

Online Appendix to “Conditional and Lenient Drug Approvals”

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This online appendix includes additional figures, facts, model calibration, and proofs.

EC.1. Institutional background

Details about drug regulation and competition provide a foundation for the model.

Fact 1: A discrete-time model is appropriate in our setting because human clinical trials typically involve pre-specified endpoints and results that are announced all at once. Typically, Phase III trials have pre-specified endpoints, followed by a comprehensive analysis by the firm before regulatory submission. However, there are exceptions, such as early-stage animal testing and the rolling review of the COVID-19 vaccine, which have a more flexible and exploratory set of protocols.

Fact 2: The government can require initiation of confirmatory testing as a condition for granting conditional approval. Beginning in fiscal year 2023, the U.S. Congress granted the FDA the authority to mandate that trials begin before a drug receives conditional approval (US Congress 2023). Previously, some firms selling conditionally-approved drugs cited difficulty enrolling enough patients in trials for rare diseases. This rationale was often a convenient excuse, as confirmatory testing could reveal that the drug should be withdrawn (Xu et al. 2021, Frank et al. 2022). Among 46 cancer drugs granted conditional approval between 2013 and 2017, 15% had not completed testing after a median of six years (Liu et al. 2024) (Figure EC.1). However, we expect that under the new policy, the share of drugs failing to do timely testing will drop.

Fact 3: The government can grant conditional approval for only a share of patients. For example, approval may be limited to those with the most severe conditions, or coverage may be restricted so that only some patients are reimbursed while others pay out-of-pocket. In the United States, the FDA determines approval scope, while Medicare sets coverage rules, both under the Secretary of Health and Human Services. After conditional approval of an Alzheimer’s drug, for instance, Medicare covered it only for patients who agreed to participate in a study of its efficacy (Centers for Medicare & Medicaid Services 2022). For tractability, we model conditional approval as applying to a share of otherwise identical patients.

Fact 4: Drugs given conditional approval are typically for rare diseases and do not have competition. From 2001 to 2025, 75% of FDA conditional approvals were for rare diseases (see Figure EC.2). At launch, drugs for rare diseases typically have no competition, due to orphan drug exclusivity of seven years in the U.S. and ten years in Europe. Even years later, markets for rare diseases are often too small to attract multiple firms.

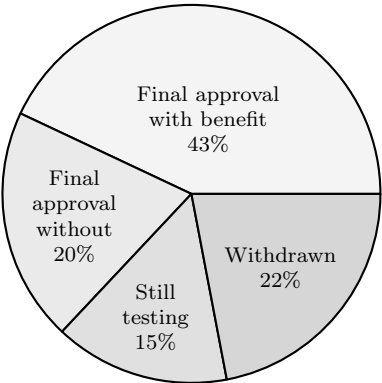


Figure EC.1 For 46 cancer indications conditionally approved in the U.S. between 2013 and 2017, 43% received final approval after demonstrating statistically significant benefits in confirmatory trials, 20% received final approval despite Liu et al. (2024) finding no evidence of statistically significant benefits, 22% were withdrawn, and 15% remained in ongoing testing. Source: Authors' figure based on data from Liu et al. (2024).

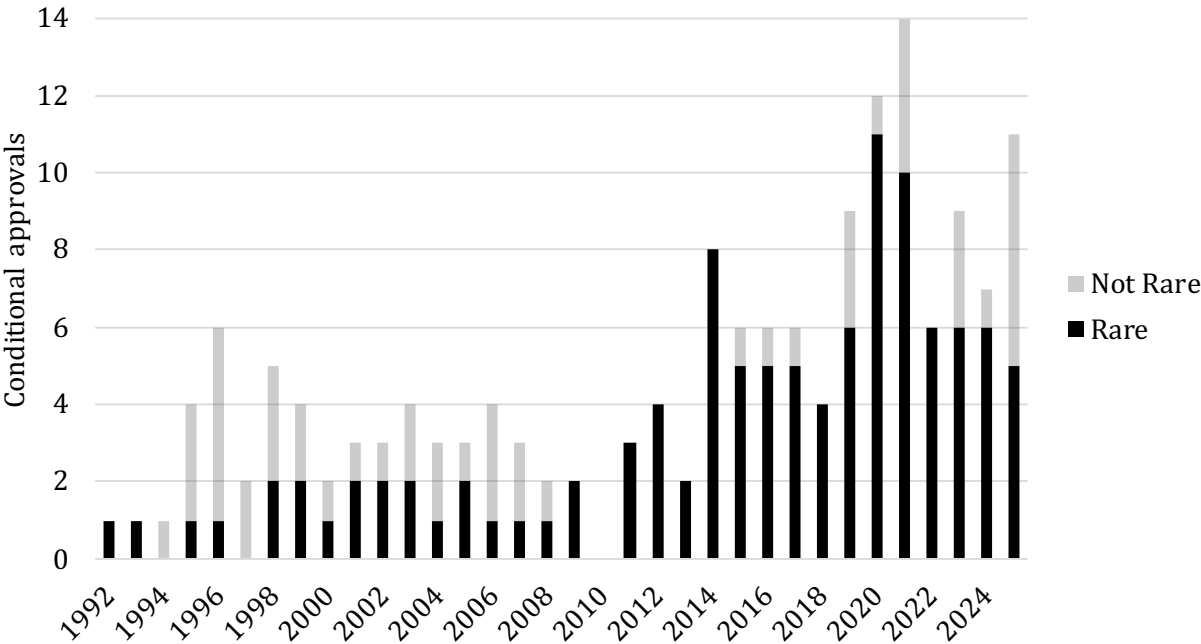


Figure EC.2 The FDA granted conditional approval to 84 drugs in the past decade. Most of the conditionally approved drugs treat rare diseases. The official FDA term for conditional approval is “accelerated approval” and for rare disease is “orphan disease.” Source: Authors’ analysis of data on approvals of new molecules by the Center for Drug Evaluation and Research.

Fact 5: Conditional approval provides earlier revenue for some firms that would otherwise not continue testing. According to the head of the FDA center responsible for gene therapies and vaccines, “The wherewithal to do a three-year study or a four-year study without having a revenue

stream, it's just beyond many companies that are startups. So having the accelerated approval process is a way to get there" (Armstrong 2024).

Fact 6: The government does not typically directly finance late-stage testing. About 3% of NIH's total research budget was devoted to financing the clinical testing of drugs ultimately approved between 2010 and 2019 (Zhou et al. 2023). Also, NIH funding represented only about 3% of Phase III testing costs for a given approved drug (Zhou et al. 2023). Rather than supporting late-stage trials for firms, NIH funding is typically directed toward lower-cost, pre-clinical research conducted at universities. A notable exception was the financing of late-stage testing for COVID-19 vaccines through Operation Warp Speed (Snyder et al. 2023). Another exception is funding for drugs for diseases with no commercial, such as Gates Foundation funding for tropical disease drug development.

Fact 7: Conditional approval increases the firm's revenue during conditional approval and several years afterward. Drug revenue typically rises during the first five years after approval (Robey and David 2016), as physicians and patients become more familiar with the drug. Demand is higher for drugs with which physicians and patients have more experience (Ridley and Lee 2020).

Fact 8: Phase II clinical testing provides limited information about a drug's efficacy. Sample sizes are typically smaller in Phase II than in Phase III and are insufficient for definitive inference. Phase I and II trials tend to be short and small to minimize patient risk and financial cost, given that most drugs entering Phase II do not progress to Phase III (DiMasi et al. 2016). We treat Phase I and II testing costs as sunk, but the model could be reframed to account for earlier decision-making by the firm.

Fact 9: The firm is legally required to share all efficacy and safety information about the drug with the government. Under FDA regulations, a developer seeking conditional approval must submit a comprehensive clinical data section covering both effectiveness and safety. The submission must present all relevant human evidence, not only studies supporting the proposed indication. In addition to detailed analyses of controlled trials, the sponsor is legally required to disclose any other information bearing on efficacy or safety, regardless of source. This includes evidence from controlled and uncontrolled studies of other uses, commercial marketing experience, published scientific reports, and unpublished studies, whether generated domestically or abroad (U.S. Food and Drug Administration 2024).

Fact 10: Information from use of the drug outside randomized trials is not very informative, as it is noisy and subject to selection bias. For example, if healthier, high-income patients are more likely to purchase a drug, observed outcomes may overstate efficacy. Accordingly, our model assumes that usage after conditional approval does not affect Phase III testing. If "real-world evidence"

were informative, it could reduce firms' testing costs by allowing smaller or shorter trials. Hence, we might underestimate the value of conditional approval.

Fact 11: Testing costs are privately known to the firm. As [Light and Warburton \(2005\)](#) observe, “this science-based industry refuses to allow independent parties to check the validity of their cost data and analyze it so that policy can be based on solid, objective, reproducible evidence.” Phase III testing costs are plausibly private for several reasons. First, firms fear the hold-up problem: disclosing true costs would lead governments to reimburse only direct testing expenses, without compensating for failed projects or the opportunity cost of capital. Second, benchmarks are limited because conditionally approved drugs are often first-in-class. Third, Phase III programs require firm-specific manufacturing scale-up that is difficult for regulators to observe. Fourth, most conditional approvals involve rare diseases, where patient recruitment is a major, typically in-house expense. Firms bear these costs through long-standing relationships with patient foundations, advocacy groups, and specialized clinicians, as well as through travel support and efforts to maintain enrollment. Fifth, clinical programs share resources across multiple drugs, making it hard for a government to audit costs.

Fact 12: The government can signal to the firm that it will grant final approval to a drug that narrowly misses the efficacy threshold. We provide evidence for both claims: (1) that the FDA sometimes grants final approval without persuasive evidence, and (2) that the FDA can signal its intended evidentiary standards. First, the FDA has granted final approval despite limited evidence of efficacy. [Liu et al. \(2024\)](#) report that 20% of cancer indications in their sample received full approval despite the absence of statistically significant evidence of benefit (Figure [EC.1](#)). In another instance, a senior FDA official acknowledged that a drug had been approved over scientific objections, describing it as “essentially a scientifically elegant placebo” ([Kaplan 2016](#)). These cases demonstrate that, in practice, decision makers sometimes commit to and implement lenient evidentiary standards. Second, the FDA can signal or commit to a lenient threshold before confirmatory testing occurs. Under the 2012 breakthrough therapy designation, the FDA provides firms with “timely advice” and “interactive communication” to guide clinical trial design and clarify what evidence will be required for approval. Such interaction effectively communicates the future approval standard.

EC.2. Calibration

We calibrate the model to illustrate the key economic drivers and to quantify the policy implications. The goal is not to make precise predictions for a specific drug, but rather to assess magnitudes using realistic parameter values.

We select a representative drug that received conditional approval: avelumab (brand name Bavenio). In 2017, the FDA granted conditional approval for avelumab to treat Merkel cell carcinoma,

a rare and aggressive skin cancer, based on evidence that the drug produced substantial tumor shrinkage in patients who had few effective treatment options. In 2023, avelumab received final approval after confirmatory studies demonstrated concrete clinical benefits, including improvements in overall survival.¹³

We bound the cost of Phase III testing based on other drugs in the therapeutic class. We assume that Phase III testing costs are incurred over the first three years, followed by regulatory review, with the final approval decision occurring at the start of year five. We then estimate doses and revenue over the drug's life cycle. We assume that the firm sells at monopoly prices for thirteen years, after which net revenue is negligible due to generic entry or price controls. Finally, we combine these inputs to compute the cumulative net present value and the implied optimal policy.

EC.2.1. Estimated testing costs

Like the government, we do not observe avelumab's testing cost. Avelumab is marketed by a large firm that does not disclose drug-level development costs. To bound the testing cost, we therefore rely on data from firms that were only developing a few drugs at a time, for which reported firm-level development costs are equivalent to drug-level development costs. Wouters et al. (2020) report Phase III cost data for 13 antineoplastic and immunomodulating drugs approved between 2009 and 2018 and marketed by small firms in the same therapeutic class as avelumab. Among these drugs, we use brigatinib's Phase III cost of \$98.1 million (2018 dollars) as a lower bound and sarilumab's cost of \$630.7 million as an upper bound. These estimates are imprecise; in particular, Wouters et al. (2020) report low confidence in the sarilumab estimate. We therefore interpret this range as capturing plausible variation in testing costs rather than precise measurement.

EC.2.2. Estimated doses and revenue

We estimate doses and revenue for the representative drug avelumab using Medicare and Medicaid data and extrapolate to total U.S. revenue.¹⁴ We report values from the first full year of sales (2018) and from the fifth year (2022), the latter being more recent and more representative of long-run outcomes. Medicare and Medicaid data imply approximately 0.4 million doses in the first year and 1.4 million doses in year five (Table EC.1). We extrapolate to the national market by dividing these totals by 0.42, reflecting that Medicare and Medicaid account for about 42% of U.S. prescription drug spending according to National Health Expenditure data. This yields roughly one million doses in the first year and three million doses in year five. We project sales beyond year five using average drug life-cycle patterns (Robey and David 2016).

¹³ Following its initial approval for Merkel cell carcinoma, avelumab was subsequently approved for bladder and kidney cancers.

¹⁴ We use sales aggregated across all indications, which overstates revenue for the initial skin cancer indication.

	2018	2022	Source
Medicaid+Medicare doses (MM)	0.4	1.4	Centers for Medicare & Medicaid Services (2024)
Medicaid+Medicare share	0.42	0.42	National Health Expenditures
Total doses (MM)	1.0	3.3	Medicaid+Medicare divided by .42
Medicaid+Medicare spending (\$MM)	35	121	Centers for Medicare & Medicaid Services (2024)
Total sales (\$MM)	83	288	Medicaid+Medicare divided by .42
Price per dose (\$)	79	86	Medicare Part B average price Centers for Medicare & Medicaid Services (2024)
Willingness to pay per dose (\$)	109	118	If buyer's surplus is half of seller's, willingness to pay is 1.375 times unit price
Production cost share of price	0.25	0.25	Wouters et al. (2024)

Table EC.1 Estimated annual doses and prices for avelumab in its first full year (2018) and its fifth year (2022).

EC.2.3. Parameter values

For a drug granted full conditional approval, discounted doses sold (q) during that four-year period total 4.36 million (Table EC.2), using a 10.5% discount rate (DiMasi et al. 2016). This is about one million discounted doses per year, consistent with avelumab's first full year of sales (Table EC.1). Although sales increase over time, doses sold later are discounted more heavily, so higher sales are largely offset.

For a drug that receives both conditional and final approval, discounted doses sold ($\lambda_0 + \lambda_1$) total about 15 million, averaging roughly 1 million per year over the 13-year period between final approval and the loss of market power.

By contrast, a drug that receives final approval but not conditional approval sells fewer discounted doses—about 11 million over the same thirteen years (λ_0). Without conditional approval, sales begin only in year five and take several years to grow, when discounting is already substantial. The values of q , λ_0 , and λ_1 are as follows:

$$\begin{cases} q = 4.36 \\ \lambda_0 = 11.34 \\ \lambda_0 + \lambda_1 = 15.43 \end{cases} \Rightarrow \begin{cases} q = 4.36 \\ \lambda_0 = 11.34 \\ \lambda_1 = 4.09 \end{cases} . \quad (\text{EC.1})$$

We normalize quantities by dividing all values by 4.36, so that $q = 1$.

To estimate revenue net of production costs, we assume a cost-of-goods-sold share of 25% (Wouters et al. 2024). In year five, the firm's net revenue per dose (π) is approximately \$65. If the drug is ineffective, the government's loss ($v(0)$) equals $-\$86$, the price paid for a drug that provides no benefit. If the drug is effective, the government's benefit ($v(1)$) equals \$32, the difference between willingness to pay and price.

We also compute the discounted quantity sold after the monopoly period,

$$\Delta = \sum_{i=18}^{\infty} \frac{3.37}{(1 + 10.5\%)^i} = 5.88, \quad (\text{EC.2})$$

Notation	Description	Value	Source
δ	Discount rate	10.5%	DiMasi et al. (2016)
c_L	Lower bound of the testing cost	\$98.1 MM	Brigatinib (Wouters et al. 2020)
c_H	Upper bound of the testing cost	\$630.7 MM	Sarilumab (Wouters et al. 2020)
$v(0)$	Government's loss from ineffective dose	\$86	Average price from Medicare
$v(1)$	Government's payoff from effective dose	\$32	Willingness to pay minus price
Γ	Price drop per dose in the competitive period	\$65	Price minus cost
q	Discounted doses with full conditional	4.36 MM	Based on average growth rates (Robey and David 2016); sum of discounted doses over the first four years
λ_0	Discounted doses if the drug is fully approved	11.34 MM	Sum of discounted doses from years 5–17 without conditional approval
λ_1	Extra total discounted doses	4.09 MM	Difference in discounted doses over years 5–17 with and without full conditional approval
Δ	Discounted doses in the competitive period	5.88 MM	Sum of discounted doses from year 18 onward

Table EC.2 Estimated model parameters for avelumab.

which captures the social value of earlier price reductions due to generic entry or regulation.

Figure EC.3 shows annual sales with full conditional approval and the parallel curve beginning at t_1 represents sales without conditional approval. Figure EC.3 illustrates doses in the first period (q), doses in the second period (λ_0), the bonus doses in the second period if the firm is granted both conditional and full approvals (λ_1), and doses in the competitive period (Δ). Hence, λ_0 denotes the number of doses sold before the competitive period under no conditional approval, and $\lambda_0 + \lambda_1$ denotes the number of doses sold under full conditional approval following conditional approval. In the model, we normalized the initial period doses (q) to 1.

EC.2.4. Estimated cumulative net present value

We now compute the firm's cumulative net present value when testing costs are high ($c_H = \$630.7$ million), evenly distributed over the first three years, and full conditional approval is granted. We set the annual discount rate to $\delta = 10.5\%$, following DiMasi et al. (2016), and assume for simplicity that firms and the government discount future payoffs at the same rate. Table EC.3 reports discounted revenue, testing costs, and cumulative profits.

Despite large early losses, the firm eventually recovers its investment as sales grow and testing costs cease. Without conditional approval, firms facing high testing costs would forgo development entirely, generating a potential social loss.

Year	Discounted net revenue (if in market) (\$MM) ^a	Discounted expected net revenue (\$MM) ^b	Annual testing cost (\$MM) ^c	Discounted annual testing cost (\$MM)	Discounted annual net profit (\$MM) ^d	Cumulative net present value (\$MM) ^e
1	22	22	210	190	-169	-169
2	55	55	210	172	-117	-286
3	93	93	210	156	-63	-348
4	111	111	0	0	111	-238
5	117	53	0	0	53	-185
6	119	54	0	0	54	-131
7	108	49	0	0	49	-83
8	98	44	0	0	44	-39
9	88	40	0	0	40	1
10	80	36	0	0	36	37
11	72	33	0	0	33	70
12	65	29	0	0	29	99
13	59	27	0	0	27	126
14	54	24	0	0	24	150
15	49	22	0	0	22	172
16	44	20	0	0	20	191
17	40	18	0	0	18	209

^a “Discounted annual doses” comes from forecasting sales based on average drug lifecycles (Robey and David 2016) times Price per dose” from Table EC.1

^b Discounted expected net revenue = Discounted net revenue if the drug is in market $\times$ Pr(the drug is in market), and the probabilistic term is 1 and 0.45 in years 1 to 4 and 5 to 17, respectively.

^c The testing cost \$ 630.7 million is evenly distributed over the first three years.

^d Discounted annual net profit = Discounted expected net revenue (column 3) - Discounted annual testing cost (column 5).

^e Cumulative NPV in year i = sum of the discounted annual net profit from years 1 to i .

Table EC.3: The firm’s cumulative net present value when its testing cost is the maximum value.

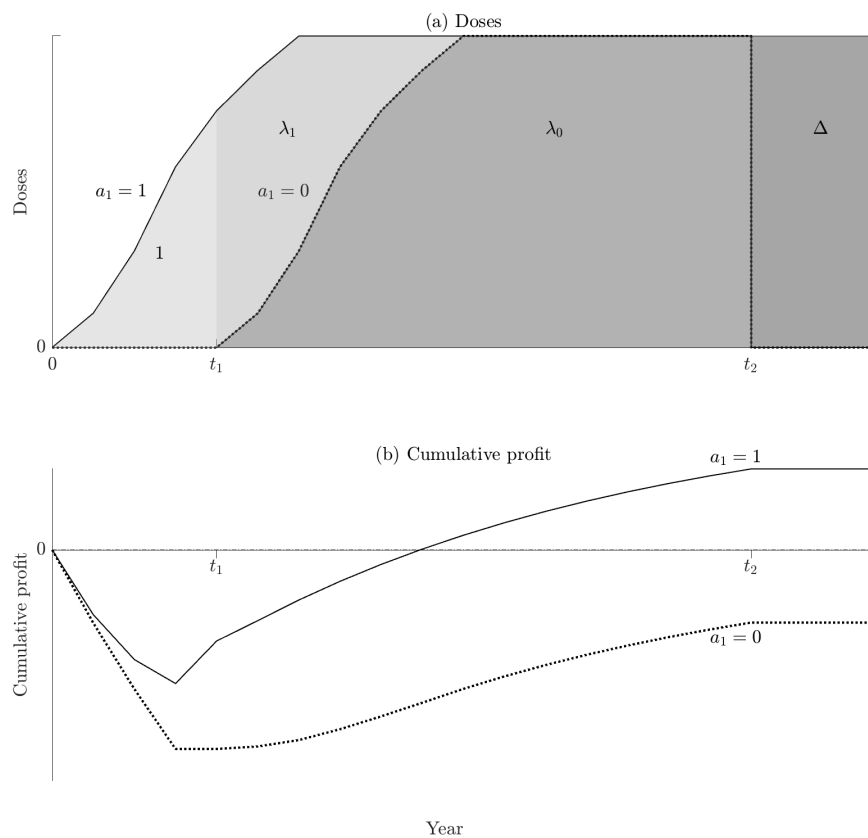


Figure EC.3 Annual revenue (top) and cumulative profit (bottom) depend on whether the government conditionally approves ($a_1 > 0$) or not ($a_1 = 0$) (assuming final approval). This figure matches Figure 2 but uses estimates for an actual drug avelumab. Source: Authors' analysis.

Figure EC.4 plots cumulative profit net of testing costs under different assumptions about conditional approval (rows), final approval (columns), and testing costs (curves). Lower curves correspond to higher testing costs. In all cases, firms incur losses early on, but some reach positive cumulative profits depending on the approval regime. For drugs that do not receive conditional approval, cumulative profits continue to decline until one year before t_1 , when the firm submits its testing data for regulatory review. Starting at t_1 , cumulative profits either rise following final approval or are flat if the government denies final approval (and hence no new profits are cumulated).

Without conditional approval, only drugs with sufficiently low testing costs are profitable, and only if final approval is guaranteed (top left panel). With partial conditional approval, even high-cost drugs can eventually earn positive cumulative profits, if they are also granted final approval (middle left panel). Thus, conditional approval expands the set of drugs for which testing is privately worthwhile.

The left column of Figure EC.4 assumes final approval. When the possibility of failing confirmatory testing is introduced (right column), expected returns fall, which further strengthens the role of conditional approval in motivating firms to test.

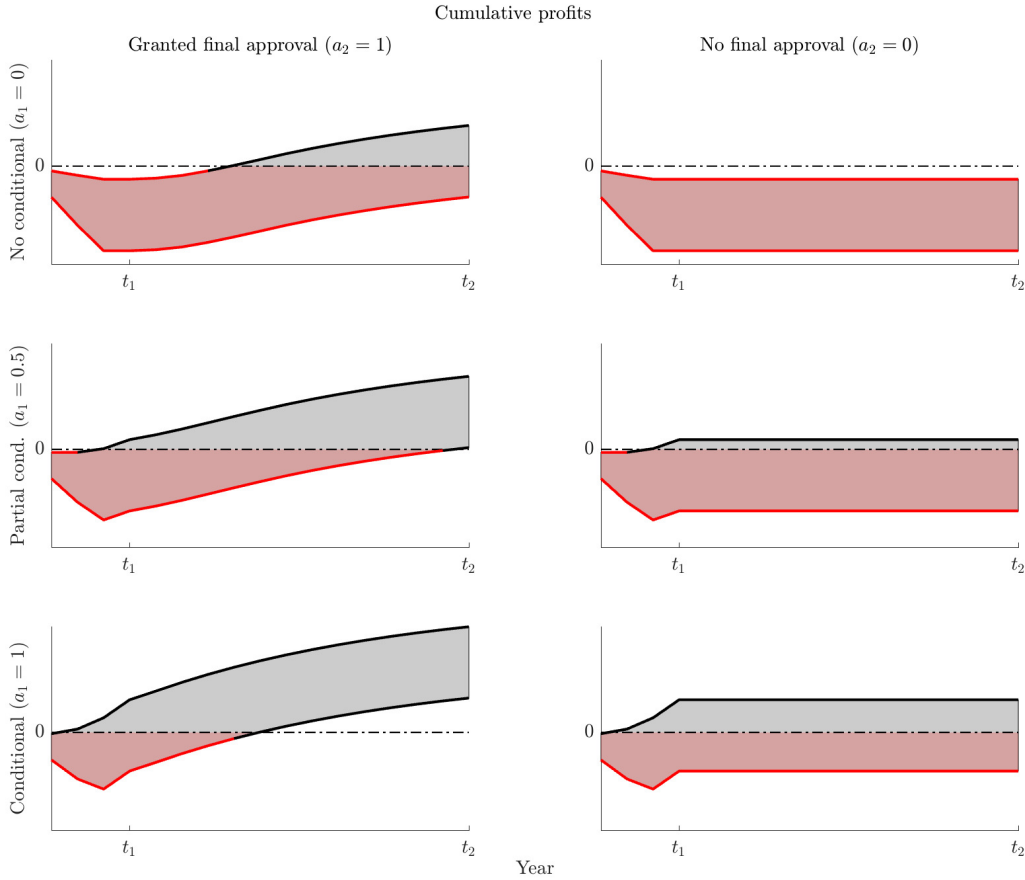


Figure EC.4 Cumulative net profit under different assumptions about conditional approval (rows), final approval (columns), and testing costs (curves). Lower curves correspond to higher testing costs. Source: Authors' analysis using a representative drug, avelumab.

EC.2.5. Calibrated optimal policy

Finally, we estimate the optimal withdrawal rate: the fraction of conditionally approved drugs that are later denied final approval. Given a final approval threshold $\hat{\eta}$, the withdrawal probability is $G(\hat{\eta})$.

Figure EC.5 plots the withdrawal rate as a function of the drug's expected value. Two discontinuities arise. First, drugs with very low expected value ($v_E < v_2$) are not granted conditional approval. Second, at v_4 , the final approval threshold jumps, increasing withdrawals.

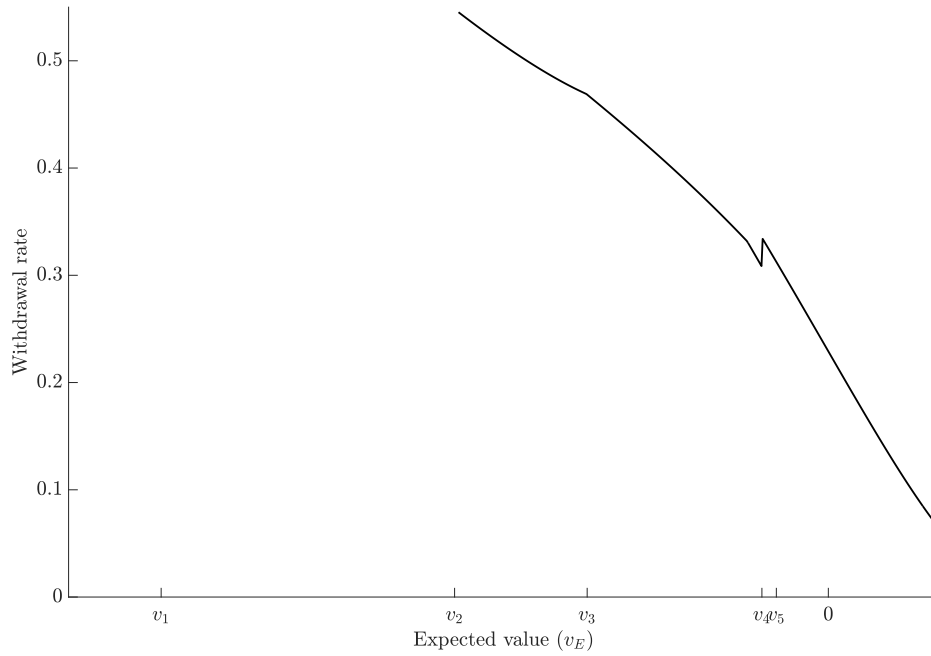


Figure EC.5 Optimal proportion of drugs withdrawn after receiving conditional approval. Source: Authors' analysis based on a representative drug.

