R) - rf_t \sim (\Sigma_c + (\psi - 1)\Sigma_v)\Sigma_R'$$

if $((\varphi - 1)(1-\alpha)) < \psi - 1$

$$\bar{r}_t(\Sigma_R) - rf_t \sim (\Sigma_c + (\varphi - 1)(1-\alpha)\Sigma_v)\Sigma_R'$$

For $\Sigma_R\Sigma_v' \to -\infty$,
if $\varphi > \psi$

$$\bar{r}_t(\Sigma_R) - rf_t = (\Sigma_c + (\varphi - 1)\Sigma_v)\Sigma_R'$$

if $\varphi < \psi$

$$\bar{r}_t(\Sigma_R) - rf_t \sim (\Sigma_c + (\psi - 1)\Sigma_v)\Sigma_R'$$

For $\Sigma_R\Sigma_v' \to 0$, a second order approximation around $\Sigma_c = 0$, $\Sigma_v = 0$ and $\Sigma_R = 0$ (all of same magnitude) yields:

$$\bar{r}_t(\Sigma_R) - rf_t \approx \frac{\alpha}{\sqrt{2\pi}}\frac{\Sigma_R\Sigma_v'}{|\Sigma_v|} + (\Sigma_c + (\varphi - 1)(1-\alpha)\Sigma_v)\Sigma_R' \ ,$$

which does not depend on ψ. □

Appendix D Dynamic Risk Pricing extensions

Appendix D.1 Risk Prices with Stochastic Volatility

As before, the representative agent has the preferences of Definition 1, with unit elasticity of intertemporal substitution ($\rho = 1$). This time, however, both the drift and the volatility of the optimal consumption process be time varying:

$$\begin{aligned} \log C_{t+1} - \log C_t &= \mu_c + \phi_c X_t + \sigma_t \Sigma_c W_{t+1} \\ X_{t+1} &= A X_t + \sigma_t \Sigma_X W_{t+1} \\ \sigma_{t+1} &= (1-a) + a\sigma_t + \Sigma_\sigma W_{t+1} \ , \end{aligned} \tag{D.1}$$

where $\{W_t\}$ is a three-dimension vector of shocks, iid $\mathcal{N}(0, I)$, and Eq.(D.1) is the stochastic volatility equivalent of Eq. (5). Both the immediate consumption shocks $\{\sigma_t\Sigma_c W_{t+1}\}$, and the long-run consumption shocks $\{\sigma_t\Sigma_X W_{t+1}\}$, now have time varying volatility, which is affected by the iid shocks $\{\Sigma_\sigma W_{t+1}\}$. To simplify the model, volatility shocks are supposed independent from expected consumption shocks: $\Sigma_\sigma\Sigma_X' = \Sigma_\sigma\Sigma_c' = 0$. A and a are contracting (all Eigen values have module strictly less than one): both state variables have stationary distributions, with mean zero for $\{X_t\}$ and mean one for the scalar $\{\sigma_t\}$.

The reference point parameter ψ is set to $\psi = 1$ to simplify the calculations.

For $\Sigma_c << 1$, $\Sigma_X << 1$ and $\Sigma_\sigma << 1$, the unique solution for the value function v is:

$$v_t - c_t \approx \mu_v + \phi_v X_t + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 \ . \tag{D.2}$$

where

$$\phi_v = \beta \phi_c (I - \beta A)^{-1}.$$

μ_v is a decreasing function of α.

$|\phi_{v,\sigma}|$ and $|\phi_{v,\sigma^2}|$ are increasing (i) in β; (ii) in the persistence of the volatility process a; (iii) in the risk aversion coefficient φ; and (iv) in the volatilities of the consumption process.

The value function average level decreases with α; as does the impact of changes in volatility, and thus the pro-cyclical variations in the value function.

This result extends to the risk-free rate, as can be observed in Figure D.3: the risk-free rate that is both lower and more strongly pro-cyclical when $\alpha > 0$.

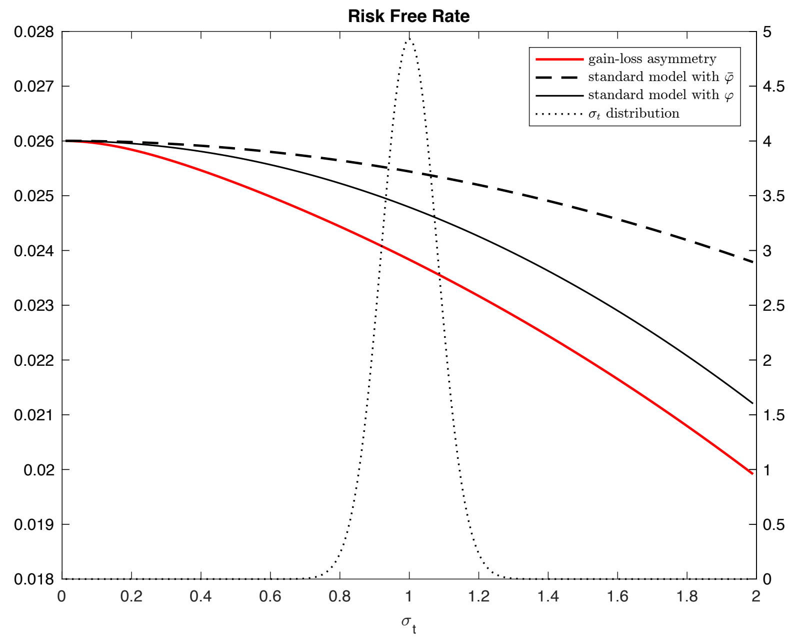

FIGURE D.3: **Risk-free rate with time-varying volatility.** THE ANNUAL RISK-FREE RATE WITH AND WITHOUT GAIN-LOSS ASYMMETRY (STANDARD RECURSIVE UTILITY MODEL FOR RISK AVERSION $\bar{\varphi}$ AND φ) ARE PLOTTED ON THE LEFT AXIS, WHERE $\bar{\varphi}-1 = (\varphi-1)(1-\alpha)$. BECAUSE THE DEPENDENCE ON THE STATE VARIABLE $\{X_t\}$ IS THE SAME WITH AND WITHOUT GAIN-LOSS ASYMMETRY, WE PLOT THE RISK-FREE RATES FOR $X_t = \mathbb{E}(X_t) = 0$. ON THE RIGHT AXIS, IS THE DISTRIBUTION OF σ_t. WE USE THE PARAMETERS FROM HANSEN ET AL. (2008) FOR THE CONSUMPTION PROCESS OF EQ.(D.1) AND $\beta = 0.999$, $\varphi = 10$.

Figures D.4 and D.5 display the risk-price elasticities of assets with conditionally log-normal returns and exposures $\Sigma_{R,t}$ to the consumption shocks.[32] The results derived in the constant volatility model extend to the model with stochastic volatility: the risk-price elasticities display asymmetrical bell shapes and both a level effect and a cross-sectional effect obtain.[33]

[32] We model the exposure $\{\Sigma_{R,t}\}$ as $\Sigma_{R,t} = \Sigma_R \begin{pmatrix} \sigma_t & & 0 \\ & \sigma_t & \\ 0 & & 1 \end{pmatrix}$, such that the aggregate volatility of the asset returns has same time dependence as the consumption risk.

[33] The risk-price elasticities are negative and the asymmetry is reversed, with a higher right-hand side asymptote, for the volatility shocks, which impact the value function negatively.

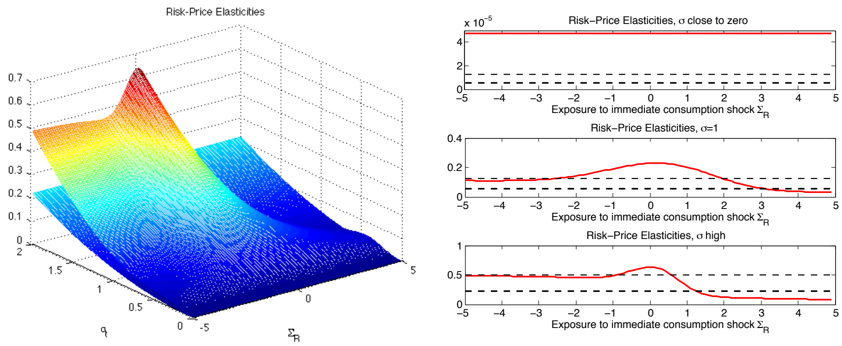

FIGURE D.4: **Risk prices with time-varying volatility.** The two graphs display the risk-price elasticities for an exposure Σ_R to the consumption level shocks, under $\alpha = 0.55$ and in the standard recursive utility model with risk aversion $\bar{\varphi}$ (the plane in the first graph and the lower dotted line in the second graph) and φ (the higher dotted line in the second graph), where $\bar{\varphi} - 1 = (\varphi - 1)(1 - \alpha)$. The second graph displays the three cases, $\sigma_t \approx 0$, $\sigma_t = 1$ (mean value), and $\sigma_t = 2$. We use the parameters from Hansen et al. (2008) for the consumption process of Eq.(D.1) and $\beta = 0.999$, $\varphi = 10$.

Compared to the standard recursive utility model, the pricing dynamics are impacted by two channels of influence. The first derives from the amplification in the value function's variations and generates additional counter-cyclicality in the pricing of risk. The second derives from the dampening effect of higher consumption volatilities on the first-order impact of gin-loss asymmetry (assets are not priced as close to the kink when the underlying volatility is high), and generates less counter-cyclicality in the pricing of risk. The first channel dominates for the pricing of assets that vary in their exposures to the consumption shocks, as in Figure D.4: for small exposures to the consumption shocks, the pricing of risk increases with σ_t at a faster rate, and is thus more strongly counter-cyclical in the model with loss aversion than in the standard recursive utility model. In contrast, the second channel can dominate for assets that vary in their exposures to the volatility shocks, as in Figure D.5, depending on the risk aversion and gain-loss asymmetry coefficients. For $\varphi \leq 20$ and $\alpha = 0.55$, the pricing of risk for the volatility shocks turns pro-cyclical, for assets with small exposures to the shocks, and counter-cyclical for assets with large exposures to the shocks. These striking predictions of the model can be contrasted with those of the standard recursive utility model, in which the risk-price elasticities are counter-cyclical for all the shocks.

