Appendix

A. Omitted Proofs of Section 2

Proof of Lemma 1: The key idea is that given $\mathsf{MMS}_i \geq 1$ for an agent a_i, then for every item $b_j \in \mathcal{M}$ we have $\mathsf{MMS}_i^{n-1}(\mathcal{M} \setminus b_j) \geq 1$. This holds since removing an item from $\mathcal{M}$ will diminish the value of at most one partition in the optimal n partitioning of the items. Therefore, at least $n-1$ partitions have a value of 1 or more to a_i and thus $\mathsf{MMS}_i^{n-1}(\mathcal{M} \setminus b_j) \geq 1$. The rest of the proof follows from the definition of α-irreducibility. If the valuation of an item b_j to an agent a_i is at least α, then the problem is α-reducible since if we allocate b_j to a_i, we have

$$\mathsf{MMS}_k^{n-1}(\mathcal{M} \setminus \{b_j\}) \geq 1$$

for every agent $a_k \neq a_i$. This contradicts the α-irreducibility assumption. □

Proof of Lemma 2: Suppose for the sake of contradiction that for every agent $a_{i'} \neq a_i$ we have $V_{i'}(\{b_j, b_k\}) \leq 1$. By this assumption, we show

$$\mathsf{MMS}_{i'}^{n-1}(\mathcal{M} \setminus \{b_j, b_k\}) \geq 1 \tag{1}$$

holds. This is true since removing two items b_j and b_k from $\mathcal{M}$ decreases the value of at most two partitions of the optimal partitioning of $\mathcal{M}$ for $\mathsf{MMS}_{i'}$. If $n-1$ partitions remain intact, then Inequality (1) trivially holds. If not, merging the two partitions that initially contained b_j and b_k results in a partition with value at least 1 to a_i. This partition together with the $n-2$ remaining partitions result in a desirable partitioning of $\mathcal{M}$ into $n-1$ partitions. Therefore, Inequality (1) holds for any agent $a_{i'}$, and this implies that by allocating $S = \{b_j, b_k\}$ to a_i, not only does $V_i(S) \geq 3/4$ hold, but also for every $a_{i'} \neq a_i$ we have

$$\mathsf{MMS}_{i'}^{n-1}(\mathcal{M} \setminus \{b_j, b_k\}) \geq 1,$$

which means the instance is 3/4-reducible, which contradicts our assumption. □

Proof of Lemma 3: To prove this lemma, we apply Lemma 2, $|T|$ times. Consider an agent $a_i \notin T$. According to the argument in Lemma 2, if we assign b_{j_1} and b_{j_2} to a_{i_1}, a_i can partition the items in $\mathcal{M} \setminus \{b_{j_1}\ b_{j_2}\}$ into $n-1$ partitions with value at least 1 to a_i, i.e.

$$\mathsf{MMS}_i^{n-1}(\mathcal{M} \setminus \{b_{j_1}, b_{j_2}\}) \geq 1.$$

By the same argument, after assigning b_{j_3} and b_{j_4} to a_{i_2}, we have

$$\mathsf{MMS}_i^{n-2}(\mathcal{M} \setminus \{b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4}\}) \geq 1.$$

Repeating above argument $|T|$ times implies $\mathsf{MMS}_i^{n-|T|}(\mathcal{M} \setminus S) \geq 1$. On the other hand, by by the second condition, every agent a_{i_k} satisfies with items $b_{j_{2k-1}}$ and $b_{j_{2k}}$. This means that we can reduce the instance by satisfying the agents in T by the items in S, which contradicts the irreducibility assumption.

□

Proof of Lemma 4: As discussed before, all the vertices in $\ddot{\mathcal{X}}$ are saturated by M. Consequently, for any subset $S \subseteq \ddot{\mathcal{X}}$, all the vertices of S are saturated by M and $|N(S)| \geq |S|$. Let $M(S)$ be the set of vertices that are matched to the vertices of S in M. We know that every vertex of S is present in at least one of the alternating paths which connect $\dot{\mathcal{Y}}$ to $\ddot{\mathcal{Y}}$. Let

$$P = \langle y_0, x_1, y_1, x_2, y_2, \ldots, x_k, y_k \rangle$$

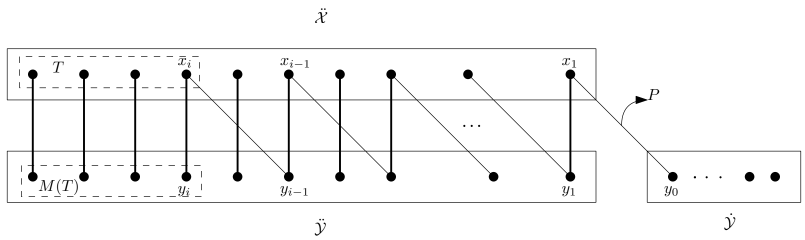

FIGURE 10. Alternating path P connects $\hat{\mathcal{Y}}_1$ to $\hat{\mathcal{Y}}_2$ and intersects S

be one of these paths that includes at least one of the vertices of S. Since P is an alternating path which connects $\dot{\mathcal{Y}}$ to $\ddot{\mathcal{Y}}$, $y_0 \in \dot{\mathcal{Y}}$ (see Figure 10). In addition, according to the definition of alternating path, every edge (x_j, y_j) of P belongs to M and every edge (x_j, y_{j-1}) does not belong to M. Let x_i be the first vertex of S that appears in P. We know that the edge (x_i, y_{i-1}) does not belong to M. On the other hand, since x_i is the first vertex of S in M, $x_{i-1} \notin S$. Note that y_{i-1} does not belong to $M(S)$, since every vertex of $M(S)$ is matched with a vertex of S in M and (x_{i-1}, y_{i-1}) is in M. The fact that $y_{i-1} \notin M(S)$ means $N(S)$ contains at least one vertex that is not in $M(S)$. Since all the vertices in $M(S)$ are in $N(S)$, $|N(S)| > |M(S)|$ and hence, $|N(S)| > |S|$. □

Proof of Lemma 5: We describe the proof for the first condition in more details. The proof for the second condition is almost the same as the first condition.

The first condition. Suppose that there exists no such vertex. Our goal is to find a new matching of G with the same size, but with a more weight. To this end, we construct a directed graph H from G as follows: for each $\hat{y}_j \in T$ we consider a vertex v_j in $V(H)$. Furthermore, there is a directed edge from v_j to v_i in H, if and only if $w(\hat{x}_j, \hat{y}_j) < w(\hat{x}_i, \hat{y}_j)$ in G.

If there exists a vertex v_j with out-degree zero in H, then $\hat{y}_j$ is the desired vertex in T, since

$$\forall \hat{y}_i \in H, w(\hat{x}_j, \hat{y}_j) \geq w(\hat{x}_i, \hat{y}_j).$$

Otherwise, the out-degree of every vertex in G is non-zero. Therefore, H has at least one cycle $L = \langle v_{l_1}, v_{l_2}, \ldots, v_{l_{|L|}} \rangle$. Now, if we change matching M by removing the set of edges

$$\{(\hat{y}_{l_1}, \hat{x}_{l_1}), (\hat{y}_{l_2}, \hat{x}_{l_2}), \ldots, (\hat{y}_{l_{|L|}}, \hat{x}_{l_{|L|}})\}$$

from M and adding

$$\{(\hat{y}_{l_1}, \hat{x}_{l_2}), (\hat{y}_{l_2}, \hat{x}_{l_3}), \ldots, (\hat{y}_{l_{|L|}}, \hat{x}_{l_1})\}$$

to M, the weight of our matching will be increased. Note that by the definition of an edge in H, we have

$$w(\hat{x}_{l_2}, \hat{y}_{l_1}) > w(\hat{x}_{l_1}, \hat{y}_{l_1}), w(\hat{x}_{l_3}, \hat{y}_{l_2}) > w(\hat{x}_{l_2}, \hat{y}_{l_2}), \ldots, w(\hat{x}_{l_1}, \hat{y}_{l_{|L|}}) > w(\hat{x}_{l_{|L|}}, \hat{y}_{l_{|L|}}).$$

But this contradicts the fact that M was MCMWM of G.

The second condition: Similar to the proof of the first condition, we construct a new directed graph H from G where we have a vertex v_j in H for each vertex $\hat{y}_j$ in T. For every pair $\hat{y}_i$ and $\hat{y}_j$ which are members of T we connect v_i to v_j with a directed edge in H if

$$w(\hat{x}_j, \hat{y}_i) > w(\hat{x}_i, \hat{y}_i)$$

in G and $(\hat{x}_j, \hat{y}_i) \in E(G)$. Note that if H contains a vertex v_i with in-degree equal to zero, then $\hat{y}_i$ is the desired loser in T. Thus, suppose that no vertex in H has in-degree zero and hence, H has a directed cycle. Let $L = \langle \hat{y}_{l_1}, \hat{y}_{l_2}, \ldots, \hat{y}_{l_{|L|}} \rangle$ be a directed cycle in H. Similar to the proof of the previous condition, we leverage L to alter M to a new matching with more weight, which is a contradiction by the maximality of M.