For drugs with $v_E = 0$, the optimal withdrawal rate is 23%. Empirically, among cancer drugs granted conditional approval between 2013 and 2017, 22% were later withdrawn (Figure EC.1). This comparison suggests that historical FDA practice was close to optimal and, if anything, slightly conservative.

By contrast, the leadership of the FDA gene therapy unit appointed in 2025 has adopted a more restrictive approach to conditional approval (The Economist 2025). Our calibration implies that such a shift will reduce social welfare.

EC.2.6. Calibration in the Three-Period Game continued

Using calibrated parameters, we compare policies with and without conditional approval and leniency. Combining both tools performs near the upper bound. It generates over \$6 billion in social value per decade, with most gains coming from conditional approval (Figure EC.6). See Section 5.3 for a more detailed explanation.

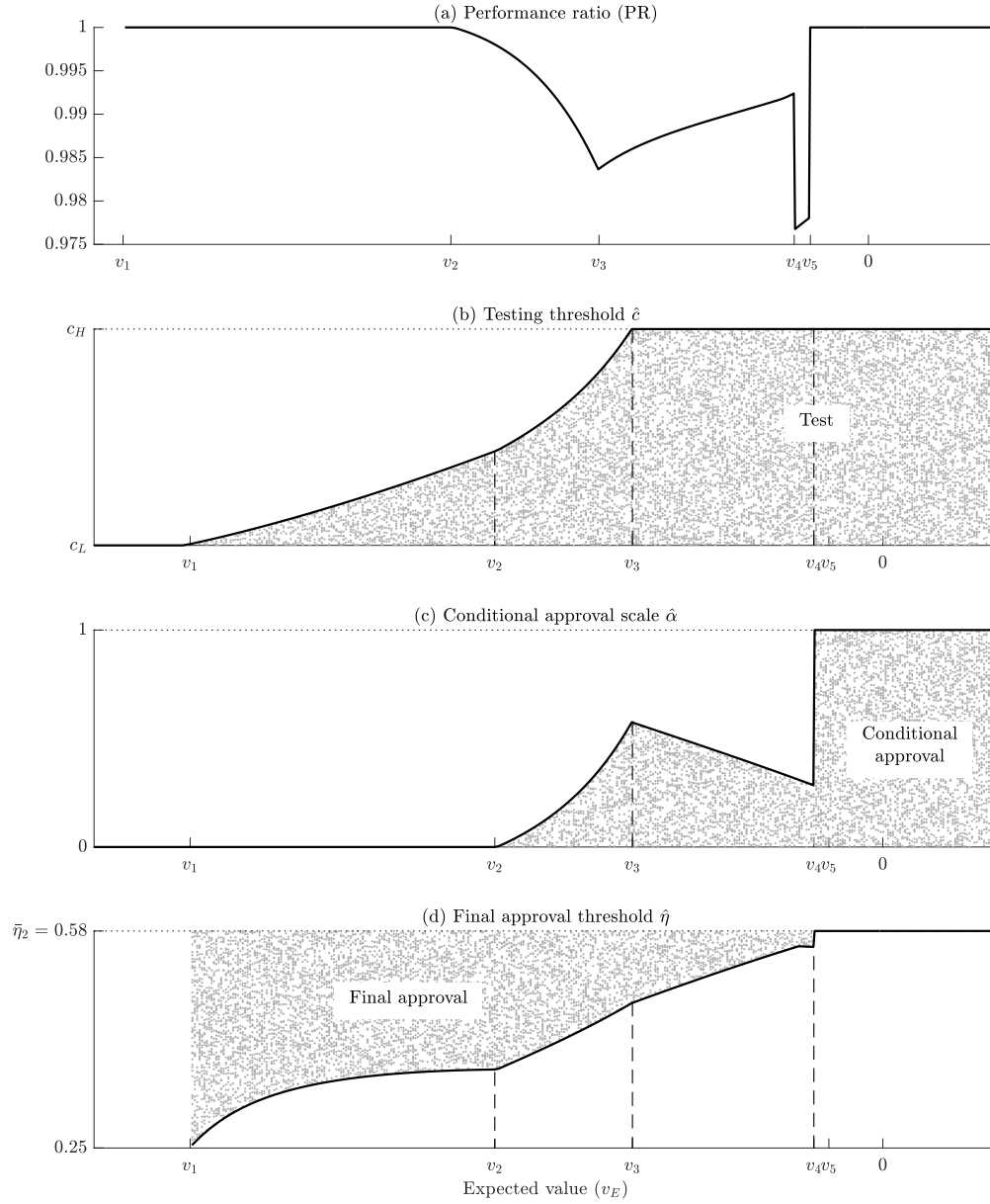


Figure EC.6 The lower-bound mechanism derived from the upper-bound problem performs well (upper panel). Drugs with high expected value and low testing cost are tested (second panel), receive conditional approval (third panel), and have a high final approval threshold, meaning less leniency (lower panel).

EC.3. Flexible exclusivity

Optimal conditional approval and leniency may depend on the length of exclusivity. Some drug classes receive extensions. For example, antibiotics gain five additional years under the GAIN Act (Kong and Zhao 2024). Longer exclusivity increases monopoly sales, boosting firm profits but delaying savings for payers. Exclusivity extensions can therefore strengthen development incentives and reduce the need for conditional approval and leniency, all else equal.

Recall that in Figure 2, the curve starting at time 0 shows annual sales with full conditional approval; the parallel curve beginning at t_1 represents sales without conditional approval. Because the competitive period begins at time t_2 , λ_0 and λ_1 correspond to the gray areas in the middle and on the left, respectively. The number of doses sold during the competitive period, denoted by Δ , is represented by the rectangle on the right.

Because annual sales peak in six years (Robey and David 2016), extending the firm's monopolistic period beyond 13 years does not affect λ_1 and only increases λ_0 . Given that $\lambda_0 + \Delta$ represents the total number of doses sold after final approval, corresponding to the area under the curve starting at t_1 , any increase in λ_0 leads to an equivalent decrease in Δ . This adjustment corresponds to a rightward shift of time t_2 in Figure 2.

Let $\hat{\lambda}_0$ and Δ denote the calibrated values of λ_0 and Δ , respectively.¹⁵ In Figure EC.7, we plot the testing threshold, the scale of conditional approval, the final approval threshold, and the expected payoffs of both the government and the firm (with testing costs \$200 and \$500 million) as functions of λ_0 , varying from $\hat{\lambda}_0$ to $\hat{\lambda}_0 + \Delta$. The value $\hat{\lambda}_0$ corresponds to a scenario in which the firm's monopoly power ends at the end of year 13, while the upper bound $\hat{\lambda}_0 + \Delta$ reflects a case where the government extends monopoly power indefinitely. Fixing λ_0 , the number of doses sold in the competitive period is $\hat{\lambda}_0 + \Delta - \lambda_0$.

As shown in panels (b) and (c), under the calibrated model parameters, when λ_0 increases, the scale of conditional approval decreases, and the final approval threshold increases. The testing threshold, illustrated in panel (a), also declines, indicating that the government induces less firm participation as the length of exclusivity increases. Finally, both the government (panel (d)) and the firm's expected payoffs (panel (e)) decrease as λ_0 increases, suggesting that it is optimal to maintain the current exclusivity timeline rather than granting additional exclusivity. Given the firm's testing cost, the monotonicity of its expected payoff in λ_0 follows from the monotonicity of the testing threshold $\hat{c}$. Specifically, since the expected payoff of a firm with production cost $\hat{c}$ is zero, the expected payoff for a firm with cost c is given by $(c - \hat{c})^+$. This observation also highlights that, although firms generally favor extended exclusivity, they could be worse off if the government reoptimizes its policy in response to the change.

¹⁵ See Appendix EC.2 for more details.

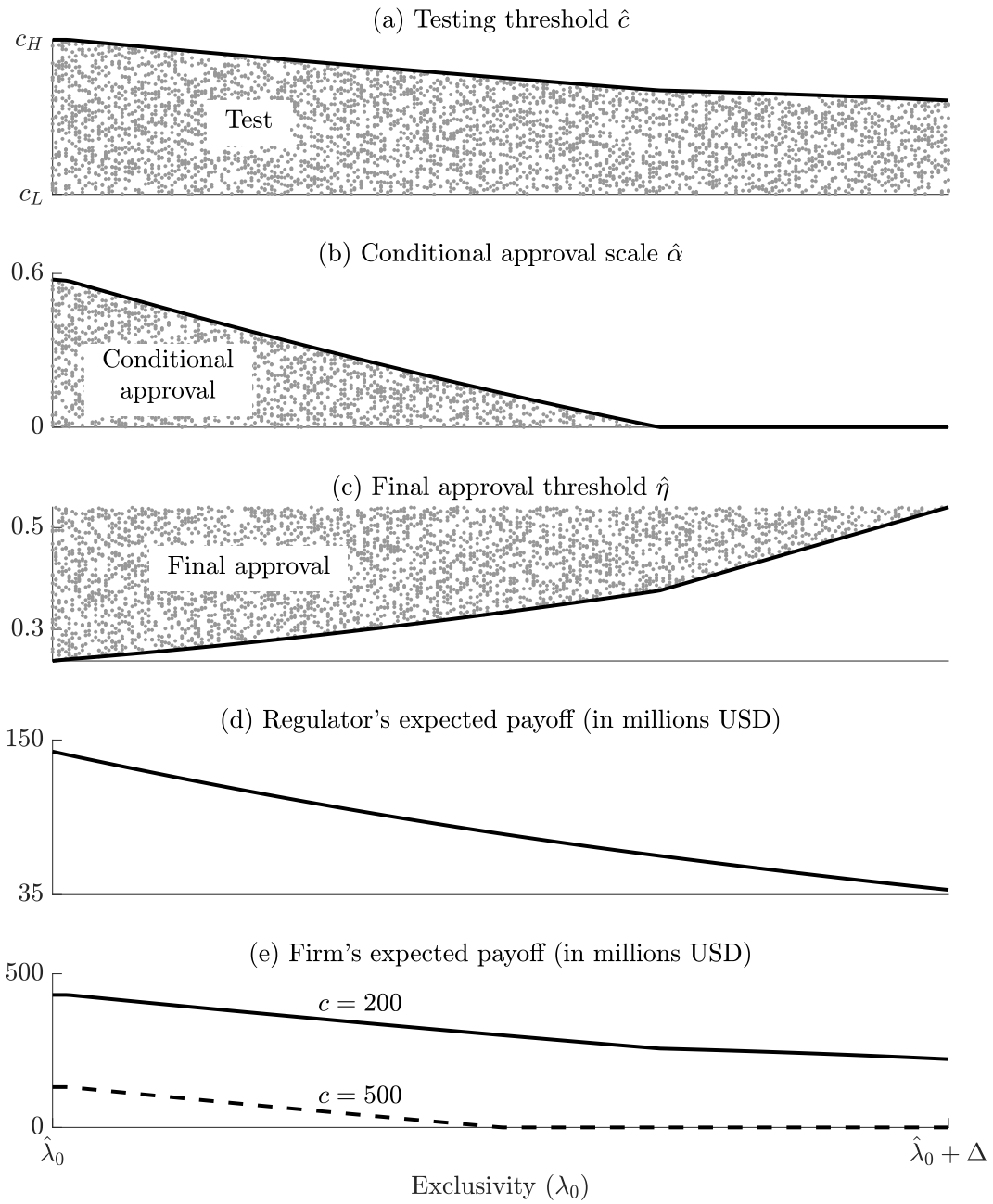


Figure EC.7 The testing threshold, scale of conditional approval, threshold for final approval, and the expected payoffs of the government and the firm (with different testing costs) under calibrated model parameters. The x -axis ranges from calibrated $\hat{\lambda}_0$ to $\hat{\lambda}_0 + \Delta$.

EC.4. Private information about the efficacy distribution

In this section, we explore the case in which the distribution of efficacy η is the firm's private information. We use k to denote this private information, and the corresponding distribution of η is denoted by G_k . We assume that the family G_k satisfies the monotone likelihood ratio property (MLRP): for any $k_1 > k_2$, the likelihood ratio $g_{k_1}(\eta)/g_{k_2}(\eta)$ is non-decreasing in η . The government believes that k is a random variable with support $[k_L, k_H]$, cumulative distribution function F , and density f . A direct mechanism again specifies the probability of inducing the firm's testing $\tau(k)$, the scale of conditional approval $a_1(k)$, and the probability of granting final approval $a_2(k, \eta)$.

Unlike the base case with the cost c being the private information, even with $\Delta = 0$, the linear optimization problem is highly intractable. To be precise, the firm's expected payoff when its private information is k but it reports k' is

$$\hat{\Pi}(k, k') = \tau(k') \left[a_1(k') - c + [\lambda_0 + \lambda_1 a_1(k')] \int_0^1 a_2(k', \eta) dG_k(\eta) \right]. \quad (\text{EC.3})$$

In equilibrium, the firm reports its type truthfully. The government's expected payoff (when $\Delta = 0$) is

$$U_R(\tau, a_1, a_2) \equiv \int_{k_L}^{k_H} \tau(k) \left\{ v_E(k) a_1(k) + [\lambda_0 + \lambda_1 a_1(k)] \int_0^1 v(\eta) a_2(k, \eta) dG_k(\eta) \right\} dF(k). \quad (\text{EC.4})$$

The government aims to maximize its expected payoff subject to the incentive constraints and feasibility constraints. The optimization problem remains (13), where the payoff functions are now given by (EC.3) and (EC.4).

The optimal solution to this problem is highly intractable and depends critically on the functional form of G_k . Figure EC.8 illustrates the optimal policy when the firm's private information k is uniformly distributed and the CDF is given by $G_k(\eta) = \eta^k$. The optimal mechanism features an infinite menu of contracts, with randomization over testing (i.e., τ strictly between 0 and 1) and a continuously varying scale of conditional approval and threshold for final approval. Such a mechanism is technically challenging to characterize analytically and difficult to implement in practice.

Therefore we again focus on a take-it-or-leave-it (TIOLI) offer. In this setup, the government offers a scale of conditional approval a_1 and commits to granting final approval if and only if $\eta \geq \eta^*$. The firm then decides whether to participate. The cutoff type, denoted by k_0 , is the type that is indifferent between testing and not testing, i.e.,

$$k_0 := \inf \{k_0 \in [k_L, k_H] : a_1 - c + (\lambda_0 + \lambda_1 a_1) [1 - G_{k_0}(\eta^*)] \geq 0\}. \quad (\text{EC.5})$$

The government's optimization problem under a TIOLI contract is therefore

$$\max_{a_1 \in [0, 1], \eta^* \in [0, 1], k_0 \in [k_L, k_H]} \int_{k_0}^{k_H} \left[v_E(k) a_1 + (\lambda_0 + \lambda_1 a_1) \int_{\eta^*}^1 v(\eta) dG_k(\eta) \right] dF(k) \quad \text{s.t. (EC.5)}. \quad (\text{EC.6})$$

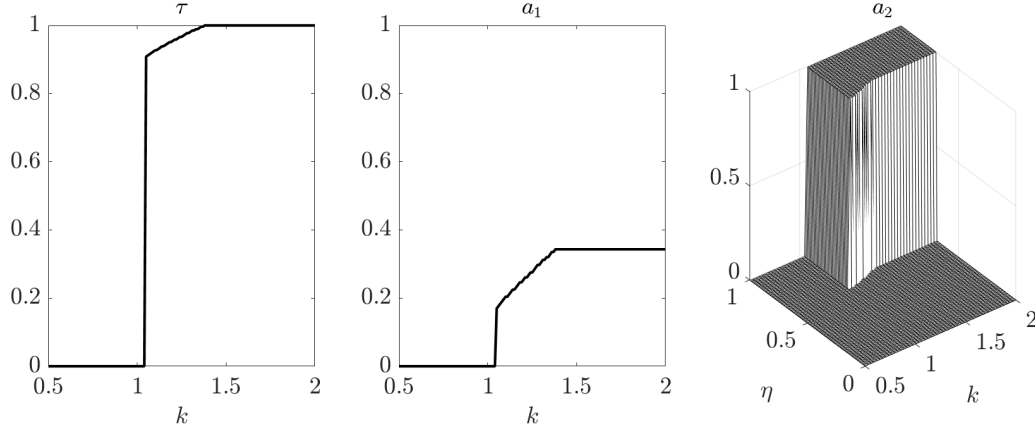


Figure EC.8 An optimal mechanism when the firm privately knows the distribution of efficacy η . We set $c = (c_L + c_H)/2$, $k_L = 0.5$, $k_H = 2$, $f(k) = 1/1.5$, and $G_k(\eta) = \eta^k$.

The optimization problem (EC.6) is straightforward to solve numerically. Specifically, we discretize the values of a_1 and η^* over the unit square $[0, 1]^2$. For each pair (a_1, η^*) , the cutoff type k_0 is uniquely determined by (EC.5), which then allows us to compute the corresponding objective value. Figure EC.9 displays the optimal TIOLI policy obtained from this numerical procedure.

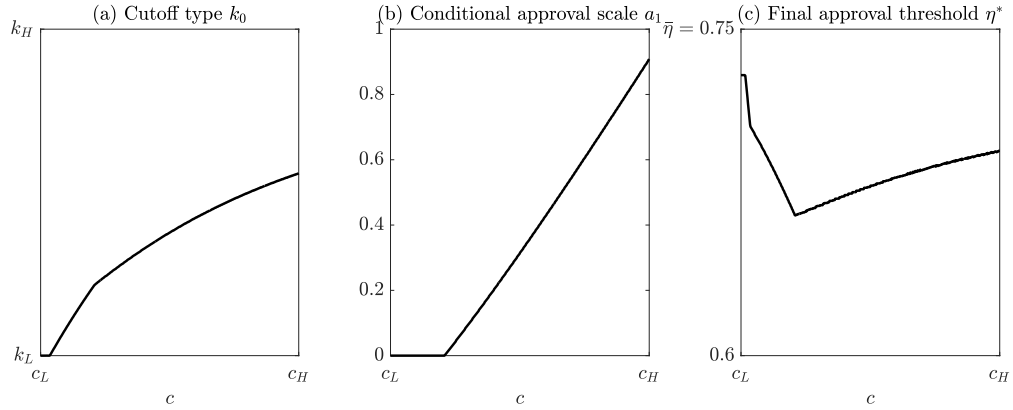


Figure EC.9 An optimal TIOLI policy when the firm privately knows the distribution of efficacy η . We set $k_L = 0.5$, $k_H = 2$, $f(k) = 1/1.5$, and $G_k(\eta) = \eta^k$.

Two features of the optimal policy coincide with our recommendations in the setting where the testing cost c is private information. First, when the testing cost is high, it is optimal for the government to offer (partial) conditional approval to incentivize testing. Second, for the same reason, it is optimal for the government to commit to a final approval threshold lower than $\bar{\eta}$. As shown in panel (c), the optimal threshold is almost always below the myopic threshold $\bar{\eta}$. Therefore, the structural properties in our model appear robust.

EC.5. Proofs

EC.5.1. Proofs for Section 2

Proof of Lemma 1. By definition of Φ , we have

$$\frac{\partial \Phi(a_1, \eta; \Delta)}{\partial \eta} = (\lambda_0 + \lambda_1 a_1 + \Delta) v'(\eta) > 0.$$

Furthermore, by assumption (5), we have

$$\Phi(a_1, 0; \Delta) = (\lambda_0 + \lambda_1 a_1) \underbrace{v(0)}_{<0} + \Delta \underbrace{[v(0) + 1]}_{\leq 0} < 0$$

and

$$\Phi(a_1, 1; \Delta) = (\lambda_0 + \lambda_1 a_1) v(1) + \Delta [v(1) + 1] > 0.$$

This completes the proof. $\square$

EC.5.2. Proofs for Section 3

Proof of Proposition 1. For any c , $a_1(c)$, and $a_2(c, \eta)$, we have

$$\begin{aligned} & v_E a_1(c) + \int_0^1 \left[[\lambda_0 + \lambda_1 a_1(c)] v(\eta) + \Delta [v(\eta) + 1] \right] a_2(c, \eta) dG(\eta) \\ & \leq v_E a_1(c) + \int_{\eta_{NL}(a_1(c))}^1 \left[[\lambda_0 + \lambda_1 a_1(c)] v(\eta) + \Delta [v(\eta) + 1] \right] dG(\eta) \\ & = \check{U}_R(a_1(c), \eta_{NL}(a_1(c))), \end{aligned}$$

because the no-leniency threshold η_{NL} maximizes the government's continuation payoff after period one.

The function $\check{U}_R$ in the previous expression is the restriction of the function defined in (14), and it is convex in a_1 because

$$\frac{d}{da_1} \check{U}_R(a_1, \eta_{NL}(a_1)) = v_E + \lambda_1 \int_{\eta_{NL}(a_1)}^1 v(\eta) dG(\eta)$$

and

$$\frac{d^2}{da_1^2} \check{U}_R(a_1, \eta_{NL}(a_1)) = -\lambda_1 v(\eta_{NL}(a_1)) g(\eta_{NL}(a_1)) \geq 0.$$

Therefore, the maximum is attained at either $a_1(c) = 0$ or $a_1(c) = 1$.

If the drug is promising and low-cost, i.e., (15) and (17) hold, we have

$$U_R^*(\Delta) \leq \max_{(a_1, \eta) \in [0, 1]^2} \check{U}_R(a_1, \eta) = \check{U}_R(1, \eta_{NL}(1)),$$

because $\tau(c) \in [0, 1]$ and the upper bound is attained by the candidate solution defined in (19).

If the drug is not promising and low-cost, i.e., (16) and (18) hold, we have

$$U_R^*(\Delta) \leq \max_{(a_1, \eta) \in [0, 1]^2} \check{U}_R(a_1, \eta) = \check{U}_R(0, \eta_{NL}(0)),$$

because $\tau(c) \in [0, 1]$ and the upper bound is attained by the candidate solution defined in (20).

This completes the proof. $\square$

Proof of Corollary 1. The proof of Proposition 1 ignores both the (IC) and (IR) constraints. Hence, the same argument applies to the first-best problem and the dictatorial problem.

This completes the proof. $\square$

EC.5.3. Proofs for Section 4

Proof of Proposition 2. When $\Delta = 0$, any feasible solution $\{\tau(c), a_1(c), a_2(c, \eta)\}$ to the optimization problem (13) yields a corresponding solution $\{\tau(c), \alpha_1(c), \alpha_2(c, \eta)\}$, with α_1 and α_2 defined in (21) and (22), respectively—that is feasible for the optimization problem (27) and attains the same objective value.

Conversely, any feasible solution $\{\tau(c), \alpha_1(c), \alpha_2(c, \eta)\}$ to the optimization problem (27) yields a corresponding solution $\{\tau(c), a_1(c), a_2(c, \eta)\}$, with a_1 and a_2 defined in (28), that is feasible for the optimization problem (13) and likewise attains the same objective value.

This completes the proof. $\square$

The following lemma is commonly used to simplify mechanism design problems involving direct payments between buyers and sellers. For completeness, we provide a proof here.

LEMMA EC.1. *The incentive constraints (IC) and (IR) can be replaced by*

$$\tau \text{ non-increasing}, \tag{EC.7}$$

$$\alpha_1(c) = \Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, \eta) dG(\eta), \tag{EC.8}$$

and

$$\Pi(c_H) \geq 0, \tag{EC.9}$$

where $\Pi(c_H) \equiv \hat{\Pi}(c_H, c_H)$.

Proof of Lemma EC.1. The incentive-compatibility constraint can be rewritten as

$$\Pi(c) \geq \Pi(c') + \tau(c') \cdot (c' - c),$$

where $\Pi(c) \equiv \hat{\Pi}(c, c)$. Flipping the role of c and c' yields

$$\Pi(c') \geq \Pi(c) + \tau(c) \cdot (c - c').$$

Combining the two inequalities above yields the monotonicity of τ .

By the envelope theorem, we have

$$\Pi'(c) = -\tau(c) \leq 0,$$

which implies that

$$\Pi(c) = \Pi(c_H) + \int_c^{c_H} \tau(y) dy.$$

Therefore, it is sufficient to impose the individual rationality constraint on type c_H and

$$\alpha_1(c) = \Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, \eta) dG(\eta).$$

To prove the other direction, for any $c < c'$, we have

$$\begin{aligned} \Pi(c) &= \Pi(c_H) + \int_c^{c_H} \tau(y) dy \\ &= \Pi(c_H) + \int_c^{c'} \tau(y) dy + \int_{c'}^{c_H} \tau(y) dy \\ &= \Pi(c') + \int_c^{c'} \tau(y) dy \\ &\geq \Pi(c') + \tau(c') \cdot (c' - c), \end{aligned}$$

where the inequality follows from the monotonicity of τ . Constraint (IR) is implied by

$$\Pi(c) \geq \Pi(c_H) \geq 0.$$

This completes the proof. $\square$

LEMMA EC.2. The optimization problem (27) is equivalent to

$$\begin{aligned} \max_{\{\tau, \alpha_2, \Pi(c_H)\} \in \Omega'_C} & v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) \\ & + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc, \end{aligned} \quad (\text{EC.10})$$

where the feasible set Ω'_C is determined by (2), (EC.7), (EC.9), and for any $(c, \eta) \in [c_L, c_H] \times [0, 1]$,

$$0 \leq \Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, \eta) dG(\eta) \leq \tau(c), \quad (\text{EC.11})$$

and

$$0 \leq \alpha_2(c, \eta) \leq \lambda_0 \tau(c) + \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, z) dG(z) \right]. \quad (\text{EC.12})$$

Proof of Lemma EC.2. Substituting the equation (EC.8) in Lemma EC.1 into the objective function yields the desired result. $\square$

Proof of Lemma 2. We first reformulate the definitions of $\check{c}$, $\tilde{\eta}$, and $\underline{\eta}$ given in (36)–(38) in a more explicit form:

$$\begin{aligned} \check{c} &= \sup \left\{ c \in [c_L, c_H] : v(\tilde{\eta}) \left[c + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\tilde{\eta}}^1 [v(y) - v(\tilde{\eta})] dG(y) \geq 0 \right\}, \\ v(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 v(y) dG(y) &= 0, \text{ and} \\ \underline{\eta} &= \inf \left\{ \eta \in [0, \tilde{\eta}] : v(\eta) \frac{F(\bar{c}(\eta))}{f(\bar{c}(\eta))} + v_E + (\lambda_0 + \lambda_1) \int_{\eta}^1 v(y) dG(y) \geq 0 \right\}. \end{aligned}$$

We start with the first part of the lemma. Because $\int_{\eta}^1 [v(y) - v(\eta)] dG(y)$ and $v(\eta)$ are decreasing and increasing in η , respectively, we know that function h is decreasing. Therefore, the unique existence of $\tilde{\eta} \in [0, \bar{\eta}]$ such that $h(\tilde{\eta}) = 0$ is implied by $h(\bar{\eta}) \leq 0 \leq h(0)$, or equivalently,

$$\lambda_1 \int_{\bar{\eta}}^1 v(y) dG(y) + v_E \leq 0 \leq \lambda_1 (v_E - v(0)) - v(0) + v_E.$$

The second part of the inequality holds by $v_E - v(0) > 0$.