Appendix D.2 History Dependence in the Updating of the Reference Point

The reference point model of Definition 1 can also be generalized to the referent as a weighted average of current and past expectations:

$$\mathcal{R}_t\left(V_{t+1}\right) = \left(\prod_{n=0}^{T}\left(\left(\mathbb{E}\left(V_{t+1}^{1-\psi} \mid \mathcal{F}_{t-n}\right)\right)^{\frac{1}{1-\psi}}\right)^{\xi^n}\right)^{\frac{1}{\sum_0^T \xi^n}},$$

where $\xi \in [0, 1)$ and T is the number of past periods impacting the reference point. The case $\xi = 0$ reverts to the model where the reference point is fully updated at each period and only current expectations matter, as in Equation (3). Cases $\xi > 0$ follow Heffetz (2021), who suggests the reference point is a slowly updated expectation, and are in the spirit of Barberis et al. (2001) and Yogo (2008). We analyze this generalized model and show the past-dependency may be useful

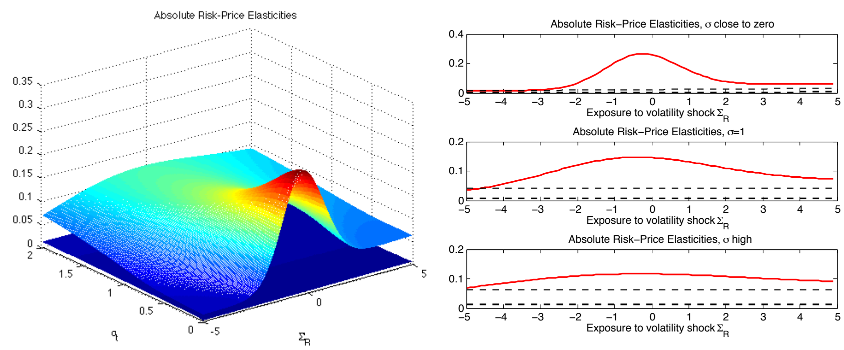

FIGURE D.5: **Risk prices with time-varying volatility.** THE TWO GRAPHS DISPLAY THE RISK-PRICE ELASTICITIES FOR AN EXPOSURE $-\Sigma_R$ TO THE CONSUMPTION VOLATILITY SHOCK, UNDER $\alpha = 0.55$ AND IN THE STANDARD RECURSIVE UTILITY MODEL WITH RISK AVERSION $\bar{\varphi}$ (THE PLANE IN THE FIRST GRAPH AND THE LOWER DOTTED LINE IN THE SECOND GRAPH) AND φ (THE HIGHER DOTTED LINE IN THE SECOND GRAPH), WHERE $\bar{\varphi} - 1 = (\varphi - 1)(1 - \alpha)$. THE SECOND GRAPH DISPLAYS THE THREE CASES, $\sigma_t \approx 0$, $\sigma_t = 1$ (MEAN VALUE), AND $\sigma_t = 2$. WE USE THE PARAMETERS FROM HANSEN ET AL. (2008) FOR THE CONSUMPTION PROCESS OF EQ.(D.1) AND $\beta = 0.999$, $\varphi = 10$.

to create counter-cyclicality in the pricing of risk, as supported by the empirical evidence, even when the consumption process has constant volatility, similar to the habit model of Campbell and Cochrane (1999).

We restrict the analysis to the constant consumption volatility case of Eq. (5), which allows me to disentangle the time variations due to the history dependence in the reference point from those due to time-varying consumption volatilities.

The reference point parameter ψ is set to $\psi = 1$ to simplify the calculations.

Under a unit intertemporal elasticity of substitution ($EIS = 1$), and time dependence on the past two periods ($T = 1$), the unique solution for the value function v is:

$$v_t - c_t = \mu_{v,s} + \phi_v X_t + f\left(\frac{\xi}{1+\xi}\Sigma_s W_t\right) \ , \tag{D.3}$$

where ϕ_v and Σ_s are independent from α and ξ, in contrast to $\mu_{v,s}$ and f, and f has the following properties:

1) f is decreasing
2) $f(0) = 0$
3) f converges to a constant in $-\infty$
4) $f(x) \sim_{+\infty} -\alpha\beta x$.

W_t, the shocks to consumption between $t-1$ and t, affect the time t value function through two channels, one increasing, $\phi_v X_t$, and one decreasing, $f\left(\frac{\xi}{1+\xi}\Sigma_s W_t\right)$. We find the first channel dominates (the value function increases following positive shocks to consumption), and Eq.(D.3) can be rewritten as $v_t - c_{t-1} = g(X_{t-1}, W_t)$, with g increasing in both X_{t-1} and W_t.

The pricing of risk is time varying and Figure D.6 displays the risk-free rate and the expected excess returns of a risky asset, as they vary with the past consumption shocks.

Like the value function, the risk-free rate can be written as $g_{r_f}(X_{t-1}, W_t)$, with g_{r_f} increasing in both terms. When $X_t = 0$, strictly positive and strictly negative shocks to consumption ($W_t \neq 0$)

both decrease the probability of being close to the reference point, and thus diminish the impact of gain-loss asymmetry. For this reason, I find the risk-free rate is at a local minimum when $W_t = 0$.

Following negative shocks to consumption, the probability of disappointment increases. Following positive shocks to consumption, the probability of disappointment decreases. The model with history dependence in the reference point thus yields mostly counter-cyclical expected excess returns. Non-zero shocks, however, decrease the probability of being close to the reference point, and thus diminish the impact of gain-loss asymmetry. We find this effect dominates over the increase in the probability of disappointment following unusually large and negative shocks (more than two standard deviations below zero), thus resulting in lower expected excess returns.[34]

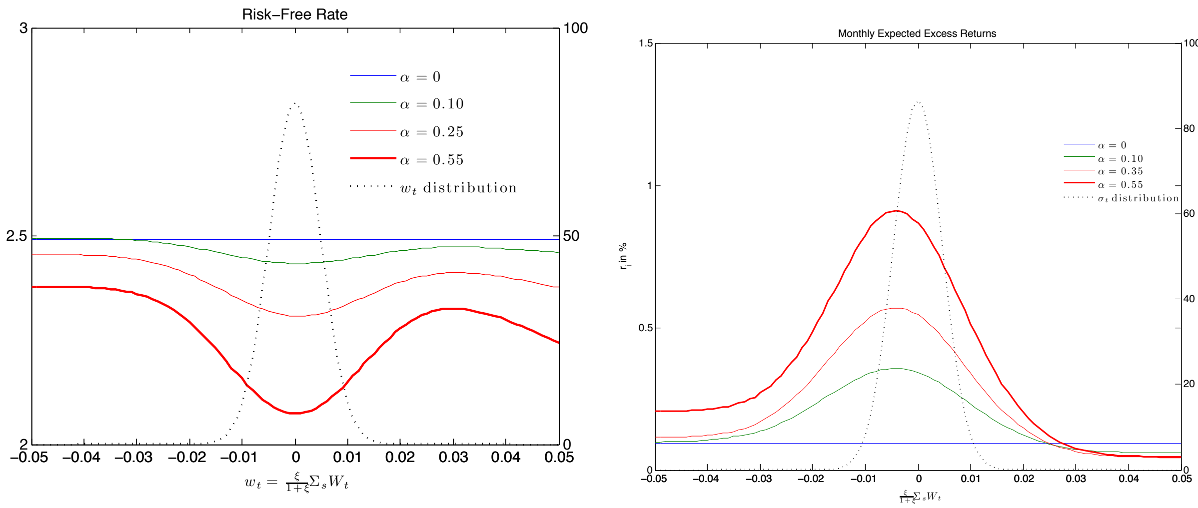

FIGURE D.6: **Risk prices with history-dependent reference point.** THE TWO GRAPHS DISPLAY THE ANNUAL (IN %) RISK-FREE RATE AND THE EXPECTED EXCESS RETURNS OF AN ASSET WITH SAME CHARACTERISTICS (VOLATILITY AND SKEWNESS) AS THE MARKET PORTFOLIO, FOR VARIOUS VALUES OF α (THE CASE $\alpha = 0$ REVERTS TO THE STANDARD RECURSIVE UTILITY MODEL), AS FUNCTIONS OF THE PAST SHOCK $w_t = \frac{\xi}{1+\xi}\Sigma_s W_t$. FOR THE RISK-FREE RATE, $X_t = \mathbb{E}(X_t) = 0$ IS ASSUMED. THE PARAMETERS FOR THE CONSUMPTION PROCESS (5) ARE FROM HANSEN ET AL. (2008) AND $\beta = 0.999$, $(\varphi - 1)(1 - \alpha) = 4$, $\xi = 0.5$. OTHER VALUES OF ξ YIELD SIMILAR ASSET PRICING RESULTS.

In both dynamic models, with stochastic volatility or with "sticky" updating of the reference point, gain-loss asymmetry has non-trivial implications for the pricing of risk. The counter-cyclical influence on pricing is occasionally reversed when assets are priced further and further away from the reference point. This feature is unique to the model with gain-loss asymmetry and would not obtain in either the standard recursive utility model or in habit formation-type models. Exploring its empirical application is left for future research.

[34] For the parameters used in Figure D.6.

Appendix E Robustness around $\rho = 1$

To simplify the calculations, the case $\psi = 1$ is chosen. We follow Hansen et al. (2007) and obtain:

$$
\begin{aligned}
v_t =& v_t^1 + (\rho - 1) Dv_t^1 \\
v_t^1 =& (1-\beta) c_t + \beta \mathcal{Q}_t \left(\overline{v_{t+1}^1} \right) \\
\mathcal{Q}_t \left(\overline{v_{t+1}} \right) =& \frac{1}{1-\varphi} \log \mathbb{E}_t \exp (1-\varphi) \left(v_{t+1} - \alpha \max \left(0, v_{t+1} - \mathbb{E}_t v_{t+1} \right) \right) \\
Dv_t^1 =& -\frac{(1-\beta) \left(v_t^1 - c_t \right)^2}{2\beta} + \beta \overline{\mathbb{E}_t} \left(\overline{Dv_{t+1}^1} \right) \\
\overline{Dv_{t+1}^1} =& Dv_{t+1}^1 - \alpha 1_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \left(Dv_{t+1}^1 - \mathbb{E}_t Dv_{t+1}^1 \right) \ ,
\end{aligned}
$$

where $v_t^1 = v_t \mid_{\rho=1}$, and $\overline{\mathbb{E}_t}$ is the distorted expectation operators associated with the Radon-Nicodym derivative:

$$
\overline{M_{t,t+1}} = \frac{\exp (1-\varphi) \overline{v_{t+1}^1}}{\mathbb{E}_t \exp (1-\varphi) \overline{v_{t+1}^1}} \ .
$$

for $\overline{v_{t+1}} = v_{t+1} - \alpha \max \left(0, v_{t+1} - \mathbb{E}_t v_{t+1} \right)$.

Observe $Dv_t^1 \leq 0$ for all t. With a higher elasticity of intertemporal substitution (ρ less than one), the future consumption stream has more immediate value and thus the value function increases. The opposite occurs when the elasticity decreases.

Approximating the SDF around $\rho = 1$ yields:

$$
s_{t,t+1} = s_{t,t+1}^1 + (\rho - 1) Ds_{t,t+1}^1 \ ,
$$

where $s_{t,t+1}^1 = s_{t,t+1} \mid_{\rho=1}$ and:

$$
\begin{aligned}
Ds_{t,t+1}^1 =& v_{t+1}^1 - \mathcal{Q}_t \left(\overline{v_{t+1}^1} \right) - (c_{t+1} - c_t) + (1-\varphi) \left(Dv_{t+1}^1 - \overline{\mathbb{E}_t} \left(\overline{Dv_{t+1}^1} \right) \right) \\
& - (1-\varphi) \alpha \mathbf{1}_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \left(Dv_{t+1}^1 - \mathbb{E}_t Dv_{t+1}^1 \right) \\
& - \alpha (1-\varphi) \frac{\left[\overline{Dv_{t+1}^1} \mathbb{E}_t \left(\mathbf{1}_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \overline{V_{t+1}^1}^{1-\varphi} \right) - \mathbb{E}_t \left(\mathbf{1}_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \overline{Dv_{t+1}^1 V_{t+1}^1}^{1-\varphi} \right) \right]}{\left(1 - \alpha \mathbf{1}_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \right) \overline{V_{t+1}^1}^{1-\varphi} + \alpha \mathbb{E}_t \left(\mathbf{1}_{v_{t+1}^1 \geq \mathbb{E}_t v_{t+1}^1} \overline{V_{t+1}^1}^{1-\varphi} \right)} \ .
\end{aligned}
$$

Suppose the optimal consumption has constant volatility (the solutions with stochastic volatility are cumbersome to derive), so that:

$$
\begin{aligned}
\log C_{t+1} - \log C_t =& \mu_c + \phi_c X_t + \Sigma_c W_{t+1} \\
X_{t+1} =& A X_t + \Sigma_X W_{t+1} \ .
\end{aligned}
$$