The third condition: If $w(\hat{x}_i, \hat{y}_i) < w(\hat{x}_j, \hat{y}_i)$, we can replace the edge between $\hat{x}_i$ and $\hat{y}_i$ by $(\hat{x}_j, \hat{y}_i)$ in M which yields a matching with a greater weight. This contradicts the maximality of M. □

B. Omitted Proofs of Section 3.2

Proof of Lemma 6: First, we prove Lemma 16. This lemma ensures that there exists a matching in G_1 that saturates all the vertices in U. Lemma 16 is a consequence of irreducibility. In fact, we show that if the condition in Lemma 16 does not hold, the instance is reducible.

LEMMA 16. *For all $R \subseteq U$ we have $|N(R)| > |R|$.*

Proof. Let M_1 a matching with the maximum number of edges in G_1 . By Lemma 4, it only suffices to show that all the vertices in U belong to $\ddot{U}$. In other words, we must show that $\tilde{U}$ is empty. Recall that by the way we select the vertices of U, we know that no vertex in U is isolated. For the sake of contradiction, suppose that $\tilde{U}$ is not empty. As mentioned before, there exists a matching between $\tilde{U}$ and $N(\tilde{U})$ that saturates all the vertices in $N(\tilde{U})$. Let

$$M_S = \{(x_{j_1}, y_{i_1}), (x_{j_2}, y_{i_2}), \ldots, (x_{j_k}, y_{i_k})\}$$

be this matching. We show that the set of agents $T = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ and the set of items $S = \{f_{i_1}, b_{j_1}, f_{i_2}, b_{j_2}, \ldots, f_{i_k}, b_{j_k}\}$. have all three conditions of Lemma 3 (Note that f_{i_l} contains exactly one item). The first condition is trivial: $|S| = 2|T|$. Regarding the definition of an edge in G_1, we know that $f_{i_l} \cup \{b_{j_l}\}$ satisfy a_{i_l} and hence, the second condition is held as well. For the third condition, we should prove that for every agent a_{i_l} in T,

$$V_{i'}(f_{i_l} \cup \{b_{j_l}\}) < 1 \qquad \forall a_{i'} \notin T.$$

To show this, we consider two cases separately. First, if $a_{i'} \notin \mathcal{C}_1$, by Lemma 3, $V_{i'}(f_{i_l}) < 1/2$. Furthermore, we have $V_{i'}(\{b_{j_l}\}) < 1/2$, which means $V_{i'}(f_{i_l} \cup \{b_{j_l}\}) < 1$.

Now, consider the case that $a_{i'} \in \mathcal{C}_1$. Since $a_{i'} \notin T$, it's corresponding vertex $y_{i'}$ is not in $N(\tilde{U})$, which means $y_{i'} \in \tilde{\mathcal{Y}} \setminus N(\tilde{U})$. By definition, there is no edge between $y_{i'}$ and x_{j_l} and hence, $V_{i'}(\{b_{j_l}\}) < \epsilon_{i'} \leq 1/4$. On the other hand, by the irreducibility assumption and the fact that f_{i_l} contains exactly one item, $V_{i'}(f_{i_l}) < 3/4$. Thus, $V_{i'}(f_{i_l} \cup \{b_{j_l}\}) < 1$.

As a result, $V_{i'}(f_{i_l} \cup \{b_{j_l}\}) < 1$ for every agent $a_{i'} \notin T$ which means the third condition of Lemma 3 is held as well. Thus, regarding Lemma 3, the instance is reducible which contradicts our irreducibility assumption. □

The rest of the proof of Lemma 6 is as follows. Since we used MCMWM to build cluster $\mathcal{C}_1$, regarding Lemma 5, $\mathcal{C}_1$ is envy-cycle-free. Consider the topological ordering of $\mathcal{C}_1$ and let p_{a_i} be the position of a_i in this ordering. More precisely, $p_{a_i} = k$ if a_i is the k-th agent in the topological ordering of $\mathcal{C}_1$. According to Lemma 16, the condition of Hall's Theorem holds for graph G_1 and as a result there exists a matching in G_1 that saturates all the vertices in U. Among all possible maximum matchings of G_1, let M_1 be a maximum matching that minimizes

$$p_{M_1} = \sum_{y_i \in M_1} p_{a_i}.$$

We claim that M_1 is the desired matching described in Lemma 6. To prove our claim, we must show that for any edge $(x_i, y_j) \in M_1$ and any unsaturated vertex $y_k \in N(x_i)$, agent a_k does not envy

a_j. Note that if a_k envies a_j, a_k appears before a_j in the topological ordering of $\mathcal{C}_1$ which means $p_{a_k} < p_{a_j}$. Therefore, if we replace (x_i, y_j) by (x_i, y_k) in M_1, p_{M_1} will be decreased that contradicts the minimality of p_{M_1}. □

Proof of Lemma 7: Let a_j be an agent in $\mathcal{S}_1^r$. First, note that $|f_j| = |g_j| = 1$. Lemma 3 together with Lemma 4 state that $V_i(f_j \cup g_j) < 1$. According to Inequality (1), we have

$$\mathsf{MMS}_i^{|\mathcal{N} \setminus a_j|}(\mathcal{M} \setminus f_j \cup g_j) \geq 1. \tag{2}$$

Note that Equation (2) holds for every agent in $\mathcal{S}_1^r$. Applying Equation (2) to all the agents of $\mathcal{S}_1^r$ yields

$$\mathsf{MMS}_i^{|\mathcal{N} \setminus \mathcal{S}_1^r|}(\mathcal{M} \setminus \bigcup_{y_i \in \mathcal{S}_1^r} f_i \cup g_i) \geq 1.$$

□

Proof of Lemma 8: Suppose for the sake of contradiction that the problem is 3/4-irreducible, and there exists a vertex $y_k \in \mathcal{Y}$ such that $V_k(\{b_i, b_j\}) \geq 3/4$. According to Lemma 2 there exists an agent $a_{k'} \neq a_k$ such that $V_{k'}(\{b_i, b_j\}) \geq 1$. Since the valuations are additive, we know that one of the inequalities $V_{k'}(\{b_i\}) \geq 1/2$ or $V_{k'}(\{b_j\}) \geq 1/2$ holds, which is contradiction, since we know both x_i and x_j are isolated vertices in $\dot{\mathcal{X}}'$ □

C. Omitted Proofs of Section 3.3

Proof of Lemma 11: At this point, for every agent $a_i \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s$, $|f_j| \leq 2$. If $|f_i| = 1$ holds, then according to Lemma 1, value of the item in f_i is less than 3/4 to all other agents. Moreover, if $|f_i| = 2$, then f_i corresponds to a merged vertex. In this case, by Lemmas 5 and 8, value of f_i is less than 3/4 to all other agents. □

Proof of Lemma 9: According to Lemma 10, value of every pair of items in $\mathcal{F}$ is less than 1/2 to a_i. Therefore, f_i contains at least three items. Let b_k be an arbitrary item in f_i. Since $|f_i| \geq 3$, $f_i \setminus \{b_k\}$ is non-empty. On the other hand, S is minimal and hence, none of the sets $f_i \setminus b_k$ and $\{b_k\}$ is feasible for any agent. According to the definition of feasibility for the agents of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s \cup \mathcal{C}_3^b$, we have

$$\forall a_j \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s \cup \mathcal{C}_3^b \qquad V_j(f_i \setminus \{b_k\}) < \epsilon_j$$

and

$$\forall a_j \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s \cup \mathcal{C}_3^b \qquad V_j(\{b_k\}) < \epsilon_j$$

which means

$$\forall a_j \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s \cup \mathcal{C}_3^b \qquad V_j(f_i) < 2\epsilon_j.$$

□

Proof of Lemma 10: The Lemma trivially holds for $\mathcal{C}_1$ and $\mathcal{C}_2$, since removing an agent from a envy-cycle-free set preserves this property. For $\mathcal{C}_3^s$, there may be multiple rounds that an agent is added to $\mathcal{C}_3^s$. We show that adding an agent to $\mathcal{C}_3^s$ preserves envy-cycle-freeness as well.