To prove the second part, we first verify that $v_E \leq 0$, which follows from

$$\tilde{\eta} \text{ uniquely exists} \Rightarrow \underbrace{\lambda_1 \int_{\bar{\eta}}^1 v(y) dG(y) + v_E}_{>0} \leq 0 \Rightarrow v_E < 0.$$

Define

$$t_1(\eta) \equiv v(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} + \lambda_0 \int_{\eta}^1 v(y) dG(y), \quad \forall \eta \in [0, \bar{\eta}].$$

Taking derivative yields

$$\begin{aligned} t_1'(\eta) &= v'(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} + v(\eta) \frac{d}{d\eta} \left[\frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] - \lambda_0 v(\eta) g(\eta) \\ &= v'(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} - \lambda_0 v(\eta) g(\eta) \frac{d}{dc} \left[c + \frac{F(c)}{f(c)} \right] \Big|_{c=\underline{c}(\eta)}, \end{aligned}$$

which is positive for any $\eta \in [0, \bar{\eta}]$. Therefore, to prove the unique existence of $\tilde{\eta}$, it suffices to verify that $t_1(\tilde{\eta}) \leq 0 \leq t_1(\bar{\eta})$, where the second inequality follows from

$$t_1(\bar{\eta}) = \lambda_0 \int_{\bar{\eta}}^1 v(y) dG(y) > 0.$$

Because $\check{c} < \underline{c}(\tilde{\eta})$, by definition of $\check{c}$, we have

$$v(\tilde{\eta}) \left[\underline{c}(\tilde{\eta}) + \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} \right] + \lambda_0 \int_{\tilde{\eta}}^1 [v(y) - v(\tilde{\eta})] dG(y) < 0,$$

which implies that

$$\begin{aligned} t_1(\tilde{\eta}) &= v(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 v(y) dG(y) \\ &< -v(\tilde{\eta}) \underline{c}(\tilde{\eta}) - \lambda_0 \int_{\tilde{\eta}}^1 [v(y) - v(\tilde{\eta})] dG(y) + \lambda_0 \int_{\tilde{\eta}}^1 v(y) dG(y) \\ &= -v(\tilde{\eta}) \underline{c}(\tilde{\eta}) + \lambda_0 v(\tilde{\eta}) [1 - G(\tilde{\eta})] \\ &= 0. \end{aligned}$$

Therefore, we know the unique existence of $\tilde{\eta} \in [\tilde{\eta}, \bar{\eta}]$, and the monotonicity property implies the desired result, i.e.,

$$h(\tilde{\eta}) \leq h(\tilde{\eta}) = 0.$$

It remains to prove the last part of the lemma. Define

$$t_2(\eta) \equiv v(\eta) \frac{F(\bar{c}(\eta))}{f(\bar{c}(\eta))} + \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta}^1 v(y) dG(y) \right], \quad \forall \eta \in [0, \bar{\eta}].$$

Taking derivative yields

$$\begin{aligned} t_2'(\eta) &= v'(\eta) \frac{F(\bar{c}(\eta))}{f(\bar{c}(\eta))} + v(\eta) \frac{d}{d\eta} \left[\frac{F(\bar{c}(\eta))}{f(\bar{c}(\eta))} \right] - (\lambda_0 + \lambda_1) v(\eta) g(\eta) \\ &= v'(\eta) \frac{F(\bar{c}(\eta))}{f(\bar{c}(\eta))} - (\lambda_0 + \lambda_1) v(\eta) g(\eta) \frac{d}{dc} \left[c + \frac{F(c)}{f(c)} \right] \Big|_{c=\bar{c}(\eta)}, \end{aligned}$$

which is positive for any $\eta \in [0, \bar{\eta}]$. When $h(\bar{\eta}) > 0$, we have

$$t_2(\bar{\eta}) = v_E + (\lambda_0 + \lambda_1) \int_{\bar{\eta}}^1 v(y) dG(y) > \lambda_0 \int_{\bar{\eta}}^1 v(y) dG(y) > 0.$$

When $h(\bar{\eta}) \leq 0$ and $\check{c} > \bar{c}(\bar{\eta})$, by definition of $\check{c}$, we have

$$v(\check{\eta}) \left[\bar{c}(\check{\eta}) + \frac{F(\bar{c}(\check{\eta}))}{f(\bar{c}(\check{\eta}))} \right] + \lambda_0 \int_{\check{\eta}}^1 [v(y) - v(\check{\eta})] dG(y) > 0,$$

which can be rewritten as

$$v(\check{\eta}) \frac{F(\bar{c}(\check{\eta}))}{f(\bar{c}(\check{\eta}))} + \lambda_0 \int_{\check{\eta}}^1 v(y) dG(y) > -v(\check{\eta}) [1 + \lambda_1 [1 - G(\check{\eta})]].$$

Because of the monotonicity of t_2 , to prove the desired result, it suffices to verify that $t_2(\check{\eta}) \geq 0$, which follows from

$$\begin{aligned} t_2(\check{\eta}) &= v(\check{\eta}) \frac{F(\bar{c}(\check{\eta}))}{f(\bar{c}(\check{\eta}))} + \left[v_E + (\lambda_0 + \lambda_1) \int_{\check{\eta}}^1 v(y) dG(y) \right] \\ &> -v(\check{\eta}) [1 + \lambda_1 [1 - G(\check{\eta})]] + v_E + \lambda_1 \int_{\check{\eta}}^1 v(y) dG(y) \\ &= \underbrace{v_E - v(\check{\eta})}_{= -\lambda_1 \int_{\check{\eta}}^1 [v(y) - v(\check{\eta})] dG(y)} + \lambda_1 \left[\int_{\check{\eta}}^1 v(y) dG(y) - v(\check{\eta}) [1 - G(\check{\eta})] \right] \\ &= 0. \end{aligned}$$

Therefore, the threshold $\underline{\eta}$ is defined as

$$\underline{\eta} = \begin{cases} \check{\eta}, & \text{if } t_2(0) \leq 0 \\ 0, & \text{if } t_2(0) > 0, \end{cases}$$

where $\check{\eta} \in [0, \bar{\eta}]$ is the unique zero of t_2 .

To prove $h(\underline{\eta}) \geq 0$, note that when $h(\bar{\eta}) > 0$, we have

$$h(\underline{\eta}) \geq h(\bar{\eta}) > 0.$$

When $\check{c} > \bar{c}(\check{\eta})$, the monotonicity of h implies that

$$h(\underline{\eta}) \geq h(\check{\eta}) = 0.$$

This completes the proof. □

Proof of Proposition 3. It is straightforward to verify that our candidate solutions are feasible in all cases.

To prove the optimality, we start with the first case. Define

$$\kappa = -v(\check{\eta}),$$

$$\nu(c, \eta) = [v(\check{\eta}) - v(\eta)] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \check{\eta}],$$

$$\mu(c, \eta) = [v(\eta) - v(\check{\eta})] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}, 1],$$

$$\sigma(c) = v_E [c f(c) + F(c)] + (\lambda_0 + \lambda_1 c) \int_{\check{\eta}}^1 \mu(c, \eta) d\eta + \lambda_1 \int_{c_L}^c \int_{\check{\eta}}^1 \mu(y, \eta) d\eta dy, \quad \forall c \in \mathcal{C}_1, \text{ and}$$

$$\beta(c) = -v_E [c f(c) + F(c)] - (\lambda_0 + \lambda_1 c) \int_{\check{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\check{\eta}}^1 \mu(y, \eta) d\eta dy, \quad \forall c \in \mathcal{C}_2,$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are subsets of $[c_L, c_H]$ satisfying: (i) $\mathcal{C}_1 \cup \mathcal{C}_2 = [c_L, c_H]$; and (ii) the measure of $\mathcal{C}_1 \cap \mathcal{C}_2$ is 0. These subsets will be specified later.

The nonnegativity of κ , ν , and μ follows from the definition of $\check{\eta}$ and the monotonicity of v directly.

The expressions of σ and β can be rewritten as

$$\sigma(c) = \underbrace{\left[v(\check{\eta}) \left[c + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\check{\eta}}^1 [v(\eta) - v(\check{\eta})] g(\eta) d\eta \right]}_{\text{decreasing in } c} f(c), \quad \forall c \in \mathcal{C}_1,$$

and

$$\beta(c) = - \left[v(\check{\eta}) \left[c + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\check{\eta}}^1 [v(\eta) - v(\check{\eta})] g(\eta) d\eta \right] f(c), \quad \forall c \in \mathcal{C}_2.$$

In the two possible cases, we have:

1. if $c_L \leq \check{c} < c_H$, we know that σ and β are nonnegative on $[c_L, \check{c}]$ and $[\check{c}, c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \check{c}]$ and $\mathcal{C}_2 = [\check{c}, c_H]$; and
2. if $\check{c} = c_H$, we know that σ is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = [c_L, c_H]$ and $\mathcal{C}_2 = \emptyset$.

Because

$$-\kappa - \lambda_1 \int_{c_L}^{c_H} \int_{\check{\eta}}^1 \mu(c, \eta) d\eta dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \check{\eta}]:$$

$$-\nu(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}}^1 \mu(c, z) dz = [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}, 1]:$$

$$\mu(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}}^1 \mu(c, z) dz = [v(\eta) - v_E] g(\eta) f(c),$$

$\forall c \in \mathcal{C}_1$:

$$\sigma(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy = v_E [c f(c) + F(c)],$$

$\forall c \in \mathcal{C}_2$:

$$-\beta(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy = v_E [c f(c) + F(c)],$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{\mathcal{C}_1} \sigma(c) \tau(c) dc + \int_{\mathcal{C}_2} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} \nu(c, \eta) [-\alpha_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) \left[\alpha_2(c, \eta) - \lambda_0 \tau(c) - \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \int_0^1 \alpha_2(c, z) dG(z) \right] \right] d\eta dc \\ & = \left[-\kappa - \lambda_1 \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{\mathcal{C}_1} \left[\sigma(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{\mathcal{C}_2} \left[-\beta(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} \left[-\nu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \left[\mu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\ & = v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc. \end{aligned}$$

Therefore, the objective function is bounded by

$$\int_{\mathcal{C}_1} \sigma(c) dc.$$

Substituting our candidate solution into the objective function achieves the upper bound in all the subcases, which follows from:

$$\begin{aligned} & \int_{c_L}^{\check{c}} \sigma(c) dc - \left[v_E \alpha^* + (\lambda_0 + \lambda_1 \alpha^*) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \right] F(\check{c}) \\ & = \int_{c_L}^{\check{c}} \left[v(\tilde{\eta}) \left[c + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\tilde{\eta}}^1 [v(\eta) - v(\tilde{\eta})] g(\eta) d\eta \right] f(c) dc \\ & \quad - \left[v_E \alpha^* + (\lambda_0 + \lambda_1 \alpha^*) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \right] F(\check{c}) \\ & = v(\tilde{\eta}) \int_{c_L}^{\check{c}} [(c - \check{c}) f(c) + F(c)] dc \\ & = 0, \end{aligned}$$

where $\alpha^* = \frac{\bar{c} - \lambda_0 [1 - G(\tilde{\eta})]}{1 + \lambda_1 [1 - G(\tilde{\eta})]}$.

This establishes the optimality in the first case.

To prove the optimality in the second case, we first define

$$\begin{aligned}
\kappa &= -v(\tilde{\eta}), \\
\nu(c, \eta) &= [v(\tilde{\eta}) - v(\eta)] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}], \\
\mu(c, \eta) &= [v(\eta) - v(\tilde{\eta})] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}, 1], \\
\phi(c) &= -\lambda_1 \int_{\tilde{\eta}}^1 \mu(c, y) dy + [v(\tilde{\eta}) - v_E] f(c), & \forall c \in [c_L, c_H], \\
\sigma(c) &= v_E [c f(c) + F(c)] + (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta \\
&\quad + \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy + \int_{c_L}^c \phi(y) dy + c \phi(c), & \forall c \in \mathcal{C}_1, \\
\beta(c) &= -v_E [c f(c) + F(c)] - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) d\eta \\
&\quad - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) d\eta dy - \int_{c_L}^c \phi(y) dy - c \phi(c), & \forall c \in \mathcal{C}_2,
\end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are subsets of $[c_L, c_H]$ satisfying: (i) $\mathcal{C}_1 \cup \mathcal{C}_2 = [c_L, c_H]$; and (ii) the measure of $\mathcal{C}_1 \cap \mathcal{C}_2$ is 0. These subsets will be specified later.

The nonnegativity of κ , ν , and μ follows from the definition of $\tilde{\eta}$ and the monotonicity of v directly.

The nonnegativity of ϕ follows from the second part of Lemma [2](#).

The expressions for σ and β can be simplified as

$$\sigma(c) = v(\tilde{\eta}) [(c - \underline{c}(\tilde{\eta})) f(c) + F(c)] + \lambda_0 f(c) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta), \quad \forall c \in \mathcal{C}_1,$$

and

$$\beta(c) = -v(\tilde{\eta}) [(c - \underline{c}(\tilde{\eta})) f(c) + F(c)] - \lambda_0 f(c) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta), \quad \forall c \in \mathcal{C}_2.$$

To prove the nonnegativity of σ , taking derivative yields

$$\begin{aligned}
\sigma'(c) &= v(\tilde{\eta}) [(c - \underline{c}(\tilde{\eta})) f'(c) + 2 f(c)] + f'(c) \underbrace{\lambda_0 \int_{\tilde{\eta}}^1 v(\eta) dG(\eta)}_{= -v(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} \geq 0} \leq 0, \quad \forall c \leq \underline{c}(\tilde{\eta}).
\end{aligned}$$

Meanwhile, we have

$$\sigma(\underline{c}(\tilde{\eta})) = \left[v(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \right] f(\underline{c}(\tilde{\eta})) = 0.$$

In the first two possible cases, we have:

1. if $\underline{c}(\tilde{\eta}) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$; and
2. if $c_L \leq \underline{c}(\tilde{\eta}) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \underline{c}(\tilde{\eta})]$ and $[\underline{c}(\tilde{\eta}), c_H]$, respectively.

In this case, we have $\mathcal{C}_1 = [c_L, \underline{c}(\tilde{\eta})]$ and $\mathcal{C}_2 = [\underline{c}(\tilde{\eta}), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa - \int_{c_L}^{c_H} \phi(c) \, dc - \lambda_1 \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta \, dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}]:$$

$$\phi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) \, dz = [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}, 1]:$$

$$\phi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) \, dz = [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall c \in \mathcal{C}_1:$$

$$-\int_{c_L}^c \phi(y) \, dy - c\phi(c) + \sigma(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) \, d\eta \, dy = v_E [c f(c) + F(c)],$$

$$\forall c \in \mathcal{C}_2:$$

$$-\int_{c_L}^c \phi(y) \, dy - c\phi(c) - \beta(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) \, d\eta \, dy = v_E [c f(c) + F(c)],$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{\mathcal{C}_1} \sigma(c) \tau(c) \, dc + \int_{\mathcal{C}_2} \beta(c) [-\tau(c)] \, dc \\ & + \int_{c_L}^{c_H} \phi(c) \left[-\Pi(c_H) - \int_c^{c_H} \tau(y) \, dy - c\tau(c) + \int_0^1 \alpha_2(c, \eta) \, dG(\eta) \right] \, dc \\ & + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} \nu(c, \eta) [-\alpha_2(c, \eta)] \, d\eta \, dc \\ & + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) \left[\alpha_2(c, \eta) - \lambda_0 \tau(c) - \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) \, dy + c\tau(c) - \int_0^1 \alpha_2(c, z) \, dG(z) \right] \right] \, d\eta \, dc \\ = & \left[-\kappa - \int_{c_L}^{c_H} \phi(c) \, dc - \lambda_1 \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta \, dc \right] \Pi(c_H) \\ & + \int_{\mathcal{C}_1} \left[\sigma(c) - \int_{c_L}^c \phi(y) \, dy - c\phi(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) \, d\eta \, dy \right] \tau(c) \, dc \\ & + \int_{\mathcal{C}_2} \left[-\beta(c) - \int_{c_L}^c \phi(y) \, dy - c\phi(c) - (\lambda_0 + \lambda_1 c) \int_{\tilde{\eta}}^1 \mu(c, \eta) \, d\eta - \lambda_1 \int_{c_L}^c \int_{\tilde{\eta}}^1 \mu(y, \eta) \, d\eta \, dy \right] \tau(c) \, dc \\ & + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} \left[\phi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) \, dz \right] \alpha_2(c, \eta) \, d\eta \, dc \end{aligned}$$

$$\begin{aligned}
& + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \left[\phi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\
& = v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc.
\end{aligned}$$

Therefore, the objective function is bounded by

$$\int_{c_1} \sigma(c) dc.$$

Substituting our candidate solution into the objective function achieves the upper bound in the first two cases, which follows from:

1. if $\underline{c}(\tilde{\eta}) \leq c_L$, we have $c^* = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff generated by our candidate policy and its upper bound are both 0;
2. if $c_L < \underline{c}(\tilde{\eta}) \leq c_H$, we have

$$\begin{aligned}
& \int_{c_L}^{\underline{c}(\tilde{\eta})} \sigma(c) dc - \lambda_0 F(\underline{c}(\tilde{\eta})) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \\
& = \int_{c_L}^{\underline{c}(\tilde{\eta})} \left[v(\tilde{\eta}) [[c - \underline{c}(\tilde{\eta})] f(c) + F(c)] + \lambda_0 f(c) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \right] dc - \lambda_0 F(\underline{c}(\tilde{\eta})) \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \\
& = v(\tilde{\eta}) \int_{c_L}^{\underline{c}(\tilde{\eta})} [[c - \underline{c}(\tilde{\eta})] f(c) + F(c)] dc \\
& = 0.
\end{aligned}$$

To verify the optimality in case (2)(c), define

$$\kappa = -v(\eta^*),$$

$$\nu(c, \eta) = [v(\eta^*) - v(\eta)] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \eta^*],$$

$$\mu(c, \eta) = [v(\eta) - v(\eta^*)] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [\eta^*, 1],$$

$$\phi(c) = -\lambda_1 \int_{\eta^*}^1 \mu(c, y) dy + [v(\eta^*) - v_E] f(c), \quad \forall c \in [c_L, c_H],$$

$$\begin{aligned}
\sigma(c) & = v_E [c f(c) + F(c)] + (\lambda_0 + \lambda_1 c) \int_{\eta^*}^1 \mu(c, \eta) d\eta \\
& \quad + \lambda_1 \int_{c_L}^c \int_{\eta^*}^1 \mu(y, \eta) d\eta dy + \int_{c_L}^c \phi(y) dy + c \phi(c), \quad \forall c \in [c_L, c_H],
\end{aligned}$$

where $\eta^* \equiv \sup \{ \eta \in [0, 1] : \underline{c}(\eta) \geq c_H \}$. Because $\underline{c}(\tilde{\eta}) > c_H > \underline{c}(\bar{\eta})$, we know that $\tilde{\eta} < \eta^* < \bar{\eta}$. The nonnegativity of κ , ν , and μ follows from the corresponding definitions directly. Because

$$\phi(c) \geq -\lambda_1 \int_{\tilde{\eta}}^1 [v(y) - v(\tilde{\eta})] f(c) g(y) dy + [v(\tilde{\eta}) - v_E] f(c) \geq 0, \quad \forall c \in [c_L, c_H],$$

the desired nonnegativity condition holds automatically. Finally, we have

$$\sigma(c) = v(\eta^*) [(c - c_H) f(c) + F(c)] + \lambda_0 f(c) \int_{\eta^*}^1 v(\eta) dG(\eta), \quad \forall c \in [c_L, c_H].$$

The monotonicity of σ still holds, and we have

$$\begin{aligned} \sigma(c_H) &= \left[v(\eta^*) \frac{F(c_H)}{f(c_H)} + \lambda_0 \int_{\eta^*}^1 v(\eta) dG(\eta) \right] f(c_H) \\ &\geq \left[v(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 v(\eta) dG(\eta) \right] f(c_H) \\ &= 0, \end{aligned}$$

where the inequality follows from the monotonicity property of v and the fact that $\eta^* > \tilde{\eta}$.

Because

$$\begin{aligned} &\kappa [-\Pi(c_H)] + \int_{c_L}^{c_H} \sigma(c) \tau(c) dc \\ &+ \int_{c_L}^{c_H} \phi(c) \left[-\Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 \alpha_2(c, \eta) dG(\eta) \right] dc \\ &+ \int_{c_L}^{c_H} \int_0^{\eta^*} \nu(c, \eta) [-\alpha_2(c, \eta)] d\eta dc \\ &+ \int_{c_L}^{c_H} \int_{\eta^*}^1 \mu(c, \eta) \left[\alpha_2(c, \eta) - \lambda_0 \tau(c) - \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \int_0^1 \alpha_2(c, z) dG(z) \right] \right] d\eta dc \\ &= \left[-\kappa - \int_{c_L}^{c_H} \phi(c) dc - \lambda_1 \int_{c_L}^{c_H} \int_{\eta^*}^1 \mu(c, \eta) d\eta dc \right] \Pi(c_H) \\ &+ \int_{c_L}^{c_H} \left[\sigma(c) - \int_{c_L}^c \phi(y) dy - c \phi(c) - (\lambda_0 + \lambda_1 c) \int_{\eta^*}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\eta^*}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\ &+ \int_{c_L}^{c_H} \int_0^{\eta^*} \left[\phi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_{\eta^*}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\ &+ \int_{c_L}^{c_H} \int_{\eta^*}^1 \left[\phi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_{\eta^*}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\ &= v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc, \end{aligned}$$

the objective function is bounded by

$$\int_{c_L}^{c_H} \sigma(c) dc.$$

Substituting our candidate solution into the objective function achieves the upper bound, which follows from

$$\begin{aligned} &\int_{c_L}^{c_H} \sigma(c) dc - \lambda_0 \int_{\eta^*}^1 v(\eta) dG(\eta) \\ &= \int_{c_L}^{c_H} \left[v(\eta^*) [(c - c_H) f(c) + F(c)] + f(c) \lambda_0 \int_{\eta^*}^1 v(\eta) dG(\eta) \right] dc - \lambda_0 \int_{\eta^*}^1 v(\eta) dG(\eta) \end{aligned}$$

$$\begin{aligned}
&= v(\eta^*) \int_{c_L}^{c_H} [(c - c_H) f(c) + F(c)] dc \\
&= 0.
\end{aligned}$$

It remains to prove the last part of the proposition, where the optimal mechanism is to grant full conditional approval to firms whose types are below the threshold c^* . In the first two subcases, we have $\bar{c}(\underline{\eta}) < c_H \leq \bar{c}(0)$, implying that $\underline{\eta} = \eta^*$.

Define

$$\begin{aligned}
\kappa &= -v(\underline{\eta}), \\
\nu(c, \eta) &= [v(\underline{\eta}) - v(\eta)] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, \underline{\eta}], \\
\mu(c, \eta) &= [v(\eta) - v(\underline{\eta})] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [\underline{\eta}, 1], \\
\psi(c) &= \lambda_1 \int_{\underline{\eta}}^1 \mu(c, y) dy - [v(\underline{\eta}) - v_E] f(c), & \forall c \in [c_L, c_H], \\
\sigma(c) &= v_E [c f(c) + F(c)] + (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta \\
&\quad + \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy - \int_{c_L}^c \psi(y) dy - (c - 1) \psi(c), & \forall c \in \mathcal{C}_1, \\
\beta(c) &= -v_E [c f(c) + F(c)] - (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta \\
&\quad - \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy + \int_{c_L}^c \psi(y) dy + (c - 1) \psi(c), & \forall c \in \mathcal{C}_2,
\end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are subsets of $[c_L, c_H]$ satisfying: (i) $\mathcal{C}_1 \cup \mathcal{C}_2 = [c_L, c_H]$; and (ii) the measure of $\mathcal{C}_1 \cap \mathcal{C}_2$ is 0. These subsets will be specified later.

The nonnegativity of κ , ν , and μ follows from the definition of $\underline{\eta}$ and the monotonicity of v directly.

The nonnegativity of ψ follows from the last part of Lemma [2](#).

The expressions of σ and β can be rewritten as

$$\sigma(c) = v(\underline{\eta}) [(c - \bar{c}(\underline{\eta})) f(c) + F(c)] + f(c) \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(y) dG(y) \right]$$

and

$$\beta(c) = -v(\underline{\eta}) [(c - \bar{c}(\underline{\eta})) f(c) + F(c)] - f(c) \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(y) dG(y) \right].$$

Differentiating σ yields

$$\begin{aligned}
\sigma'(c) &= v(\underline{\eta}) [(c - \bar{c}(\underline{\eta})) f'(c) + 2f(c)] + f'(c) \underbrace{\left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(y) dG(y) \right]}_{=-v(\underline{\eta}) \frac{F(\bar{c}(\underline{\eta}))}{f(\bar{c}(\underline{\eta}))}} \leq 0, \quad \forall c \leq \bar{c}(\underline{\eta}).
\end{aligned}$$

Meanwhile, we have

$$\sigma(\bar{c}(\underline{\eta})) = \left[v(\underline{\eta}) \frac{F(\bar{c}(\underline{\eta}))}{f(\bar{c}(\underline{\eta}))} + \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(y) dG(y) \right] \right] f(\bar{c}(\underline{\eta})) = 0.$$

In the first two possible cases, we have:

1. if $\bar{c}(\underline{\eta}) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$;
2. if $c_L \leq \bar{c}(\underline{\eta}) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \bar{c}(\underline{\eta})]$ and $[\bar{c}(\underline{\eta}), c_H]$, respectively.

In this case, we have $\mathcal{C}_1 = [c_L, \bar{c}(\underline{\eta})]$ and $\mathcal{C}_2 = [\bar{c}(\underline{\eta}), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \lambda_1 \int_{c_L}^{c_H} \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta dc = v_E,$$

$\forall (c, \eta) \in [c_L, c_H] \times [0, \underline{\eta}]$:

$$-\psi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_{\underline{\eta}}^1 \mu(c, y) dy = [v(\eta) - v_E] g(\eta) f(c),$$

$\forall (c, \eta) \in [c_L, c_H] \times [\underline{\eta}, 1]$:

$$-\psi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_{\underline{\eta}}^1 \mu(c, y) dy = [v(\eta) - v_E] g(\eta) f(c),$$

$\forall c \in \mathcal{C}_1$:

$$\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) + \sigma(c) - (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy = v_E [c f(c) + F(c)],$$

$\forall c \in \mathcal{C}_2$:

$$\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - \beta(c) - (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy = v_E [c f(c) + F(c)],$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{\mathcal{C}_1} \sigma(c) \tau(c) dc + \int_{\mathcal{C}_2} \beta(c) [-\tau(c)] dc \\ & + \int_{c_L}^{c_H} \psi(c) \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, \eta) dG(\eta) - \tau(c) \right] dc \\ & + \int_{c_L}^{c_H} \int_0^{\underline{\eta}} \nu(c, \eta) [-\alpha_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\underline{\eta}}^1 \mu(c, \eta) \left[\alpha_2(c, \eta) - \lambda_0 \tau(c) - \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \int_0^1 \alpha_2(c, z) dG(z) \right] \right] d\eta dc \\ & = \left[-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \lambda_1 \int_{c_L}^{c_H} \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta dc \right] \Pi(c_H) \end{aligned}$$

$$\begin{aligned}
& + \int_{c_1} \left[\sigma(c) + \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\
& + \int_{c_2} \left[-\beta(c) + \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - (\lambda_0 + \lambda_1 c) \int_{\underline{\eta}}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\underline{\eta}}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\
& + \int_{c_L}^{c_H} \int_0^{\underline{\eta}} \left[-\psi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_0^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\underline{\eta}}^1 \left[-\psi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_0^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\
& = v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc.
\end{aligned}$$

Therefore, the objective function is bounded by

$$\int_{c_1} \sigma(c) dc.$$

Substituting our candidate solution into the objective function achieves the upper bound in both cases, which follows from

1. if $\bar{c}(\underline{\eta}) < c_L$, we have $c^* = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff generated by our candidate policy and its upper bound are both 0;
2. if $c_L \leq \bar{c}(\underline{\eta}) \leq c_H$, we have

$$\begin{aligned}
& \int_{c_L}^{\bar{c}(\underline{\eta})} \sigma(c) dc - \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(\eta) dG(\eta) \right] F(\bar{c}(\underline{\eta})) \\
& = \int_{c_L}^{\bar{c}(\underline{\eta})} \left[v(\underline{\eta}) [(c - \bar{c}(\underline{\eta})) f(c) + F(c)] + f(c) \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(\eta) dG(\eta) \right] \right] dc \\
& \quad - \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(\eta) dG(\eta) \right] F(\bar{c}(\underline{\eta})) \\
& = v(\underline{\eta}) \int_{c_L}^{\bar{c}(\underline{\eta})} [(c - \bar{c}(\underline{\eta})) f(c) + F(c)] dc \\
& = 0.
\end{aligned}$$

To prove the optimality in case (3)(c), define

$$\kappa = -v(\eta^*),$$

$$\nu(c, \eta) = [v(\eta^*) - v(\eta)] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \eta^*],$$

$$\mu(c, \eta) = [v(\eta) - v(\eta^*)] f(c) g(\eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [\eta^*, 1],$$

$$\psi(c) = \lambda_1 \int_{\eta^*}^1 \mu(c, y) dy - [v(\eta^*) - v_E] f(c), \quad \forall c \in [c_L, c_H],$$

$$\begin{aligned}\sigma(c) &= v_E [c f(c) + F(c)] + (\lambda_0 + \lambda_1 c) \int_{\eta^*}^1 \mu(c, \eta) d\eta \\ &\quad + \lambda_1 \int_{c_L}^c \int_{\eta^*}^1 \mu(y, \eta) d\eta dy - \int_{c_L}^c \psi(y) dy - (c-1) \psi(c), \quad \forall c \in [c_L, c_H].\end{aligned}$$

The monotonicity of $\bar{c}$ implies that $\underline{\eta} \leq \eta^* < \bar{\eta}$. The nonnegativity of κ , ν , and μ follows from their definitions directly.