We solve:

$$
\begin{aligned}
\overline{M_{t,t+1}} &= \frac{\exp\left(1-\varphi\right)\overline{v^1_{t+1}}}{\mathbb{E}_t\exp\left(1-\varphi\right)\overline{v^1_{t+1}}} \\
&= \frac{\exp\left(1-\varphi\right)\left(\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}-\alpha 1_{\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}\geq 0}\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}\right)}{\mathbb{E}_t\exp\left(1-\varphi\right)\left(\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}-\alpha 1_{\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}\geq 0}\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}\right)}\ .
\end{aligned}
$$

We define $w_{t+1}=\frac{\left(\Sigma_c+\phi_v\Sigma_X\right)}{\left|\Sigma_c+\phi_v\Sigma_X\right|}W_{t+1}=\frac{\Sigma_v}{\left|\Sigma_v\right|}W_{t+1}$.
We find:

$$
\overline{\mathbb{E}_t}\left(1_{w_{t+1}\geq 0}\right)=\frac{\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}{\Phi\left(-\left(1-\varphi\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\left(1-\varphi\right)^2\alpha\left(2-\alpha\right)\left|\Sigma_v\right|^2}{2}\right)+\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}\ ,
$$

$$
\overline{\mathbb{E}_t}\left(w_{t+1}\right)=\left(1-\varphi\right)\left|\Sigma_v\right|\left(1-\alpha\frac{\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}{\Phi\left(-\left(1-\varphi\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\left(1-\varphi\right)^2\alpha\left(2-\alpha\right)\left|\Sigma_v\right|^2}{2}\right)+\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}\right)\ ,
$$

$$
\begin{aligned}
\overline{\mathbb{E}_t}\left(w^2_{t+1}\right) &= 1+\left(1-\varphi\right)^2\left|\Sigma_v\right|^2 \\
&\quad -\left(1-\varphi\right)\alpha\left|\Sigma_v\right|\frac{\left(1-\varphi\right)\left(2-\alpha\right)\left|\Sigma_v\right|\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)+\phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}{\Phi\left(-\left(1-\varphi\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\left(1-\varphi\right)^2\alpha\left(2-\alpha\right)\left|\Sigma_v\right|^2}{2}\right)+\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}\ ,
\end{aligned}
$$

$$
\overline{\mathbb{E}_t}\left(1_{w_{t+1}\geq 0}w_{t+1}\right)=\frac{\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)+\phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}{\Phi\left(-\left(1-\varphi\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\left(1-\varphi\right)^2\alpha\left(2-\alpha\right)\left|\Sigma_v\right|^2}{2}\right)+\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}\ ,
$$

and

$$
\overline{\mathbb{E}_t}\left(1_{w_{t+1}\geq 0}w^2_{t+1}\right)=\frac{\begin{cases}\left(1+\left[\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right]^2\right)\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)\\ +\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)\end{cases}}{\Phi\left(-\left(1-\varphi\right)\left|\Sigma_v\right|\right)\exp\left(\frac{\left(1-\varphi\right)^2\alpha\left(2-\alpha\right)\left|\Sigma_v\right|^2}{2}\right)+\Phi\left(\left(1-\varphi\right)\left(1-\alpha\right)\left|\Sigma_v\right|\right)}\ .
$$

For any y_{t+1} independent from w_{t+1}:

$$
\overline{\mathbb{E}_t}\left(y_{t+1}\right)=\mathbb{E}_t\left(y_{t+1}\right)\ .
$$

Solution for Dv^1_t $\quad v^1_t-c_t=\mu_v+\phi_v X_t$.
Suppose

$$
Dv^1_t=\mu_{v,\rho}+\phi_{v,\rho}X_t+X'_t\varphi_{v,\rho}X_t\ ,
$$

with $\varphi_{v,\rho}$ symmetric matrix.

Then

$$
\begin{aligned}
\overline{Dv^1_{t+1}} &= Dv^1_{t+1} - \alpha 1_{v^1_{t+1} \geq \mathbb{E}_t v^1_{t+1}} \left(Dv^1_{t+1} - \mathbb{E}_t Dv^1_{t+1} \right) \\
&= \mu_{v,\rho} + \phi_{v,\rho} X_{t+1} + X'_{t+1} \varphi_{v,\rho} X_{t+1} \\
&- \alpha 1_{w_{t+1} \geq 0} \left(\left(\phi_{v,\rho} + 2X'_t A' \varphi_{v,\rho} \right) \Sigma_X W_{t+1} + W_{t+1} \Sigma'_X \varphi_{v,\rho} \Sigma_X W_{t+1} - \mathrm{Tr} \left(\Sigma'_X \varphi_{v,\rho} \Sigma_X \right) \right) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu_{v,\rho} + \phi_{v,\rho} X_t + X'_t \varphi_{v,\rho} X_t = \\
&-\frac{(1-\beta)}{2\beta} \left(\mu_v^2 + (\phi_v X_t)^2 + 2\mu_v \phi_v X_t \right) + \beta \overline{\mathbb{E}_t} \left(\mu_{v,\rho} + \phi_{v,\rho} X_{t+1} + X'_{t+1} \varphi_{v,\rho} X_{t+1} \right) \\
&+\beta \overline{\mathbb{E}_t} \left(-\alpha 1_{w_{t+1} \geq 0} \left(\left(\phi_{v,\rho} + 2X'_t A' \varphi_{v,\rho} \right) \Sigma_X W_{t+1} + W_{t+1} \Sigma'_X \varphi_{v,\rho} \Sigma_X W_{t+1} - \mathrm{Tr} \left(\Sigma'_X \varphi_{v,\rho} \Sigma_X \right) \right) \right) \ .
\end{aligned}
$$

Group the quadratic terms:

$$
\varphi_{v,\rho} = -\frac{(1-\beta)}{2\beta} \phi'_v \phi_v + \beta A' \varphi_{v,\rho} A \ .
$$

This is a Sylvester equation, for which the solution for $\varphi_{v,\rho}$ is the same as in the standard recursive model:

$$
\varphi_{v,\rho} = -\frac{(1-\beta)}{2\beta} \sum_{t=0}^{+\infty} \beta^t A'^t \phi'_v \phi_v A^t \ ,
$$

which, because it is negative semi-definite, can be written, using Cholesky decomposition, as $\varphi_{v,\rho} = -\widetilde{\varphi_{v,\rho}}' \widetilde{\varphi_{v,\rho}}$ with $\widetilde{\varphi_{v,\rho}}$ triangular.

We decompose:

$$
W_{t+1} = \frac{\Sigma'_v}{|\Sigma_v|} w_{t+1} + Y_{t+1} \ ,
$$

with Y_{t+1}, vector of variables independent from w_{t+1}:

$$
\overline{\mathbb{E}_t} \left(Y_{t+1} \right) = \mathbb{E}_t \left(Y_{t+1} \right) = 0 \ .
$$

We group the terms in X_t:

$$
\phi_{v,\rho} X_t = -\frac{(1-\beta)}{\beta} \mu_v \phi_v X_t + \beta \phi_{v,\rho} A X_t + 2\beta X'_t A' \varphi_{v,\rho} \Sigma_X \overline{\mathbb{E}_t} \left(W_{t+1} - \alpha 1_{w_{t+1} \geq 0} W_{t+1} \right) \ ,
$$

and so

$$
\begin{aligned}
\phi_{v,\rho} = &\left[-\frac{(1-\beta)}{\beta} \mu_v \phi_v + 2\beta (1-\varphi)(1-\alpha) \Sigma_v \Sigma'_X \varphi_{v,\rho} A \right] (I - \beta A)^{-1} \\
&+ 2\beta\alpha \Sigma_v \Sigma'_X \varphi_{v,\rho} A (I - \beta A)^{-1} \\
&\times \left(\frac{1-\varphi}{1-\alpha} - \frac{(1-\varphi)(2-\alpha) \Phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right) + \phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right) / |\Sigma_v|}{\Phi\left(-(1-\varphi)|\Sigma_v|\right) \exp\left(\frac{(1-\varphi)^2 \alpha (2-\alpha) |\Sigma_v|^2}{2} \right) + \Phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right)} \right) \ .
\end{aligned}
$$

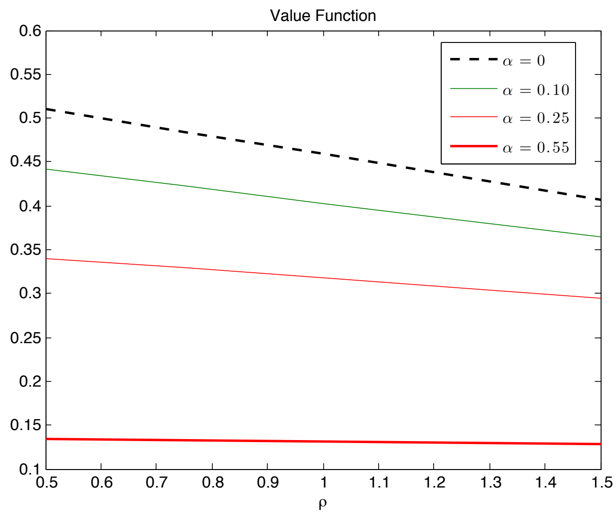

FIGURE E.7: VALUE FUNCTION DEPENDENCE ON ρ

Average $v_t - c_t$ (calculated for $X_t = \mathbb{E}(X_t) = 0$) in the model with gain-loss asymmetry and in the standard recursive utility model (case $\alpha = 0$). We use the parameters from Hansen et al. (2008) for the consumption process and $\beta = 0.99$, $(1-\varphi)(1-\alpha) = 10$.

Finally the constant terms are:

$$\begin{aligned}\mu_{v,\rho}(1-\beta) = &-\frac{(1-\beta)}{2\beta}\mu_v^2 + \beta\phi_{v,\rho}\Sigma_X\overline{\mathbb{E}_t}\left(W_{t+1} - \alpha 1_{w_{t+1}\geq 0}W_{t+1}\right) - \beta\overline{\mathbb{E}_t}\left(\left|\widetilde{\varphi_{v,\rho}}\Sigma_X W_{t+1}\right|^2\right) \\ &+ \beta\alpha\overline{\mathbb{E}_t}\left(1_{w_{t+1}\geq 0}\left(\left|\widetilde{\varphi_{v,\rho}}\Sigma_X W_{t+1}\right|^2 - \mathrm{Tr}\left(\widetilde{\varphi_{v,\rho}}\Sigma_X\right)'\widetilde{\varphi_{v,\rho}}\Sigma_X\right)\right) \ ,\end{aligned}$$

$$\begin{aligned}\mu_{v,\rho} = &-\frac{1}{2\beta}\mu_v^2 + \frac{\beta}{1-\beta}(1-\varphi)\phi_{v,\rho}\Sigma_X\Sigma_v' - \frac{\beta}{1-\beta}\left((1-\varphi)^2\left|\widetilde{\varphi_{v,\rho}}\Sigma_X\Sigma_v'\right|^2 + \mathrm{Tr}\left(\widetilde{\varphi_{v,\rho}}\Sigma_X\right)'\widetilde{\varphi_{v,\rho}}\Sigma_X\right) \\ &+ \frac{\beta}{1-\beta}\alpha\left[\left\{\begin{array}{l} -\frac{\phi_{v,\rho}\Sigma_X\Sigma_v'[(1-\varphi)(2-\alpha)\Phi((1-\varphi)(1-\alpha)|\Sigma_v|)+\phi((1-\varphi)(1-\alpha)|\Sigma_v|)/|\Sigma_v|]}{\Phi(-(1-\varphi)|\Sigma_v|)\exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)|\Sigma_v|^2}{2}\right)+\Phi((1-\varphi)(1-\alpha)|\Sigma_v|)} \\ +\frac{(1-\varphi)\left|\widetilde{\varphi_{v,\rho}}\Sigma_X\Sigma_v'\right|^2\left[\left\{\begin{array}{l}(1-\varphi)\Phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right)\left[(2-\alpha)+(1-\alpha)^2\right]+ \\ (2-\alpha)\phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right)/|\Sigma_v|\end{array}\right.\right]}{\Phi(-(1-\varphi)|\Sigma_v|)\exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)|\Sigma_v|^2}{2}\right)+\Phi((1-\varphi)(1-\alpha)|\Sigma_v|)}\end{array}\right.\right] .\end{aligned}$$

As mentioned above, $Dv_t^1 \leq 0$ for all t. We find $\left|Dv_t^1\right|$ is higher in good times ($\phi_c X_t \geq 0$) than in bad times ($\phi_c X_t \leq 0$). This enhances the pro-cyclical variations of the value function when the elasticity of intertemporal substitution is greater than one ($\rho \leq 1$) and reduces them when the elasticity of intertemporal substitution is less than one ($\rho \geq 1$). $|\phi_{v,\rho}|$ and $|\mu_{v,\rho}|$ are lower than in the standard recursive utility model, and the solution for $\varphi_{v,\rho}$ is unchanged. Therefore, the impact of a change in the elasticity of intertemporal substitution is reduced by $\alpha > 0$. In Figure E.7, we display the value of the consumption stream as it varies with ρ. Observe, in the model with $\alpha = 0.55$, the value of the elasticity of intertemporal substitution has little impact on the value function.

