Technical Appendix

Effective ZIO Policies for the Single-Warehouse Multi-Retailer Problem with Piecewise Linear Cost Structures

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1 Proof of Lemma 3.8

The first property is identical to that of Lemma 3.2. In what follows, we show that if the second property is not satisfied, i.e. $B_k^i + H_k^i \leq B_{k+1}^i$ for some k, then orders at periods t_k and t_{k+1}^i can be combined and shipped at period t_k^i without increasing total cost.

The cost associated with shipping the combined order to retailer i in period t_k^i and holding the $Q^i_{t^i_{k+1}}$ units in inventory until period t_{k+1}^i is

$$
\begin{aligned}
& K^0(Q^0_{t^i_k} + Q^i_{t^i_{k+1}}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + K^i(Q^i_{t^i_k} + Q^i_{t^i_{k+1}}) + H^i_k Q^i_{t^i_{k+1}} \\
= & K^0(Q^0_{t^i_k} + Q^i_{t^i_{k+1}}) - K^0(Q^0_{t^i_k}) + K^0(Q^0_{t^i_k}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + \\
& K^i(Q^i_{t^i_k} + Q^i_{t^i_{k+1}}) - K^i(Q^i_{t^i_k}) + K^i(Q^i_{t^i_k}) + H^i_k Q^i_{t^i_{k+1}} \\
= & Q^i_{t^i_{k+1}} \Big[\frac{K^0(Q^0_{t^i_k} + Q^i_{t^i_{k+1}}) - K^0(Q^0_{t^i_k}) + K^i(Q^i_{t^i_k} + Q^i_{t^i_{k+1}}) - K^i(Q^i_{t^i_k})}{Q^i_{t^i_{k+1}}} \Big] + \\
& K^0(Q^0_{t^i_k}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + K^i(Q^i_{t^i_k}) + H^i_k Q^i_{t^i_{k+1}} \\
\leq & (B^i_k + H^i_k) Q^i_{t^i_{k+1}} + K^0(Q^0_{t^i_k}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + K^i(Q^i_{t^i_k}) \\
\leq & B^i_{k+1} Q^i_{t^i_{k+1}} + K^0(Q^0_{t^i_k}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + K^i(Q^i_{t^i_k}) \\
\leq & \Big(K^0(Q^0_{t^i_{k+1}}) - K^0(Q^0_{t^i_{k+1}} - Q^i_{t^i_{k+1}}) + K^i(Q^i_{t^i_{k+1}}) \Big) + \Big(K^0(Q^0_{t^i_k}) - K^0(Q^0_{t^i_k} - Q^i_{t^i_k}) + K^i(Q^i_{t^i_k}) \Big).
\end{aligned}
$$

The first inequality is derived from the concavity of the function $K_0(\cdot)$ and the fact that, since $K^i(\cdot)$ exhibits economies of scale, $\frac{K^i(Q^i_{t^i_k} + Q^i_{t^i_{k+1}}) - K^i(Q^i_{t^i_k})}{Q^i_{t^i_{k+1}}} \leq \frac{K^i(Q^i_{t^i_k})}{Q^i_{t^i_k}}$. The last inequality is true since B^i_{k+1} is a lower bound on the cost per unit associated with shipping the $Q^i_{t^i_{k+1}}$ units in period t^i_{k+1}, see Observation 3.7. Observe that the final expression represents the cost associated with the kth and $k+1$th shipments to retailer i in the current policy S, showing that combining the two orders does not increase total costs.

2 Proof of Theorem 2.3

As explained in the paper, it suffices to show that at any iteration of the transformation procedure the current solution S satisfies the condition,

$$Z(S) \leq Z(S,i,k) + \frac{5.6}{4.6}[Z^* - Z(S,i,k)]. \tag{1}$$

Lemmas 3.9 and 3.10, proven below, show that Step 2' and Step 3, respectively, of the transformation procedure preserve this condition.