When $h(\bar{\eta}) > 0$, we have

$$\psi(c) = h(\eta^*) f(c) \geq h(\bar{\eta}) f(c) > 0.$$

When $\check{c} > \bar{c}(\check{\eta})$, by Lemma 2, we know that ψ is nonnegative if and only if $\eta^* \leq \check{\eta}$, which follows from

$$\bar{c}(\check{\eta}) < \check{c} \leq c_H \quad \Rightarrow \quad \check{\eta} > \eta^*.$$

Similar to the previous case, the expression of σ can be simplified as

$$\sigma(c) = v(\eta^*) [(c - c_H) f(c) + F(c)] + f(c) \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta^*}^1 v(y) dG(y) \right], \quad \forall c \in [c_L, c_H].$$

The monotonicity of σ still holds, and we have

$$\begin{aligned}\sigma(c_H) &= v(\eta^*) + f(c_H) \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta^*}^1 v(y) dG(y) \right] \\ &\geq \left[v(\underline{\eta}) \frac{F(\bar{c}(\underline{\eta}))}{f(\bar{c}(\underline{\eta}))} + \left[v_E + (\lambda_0 + \lambda_1) \int_{\underline{\eta}}^1 v(y) dG(y) \right] \right] f(c_H) \\ &\geq 0,\end{aligned}$$

implying the nonnegativity of σ . Because

$$\begin{aligned}&\kappa [-\Pi(c_H)] + \int_{c_L}^{c_H} \sigma(c) \tau(c) dc \\ &\quad + \int_{c_L}^{c_H} \psi(c) \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \int_0^1 \alpha_2(c, \eta) dG(\eta) - \tau(c) \right] dc \\ &\quad + \int_{c_L}^{c_H} \int_0^{\eta^*} \nu(c, \eta) [-\alpha_2(c, \eta)] d\eta dc \\ &\quad + \int_{c_L}^{c_H} \int_{\eta^*}^1 \mu(c, \eta) \left[\alpha_2(c, \eta) - \lambda_0 \tau(c) - \lambda_1 \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \int_0^1 \alpha_2(c, z) dG(z) \right] \right] d\eta dc \\ &= \left[-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \lambda_1 \int_{c_L}^{c_H} \int_{\eta^*}^1 \mu(c, \eta) d\eta dc \right] \Pi(c_H) \\ &\quad + \int_{c_L}^{c_H} \left[\sigma(c) + \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - (\lambda_0 + \lambda_1 c) \int_{\eta^*}^1 \mu(c, \eta) d\eta - \lambda_1 \int_{c_L}^c \int_{\eta^*}^1 \mu(y, \eta) d\eta dy \right] \tau(c) dc \\ &\quad + \int_{c_L}^{c_H} \int_0^{\eta^*} \left[-\psi(c) g(\eta) - \nu(c, \eta) + \lambda_1 g(\eta) \int_{\eta^*}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc\end{aligned}$$

$$\begin{aligned}
& + \int_{c_L}^{c_H} \int_{\eta^*}^1 \left[-\psi(c) g(\eta) + \mu(c, \eta) + \lambda_1 g(\eta) \int_{\eta^*}^1 \mu(c, z) dz \right] \alpha_2(c, \eta) d\eta dc \\
& = v_E \Pi(c_H) + \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \alpha_2(c, \eta) dG(\eta) dF(c) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc,
\end{aligned}$$

the objective function is bounded by

$$\int_{c_L}^{c_H} \sigma(c) dc.$$

Substituting our candidate solution into the objective function achieves the upper bound, which follows from

$$\begin{aligned}
& \int_{c_L}^{c_H} \sigma(c) dc - \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta^*}^1 v(\eta) dG(\eta) \right] \\
& = \int_{c_L}^{c_H} \left[v(\eta^*) [(c - c_H) f(c) + F(c)] + f(c) \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta^*}^1 v(\eta) dG(\eta) \right] \right] dc \\
& \quad - \left[v_E + (\lambda_0 + \lambda_1) \int_{\eta^*}^1 v(\eta) dG(\eta) \right] \\
& = v(\eta^*) \int_{c_L}^{c_H} [(c - c_H) f(c) + F(c)] dc \\
& = 0.
\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 4. When the testing cost is high and the government induces testing, the firm's individual rationality constraint (IR) must be binding. Otherwise, the government could strictly increase its expected payoff by reducing the costly incentives offered to the firm. That is, the government's first-best optimization problem can be written as follows:

$$\begin{aligned}
& \max_{\eta} v_E \tilde{a}_1(\eta, c) + (\lambda_0 + \lambda_1) \tilde{a}_1(\eta, c) \int_{\eta}^1 v(y) dG(y), \\
& \text{subject to } 0 \leq \tilde{a}_1(\eta, c) \leq 1,
\end{aligned}$$

where $\tilde{a}_1$ is defined by (31). We refer to the above optimization problem without the feasibility constraint on a_1 as the “relaxed problem”.

Since the firm's revenue across both periods is bounded above by $\bar{c}(0) = 1 + \lambda_0 + \lambda_1$, the government cannot induce testing when the testing cost exceeds this value. In the remainder of the proof, we focus on the cases where $c \leq \bar{c}(0)$.

As established in Section 4.1, the derivative of the objective function with respect to η has the same sign as the function h defined in (33). If the drug is promising ($h(\bar{\eta}) \geq 0$) and $c > 1 + (\lambda_0 + \lambda_1) [1 - G(\bar{\eta})]$, since $\tilde{a}_1(\bar{\eta}, c) > 1$, the optimal solution to the relaxed problem is infeasible. Because η is increasing in a_1 , it is optimal to set $a_1 = 1$, and the corresponding value of η is determined such that the individual rationality constraint (IR) binds.

Next, we consider the case in which the drug is not promising ($h(\bar{\eta}) \leq 0$) and $c > \lambda_0 [1 - G(\bar{\eta})]$. Recall that $\check{\eta}$ is the unique root of the function h on $(0, \bar{\eta}]$. If $\tilde{a}_1(\check{\eta}, c) \in [0, 1]$, then the optimal solution to the relaxed problem is feasible, and hence also optimal for the original problem. Instead, if $\tilde{a}_1(\check{\eta}, c) < 0$, then any feasible value of η must lie above $\check{\eta}$. Since η is increasing in a_1 and the objective function is decreasing in η for all $\eta \in [\check{\eta}, \bar{\eta}]$, it is optimal to set $a_1 = 0$, which uniquely determines the final approval threshold $\eta^* = G^{-1} \left(1 - \frac{c}{\lambda_0} \right)$. Finally, if $\tilde{a}_1(\check{\eta}, c) > 1$, then any feasible value of η must lie below $\check{\eta}$. Since η is increasing in a_1 and the objective function is increasing in η for all $\eta \in [0, \check{\eta}]$, it is optimal to set $a_1 = 1$, and the corresponding final approval threshold is given by $\eta^* = G^{-1} \left(1 - \frac{c-1}{\lambda_0 + \lambda_1} \right)$.

When the government's expected payoff from inducing testing ($\tau = 1$) is negative, it is optimal to forgo testing by setting $\tau = 0$.

This completes the proof. $\square$

EC.5.4. Proofs for Section 5

Proof of Proposition 5. To prove the desired result, it suffices to show that for any feasible solution to (13), we can construct a feasible solution to (48) that yields the same objective function value.

Let $\mathcal{M} \equiv \{\tau, a_1, a_2\}$ denote a feasible solution to the optimization problem (13). By construction, it is straightforward to verify that $\bar{\mathcal{M}} \equiv \{\tau, \alpha_1, \alpha_2, \alpha_3\}$, where

$$\begin{aligned} \alpha_1(c) &= \tau(c) a_1(c), & \forall c \in [c_L, c_H], \\ \alpha_2(c, \eta) &= \tau(c) a_2(c, \eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, 1], \\ \alpha_3(c, \eta) &= \tau(c) a_1(c) a_2(c, \eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, 1], \end{aligned}$$

is feasible to the upper-bound problem (48) and yields the same objective function value.

This completes the proof. $\square$

Similar to Lemma EC.1 after eliminating α_1 using the envelope theorem, the government's upper-bound problem can be formulated as follows:

$$\begin{aligned} \max_{\{\tau, \hat{\alpha}_2, \hat{\alpha}_3, \Pi(c_H)\} \in \Omega'_R} & v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc \\ & + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\ & + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c), \end{aligned} \quad (\text{EC.13})$$

where the feasible set Ω'_R is determined by (2), (45), (47), (EC.7), (EC.9), and for any $(c, \eta) \in [c_L, c_H] \times [0, 1]$,

$$0 \leq \Pi(c_H) + \int_c^{c_H} \tau(y) dy + c\tau(c) - \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) - \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta) \leq \tau(c), \quad (\text{EC.14})$$

and

$$\hat{\alpha}_3(c, \eta) \leq \Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) - \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta). \quad (\text{EC.15})$$

With a slight abuse of notation, in this subsection, let

$$\bar{c}(\eta_1, \eta_2) \equiv 1 + \lambda_0 [1 - G(\eta_1)] + \lambda_1 [1 - G(\eta_2)], \quad \forall \eta_1, \eta_2 \in [0, 1]. \quad (\text{EC.16})$$

By Lemma 2(1), we know that there exists a unique $\check{\eta}_2 \in [0, \bar{\eta}]$ such that $h(\check{\eta}_2) = 0$ if $h(\bar{\eta}) \leq 0$.
Let

$$\Psi_1(\eta_1, \eta_2) \equiv (\lambda_0 + \Delta) v(\eta_1) + \Delta - \lambda_0 v(\eta_2) \quad (\text{EC.17})$$

and

$$\Psi_2(\eta_1, \eta_2) \equiv (\lambda_0 + \lambda_1 + \Delta) [v(\eta_2) - v(\eta_1)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\eta_2) + 1]. \quad (\text{EC.18})$$

In order to prove Proposition 6, we need the following technical results.

LEMMA EC.3. *When $h(\bar{\eta}) \leq 0$, we have the following results.*

- (1) *If $v(\check{\eta}_2) + 1 \geq 0$, there exists a unique $\check{\eta}_1 \in [0, \check{\eta}_2]$ such that $\Psi_1(\check{\eta}_1, \check{\eta}_2) = 0$.*
- (2) *If $v(\check{\eta}_2) + 1 < 0$, there uniquely exist $(\check{\eta}_1, \check{\eta}_2)$ such that $0 \leq \check{\eta}_1 \leq \check{\eta}_2 \leq \bar{\eta}$, $\Psi_2(\check{\eta}_1, \check{\eta}_2) = 0$, and $\iota(\check{\eta}_1, \check{\eta}_2) = 0$, where*

$$\eta^\dagger \equiv \inf \{ \eta \in [0, 1] : v(\eta) + 1 \geq 0 \} \quad (\text{EC.19})$$

and

$$\begin{aligned} \iota(\eta_1, \eta_2) &\equiv (\lambda_0 + \lambda_1 + \Delta) \int_{\eta_1}^{\eta_2} [v(\eta_2) - v(z)] g(z) dz - \lambda_1 \int_{\eta_2}^1 [v(z) - v(\eta_2)] g(z) dz \\ &\quad + \frac{\Delta}{\lambda_0} [v(\eta_2) + 1] [1 + \lambda_1 [1 - G(\eta_1)]] + v(\eta_2) - v_E. \end{aligned} \quad (\text{EC.20})$$

Proof of Lemma EC.3. When $\check{\eta}_2$ exists and $v(\check{\eta}_2) + 1 \geq 0$, we have

$$\Psi_1(0, \check{\eta}_2) = \lambda_0 [v(0) - v(\check{\eta}_2)] + \Delta [v(0) + 1] \leq 0,$$

and

$$\Psi_1(\check{\eta}_2, \check{\eta}_2) = \Delta [v(\check{\eta}_2) + 1] \geq 0.$$

The unique existence of $\check{\eta}_1$ follows directly from the monotonicity of Ψ_1 in η_1 .

If $v(\check{\eta}_2) + 1 < 0$, define

$$\begin{aligned} \Xi(\eta_2) &\equiv -\lambda_0 \iota(\eta_1(\eta_2), \eta_2) \\ &= \lambda_0 \left[-(\lambda_0 + \lambda_1 + \Delta) \int_{\eta_1(\eta_2)}^{\eta_2} [v(\eta_2) - v(z)] g(z) dz + \lambda_1 \int_{\eta_2}^1 [v(z) - v(\eta_2)] g(z) dz \right] \\ &\quad - \Delta [v(\eta_2) + 1] [1 + \lambda_1 [1 - G(\eta_1(\eta_2))]] - \lambda_0 [v(\eta_2) - v_E], \end{aligned} \quad (\text{EC.21})$$

where $\eta_1(\eta_2)$ is determined by

$$\Psi_2(\eta_1(\eta_2), \eta_2) = 0 \quad \Leftrightarrow \quad v(\eta_1(\eta_2)) = v(\eta_2) + \frac{\lambda_1 \Delta [v(\eta_2) + 1]}{\lambda_0 (\lambda_0 + \lambda_1 + \Delta)}.$$

Note that $\eta_1(\eta_2)$ is increasing.

The monotonicity of v implies that $\check{\eta}_2 < \eta^\dagger$. We have $\eta_1(\eta^\dagger) = \eta^\dagger$, implying that

$$\begin{aligned} \Xi(\eta^\dagger) &\equiv \lambda_0 \left[\lambda_1 \int_{\eta^\dagger}^1 [v(z) - v(\eta^\dagger)] g(z) dz - v(\eta^\dagger) + v_E \right] \\ &\leq \lambda_0 \left[\lambda_1 \int_{\check{\eta}_2}^1 [v(z) - v(\check{\eta}_2)] g(z) dz - v(\check{\eta}_2) + v_E \right] \\ &= 0, \end{aligned}$$

where the inequality follows from the monotonicity of the function h . To prove the desired result, it suffices to show that $\Xi(\check{\eta}_2) \geq 0$ and Ξ is non-increasing.

Taking derivative yields

$$\begin{aligned} \Xi'(\eta_2) &= -\lambda_0 (\lambda_0 + \lambda_1 + \Delta) \int_{\eta_1(\eta_2)}^{\eta_2} v'(\eta_2) g(z) dz + \lambda_0 (\lambda_0 + \lambda_1 + \Delta) [v(\eta_2) - v(\eta_1(\eta_2))] g(\eta_1(\eta_2)) \eta_1'(\eta_2) \\ &\quad - \lambda_0 \lambda_1 \int_{\eta_2}^1 v'(\eta_2) g(z) dz - \Delta v'(\eta_2) [1 + \lambda_1 (1 - G(\eta_1(\eta_2)))] \\ &\quad + \Delta [v(\eta_2) + 1] \lambda_1 g(\eta_1(\eta_2)) \eta_1'(\eta_2) - \lambda_0 v'(\eta_2) \\ &= \left\{ -\lambda_0 (\lambda_0 + \lambda_1 + \Delta) [G(\eta_2) - G(\eta_1(\eta_2))] - \lambda_0 \lambda_1 [1 - G(\eta_2)] \right. \\ &\quad \left. - \lambda_0 - \Delta [1 + \lambda_1 (1 - G(\eta_1(\eta_2)))] \right\} v'(\eta_2) \\ &\leq 0, \end{aligned}$$

where the second equality follows from the definition of $\eta_1(\eta_2)$, i.e., $\Psi_2(\eta_1(\eta_2), \eta_2) = 0$.

Since $h(\check{\eta}_2) = 0$, the desired condition $\Xi(\check{\eta}_2) \geq 0$ reduces to $\tilde{\Xi}(\check{\eta}_2) \geq 0$, where

$$\tilde{\Xi}(\eta) \equiv -\lambda_0 (\lambda_0 + \lambda_1 + \Delta) \int_{\eta_1(\eta)}^{\eta} [v(\eta) - v(z)] g(z) dz - \Delta [v(\eta) + 1] [1 + \lambda_1 [1 - G(\eta_1(\eta))]].$$

Taking derivative yields

$$\begin{aligned} \tilde{\Xi}'(\eta) &= -\lambda_0 (\lambda_0 + \lambda_1 + \Delta) \left[\int_{\eta_1(\eta)}^{\eta} v'(\eta) g(z) dz - [v(\eta) - v(\eta_1(\eta))] g(\eta_1(\eta)) \eta_1'(\eta) \right] \\ &\quad - \Delta v'(\eta) [1 + \lambda_1 [1 - G(\eta_1(\eta))]] + \Delta [v(\eta) + 1] \lambda_1 g(\eta_1(\eta)) \eta_1'(\eta) \\ &= -\{\lambda_0 (\lambda_0 + \lambda_1 + \Delta) [G(\eta) - G(\eta_1(\eta))] + \Delta [1 + \lambda_1 [1 - G(\eta_1(\eta))]]\} v'(\eta) \\ &\leq 0. \end{aligned}$$

Therefore, the desired result is implied by

$$\tilde{\Xi}(\check{\eta}_2) \geq \tilde{\Xi}(\eta^\dagger) = 0.$$

This completes the proof. □

Let

$$\check{c}_1 \equiv \sup \left\{ c \in [c_L, c_H] : (\lambda_0 + \Delta) \int_{\tilde{\eta}_1}^1 [v(\eta) - v(\check{\eta}_1)] g(\eta) d\eta + v(\check{\eta}_2) \left[c + \frac{F(c)}{f(c)} \right] \geq 0 \right\} \quad (\text{EC.22})$$

and

$$\check{c}_2 \equiv \sup \left\{ c \in [c_L, c_H] : (\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] \geq 0 \right\}. \quad (\text{EC.23})$$

By Lemma [EC.3](#), we know that $\check{c}_1$ and $\check{c}_2$ are well-defined when $v(\check{\eta}_2) + 1 \geq 0$ and $v(\check{\eta}_2) + 1 < 0$, respectively, with the existence of $\check{\eta}_2$ requiring $h(\bar{\eta}) \leq 0$.

LEMMA EC.4. *When $h(\bar{\eta}) \leq 0$, we have the following results.*

(1) *If $\check{c}_2 < \underline{c}(\check{\eta}_2)$, there exists a unique $\tilde{\eta}_2 \in [\check{\eta}_2, \bar{\eta}']$ such that*

$$(\lambda_0 + \Delta) \int_{\tilde{\eta}_2}^1 [v(\eta) - v(\tilde{\eta}_2)] g(\eta) d\eta + \left[\underline{c}(\tilde{\eta}_2) + \frac{F(\underline{c}(\tilde{\eta}_2))}{f(\underline{c}(\tilde{\eta}_2))} \right] \left[v(\tilde{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\tilde{\eta}_2) + 1] \right] = 0, \quad (\text{EC.24})$$

where $\bar{\eta}'$ is defined by [55](#). If $v(\tilde{\eta}_2) + 1 \leq 0$, there exists $\tilde{\eta}_1 \leq \tilde{\eta}_2$ such that $\Psi_2(\tilde{\eta}_1, \tilde{\eta}_2) = 0$. In this case, we have $\iota(\tilde{\eta}_1, \tilde{\eta}_2) \geq 0$, where ι is defined by [EC.20](#).

(2) *If (i) $v(\tilde{\eta}_2) + 1 > 0$, there exists a unique $\hat{\eta}_1 \in [\hat{\eta}_2, \bar{\eta}']$, or if (ii) $\check{c}_1 < \underline{c}(\tilde{\eta}_1)$, there exists a unique $\hat{\eta}_1 \in [\tilde{\eta}_1, \bar{\eta}']$, such that*

$$v(\hat{\eta}_2) \left[\underline{c}(\hat{\eta}_1) + \frac{F(\underline{c}(\hat{\eta}_1))}{f(\underline{c}(\hat{\eta}_1))} \right] + (\lambda_0 + \Delta) \int_{\hat{\eta}_1}^1 [v(\eta) - v(\hat{\eta}_1)] g(\eta) d\eta = 0 \quad (\text{EC.25})$$

and $\Psi_1(\hat{\eta}_1, \hat{\eta}_2) = 0$, respectively. Moreover, we have $h(\hat{\eta}_2) \leq 0$.

Proof of Lemma [EC.4](#). To prove the first part, define

$$h_2(\eta) \equiv (\lambda_0 + \Delta) \int_{\eta}^1 [v(z) - v(\eta)] g(z) dz + \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] \left[v(\eta) + \frac{\Delta}{\lambda_0} [v(\eta) + 1] \right]. \quad (\text{EC.26})$$

Taking derivative yields

$$\begin{aligned} h'_2(\eta) &= -(\lambda_0 + \Delta) v'(\eta) [1 - G(\eta)] + \frac{d}{d\eta} \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] \left[v(\eta) + \frac{\Delta}{\lambda_0} [v(\eta) + 1] \right] \\ &\quad + \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] \left(\frac{\Delta}{\lambda_0} + 1 \right) v'(\eta) \\ &= \underbrace{\frac{d}{d\eta} \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right]}_{\leq 0} \left[v(\eta) + \frac{\Delta}{\lambda_0} [v(\eta) + 1] \right] + \left(\frac{\Delta}{\lambda_0} + 1 \right) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} v'(\eta), \end{aligned}$$

which is nonnegative if $\eta \leq \bar{\eta}'$. We have

$$h_2(\bar{\eta}') = (\lambda_0 + \Delta) \int_{\bar{\eta}'}^1 [v(z) - v(\bar{\eta}')] g(z) dz \geq 0$$

and

$$h_2(\dot{\eta}_2) = (\lambda_0 + \Delta) \int_{\tilde{\eta}_2}^1 [v(z) - v(\dot{\eta}_2)] g(z) dz + \left[\underline{c}(\dot{\eta}_2) + \frac{F(\underline{c}(\dot{\eta}_2))}{f(\underline{c}(\dot{\eta}_2))} \right] \left[v(\dot{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\dot{\eta}_2) + 1] \right] < 0,$$

where the inequality follows from $\check{c}_2 < \underline{c}(\dot{\eta}_2)$, implying the unique existence of $\tilde{\eta}_2 \in [\dot{\eta}_2, \bar{\eta}']$. The existence of $\tilde{\eta}_1 \in [0, \tilde{\eta}_2]$ is ensured by $v(\tilde{\eta}_2) + 1 \leq 0$. It remains to verify that $\iota(\tilde{\eta}_1, \tilde{\eta}_2) \geq 0$, which follows from

$$\iota(\tilde{\eta}_1, \tilde{\eta}_2) = -\frac{\Xi(\tilde{\eta}_2)}{\lambda_0} \geq -\frac{\Xi(\dot{\eta}_2)}{\lambda_0} = 0.$$

To prove the second part, we first verify that $v_E \leq 0$, which follows from

$$h(\bar{\eta}) \leq 0 \quad \Rightarrow \quad \lambda_1 \underbrace{\int_{\bar{\eta}}^1 v(y) dG(y)}_{>0} + v_E \leq 0 \quad \Rightarrow \quad v_E < 0.$$

Define

$$h_1(\eta) \equiv \frac{1}{\lambda_0} [(\lambda_0 + \Delta) v(\eta) + \Delta] \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] + (\lambda_0 + \Delta) \int_{\eta}^1 [v(z) - v(\eta)] g(z) dz. \quad (\text{EC.27})$$

Taking derivative yields

$$\begin{aligned} h_1'(\eta) &= \frac{1}{\lambda_0} (\lambda_0 + \Delta) v'(\eta) \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] + \frac{1}{\lambda_0} [(\lambda_0 + \Delta) v(\eta) + \Delta] \underbrace{\frac{d}{d\eta} \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right]}_{\leq 0} \\ &\quad - (\lambda_0 + \Delta) v'(\eta) [1 - G(\eta)] \\ &= \frac{1}{\lambda_0} \left\{ (\lambda_0 + \Delta) v'(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} + [(\lambda_0 + \Delta) v(\eta) + \Delta] \frac{d}{d\eta} \left[\underline{c}(\eta) + \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} \right] \right\}. \end{aligned}$$

We know that function h_1 is increasing for any $\eta \in [0, \bar{\eta}']$.

To prove the unique existence of $\dot{\eta}_1$, it suffices to show that, in cases (i) and (ii), we have $h_1(\dot{\eta}_2) \leq 0 \leq h_1(\bar{\eta}')$ and $h_1(\tilde{\eta}_1) \leq 0 \leq h_1(\bar{\eta}')$, respectively. The nonnegative condition follows from

$$h_1(\bar{\eta}') = (\lambda_0 + \Delta) \int_{\bar{\eta}'}^1 [v(z) - v(\bar{\eta}')] g(z) dz \geq 0.$$

It remains to prove the other desired inequality condition.

- When $v(\tilde{\eta}_2) + 1 > 0$, we have

$$\begin{aligned} h_1(\dot{\eta}_2) &= \left[v(\dot{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\dot{\eta}_2) + 1] \right] \left[\underline{c}(\dot{\eta}_2) + \frac{F(\underline{c}(\dot{\eta}_2))}{f(\underline{c}(\dot{\eta}_2))} \right] + (\lambda_0 + \Delta) \int_{\dot{\eta}_2}^1 [v(z) - v(\dot{\eta}_2)] g(z) dz \\ &< 0, \end{aligned}$$

where the inequality follows from $\check{c}_2 < \underline{c}(\dot{\eta}_2)$.

- When $v(\check{\eta}_2) + 1 \geq 0$ and $\check{c}_1 < \underline{c}(\check{\eta}_1)$, we have

$$\begin{aligned} h_1(\check{\eta}_1) &= \left[v(\check{\eta}_1) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_1) + 1] \right] \left[\underline{c}(\check{\eta}_1) + \frac{F(\underline{c}(\check{\eta}_1))}{f(\underline{c}(\check{\eta}_1))} \right] + (\lambda_0 + \Delta) \int_{\check{\eta}_1}^1 [v(z) - v(\check{\eta}_1)] g(z) dz \\ &< \left[v(\check{\eta}_1) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_1) + 1] \right] \left[\underline{c}(\check{\eta}_1) + \frac{F(\underline{c}(\check{\eta}_1))}{f(\underline{c}(\check{\eta}_1))} \right] - v(\check{\eta}_2) \left[\underline{c}(\check{\eta}_1) + \frac{F(\underline{c}(\check{\eta}_1))}{f(\underline{c}(\check{\eta}_1))} \right] \\ &= 0, \end{aligned}$$

where the inequality follows from $\check{c}_1 < \underline{c}(\check{\eta}_1)$.

Therefore, the given conditions guarantee the unique existence of $\check{\eta}_1$.

It remains to verify that $h(\check{\eta}_2) \leq 0$, which follows from

$$\begin{aligned} (i) \quad \check{\eta}_2 \geq \check{\eta}_1 \geq \check{\eta}_2 \geq \check{\eta}_2 &\Rightarrow h(\check{\eta}_2) \leq 0, \text{ and} \\ (ii) \quad \check{\eta}_1 \geq \check{\eta}_1 &\Rightarrow \check{\eta}_2 \geq \check{\eta}_2 \Rightarrow h(\check{\eta}_2) \leq 0, \end{aligned}$$

where $\check{\eta}_2 \geq \check{\eta}_2$ follows from the proof of Lemma [EC.3](#).

This completes the proof. $\square$

LEMMA EC.5. If $\check{c}_2 > \bar{c}(\check{\eta}_1, \check{\eta}_1)$, we have $t_1(\eta^\dagger) \geq 0$, where $\eta^\dagger$ is defined by [\(EC.19\)](#),

$$\begin{aligned} t_1(\eta_2) &\equiv v(\eta_2) \left[\bar{c}(\eta_1(\eta_2), \eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} \right] + (\lambda_0 + \Delta) \int_{\eta_1(\eta_2)}^1 [v(z) - v(\eta_1(\eta_2))] g(z) dz \\ &\quad + \lambda_1 \int_{\eta_2}^1 [v(z) - v(\eta_2)] g(z) dz - v(\eta_2) + v_E \end{aligned} \tag{EC.28}$$

and function $\eta_1(\eta_2)$ is determined by $\Psi_1(\eta_1(\eta_2), \eta_2) = 0$.

Proof of Lemma [EC.5](#). The given condition $\check{c}_2 > \bar{c}(\check{\eta}_1, \check{\eta}_1)$ can be rewritten as

$$(\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] > 0.$$

Recall that the two thresholds $\check{\eta}_1$ and $\check{\eta}_2$ satisfy $\iota(\check{\eta}_1, \check{\eta}_2) = 0$, we have

$$\begin{aligned} &(\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] \\ &= (\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] v(\check{\eta}_2) \\ &\quad + \frac{\Delta}{\lambda_0} \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] [v(\check{\eta}_2) + 1] \\ &\leq (\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] v(\check{\eta}_2) \\ &\quad + \frac{\Delta}{\lambda_0} [1 + \lambda_1 [1 - G(\check{\eta}_1)]] [v(\check{\eta}_2) + 1] + \Delta [1 - G(\check{\eta}_1)] [v(\check{\eta}_2) + 1] \\ &= (\lambda_0 + \lambda_1 + \Delta) \int_{\check{\eta}_1}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\check{\eta}_1, \check{\eta}_1) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_1))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_1))} \right] v(\check{\eta}_2) - v(\check{\eta}_2) + v_E \end{aligned}$$

$$\begin{aligned}
& + \Delta [1 - G(\dot{\eta}_1)] [v(\dot{\eta}_2) + 1] \\
& \leq (\lambda_0 + \lambda_1 + \Delta) \int_{\dot{\eta}_2}^1 [v(\eta) - v(\dot{\eta}_2)] g(\eta) d\eta + \left[\bar{c}(\dot{\eta}_1, \dot{\eta}_1) + \frac{F(\bar{c}(\dot{\eta}_1, \dot{\eta}_1))}{f(\bar{c}(\dot{\eta}_1, \dot{\eta}_1))} \right] v(\dot{\eta}_2) - v(\dot{\eta}_2) + v_E \\
& + \Delta [1 - G(\dot{\eta}_2)] [v(\dot{\eta}_2) + 1],
\end{aligned}$$

where the first inequality and the second equality follow from $v(\dot{\eta}_2) + 1 \leq v(\eta^\dagger) + 1 = 0$ and $\iota(\dot{\eta}_1, \dot{\eta}_2) = 0$, respectively.