Suppose that at some round $\mathbb{R}_z$, an agent a_i is added to $\mathcal{C}_3^s$. By Lemma 9 we have:

$$\forall a_j \in \mathcal{C}_3^s, j \neq i, \qquad V_j(f_i) < 2\epsilon_j. \tag{3}$$

Since a_i previously belonged to $\mathcal{C}_3^f$, we have

$$\forall a_j \in \mathcal{C}_3^s, j \neq i, \qquad V_i(f_j) < 1/2. \tag{4}$$

Inequalities (3) and (4) together imply that a_i does not envy any agent in $\mathcal{C}_3^s$, and no agent in $\mathcal{C}_3^s$ envies a_i, which implies that $\mathcal{C}_3^s$ remains envy-cycle-free. □

Proof of Lemma 11: By the priority rule we select the agents from $\Phi(S)$, If $a_j \prec_{pr} a_i$, then g_i is not feasible for a_j. Thus, $V_j(g_i) < \epsilon_j$. For the case where $a_i \prec_{pr} a_j$, let b_k be an arbitrary item of g_i. According to the fact that g_i is minimal, $g_i \setminus \{b_k\}$ is not feasible for any agent. Hence, $V_j(g_i \setminus \{b_k\}) < \epsilon_j$. On the other hand, by Observations 5 and 7, $V_j(\{b_k\}) < \epsilon_j$. Therefore, $V_j(g_i) < 2\epsilon_j$. □

Proof of Lemma 12: Let $\mathbb{R}_z$ be the round, in which a_i is satisfied. At that point, if $a_j \in \mathcal{C}_3^f$ then $V_j(g_i) < 1/2$ trivially holds. Since in round $\mathbb{R}_z$, $a_j \prec_{pr} a_i$ holds, g_i was not feasible for a_j in the first place. Recall that in each round, we select the smallest agent in $\Phi(S)$ with respect to $\prec_{pr}$.

Furthermore, if in round $\mathbb{R}_z$, a_j was in $\mathcal{C}_3^s \cup \mathcal{C}_3^b$, according to Observations 5 and 7, $|S| \geq 2$, since no item alone can satisfy a_i. If $|S| = 2$, then by Observation 10, $V_j(g_i) < 1/2$. For the case of $|S| > 2$, let b_k be the item in S with the minimum value to a_j. According to Corollary 2, $V_j(\{b_k\}) < 1/4$. Also, since S is minimal, $S \setminus \{b_k\}$ is not feasible for any agent and hence, $V_j(S \setminus \{b_k\}) < \epsilon_j \leq 1/4$. Thus, $V_j(S) < 1/2$. □

D. Omitted Proofs of Section 3.4

Before proceeding to the proof of Lemma 13, we prove Lemmas 17, 18 and 19.

Lemma 17. *Let a_i be an agent in $\mathcal{S}_3$ and let $\mathbb{R}_z$ be the round of the second phase in which a_i is satisfied. Then, for any other agent a_j that is in $\mathcal{C}_3^f$ in $\mathbb{R}_z$, $V_j(g_i) < 1/2$ holds.*

Proof. In $\mathbb{R}_z$, a_i either belongs to $\mathcal{C}_3^s$ or $\mathcal{C}_3^b$. Thus, $a_j \prec_{pr} a_i$, and g_i is not feasible for a_j in that round. Therefore, $V_j(g_i) < 1/2$. □

Lemma 18. *Let $a_i \in \mathcal{S}_3$ be a satisfied agent and let $\mathbb{R}_z$ be the round in which a_i is satisfied. Then, for every other agent a_j that belongs to $\mathcal{C}_3^s \cup \mathcal{C}_3^b$ in that round, either $V_j(g_i) < \epsilon_j$ or $V_j(f_i) \leq 3/4 - \epsilon_j$.*

Proof. If g_i is not feasible for a_j, then the condition trivially holds. Moreover, by the definition, the statement is correct for the agents of $\mathcal{C}_3^b$. Therefore, it only suffices to consider the case that $a_j \in \mathcal{C}_3^s$ and g_i is feasible for a_j. Due to the priority rules for satisfying the agents in the second phase, $a_i \prec_{pr} a_j$ and hence, a_i cannot be in $\mathcal{C}_3^b$. Thus, $a_i \in \mathcal{C}_3^s$. According to Observation 2 and the fact that $\prec_{pr}$ is equivalent to $\prec_o$ for the agents in $\mathcal{C}_3^s$, we have $V_j(f_i) \leq 3/4 - \epsilon_j$. □

Lemma 19. *During the second phase, for any agent a_i in $\mathcal{C}_3$, we have:*

$$\sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) < |\mathcal{S}_3| + 1/4.$$

Proof. To show Lemma 19, we show that for all the agents $a_j \in \mathcal{S}_3$ except at most one agent, $V_i(f_j \cup g_j) < 1$ holds. To show this, let $\mathbb{R}_z$ be an arbitrary round of the second phase, in which an agent $a_j \in \mathcal{C}_3$ is satisfied. First, note that in $\mathbb{R}_z$, a_j belongs to $\mathcal{C}_3^s \cup \mathcal{C}_3^b$. Also, in round $\mathbb{R}_z$, a_i belongs to one of $\mathcal{C}_3^s, \mathcal{C}_3^b$, or $\mathcal{C}_3^f$.

If $a_i \in \mathcal{C}_3^f$, then by Lemma 17, $V_i(g_j) < 1/2$ holds. On the other hand, by definition, $V_i(f_j) < 1/2$ and hence, $V_i(f_j \cup g_j) < 1$.

Now, consider the case, where $a_i \in \mathcal{C}_3^b \cup \mathcal{C}_3^s$. Note that by Lemma 18, either $V_i(f_j) \leq 3/4 - \epsilon_i$ or $V_i(g_j) < \epsilon_i$. If $V_i(g_j) < \epsilon_i$, then by Lemmas 11 and 9, we know $V_i(f_j) < 3/4$ and hence, $V_i(f_j \cup g_j) < 3/4 + \epsilon_i < 1$.

For the case that $V_i(f_j) \leq 3/4 - \epsilon_i$, let b_l be the item in g_j with the maximum value to a_i. Since $|g_j|$ is minimal, $g_j \setminus \{b_l\}$ is not feasible for any agent, including a_i and thus, $V_i(g_j \setminus \{b_l\}) < \epsilon_i$. Recall that by Corollary 2, there is at most one item b_k in $\mathcal{F}$, such that $V_i(b_k) \geq 1/4$. In addition to this, $V_i(b_k) < 1/2$ trivially holds, since b_k is not assigned to any agent during the clustering phase. If $b_l \neq b_k$, $V_i(g_j) < 1/4 + \epsilon_i$ holds and hence,

$$V_i(f_j \cup g_j) < 3/4 - \epsilon_i + 1/4 + \epsilon_i < 1.$$

Moreover, If $b_l = b_k$, $V_i(g_j) < 1/2 + \epsilon_i$ holds and thus, $V_i(f_j \cup g_j) < 3/4 - \epsilon_i + 1/2 + \epsilon_i < 5/4$. But, this can happen at most one round. Therefore, for all the agents $a_j \in \mathcal{S}_3$ except at most one, $V_i(f_j \cup g_j) < 1$. Also, for at most one agent $a_j \in \mathcal{S}_3$, $V_i(f_j \cup g_j) < 5/4$. Thus,

$$\sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) < |\mathcal{S}_3| + 1/4.$$

□

Proof of Lemma 13: Suppose for the sake of contradiction that $\mathcal{C}_3 \neq \emptyset$. Note that, by the definition of $\mathcal{C}_3^b$, if $\mathcal{C}_3^s = \emptyset$ holds, then consequently $\mathcal{C}_3^b = \emptyset$. Therefore, since we have $\mathcal{C}_3 = \mathcal{C}_3^s \cup \mathcal{C}_3^b \cup \mathcal{C}_3^f$, if $\mathcal{C}_3$ is non-empty, at least either of the two sets $\mathcal{C}_3^s$ or $\mathcal{C}_3^f$ is non-empty. In case $\mathcal{C}_3^s$ is non-empty, let a_i be an agent of $\mathcal{C}_3^s$ which does not envy any other agent in $\mathcal{C}_3^s$. Otherwise let a_i be an arbitrary agent of $\mathcal{C}_3^f$.

According to Lemma 12, for every agent $a_j \in \mathcal{S}_1^s \cup \mathcal{S}_2^s$, $V_i(g_j) < 1/2$ holds. Also, by Lemmas 4 and 9, for every agent $a_j \in \mathcal{S}_1^r \cup \mathcal{S}_2^r$, we have $V_i(g_j) < 1/2$. Therefore,

$$\forall a_j \in \mathcal{S}_1 \cup \mathcal{S}_2 \qquad V_i(g_j) < 1/2.$$

Also, by Lemmas 3 and 6 we know that $V_i(f_j) < 1/2$ for every $a_j \in \mathcal{S}_1 \cup \mathcal{S}_2$. Thus, for every satisfied agent $a_j \in \mathcal{S}_1 \cup \mathcal{S}_2$, $V_i(f_j \cup g_j) < 1$ holds, and hence

$$\sum_{a_j \in \mathcal{S}_1 \cup \mathcal{S}_2} V_i(f_j \cup g_j) < |\mathcal{S}_1 \cup \mathcal{S}_2|. \tag{5}$$

Moreover, by Lemma 19, the total value of items assigned to the agents in $\mathcal{S}_3$ to a_i is less than $|\mathcal{S}_3| + 1/4$. More precisely,

$$\sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) \leq |\mathcal{S}_3| + 1/4. \tag{6}$$