2.1 Proof of Lemma 3.9

Let S_1 be the on-hand policy before the execution of Step 3 and S_2 be the policy obtained after Step 3 has been performed. Moreover, let k^1 (k^2) be the index of the earliest period in which an order is placed while there are some units in inventory in solution S_1 (resp. S_2). Since the current solution satisfies condition (1) we have that,

$$Z(S_1) \leq Z(S_1,i,k^1) + \frac{5.6}{4.6}[Z^* - Z(S_1,i,k^1)].$$

Observe that $t^i_{k^2} \geq t^i_{k^1}$ since at Step 3 we may consolidate some orders. Moreover, the total cost of satisfying each period's demand in S_2 is no greater than that in S_1 by Lemma 3.8. Hence,

$$Z(S_2,i,k^2) \leq Z(S_1,i,k^2) \leq Z(S_1,i,k^1)$$

and

$$\begin{aligned} Z(S_2) \leq Z(S_1) &\leq Z(S_1,i,k^1) + \frac{5.6}{4.6}[Z^* - Z(S_1,i,k^1)] \\ &\leq Z(S_2,i,k^2) + \frac{5.6}{4.6}[Z^* - Z(S_2,i,k^2)]. \end{aligned}$$

2.2 Proof of Lemma 3.10

In order to show that Step 2' preserves condition (1), we need the following lemma.

Lemma 2.1 *At least one of the following inequalities must hold.*

$$4.6\alpha(K_2 - H) \leq \alpha K_1 + (1-\alpha)K_2$$

$$4.6(1-\alpha)(K_1 - K_2) \leq \alpha K_1 + (1-\alpha)K_2 + \beta K_2$$

$$4.6\alpha(K_2 - K_1) + 4.6(1+\beta)H \leq \alpha K_1 + (1-\alpha)K_2 - \beta H$$

Proof. See Chan et al. (1999b). ∎

Now we are ready to prove Lemma 3.10: If the current solution S satisfies (1), then it continues to hold after Step 2' has been executed.

Proof. Let S_1 and S_2 be the on-hand solutions before and after an iteration of Step 2'. Let k be the index of the earliest period in which an order is placed while there are some units in inventory in solution S_1. In addition, we let H' be the cost of holding the units to satisfy the demands at periods $t_k^i, t_k^i + 1, \ldots, t_{k+1}^i - 1$ from period t_k^i until their consumption.

By assumption, the current solution satisfies condition (1), that is,

$$Z(S_1) \leq Z(S_1, i, k) + \frac{5.6}{4.6}[Z^* - Z(S_1, i, k)]. \tag{2}$$

We prove the lemma by showing that condition (1) is preserved when executing Step 2' under each of the three possible situations.

1. If $4.6\alpha(K_2 - H) \leq \alpha K_1 + (1-\alpha)K_2$ then **Combine 1** is executed.

   In this case, in policy S_2, t_{k+1}^i is the earliest period in which an order is placed while there is remaining inventory at retailer i, and

$$Z(S_1, i, k) \geq \alpha K_1 D + (1-\alpha)K_2 D + H' + Z(S_1, i, k+1).$$

The inequality is true since K_1 is a lower bound on the cost per unit associated with shipping at period t_{k-1} (see Observations 3.6 and 3.7) and holding the units in inventory until period t_k.

Observe that, using a similar argument as that in Lemma 3.4, since K_2 is an upper bound on the total transportation cost associated with sending one more unit in period t^i_k, and $H^i_{k-1} + K^{0'}(Q^0_{t^i_{k-1}})$ is a lower bound on the savings in holding and supplier-warehouse transportation cost at time t^i_{k-1}, $\alpha D(K_2 - (H^i_{k-1} + K^{0'}(Q^0_{t^i_{k-1}})))$ is an upper bound on the increase in cost incurred when executing Combine 1. Thus,

$$\begin{aligned}
Z(S_2) &\le Z(S_1) + \alpha(K_2 - H)D \\
&\le Z(S_1) + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2]D \\
&\le Z(S_1, i, k) + \frac{5.6}{4.6}[Z^* - Z(S_1, i, k)] + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2]D \\
&\le \frac{5.6}{4.6}Z^* - \frac{1}{4.6}[\alpha K_1 D + (1-\alpha)K_2 D + H' + Z(S_1, i, k+1)] \\
&\quad + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2]D \\
&\le Z(S_1, i, k+1) + \frac{5.6}{4.6}[Z^* - Z(S_1, i, k+1)].
\end{aligned}$$