Define

$$\begin{aligned}
\Upsilon(\eta_2) \equiv & (\lambda_0 + \lambda_1 + \Delta) \int_{\eta_2}^1 [v(\eta) - v(\eta_2)] g(\eta) d\eta + \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] v(\eta_2) \\
& - v(\eta_2) + v_E + \Delta [1 - G(\eta_2)] [v(\eta_2) + 1], \quad \forall \eta_2 \in [0, \eta^\dagger],
\end{aligned}$$

where $\eta_1(\eta_2)$ is determined by $\Psi_2(\eta_1(\eta_2), \eta_2) = 0$. Taking derivative yields

$$\begin{aligned}
\Upsilon'(\eta_2) = & -(\lambda_0 + \lambda_1 + \Delta) v'(\eta_2) [1 - G(\eta_2)] + \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] v'(\eta_2) \\
& + \frac{d}{d\eta_2} \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] v(\eta_2) - v'(\eta_2) \\
& + \Delta [-[v(\eta_2) + 1] g(\eta_2) + [1 - G(\eta_2)] v'(\eta_2)] \\
\geq & \frac{d}{d\eta_2} \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] v(\eta_2) - \Delta [v(\eta_2) + 1] g(\eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} v'(\eta_2) \\
\geq & 0,
\end{aligned}$$

where the first inequality follows from $\eta_1(\eta_2) \leq \eta_2$ for any $\eta_2 \in [0, \eta^\dagger]$, as established in Lemma [EC.3\(2\)](#). Therefore, we have

$$t_1(\eta^\dagger) = \Upsilon(\eta^\dagger) \geq \Upsilon(\dot{\eta}_2) > 0,$$

which completes the proof. $\square$

LEMMA EC.6. *When one of the following conditions holds, (i) $h(\bar{\eta}) \leq 0$, $\check{c}_2 > \bar{c}(\dot{\eta}_1, \dot{\eta}_1)$, (ii) $h(\bar{\eta}) \leq 0$, $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, or (iii) $h(\bar{\eta}) > 0$, we have the following results.*

(1) *If $t_1(\eta^\dagger) \leq 0$, there uniquely exist $\eta^\dagger \leq \dot{\eta}_1 \leq \dot{\eta}_2 \leq \bar{\eta}$ such that $t_1(\dot{\eta}_2) = 0$, where function t_1 is defined by [\(EC.28\)](#), $\dot{\eta}_1 = \eta_1(\dot{\eta}_2)$, and function $\eta_1(\eta_2)$ is determined by $\Psi_1(\eta_1(\eta_2), \eta_2) = 0$. Moreover, we have $h(\dot{\eta}_2) \geq 0$.*

(2) *If $t_2(\check{\eta}) \leq 0 \leq t_1(\eta^\dagger) = t_2(\eta^\dagger)$, there uniquely exist $\check{\eta} \leq \dot{\eta}_1 \leq \dot{\eta}_2 \leq \eta^\dagger$ such that $t_2(\dot{\eta}_2) = 0$, where $\dot{\eta}_1 = \eta_1(\dot{\eta}_2)$, $\check{\eta}$ is determined by $\eta_1(\check{\eta}) = 0$,*

$$\begin{aligned}
t_2(\eta_2) \equiv & \left[v(\eta_2) + \frac{\Delta}{\lambda_0} [v(\eta_2) + 1] \right] \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] \\
& + (\lambda_0 + \Delta) \int_{\eta_2}^1 [v(\eta) - v(\eta_2)] g(\eta) d\eta - \iota(\eta_1(\eta_2), \eta_2),
\end{aligned} \tag{EC.29}$$

and function $\eta_1(\eta_2)$ is determined by $\Psi_2(\eta_1(\eta_2), \eta_2) = 0$. Moreover, we have $\iota(\dot{\eta}_1, \dot{\eta}_2) \leq 0$, where ι is defined by [\(EC.20\)](#).

Proof of Lemma EC.6. To prove the result in case (1), taking derivative yields

$$\begin{aligned}
t'_1(\eta_2) &= v'(\eta_2) \left[\bar{c}(\eta_1(\eta_2), \eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} \right] + v(\eta_2) \frac{d}{d\eta_2} \left(\bar{c}(\eta_1(\eta_2), \eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} \right) \\
&\quad - (\lambda_0 + \Delta) v'(\eta_1(\eta_2)) [1 - G(\eta_1(\eta_2))] \eta'_1(\eta_2) - \lambda_1 v'(\eta_2) [1 - G(\eta_2)] - v'(\eta_2) \\
&= v'(\eta_2) \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} + v(\eta_2) \frac{d}{d\eta_2} \left(\bar{c}(\eta_1(\eta_2), \eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} \right) \\
&\quad + [\lambda_0 v'(\eta_2) - (\lambda_0 + \Delta) v'(\eta_1(\eta_2)) \eta'_1(\eta_2)] [1 - G(\eta_1(\eta_2))] \\
&= v'(\eta_2) \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} + v(\eta_2) \frac{d}{d\eta_2} \left(\bar{c}(\eta_1(\eta_2), \eta_2) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_2))}{f(\bar{c}(\eta_1(\eta_2), \eta_2))} \right),
\end{aligned}$$

which is positive for any $\eta_2 \in [0, \bar{\eta}]$. The unique existence of $\dot{\eta}_2$ is ensured by the monotonicity of t_1 , $t_1(\eta^\dagger) \leq 0$, and the following:

- In Lemma EC.5, we have proved that given $t_1(\eta^\dagger) \leq 0$, it is impossible to have $\check{c}_2 > \bar{c}(\check{\eta}_1, \check{\eta}_1)$.
- If $h(\bar{\eta}) \leq 0$ and $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, we have

$$\begin{aligned}
t_1(\check{\eta}_2) &= v(\check{\eta}_2) \left[\bar{c}(\check{\eta}_1, \check{\eta}_2) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_2))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_2))} \right] + (\lambda_0 + \Delta) \int_{\check{\eta}_1}^1 [v(z) - v(\check{\eta}_1)] g(z) dz \\
&\quad + \lambda_1 \int_{\check{\eta}_2}^1 [v(z) - v(\check{\eta}_2)] g(z) dz - v(\check{\eta}_2) + v_E \\
&= v(\check{\eta}_2) \left[\bar{c}(\check{\eta}_1, \check{\eta}_2) + \frac{F(\bar{c}(\check{\eta}_1, \check{\eta}_2))}{f(\bar{c}(\check{\eta}_1, \check{\eta}_2))} \right] + (\lambda_0 + \Delta) \int_{\check{\eta}_1}^1 [v(z) - v(\check{\eta}_1)] g(z) dz \\
&\geq 0,
\end{aligned}$$

where the inequality follows from $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$.

- If $h(\bar{\eta}) > 0$, or equivalently,

$$\lambda_1 \int_{\bar{\eta}}^1 v(z) g(z) dz + v_E > 0,$$

we have

$$t_1(\bar{\eta}) = (\lambda_0 + \Delta) \int_{\eta_1(\bar{\eta})}^1 [v(z) - v(\eta_1(\bar{\eta}))] g(z) dz + \underbrace{\lambda_1 \int_{\bar{\eta}}^1 v(z) g(z) dz + v_E}_{>0} > 0.$$

The remaining condition $h(\dot{\eta}_2) \geq 0$ follows from either $\dot{\eta}_2 \leq \check{\eta}_2$ or $h(\bar{\eta}) > 0$.

To prove the second part, differentiating t_2 yields

$$\begin{aligned}
t'_2(\eta_2) &= \left(\frac{\Delta}{\lambda_0} + 1 \right) v'(\eta_2) \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] \\
&\quad + \left[v(\eta_2) + \frac{\Delta}{\lambda_0} [v(\eta_2) + 1] \right] \frac{d}{d\eta_2} \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] \\
&\quad - (\lambda_0 + \Delta) v'(\eta_2) [1 - G(\eta_2)] + \frac{\Xi'(\eta_2)}{\lambda_0} \\
&= \left(\frac{\Delta}{\lambda_0} + 1 \right) v'(\eta_2) \left[1 + (\lambda_0 + \lambda_1) [1 - G(\eta_1(\eta_2))] - \lambda_0 [1 - G(\eta_2)] + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[v(\eta_2) + \frac{\Delta}{\lambda_0} [v(\eta_2) + 1] \right] \frac{d}{d\eta_2} \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] \\
& - \left\{ (\lambda_0 + \lambda_1 + \Delta) [G(\eta_2) - G(\eta_1(\eta_2))] + \lambda_1 [1 - G(\eta_2)] + \frac{\Delta}{\lambda_0} [1 + \lambda_1 [1 - G(\eta_1(\eta_2))]] + 1 \right\} v'(\eta_2) \\
& = \left[v(\eta_2) + \frac{\Delta}{\lambda_0} [v(\eta_2) + 1] \right] \frac{d}{d\eta_2} \left[\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)) + \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))} \right] \\
& + \left(\frac{\Delta}{\lambda_0} + 1 \right) v'(\eta_2) \frac{F(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}{f(\bar{c}(\eta_1(\eta_2), \eta_1(\eta_2)))}.
\end{aligned}$$

For any $\eta_2 \in [0, \bar{\eta}']$, we have $(\lambda_0 + \Delta) v(\eta_2) + \Delta \leq 0$, implying that $t'_2(\eta_2) \geq 0$. The unique existence of $\hat{\eta}_2 \in [\check{\eta}, \eta^\dagger]$ is ensured by $t_2(\check{\eta}) \leq 0 \leq t_2(\eta^\dagger)$ and the monotonicity of t_2 .

By definition of Ξ in Lemma [EC.3](#), to prove the non-positivity of $\iota(\hat{\eta}_1, \hat{\eta}_2)$, it is equivalent to showing that $\Xi(\hat{\eta}_2) \geq 0$. We prove the desired result in the following cases:

- If $h(\bar{\eta}) \leq 0$ and $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, the desired result follows from

$$\iota(\hat{\eta}_1, \hat{\eta}_2) = -\frac{\Xi(\hat{\eta}_2)}{\lambda_0} \leq -\frac{\Xi(\eta^\dagger)}{\lambda_0} \leq -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \leq 0,$$

where the three inequalities follow from $\hat{\eta}_2 \leq \eta^\dagger$, $\eta^\dagger \leq \check{\eta}_2$, and $\Xi(\check{\eta}_2) \geq 0$, respectively;

- If $h(\bar{\eta}) \leq 0$ and $\check{c}_2 > \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$, since $\Xi(\hat{\eta}_2) = 0$, it suffices to verify that $\hat{\eta}_2 \leq \hat{\eta}_2$, or equivalently, $t_2(\hat{\eta}_2) \geq 0$. The desired result follows from

$$\begin{aligned}
t_2(\hat{\eta}_2) &= \left[v(\hat{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\hat{\eta}_2) + 1] \right] \left[\bar{c}(\hat{\eta}_1, \hat{\eta}_1) + \frac{F(\bar{c}(\hat{\eta}_1, \hat{\eta}_1))}{f(\bar{c}(\hat{\eta}_1, \hat{\eta}_1))} \right] \\
&+ (\lambda_0 + \Delta) \int_{\hat{\eta}_2}^1 [v(\eta) - v(\hat{\eta}_2)] g(\eta) d\eta - \underbrace{\iota(\hat{\eta}_1, \hat{\eta}_2)}_{=0} \\
&\geq 0,
\end{aligned}$$

where the inequality is implied by $\check{c}_2 > \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$.

- If $h(\bar{\eta}) > 0$, the desired result follows from

$$\iota(\hat{\eta}_1, \hat{\eta}_2) = -\frac{\Xi(\hat{\eta}_2)}{\lambda_0} \leq -\frac{\Xi(\eta^\dagger)}{\lambda_0} \leq 0,$$

where the second inequality is implied by $\Xi(\eta^\dagger) = \lambda_0 h(\eta^\dagger) \geq \lambda_0 h(\bar{\eta}) \geq 0$.

This completes the proof. $\square$

The following proposition summarizes the values of $\hat{c}$, $\hat{\alpha}$, $\hat{\eta}_1$, and $\hat{\eta}_2$ in all possible cases. We assume $t_2(\check{\eta}) \leq 0$ to exclude scenarios where the optimal $\hat{\eta}_1 = 0$, meaning final approval is granted regardless of the revealed efficacy level. Such cases imply that testing is not utilized, which is unrealistic in practice.

PROPOSITION EC.1. *The values of $\hat{c}$, $\hat{\alpha}$, $\hat{\eta}_1$, and $\hat{\eta}_2$, depend on model parameters in the following cases.*

- (1) If $\underline{c}(\tilde{\eta}_1) \leq \check{c}_1 \leq \bar{c}(\tilde{\eta}_1, \tilde{\eta}_2)$, the optimal policy follows case 1, as defined in (49). We have $\hat{c} = \check{c}_1$, $\hat{\alpha} = \frac{\hat{c} - \lambda_0 [1 - G(\hat{\eta}_1)]}{1 + \lambda_1 [1 - G(\hat{\eta}_2)]}$, $\hat{\eta}_1 = \tilde{\eta}_1$, and $\hat{\eta}_2 = \tilde{\eta}_2$.
- (2) If $\underline{c}(\hat{\eta}_2) \leq \check{c}_2 \leq \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$, the optimal policy follows case 2, as defined in (50). We have $\hat{c} = \check{c}_2$, $\hat{\alpha} = \frac{\hat{c} - \lambda_0 [1 - G(\hat{\eta}_2)]}{1 + \lambda_0 [G(\hat{\eta}_2) - G(\hat{\eta}_1)] + \lambda_1 [1 - G(\hat{\eta}_1)]}$, $\hat{\eta}_1 = \hat{\eta}_1$, and $\hat{\eta}_2 = \hat{\eta}_2$.
- (3) If $\check{c}_1 < \underline{c}(\tilde{\eta}_1)$ or $\check{c}_2 < \underline{c}(\hat{\eta}_2)$, the government never grants conditional approval, i.e., $\hat{\alpha} = 0$, implying that $\hat{\alpha}_3(c, \eta) = 0$ for any $(c, \eta) \in [c_L, c_H] \times [0, 1]$. The thresholds $\hat{c}$, $\hat{\eta}_1$, and $\hat{\eta}_2$ are determined as follows.
- (a) When $\check{c}_2 < \underline{c}(\hat{\eta}_2)$ and $v(\tilde{\eta}_2) + 1 \leq 0$, the optimal policy follows case 2, as defined in (50),
- (i) if $\underline{c}(\tilde{\eta}_2) < c_L$, we have $\hat{c} = c_L$ and $\hat{\eta}_2 = 1$;
 - (ii) if $c_L \leq \underline{c}(\tilde{\eta}_2) < c_H$, we have $\hat{c} = \underline{c}(\tilde{\eta}_2)$ and $\hat{\eta}_2 = \tilde{\eta}_2$;
 - (iii) if $\underline{c}(\tilde{\eta}') < c_H \leq \underline{c}(\tilde{\eta}_2)$, we have $\hat{c} = c_H$ and $\hat{\eta}_2 = \sup \{\eta \in [0, 1] : \underline{c}(\eta) \geq c_H\}$; and
 - (iv) if $c_H \leq \underline{c}(\tilde{\eta}')$, we have $\hat{c} = c_H$ and $\hat{\eta}_2 = \tilde{\eta}'$.
- The other threshold $\hat{\eta}_1$ can take any values between 0 and $\hat{\eta}_2$.
- (b) When (i) $\check{c}_2 < \underline{c}(\hat{\eta}_2)$ and $v(\tilde{\eta}_2) + 1 > 0$, or (ii) $\check{c}_1 < \underline{c}(\tilde{\eta}_1)$, the optimal policy follows case 1, as defined in (49),
- (i) if $\underline{c}(\hat{\eta}_1) < c_L$, we have $\hat{c} = c_L$ and $\hat{\eta}_1 = 1$;
 - (ii) if $c_L \leq \underline{c}(\hat{\eta}_1) < c_H$, we have $\hat{c} = \underline{c}(\hat{\eta}_1)$ and $\hat{\eta}_1 = \hat{\eta}_1$;
 - (iii) if $\underline{c}(\tilde{\eta}') < c_H \leq \underline{c}(\hat{\eta}_1)$, we have $\hat{c} = c_H$ and $\hat{\eta}_1 = \sup \{\eta \in [0, 1] : \underline{c}(\eta) \geq c_H\}$; and
 - (iv) if $c_H \leq \underline{c}(\tilde{\eta}')$, we have $\hat{c} = c_H$ and $\hat{\eta}_1 = \tilde{\eta}'$.
- The other threshold $\hat{\eta}_2$ can take any values between $\hat{\eta}_1$ and 1.
- (4) If $h(\bar{\eta}) > 0$, or $\check{c}_1 > \bar{c}(\tilde{\eta}_1, \tilde{\eta}_2)$, or $\check{c}_2 > \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$, the government grants full conditional approval to firms whose types are below the threshold, i.e., $\hat{\alpha} = 1$. The thresholds $\hat{c}$, $\hat{\eta}_1$, and $\hat{\eta}_2$ are determined as follows.
- (a) When $t_1(\eta^\dagger) \leq 0$, the optimal policy follows case 1, as defined in (49),
- (i) if $\bar{c}(\hat{\eta}_1, \hat{\eta}_2) < c_L$, we have $\hat{c} = c_L$ and $\hat{\eta}_1 = \hat{\eta}_2 = 1$;
 - (ii) if $c_L \leq \bar{c}(\hat{\eta}_1, \hat{\eta}_2) < c_H$, we have $\hat{c} = \bar{c}(\hat{\eta}_1, \hat{\eta}_2)$, $\hat{\eta}_1 = \hat{\eta}_1$, and $\hat{\eta}_2 = \hat{\eta}_2$;
 - (iii) if $\bar{c}(\tilde{\eta}', \bar{\eta}) < c_H \leq \bar{c}(\hat{\eta}_1, \hat{\eta}_2)$, we have $\hat{c} = c_H$ and $(\hat{\eta}_1, \hat{\eta}_2)$ such that $\bar{c}(\hat{\eta}_1, \hat{\eta}_2) = c_H$ and $\Psi_1(\hat{\eta}_1, \hat{\eta}_2) = 0$; and
 - (iv) if $c_H \leq \bar{c}(\tilde{\eta}', \bar{\eta})$, we have $\hat{c} = c_H$, $\hat{\eta}_1 = \tilde{\eta}'$, and $\hat{\eta}_2 = \bar{\eta}$.
- (b) When $0 \leq t_1(\eta^\dagger) = t_2(\eta^\dagger)$, for cases (i)–(iii), the optimal policy follows case 2, as defined in (50); and for cases (iv) and (v), the optimal policy follows case 1, as defined in (49),
- (i) if $\bar{c}(\hat{\eta}_1, \hat{\eta}_1) < c_L$, we have $\hat{c} = c_L$ and $\hat{\eta}_1 = 1$;
 - (ii) if $c_L \leq \bar{c}(\hat{\eta}_1, \hat{\eta}_1) < c_H$, we have $\hat{c} = \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$ and $\hat{\eta}_1 = \hat{\eta}_1$;
 - (iii) if $\bar{c}(\eta^\dagger, \eta^\dagger) \leq c_H \leq \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$, we have $\hat{c} = c_H$ and $\hat{\eta}_1 = \sup \{\eta \in [0, 1] : \bar{c}(\eta, \eta) \geq c_H\}$;

(iv) if $\bar{c}(\bar{\eta}', \bar{\eta}) < c_H \leq \bar{c}(\eta^\dagger, \eta^\dagger)$, we have $\hat{c} = c_H$ and $(\hat{\eta}_1, \hat{\eta}_2)$ such that $\bar{c}(\hat{\eta}_1, \hat{\eta}_2) = c_H$ and $\Psi_1(\hat{\eta}_1, \hat{\eta}_2) = 0$; and

(v) if $c_H \leq \bar{c}(\bar{\eta}', \bar{\eta})$, we have $\hat{c} = c_H$, $\hat{\eta}_1 = \bar{\eta}'$, and $\hat{\eta}_2 = \bar{\eta}$.

For cases (i)–(iii), the other threshold $\hat{\eta}_2$ can take any values between $\hat{\eta}_1$ and 1.

Proof of Proposition EC.1. The mechanism is a “take-it-or-leave-it” offer, and a firm of threshold type $\hat{c} < c_H$ ’s expected payoff is always 0. When $\hat{c} = c_H$, the mechanism is pooling and clearly feasible. Moreover, by Lemmas EC.3–EC.6, we have $\hat{\eta}_1 \leq \hat{\eta}_2$, ensuring the feasibility of our candidate solutions in all cases.

We start with proving the optimality in case (1). Define

$$\begin{aligned} \kappa &\equiv -v(\tilde{\eta}_2), \\ \chi(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\tilde{\eta}_1) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1], \\ \omega(c, \eta) &\equiv \begin{cases} \{(\lambda_0 + \lambda_1 + \Delta) [v(\tilde{\eta}_1) - v(\eta)] + \lambda_1 [v(\tilde{\eta}_2) - v(\tilde{\eta}_1)]\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1], \\ \lambda_1 [v(\tilde{\eta}_2) - v(\eta)] g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_1, \tilde{\eta}_2], \end{cases} \\ \mu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\eta) - v(\tilde{\eta}_1)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_1, 1], \\ \xi(c, \eta) &\equiv \lambda_1 [v(\eta) - v(\tilde{\eta}_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_2, 1], \\ \sigma(c) &\equiv \left\{ (\lambda_0 + \Delta) \int_{\tilde{\eta}_1}^1 [v(\eta) - v(\tilde{\eta}_1)] g(\eta) d\eta + v(\tilde{\eta}_2) \left[c + \frac{F(c)}{f(c)} \right] \right\} f(c), \quad \forall c \in [c_L, \hat{c}], \text{ and} \\ \beta(c) &\equiv -\sigma(c), \quad \forall c \in [\hat{c}, c_H]. \end{aligned}$$

The nonnegativity of κ , χ , the second part of ω , μ , and ξ follows directly from their respective definitions, while the nonnegativity of the first part of ω follows from its monotonicity in η . The nonnegativity of σ and β is ensured by the monotonicity of $\sigma(c)/f(c)$ and $\beta(c)/f(c)$ (Assumption 1), along with the definition of the testing threshold $\hat{c} = \tilde{c}_1$. Because

$$-\kappa - \int_{c_L}^{c_H} \int_{\tilde{\eta}_2}^1 \xi(c, \eta) d\eta dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1]:$$

$$-\chi(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_1, 1]:$$

$$\mu(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1]:$$

$$-\omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}_1, \check{\eta}_2]:$$

$$-\omega(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}_2, 1]:$$

$$\xi(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall c \in [c_L, \hat{c}]:$$

$$\sigma(c) - \int_{\check{\eta}_1}^1 \mu(c, \eta) d\eta - \int_{c_L}^c \int_{\check{\eta}_2}^1 \xi(y, \eta) d\eta dy - c \int_{\check{\eta}_2}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)],$$

$$\forall c \in [\hat{c}, c_H]:$$

$$-\beta(c) - \int_{\check{\eta}_1}^1 \mu(c, \eta) d\eta - \int_{c_L}^c \int_{\check{\eta}_2}^1 \xi(y, \eta) d\eta dy - c \int_{\check{\eta}_2}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)],$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_L}^{\hat{c}} \sigma(c) \tau(c) dc + \int_{\hat{c}}^{c_H} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_{\check{\eta}_1}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\check{\eta}_2} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\check{\eta}_1} \chi(c, \eta) [\hat{\alpha}_3(c, \eta) - \hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\check{\eta}_2}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & = \left[-\kappa - \int_{c_L}^{c_H} \int_{\check{\eta}_2}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{c_L}^{c_H} \int_0^{\check{\eta}_1} \left[-\chi(c, \eta) + \lambda_0 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\check{\eta}_1}^1 \left[\mu(c, \eta) + \lambda_0 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\check{\eta}_1} \left[-\omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\check{\eta}_1}^{\check{\eta}_2} \left[-\omega(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\check{\eta}_2}^1 \left[\xi(c, \eta) + \lambda_1 g(\eta) \int_{\check{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \left[\sigma(c) - \int_{\check{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\check{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\check{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{c_L}^{c_H} \left[-\beta(c) - \int_{\check{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\check{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\check{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\ & + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c). \end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_L}^{\hat{c}} \sigma(c) dc$, which is attained by our candidate primal solution, i.e.,

$$\begin{aligned}
& \int_{c_L}^{\hat{c}} \sigma(c) dc - \left[v_E \hat{\alpha} + \lambda_0 \int_{\check{\eta}_1}^1 v(\eta) dG(\eta) + \lambda_1 \hat{\alpha} \int_{\check{\eta}_2}^1 v(\eta) dG(\eta) + \Delta \int_{\check{\eta}_1}^1 [v(\eta) + 1] dG(\eta) \right] F(\hat{c}) \\
&= (\lambda_0 + \Delta) F(\hat{c}) \int_{\check{\eta}_1}^1 v(\eta) g(\eta) d\eta - (\lambda_0 + \Delta) F(\hat{c}) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] + \hat{c} F(\hat{c}) v(\check{\eta}_2) \\
&\quad - \left[v_E \hat{\alpha} + \lambda_0 \int_{\check{\eta}_1}^1 v(\eta) dG(\eta) + \lambda_1 \hat{\alpha} \int_{\check{\eta}_2}^1 v(\eta) dG(\eta) + \Delta \int_{\check{\eta}_1}^1 [v(\eta) + 1] dG(\eta) \right] F(\hat{c}) \\
&= -(\lambda_0 + \Delta) F(\hat{c}) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] + \hat{c} F(\hat{c}) v(\check{\eta}_2) \\
&\quad - \left[v_E \hat{\alpha} + \underbrace{\lambda_1 \hat{\alpha} \int_{\check{\eta}_2}^1 v(\eta) dG(\eta)}_{=\hat{\alpha} [v(\check{\eta}_2) - v_E + \lambda_1 v(\check{\eta}_2) [1 - G(\check{\eta}_2)]]} + \Delta [1 - G(\check{\eta}_1)] \right] F(\hat{c}) \\
&= -(\lambda_0 + \Delta) F(\hat{c}) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] + \hat{c} F(\hat{c}) v(\check{\eta}_2) - [[\hat{c} - \lambda_0 [1 - G(\check{\eta}_1)]] v(\check{\eta}_2) + \Delta [1 - G(\check{\eta}_1)]] F(\hat{c}) \\
&= [-(\lambda_0 + \Delta) v(\check{\eta}_1) + \lambda_0 v(\check{\eta}_2) - \Delta] [1 - G(\check{\eta}_1)] F(\hat{c}) \\
&= 0.
\end{aligned}$$

To prove the optimality in case (2), define

$$\begin{aligned}
\kappa &\equiv -\frac{1}{\lambda_0} [(\lambda_0 + \Delta) v(\check{\eta}_2) + \Delta], \\
\chi(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\check{\eta}_2) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \check{\eta}_2], \\
\mu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\eta) - v(\check{\eta}_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}_2, 1], \\
\omega(c, \eta) &\equiv \left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\check{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\check{\eta}_2) + 1] \right\} g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \check{\eta}_1], \\
\xi(c, \eta) &\equiv \begin{cases} -\left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\check{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\check{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}_1, \check{\eta}_2], \\ \left\{ \lambda_1 [v(\eta) - v(\check{\eta}_2)] - \frac{\lambda_1}{\lambda_0} \Delta [v(\check{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\check{\eta}_2, 1], \end{cases} \\
\sigma(c) &\equiv \left\{ (\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] \right\} f(c), \quad \forall c \in [c_L, \hat{c}], \\
\beta(c) &\equiv -\sigma(c), \quad \forall c \in [\hat{c}, c_H].
\end{aligned}$$