Inequality (5) along with Inequality (6) implies:

$$\begin{aligned}\sum_{a_j \in \mathcal{S}} V_i(f_j \cup g_j) &= \sum_{a_j \in \mathcal{S}_1 \cup \mathcal{S}_2} V_i(f_j \cup g_j) + \sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) \\ &< |\mathcal{S}_1 \cup \mathcal{S}_2| + |\mathcal{S}_3| + 1/4 \\ &= |\mathcal{S}| + 1/4\end{aligned} \tag{7}$$

Recall that the total sum of the item values for a_i is equal to n. In addition to this, since every agent belongs to either of the Clusters $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, or $\mathcal{S}$ we have

$$|\mathcal{S}| + |\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3| = n.$$

Furthermore, every item $b_j \in \mathcal{M}$ either belongs to $\mathcal{F}$ or one of the sets $f_{j'}$ and $g_{j'}$ for an agent $a_{j'}$. More precisely,

$$\mathcal{F} = \mathcal{M} \setminus \Big[\bigcup_{a_j \in \mathcal{S} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3^s} f_j \cup \bigcup_{a_j \in \mathcal{S}} g_j \Big].$$

Therefore

$$\begin{aligned}\sum_{a_j\in\mathcal{C}_1} V_i(f_j)+\sum_{a_j\in\mathcal{C}_2} V_i(f_j)+\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j)+V_i(\mathcal{F}) &= V_i(\mathcal{M})-\sum_{a_j\in\mathcal{S}} V_i(f_j\cup g_j)\\ &= n-\sum_{a_j\in\mathcal{S}} V_i(f_j\cup g_j)\\ &\geq n-(|\mathcal{S}|+1/4)\\ &= |\mathcal{C}_1|+|\mathcal{C}_2|+|\mathcal{C}_3|-1/4\end{aligned} \tag{8}$$

According to Lemmas 3 and 7,

$$\sum_{a_j\in\mathcal{C}_1} V_i(f_j) < 1/2|\mathcal{C}_1| \tag{9}$$

and

$$\sum_{a_j\in\mathcal{C}_2} V_i(f_j) < 1/2|\mathcal{C}_2| \tag{10}$$

hold. Inequalities (8), (9), and (10) together prove

$$\begin{aligned}V_i(\mathcal{F}) &\geq |\mathcal{C}_1|+|\mathcal{C}_2|+|\mathcal{C}_3|-1/4-\Big[\sum_{a_j\in\mathcal{C}_1} V_i(f_j)+\sum_{a_j\in\mathcal{C}_2} V_i(f_j)+\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j)\Big]\\ &\geq |\mathcal{C}_1|+|\mathcal{C}_2|+|\mathcal{C}_3|-1/4-\Big[1/2|\mathcal{C}_1|+1/2|\mathcal{C}_2|+\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j)\Big]\\ &\geq 1/2|\mathcal{C}_1|+1/2|\mathcal{C}_2|+|\mathcal{C}_3|-1/4-\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j).\end{aligned} \tag{11}$$

Now, we consider two cases separately: (i) $a_i\in\mathcal{C}_3^s$ and (ii) $a_i\in\mathcal{C}_3^f$.

In case $a_i\in\mathcal{C}_3^s$, since a_i does not envy any other agent in $\mathcal{C}_3^s$, we have

$$\begin{aligned}\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j) &\leq \sum_{a_j\in\mathcal{C}_3^s} V_i(f_i)\\ &= \sum_{a_j\in\mathcal{C}_3^s} 3/4-\epsilon_i\\ &= (3/4-\epsilon_i)|\mathcal{C}_3^s|.\end{aligned} \tag{12}$$

This combined with Inequality (11) concludes

$$\begin{aligned}V(\mathcal{F}) &\geq 1/2|\mathcal{C}_1|+1/2|\mathcal{C}_2|+|\mathcal{C}_3|-1/4-\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j)\\ &\geq 1/2|\mathcal{C}_1|+1/2|\mathcal{C}_2|+|\mathcal{C}_3|-1/4-(3/4-\epsilon_i)|\mathcal{C}_3^s|\\ &\geq 1/2|\mathcal{C}_1|+1/2|\mathcal{C}_2|+(1/4+\epsilon)|\mathcal{C}_3|-1/4.\end{aligned}$$

On the other hand, since $a_i\in\mathcal{C}_3^s$, $|\mathcal{C}_3|\geq 1$ and hence, $V_i(\mathcal{F})\geq 1/4+\epsilon_j-1/4=\epsilon_j$. This means that $\mathcal{F}$ is feasible for a_i, which contradicts the termination of the algorithm.

In case $a_i\in\mathcal{C}_3^f$, by the definition of $\mathcal{C}_3^f$ we know that $\sum_{a_j\in\mathcal{C}_3^s} V_i(f_j)<1/2|\mathcal{C}_3^s|$, which by Inequality (11) implies:

$$V_i(\mathcal{F}) > 1/2|\mathcal{C}_3^s|+|\mathcal{C}_3^b|+|\mathcal{C}_3^f|+1/2|\mathcal{C}_2|+1/2|\mathcal{C}_1|-1/4.$$

Since $a_i\in\mathcal{C}_3^f$, we have $|\mathcal{C}_3^f|\geq 1$ and hence, $V_i(\mathcal{F})>3/4$. Again, this contradicts the termination of the algorithm since $\mathcal{F}$ is feasible for a_i. □

Proof of Lemma 14: By Lemma 13, we already know $\mathcal{C}_3=\emptyset$. Now, let a_i be an agent of $\mathcal{C}_1$, which does not envy any other agent in $\mathcal{C}_1$. For convenience, we color the items in either blue or white.

Intuitively, blue items may have a high value for a_i whereas white items are always of lower value to a_i. Initially, all items are colored in white. For each $a_j \in \mathcal{N}$, if $|f_j| = 1$, then we color the item in f_j in blue. Moreover, for every $a_j \in \mathcal{S}$, if $|g_j| = 1$ and $V_i(g_j) \geq \epsilon_i$, then we color the item in g_j in blue.

Now, let $\mathcal{P} = \langle P_1, P_2, \ldots, P_n \rangle$ be the optimal n-partitioning of the items in $\mathcal{M}$ for a_i, that is, the value of every partition P_k to a_i is at least 1. Based on the coloring procedure, we have three types of partitions in $\mathcal{P}$:

- B_2: the set of partitions with at least two blue items
- B_1: the set of partitions with exactly one blue item
- B_0: the set of partitions without any blue items

Note that every partition in $\mathcal{P}$ belongs to one of B_0, B_1 or B_2. Hence,

$$|B_0| + |B_1| + |B_2| = n \tag{13}$$

As declared, all the items in the partitions of B_0 are white. The total value of these items to a_i is at least $|B_0| \geq 4\epsilon_i |B_0|$, which is

$$\sum_{P_k \in B_0} \sum_{b_j \in P_k} V_i(b_j) \geq 4\epsilon_i |B_0|. \tag{14}$$

Also, each partition in B_2 has at least two blue items, each of which is singly assigned to another agent. We decompose the partitions of B_1 into two disjoint sets, namely $\hat{B}_1$ and $\tilde{B}_1$. More precisely, let $\hat{B}_1$ be the partitions in B_1, in which the blue item is worth more than $V_i(f_i)$ to a_i and $\tilde{B}_1 = B_1 \setminus \hat{B}_1$. As such, for each partition $P_k \in \tilde{B}_1$, the white items in P_k are worth at least

$$\begin{aligned} 1 - V_i(f_i) &= 1 - (3/4 - \epsilon_i) \\ &= 1/4 + \epsilon_i \\ &\geq 2\epsilon_i \end{aligned}$$

to a_i. Therefore,

$$\sum_{P_k \in \tilde{B}_1} V_i(\mathcal{W}(P_k)) \geq 2|\tilde{B}_1|\epsilon_i \tag{15}$$

where $\mathcal{W}(S)$ stands for the set of white items in a set S of items. On the other hand, since the problem is 3/4-irreducible, by Lemma 1, no item alone is of worth 3/4 to a_i and thus for each partition $P_k \in \hat{B}_1$, the white items in P_k have a value of at least $1/4 \geq \epsilon_i$ to a_i. This implies that

$$\sum_{P_k \in \hat{B}_1} V_i(\mathcal{W}(P_k)) \geq |\hat{B}_1|\epsilon_i. \tag{16}$$

By Inequalities (13),(14), (15), and (16) we have

$$\begin{aligned} V_i(\mathcal{W}(\mathcal{M})) &= \sum_{P_j \in B_0} V_i(\mathcal{W}(P_j)) + \sum_{P_j \in B_1} V_i(\mathcal{W}(P_j)) + \sum_{P_j \in B_2} V_i(\mathcal{W}(P_j)) \\ &\geq \sum_{P_j \in B_0} V_i(\mathcal{W}(P_j)) + \sum_{P_j \in B_1} V_i(\mathcal{W}(P_j)) \\ &\geq \sum_{P_j \in B_0} V_i(\mathcal{W}(P_j)) + \sum_{P_j \in \hat{B}_1} V_i(\mathcal{W}(P_j)) + \sum_{P_j \in \tilde{B}_1} V_i(\mathcal{W}(P_j)) \\ &\geq |B_0|4\epsilon_i + |\hat{B}_1|\epsilon_i + |\tilde{B}_1|2\epsilon_i \\ &\geq |B_0|4\epsilon_i + |B_1|2\epsilon_i - |\hat{B}_1|\epsilon_i \\ &\geq |B_0|4\epsilon_i + |B_1|4\epsilon_i + |B_2|4\epsilon_i - |B_1|2\epsilon_i - |B_2|4\epsilon_i - |\hat{B}_1|\epsilon_i \\ &= (2n - 2|B_2| - |B_1| - |\hat{B}_1|)2\epsilon_i + (|\hat{B}_1|)\epsilon_i \end{aligned} \tag{17}$$