Observe now that $Z(S_2, i, k+1) \le Z(S_1, i, k+1)$. This is true because when executing Combine 1 the order placed at period t^i_k is increased, leading to lower transportation cost per unit ordered at that period, while all later orders remain identical in both S_1 and S_2. Hence

$$Z(S_2) \le Z(S_2, i, k+1) + \frac{5.6}{4.6}[Z^* - Z(S_2, i, k+1)].$$

2. Else if $4.6(1-\alpha)(K_1 - K_2) \le \alpha K_1 + (1-\alpha)K_2 + \beta K_2$ **Combine 2** is executed.

   In this case, following policy S_2, t^i_{k+2} is the earliest period in which an order is placed while there is remaining inventory at retailer i. Let K' be the total ordering and holding costs associated with the quantity $\alpha^i_{k+1}Q^i_{t^i_{k+1}}$ ordered at period t^i_{k+1} in policy S_1 to satisfy demand of periods earlier than t^i_{k+2}. Then

$$Z(S_1, i, k) \geq \alpha K_1 D + (1-\alpha)K_2 D + H' + \beta K_2 D + K' + Z(S_1, i, k+2).$$

Again, the inequality is due to the fact that K_1 and K_2 are lower bounds on the cost per unit associated with ordering at retailer, in periods t^i_{k-1} and t^i_k respectively, and holding the units in inventory up to period t^i_k, in the current policy S_1.

Observe that,

- K_1 is also an upper bound on the cost of sending one more unit at period t^i_{k-1} and holding it in inventory until period t_k,
- K_2 is a lower bound on the transportation cost per unit associated with the current shipment to retailer i at period t^i_k, and
- there is no increase in cost per unit associated with transferring units from t^i_k to t^i_{k+1} because the current policy satisfies Lemma 3.8, $B^i_k + H^i_k > B^i_{k+1}$. (Note that we are using the fact that $B^i_k + H^i_k$ is a lower bound on the cost that those units were incurring in period t^i_k and B^i_{k+1} is an upper bound on the cost that they will incur when added to t^i_{k+1}).

Thus,

$$(1-\alpha)(K_1 - K_2)D$$

is an upper bound on the increase in cost incurred when executing Combine 2, and

$$
\begin{aligned}
Z(S_2) &\le Z(S_1) + (1-\alpha)(K_1 - K_2)D \\
&\le Z(S_1) + \frac{1}{4.6}[\alpha K_1 + (1-\alpha+\beta)K_2]D \\
&\le Z(S_1,i,k) + \frac{5.6}{4.6}[Z^* - Z(S_1,i,k)] + \frac{1}{4.6}[\alpha K_1 + (1-\alpha+\beta)K_2]D \\
&\le \frac{5.6}{4.6}Z^* - \frac{1}{4.6}[\alpha K_1 D + (1-\alpha)K_2 D + H' + \beta K_2 D + K' + Z(S_1,i,k+2)] \\
&\quad + \frac{1}{4.6}[\alpha K_1 + (1-\alpha+\beta)K_2]D \\
&\le Z(S_1,i,k+2) + \frac{5.6}{4.6}[Z^* - Z(S_1,i,k+2)].
\end{aligned}
$$