Solution for Dv_t^1 From the derivation of the SDF,

$$
\begin{aligned}
Ds^1_{t,t+1} = & \left(-\frac{1-\beta}{\beta}\mu_v - \phi_c X_t + \phi_v \Sigma_X W_{t+1}\right) \\
& + (1-\varphi)\left(-\frac{1-\beta}{\beta}\left(\mu_{v,\rho} + \frac{\mu_v^2}{2\beta}\right) - \frac{1}{\beta}\left(\phi_{v,\rho}(I-\beta A) + \frac{1-\beta}{\beta}\mu_v\phi_v\right) X_t\right) \\
& + (1-\varphi)\left(\left(\phi_{v,\rho} + 2X_t' A' \varphi_{v,\rho}\right)\Sigma_X W_{t+1} + W_{t+1}'\Sigma_X' \varphi_{v,\rho}\Sigma_X W_{t+1}\right) \\
& - (1-\varphi)\,\alpha \mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\left(\left(\phi_{v,\rho} + 2X_t' A' \varphi_{v,\rho}\right)\Sigma_X W_{t+1} + W_{t+1}'\Sigma_X' \varphi_{v,\rho}\Sigma_X W_{t+1} - \operatorname{Tr}\left(\Sigma_X' \varphi_{v,\rho}\Sigma_X\right)\right) \\
& - \frac{\alpha(1-\varphi)\begin{cases} \widetilde{Dv^1_{t+1}}\,\Phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right)\exp\left(\frac{1}{2}\left|(1-\varphi)(1-\alpha)|\Sigma_v|\right|^2\right) - \\ \mathbb{E}_t\left(\mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\widetilde{Dv^1_{t+1}}\exp\left((1-\varphi)(1-\alpha)\Sigma_v W_{t+1}\right)\right)\end{cases}}{\begin{cases}\left(1-\alpha\mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\right)\exp\left((1-\varphi)\left(1-\alpha\mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\right)\Sigma_v W_{t+1}\right) + \\ \alpha\Phi\left((1-\varphi)(1-\alpha)|\Sigma_v|\right)\exp\left(\frac{1}{2}(1-\varphi)(1-\alpha)^2|\Sigma_v|^2\right)\end{cases}},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{Dv^1_{t+1}} = & \left(\left(\phi_{v,\rho} + 2X_t' A' \varphi_{v,\rho}\right)\Sigma_X W_{t+1} + W_{t+1}\Sigma_X' \varphi_{v,\rho}\Sigma_X W_{t+1}\right)\left(1-\alpha\mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\right) \\
& + \alpha\mathbf{1}_{\Sigma_v W_{t+1}\geq 0}\operatorname{Tr}\left(\Sigma_X' \varphi_{v,\rho}\Sigma_X\right) \ .
\end{aligned}
$$

As evidenced by Figure E.8, the variations in the risk-free rate due to variations in the elasticity of intertemporal substitution are lowered by the gain-loss asymmetry model relative to the standard recursive utility model. Note $\alpha > 0$ lowers the risk-free rate for ρ around one and higher. As the elasticity of intertemporal substitution increases away from one (ρ is lowered away from one), the effect is reversed and the risk-free rate is higher in the model with $\alpha > 0$ than in the standard model.

As seen in Figure E.9, the expected excess returns are very little impacted by changes in the elasticity of intertemporal substitution, in either the standard recursive utility model or the model with gain-loss asymmetry.

Appendix F Stochastic Volatility in Consumption—Proofs of Extension Appendix D.1

As in Appendix E, $\psi = 1$ to simplify the calculations. We analyze the model:

$$
\begin{aligned}
v_t &= (1-\beta)c_t + \frac{\beta}{1-\varphi}\log\mathbb{E}_t\left[\exp(1-\varphi)\log U\left(V_{t+1}\mid\mathcal{F}_t\right)\right] \\
\log U\left(V_{t+1}\mid\mathcal{F}_t\right) &= v_{t+1} - \alpha\max\left(0, v_{t+1} - \mathbb{E}_t\left(v_{t+1}\right)\right) \ ,
\end{aligned}
$$

with

$$
\begin{aligned}
\log C_{t+1} - \log C_t &= \mu_c + \phi_c X_t + \sigma_t\Sigma_c W_{t+1} \\
X_{t+1} &= AX_t + \sigma_t\Sigma_X W_{t+1} \\
\sigma_{t+1} &= (1-a) + a\sigma_t + \Sigma_\sigma W_{t+1} \ ,
\end{aligned}
$$

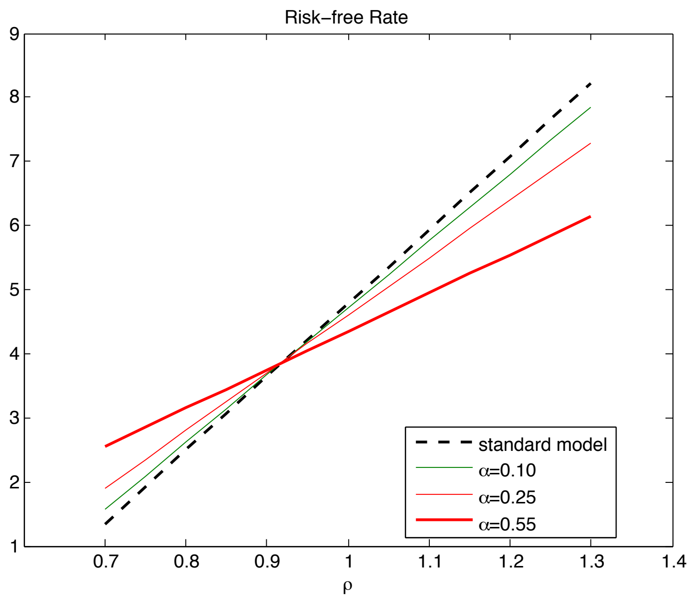

FIGURE E.8: RISK-FREE RATE DEPENDENCE ON ρ

Annual risk-free rate for X_t chosen randomly in its stationary distribution, in the model with gain-loss asymmetry and in the standard recursive utility model (case $\alpha = 0$). We use the parameters from Hansen et al. (2008) for the consumption process and $\beta = 0.999$, $(1 - \varphi)(1 - \alpha) = 10$.

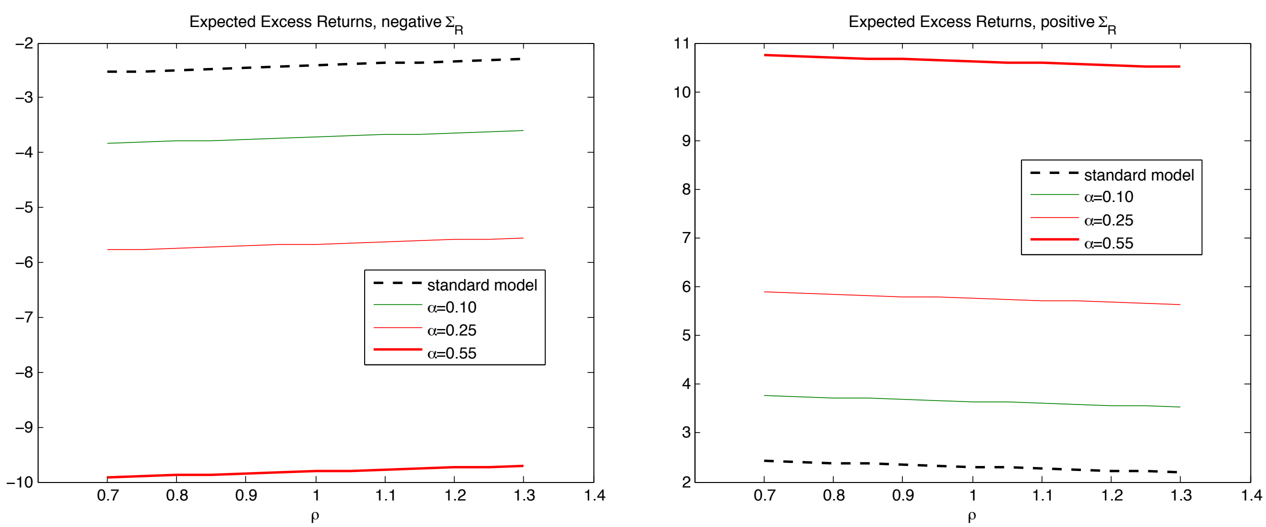

FIGURE E.9: EXPECTED EXCESS RETURNS DEPENDENCE ON ρ

Annual expected returns for X_t chosen randomly in its stationary distribution, in the model with gain-loss asymmetry and in the standard recursive utility model (case $\alpha = 0$). We plot the returns for an asset with negative loadings on the consumption shocks on the left graph, and an asset with positive loadings on the consumption shocks on the right hand graph. We use the parameters from Hansen et al. (2008) for the consumption process and $\beta = 0.999$, $(1 - \varphi)(1 - \alpha) = 10$.

W_t iid $\mathcal{N}(0,I)$, σ_t scalar. For simplification, we assume that the shocks to volatility are independent from the shocks to the immediate and future consumption: $\Sigma_\sigma \Sigma_X^{'} = 0$ and $\Sigma_\sigma \Sigma_c^{\prime} = 0$.

We assume $v_t - c_t = f(X_t, \sigma_t)$. The recursive problem is:

$$f(X_t,\sigma_t) = \frac{\beta}{1-\varphi} \log \mathbb{E}_t \exp\left[(1-\varphi)\left[f(X_{t+1},\sigma_{t+1}) + c_{t+1} - c_t - \alpha \max\left(0, \Sigma_c W_{t+1} + (1-\mathbb{E}_t) f(X_{t+1},\sigma_{t+1})\right)\right]\right] ,$$

which satisfies Blackwell conditions for bounded functions. While $f(X_t)$ is not necessarily bounded, it can be approximated to a bounded function (by truncating the domain of (X_t,σ_t)). In that case, there is a unique solution which can be obtained by iterating the process starting from any initial function.