The nonnegativity of κ follows from $\check{\eta}_2 \leq \eta^\dagger$, as established in Lemma [EC.3\(2\)](#). The nonnegativity of χ and μ follows from their definitions directly. The nonnegativity of ω and ξ follows from the relationship between $\check{\eta}_1$ and $\check{\eta}_2$ (i.e., $\Psi_2(\check{\eta}_1, \check{\eta}_2) = 0$, implying $\omega(c, \check{\eta}_1) = \xi(c, \check{\eta}_1) = 0$ for any $c \in [c_L, c_H]$) and their monotonicity in η . The nonnegativity of σ and β is ensured by the monotonicity of $\sigma(c)/f(c)$ and $\beta(c)/f(c)$, along with the definition of the testing threshold $\hat{c} = \check{c}_2$. Because

$$-\kappa - \int_{c_L}^{c_H} \int_{\check{\eta}_1}^1 \xi(c, \eta) d\eta dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \mathring{\eta}_2]:$$

$$-\chi(c, \eta) + \lambda_0 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\mathring{\eta}_2, 1]:$$

$$\mu(c, \eta) + \lambda_0 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \mathring{\eta}_1]:$$

$$-\omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\mathring{\eta}_1, \mathring{\eta}_2]:$$

$$\chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\mathring{\eta}_2, 1]:$$

$$\xi(c, \eta) + \lambda_1 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall c \in [c_L, \hat{c}]:$$

$$\sigma(c) - \int_{\mathring{\eta}_2}^1 \mu(c, \eta) d\eta - \int_{c_L}^c \int_{\mathring{\eta}_1}^1 \xi(y, \eta) d\eta dy - c \int_{\mathring{\eta}_1}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)],$$

$$\forall c \in [\hat{c}, c_H]:$$

$$-\beta(c) - \int_{\mathring{\eta}_2}^1 \mu(c, \eta) d\eta - \int_{c_L}^c \int_{\mathring{\eta}_1}^1 \xi(y, \eta) d\eta dy - c \int_{\mathring{\eta}_1}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)],$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_L}^{\hat{c}} \sigma(c) \tau(c) dc + \int_{\hat{c}}^{c_H} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_{\mathring{\eta}_2}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\mathring{\eta}_1} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\mathring{\eta}_2} \chi(c, \eta) [\hat{\alpha}_3(c, \eta) - \hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\mathring{\eta}_1}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & = \left[-\kappa - \int_{c_L}^{c_H} \int_{\mathring{\eta}_1}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{c_L}^{c_H} \int_0^{\mathring{\eta}_2} \left[-\chi(c, \eta) + \lambda_0 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\mathring{\eta}_2}^1 \left[\mu(c, \eta) + \lambda_0 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\mathring{\eta}_1} \left[-\omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\mathring{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \end{aligned}$$

$$\begin{aligned}
& + \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^{\hat{\eta}_2} \left[\chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \left[\xi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \left[\sigma(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\
& + \int_{c_L}^{c_H} \left[-\beta(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\
& = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\
& + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c).
\end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_L}^{\hat{c}} \sigma(c) dc$, which is attained by our candidate primal solution, i.e.,

$$\begin{aligned}
& \int_{c_L}^{\hat{c}} \sigma(c) dc - \left[v_E \hat{\alpha} + \hat{\alpha} \int_{\hat{\eta}_1}^{\hat{\eta}_2} [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \right. \\
& \quad \left. + \int_{\hat{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \hat{\alpha} \int_{\hat{\eta}_1}^1 v(\eta) dG(\eta) \right] F(\hat{c}) \\
& = (\lambda_0 + \Delta) F(\hat{c}) \int_{\hat{\eta}_2}^1 [v(\eta) - v(\hat{\eta}_2)] g(\eta) d\eta + \hat{c} F(\hat{c}) \left[v(\hat{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\hat{\eta}_2) + 1] \right] \\
& \quad - \left[v_E \hat{\alpha} + \hat{\alpha} \int_{\hat{\eta}_1}^{\hat{\eta}_2} [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \right. \\
& \quad \left. + \int_{\hat{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \hat{\alpha} \int_{\hat{\eta}_1}^1 v(\eta) dG(\eta) \right] F(\hat{c}) \\
& = \left\{ -(\lambda_0 + \Delta) v(\hat{\eta}_2) [1 - G(\hat{\eta}_2)] + \hat{c} \left[v(\hat{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\hat{\eta}_2) + 1] \right] \right. \\
& \quad \left. - \left[v_E \hat{\alpha} + \hat{\alpha} \int_{\hat{\eta}_1}^{\hat{\eta}_2} [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \Delta [1 - G(\hat{\eta}_2)] + \lambda_1 \hat{\alpha} \int_{\hat{\eta}_1}^1 v(\eta) dG(\eta) \right] \right\} F(\hat{c}),
\end{aligned}$$

where $\hat{\alpha} = \frac{\hat{c} - \lambda_0 [1 - G(\hat{\eta}_2)]}{1 + \lambda_0 [G(\hat{\eta}_2) - G(\hat{\eta}_1)] + \lambda_1 [1 - G(\hat{\eta}_1)]}$ represents the scale of conditional approval. Because

$$\begin{aligned}
\lambda_1 \int_{\hat{\eta}_1}^1 v(\eta) dG(\eta) & = \lambda_1 \left[\int_{\hat{\eta}_1}^{\hat{\eta}_2} v(\eta) dG(\eta) + \int_{\hat{\eta}_2}^1 v(\eta) dG(\eta) \right] \\
& = \lambda_1 \int_{\hat{\eta}_1}^{\hat{\eta}_2} v(\eta) dG(\eta) + [v(\hat{\eta}_2) - v_E] + \frac{\Delta}{\lambda_0} [v(\hat{\eta}_2) + 1] [1 + \lambda_1 [1 - G(\hat{\eta}_1)]] \\
& \quad + (\lambda_0 + \lambda_1 + \Delta) \int_{\hat{\eta}_1}^{\hat{\eta}_2} [v(\hat{\eta}_2) - v(z)] g(z) dz + \lambda_1 v(\hat{\eta}_2) [1 - G(\hat{\eta}_2)],
\end{aligned}$$

where the second equality follows from $\iota(\hat{\eta}_1, \hat{\eta}_2) = 0$, and

$$\hat{c} = \hat{\alpha} [\lambda_0 [G(\hat{\eta}_2) - G(\hat{\eta}_1)] + \lambda_1 [1 - G(\hat{\eta}_1)] + 1] + \lambda_0 [1 - G(\hat{\eta}_2)],$$

the duality gap boils down to

$$-(\lambda_0 + \Delta) v(\hat{\eta}_2) [1 - G(\hat{\eta}_2)] - \Delta [1 - G(\hat{\eta}_2)] + \hat{\alpha} [G(\hat{\eta}_2) - G(\hat{\eta}_1)],$$

$$\begin{aligned}
& + [\hat{\alpha} [\lambda_0 [G(\tilde{\eta}_2) - G(\tilde{\eta}_1)] + \lambda_1 [1 - G(\tilde{\eta}_1)]] + \lambda_0 [1 - G(\tilde{\eta}_2)] v(\tilde{\eta}_2) \\
& + [\hat{\alpha} \lambda_0 [G(\tilde{\eta}_2) - G(\tilde{\eta}_1)] + \lambda_0 [1 - G(\tilde{\eta}_2)]] \frac{\Delta}{\lambda_0} [v(\tilde{\eta}_2) + 1] \\
& - \hat{\alpha} [(\lambda_0 + \lambda_1 + \Delta) v(\tilde{\eta}_2) [G(\tilde{\eta}_2) - G(\tilde{\eta}_1)] + \lambda_1 v(\tilde{\eta}_2) [1 - G(\tilde{\eta}_2)]] \\
& = 0.
\end{aligned}$$

To prove the optimality of cases (3)(a)(i) and (3)(a)(ii), define

$$\begin{aligned}
\kappa & \equiv -\frac{1}{\lambda_0} [(\lambda_0 + \Delta) v(\tilde{\eta}_2) + \Delta], \\
\chi(c, \eta) & \equiv (\lambda_0 + \Delta) [v(\tilde{\eta}_2) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_2], \\
\mu(c, \eta) & \equiv (\lambda_0 + \Delta) [v(\eta) - v(\tilde{\eta}_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_2, 1], \\
\omega(c, \eta) & \equiv \left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\tilde{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\tilde{\eta}_2) + 1] \right\} g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1], \\
\xi(c, \eta) & \equiv \begin{cases} -\left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\tilde{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\tilde{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_1, \tilde{\eta}_2], \\ \left\{ \lambda_1 [v(\eta) - v(\tilde{\eta}_2)] - \frac{\lambda_1}{\lambda_0} \Delta [v(\tilde{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_2, 1], \end{cases} \\
\phi(c) & \equiv \iota(\tilde{\eta}_1, \tilde{\eta}_2) f(c), \quad \forall c \in [c_L, c_H], \\
\sigma(c) & \equiv \left\{ (\lambda_0 + \Delta) \int_{\tilde{\eta}_2}^1 [v(\eta) - v(\tilde{\eta}_2)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\tilde{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\tilde{\eta}_2) + 1] \right] \right\} f(c), \quad \forall c \in \mathcal{C}_1, \\
\beta(c) & \equiv -\sigma(c), \quad \forall c \in \mathcal{C}_2,
\end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are subsets of $[c_L, c_H]$ satisfying: (i) $\mathcal{C}_1 \cup \mathcal{C}_2 = [c_L, c_H]$; and (ii) the measure of $\mathcal{C}_1 \cap \mathcal{C}_2$ is 0. These subsets will be specified later.

The nonnegativity of κ is implied by $\tilde{\eta}_2 \leq \tilde{\eta}'$, as established in Lemma [EC.4](#)(1). The nonnegativity of χ and μ follows from their definitions directly. The nonnegativity of ω and ξ follows from the relationship between $\tilde{\eta}_1$ and $\tilde{\eta}_2$ (i.e., $\Psi_2(\tilde{\eta}_1, \tilde{\eta}_2) = 0$, implying $\omega(c, \tilde{\eta}_1) = \xi(c, \tilde{\eta}_1) = 0$ for any $c \in [c_L, c_H]$) and their monotonicity in η . The nonnegativity of ϕ follows from the fact that $\iota(\tilde{\eta}_1, \tilde{\eta}_2) \geq 0$, as established in Lemma [EC.4](#)(1).

In the first two possible cases, we have:

1. if $\underline{c}(\tilde{\eta}_2) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$; and
2. if $c_L \leq \underline{c}(\tilde{\eta}_2) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \underline{c}(\tilde{\eta}_2)]$ and $[\underline{c}(\tilde{\eta}_2), c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \underline{c}(\tilde{\eta}_2)]$ and $\mathcal{C}_2 = [\underline{c}(\tilde{\eta}_2), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa - \int_{c_L}^{c_H} \phi(c) dc - \int_{c_L}^{c_H} \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta dc = v_E,$$

$$\forall(c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_2]:$$

$$\lambda_0 \phi(c) g(\eta) - \chi(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall(c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_2, 1]:$$

$$\lambda_0 \phi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall(c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}_1]:$$

$$\lambda_1 \phi(c) g(\eta) - \omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall(c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_1, \tilde{\eta}_2]:$$

$$\lambda_1 \phi(c) g(\eta) + \chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall(c, \eta) \in [c_L, c_H] \times [\tilde{\eta}_2, 1]:$$

$$\lambda_1 \phi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall c \in \mathcal{C}_1:$$

$$\begin{aligned} & - \int_{c_L}^c \phi(y) dy - c \phi(c) + \sigma(c) - c \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\tilde{\eta}_1}^1 \xi(y, \eta) d\eta dy - \int_{\tilde{\eta}_2}^1 \mu(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

$$\forall c \in \mathcal{C}_2:$$

$$\begin{aligned} & - \int_{c_L}^c \phi(y) dy - c \phi(c) - \beta(c) - c \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\tilde{\eta}_1}^1 \xi(y, \eta) d\eta dy - \int_{\tilde{\eta}_2}^1 \mu(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_1} \sigma(c) \tau(c) dc + \int_{c_2} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_{\tilde{\eta}_2}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}_1} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}_2} \chi(c, \eta) [\hat{\alpha}_3(c, \eta) - \hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\tilde{\eta}_1}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & + \int_{c_L}^{c_H} \phi(c) \left[-\Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) + \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta) \right] dc \\ & = \left[-\kappa - \int_{c_L}^{c_H} \phi(c) dc - \int_{c_L}^{c_H} \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \end{aligned}$$

$$\begin{aligned}
& + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}_2} \left[\lambda_0 \phi(c) g(\eta) - \chi(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\tilde{\eta}_2}^1 \left[\lambda_0 \phi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}_1} \left[\lambda_1 \phi(c) g(\eta) - \omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} \left[\lambda_1 \phi(c) g(\eta) + \chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\tilde{\eta}_2}^1 \left[\lambda_1 \phi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\tilde{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\
& + \int_{\mathcal{C}_1} \left[- \int_{c_L}^c \phi(y) dy - c \phi(c) + \sigma(c) - \int_{\tilde{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\tilde{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\
& + \int_{\mathcal{C}_2} \left[- \int_{c_L}^c \phi(y) dy - c \phi(c) - \beta(c) - \int_{\tilde{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\tilde{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\tilde{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\
& = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\
& + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c).
\end{aligned}$$

Therefore, the objective function is bounded by $\int_{\mathcal{C}_1} \sigma(c) dc$. Substituting our candidate solution into the objective function achieves the upper bound in the first two cases:

1. If $\underline{c}(\tilde{\eta}_2) \leq c_L$, we have $\hat{c} = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff from our candidate policy and its upper bound are both 0.
2. If $c_L < \underline{c}(\tilde{\eta}_2) \leq c_H$, we have

$$\begin{aligned}
& \int_{c_L}^{\underline{c}(\tilde{\eta}_2)} \sigma(c) dc - F(\underline{c}(\tilde{\eta}_2)) \int_{\tilde{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
& = \int_{c_L}^{\underline{c}(\tilde{\eta}_2)} \left\{ (\lambda_0 + \Delta) \int_{\tilde{\eta}_2}^1 [v(\eta) - v(\tilde{\eta}_2)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\tilde{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\tilde{\eta}_2) + 1] \right] \right\} f(c) dc \\
& \quad - F(\underline{c}(\tilde{\eta}_2)) \int_{\tilde{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
& = \left\{ -(\lambda_0 + \Delta) v(\tilde{\eta}_2) [1 - G(\tilde{\eta}_2)] + \underline{c}(\tilde{\eta}_2) \left[v(\tilde{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\tilde{\eta}_2) + 1] \right] - \Delta [1 - G(\tilde{\eta}_2)] \right\} F(\underline{c}(\tilde{\eta}_2)) \\
& = 0,
\end{aligned}$$

establishing the optimality.

In case (3)(a)(iii), we set $\mathcal{C}_1 = [c_L, c_H]$ and $\mathcal{C}_2 = \emptyset$, and replace the thresholds $\tilde{\eta}_1$ and $\tilde{\eta}_2$ with $\eta_1(\tilde{\eta}_2)$ and $\check{\eta}_2$, respectively, where $\underline{c}(\check{\eta}_2) = c_H$, and $\eta_1(\eta_2)$ is obtained by solving $\Psi_2(\eta_1(\eta_2), \eta_2) = 0$. In this case, the nonnegativity of κ , χ , μ , ω , and ξ follows from their definitions directly. The desired nonnegativity of ϕ follows from

$$\iota(\check{\eta}_1, \check{\eta}_2) = -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \geq -\frac{\Xi(\tilde{\eta}_2)}{\lambda_0} = \iota(\tilde{\eta}_1, \tilde{\eta}_2) \geq 0,$$

where function Ξ is defined by (EC.21). The first inequality follows from the monotonicity of Ξ and the fact that $\check{\eta}_2 \geq \tilde{\eta}_2$. Due to the monotonicity of σ , its desired nonnegativity condition boils down to $\sigma(c_H) \geq 0$, which follows from

$$\sigma(c_H) = h_2(\check{\eta}_2) f(c_H) \geq h_2(\tilde{\eta}_2) f(c_H) = 0,$$

where the function h_2 is defined in (EC.26), and the inequality follows from the monotonicity of h_2 , as established in Lemma EC.4. Because all the multipliers remain nonnegative, the objective function is bounded by $\int_{c_L}^{c_H} \sigma(c) dc$, which is attained by our candidate solution, i.e.,

$$\begin{aligned} & \int_{c_L}^{c_H} \sigma(c) dc - \int_{\check{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\ &= \int_{c_L}^{c_H} \left\{ (\lambda_0 + \Delta) \int_{\check{\eta}_2}^1 [v(\eta) - v(\check{\eta}_2)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] \right\} f(c) dc \\ & \quad - \int_{\check{\eta}_2}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\ &= -(\lambda_0 + \Delta) v(\check{\eta}_2) [1 - G(\check{\eta}_2)] + c_H \left[v(\check{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] \right] - \Delta [1 - G(\check{\eta}_2)] \\ &= 0, \end{aligned}$$

where the last step follows from $c_H = \lambda_0 [1 - G(\check{\eta}_2)]$.

In case (3)(a)(iv), note that the integrand of the objective function (43) is bounded above by

$$\begin{aligned} & v_E \alpha_1(c) + \int_0^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] \hat{\alpha}_2(c, \eta) dG(\eta) + \lambda_1 \int_0^1 v(\eta) \hat{\alpha}_3(c, \eta) dG(\eta) \\ &\leq v_E \alpha_1(c) + \int_{\bar{\eta}'}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] \hat{\alpha}_2(c, \eta) dG(\eta) + \lambda_1 \int_{\bar{\eta}}^1 v(\eta) \alpha_1(c) dG(\eta) \\ &= \left[v_E + \lambda_1 \int_{\bar{\eta}}^1 v(\eta) dG(\eta) \right] \alpha_1(c) + \int_{\bar{\eta}'}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] \hat{\alpha}_2(c, \eta) dG(\eta) \\ &\leq \int_{\bar{\eta}'}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] \hat{\alpha}_2(c, \eta) dG(\eta), \end{aligned}$$

where the first inequality follows from the definitions of $\bar{\eta}$ and $\bar{\eta}'$ and the upper bound of $\hat{\alpha}_3(c, \eta)$, while the second inequality follows from the given condition $h(\bar{\eta}) \leq 0$. This upper bound is attained by committing to no conditional approval and adopting the myopic threshold ($\hat{\eta}_2 = \bar{\eta}'$). When $c_H \leq \underline{c}(\bar{\eta}')$, this policy enables the government to induce firms of all types to participate. Therefore, the mechanism is optimal under the given conditions.

To prove the optimality of cases (3)(b)(i) and (3)(b)(ii), define

$$\begin{aligned} \kappa &\equiv -v(\hat{\eta}_2), \\ \nu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\hat{\eta}_1) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \hat{\eta}_1], \end{aligned}$$

$$\begin{aligned}
\mu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\eta) - v(\eta_1)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\eta_1, 1], \\
\omega(c, \eta) &\equiv \lambda_1 [v(\eta_2) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \eta_2], \\
\xi(c, \eta) &\equiv \lambda_1 [v(\eta) - v(\eta_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\eta_2, 1], \\
\phi(c) &\equiv \left\{ -\lambda_1 \int_{\eta_2}^1 [v(z) - v(\eta_2)] g(z) dz + v(\eta_2) - v_E \right\} f(c), \quad \forall c \in [c_L, c_H], \\
\sigma(c) &\equiv \left\{ v(\eta_2) \left[c + \frac{F(c)}{f(c)} \right] + (\lambda_0 + \Delta) \int_{\eta_1}^1 [v(\eta) - v(\eta_1)] g(\eta) d\eta \right\} f(c), \quad \forall c \in \mathcal{C}_1, \\
\beta(c) &\equiv -\sigma(c), \quad \forall c \in \mathcal{C}_2.
\end{aligned}$$

The nonnegativity of κ follows from the fact that $\eta_2 \leq \bar{\eta}' \leq \bar{\eta}$. The nonnegativity of ν , μ , ω , and ξ follows from their respective definitions directly. The nonnegativity of ϕ follows from Lemma [EC.4\(2\)](#).

In the first two possible cases, we have:

1. if $\underline{c}(\eta_1) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$; and
2. if $c_L \leq \underline{c}(\eta_1) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \underline{c}(\eta_1)]$ and $[\underline{c}(\eta_1), c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \underline{c}(\eta_1)]$ and $\mathcal{C}_2 = [\underline{c}(\eta_1), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa - \int_{c_L}^{c_H} \phi(c) dc - \int_{c_L}^{c_H} \int_{\eta_2}^1 \xi(c, \eta) d\eta dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \eta_1]:$$

$$\lambda_0 \phi(c) g(\eta) - \nu(c, \eta) + \lambda_0 g(\eta) \int_{\eta_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\eta_1, 1]:$$

$$\lambda_0 \phi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\eta_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \eta_2]:$$

$$\lambda_1 \phi(c) g(\eta) - \omega(c, \eta) + \lambda_1 g(\eta) \int_{\eta_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\eta_2, 1]:$$

$$\lambda_1 \phi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\eta_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$\forall c \in \mathcal{C}_1$:

$$\begin{aligned} & - \int_{c_L}^c \phi(y) dy - c\phi(c) + \sigma(c) - c \int_{\hat{\eta}_2}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\hat{\eta}_2}^1 \xi(y, \eta) d\eta dy - \int_{\hat{\eta}_1}^1 \mu(c, \eta) d\eta = v_E [cf(c) + F(c)], \end{aligned}$$

$\forall c \in \mathcal{C}_2$:

$$\begin{aligned} & - \int_{c_L}^c \phi(y) dy - c\phi(c) - \beta(c) - c \int_{\hat{\eta}_2}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\hat{\eta}_2}^1 \xi(y, \eta) d\eta dy - \int_{\hat{\eta}_1}^1 \mu(c, \eta) d\eta = v_E [cf(c) + F(c)], \end{aligned}$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_1} \sigma(c) \tau(c) dc + \int_{c_2} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_1} \nu(c, \eta) [-\hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_2} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c\tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & + \int_{c_L}^{c_H} \phi(c) \left[-\Pi(c_H) - \int_c^{c_H} \tau(y) dy - c\tau(c) + \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) + \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta) \right] dc \\ & = \left[-\kappa - \int_{c_L}^{c_H} \phi(c) dc - \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_1} \left[\lambda_0 \phi(c) g(\eta) - \nu(c, \eta) + \lambda_0 g(\eta) \int_{\hat{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^1 \left[\lambda_0 \phi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\hat{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_2} \left[\lambda_1 \phi(c) g(\eta) - \omega(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \left[\lambda_1 \phi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_1} \left[- \int_{c_L}^c \phi(y) dy - c\phi(c) + \sigma(c) - \int_{\hat{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{c_2} \left[- \int_{c_L}^c \phi(y) dy - c\phi(c) - \beta(c) - \int_{\hat{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\ & + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c). \end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_1} \sigma(c) dc$. Substituting our candidate solution into the objective function achieves the upper bound in the first two cases:

1. If $\underline{c}(\hat{\eta}_1) \leq c_L$, we have $\hat{c} = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff from our candidate policy and its upper bound are both 0.

2. If $c_L < \underline{c}(\eta_1) \leq c_H$, we have

$$\begin{aligned}
& \int_{c_L}^{\underline{c}(\eta_1)} \sigma(c) dc - F(\underline{c}(\eta_1)) \int_{\eta_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
&= \int_{c_L}^{\underline{c}(\eta_1)} \left\{ v(\eta_2) \left[c + \frac{F(c)}{f(c)} \right] + (\lambda_0 + \Delta) \int_{\eta_1}^1 [v(\eta) - v(\eta_1)] g(\eta) d\eta \right\} f(c) dc \\
&\quad - F(\underline{c}(\eta_1)) \int_{\eta_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
&= [\lambda_0 v(\eta_2) - (\lambda_0 + \Delta) v(\eta_1) - \Delta] [1 - G(\eta_1)] F(\underline{c}(\eta_1)) \\
&= 0,
\end{aligned}$$

where the last step follows from $\Psi_1(\eta_1, \eta_2) = 0$.

In case (3)(b)(iii), we set $\mathcal{C}_1 = [c_L, c_H]$ and $\mathcal{C}_2 = \emptyset$, and replace the thresholds η_1 and η_2 with $\check{\eta}_1$ and $\eta_2(\check{\eta}_1)$, respectively, where $\check{\eta}_1$ is determined by $\underline{c}(\check{\eta}_1) = c_H$, and $\eta_2(\eta_1)$ is obtained by solving $\Psi_1(\eta_1, \eta_2(\eta_1)) = 0$. In this case, the nonnegativity of κ , ν , μ , ω , and ξ follows from their definitions directly. The desired nonnegativity of ϕ follows from

$$h(\check{\eta}_2) \leq h(\eta_2) \leq 0.$$

The nonnegativity of σ follows from its monotonicity and

$$\sigma(c_H) = h_1(\check{\eta}_1) f(c_H) \geq h_1(\eta_1) f(c_H) = 0,$$

where the function h_1 is defined in [\(EC.27\)](#), and the inequality follows from the monotonicity of function h_1 , as established in Lemma [EC.4](#). Because all the multipliers remain nonnegative, the objective function is bounded by $\int_{c_L}^{c_H} \sigma(c) dc$, which is attained by our candidate solution, i.e.,

$$\begin{aligned}
& \int_{c_L}^{c_H} \sigma(c) dc - \int_{\check{\eta}_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
&= \int_{c_L}^{c_H} \left\{ (\lambda_0 + \Delta) \int_{\check{\eta}_1}^1 [v(\eta) - v(\check{\eta}_1)] g(\eta) d\eta + \left[c + \frac{F(c)}{f(c)} \right] \left[v(\check{\eta}_1) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_1) + 1] \right] \right\} f(c) dc \\
&\quad - \int_{\check{\eta}_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) \\
&= -(\lambda_0 + \Delta) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] + c_H \left[v(\check{\eta}_1) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_1) + 1] \right] - \Delta [1 - G(\check{\eta}_1)] \\
&= 0,
\end{aligned}$$

where the last step follows from $c_H = \lambda_0 [1 - G(\check{\eta}_1)]$.

In case (3)(b)(iv), similar to case (3)(a)(iv), committing to no conditional approval and adopting the myopic threshold ($\hat{\eta}_1 = \bar{\eta}'$) enables the government to induce all firms to participate. Therefore, this mechanism is optimal under the given conditions.

It remains to prove the optimality of the last case. For (4)(a)(i) and (4)(a)(ii), define

$$\begin{aligned}
\kappa &\equiv -v(\dot{\eta}_2), \\
\nu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\dot{\eta}_1) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_1], \\
\mu(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\eta) - v(\dot{\eta}_1)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_1, 1], \\
\omega(c, \eta) &\equiv \lambda_1 [v(\dot{\eta}_2) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_2], \\
\xi(c, \eta) &\equiv \lambda_1 [v(\eta) - v(\dot{\eta}_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1], \\
\psi(c) &\equiv \left\{ \lambda_1 \int_{\dot{\eta}_2}^1 [v(z) - v(\dot{\eta}_2)] g(z) dz - v(\dot{\eta}_2) + v_E \right\} f(c), \\
\sigma(c) &\equiv \left\{ v(\dot{\eta}_2) \left[c + \frac{F(c)}{f(c)} \right] + (\lambda_0 + \Delta) \int_{\dot{\eta}_1}^1 [v(\eta) - v(\dot{\eta}_1)] g(\eta) d\eta \right. \\
&\quad \left. + \lambda_1 \int_{\dot{\eta}_2}^1 [v(z) - v(\dot{\eta}_2)] g(z) dz - v(\dot{\eta}_2) + v_E \right\} f(c), \quad \forall c \in \mathcal{C}_1, \\
\beta(c) &\equiv -\sigma(c), \quad \forall c \in \mathcal{C}_2.
\end{aligned}$$

The nonnegativity of κ follows from the fact that $\dot{\eta}_2 \leq \bar{\eta}$. The nonnegativity of ν , μ , ω , and ξ follows from their respective definitions directly. The nonnegativity of ψ follows from Lemma [EC.6\(1\)](#).