Observe finally that, since the quantities ordered at periods t_{k+2} to T remain the same and that at t_{k+1} increases by executing Combine 2, $Z(S_2,i,k+2) \le Z(S_1,i,k+2)$. Hence

$$
Z(S_2) \le Z(S_2,i,k+2) + \frac{5.6}{4.6}[Z^* - Z(S_2,i,k+2)].
$$

3. Else **Combine 3** is executed and Lemma 2.1 implies that

$$
4.6\alpha(K_2 - K_1) + 4.6(1+\beta)H \le \alpha K_1 + (1-\alpha)K_2 - \beta H.
$$

In this case, t^i_{k+1} is the earliest period in which an order is placed while there is remaining inventory at retailer i following policy S_2, and, as in the first case,

$$
Z(S_1,i,k) \ge \alpha K_1 D + (1-\alpha)K_2 D + H' + Z(S_1,i,k+1).
$$

The increase in transportation cost in the supplier-warehouse link after moving the Q^0_k units ordered at period t_k to period t_{k-1} is at most $K^{0'}(Q^0_{t^i_{k-1}}) - K^{0'}(Q^0_{t^i_k})$ since the first is an upper bound on the cost of sending an additional unit and the second is a lower bound on the cost per unit associated with sending the current quantity. On the other hand, in the warehouse-retailer link we save in transportation cost by consolidating the two orders. In fact, if $\frac{K^i(Q^i_{t^i_{k-1}})}{Q^i_{t^i_{k-1}}} > \frac{K^i(Q^i_{t^i_k})}{Q^i_{t^i_k}}$, then the ordering cost decreases by at least

$$(1-\alpha^i_{k-1})Q^i_{t^i_{k-1}}\Big(\frac{K^i(Q^i_{t^i_{k-1}})}{Q^i_{t^i_{k-1}}} - \frac{K^i(Q^i_{t^i_k})}{Q^i_{t^i_k}}\Big).$$

However, holding cost increases by $Q^i_{t^i_k}H^i_{k-1}$. Hence the total increase in cost incurred when performing Combine 3 is no more than

$$Q^i_{t^i_k}H^i_{k-1} + Q^i_{t^i_k}(K^{0'}(Q_{t^i_{k-1}}) - K^{0'}(Q^0_{t^i_k})) + (1-\alpha^i_{k-1})Q^i_{t^i_{k-1}}\Big(\frac{K^i(Q^i_{t^i_k})}{Q^i_{t^i_k}} - \frac{K^i(Q^i_{t^i_{k-1}})}{Q^i_{t^i_{k-1}}}\Big)$$

$$= \alpha(K_2 - K_1)D + (1+\beta)HD.$$

Hence,

$$\begin{aligned}
Z(S_2) &\le Z(S_1) + \alpha(K_2-K_1)D + (1+\beta)HD \\
&\le Z(S_1) + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2 - \beta H]D \\
&\le Z(S_1,i,k) + \frac{5.6}{4.6}[Z^* - Z(S_1,i,k)] + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2 - \beta H]D \\
&\le \frac{5.6}{4.6}Z^* - \frac{1}{4.6}[\alpha K_1 D + (1-\alpha)K_2 D + H' + Z(S_1,i,k+1)] \\
&\quad + \frac{1}{4.6}[\alpha K_1 + (1-\alpha)K_2 - \beta H]D \\
&\le Z(S_1,i,k+1) + \beta HD + \frac{5.6}{4.6}\{Z^* - [Z(S_1,i,k+1) + \beta HD]\}.
\end{aligned}$$