Suppose

$$v_t - c_t \approx \mu_v + \phi_v X_t + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 ,$$

then

$$\begin{aligned} v_{t+1} - \mathbb{E}_t v_{t+1} = &\left(\sigma_t\left(\Sigma_c + \phi_v \Sigma_X + 2\phi_{v,\sigma^2} a \Sigma_\sigma\right) + \left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1-a)\right)\Sigma_\sigma\right) W_{t+1} \\ &+ \phi_{v,\sigma^2}\left(\left(\Sigma_\sigma W_{t+1}\right)^2 - \Sigma_\sigma \Sigma_\sigma^{\prime}\right) . \end{aligned}$$

Because Σ_σ is small, we ignore the second order term:

$$v_{t+1} - \mathbb{E}_t v_{t+1} \approx \Sigma_v(\sigma_t) W_{t+1} ,$$

with

$$\Sigma_v(\sigma_t) = \sigma_t\left(\Sigma_c + \phi_v \Sigma_X + 2\phi_{v,\sigma^2} a \Sigma_\sigma\right) + \left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1-a)\right)\Sigma_\sigma .$$

The recursive problem is:

$$\begin{aligned} &\mu_v + \phi_v X_t + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 = \\ &\beta\left(\mu_v + \mu_c + \phi_{v,\sigma}(1-a) + (\phi_c + \phi_v A) X_t + \phi_{v,\sigma} a \sigma_t + \phi_{v,\sigma^2}\left(1 - a + a\sigma_t\right)^2\right) + \\ &\frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp(1-\varphi)\left[\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}\left(\Sigma_\sigma W_{t+1}\right)^2 - \alpha \max\left(0, \Sigma_v(\sigma_t) W_{t+1}\right)\right] . \end{aligned}$$

Grouping the X_t terms, we find, as in the standard case:

$$\phi_v = \beta \phi_c (I - \beta A)^{-1} .$$

Define $w_{t+1} = \frac{\Sigma_v(\sigma_t)}{|\Sigma_v(\sigma_t)|} W_{t+1}$. Then

$$\Sigma_\sigma W_{t+1} = \delta_1(\sigma_t) w_{t+1} + \delta_2(\sigma_t) z_{t+1} ,$$

with

$$\mathbb{E}\left(w_{t+1} z_{t+1}\right) = 0 .$$

and $z_{t+1} \sim \mathcal{N}(0,1)$.

$$\mathbb{E}_t \exp(1-\varphi)\left[\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2 - \alpha \mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t) W_{t+1}\right] =$$

$$\mathbb{E}_t \exp(1-\varphi)\left[|\Sigma_v(\sigma_t)| w_{t+1} + \phi_{v,\sigma^2}\left(\delta_1^2 w_{t+1}^2 + \delta_2^2 z_{t+1}^2 + 2\delta_1 w_{t+1}\delta_2 z_{t+1}\right) - \alpha \mathbf{1}_{w_{t+1}\geq 0}|\Sigma_v(\sigma_t)| w_{t+1}\right],$$

and, if $1-2\phi_{v,\sigma^2}(1-\varphi)\delta_2^2 > 0$

$$\mathbb{E}_t \exp(1-\varphi)\left[\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2 - \alpha \mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t) W_{t+1}\right] =$$

$$\frac{1}{\sqrt{1-2(1-\varphi)\phi_{v,\sigma^2}\delta_2^2}}\mathbb{E}_t \exp\left[(1-\varphi)|\Sigma_v(\sigma_t)| w_{t+1}\left(1-\alpha\mathbf{1}_{w_{t+1}\geq 0}\right) + \frac{(1-\varphi)\phi_{v,\sigma^2}\delta_1^2}{1-2(1-\varphi)\phi_{v,\sigma^2}\delta_2^2} w_{t+1}^2\right].$$

This expectation is defined for $1-2\phi_{v,\sigma^2}(1-\varphi)\left(\delta_2^2+\delta_1^2\right) > 0$ (since we already assume $1-2\phi_{v,\sigma^2}(1-\varphi)\delta_2^2 > 0$). If it is verified, then we automatically have $1-2\phi_{v,\sigma^2}(1-\varphi)\delta_2^2 > 0$ as well.

Observe that $\left(\delta_2^2+\delta_1^2\right) = \Sigma_\sigma\Sigma_\sigma'$. A solution for ϕ_{v,σ^2} exists if and only if $1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma' > 0$.

Let's define $\Sigma(\sigma_t) = \sqrt{\frac{1-2(1-\varphi)\phi_{v,\sigma^2}\delta_2^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}}$ and $\Sigma_v(\sigma_t)\Sigma(\sigma_t) = \widetilde{\Sigma_v}(\sigma_t)$.

Then:

$$\begin{aligned}\mu_v + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 = \beta\left(\mu_v + \mu_c + \phi_{v,\sigma}(1-a) + \phi_{v,\sigma}a\sigma_t + \phi_{v,\sigma^2}(1-a+a\sigma_t)^2\right) \\ -\frac{1}{2}\frac{\beta}{1-\varphi}\log\left(1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'\right) + \frac{\beta(1-\varphi)\left|\widetilde{\Sigma_v}(\sigma_t)\right|^2}{2} + \\ \frac{\beta}{1-\varphi}\log\left[\begin{cases}\Phi\left(-(1-\varphi)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right) \\ +\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right)\exp\left(-\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|^2}{2}\right)\end{cases}\right],\end{aligned}$$

and

$$\left|\widetilde{\Sigma_v}(\sigma_t)\right| = \sqrt{\sigma_t^2|\Sigma_c + \phi_v\Sigma_X|^2 + \frac{\Sigma_\sigma\Sigma_\sigma'\left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1-a+a\sigma_t)\right)^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}}.$$

We linearize around $\left|\widetilde{\Sigma_v}(\sigma_t)\right| = 0$ up to the second order:

$$\begin{aligned}\mu_v + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 = \beta\left(\mu_v + \mu_c + \phi_{v,\sigma}(1-a) + \phi_{v,\sigma}a\sigma_t + \phi_{v,\sigma^2}(1-a+a\sigma_t)^2\right) \\ -\frac{1}{2}\frac{\beta}{1-\varphi}\log\left(1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'\right) \\ +\frac{1}{2}\beta(1-\varphi)\left(\sigma_t^2|\Sigma_c+\phi_v\Sigma_X|^2 + \frac{\Sigma_\sigma\Sigma_\sigma'\left(\phi_{v,\sigma}+2\phi_{v,\sigma^2}(1-a+a\sigma_t)\right)^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}\right)\left(1-\alpha+\frac{\alpha^2}{2}\left(1-\frac{1}{\pi}\right)\right) \\ -\frac{\beta\alpha\sqrt{\sigma_t^2|\Sigma_c+\phi_v\Sigma_X|^2 + \frac{\Sigma_\sigma\Sigma_\sigma'\left(\phi_{v,\sigma}+2\phi_{v,\sigma^2}(1-a+a\sigma_t)\right)^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}}}{\sqrt{2\pi}}.\end{aligned}$$

Let's group the terms in σ_t^2:

$$
\begin{aligned}
0 = {} & 2\phi_{v,\sigma^2}^2 \left|\Sigma_\sigma\right|^2 \left(\frac{1}{\beta} - a^2\alpha\left(1 - \frac{\alpha}{2}\left(1 - \frac{1}{\pi}\right)\right)\right) \\
& + \phi_{v,\sigma^2}\left(\frac{a^2 - \frac{1}{\beta}}{1-\varphi} - (1-\varphi)\left|\Sigma_c + \phi_v \Sigma_X\right|^2 \left|\Sigma_\sigma\right|^2 \left(1 - \alpha + \frac{\alpha^2}{2}\left(1 - \frac{1}{\pi}\right)\right)\right) \\
& + \frac{1}{2}\left|\Sigma_c + \phi_v \Sigma_X\right|^2 \left(1 - \alpha + \frac{\alpha^2}{2}\left(1 - \frac{1}{\pi}\right)\right) \; .
\end{aligned}
$$

This is a second order equation in ϕ_{v,σ^2} where ϕ_{v,σ^2} is the only unknown. The solution has to satisfy $1 - 2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma' > 0$. There is a unique choice of ϕ_{v,σ^2} such that (X_t, σ_t) is stable under the $^-$distribution.

We suppose that:

$$
\begin{aligned}
& \sigma_t^2 \left(\left|\Sigma_c + \phi_v \Sigma_X\right|^2 + \frac{4\phi_{v,\sigma^2}^2 a^2 \left|\Sigma_\sigma\right|^2}{1 - 2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}\right) >> \\
& \frac{\Sigma_\sigma\Sigma_\sigma'\left(\left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1-a)\right)^2 + 4\phi_{v,\sigma^2} a\sigma_t\left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1-a)\right)\right)}{1 - 2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'} \; ,
\end{aligned}
$$

and use the simplification:

$$
\begin{aligned}
& \sqrt{\sigma_t^2\left|\Sigma_c + \phi_v \Sigma_X\right|^2 + \frac{\Sigma_\sigma\Sigma_\sigma'\left(\phi_{v,\sigma} + 2\phi_{v,\sigma^2}(1 - a + a\sigma_t)\right)^2}{1 - 2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}} = \\
& \qquad \sigma_t \sqrt{\left(\left|\Sigma_c + \phi_v \Sigma_X\right|^2 + \frac{4\phi_{v,\sigma^2}^2 a^2 \left|\Sigma_\sigma\right|^2}{1 - 2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}\right)} \; .
\end{aligned}
$$

Let's group the terms in σ_t:

$$
\phi_{v,\sigma} = \frac{\left(-\frac{\alpha}{a\sqrt{2\pi}}\sqrt{\left(\left|\Sigma_c + \phi_v\Sigma_X\right|^2 + \frac{4\phi_{v,\sigma^2}^2 a^2 |\Sigma_\sigma|^2}{1-2\phi_{v,\sigma^2}(1-\varphi)|\Sigma_\sigma|^2}\right)} + \frac{2\phi_{v,\sigma^2}(1-a)\left(1-\alpha 2\phi_{v,\sigma^2}(1-\varphi)|\Sigma_\sigma|^2\left(1-\frac{\alpha}{2}\left(1-\frac{1}{\pi}\right)\right)\right)}{1-2\phi_{v,\sigma^2}(1-\varphi)|\Sigma_\sigma|^2}\right)}{\frac{1}{\beta a} - \frac{1-\alpha 2\phi_{v,\sigma^2}(1-\varphi)|\Sigma_\sigma|^2\left(1-\frac{\alpha}{2}\left(1-\frac{1}{\pi}\right)\right)}{1-2\phi_{v,\sigma^2}(1-\varphi)|\Sigma_\sigma|^2}} \; .
$$

The changes due to the gain-loss asymmetry specification enters both through the modified value for ϕ_{v,σ^2} and through the extra terms in $\phi_{v,\sigma}$.

Let's group the constant terms:

$$\frac{1-\beta}{\beta}\mu_v = \mu_c + \phi_{v,\sigma}\left(1-a\right) + \phi_{v,\sigma^2}\left(1-a\right)^2 - \frac{1}{2}\frac{1}{1-\varphi}\log\left(1-2\phi_{v,\sigma^2}\left(1-\varphi\right)\Sigma_\sigma\Sigma_\sigma'\right)$$
$$-\frac{\alpha}{\sqrt{2\pi}}\frac{\frac{|\Sigma_\sigma|^2\left(\phi_{v,\sigma}+2\phi_{v,\sigma^2}(1-a)\right)^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}\left[1+\frac{4a\phi_{v,\sigma^2}}{\left(\phi_{v,\sigma}+2\phi_{v,\sigma^2}(1-a)\right)}\right]}{2\sqrt{\left(\left|\Sigma_c+\phi_v\Sigma_X\right|^2+\frac{4\phi_{v,\sigma^2}^2a^2|\Sigma_\sigma|^2}{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}\right)}}$$
$$+\frac{1}{2}\left(1-\varphi\right)\frac{\left|\Sigma_\sigma\right|^2\left(\phi_{v,\sigma}+2\phi_{v,\sigma^2}\left(1-a\right)\right)^2}{1-2\phi_{v,\sigma^2}\left(1-\varphi\right)\Sigma_\sigma\Sigma_\sigma'}\left(1-\alpha+\frac{\alpha^2}{2}\left(1-\frac{1}{\pi}\right)\right) .$$

The changes due to the gain-loss asymmetry specification enters through changes in the values of ϕ_{v,σ^2} and $\phi_{v,\sigma}$ and through the extra terms in μ_v.