In the first two possible cases, we have:

1. if $\bar{c}(\dot{\eta}_1, \dot{\eta}_2) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$; and
2. if $c_L \leq \bar{c}(\dot{\eta}_1, \dot{\eta}_2) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \bar{c}(\dot{\eta}_1, \dot{\eta}_2)]$ and $[\bar{c}(\dot{\eta}_1, \dot{\eta}_2), c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \bar{c}(\dot{\eta}_1, \dot{\eta}_2)]$ and $\mathcal{C}_2 = [\bar{c}(\dot{\eta}_1, \dot{\eta}_2), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \int_{c_L}^{c_H} \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta dc = v_E,$$

$\forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_1]$:

$$-\lambda_0 \psi(c) g(\eta) - \nu(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$\forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_1, 1]$:

$$-\lambda_0 \psi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$\forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_2]$:

$$-\lambda_1 \psi(c) g(\eta) - \omega(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1]:$$

$$-\lambda_1 \psi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall c \in [c_L, \hat{c}]:$$

$$\begin{aligned} & \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) + \sigma(c) - c \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\dot{\eta}_2}^1 \xi(y, \eta) d\eta dy - \int_{\dot{\eta}_1}^1 \mu(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

$$\forall c \in [\hat{c}, c_H]:$$

$$\begin{aligned} & \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - \beta(c) - c \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\dot{\eta}_2}^1 \xi(y, \eta) d\eta dy - \int_{\dot{\eta}_1}^1 \mu(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_1} \sigma(c) \tau(c) dc + \int_{c_2} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_0^{\dot{\eta}_1} \nu(c, \eta) [-\hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\dot{\eta}_1}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\dot{\eta}_2} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\dot{\eta}_2}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & + \int_{c_L}^{c_H} \psi(c) \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) - \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta) - \tau(c) \right] dc \\ & = \left[-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \int_{c_L}^{c_H} \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{c_L}^{c_H} \int_0^{\dot{\eta}_1} \left[-\lambda_0 \psi(c) g(\eta) - \nu(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\dot{\eta}_1}^1 \left[-\lambda_0 \psi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\dot{\eta}_2} \left[-\lambda_1 \psi(c) g(\eta) - \omega(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_2}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\dot{\eta}_2}^1 \left[-\lambda_1 \psi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_1} \left[\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) + \sigma(c) - \int_{\dot{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\dot{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{c_2} \left[\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - \beta(c) - \int_{\dot{\eta}_1}^1 \mu(c, \eta) d\eta - c \int_{\dot{\eta}_2}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\dot{\eta}_2}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\ & + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c). \end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_1} \sigma(c) dc$. Substituting our candidate solution into the objective function achieves the upper bound in the first two cases, which follows from:

1. if $\bar{c}(\dot{\eta}_1, \dot{\eta}_2) \leq c_L$, we have $\hat{c} = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff from our candidate policy and its upper bound are both 0;
2. if $c_L < \bar{c}(\dot{\eta}_1, \dot{\eta}_2) \leq c_H$, the government's expected payoff from our candidate policy attains this upper bound, i.e.,

$$\begin{aligned}
& \int_{c_L}^{\bar{c}(\dot{\eta}_1, \dot{\eta}_2)} \sigma(c) dc - \left[v_E + \int_{\dot{\eta}_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \int_{\dot{\eta}_2}^1 v(\eta) dG(\eta) \right] F(\bar{c}(\dot{\eta}_1, \dot{\eta}_2)) \\
&= \{v(\dot{\eta}_2) [1 + \lambda_0 [1 - G(\dot{\eta}_1)] + \lambda_1 [1 - G(\dot{\eta}_2)]] - \Delta [1 - G(\dot{\eta}_1)] \\
&\quad - (\lambda_0 + \Delta) v(\dot{\eta}_1) [1 - G(\dot{\eta}_1)] - \lambda_1 v(\dot{\eta}_2) [1 - G(\dot{\eta}_2)] - v(\dot{\eta}_2)\} F(\bar{c}(\dot{\eta}_1, \dot{\eta}_2)) \\
&= [\lambda_0 v(\dot{\eta}_2) - \Delta - (\lambda_0 + \Delta) v(\dot{\eta}_1)] [1 - G(\dot{\eta}_1)] F(\bar{c}(\dot{\eta}_1, \dot{\eta}_2)) \\
&= 0,
\end{aligned}$$

where the last step follows from $\Psi_1(\dot{\eta}_1, \dot{\eta}_2) = 0$.

In case (4)(a)(iii), we set $\mathcal{C}_1 = [c_L, c_H]$ and $\mathcal{C}_2 = \emptyset$, and replace the thresholds $\dot{\eta}_1$ and $\dot{\eta}_2$ with $\check{\eta}_1$ and $\check{\eta}_2$, respectively, where the two thresholds are determined by $\bar{c}(\check{\eta}_1, \check{\eta}_2) = c_H$ and $\Psi_1(\check{\eta}_1, \check{\eta}_2) = 0$. In this case, the nonnegativity of κ , ν , μ , ω , and ξ follows from their definitions directly. The nonnegativity of ψ follows as: if $h(\bar{\eta}) \leq 0$, we have $h(\check{\eta}_2) \geq h(\bar{\eta}_2) = 0$; otherwise, we have $h(\check{\eta}) \geq h(\bar{\eta}) > 0$. The nonnegativity of σ follows from the fact that for any $c \in [c_L, c_H]$, we have

$$\sigma(c) \geq \sigma(c_H) = t_1(\check{\eta}_2) f(c_H) \geq t_1(\dot{\eta}_2) f(c_H) = 0,$$

where the function t_1 is defined in (EC.28), and the inequality follows from the monotonicity of t_1 , as established in Lemma EC.6. Because all the multipliers remain nonnegative, the objective function is bounded by $\int_{c_L}^{c_H} \sigma(c) dc$, which is attained by our candidate solution, i.e.,

$$\begin{aligned}
& \int_{c_L}^{c_H} \sigma(c) dc - \left[v_E + \int_{\check{\eta}_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \int_{\check{\eta}_2}^1 v(\eta) dG(\eta) \right] F(c_H) \\
&= c_H v(\check{\eta}_2) + (\lambda_0 + \Delta) \int_{\check{\eta}_1}^1 [v(\eta) - v(\check{\eta}_1)] g(\eta) d\eta + \lambda_1 \int_{\check{\eta}_2}^1 [v(z) - v(\check{\eta}_2)] g(z) dz - v(\check{\eta}_2) + v_E \\
&\quad - \left[v_E + \int_{\check{\eta}_1}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \int_{\check{\eta}_2}^1 v(\eta) dG(\eta) \right] \\
&= c_H v(\check{\eta}_2) - (\lambda_0 + \Delta) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] - \lambda_1 v(\check{\eta}_2) [1 - G(\check{\eta}_2)] - v(\check{\eta}_2) - \Delta [1 - G(\check{\eta}_1)] \\
&= [c_H - \lambda_1 [1 - G(\check{\eta}_2)]] - 1 \left[v(\check{\eta}_1) + \frac{\Delta}{\lambda_0} [v(\check{\eta}_1) + 1] \right] - (\lambda_0 + \Delta) v(\check{\eta}_1) [1 - G(\check{\eta}_1)] - \Delta [1 - G(\check{\eta}_1)] \\
&= 0,
\end{aligned}$$

where the last equality follows from

$$c_H = 1 + \lambda_0 [1 - G(\check{\eta}_1)] + \lambda_1 [1 - G(\check{\eta}_2)].$$

In case (4)(a)(iv), we replace the two thresholds $\dot{\eta}_1$ and $\dot{\eta}_2$ with $\bar{\eta}'$ and $\bar{\eta}$, respectively. The desired nonnegativity condition for ψ boils down to

$$\lambda_1 \int_{\bar{\eta}}^1 v(z) dG(z) + v_E \geq 0 \quad \Leftrightarrow \quad h(\bar{\eta}) \geq 0.$$

Suppose $h(\bar{\eta}) < 0$. We have

$$\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2) \geq 1 + \lambda_0 [1 - G(\bar{\eta}')] + \lambda_1 [1 - G(\bar{\eta})]$$

or

$$\check{c}_2 > \bar{c}(\dot{\eta}_1, \dot{\eta}_1) \geq 1 + (\lambda_0 + \lambda_1) [1 - G(\eta^\dagger)] \geq 1 + (\lambda_0 + \lambda_1) [1 - G(\bar{\eta}')],$$

which contradicts the condition $c_H \leq \bar{c}(\bar{\eta}', \bar{\eta})$. Therefore, we know that $h(\bar{\eta}) > 0$, guaranteeing the nonnegativity of ψ and σ , where the latter one follows from

$$\sigma(c) = \left\{ (\lambda_0 + \Delta) \int_{\bar{\eta}'}^1 [v(\eta) - v(\bar{\eta}')] g(\eta) d\eta + \underbrace{\lambda_1 \int_{\bar{\eta}}^1 v(z) g(z) dz + v_E}_{=h(\bar{\eta}) \geq 0} \right\} f(c) \geq 0.$$

The candidate primal solution attains the upper bound $\int_{c_L}^{c_H} \sigma(c) dc$, i.e.,

$$\begin{aligned} & \int_{c_L}^{c_H} \sigma(c) dc - \left[v_E + \int_{\bar{\eta}'}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \int_{\bar{\eta}}^1 v(\eta) dG(\eta) \right] F(c_H) \\ &= (\lambda_0 + \Delta) \int_{\bar{\eta}'}^1 [v(\eta) - v(\bar{\eta}')] g(\eta) d\eta + \lambda_1 \int_{\bar{\eta}}^1 v(z) g(z) dz + v_E \\ & \quad - \left[v_E + \int_{\bar{\eta}'}^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] dG(\eta) + \lambda_1 \int_{\bar{\eta}}^1 v(\eta) dG(\eta) \right] \\ &= - [(\lambda_0 + \Delta) v(\bar{\eta}') + \Delta] [1 - G(\bar{\eta}')] \\ &= 0, \end{aligned}$$

where the last step follows from the definition of $\bar{\eta}'$.

It remains to prove the optimality in case (4)(b). For the first two subcases, define

$$\begin{aligned} \kappa &\equiv -\frac{1}{\lambda_0} [(\lambda_0 + \Delta) v(\dot{\eta}_2) + \Delta], \\ \chi(c, \eta) &\equiv (\lambda_0 + \Delta) [v(\dot{\eta}_2) - v(\eta)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_2], \\ \mu(c, \eta) &= (\lambda_0 + \Delta) [v(\eta) - v(\dot{\eta}_2)] g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1], \\ \omega(c, \eta) &= \left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\dot{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\dot{\eta}_2) + 1] \right\} g(\eta) f(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_1], \\ \xi(c, \eta) &\equiv \begin{cases} - \left\{ (\lambda_0 + \lambda_1 + \Delta) [v(\dot{\eta}_2) - v(\eta)] + \frac{\lambda_1}{\lambda_0} \Delta [v(\dot{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_1, \dot{\eta}_2], \\ \left\{ \lambda_1 [v(\eta) - v(\dot{\eta}_2)] - \frac{\lambda_1}{\lambda_0} \Delta [v(\dot{\eta}_2) + 1] \right\} g(\eta) f(c), & \forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1], \end{cases} \end{aligned}$$

$$\begin{aligned}
\psi(c) &\equiv -\iota(\dot{\eta}_1, \dot{\eta}_2) f(c), \quad \forall c \in [c_L, c_H], \\
\sigma(c) &\equiv \left\{ \left[v(\dot{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\dot{\eta}_2) + 1] \right] \left[c + \frac{F(c)}{f(c)} \right] \right. \\
&\quad \left. + (\lambda_0 + \Delta) \int_{\dot{\eta}_2}^1 [v(\eta) - v(\dot{\eta}_2)] g(\eta) d\eta + \frac{\psi(c)}{f(c)} \right\} f(c), \quad \forall c \in \mathcal{C}_1, \\
\beta(c) &\equiv -\sigma(c), \quad \forall c \in \mathcal{C}_2.
\end{aligned}$$

The nonnegativity of κ follows from the fact that $\dot{\eta}_2 \leq \eta^\dagger \leq \dot{\eta}'$. The nonnegativity of χ and μ follows from their respective definitions directly. The nonnegativity of ω and ξ follows from the relationship between $\dot{\eta}_1$ and $\dot{\eta}_2$ (i.e., $\Psi_2(\dot{\eta}_1, \dot{\eta}_2) = 0$, implying $\omega(c, \dot{\eta}_1) = \xi(c, \dot{\eta}_1) = 0$ for any $c \in [c_L, c_H]$) and their monotonicity in η . The nonnegativity of ψ follows from Lemma [EC.6](#)(2).

In the first two possible cases, we have:

1. if $\bar{c}(\dot{\eta}_1, \dot{\eta}_1) < c_L$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$; and
2. if $c_L \leq \bar{c}(\dot{\eta}_1, \dot{\eta}_1) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \bar{c}(\dot{\eta}_1, \dot{\eta}_1)]$ and $[\bar{c}(\dot{\eta}_1, \dot{\eta}_1), c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \bar{c}(\dot{\eta}_1, \dot{\eta}_1)]$ and $\mathcal{C}_2 = [\bar{c}(\dot{\eta}_1, \dot{\eta}_1), c_H]$.

Therefore, the nonnegativity of σ and β holds automatically.

Because

$$-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \int_{c_L}^{c_H} \int_{\dot{\eta}_1}^1 \xi(c, \eta) d\eta dc = v_E,$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_2]:$$

$$-\lambda_0 \psi(c) g(\eta) - \chi(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1]:$$

$$-\lambda_0 \psi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz = [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [0, \dot{\eta}_1]:$$

$$-\lambda_1 \psi(c) g(\eta) - \omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_1, \dot{\eta}_2]:$$

$$-\lambda_1 \psi(c) g(\eta) + \chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$$\forall (c, \eta) \in [c_L, c_H] \times [\dot{\eta}_2, 1]:$$

$$-\lambda_1 \psi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\dot{\eta}_1}^1 \xi(c, z) dz = \lambda_1 [v(\eta) - v_E] g(\eta) f(c),$$

$\forall c \in [c_L, \hat{c}]$:

$$\begin{aligned} & \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) + \sigma(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

$\forall c \in [\hat{c}, c_H]$:

$$\begin{aligned} & \int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - \beta(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta \\ & - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta = v_E [c f(c) + F(c)], \end{aligned}$$

we have

$$\begin{aligned} & \kappa [-\Pi(c_H)] + \int_{c_1} \sigma(c) \tau(c) dc + \int_{c_2} \beta(c) [-\tau(c)] dc + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_2} \chi(c, \eta) [\hat{\alpha}_3(c, \eta) - \hat{\alpha}_2(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \mu(c, \eta) [\hat{\alpha}_2(c, \eta) - \tau(c)] d\eta dc + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_1} \omega(c, \eta) [-\hat{\alpha}_3(c, \eta)] d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^1 \xi(c, \eta) \left[\hat{\alpha}_3(c, \eta) - \Pi(c_H) - \int_c^{c_H} \tau(y) dy - c \tau(c) + \int_0^1 [\lambda_0 \hat{\alpha}_2(c, z) + \lambda_1 \hat{\alpha}_3(c, z)] dG(z) \right] d\eta dc \\ & + \int_{c_L}^{c_H} \psi(c) \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + c \tau(c) - \lambda_0 \int_0^1 \hat{\alpha}_2(c, \eta) dG(\eta) - \lambda_1 \int_0^1 \hat{\alpha}_3(c, \eta) dG(\eta) - \tau(c) \right] dc \\ & = \left[-\kappa + \int_{c_L}^{c_H} \psi(c) dc - \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta dc \right] \Pi(c_H) \\ & + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_2} \left[-\lambda_0 \psi(c) g(\eta) - \chi(c, \eta) + \lambda_0 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \left[-\lambda_0 \psi(c) g(\eta) + \mu(c, \eta) + \lambda_0 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_2(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_0^{\hat{\eta}_1} \left[-\lambda_1 \psi(c) g(\eta) - \omega(c, \eta) + \chi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_1}^{\hat{\eta}_2} \left[-\lambda_1 \psi(c) g(\eta) + \chi(c, \eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_L}^{c_H} \int_{\hat{\eta}_2}^1 \left[-\lambda_1 \psi(c) g(\eta) + \xi(c, \eta) + \lambda_1 g(\eta) \int_{\hat{\eta}_1}^1 \xi(c, z) dz \right] \hat{\alpha}_3(c, \eta) d\eta dc \\ & + \int_{c_1} \left[\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) + \sigma(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & + \int_{c_2} \left[\int_{c_L}^c \psi(y) dy + (c-1) \psi(c) - \beta(c) - \int_{\hat{\eta}_2}^1 \mu(c, \eta) d\eta - c \int_{\hat{\eta}_1}^1 \xi(c, \eta) d\eta - \int_{c_L}^c \int_{\hat{\eta}_1}^1 \xi(y, \eta) d\eta dy \right] \tau(c) dc \\ & = v_E \Pi(c_H) + v_E \int_{c_L}^{c_H} \left[c + \frac{F(c)}{f(c)} \right] \tau(c) f(c) dc + \int_{c_L}^{c_H} \int_0^1 [\lambda_0 [v(\eta) - v_E] + \Delta [v(\eta) + 1]] \hat{\alpha}_2(c, \eta) dG(\eta) dF(c) \\ & + \lambda_1 \int_{c_L}^{c_H} \int_0^1 [v(\eta) - v_E] \hat{\alpha}_3(c, \eta) dG(\eta) dF(c). \end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_1} \sigma(c) dc$. Substituting our candidate solution into the objective function achieves the upper bound in the first two cases, which follows from:

1. if $\bar{c}(\hat{\eta}_1, \hat{\eta}_1) \leq c_L$, we have $\hat{c} = c_L$ and the measure of $\mathcal{C}_1$ is 0. The government's expected payoff from our candidate policy and its upper bound are both 0;
2. if $c_L < \bar{c}(\hat{\eta}_1, \hat{\eta}_1) \leq c_H$, we have

$$\begin{aligned}
& \int_{c_L}^{\bar{c}(\hat{\eta}_1, \hat{\eta}_1)} \sigma(c) dc - \left[v_E + \int_{\hat{\eta}_1}^1 [(\lambda_0 + \lambda_1 + \Delta) v(\eta) + \Delta] dG(\eta) \right] F(\bar{c}(\hat{\eta}_1, \hat{\eta}_1)) \\
&= \left\{ \bar{c}(\hat{\eta}_1, \hat{\eta}_1) \left[v(\hat{\eta}_2) + \frac{\Delta}{\lambda_0} [v(\hat{\eta}_2) + 1] \right] + (\lambda_0 + \Delta) \int_{\hat{\eta}_2}^1 [v(\eta) - v(\hat{\eta}_2)] g(\eta) d\eta \right. \\
&\quad \left. - \iota(\hat{\eta}_1, \hat{\eta}_2) - \left[v_E + \int_{\hat{\eta}_1}^1 [(\lambda_0 + \lambda_1 + \Delta) v(\eta) + \Delta] dG(\eta) \right] \right\} F(\bar{c}(\hat{\eta}_1, \hat{\eta}_1)) \\
&= 0.
\end{aligned}$$

In case (4)(b)(iii), we set $\mathcal{C}_1 = [c_L, c_H]$ and $\mathcal{C}_2 = \emptyset$, and replace the thresholds $\hat{\eta}_1$ and $\hat{\eta}_2$ with $\check{\eta}_1$ and $\check{\eta}_2$, respectively, where the two thresholds are determined by $\bar{c}(\check{\eta}_1, \check{\eta}_1) = c_H$ and $\Psi_2(\check{\eta}_1, \check{\eta}_2) = 0$. In this case, the nonnegativity of κ , χ , μ , ω , and ξ follows from their definitions directly. Given the monotonicity of the function Ξ (defined by (EC.21)), the desired nonnegativity of ψ follows from the following argument:

- If $h(\bar{\eta}) \leq 0$ and $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, we have

$$\iota(\check{\eta}_1, \check{\eta}_2) = -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \leq -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \leq 0.$$

The inequalities above follow from the fact that $\check{\eta}_2 \leq \check{\eta}_2$ and $\Xi(\check{\eta}_2) \geq 0$, as established in Lemma EC.3.

- If $h(\bar{\eta}) \leq 0$ and $\check{c}_2 > \bar{c}(\hat{\eta}_1, \hat{\eta}_1)$, we have

$$\iota(\check{\eta}_1, \check{\eta}_2) = -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \leq -\frac{\Xi(\hat{\eta}_2)}{\lambda_0} = \iota(\hat{\eta}_1, \hat{\eta}_2) = 0.$$

The inequalities above follow from the fact that $\check{\eta}_2 \leq \hat{\eta}_2$.

- If $h(\bar{\eta}) > 0$, we have

$$\iota(\check{\eta}_1, \check{\eta}_2) = -\frac{\Xi(\check{\eta}_2)}{\lambda_0} \leq -\frac{\Xi(\eta^\dagger)}{\lambda_0} \leq 0.$$

The inequalities above follow from the fact that $\check{\eta}_2 \leq \eta^\dagger$ and $\Xi(\eta^\dagger) = \lambda_0 h(\eta^\dagger) \geq \lambda_0 h(\bar{\eta}) > 0$.

The nonnegativity of σ follows from the fact that for any $c \in [c_L, c_H]$, we have

$$\sigma(c) \geq \sigma(c_H) = t_2(\check{\eta}_2) f(c_H) \geq t_2(\hat{\eta}_2) f(c_H) = 0,$$

where the function t_2 is defined in (EC.29), and the inequality follows from the monotonicity of t_2 , as established in Lemma EC.6. Because all the multipliers remain nonnegative, the objective function is bounded by $\int_{c_L}^{c_H} \sigma(c) dc$, which is attained by our candidate solution, i.e.,

$$\int_{c_L}^{c_H} \sigma(c) dc - \left[v_E + \int_{\hat{\eta}_1}^1 [(\lambda_0 + \lambda_1 + \Delta) v(\eta) + \Delta] dG(\eta) \right]$$

$$\begin{aligned}
&= \frac{\Delta}{\lambda_0} [v(\check{\eta}_2) + 1] [c_H - [1 + \lambda_1 [1 - G(\check{\eta}_1)]]] + [c_H - (\lambda_0 + \lambda_1 + \Delta) [1 - G(\check{\eta}_1)] - 1] v(\check{\eta}_2) \\
&\quad - \Delta [1 - G(\check{\eta}_1)] \\
&= \Delta [v(\check{\eta}_2) + 1] [1 - G(\check{\eta}_1)] - \Delta [1 - G(\check{\eta}_1)] v(\check{\eta}_2) - \Delta [1 - G(\check{\eta}_1)] \\
&= 0,
\end{aligned}$$

where the second equality follows from

$$1 + (\lambda_0 + \lambda_1) [1 - G(\check{\eta}_1)] = c_H.$$

When $h(\bar{\eta}) \leq 0$ and $\check{c}_2 > \bar{c}(\check{\eta}_1, \check{\eta}_1)$, we have

$$c_H \geq \check{c}_2 > \bar{c}(\check{\eta}_1, \check{\eta}_1) \geq \bar{c}(\eta^\dagger, \eta^\dagger).$$

Therefore, it remains to prove cases (4)(b)(iv) and (4)(b)(v) for either (1) $h(\bar{\eta}) \leq 0$ and $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, or (2) $h(\bar{\eta}) > 0$.

The proof of case (4)(b)(iv) closely follows that of case (4)(a)(iv). The nonnegativity of ψ follows as: if $h(\bar{\eta}) \leq 0$ and $\check{c}_1 > \bar{c}(\check{\eta}_1, \check{\eta}_2)$, we have $h(\check{\eta}_2) \geq h(\eta^\dagger) \geq h(\check{\eta}_2) = 0$; otherwise, we have $h(\check{\eta}) \geq h(\bar{\eta}) > 0$. The nonnegativity of σ follows from the fact that for any $c \in [c_L, c_H]$, we have

$$\sigma(c) \geq \sigma(c_H) = t_1(\check{\eta}_2) f(c_H) \geq t_1(\eta^\dagger) f(c_H) \geq 0.$$

Since all the multipliers are nonnegative, optimality is established.

The proof of case (4)(b)(v) is identical to that of case (4)(a)(iv).

This completes the proof. □

Proof of Corollary 3. When $\Delta = 0$, we have

$$\Psi_1(\eta_1, \eta_2) = 0 \text{ or } \Psi_2(\eta_1, \eta_2) = 0 \quad \Rightarrow \quad \eta_1 = \eta_2.$$

Therefore, the optimal solution to the upper-bound problem (48) is reduced to

$$\hat{\alpha}_2(c, \eta) = \begin{cases} 1, & \forall (c, \eta) \in [c_L, \hat{c}] \times [\hat{\eta}, 1], \\ 0, & \text{otherwise,} \end{cases} \quad \hat{\alpha}_3(c, \eta) = \begin{cases} \hat{\alpha}, & \forall (c, \eta) \in [c_L, \hat{c}] \times [\hat{\eta}, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Given any feasible mechanism $\{\hat{\tau}(c), \hat{\alpha}_1(c), \hat{\alpha}_2(c, \eta), \hat{\alpha}_3(c, \eta)\}$, the following mechanism $\{\hat{\tau}(c), a_1(c), a_2(c, \eta)\}$ is feasible for the original problem (13) and yields the same objective function value, where a_1 and a_2 are constructed as follows:

$$a_1(c) = \begin{cases} \frac{\hat{\alpha}_1(c)}{\hat{\tau}(c)}, & \text{if } \hat{\tau}(c) > 0, \\ 0, & \text{if } \hat{\tau}(c) = 0, \end{cases} \quad a_2(c, \eta) = \begin{cases} \frac{\hat{\alpha}_2(c, \eta)}{\hat{\tau}(c)}, & \text{if } \hat{\tau}(c) > 0, \\ 0, & \text{if } \hat{\tau}(c) = 0. \end{cases}$$

This completes the proof. □

Proof of Proposition 7. By construction, we have

$$\int_0^1 \hat{\tau}(c) [\lambda_0 + \lambda_1 a'_1(c)] a'_2(c, \eta) dG(\eta) = \int_0^1 [\lambda_0 \hat{\alpha}_2(c, \eta) + \lambda_1 \hat{\alpha}_3(c, \eta)] dG(\eta), \quad \forall c \in [c_L, c_H].$$

Hence, under the following mechanism

$$\mathcal{M}_1 \equiv \left\{ \hat{\tau}(c) = \mathbb{1}_{c < \hat{c}}, \quad a'_1(c) = \hat{\alpha} \cdot \mathbb{1}_{c < \hat{c}}, \quad a'_2(c, \eta) = \mathbb{1}_{(c, \eta) \in [c_L, \hat{c}] \times [\eta_1, 1]} \right\},$$

constraints (IC) and (IR) hold automatically. The remaining constraints (2), (3), and (4) hold trivially by their definitions. When $\eta_1 > \eta_2$, by definition of η_2 , we have

$$\int_{\eta_2}^1 [(\lambda_0 + \lambda_1 \hat{\alpha} + \Delta) v(\eta) + \Delta] dG(\eta) \geq \int_{\eta_1}^1 [(\lambda_0 + \lambda_1 \hat{\alpha} + \Delta) v(\eta) + \Delta] dG(\eta).$$

Furthermore, the testing threshold satisfies

$$\hat{c} \leq \hat{\alpha} + (\lambda_0 + \lambda_1 \hat{\alpha}) [1 - G(\eta_1)] < \hat{\alpha} + (\lambda_0 + \lambda_1 \hat{\alpha}) [1 - G(\eta_2)] = c_2,$$

implying that the government is better off under mechanism

$$\mathcal{M}_2 \equiv \left\{ \tau''(c) = \mathbb{1}_{c \leq c_2}, \quad a''_1(c) = \hat{\alpha} \cdot \mathbb{1}_{c \leq c_2}, \quad a''_2(c, \eta) = \mathbb{1}_{(c, \eta) \in [c_L, c_2] \times [\eta_2, 1]} \right\}.$$

It is straightforward to verify the feasibility of $\mathcal{M}_2$. Because U_R^* is the optimal value of the objective function, the feasibility property implies that

$$\tilde{U}_R^* \leq U_R^*.$$

This completes the proof. □

EC.5.5. Proofs for Section 5: either conditional approval or leniency unavailable

In this section, we provide the optimal mechanisms in closed form when either condition approval ($a_1(c) = 0$ for any $c \in [c_L, c_H]$) or leniency ($a_2(c, \eta) = \mathbb{1}_{\eta \in [\bar{\eta}', 1]}$ for any $(c, \eta) \in [c_L, c_H] \times [0, 1]$) is unavailable, for completeness.

With a slight abuse of notation, in this section, we introduce a new decision variable

$$\alpha_2(c, \eta) \equiv \tau(c) a_2(c, \eta), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, 1].$$

Both parties' payoff functions boil down to

$$\hat{\Pi}(c, c') = -c \tau(c') + \lambda_0 \int_0^1 \alpha_2(c', \eta) dG(\eta) \tag{EC.30}$$

and

$$U_R(\tau, \alpha_2) = \int_{c_L}^{c_H} \int_0^1 [(\lambda_0 + \Delta) v(\eta) + \Delta] \alpha_2(c, \eta) dG(\eta) dF(c). \tag{EC.31}$$

We redefine the government's utility function as

$$\tilde{v}(\eta) \equiv (\lambda_0 + \Delta) v(\eta) + \Delta, \quad \forall \eta \in [0, 1],$$

and denote $\tilde{v}(0)$ and $\mathbb{E}[\tilde{v}(\eta)]$ by $\tilde{v}_L$ and $\tilde{v}_E$, respectively. With a slight abuse of notation, in this section, we define

$$\bar{\eta} \equiv \inf \{ \eta \in [0, 1] : \tilde{v}(\eta) \geq 0 \}.$$

The government's problem can be formulated as follows:

$$\max_{\{\tau, \alpha_2\} \in \Omega'_1} U_R(\tau, \alpha_2), \quad (\text{EC.32})$$

where the feasible set Ω'_1 is determined by (IC), (IR), (2), and

$$0 \leq \alpha_2(c, \eta) \leq \tau(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, 1]. \quad (\text{EC.33})$$

The optimal solution to this restricted problem remains a “take-it-or-leave-it” offer, as stated in the following proposition.