Note that the total value of white items that are assigned to the agents during the algorithm is equal to $V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F}))$. The rest of the white items are still in $\mathcal{F}$. Thus, we have

$$V_i(\mathcal{W}(\mathcal{M})) = V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F})) + V_i(\mathcal{F}) \tag{18}$$

Now, we provide an upper bound on the value of $V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F}))$. As a warm up, one can trivially prove an upper bound of $2\epsilon_i(2n-1-|B_1|-2|B_2|)$ on $V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F}))$. This follows from the fact that two sets of items are assigned to any agent and hence we have a total of $2n$ disjoint sets. Among these $2n$ sets, at least one of them is empty (since $g_i=\emptyset$) and at least $|B_1|+2|B_2|$ of the sets contain a single blue item. On the other hand, by Lemmas 5, 8, 9 and 11 every set with white items is of worth at most $2\epsilon_i$ to a_i. Therefore, the total value of the white items in $\mathcal{M}\setminus\mathcal{F}$ to a_i is less than $2\epsilon_i(2n-1-|B_1|-2|B_2|)$ and thus

$$V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F})) \le 2\epsilon_i(2n-1-|B_1|-2|B_2|).$$

However, in order to complete the proof, we need a stronger upper bound on $V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F}))$. To this end, we provide the following auxiliary lemma.

LEMMA 20. *Let a_j be an agent such that $|f_j|=1$ and $V_i(f_j)>V_i(f_i)$. Then, $V_i(g_j)<\epsilon_i$.*

Proof. First, note that if a_j is not satisfied yet, then $g_j=\emptyset$ and therefore $V_i(g_j)<\epsilon_i$. Otherwise, we argue that agent a_j is either satisfied in the second phase, or in the refinement phases of $\mathcal{C}_1$ or $\mathcal{C}_2$.

Consider the case that a_j is satisfied in the second phase. If $a_j\in\mathcal{S}_2^s\cup\mathcal{S}_3$, then by Lemma 11, $V_i(g_j)<\epsilon$ holds. Also, if $a_j\in\mathcal{S}_1^s$, considering the fact that a_i envies a_j, $a_i\prec_{pr}a_j$. Thus, by Lemma 11, we have $V_i(g_j)<\epsilon_i$.

Next, consider the case that a_j is in $\mathcal{S}_1^r\cup\mathcal{S}_2^r$. Note that the matching of the refinement phase of $\mathcal{C}_1$ preserves the property described in Lemma 6. Hence, if a_j belongs to $\mathcal{S}_1^r$, then either $a_j\prec_{pr}a_i$ or there is no edge between y_i and $M_1(y_j)$ in G_1, where $M_1(y_j)$ is the vertex matched with y_j in M_1. If $a_j\prec_{pr}a_i$, according to Observation 2, $V_i(f_j)\le 3/4-\epsilon_i$ holds. On the other hand, by the definition, if no edge exists between y_i and $M_1(y_j)$ in G_1, $V_i(g_j)<\epsilon_i$. In addition to this, if a_j belongs to $\mathcal{S}_2^r$, according to Lemma 8, $V_i(g_j)<\epsilon_i$ holds. Therefore, Lemma 20 holds for the agents in $\mathcal{S}_1^r\cup\mathcal{S}_2^r$. □

Note that since matching M of $G_{1/2}$ for building Cluster $\mathcal{C}_1$ is MCMWM, according to the third condition of Lemma 5, there exists no agent a_k, such that $|g_k|=1$ and $V_i(g_k)>3/4-\epsilon_i$. Otherwise, by assigning the item in g_j to a_i instead of the item in f_i, we can increase the total weight of the matching, that contradicts the maximality of M.

According to Lemma 20, for all the agents a_j with the property that f_j is a blue item that belongs to a partition in $\hat{B}_1$, $V_i(g_j)<\epsilon_i$ holds. The number of such agents is at least $|\hat{B}_1|$. Therefore, the total value of $V_i(\mathcal{W}(\mathcal{M}\setminus\mathcal{F}))$ is less than $2\epsilon_i\cdot(2n-1-|B_1|-2|B_2|-|\hat{B}_1|)+\epsilon_i\cdot|\hat{B}_1|$. Combining the bounds obtained in Observation 17 and Lemma 20 by Inequality (18), we have:

$$V_i(\mathcal{F}) \ge 2\epsilon_i\cdot(2n-2|B_2|-|B_1|-|\hat{B}_1|)+\epsilon_i\cdot(|\hat{B}_1|)-2\epsilon_i\cdot(2n-1-|B_1|-2|B_2|-|\hat{B}_1|)-\epsilon_i\cdot|\hat{B}_1|$$

That is:

$$V_i(\mathcal{F})\ge 2\epsilon_i$$

This contradicts the fact that the set $\mathcal{F}$ is not feasible for a_i.

□

Proof of Lemma 15: Lemmas 13 and 14 state that at the end of the algorithm, $\mathcal{C}_1=\mathcal{C}_3=\emptyset$. Now, let a_i an agent in $\mathcal{C}_2$ which does not envy any other agent of $\mathcal{C}_2$. We consider two cases separately: $\epsilon_i\ge 1/8$ and $\epsilon_i<1/8$.

If $\epsilon_i\ge 1/8$, the proof follows from a similar argument we used to prove Lemma 14.

LEMMA 21. *If $\epsilon_i \geq 1/8$, then the following inequality holds:*

$$\sum_{a_j \in \mathcal{S}} V_i(f_j \cup g_j) \leq |\mathcal{S}| + 1/8.$$

Proof. We know $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. For every agent a_j in $\mathcal{S}_3$, by Lemmas 11 and 9, we know $V_i(f_j) < 3/4$. Also, according to Lemma 11, $V_i(g_j) < \epsilon_i \leq 1/4$. Therefore,

$$\sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) \leq \sum_{a_j \in \mathcal{S}_3} (3/4 + 1/4) = |\mathcal{S}_3|. \tag{19}$$

Now, consider an agent $a_j \in \mathcal{S}_1$. Note that by Lemma 3, $V_i(f_j) < 1/2$. Also, remark that either $a_j \in \mathcal{S}_1^r$ or $a_j \in \mathcal{S}_1^s$. If $a_j \in \mathcal{S}_1^r$ then according to Lemma 4, $V_i(g_j) < 1/2$ holds and hence $V_i(f_j \cup g_j) < 1$. Also, If $a_j \in \mathcal{S}_1^s$, then according to Lemma 11, $V_i(g_j) < 2\epsilon_i < 1/2$. Thus, in both cases, $V_i(f_j \cup g_j) < 1$ and hence:

$$\sum_{a_j \in \mathcal{S}_1} V_i(f_j \cup g_j) \leq \sum_{a_j \in \mathcal{S}_1} 1 = |\mathcal{S}_1|. \tag{20}$$

Finally consider a satisfied agent $a_j \in \mathcal{S}_2$. Again, remark that either $a_j \in \mathcal{S}_2^r$ or $a_j \in \mathcal{S}_2^s$ holds.

Consider the case that $a_j \in \mathcal{S}_2^s$. If $a_j \prec_{pr} a_i$, then by Observation 2, $V_i(f_j) \leq 3/4 - \epsilon_i$ and by Lemma 11, $V_i(g_j) < 2\epsilon_i \leq 1/4 + \epsilon_i$ which means $V_i(f_j \cup g_j) < 1$. Moreover, if $a_i \prec_{pr} a_j$, according to Lemmas 11 and 11, $V_i(f_j \cup g_j) < 3/4 + \epsilon_i \leq 1$. Thus, we have:

$$\sum_{a_j \in \mathcal{S}_2^s} V_i(f_j \cup g_j) \leq \sum_{a_j \in \mathcal{S}_2^s} 1 = |\mathcal{S}_1|. \tag{21}$$

It only remains to investigate the case where $a_j \in \mathcal{S}_2^r$. Note that since a_i is not satisfied in the refinement phase of $\mathcal{C}_2$, if $a_i \prec_{pr} a_j$, then $V_i(g_j) < \epsilon_i \leq 1/4$. Otherwise, we could assign the item in g_j to a_i in the refinement of $\mathcal{C}_2$. Also, by Lemma 11, $V_i(f_j) < 3/4$ holds which yields $V_i(f_j \cup g_j) < 1$.