Since it was obtained through several steps of approximation, this linear solution is not the exact solution to the recursive problem

$$f\left(\sigma_t\right) = \frac{\beta}{1-\varphi}\log\mathbb{E}_t\exp\left[\left(1-\varphi\right)\begin{cases} f\left(\sigma_{t+1}\right)+\mu_c+\sigma_t\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1} \\ -\alpha\max\left(0,\sigma_t\left(\Sigma_c+\phi_v\Sigma_X\right)W_{t+1}+\left(1-\mathbb{E}_t\right)f\left(\sigma_{t+1}\right)\right)\end{cases}\right] ,$$

where $v_t - c_t = \phi_v X_t + f\left(\sigma_t\right)$.

To check the robustness of my approximate solution, we numerically derive the unique solution to the recursive problem. In Figure F.10, we compare the approximate closed-form solution and the exact solution to the recursive problem. The distribution of σ_t illustrates that, for realistic volatility levels, the closed-form solution of Eq.(D.2) is a good approximation of the exact solution. For the rest of this section, the results are derived using the approximate closed-form solution of Eq.(D.2).

SDF

$$\begin{aligned} s_{t,t+1} = {} & \log\beta + \left(1-\varphi\right)\left(v_{t+1}^1 - \mathcal{Q}_t\left(\overline{v_{t+1}^1}\right)\right) - \left(c_{t+1}-c_t\right) \\ & - \left(1-\varphi\right)\alpha\max\left(0, v_{t+1}^1 - \mathbb{E}_t v_{t+1}^1\right) \\ & + \log\left(\left(1-\alpha\mathbf{1}_{v_{t+1}^1\geq\mathbb{E}_t v_{t+1}^1}\right) + \alpha\frac{\mathbb{E}_t\left(\mathbf{1}_{v_{t+1}^1\geq\mathbb{E}_t v_{t+1}^1}\overline{V_{t+1}^1}^{1-\varphi}\right)}{\overline{V_{t+1}^1}^{1-\varphi}}\right) \end{aligned}$$

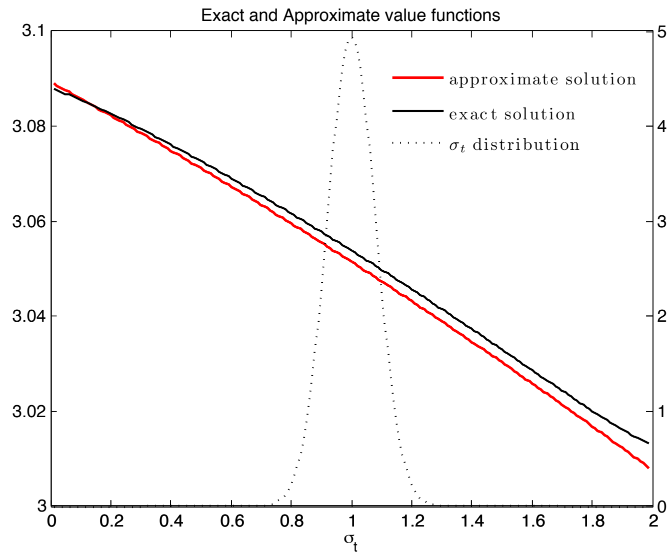

Figure F.10: Approximation of the Value Function

On the left axis, the exact (dotted line) and closed-form approximate (bold line) solutions for $v_t^1 - c_t$ are displayed for $X_t = 0$, at its mean value. On the right axis, the distribution of σ_t illustrates how close the approximate solution is to the exact solution for realistic values of the volatility. We use the parameters of Hansen et al. (2008) for the consumption process and $\beta = 0.99$, $(1-\varphi)(1-\alpha) = 10$, $\alpha = 0.55$.

becomes

$$
\begin{aligned}
s_{t,t+1} &= \log\beta + (1-\varphi)\left(-\frac{1-\beta}{\beta}\mu_v - \frac{\varphi}{(1-\varphi)}\mu_c + \phi_{v,\sigma}(1-a) + \phi_{v,\sigma^2}(1-a)^2\right) \\
&+ (1-\varphi)\left(\left(\phi_{v,\sigma}\left(a - \frac{1}{\beta}\right) + 2a\phi_{v,\sigma^2}(1-a)\right)\sigma_t + \phi_{v,\sigma^2}\left(a^2 - \frac{1}{\beta}\right)\sigma_t^2\right) - \phi_c X_t \\
&- \sigma_t \Sigma_c W_{t+1} + (1-\varphi)\left(\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - (1-\varphi)\alpha \mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t) W_{t+1} \\
&+ \log\left(\left\{\begin{array}{l}
\left(1 - \alpha\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\right) + \\
\alpha \exp\left(-(1-\varphi)\left[\left(1-\alpha\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\right)\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right]\right) \times \\
\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right) \dfrac{\exp\left(\frac{(1-\varphi)(1-\alpha)^2\left|\widetilde{\Sigma_v}(\sigma_t)\right|^2}{2}\right)}{\sqrt{1-2\phi_{v,\sigma^2}(1-\varphi)\Sigma_\sigma\Sigma_\sigma'}}
\end{array}\right.\right).
\end{aligned}
$$

The Euler Equation:

$$
\begin{aligned}
ee_{t,t+1} &= r_{t+1} + \log\beta + (1-\varphi)\left(v_{t+1}^1 - \mathcal{Q}_t\left(\overline{v_{t+1}^1}\right)\right) - (c_{t+1} - c_t) \\
&- (1-\varphi)\alpha\max\left(0, v_{t+1}^1 - \mathbb{E}_t v_{t+1}^1\right) \\
&+ 1_{v_{t+1}^1 \geq E_t v_{t+1}^1}\log\left((1-\alpha) + \alpha\frac{\mathbb{E}_t\left(R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\right)}{R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-1}}\right)
\end{aligned}
$$

becomes

$$
\begin{aligned}
ee_{t,t+1} &= r_{t+1} + \log\beta + (1-\varphi)\left(-\frac{1-\beta}{\beta}\mu_v - \frac{\varphi}{(1-\varphi)}\mu_c + \phi_{v,\sigma}(1-a) + \phi_{v,\sigma^2}(1-a)^2\right) \\
&+ (1-\varphi)\left(\left(\phi_{v,\sigma}\left(a-\frac{1}{\beta}\right) + 2a\phi_{v,\sigma^2}(1-a)\right)\sigma_t + \phi_{v,\sigma^2}\left(a^2-\frac{1}{\beta}\right)\sigma_t^2\right) - \phi_c X_t \\
&- \sigma_t\Sigma_c W_{t+1} + (1-\varphi)\left(\Sigma_v(\sigma_t)W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - (1-\varphi)\alpha\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t)W_{t+1} \\
&+ \mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\log\left((1-\alpha) + \alpha\frac{E_t\left(R_{t+1}\exp\left(-\sigma_t\Sigma_c W_{t+1}\right)\right)}{R_{t+1}\exp\left(-\sigma_t\Sigma_c W_{t+1}\right)}\right) \; .
\end{aligned}
$$

Risk Free Rate The risk free rate is given by

$$
rf_t = -\log E_t\left(\exp ee_{t,t+1}(0)\right) \; ,
$$

so

$$
\begin{aligned}
rf_t &= -\log\beta + (1-\varphi)\left(\frac{1-\beta}{\beta}\mu_v + \frac{\varphi}{(1-\varphi)}\mu_c - \phi_{v,\sigma}(1-a) - \phi_{v,\sigma^2}(1-a)^2\right) + \phi_c X_t \\
&- (1-\varphi)\left(\left(\phi_{v,\sigma}\left(a-\frac{1}{\beta}\right) + 2a\phi_{v,\sigma^2}(1-a)\right)\sigma_t + \phi_{v,\sigma^2}\left(a^2-\frac{1}{\beta}\right)\sigma_t^2\right) \\
&- \log\mathbb{E}_t\exp\begin{cases} (1-\varphi)\left(\Sigma_v(\sigma_t)W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - \sigma_t\Sigma_c W_{t+1} \\ +\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\times \\ \left[-(1-\varphi)\alpha\Sigma_v(\sigma_t)W_{t+1} + \log\left((1-\alpha) + \alpha\exp\left(\sigma_t\Sigma_c W_{t+1} + \frac{1}{2}\sigma_t^2\left|\Sigma_c\right|^2\right)\right)\right] \end{cases} .
\end{aligned}
$$

Remember

$$
\begin{aligned}
\mu_v + \phi_{v,\sigma}\sigma_t + \phi_{v,\sigma^2}\sigma_t^2 &= \beta\left(\mu_v + \mu_c + \phi_{v,\sigma}(1-a) + \phi_{v,\sigma}a\sigma_t + \phi_{v,\sigma^2}(1-a+a\sigma_t)^2\right) + \\
\frac{\beta}{1-\varphi}\log\mathbb{E}_t\exp(1-\varphi)&\left[\Sigma_v(\sigma_t)W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2 - \alpha 1_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t)W_{t+1}\right] \; ,
\end{aligned}
$$

so

$$
\begin{aligned}
rf_t &= -\log\beta + \mu_c + \phi_c X_t \\
&+ \log\mathbb{E}_t\exp(1-\varphi)\left[\Sigma_v(\sigma_t)W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2 - \alpha 1_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\Sigma_v(\sigma_t)W_{t+1}\right] \\
&- \log\mathbb{E}_t\exp\begin{cases} (1-\varphi)\left(\Sigma_v(\sigma_t)W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - \sigma_t\Sigma_c W_{t+1} \\ -\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0}\times \\ \left[(1-\varphi)\alpha\Sigma_v(\sigma_t)W_{t+1} - \log\left((1-\alpha) + \alpha\exp\left(\sigma_t\Sigma_c W_{t+1} + \frac{1}{2}\sigma_t^2\left|\Sigma_c\right|^2\right)\right)\right] \end{cases} ,
\end{aligned}
$$

and

$$
\begin{aligned}
rf_t =& -\log\beta + \mu_c + \phi_c X_t - \frac{1}{2}\sigma_t^2 \left|\Sigma_c\right|^2 + \frac{1}{2}\sigma_t^2 \left|\frac{\Sigma_c \Sigma_v(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right|^2 \left(1 - \left(\Sigma(\sigma_t)\right)^2\right) \\
&+ (1-\varphi)(1-\alpha)\left(\Sigma(\sigma_t)\right)^2 \sigma_t \Sigma_v(\sigma_t)\Sigma_c' \\
&+ \log\left[\begin{cases} \Phi\left(-(1-\varphi)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right)\exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|^2}{2}\right) \\ +\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right) \end{cases}\right] \\
&- \log\begin{cases} \Phi\left(\left(-(1-\varphi)\left|\widetilde{\Sigma_v}(\sigma_t)\right| + \sigma_t\frac{\Sigma_c\widetilde{\Sigma_v}(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right)\right)\times \\ \exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|^2}{2} - \left(\Sigma(\sigma_t)\right)^2\alpha(1-\varphi)\sigma_t\Sigma_v(\sigma_t)\Sigma_c'\right) \\ +(1-\alpha)\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right| - \sigma_t\frac{\Sigma_c\widetilde{\Sigma_v}(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right) \\ +\alpha\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}(\sigma_t)\right|\right)\exp\left(\left(\Sigma(\sigma_t)\right)^2(1-\varphi)(1-\alpha)\sigma_t\Sigma_v(\sigma_t)\Sigma_c'\right)\times \\ \exp\left(\frac{1}{2}\sigma_t^2\left|\frac{\Sigma_c\Sigma_v(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right|^2\left(1-\left(\Sigma(\sigma_t)\right)^2\right)\right) \end{cases} .
\end{aligned}
$$