Observe now that the quantities ordered at periods t^i_{k+1} to T remain the same in S_1 and S_2. However, the portion of the order at period t^i_k used to satisfy demand occurring after t^i_{k+1}, that is, $(1-\alpha^i_k)Q^i_{t^i_k} = \beta D$ units, are ordered at period t_{k-1} in S_2 while at period t_k in S_1. The ordering cost associated with those units is no larger in S_2 than in S_1; on the other hand, holding cost increases by βDH. Thus, $Z(S_2,i,k+1) \le Z(S_1,i,k+1) + \beta DH$, and hence

$$Z(S_2) \le Z(S_2,i,k+1) + \frac{5.6}{4.6}[Z^* - Z(S_2,i,k+1)].$$

∎

3 Extension to Systems with Central Stocks

The bounds on the performance of ZIO policies developed in the paper can be easily extended to a more general distribution problem with central stocks, in which the warehouse is allowed to carry inventory. To show this, we first observe that, since the transportation charges from supplier to warehouse are concave, it is optimal for the warehouse to follow a ZIO policy. Thus, an order, $Q_t^0 > 0$, placed by the warehouse in period t, will cover all of the retailers' orders from a certain period $a(t) \geq t$ to a period $b(t) \geq a(t)$ and can be expressed as

$$Q_t^0 = \sum_{i=1}^{n} \sum_{p=a(t)}^{b(t)} Q_p^i.$$

In this way, each order placed by the retailers in periods $a(t)$ through $b(t)$ is associated with the warehouse order at time t.

Consider an optimal solution, S^*, to the Single-Warehouse Multi-Retailer Problem with Central Stocks. In addition, consider the kth shipment from the warehouse to retailer i and let w_k^i be the time at which the warehouse ordered the units in that shipment. Then, $a(w_k^i)$ ($b(w_k^i)$) denotes the first (last) period in which retailers orders are satisfied by the order placed at the warehouse at time w_k^i. If we apply the same transformation procedure as in the previous section, we observe that before the execution of Step 2 (or Step 2'), for a certain retailer i and index k, the total order quantity at the warehouse associated with a period $t \geq w_{k-1}^i$ in the current policy S satisfies

$$Q_t^0 \geq \sum_{p=max(a(t),t_{k-1}^i)}^{b(t)} Q_p^{*i} + \sum_{y=i+1}^{n} \sum_{p=a(t)}^{b(t)} Q_p^{*y}.$$

Using this, the proofs of Theorem 2.2 and Theorem 2.3 follow using similar arguments. In this case, the definition of C_j^i is slightly different. For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$, we let $C_j^i =$

$$\frac{K^0_{w^i_j}(\sum_{t=t^i_j}^{b(w^i_j)} Q^i_t + \sum_{y=i+1}^n \sum_{t=a(w^i_j)}^{b(w^i_j)} Q^y_t) - K^0_{w^i_j}(\sum_{t=t^i_{j+1}}^{b(w^i_j)} Q^i_t + \sum_{y=i+1}^n \sum_{t=a(w^i_j)}^{b(w^i_j)} Q^y_t) + K^i_{t^i_j}(Q^i_{t^i_j})}{Q^i_{t^i_j}}.$$

To put this extension in perspective, it is appropriate to point out that the model with central stock is directly related to the seminal work of Roundy (1985). In his work, Roundy analyzed the Single-Warehouse Multi-Retailer model with concave ordering cost functions, infinite time horizon and constant demand rates. For this problem, Roundy shows that Power-of-Two policies, which belong to the class of ZIO policies, are highly effective. Our results indicate that in the case of a finite horizon, time varying demand and modified all unit discount costs, ZIO policies are very effective as well. Indeed, by restricting the solution set to ZIO policies we can obtain a solution whose cost is no higher than 4/3 times the optimal cost and this bound is tight. If the transportation cost functions do not change from period to period, then there exists a ZIO policy whose cost is no higher than $\frac{5.6}{4.6}$ times the optimal cost. Unfortunately, finding the optimal ZIO policy in our case is an NP-hard problem. This is in contrast to the model analyzed by Roundy where finding the best Power-of-Two policy can be done very efficiently.

4 References

Chan, L. M. A., A. Muriel, Z. J. Shen and D. Simchi-Levi. 1999b. On the Effectiveness of Zero-Inventory-Ordering Policies for the Economic Lot Sizing Problem with a Class of Piece-Wise Linear Cost Structures. To appear in *Operations Research*.

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