Finally, if $a_j \prec_{pr} a_i$, by Observation 2 $V_i(f_j) \leq 3/4 - \epsilon_i$ holds. Corollary 1 states that there is at most one item b_k with $\mathcal{M}_k \in \mathcal{X}' \setminus \mathcal{X}'_{1/2}$ and $V_i(b_k) \geq 3/8$. Also, note that since b_k belongs to $\mathcal{X}' \setminus \mathcal{X}'_{1/2}$, $V_i(\{b_k\}) < 1/2$ holds. For agent a_j, let b_l be the item that is assigned to a_j in the refinement of $\mathcal{C}_2$, i.e., $g_j = \{b_l\}$. We have

$$V_i(f_j \cup g_j) \leq 3/4 - \epsilon_i + V_i(\{b_l\}).$$

If $b_l \neq b_k$, $V_i(f_j \cup g_j) \leq 3/4 - \epsilon_i + 3/8$ holds which by the fact that $\epsilon_i \geq 1/8$, implies $V_i(f_j \cup g_j) \leq 3/4 - 1/8 + 3/8 \leq 1$. In addition to this, If $b_l = b_k$, $V_i(f_j \cup g_j) \leq 3/4 - 1/8 + 4/8 \leq 1 + 1/8$. But this can happen for at most one agent. Thus, for every agent a_j in $\mathcal{S}_2^r$, $V_i(f_j \cup g_j) \leq 1$ holds and for at most one agent $a_j \in \mathcal{S}_2^r$, $V_i(f_j \cup g_j) \leq 1 + 1/8$. Thus, we have

$$\sum_{a_j \in \mathcal{S}_2^r} V_i(f_j \cup g_j) \leq |\mathcal{S}_2^r| + 1/8. \tag{22}$$

Inequality (22) together with Inequality (21) yields

$$\sum_{a_j \in \mathcal{S}_2} V_i(f_j \cup g_j) \leq |\mathcal{S}_2| + 1/8. \tag{23}$$

Furthermore, by Inequalities (19), (20) and (23) we have

$$\begin{aligned} \sum_{a_j \in \mathcal{S}} V_i(f_j \cup g_j) &= \sum_{a_j \in \mathcal{S}_1} V_i(f_j \cup g_j) + \sum_{a_j \in \mathcal{S}_2} V_i(f_j \cup g_j) + \sum_{a_j \in \mathcal{S}_3} V_i(f_j \cup g_j) \\ &\leq |\mathcal{S}_1| + |\mathcal{S}_2| + 1/8 + |\mathcal{S}_3| \\ &\leq |\mathcal{S}| + 1/8. \end{aligned} \tag{24}$$

□

By Lemma 21, value of agent a_i for the items allocated to the satisfied agents is less than $|\mathcal{S}|+1/8$. Recall that $\mathcal{C}_2=\mathcal{C}_3=\emptyset$ and hence $|\mathcal{S}|=n-|\mathcal{C}_2|$. Therefore,

$$\sum_{a_j\in\mathcal{S}} V_i(f_j\cup g_j)\leq n-|\mathcal{C}_2|+1/8. \tag{25}$$

Since a_i does not envy any agent of $\mathcal{C}_2$, for all $a_j\in\mathcal{C}_2$, we have $V_i(f_j)\leq V_i(f_i)$. On the other hand, since the total value of all items for a_i is equal to n we have

$$\begin{aligned} V_i(\mathcal{F}) &= V_i(\mathcal{M})-\sum_{a_j\in\mathcal{C}_2} V_i(f_j)-\sum_{a_j\in\mathcal{S}} V_i(f_j\cup g_j)\\ &= n-\sum_{a_j\in\mathcal{C}_2} V_i(f_j)-\sum_{a_j\in\mathcal{S}} V_i(f_j\cup g_j)\\ &\geq n-\sum_{a_j\in\mathcal{C}_2} V_i(f_j)-\left[n-|\mathcal{C}_2|+1/8\right]\\ &= |\mathcal{C}_2|-1/8-\sum_{a_j\in\mathcal{C}_2} V_i(f_j). \end{aligned} \tag{26}$$

Also, $V_i(f_i)=3/4-\epsilon_i$ holds and $V_i(f_j)\leq V_i(f_i)$ for any $a_j\in\mathcal{C}_2$ follows from the fact that a_i does not envy other agents of $\mathcal{C}_2$. Therefore by Inequality (26) we have

$$\begin{aligned} V_i(\mathcal{F}) &\geq |\mathcal{C}_2|-1/8-\sum_{a_j\in\mathcal{C}_2} V_i(f_j)\\ &\geq |\mathcal{C}_2|-1/8-\sum_{a_j\in\mathcal{C}_2} V_i(f_i)\\ &= |\mathcal{C}_2|-1/8-|\mathcal{C}_2|V_i(f_i)\\ &= |\mathcal{C}_2|-1/8-|\mathcal{C}_2|(3/4-\epsilon_i)\\ &= |\mathcal{C}_2|(1/4+\epsilon_i)-1/8. \end{aligned}$$

Recall that by definition $\epsilon_i\geq 1/8$ holds. Moreover, $\epsilon_i\leq 1/4$, and thus

$$\begin{aligned} V_i(\mathcal{F}) &\geq |\mathcal{C}_2|(1/4+\epsilon_i)-1/8\\ &\geq |\mathcal{C}_2|2\epsilon_i-1/8\\ &\geq |\mathcal{C}_2|2\epsilon_i-\epsilon_i \end{aligned}$$

and since $|\mathcal{C}_2|\geq 1$,

$$\begin{aligned} V_i(\mathcal{F}) &\geq |\mathcal{C}_2|2\epsilon_i-\epsilon_i\\ &\geq 2\epsilon_i-\epsilon_i\\ &\geq \epsilon_i \end{aligned}$$

and thus $\mathcal{F}$ is feasible for a_i. This contradicts the termination of the algorithm.

Next, we investigate the case that $\epsilon<1/8$. Our proof for this case is similar to the one for $\mathcal{C}_1$. Let $\mathcal{S}_1^r$ be the agents in $\mathcal{S}_1$ that are satisfied in the refinement phase and let

$$\mathcal{M}_1^r=\bigcup_{a_j\in\mathcal{S}_1^r} f_j\cup g_j.$$

Lemma 7 states that the maximin value of the agents in $\mathcal{C}_2\cup\mathcal{C}_3$ for the items in $\mathcal{M}'=\mathcal{M}\setminus\mathcal{M}_1^r$ is at least 1. More precisely for every $a_j\in\mathcal{C}_2$:

$$\mathsf{MMS}_j^{n-|\mathcal{S}_1^r|}(\mathcal{M}\setminus\mathcal{M}_1^r)\geq 1 \tag{27}$$

We color the items of $\mathcal{M}'$ in one of four colors blue, red, green, or white. Initially, all the items are colored in white. For each agent $a_j \in \mathcal{N} \setminus \mathcal{S}_1^r$, if $|f_j| = 1$, then we color the item in f_j in blue. Also, if $|f_j| = 2$ (which means f_j is corresponding to a merged vertex), color both the elements of f_j in red. In addition to this, if $|g_j| = 1$ then color the item in g_j in green. For any set $S \subseteq \mathcal{M}$, we denote the subset of blue, red, green, and white items in S by $\mathcal{B}(S)$,$\mathcal{R}(S)$, $\mathcal{G}(S)$, and $\mathcal{W}(S)$, respectively. Recall that by Lemma 8, every pair of items in red or green are worth less that 3/4 in total to a_i. In other words, for any two different items $b_j, b_k \in \mathcal{B}(\mathcal{M}) \cup \mathcal{G}(\mathcal{M})$ we have

$$V_i(\{b_j, b_k\}) \leq 3/4.$$

Also, By Lemmas 11 and 9, every set including white items is worth less than $2\epsilon_i < 1/4$ to a_i. Let $n' = n - |\mathcal{S}_1^r|$. Let $\mathcal{P} = \langle P_1, P_2, \ldots, P_{n'} \rangle$ be the optimal $n'-$partitioning of $\mathcal{M}'$ for a_i. Recall that by Inequality (27), the value of every partition in $\mathcal{P}$ is at least 1 for a_i. Based on the number of blue and red items in every partition, we define three sets of partitions:

- B_{00} : Partitions with no red or blue items.
- B_{10} : Partitions with blue items, but without any red items.
- B_{01} : Partitions that contain at least one red item.

Next we prove Lemmas 22 and 23 to be used later in the proof.