As a second order approximation around $\widetilde{\Sigma_v}(\sigma_t) = 0$, and $\Sigma_c = 0$:

$$
\begin{aligned}
rf_t \approx& -\log\beta + \mu_c + \phi_c X_t - \frac{1}{2}\sigma_t^2\left|\Sigma_c\right|^2 + \frac{1}{2}\sigma_t^2\left|\frac{\Sigma_c\Sigma_v(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right|^2\left(1-\left(\Sigma(\sigma_t)\right)^2\right) \\
&+ (1-\varphi)(1-\alpha)\left(\Sigma(\sigma_t)\right)^2\sigma_t\Sigma_v(\sigma_t)\Sigma_c' + \\
&+ \alpha\begin{cases} -\frac{1}{\sqrt{2\pi}}\sigma_t\frac{\Sigma_c\widetilde{\Sigma_v}(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|} + \frac{1}{2}\alpha\left(\Sigma(\sigma_t)\right)^2\left(1-\frac{1}{\pi}\right)(1-\varphi)\sigma_t\Sigma_v(\sigma_t)\Sigma_c' + \\ -\frac{1}{4}\sigma_t^2\left|\frac{\Sigma_c\Sigma_v(\sigma_t)'}{\left|\Sigma_v(\sigma_t)\right|}\right|^2\left(1-\left(\Sigma(\sigma_t)\right)^2\left(1+\frac{\alpha}{\pi}\right)\right) \end{cases} .
\end{aligned}
$$

Expected Excess Returns For returns $\log R_{t+1} = \left(\bar{r}_t - \frac{1}{2}\left|\Sigma_R\right|^2 - \frac{1}{2}\left|\widetilde{\Sigma_R}\right|^2\right) + \Sigma_R W_{t+1} + \widetilde{\Sigma_R}\widetilde{W_{t+1}}$, where $\Sigma_{R,t} = \Sigma_R\begin{pmatrix}\sigma_t & & 0 \\ & \sigma_t & \\ 0 & & 1\end{pmatrix}$,

$$
\bar{r}_t(\Sigma_R) = -\log\mathbb{E}_t\left(\exp ee_{t,t+1}\left(\Sigma_R W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right)\right) .
$$

The expected excess returns are thus:

$$\bar{r}_t(\Sigma_R) - rf_t =$$

$$- \log \mathbb{E}_t \exp \left[\begin{cases} \Sigma_{R,t} W_{t+1} - \frac{1}{2} \left|\Sigma_{R,t}\right|^2 + (1-\varphi)\left(\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - \sigma_t \Sigma_c W_{t+1} \\ -\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0} (1-\varphi)\alpha \Sigma_v(\sigma_t) W_{t+1} \\ +\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0} \log\left((1-\alpha) + \alpha \exp\left((\sigma_t \Sigma_c - \Sigma_{R,t}) W_{t+1} + \frac{1}{2}\left|(\sigma_t\Sigma_c - \Sigma_{R,t})\right|^2\right)\right) \end{cases} \right]$$

$$+ \log \mathbb{E}_t \exp \left[\begin{cases} (1-\varphi)\left(\Sigma_v(\sigma_t) W_{t+1} + \phi_{v,\sigma^2}(\Sigma_\sigma W_{t+1})^2\right) - \sigma_t \Sigma_c W_{t+1} \\ -\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0} (1-\varphi)\alpha \Sigma_v(\sigma_t) W_{t+1} \\ +\mathbf{1}_{\Sigma_v(\sigma_t)W_{t+1}\geq 0} \log\left((1-\alpha) + \alpha \exp\left(\sigma_t \Sigma_c W_{t+1} + \frac{1}{2}\sigma_t^2 \left|\Sigma_c\right|^2\right)\right) \end{cases} \right],$$

and

$$\bar{r}_t(\Sigma_R) - rf_t = \Sigma^2\left((\varphi-1)\Sigma_v + \sigma_t\Sigma_c\right)\Sigma_{R,t}'$$

$$+ \left(\frac{1}{2}\left|\frac{\Sigma_{R,t}\Sigma_v'}{|\Sigma_v|}\right|^2 + \sigma_t \Sigma_c \left(I - \frac{\Sigma_v'\Sigma_v}{|\Sigma_v|^2}\right)\Sigma_{R,t}'\right)\left(1-\Sigma^2\right)$$

$$+ \log \left[\begin{cases} \Phi\left(\left(-(1-\varphi)\left|\widetilde{\Sigma_v}\right| + \sigma_t \frac{\Sigma_c \widetilde{\Sigma_v}'}{|\Sigma_v|}\right)\right) \exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\widetilde{\Sigma_v}\right|^2}{2} - \Sigma^2\alpha(1-\varphi)\sigma_t\Sigma_c\Sigma_v'\right) \\ +(1-\alpha)\Phi\left(\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}\right| - \sigma_t\frac{\Sigma_c\widetilde{\Sigma_v}'}{|\Sigma_v|}\right)\right) \\ +\alpha\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}\right|\right)\exp\left(\Sigma^2(1-\varphi)(1-\alpha)\sigma_t\Sigma_c\Sigma_v'\right)\exp\left(\frac{1}{2}\sigma_t^2\left|\frac{\Sigma_c\Sigma_v'}{|\Sigma_v|}\right|^2\left(1-\Sigma^2\right)\right) \end{cases} \right]$$

$$- \log \left[\begin{cases} \Phi\left(\left(-(1-\varphi)\left|\widetilde{\Sigma_v}\right| + \frac{(\sigma_t\Sigma_c - \Sigma_{R,t})\widetilde{\Sigma_v}'}{|\Sigma_v|}\right)\right) \exp\left(\frac{(1-\varphi)^2\alpha(2-\alpha)\left|\widetilde{\Sigma_v}\right|^2}{2} - \Sigma^2\alpha(1-\varphi)(\sigma_t\Sigma_c - \Sigma_{R,t})\Sigma_v'\right) \\ +(1-\alpha)\Phi\left(\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}\right| - \frac{(\sigma_t\Sigma_c - \Sigma_{R,t})\widetilde{\Sigma_v}'}{|\Sigma_v|}\right)\right) \\ +\alpha\Phi\left((1-\varphi)(1-\alpha)\left|\widetilde{\Sigma_v}\right|\right)\exp\left(\begin{cases}\Sigma^2(1-\varphi)(1-\alpha)(\sigma_t\Sigma_c - \Sigma_{R,t})\Sigma_v' \\ +\frac{1}{2}\left|\frac{(\sigma_t\Sigma_c-\Sigma_{R,t})\Sigma_v'}{|\Sigma_v|}\right|^2\left(1-\Sigma^2\right)\end{cases}\right) \end{cases} \right].$$

Appendix G History dependent reference point—Proofs of Extension Appendix D.2

Once more, the case $\psi = 1$ simplifies the derivations.

We analyze the model:

$$v_t = (1-\beta)c_t + \frac{\beta}{1-\varphi}\log\mathbb{E}_t\left[\exp(1-\varphi)\log U(V_{t+1} \mid \mathcal{F}_t)\right]$$
$$\log U(V_{t+1} \mid \mathcal{F}_t) = v_{t+1} - \alpha\max\left(0, v_{t+1} - \frac{\mathbb{E}_t(v_{t+1}) + \xi\mathbb{E}_{t-1}(v_{t+1})}{1+\xi}\right),$$

where we suppose $\xi > 0$, and

$$\log C_{t+1} - \log C_t = \mu_c + \phi_c X_t + \Sigma_c W_{t+1}$$
$$X_{t+1} = AX_t + \Sigma_X W_{t+1}.$$

Suppose:

$$v_t - c_t = \mu_{v,s} + \phi_v X_t + f\left(W_t\right) \ ,$$

then

$$v_{t+1} - \mathcal{R}_t\left(v_{t+1}\right) = \left(\Sigma_c + \phi_v \Sigma_X\right) W_{t+1} + f\left(W_{t+1}\right) + \frac{\xi}{1+\xi}\left(\Sigma_c + \left(\phi_c + \phi_v A\right)\Sigma_X\right) W_t \ .$$

We simplifies the notations with:

$$\begin{aligned} \Sigma_{l,1} &= \left(\Sigma_c + \phi_v \Sigma_X\right) \\ \Sigma_{l,2} &= \frac{\xi}{1+\xi}\left(\Sigma_c + \left(\phi_c + \phi_v A\right)\Sigma_X\right) \ , \end{aligned}$$

and the recursive problem is given by:

$$\mu_{v,s} + \phi_v X_t + f\left(W_t\right) = \frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp\left(1-\varphi\right)\begin{cases} \mu_{v,s} + \phi_v X_{t+1} + f\left(W_{t+1}\right) \\ +\mu_c + \phi_c X_t + \Sigma_c W_{t+1} \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t \geq 0}\left(\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t\right) \end{cases} ,$$

so that

$$\begin{aligned} \mu_{v,s} + \phi_v X_t + f\left(W_t\right) &= \beta\left(\mu_{v,s} + \phi_v A X_t + \mu_c + \phi_c X_t\right) \\ &\quad + \frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp\left(1-\varphi\right)\begin{cases} \Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t \geq 0}\left(\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t\right) \end{cases} . \end{aligned}$$

Group the terms in X_t:

$$\phi_v = \beta \phi_c \left(I - \beta A\right)^{-1} \ .$$

Write: $\left(1-\beta\right)\mu_{v,s} - \beta\mu_c = \tilde{\mu}_v$. The recursion becomes:

$$\tilde{\mu}_v + f\left(W_t\right) = \frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp\left(1-\varphi\right)\begin{cases} \Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t \geq 0}\left(\Sigma_{l,1}W_{t+1} + f\left(W_{t+1}\right) + \Sigma_{l,2}W_t\right) \end{cases} ,$$

which can be rewritten as:

$$\tilde{\mu}_v + f\left(\Sigma_{l,2}W_t\right) = \frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp\left(1-\varphi\right)\begin{cases} \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + \Sigma_{l,2}W_t \geq 0} \times \\ \left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + \Sigma_{l,2}W_t\right) \end{cases} .$$

This recursion admits a unique solution f. We find

$$f\left(0\right) = 0 \ ,$$

and

$$\tilde{\mu}_v = \frac{\beta}{1-\varphi}\log \mathbb{E}_t \exp\left(1-\varphi\right)\begin{cases} \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) \geq 0}\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) \end{cases} .$$

For $\Sigma_{l,2}W_t \longrightarrow -\infty$

$$f \underset{-\infty}{\longrightarrow} \mu_- \ ,$$

where μ_- is a constant such that

$$\tilde{\mu}_v + \mu_- = \frac{\beta}{1-\varphi} \log \mathbb{E}_t \exp\left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) \ .$$

For $\Sigma_{l,2}W_t \longrightarrow +\infty$

$$f \underset{+\infty}{\longrightarrow} \mu_+ - \alpha\beta\Sigma_{l,2}W_t \ ,$$

where μ_+ is a constant such that

$$\tilde{\mu}_v + \mu_+ = \frac{\beta}{1-\varphi} \log \mathbb{E}_t \exp\left(1-\varphi\right)\left(1-\alpha\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) \ .$$

Further, we find that:

$$f'(x) = -\beta\alpha \exp\left(-\alpha\left(1-\varphi\right)x\right) \times \left(\frac{\mathbb{E}_t \begin{cases} 0 & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \leq 0 \\ \exp\left[\left(1-\varphi\right)\left(1-\alpha\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right)\right] & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \geq 0 \end{cases}}{\mathbb{E}_t \exp\left(1-\varphi\right) \begin{cases} \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \geq 0} \times \\ \left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x\right) \end{cases}} \right),$$

and thus f is decreasing.