PROPOSITION EC.2. *Consider the optimization problem (EC.32) under Assumption 1. There exists a cost threshold $\hat{c}' \in [c_L, c_H]$ and a final approval threshold $\hat{\eta}' \in [0, 1]$ such that it is optimal to request testing and grant final approval as follows:*

$$\tau^*(c) = \begin{cases} 1, & \forall c \in [c_L, \hat{c}'], \\ 0, & \forall c \in [\hat{c}', c_H], \end{cases} \quad \alpha_2^*(c, \eta) = \begin{cases} 1, & \forall (c, \eta) \in [c_L, \hat{c}') \times [\hat{\eta}', 1], \\ 0, & \text{otherwise.} \end{cases}$$

The values of $\hat{c}'$ and $\hat{\eta}'$ depend on model parameters in the following cases.

- (1) If $\tilde{v}_L \frac{F(\underline{c}(0))}{f(\underline{c}(0))} + \lambda_0 \tilde{v}_E > 0$, we have $\hat{c}' = c_H$ and $\hat{\eta}' = \min \{ \underline{c}^{-1}(c_H), \bar{\eta} \}$.
- (2) If $\tilde{v}_L \frac{F(\underline{c}(0))}{f(\underline{c}(0))} + \lambda_0 \tilde{v}_E \leq 0$, there exists a unique $\tilde{\eta} \in [0, \bar{\eta}]$ such that

$$\tilde{v}(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 \tilde{v}(y) g(y) dy = 0.$$

- (a) If $c_L > \underline{c}(\tilde{\eta})$, we have $\hat{c}' = c_L$ and $\hat{\eta}' = 1$.
- (b) If $c_L \leq \underline{c}(\tilde{\eta}) \leq c_H$, we have $\hat{c}' = \underline{c}(\tilde{\eta})$ and $\hat{\eta}' = \tilde{\eta}$.
- (c) If $\underline{c}(\tilde{\eta}) < c_H < \underline{c}(\tilde{\eta})$, we have $\hat{c}' = c_H$ and $\hat{\eta}' = \underline{c}^{-1}(c_H)$.
- (d) If $c_H \leq \underline{c}(\tilde{\eta})$, we have $\hat{c}' = c_H$ and $\hat{\eta}' = \bar{\eta}$.

Proof of Proposition EC.2. It is straightforward to verify that our candidate solutions in all the cases are feasible.

When $c_H \leq \underline{c}(\bar{\eta})$, our candidate solution attains the upper bound of the objective function $\int_{\bar{\eta}}^1 \tilde{v}(\eta) dG(\eta)$, with all constraints holding automatically. In the remainder of the proof, we focus on scenarios where $c_H > \underline{c}(\bar{\eta})$.

Invoking envelope theorem, the incentive compatibility constraint can be replaced by EC.7 and

$$c\tau(c) - \lambda_0 \int_0^1 \alpha_2(c, \eta) dG(\eta) + \Pi(c_H) + \int_c^{c_H} \tau(y) dy = 0, \quad \forall c \in [c_L, c_H]. \quad (\text{EC.34})$$

When $c_H > \underline{c}(\bar{\eta})$, to prove the optimality in cases 1 and 2(3), define

$$\begin{aligned} \kappa &= -\tilde{v}(\underline{c}^{-1}(c_H)), \\ \nu(c, \eta) &= \lambda_0 [\tilde{v}(\underline{c}^{-1}(c_H)) - \tilde{v}(\eta)] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, \underline{c}^{-1}(c_H)], \\ \mu(c, \eta) &= \lambda_0 [\tilde{v}(\eta) - \tilde{v}(\underline{c}^{-1}(c_H))] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [\underline{c}^{-1}(c_H), 1], \\ \phi(c) &= -\tilde{v}(\underline{c}^{-1}(c_H)) f(c), & \forall c \in [c_L, c_H], \\ \sigma(c) &= \left[\tilde{v}(\underline{c}^{-1}(c_H)) \left[(c - c_H) + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(y) g(y) dy \right] f(c), & \forall c \in [c_L, c_H]. \end{aligned}$$

The monotonicity of $\underline{c}$ implies that

$$c_H > \underline{c}(\bar{\eta}) \quad \Leftrightarrow \quad \underline{c}^{-1}(c_H) < \bar{\eta},$$

guaranteeing the nonnegativity of κ . The nonnegativity of ν and μ follows from their definitions directly. Because $\sigma(c)/f(c)$ is decreasing, it suffices to verify the nonnegativity of $\sigma(c_H)$. When $\tilde{v}_L \frac{F(\underline{c}(0))}{f(\underline{c}(0))} + \lambda_0 \tilde{v}_E > 0$, we have

$$\sigma(c_H) = \underbrace{\left[\tilde{v}(\underline{c}^{-1}(c_H)) \frac{1}{f(c_H)} + \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(y) g(y) dy \right]}_{>0} f(c_H) \geq 0.$$

When $\tilde{v}_L \frac{F(\underline{c}(0))}{f(\underline{c}(0))} + \lambda_0 \tilde{v}_E \leq 0$ and $c_H < \underline{c}(\tilde{\eta})$, we have $\underline{c}^{-1}(c_H) > \tilde{\eta}$, and hence,

$$\begin{aligned} \sigma(c_H) &= \left[\tilde{v}(\underline{c}^{-1}(c_H)) \frac{F(c_H)}{f(c_H)} + \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(y) g(y) dy \right] f(c_H) \\ &\geq \left[\tilde{v}(\tilde{\eta}) \frac{F(\underline{c}(\tilde{\eta}))}{f(\underline{c}(\tilde{\eta}))} + \lambda_0 \int_{\tilde{\eta}}^1 \tilde{v}(y) g(y) dy \right] f(c_H) \\ &= 0, \end{aligned}$$

where the inequality follows from the monotonicity of function

$$\Upsilon(\eta) \equiv \tilde{v}(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} + \lambda_0 \int_{\eta}^1 \tilde{v}(y) g(y) dy,$$

i.e.,

$$\Upsilon'(\eta) = \tilde{v}'(\eta) \frac{F(\underline{c}(\eta))}{f(\underline{c}(\eta))} - \lambda_0 \tilde{v}(\eta) g(\eta) \underbrace{\frac{d}{dc} \left[c + \frac{F(c)}{f(c)} \right]}_{\geq 0 \text{ by Assumption } \textcolor{red}{\boxed{A}} \Big|_{c=\underline{c}(\eta)}} \geq 0.$$

Because

$$\begin{aligned}
& \kappa [-\Pi(c_H)] + \int_{c_L}^{c_H} \phi(c) \left[c\tau(c) - \lambda_0 \int_0^1 \alpha_2(c, \eta) dG(\eta) + \Pi(c_H) + \int_c^{c_H} \tau(y) dy \right] dc \\
& + \int_{c_L}^{c_H} \sigma(c) \tau(c) dc - \int_{c_L}^{c_H} \int_0^{\underline{c}^{-1}(c_H)} \nu(c, \eta) \alpha_2(c, \eta) d\eta dc + \int_{c_L}^{c_H} \int_{\underline{c}^{-1}(c_H)}^1 \mu(c, \eta) [\alpha_2(c, \eta) - \tau(c)] d\eta dc \\
& = \left[-\kappa + \int_{c_L}^{c_H} \phi(c) dc \right] \Pi(c_H) + \int_{c_L}^{c_H} \int_0^{\underline{c}^{-1}(c_H)} [-\lambda_0 \phi(c) g(\eta) - \nu(c, \eta)] \alpha_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\underline{c}^{-1}(c_H)}^1 [-\lambda_0 \phi(c) g(\eta) + \mu(c, \eta)] \alpha_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \left[\sigma(c) + c\phi(c) + \int_{c_L}^c \phi(y) dy - \int_0^1 \mu(c, \eta) d\eta \right] \tau(c) dc \\
& = \lambda_0 \int_{c_L}^{c_H} \int_0^1 \tilde{v}(\eta) \alpha_2(c, \eta) dG(\eta) dF(c),
\end{aligned}$$

the objective function is bounded by $\int_{c_L}^{c_H} \sigma(c) dc$. Substituting the decision variables with our candidate solution attains this upper bound, which follows from

$$\begin{aligned}
& \int_{c_L}^{c_H} \sigma(c) dc - \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(\eta) dG(\eta) \\
& = \int_{c_L}^{c_H} \left[\tilde{v}(\underline{c}^{-1}(c_H)) \left[(c - c_H) + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(y) g(y) dy \right] f(c) dc - \lambda_0 \int_{\underline{c}^{-1}(c_H)}^1 \tilde{v}(\eta) dG(\eta) \\
& = \tilde{v}(\underline{c}^{-1}(c_H)) \int_{c_L}^{c_H} [(c - c_H) f(c) + F(c)] dc \\
& = 0.
\end{aligned}$$

To prove the optimality in the remaining two subcases ((2)(a) and (2)(b)), define

$$\begin{aligned}
\kappa &= -\tilde{v}(\tilde{\eta}), \\
\nu(c, \eta) &= \lambda_0 [\tilde{v}(\tilde{\eta}) - \tilde{v}(\eta)] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [0, \tilde{\eta}], \\
\mu(c, \eta) &= \lambda_0 [\tilde{v}(\eta) - \tilde{v}(\tilde{\eta})] f(c) g(\eta), & \forall (c, \eta) \in [c_L, c_H] \times [\tilde{\eta}, 1], \\
\phi(c) &= -\tilde{v}(\tilde{\eta}) f(c), & \forall c \in [c_L, c_H], \\
\sigma(c) &= \left[\tilde{v}(\tilde{\eta}) \left[[c - \underline{c}(\tilde{\eta})] + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\tilde{\eta}}^1 \tilde{v}(y) g(y) dy \right] f(c), & \forall c \in \mathcal{C}_1, \\
\beta(c) &= - \left[\tilde{v}(\tilde{\eta}) \left[[c - \underline{c}(\tilde{\eta})] + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\tilde{\eta}}^1 \tilde{v}(y) g(y) dy \right] f(c), & \forall c \in \mathcal{C}_2.
\end{aligned}$$

When $c_L > \underline{c}(\tilde{\eta})$, we know that β is nonnegative on $[c_L, c_H]$. In this case, we have $\mathcal{C}_1 = \emptyset$ and $\mathcal{C}_2 = [c_L, c_H]$. When $c_L \leq \underline{c}(\tilde{\eta}) \leq c_H$, we know that σ and β are nonnegative on $[c_L, \underline{c}(\tilde{\eta})]$ and $[\underline{c}(\tilde{\eta}), c_H]$, respectively. In this case, we have $\mathcal{C}_1 = [c_L, \underline{c}(\tilde{\eta})]$ and $\mathcal{C}_2 = [\underline{c}(\tilde{\eta}), c_H]$. This guarantees the nonnegativity of all the multipliers (except ϕ , which corresponds to an equality constraint).

Because

$$\begin{aligned}
& \kappa [-\Pi(c_H)] + \int_{c_L}^{c_H} \phi(c) \left[c\tau(c) - \lambda_0 \int_0^1 \alpha_2(c, \eta) dG(\eta) + \Pi(c_H) + \int_c^{c_H} \tau(y) dy \right] dc + \int_{c_1} \sigma(c) \tau(c) dc \\
& - \int_{c_2} \beta(c) \tau(c) dc - \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} \nu(c, \eta) \alpha_2(c, \eta) d\eta dc + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 \mu(c, \eta) [\alpha_2(c, \eta) - \tau(c)] d\eta dc \\
& = \left[-\kappa + \int_{c_L}^{c_H} \phi(c) dc \right] \Pi(c_H) + \int_{c_L}^{c_H} \int_0^{\tilde{\eta}} [-\lambda_0 \phi(c) g(\eta) - \nu(c, \eta)] \alpha_2(c, \eta) d\eta dc \\
& + \int_{c_L}^{c_H} \int_{\tilde{\eta}}^1 [-\lambda_0 \phi(c) g(\eta) + \mu(c, \eta)] \alpha_2(c, \eta) d\eta dc \\
& + \int_{c_1} \left[\sigma(c) + c\phi(c) + \int_{c_L}^c \phi(y) dy - \int_0^1 \mu(c, \eta) d\eta \right] \tau(c) dc \\
& + \int_{c_2} \left[-\beta(c) + c\phi(c) + \int_{c_L}^c \phi(y) dy - \int_0^1 \mu(c, \eta) d\eta \right] \tau(c) dc \\
& = \lambda_0 \int_{c_L}^{c_H} \int_0^1 \tilde{v}(\eta) \alpha_2(c, \eta) dG(\eta) dF(c),
\end{aligned}$$

the objective function is bounded by $\int_{c_1} \sigma(c) dc$.

1. In case (2)(a), the government's payoff generated by our candidate solution and its upper bound are both 0.
2. In case (2)(b), we have

$$\begin{aligned}
& \int_{c_L}^{\underline{c}(\tilde{\eta})} \sigma(c) dc - \lambda_0 F(\underline{c}(\tilde{\eta})) \int_{\tilde{\eta}}^1 \tilde{v}(\eta) \alpha_2(c, \eta) dG(\eta) \\
& = \int_{c_L}^{\underline{c}(\tilde{\eta})} \left[\tilde{v}(\tilde{\eta}) \left[[c - \underline{c}(\tilde{\eta})] + \frac{F(c)}{f(c)} \right] + \lambda_0 \int_{\tilde{\eta}}^1 \tilde{v}(y) g(y) dy \right] f(c) dc \\
& \quad - \lambda_0 F(\underline{c}(\tilde{\eta})) \int_{\tilde{\eta}}^1 \tilde{v}(\eta) \alpha_2(c, \eta) dG(\eta) \\
& = \tilde{v}(\tilde{\eta}) \int_{c_L}^{\underline{c}(\tilde{\eta})} [[c - \underline{c}(\tilde{\eta})] f(c) + F(c)] dc \\
& = 0,
\end{aligned}$$

establishing the optimality.

This completes the proof. $\square$

When leniency is unavailable, in which case we assume that the government will adopt the myopic threshold $\bar{\eta}'$ (defined in (55)), both parties' payoff functions are reduced to

$$\hat{\Pi}(c, c') = \alpha_1(c') - c\tau(c') + [\lambda_0 \tau(c') + \lambda_1 \alpha_1(c')] \bar{G}(\bar{\eta}')$$

and

$$U_R = \int_{c_L}^{c_H} \left[v_E \alpha_1(c) + [\lambda_0 \tau(c) + \lambda_1 \alpha_1(c)] \underbrace{\int_{\bar{\eta}'}^1 v(\eta) dG(\eta)}_{\equiv z_1} + \tau(c) \Delta \underbrace{\int_{\bar{\eta}'}^1 [v(\eta) + 1] dG(\eta)}_{\equiv z_2} \right] dF(c),$$

where $\bar{G}$ denotes the complementary cumulative distribution function of G . By envelope theorem, we have

$$\alpha_1(c) = \frac{1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + (c - \lambda_0 \bar{G}(\bar{\eta}')) \tau(c) \right].$$

After eliminating α_1 , the optimization problem can be reformulated as follows:

$$\max_{\{\tau, \alpha_1\} \in \Omega'_2} \frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \Pi(c_H) + \int_{c_L}^{c_H} w(c) \tau(c) dF(c), \quad (\text{EC.35})$$

where the virtual valuation function w is given by

$$w(c) = \frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \left[c + \frac{F(c)}{f(c)} - \lambda_0 \bar{G}(\bar{\eta}') \right] + \lambda_0 z_1 + z_2,$$

and the feasible set Ω'_2 is determined by (2), (EC.7), (EC.9) and

$$0 \leq \frac{1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + (c - \lambda_0 \bar{G}(\bar{\eta}')) \tau(c) \right] \leq \tau(c), \quad \forall (c, \eta) \in [c_L, c_H] \times [0, 1]. \quad (\text{EC.36})$$

Assumption 1 guarantees that the virtual valuation $w(c)$ is non-increasing in c when $v_E + \lambda_1 z_1$ is negative, and non-decreasing otherwise. Moreover, w changes its sign at most once as c increases in $[c_L, c_H]$. Therefore, when $v_E + \lambda_1 z_1 < 0$, w is nonnegative if $c < c_0$, and non-positive if $c_0 < c$, where the threshold c_0 is defined by

$$c_0 \equiv \sup \{c \in [c_L, c_H] : w(c) \geq 0\}. \quad (\text{EC.37})$$

To characterize the optimal mechanism for the government's problem without leniency (EC.35), we make the following assumption.

ASSUMPTION EC.1. *The density function $f(c)$ is differentiable and non-increasing.*

Assumption EC.1 implies that the virtual valuation function $c + F(c)/f(c)$ is increasing.

The optimal mechanism is summarized in the following proposition.

PROPOSITION EC.3. *Under Assumption EC.1, the optimal solution α_1^* and τ^* for the optimization problem (EC.35) is as follows:*

- (1) if $v_E + \lambda_1 z_1 \geq 0$, then $\alpha_1^*(c) = 1$ and $\tau^*(c) = 1$ for any $c \in [c_L, c_H]$;
- (2) if $v_E + \lambda_1 z_1 < 0$ and $c_H \leq \lambda_0 \bar{G}(\bar{\eta}')$, then $\alpha_1^*(c) = 0$ and $\tau^*(c) = 1$ for any $c \in [c_L, c_H]$;
- (3) if $v_E + \lambda_1 z_1 < 0$, $c_H > \lambda_0 \bar{G}(\bar{\eta}')$, and $c_0 \geq \lambda_0 \bar{G}(\bar{\eta}')$, then

$$\alpha_1^*(c) = \begin{cases} \frac{c_0 - \lambda_0 \bar{G}(\bar{\eta}')}{1 + \lambda_1 \bar{G}(\bar{\eta}')} & \forall c \in [c_L, c_0], \\ 0 & \forall c \in (c_0, c_H], \end{cases} \quad \tau^*(c) = \begin{cases} 1 & \forall c \in [c_L, c_0], \\ 0 & \forall c \in (c_0, c_H]; \end{cases}$$

- (4) if $v_E + \lambda_1 z_1 < 0$, $c_H > \lambda_0 \bar{G}(\bar{\eta}')$, and $c_0 < \lambda_0 \bar{G}(\bar{\eta}')$, then

$$\alpha_1^*(c) = 0, \quad \forall c \in [c_L, c_H], \quad \tau^*(c) = \begin{cases} 1 & \forall c \in [c_L, \lambda_0 \bar{G}(\bar{\eta}')], \\ 0 & \forall c \in (\lambda_0 \bar{G}(\bar{\eta}'), c_H], \end{cases}$$

where c_0 is defined by (EC.37).

Proof of Proposition EC.3. In case (1), the coefficient of α_1 in the integrand of the objective function is nonnegative, i.e., $v_E + \lambda_1 z_1 \geq 0$. Therefore, it is optimal to set $\alpha_1^*(c) = 1$ for any $c \in [c_L, c_H]$. In this case, the government can induce firms of all types to participate, and hence, this policy is optimal.

In case (2), the coefficient of α_1 in the integrand of the objective function is negative, i.e., $v_E + \lambda_1 z_1 < 0$. Therefore, ignoring all constraints, it is optimal to set $\alpha_1^*(c) = 0$ for any $c \in [c_L, c_H]$. In this case, the government can still induce firms of all types to participate ($c_H \leq \lambda_0 \bar{G}(\bar{\eta}')$), and hence, this policy is optimal.

In case (3), we first ignore constraints (EC.7) and (EC.36), and solving the relaxed problem point-wise yields

$$\tau(c) = \begin{cases} 1, & \forall c \in [c_L, c_0], \\ 0, & \forall c \in (c_0, c_H], \end{cases}$$

satisfying (EC.7) automatically. Plugging the expression of τ into (EC.8) yields

$$\alpha_1(c) = \begin{cases} \frac{c_0 - \lambda_0 \bar{G}(\bar{\eta}')}{1 + \lambda_1 \bar{G}(\bar{\eta}')}, & \forall c \in [c_L, c_0], \\ 0, & \forall c \in (c_0, c_H]. \end{cases}$$

When $c_0 \geq \lambda_0 \bar{G}(\bar{\eta}')$, the feasibility constraint (EC.36) holds automatically, and thus the solution above is feasible.

In case (4), it is straightforward to verify that our candidate solution is feasible. When $c_L \leq c_0 < \lambda_0 \bar{G}(\bar{\eta}')$, define

$$\begin{aligned} \phi &= -\frac{1}{c_2 - c_1} \int_{c_L}^{c_1} w(y) f(y) dy - \frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')}, \\ \nu(c) &= \frac{1}{(c_2 - c)^2} \left[\int_{c_L}^c w(y) f(y) dy + (c_2 - c) w(c) f(c) \right], & \forall c \in [c_L, c_1], \\ \sigma(c) &= \frac{1}{c_2 - c_1} \int_{c_L}^{c_1} w(y) f(y) dy + w(c) f(c), & \forall c \in [c_1, c_2], \text{ and} \\ \mu(c) &= -\frac{1}{c_2 - c_1} \int_{c_L}^{c_1} w(y) f(y) dy - w(c) f(c), & \forall c \in [c_2, c_H], \end{aligned}$$

where $c_2 = \lambda_0 \bar{G}(\bar{\eta}')$ and c_1 is determined by

$$\int_{c_L}^{c_1} w(y) f(y) dy + (c_2 - c_1) w(c_2) f(c_2) = 0.$$

We first verify the monotonicity of $w(c) f(c)$ on $[c_L, c_2]$. By definition, we have

$$w(c) f(c) = \frac{1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \left[\left[(\lambda_0 z_1 + z_2) (1 + \lambda_1 \bar{G}(\bar{\eta}')) + \underbrace{(v_E + \lambda_1 z_1) (c - \lambda_0 \bar{G}(\bar{\eta}'))}_{\text{nonnegative and decreasing in } c} \right] f(c) + \underbrace{(v_E + \lambda_1 z_1)}_{<0} F(c) \right].$$

Under Assumption [EC.1](#), we know that $w(c)f(c)$ is decreasing in c .

Then we prove the unique existence of c_1 on $[c_L, c_2]$. Let

$$\psi(c) \equiv \int_{c_L}^c w(y)f(y) dy + (c_2 - c)w(c_2)f(c_2), \quad \forall c \in [c_L, c_2].$$

Because

$$\begin{aligned} \psi'(c) &= w(c)f(c) - w(c_2)f(c_2) \geq 0, \\ \psi(c_L) &= (c_2 - c_L)w(c_2)f(c_2) < 0, \text{ and} \\ \psi(c_2) &= \int_{c_L}^{c_2} w(y)f(y) dy > 0, \end{aligned}$$

the unique existence of c_1 is guaranteed. The first inequality follows from the monotonicity of $w(c)f(c)$, and the last inequality follows from the fact that under the mechanism granting no conditional approval, the government's expected payoff is strictly positive.

Next, we will prove that ϕ , ν , σ , and μ are all nonnegative. By definition of c_1 , we have

$$\phi = (\lambda_0 z_1 + z_2)f(c_2) - \overbrace{\frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')}}^{<0} [1 - F(c_2)] \geq 0.$$

The nonnegativity of ν boils down to

$$h(c) \equiv \int_{c_L}^c w(y)f(y) dy + (c_2 - c)w(c)f(c) \geq 0, \quad \forall c \in [c_L, c_1].$$

Taking derivative yields

$$h'(c) = (c_2 - c) \frac{d}{dc} (w(c)f(c)) \leq 0, \quad \forall c \in [c_L, c_1],$$

where the inequality follows from the monotonicity of $w(c)f(c)$. Therefore, the desired property is reduced to

$$h(c_1) = \int_{c_L}^{c_1} w(y)f(y) dy + (c_2 - c_1)w(c_1)f(c_1) \geq 0,$$

which follows from

$$\int_{c_L}^{c_1} w(y)f(y) dy + (c_2 - c_1)w(c_1)f(c_1) \geq \int_{c_L}^{c_1} w(y)f(y) dy + (c_2 - c_1)w(c_2)f(c_2) = 0.$$

For any $c \in [c_1, c_2]$ and $c \in [c_2, c_H]$, we have

$$\sigma'(c) = \frac{d}{dc} (w(c)f(c)) \leq 0$$

and

$$\mu(c) \geq \mu(c_2),$$

respectively. The first and second inequalities follow from the monotonicity of $w(c)f(c)$ on $c \in [c_L, c_2]$ and

$$\begin{aligned} \mu(c) \geq \mu(c_2) &\Leftrightarrow w(c)f(c) \leq w(c_2)f(c_2) \\ &\Leftrightarrow [(\lambda_0 z_1 + z_2)(1 + \lambda_1 \bar{G}(\bar{\eta}')) + (v_E + \lambda_1 z_1)(c - \lambda_0 \bar{G}(\bar{\eta}'))] f(c) + (v_E + \lambda_1 z_1) F(c) \\ &\leq [(\lambda_0 z_1 + z_2)(1 + \lambda_1 \bar{G}(\bar{\eta}')) + (v_E + \lambda_1 z_1)(c_2 - \lambda_0 \bar{G}(\bar{\eta}'))] f(c_2) + (v_E + \lambda_1 z_1) F(c_2) \end{aligned}$$

for any $c \in [c_2, c_H]$, where the last step follows from

$$v_E + \lambda_1 z_1 < 0, \quad f(c) \leq f(c_2), \quad \text{and} \quad c - \lambda_0 \bar{G}(\bar{\eta}') > 0 = c_2 - \lambda_0 \bar{G}(\bar{\eta}').$$

The desired nonnegativity conditions are automatically implied by $\sigma(c_2) = \mu(c_2) = 0$.

Because

$$\int_{c_L}^c \nu(y) dy = \frac{1}{c_2 - c} \int_{c_L}^c w(y) f(y) dy, \quad \forall c \in [c_L, c_1],$$

we have

$$\begin{aligned} -\phi - \int_{c_L}^{c_1} \nu(y) dy &= \frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')}, \\ (c_2 - c) \nu(c) - \int_{c_L}^c \nu(y) dy &= w(c) f(c), \quad \forall c \in [c_L, c_1], \\ \sigma(c) - \int_{c_L}^{c_1} \nu(y) dy &= w(c) f(c), \quad \forall c \in [c_1, c_2], \text{ and} \\ -\mu(c) - \int_{c_L}^{c_1} \nu(y) dy &= w(c) f(c), \quad \forall c \in [c_2, c_H]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\phi [-\Pi(c_H)] - \int_{c_L}^{c_1} \left[\Pi(c_H) + \int_c^{c_H} \tau(y) dy + (c - \lambda_0 \bar{G}(\bar{\eta}')) \tau(c) \right] \nu(c) dc \\ &+ \int_{c_1}^{c_2} \tau(c) \sigma(c) dc - \int_{c_2}^{c_H} \tau(c) \mu(c) dc \\ &= \left[-\phi - \int_{c_L}^{c_1} \nu(c) dc \right] \Pi(c_H) + \int_{c_L}^{c_1} \left[(c_2 - c) \nu(c) - \int_{c_L}^c \nu(y) dy \right] \tau(c) dc \\ &+ \int_{c_1}^{c_2} \left[\sigma(c) - \int_{c_L}^{c_1} \nu(y) dy \right] \tau(c) dc + \int_{c_2}^{c_H} \left[-\mu(c) - \int_{c_L}^{c_1} \nu(y) dy \right] \tau(c) dc \\ &= \frac{v_E + \lambda_1 z_1}{1 + \lambda_1 \bar{G}(\bar{\eta}')} \Pi(c_H) + \int_{c_L}^{c_H} w(c) \tau(c) dF(c). \end{aligned}$$

Therefore, the objective function is bounded by $\int_{c_1}^{c_2} \sigma(c) dc$. Substituting our candidate primal solution into the objective function yields $\int_{c_L}^{c_2} w(y) f(y) dy$, which is the same value as the upper bound, i.e.,

$$\int_{c_1}^{c_2} \sigma(y) dy = \int_{c_1}^{c_2} \left[\frac{1}{c_2 - c_1} \int_{c_L}^{c_1} w(z) f(z) dz + w(y) f(y) \right] dy$$

$$\begin{aligned}
&= \int_{c_L}^{c_1} w(y) f(y) dy + \int_{c_1}^{c_2} w(y) f(y) dy \\
&= \int_{c_L}^{c_2} w(y) f(y) dy.
\end{aligned}$$

This establishes the optimality of our candidate solution.

When $c_0 = -\infty$, the virtual valuation function w is always non-positive on $[c_L, c_H]$. Moreover, the coefficient of $\Pi(c_H)$ is also negative; hence, it is evident that the optimal mechanism is to not request testing and to grant no conditional approval.

This completes the proof. $\square$

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