Lemma 22. *Let $|\mathcal{G}(B_{00})|$ be the number of green items in the partitions of B_{00}. Then,*

$$V_i(\mathcal{W}(B_{00})) \geq (3|B_{00}| - |\mathcal{G}(B_{00})|) \cdot 1/4.$$

Proof. Let B_{00}^j be the set of partitions in B_{00} that contain exactly j green items. We have:

$$|\mathcal{G}(B_{00})| = \sum_{1 \leq j < \infty} j|B_{00}^j| \geq |B_{00}^1| + 2|B_{00}^2| + \sum_{3 \leq j < \infty} 3|B_{00}^j| \tag{28}$$

Also, we have:

$$3|B_{00}| = \sum_{0 \leq j < \infty} 3|B_{00}^j| = 3|B_{00}^0| + 3|B_{00}^1| + 3|B_{00}^2| + \sum_{3 \leq j < \infty} 3|B_{00}^j| \tag{29}$$

Finally, we argue that the value of white items in B_{00} is at least $|B_{00}^0| + |B_{00}^1| \cdot 1/2 + |B_{00}^3| \cdot 1/4$. This follows from the fact that every green item in $P_k \in B_{00}^1$ has a value less than 1/2 and by Lemma 8, every pair of green items in $P_k \in B_{00}^2$ are worth less than 3/4 to a_j. According to the fact that the value of every partition P_k is at least 1, we have:

$$V_i(\mathcal{W}(B_{00})) \geq |B_{00}^0| + |B_{00}^1| \cdot 1/2 + |B_{00}^3| \cdot 1/4 = \left(4|B_{00}^0| + 2|B_{00}^1| + |B_{00}^2|\right) \cdot 1/4 \tag{30}$$

According to Equations (28) and (29), we have:

$$3|B_{00}| - |\mathcal{G}(B_{00})| \leq 3|B_{00}^0| + 2|B_{00}^1| + |B_{00}^2| \leq 4|B_{00}^0| + 2|B_{00}^1| + |B_{00}^2| \tag{31}$$

Next we combine Equations (30) and (31) to obtain:

$$V_i(\mathcal{W}(B_{00})) \geq \left(3|B_{00}| - |\mathcal{G}(B_{00})|\right) \cdot 1/4. \tag{32}$$

□

Lemma 23. $V_i(\mathcal{W}(B_{10})) \geq (2|B_{10}| - |\mathcal{B}(B_{10})| - |\mathcal{G}(B_{10})|) \cdot 1/4$

Proof. First, note that every partition in B_{10} contains at least one blue item. Let B_{10}^{w} be the partitions in B_{10} that contains exactly one blue item and no green item. The other items in each partition of B_{10}^{w}, are white. Since the problem is 3/4-irreducible, the value of every blue item to a_i is less than 3/4 and therefore we have:

$$V_i(\mathcal{W}(B_{10})) \geq |B_{10}^{w}| \cdot 1/4$$

or

$$4V_i(\mathcal{W}(B_{10})) \geq |B_{10}^{w}|. \tag{33}$$

Moreover, let $B_{10}^{\bar{w}} = B_{10} \setminus B_{10}^{w}$. Since every partition in B_{10} contains at least one blue item, every partition in $B_{10}^{\bar{w}}$ contains at least two items with colors blue or green. Thus, we have:

$$|\mathcal{G}(B_{10}^{\bar{w}})| + |\mathcal{B}(B_{10}^{\bar{w}})| \geq 2|B_{10}^{\bar{w}}| \tag{34}$$

Summing up Equations (33) and (34) results in

$$4V_i(\mathcal{W}(B_{10})) + |\mathcal{G}(B_{10}^{\bar{w}})| + |\mathcal{B}(B_{10}^{\bar{w}})| \geq 2|B_{10}^{\bar{w}}| + |B_{10}^{w}|$$

which implies:

$$4V_i(\mathcal{W}(B_{10})) \geq 2|B_{10}^{\bar{w}}| - |\mathcal{G}(B_{10}^{\bar{w}})| - |\mathcal{B}(B_{10}^{\bar{w}})| + |B_{10}^{w}|. \tag{35}$$

Moreover, we have $|\mathcal{B}(B_{10})| = |\mathcal{B}(B_{10}^{w})| + |\mathcal{B}(B_{10}^{\bar{w}})|$. According to the fact that every partition in B_{10}^{w} contains exactly one blue item, $|\mathcal{B}(B_{10}^{w})| = |B_{10}^{w}|$ and hence, $|\mathcal{B}(B_{10})| = |B_{10}^{w}| + |\mathcal{B}(B_{10}^{\bar{w}})|$. By Equation (35), we have:

$$4V_i(\mathcal{W}(B_{10})) \geq 2|B_{10}^{\bar{w}}| - |\mathcal{G}(B_{10}^{\bar{w}})| - |\mathcal{B}(B_{10})| + |B_{10}^{w}| + |B_{10}^{w}|.$$

Finally by the fact that $2|B_{10}^{w}| + 2|B_{10}^{\bar{w}}| = 2|B_{10}|$, we have:

$$4V_i(\mathcal{W}(B_{10})) \geq 2|B_{10}| - |\mathcal{G}(B_{10}^{\bar{w}})| - |\mathcal{B}(B_{10})|$$

which is:

$$V_i(\mathcal{W}(B_{10})) \geq \left(2|B_{10}| - |\mathcal{B}(B_{10})| - |\mathcal{G}(B_{10}^{\bar{w}})|\right) \cdot 1/4$$

. □

For the partitions in B_{01}, we construct a graph $G_{01}\langle V_{01}, E_{01}\rangle$, where every vertex $v_j \in V_{01}$ corresponds to a partition $P_j \in B_{01}$. Consider an agent a_j such that f_j consists of a pair of red items $b_k, b_{k'}$ and let $b_k \in P_l$ and $b_{k'} \in P_{l'}$. We add an edge $(v_l, v_{l'})$ to E_{01}. By the definition of B_{01}, $P_l, P_{l'} \in B_{01}$ holds. Note that b_k and $b_{k'}$ might belong to the same partition, i.e., $P_l = P_{l'}$. In this case, we add a loop to G_{01}. Furthermore, for every item $b_k \in \mathcal{B}(B_{01})$, we add a loop to the vertex v_l, where $b_k \in P_l$.

Next, define R_j as the set of partitions in B_{01}, such that the degree of their corresponding vertices in V_{01} are equal to j. In other words:

$$P_k \in R_j \iff d(v_k) = j$$

We now prove Lemma 24.

Lemma 24. *For R_1, we have:*

$$V_i(\mathcal{W}(R_1)) \geq (2|R_1| - |\mathcal{G}(R_1)|) \cdot 1/4.$$

Proof. Consider a partition $P_j \in R_1$. Since $d(v_j) = 1$, P_j contains exactly one red item and no blue item. Thus, other items in P_j are either green or white. We show that

$$|\mathcal{G}(P_j)| + 4.V_i(\mathcal{W}(P_j)) \geq 2. \tag{36}$$

First, note that if $|\mathcal{G}(P_j)| \geq 2$, then Inequality (36) holds. Also, if $|\mathcal{G}(P_j)| = 0$, then $V_i(\mathcal{W}(P_j)) \geq 1/2$, because the value of the red item in P_j is less than $1/2$ (recall that all the red items correspond to the vertices in $\mathcal{X}' \setminus \mathcal{X}'_{1/2}$). This immediately implies the fact that $4.V_i(\mathcal{W}(P_j)) \geq 2$. Finally, if $|\mathcal{G}(P_j)| = 1$, then by Lemma 8, the total value of the green and red items in P_j is less than $3/4$ and hence, $V_i(\mathcal{W}(P_j)) \geq 1/4$ which means $|\mathcal{G}(P_j)| + 4.V_i(\mathcal{W}(P_j)) \geq 2$.

Since Inequality (36) holds for every partition $P_j \in R_1$, we have:

$$\sum_{P_j \in R_1} \big(|\mathcal{G}(P_j)| + 4.V_i(\mathcal{W}(P_j))\big) \geq 2|R_1|.$$

Therefore,

$$|\mathcal{G}(R_1)| + 4.V_i(\mathcal{W}(R_1)) \geq 2|R_1|$$

and hence,

$$V_i(\mathcal{W}(R_1)) \geq (2|R_1| - |\mathcal{G}(R_1)|) \cdot 1/4.$$

□

Lemma 25. *For R_2, we have $V_i(\mathcal{W}(R_2)) \geq (|R_2| - |\mathcal{G}(R_2)|) \cdot 1/4$.*