This can be rewritten as:

$$f'(x) = -\beta\alpha \left(\frac{\mathbb{E}_t \begin{cases} 0 & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \leq 0 \\ \exp\left[\left(1-\varphi\right)\left(1-\alpha\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right)\right] & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \geq 0 \end{cases}}{\mathbb{E}_t \exp \begin{cases} \left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + \alpha x\right) & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \leq 0 \\ \left(1-\varphi\right)\left(1-\alpha\right)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) & \text{if } \Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right) + x \geq 0 \end{cases}} \right),$$

and therefore $|f'(x)| \leq -\beta\alpha$. This implies that v_t can be re-written as $v_t - c_{t-1} = g\left(X_{t-1}, W_t\right)$ with g increasing in both X_{t-1} and W_t.

Euler Equation When $\rho = 1$, the Euler Equation is given by:

$$\begin{aligned} ee_{t,t+1} = {} & r_{t+1} + \log\beta + \left(1-\varphi\right)\left(v_{t+1} - \mathcal{Q}_t\left(U\left(V_{t+1} \mid \mathcal{F}_t\right)\right)\right) - \left(c_{t+1} - c_t\right) \\ & - \left(1-\varphi\right)\alpha \max\left(0, v_{t+1} - \widetilde{\mathbb{E}}_t v_{t+1}\right) \\ & + 1_{v_{t+1} \geq \widetilde{\mathbb{E}}_t v_{t+1}} \log\left(\left(1-\alpha\right) + \frac{\alpha}{1+\xi} \frac{\left(\mathbb{E}_t\left(R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\right) + \xi\mathbb{E}_{t-1}\left(R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\right)\right)}{R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-1}} \right), \end{aligned}$$

and so

$$
\begin{aligned}
ee_{t,t+1} &= r_{t+1} + \log\beta - \mu_c - \phi_c X_t - \frac{(1-\varphi)}{\beta} f\left(\Sigma_{l,2} W_t\right) \\
&\quad - \log \mathbb{E}_t \exp\left(1-\varphi\right) \begin{cases} \Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) \geq 0} \left(\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right)\right) \end{cases} \\
&\quad + \left(1-\varphi\right)\left(1 - \alpha \mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t \geq 0 \geq 0}\right)\left(\Sigma_v W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right)\right) - \Sigma_c W_{t+1} \\
&\quad - \left(1-\varphi\right)\alpha \mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t \geq 0 \geq 0} \Sigma_{l,2} W_t \\
&\quad + \mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t \geq 0 \geq 0} \times \\
&\quad \log\left(\left(1-\alpha\right) + \frac{\alpha}{1+\xi}\left(\frac{\mathbb{E}_t\left(R_{t+1} \exp^{-\Sigma_c W_{t+1}}\right)}{R_{t+1} \exp^{-\Sigma_c W_{t+1}}} + \xi \frac{\mathbb{E}_{t-1}\left(R_{t+1} \exp^{-\phi_c \Sigma_X W_t - \Sigma_c W_{t+1}}\right)}{R_{t+1} \exp^{-\phi_c \Sigma_X W_t - \Sigma_c W_{t+1}}}\right)\right) .
\end{aligned}
$$

Risk Free Rate: The risk free rate is given by

$$
rf_t = -\log \mathbb{E}_t\left(\exp ee_{t,t+1}\left(0\right)\right) ,
$$

$$
\begin{aligned}
rf_t &= -\log\beta + \mu_c + \phi_c A X_{t-1} + \phi_c \Sigma_X W_t \\
&\quad + \log \mathbb{E}_t \exp\left(1-\varphi\right) \begin{cases} \Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) \\ -\alpha \mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t \geq 0} \left(\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t\right) \end{cases} \\
&\quad - \log \mathbb{E}_t \exp\left[\begin{cases} \left(1-\varphi\right)\left(\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right)\right) - \Sigma_c W_{t+1} \\ -\mathbf{1}_{\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t \geq 0} \times \\ \begin{cases} \left(1-\varphi\right)\alpha\left(\Sigma_{l,1} W_{t+1} + f\left(\Sigma_{l,2} W_{t+1}\right) + \Sigma_{l,2} W_t\right) \\ -\log\left(\left(1-\alpha\right) + \alpha \exp\left(\Sigma_c W_{t+1} + \frac{1}{2}\left|\Sigma_c\right|^2\right)\left(\frac{1+\xi \exp\left(\phi_c \Sigma_X W_t + \frac{1}{2}\left|\phi_c \Sigma_X\right|^2\right)}{1+\xi}\right)\right) \end{cases} \end{cases}\right] .
\end{aligned}
$$

Expected Excess Returns: For returns $\log R_{t+1} = \left(\bar{r}_t - \frac{1}{2}\left|\Sigma_R\right|^2 - \frac{1}{2}\left|\widetilde{\Sigma_R}\right|^2\right) + \Sigma_R W_{t+1} + \widetilde{\Sigma_R}\widetilde{W_{t+1}}$,

$$
\bar{r}_t\left(\Sigma_R\right) = -\log \mathbb{E}_t\left(\exp ee_{t,t+1}\left(\Sigma_R W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right)\right) .
$$

Because

$$ee_{t,t+1}\left(\Sigma_R W_{t+1}-\frac{1}{2}\left|\Sigma_R\right|^2\right)=\Sigma_R W_{t+1}-\frac{1}{2}\left|\Sigma_R\right|^2+\log\beta-\mu_c-\phi_c X_t-\frac{(1-\varphi)}{\beta}f\left(\Sigma_{l,2}W_t\right)$$
$$-\log\mathbb{E}_t\exp\left(1-\varphi\right)\begin{cases}\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\\ -\alpha\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\geq 0}\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)\end{cases}$$
$$+\left(1-\varphi\right)\left(1-\alpha\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)+\Sigma_{l,2}W_t\geq 0\geq 0}\right)\left(\Sigma_v W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)$$
$$-\Sigma_c W_{t+1}-\left(1-\varphi\right)\alpha\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)+\Sigma_{l,2}W_t\geq 0\geq 0}\Sigma_{l,2}W_t$$
$$+\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)+\Sigma_{l,2}W_t\geq 0\geq 0}\times$$
$$\log\left(\left(1-\alpha\right)+\alpha\begin{cases}\left(\frac{1+\xi\exp\left(\phi_c\Sigma_X W_t+\frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\\ \exp\left(\left(\Sigma_c-\Sigma_R\right)W_{t+1}+\frac{1}{2}\left|\Sigma_c-\Sigma_R\right|^2\right)\end{cases}\right),$$

the solution for the excess returns is:

$$\bar{r}_t\left(\Sigma_R\right)-rf_t=$$
$$+\log\mathbb{E}_t\exp\begin{cases}\left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)-\Sigma_c W_{t+1}\\ -\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f(\Sigma_{l,2}W_{t+1})+\Sigma_{l,2}W_t\geq 0}\begin{cases}\left(1-\varphi\right)\alpha\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)+\Sigma_{l,2}W_t\right)\\ -\log\left(\left(1-\alpha\right)+\alpha\begin{cases}\left(\frac{1+\xi\exp\left(\phi_c\Sigma_X W_t+\frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\\ \exp\left(\Sigma_c W_{t+1}+\frac{1}{2}\left|\Sigma_c\right|^2\right)\end{cases}\right)\end{cases}\end{cases}$$
$$-\log\mathbb{E}_t\exp\begin{cases}\left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)-\left(\Sigma_c-\Sigma_R\right)W_{t+1}-\frac{1}{2}\left|\Sigma_R\right|^2\\ -\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f(\Sigma_{l,2}W_{t+1})+\Sigma_{l,2}W_t\geq 0}\begin{cases}\left(1-\varphi\right)\alpha\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)+\Sigma_{l,2}W_t\right)\\ -\log\left(\left(1-\alpha\right)+\alpha\begin{cases}\left(\frac{1+\xi\exp\left(\phi_c\Sigma_X W_t+\frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\\ \exp\left(\left(\Sigma_c-\Sigma_R\right)W_{t+1}+\frac{1}{2}\left|\Sigma_c-\Sigma_R\right|^2\right)\end{cases}\right)\end{cases}\end{cases}.$$

If $\Sigma_{l,2}W_t=0$,

$$\bar{r}_t\left(\Sigma_R\right)-rf_t=$$
$$+\log\mathbb{E}_t\exp\begin{cases}\left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)-\Sigma_c W_{t+1}\\ -\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f(\Sigma_{l,2}W_{t+1})\geq 0}\begin{cases}\left(1-\varphi\right)\alpha\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)\\ -\log\left(\left(1-\alpha\right)+\alpha\begin{cases}\left(\frac{1+\xi\exp\left(\frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\\ \exp\left(\Sigma_c W_{t+1}+\frac{1}{2}\left|\Sigma_c\right|^2\right)\end{cases}\right)\end{cases}\end{cases}$$
$$-\log\mathbb{E}_t\exp\begin{cases}\left(1-\varphi\right)\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)-\left(\Sigma_c-\Sigma_R\right)W_{t+1}-\frac{1}{2}\left|\Sigma_R\right|^2\\ -\mathbf{1}_{\Sigma_{l,1}W_{t+1}+f(\Sigma_{l,2}W_{t+1})\geq 0}\begin{cases}\left(1-\varphi\right)\alpha\left(\Sigma_{l,1}W_{t+1}+f\left(\Sigma_{l,2}W_{t+1}\right)\right)\\ -\log\left(\left(1-\alpha\right)+\alpha\begin{cases}\left(\frac{1+\xi\exp\left(\frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\\ \exp\left(\left(\Sigma_c-\Sigma_R\right)W_{t+1}+\frac{1}{2}\left|\Sigma_c-\Sigma_R\right|^2\right)\end{cases}\right)\end{cases}\end{cases}.$$

If $\Sigma_{l,2}W_t \longrightarrow +\infty$,

$$\bar{r}_t(\Sigma_R) - rf_t =$$
$$+\log\begin{cases}(1-\alpha)\,\mathbb{E}_t\exp\left[(1-\varphi)(1-\alpha)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) - \Sigma_c W_{t+1}\right]\\ +\alpha\left(\frac{1+\xi\exp\left(\phi_c\Sigma_X W_t + \frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\exp\left(+\frac{1}{2}\left|\Sigma_c\right|^2\right)\times\\ \mathbb{E}_t\exp\left[(1-\varphi)(1-\alpha)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right)\right]\end{cases}$$
$$-\log\begin{cases}(1-\alpha)\,\mathbb{E}_t\exp\left[(1-\varphi)(1-\alpha)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) - \left(\Sigma_c - \Sigma_R\right)W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right]\\ +\alpha\left(\frac{1+\xi\exp\left(\phi_c\Sigma_X W_t + \frac{1}{2}\left|\phi_c\Sigma_X\right|^2\right)}{1+\xi}\right)\exp\left(+\frac{1}{2}\left(\left|\Sigma_c - \Sigma_R\right|^2 - \left|\Sigma_R\right|^2\right)\right)\times\\ \mathbb{E}_t\exp\left[(1-\varphi)(1-\alpha)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right)\right]\end{cases}.$$

If $\Sigma_{l,2}W_t \longrightarrow -\infty$,

$$\begin{aligned}&\bar{r}_t(\Sigma_R) - rf_t =\\ &+\log\mathbb{E}_t\exp\left[(1-\varphi)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) - \Sigma_c W_{t+1}\right]\\ &-\log\mathbb{E}_t\exp\left[(1-\varphi)\left(\Sigma_{l,1}W_{t+1} + f\left(\Sigma_{l,2}W_{t+1}\right)\right) - \left(\Sigma_c - \Sigma_R\right)W_{t+1} - \frac{1}{2}\left|\Sigma_R\right|^2\right].\end{aligned}$$

The expected excess returns converge to a constant in both $+\infty$ and $-\infty$.

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