Proof. Let P_j be a partition in R_2. First, we show the following inequality holds:

$$4V_i(\mathcal{W}(P_j)) + |\mathcal{G}(P_j)| \geq 1. \tag{37}$$

By the definition of R_2, degree of v_j is 2. Therefore, P_j contains two red items. Note that the degree of the partitions in B_{01} that contain blue items is at least 3. Thus, P_j contains no blue items. By Lemma 8, the total value of the red items in P_j is less than $3/4$. The rest of the items in P_j are either green or white. If P_j contains a green item, then Inequality (37) holds. On the other hand, if P_j contains no green items, then $V_i(\mathcal{W}(P_j)) \geq 1/4$ and hence, $4V_i(\mathcal{W}(P_j)) \geq 1$. Therefore, Inequality (37) holds in both cases. Summing up Inequality (37) for all the partitions in R_2, we have:

$$\sum_{P_j \in R_2} 4V_i(\mathcal{W}(P_j)) + |\mathcal{G}(P_j)| \geq |R_2|$$

which means:

$$4V_i(\mathcal{W}(R_2)) + |\mathcal{G}(R_2)| \geq |R_2|$$

That is:

$$V_i(\mathcal{W}(R_2)) \geq \big(|R_2| - |\mathcal{G}(R_2)|\big) \cdot 1/4$$

□

Putting together Lemmas 22,23,24, and 25 we obtain the following lower bound on the valuation of a_i for all white items:

$$\begin{aligned}
V_i(\mathcal{W}(\mathcal{M}')) &= V_i(\mathcal{W}(B_{00})) + V_i(\mathcal{W}(B_{01})) + V_i(\mathcal{W}(B_{10}))\\
&\geq \Big(3|B_{00}| - |\mathcal{G}(B_{00})|\Big)\cdot 1/4 + \Big(2|B_{10}| - |\mathcal{B}(B_{10})| - |\mathcal{G}(B_{10})|\Big)\cdot 1/4\\
&\quad + \Big(2|R_1| - |\mathcal{G}(R_1)|\Big)\cdot 1/4 + \Big(|R_2| - |\mathcal{G}(R_2)|\Big)\cdot 1/4\\
&= \Big((3|B_{00}| + 2|B_{10}| + 2|R_1| + |R_2| - |\mathcal{B}(B_{10})|) - (|\mathcal{G}(B_{00})| + |\mathcal{G}(B_{10})| + |\mathcal{G}(R_1)| + |\mathcal{G}(R_2)|)\Big)\cdot 1/4\\
&\geq \Big((3|B_{00}| + 2|B_{10}| + 2|R_1| + |R_2|) - |\mathcal{B}(B_{10})| - |\mathcal{G}(\mathcal{M}')|\Big)\cdot 1/4
\end{aligned} \tag{38}$$

where $|\mathcal{G}(\mathcal{M}')|$ is the total number of green items.

The items in $\mathcal{W}(\mathcal{M}')$ are either allocated to an agent during the second phase, or are still in $\mathcal{F}$. Let $\mathcal{W}_2$ be the white items that are allocated to an agent during the second phase. We have:

$$V_i(\mathcal{W}(\mathcal{M}')) = V_i(\mathcal{W}_2) + V_i(\mathcal{F}). \tag{39}$$

Now, we present an upper bound on the value of $V_i(\mathcal{W}_2)$. First, note that the number of agents in $\mathcal{S}\setminus\mathcal{S}_1^r$ is n'. Each of these n' agents has two sets f_j and g_j, that leaves us $2n'$ sets. Since $g_i = \emptyset$ we know that at least one of these sets is empty. Moreover, all of these $|\mathcal{G}(\mathcal{M}')|$ sets contain a single green item and $|\mathcal{B}(B_{10})| + |E_{01}|$ of the sets contain either a single blue item, or a pair of red items (recall that each edge of G_{01} refers to a blue item or a pair of red items). Therefore, the number of the sets that contain only white items is at most:

$$2n' - 1 - |\mathcal{G}(\mathcal{M}')| - |\mathcal{B}(B_{10})| - |E_{01}|.$$

By Lemmas 11 and 9, the value of every set with white items to a_i is less than $2\epsilon_i < 1/4$ and hence:

$$V_i(\mathcal{W}_2) \leq (2n' - 1 - |\mathcal{G}(\mathcal{M}')| - |\mathcal{B}(B_{10})| - |E_{01}|)\cdot 1/4. \tag{40}$$

Subtracting the lower bound obtained for $V_i(\mathcal{W}(\mathcal{M}'))$ in (38) from the upper bound for $V_i(\mathcal{W}_2)$ in (40) gives us a lower bound on the value of $\mathcal{F}$:

$$\begin{aligned}
V_i(\mathcal{F}) &= V_i(\mathcal{W}(\mathcal{M}')) - V_i(\mathcal{W}_2)\\
&\geq \Big((3|B_{00}| + 2|B_{10}| - |\mathcal{B}(B_{10})| + 2|R_1| + |R_2|) - |\mathcal{G}(\mathcal{M}')|\Big)\cdot 1/4 - V_i(\mathcal{W}_2)\\
&\geq \Big((3|B_{00}| + 2|B_{10}| - |\mathcal{B}(B_{10})| + 2|R_1| + |R_2|) - |\mathcal{G}(\mathcal{M}')|\Big)\cdot 1/4\\
&\qquad - \Big(2n' - 1 - |\mathcal{G}(\mathcal{M}')| - |\mathcal{B}(B_{10})| - |E_{01}|\Big)\cdot 1/4\\
&= \Big(3|B_{00}| + 2|B_{10}| + 2|R_1| + |R_2| - 2n' + 1 + |E_{01}|\Big)\cdot 1/4\\
&= \Big(2|B_{00}| + 2|B_{10}| + |B_{00}| + |E_{01}| + 2|R_1| + |R_2| - 2n' + 1\Big)\cdot 1/4
\end{aligned} \tag{41}$$

Next we provide Lemmas 26, 27, and 28 to complete the proof.

Lemma 26. $|B_{00}| \geq |E_{01}| - |B_{01}|$

Proof. First, note that $|B_{00}|+|B_{10}|+|B_{01}|=n'$. Moreover we have $|\mathcal{B}(B_{10})|+|E_{01}|\leq n'$. To show this Lemma, note that each edge in G_{01} corresponds to the first set of an agent in $\mathcal{S}\setminus\mathcal{S}_1^r$. Also, every blue item in B_{10} corresponds to the first set of an agent in $\mathcal{S}\setminus\mathcal{S}_1^r$. Therefore, the total number of the agents must be more than this number. By the definition of B_{10}, we know that $|\mathcal{B}(B_{10})|\geq|B_{10}|$. Therefore, we have:

$$\begin{aligned}|B_{00}|+|B_{10}|+|B_{01}| &\geq |\mathcal{B}(B_{10})|+|E_{01}| \\ &\geq |(B_{10})|+|E_{01}|\end{aligned} \tag{42}$$

This means:

$$|B_{00}|\geq|E_{01}|-|B_{01}|$$

□

LEMMA 27. $|E_{01}|\geq 3/2\sum_{j\geq 3}|R_j|+|R_2|+|R_1|/2$.

Proof. $|E_{01}|=\frac{\sum_{v_j\in V_{01}}d(v_j)}{2}=\frac{\sum_j j|R_j|}{2}\geq 3/2\sum_{j\geq 3}|R_j|+|R_2|+|R_1|/2$. □

LEMMA 28. $|B_{00}|\geq\frac{\sum_{j\geq 3}|R_j|-|R_1|}{2}$

Proof. Lemma 26, $|B_{00}|\geq|E_{01}|-|B_{01}|$. Furthermore, by Lemma 27,

$$|E_{01}|\geq 3/2\sum_{j\geq 3}|R_j|+|R_2|+|R_1|/2.$$

Combining these two inequalities, we have:

$$|B_{00}|\geq 3/2\sum_{j\geq 3}|R_j|+|R_2|+|R_1|/2-|B_{01}|. \tag{43}$$

Also, since there is a one-to-one correspondence between B_{01} and V_{01}, $|B_{01}|=|V_{01}|$ holds. By the definition of R_j, we have:

$$|V_{01}|=\sum_j|R_j|. \tag{44}$$

By replacing the value obtained for B_{01} from (44) into Inequality (43), we have:

$$\begin{aligned}|B_{00}| &\geq 1/2\sum_{j\geq 3}|R_j|-|R_1|/2 \\ &= \frac{\sum_{j\geq 3}|R_j|-|R_1|}{2}.\end{aligned} \tag{45}$$

□

By applying Lemmas 28 and 27 to Inequality (41) we have:

$$\begin{aligned}V_i(\mathcal{F}) &= \Big(2|B_{00}|+2|B_{10}|+|B_{00}|+|E_{01}|+2|R_1|+|R_2|-2n'+1\Big)\cdot 1/4 \\ &\geq \Big(2|B_{00}|+2|B_{10}|+\frac{\sum_{j\geq 3}|R_j|-|R_1|}{2}+3/2\sum_{j\geq 3}|R_j|+|R_2|+|R_1|/2+2|R_1|+|R_2|-2n'+1\Big)\cdot 1/4 \\ &= \Big(2|B_{00}|+2|B_{10}|+\sum_{j\geq 3}2|R_j|+2|R_2|+2|R_1|-2n'+1\Big)\cdot 1/4\end{aligned}$$

Finally, note that $\sum_{j\geq 3}2|R_j|+2|R_2|+2|R_1|=2|V_{01}|=2|B_{01}|$. This, together with the fact that $|B_{00}|+|B_{01}|+|B_{10}|=n'$, yields $V_i(\mathcal{F})\geq(2n'-2n'+1)\cdot 1/4$. This means $V_i(\mathcal{F})\geq 1/4$ which is a contradiction since $\mathcal{F}$ is feasible for a_i. □