Online Appendix for "Managing Customer Churn via Service Mode Control"

This online appendix is an e-companion for paper "Managing Customer Churn via Service Mode Control" and provides additional proofs and materials in supplement to it. This online appendix is subdivided into six sections. In Online Appendix A, we prove results related to the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and its complement $\text{erfc}(x) = 1 - \text{erf}(x)$. These are used to support and complete the proof of Lemma 5 in the paper. Online appendix B provides the remaining proofs of the Base model, including the remaining proof of Proposition 3 and the proof of Corollary 2. Online appendices C–F demonstrate robustness of our main findings by analyzing four variations of the Base model: allowing mixed policies, the reward process being a geometric Brownian motion, the inclusion of positive switching costs, and alternative hazard rate functions, respectively.

A Properties of the error function

This section provides the proofs for properties 4, 6, 7, and 9 − 16 in Lemma 5.

Proof of Lemma 5 (Part II). We provide the proofs for properties 4, 6, 7, 9 − 16.

To prove 4, it is equivalent to show $f(x) \triangleq x\text{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} > 0$ for all $x \in \mathbb{R}$. In fact, $f'(x) = \text{erf}(x)$. Therefore $f(x)$ is decreasing on the interval $(-\infty, 0)$, increasing on the interval $(0, \infty)$, and the minimum value is obtained at $x = 0$, with $f(0) = \frac{1}{\sqrt{\pi}} > 0$.

Now we prove 6. Denote $x_1 \triangleq \frac{\alpha(\beta-2)+\sqrt{\alpha^2\beta^2+2\beta(1-\beta)}}{2(1-\beta)}$. Since $\alpha > 0$ and $0 < \beta < 1$, we have $x_1 > \frac{\alpha(\beta-2)+\alpha\beta}{2(1-\beta)} = -\alpha$. Also by property 5 (ii), $\{x > -\alpha : r(x) \geq \frac{\beta}{\sqrt{\pi}}\} = [x_0, \infty)$. Hence to show $x_1 \geq x_0$, we only need to show $r(x_1) \geq \frac{\beta}{\sqrt{\pi}}$. Since $x_1 > -\alpha$, we can divide both sides by $(x_1 + \alpha)e^{x_1^2}$ and equivalently show $f(\beta) \geq 0$ for any $\beta \in (0, 1)$, where

$$f(\beta) \triangleq -\frac{\left(\alpha\beta + \sqrt{\alpha^2\beta^2 + 2\beta(1-\beta)}\right) e^{-\frac{\left(\alpha(\beta-2)+\sqrt{\alpha^2\beta^2+2\beta(1-\beta)}\right)^2}{4(\beta-1)^2}}}{\sqrt{\pi}} + \text{erfc}\left(\frac{\alpha(\beta-2) + \sqrt{\alpha^2\beta^2 + 2\beta(1-\beta)}}{2(1-\beta)}\right).$$

One can verify that $\lim_{\beta\to 1^-} \frac{\alpha(\beta-2)+\sqrt{\alpha^2\beta^2+2\beta(1-\beta)}}{2(\beta-1)} = \alpha - \frac{1}{\alpha}$. Therefore, $g(\alpha) \triangleq \lim_{\beta\to 1^-} f(\beta) = \text{erfc}(\frac{1}{2\alpha} - \alpha) - \frac{2\alpha e^{-\left(\alpha-\frac{1}{2\alpha}\right)^2}}{\sqrt{\pi}}$. In fact, $g(\alpha) > 0$ for all $\alpha > 0$, since $\lim_{\alpha\to 0^+} g(\alpha) = 0$ and $g'(\alpha) = \frac{4\alpha^2 e^{-\left(\alpha-\frac{1}{2\alpha}\right)^2}}{\sqrt{\pi}} > 0$. Also, for any $\alpha > 0$ and $\beta \in (0, 1)$, obviously

$$f'(\beta) = -\frac{\left(\alpha\beta + \sqrt{\alpha^2\beta^2 + 2\beta(1-\beta)}\right) e^{-\frac{\left(\alpha(\beta-2)+\sqrt{\alpha^2\beta^2+2\beta(1-\beta)}\right)^2}{4(\beta-1)^2}}}{\beta\sqrt{\pi}} < 0.$$

Therefore $f(\beta)$ is decreasing on $(0, 1)$ with $\lim_{\beta\to 1^-} f(\beta) > 0$, and the result follows.

Now we prove 7. Define $F(x) \triangleq \frac{C}{\sqrt{\pi}} + xe^{x^2}(1 + C\text{erf}(x))$, with $F'(x) = e^{x^2}(2x^2+1)(1+C\text{erf}(x)) + \frac{2Cx}{\sqrt{\pi}}$. If $C \in \{0, 1\}$, then the result trivially holds. If $C = -1$, $F'(x) \geq 0$ follows from property 1. It only remains to consider $C \in (-1, 1)\backslash\{0\}$. Let $f(x) \triangleq F'(x)e^{-x^2}$. We have $f''(x) = 4C\text{erf}(x) + 4 > 0$, $\lim_{x\to-\infty} f'(x) = -\infty$, and $\lim_{x\to\infty} f'(x) = \infty$ if $C \in (-1, 1)$. Thus $f(x)$ is first decreasing then increasing on $\mathbb{R}$ if $C \in (-1, 1)$. Consider its global minimizer x_0, with

$$f'(x_0) = 4\left(Cx_0\text{erf}(x_0) + \frac{Ce^{-x_0^2}}{\sqrt{\pi}} + x_0\right) = 0. \tag{43}$$

Also one can check that $f'(0) = \frac{4C}{\sqrt{\pi}}$, so that $f'(0) > 0$ if $C \in (0,1)$, and $f'(0) < 0$ if $C \in (-1,0)$. Recall that $f''(x) > 0$, hence $x_0 < 0$ if $C \in (0,1)$ and $x_0 > 0$ if $C \in (-1,0)$. In both cases $Cx_0 < 0$, and thus we can divide both sides of Eq. (43) by Cx_0 and obtain $\operatorname{erf}(x_0) = \frac{\frac{Ce^{-x_0^2}}{\sqrt{\pi}} + x_0}{-Cx_0}$. Then we have $\min_{x\in\mathbb{R}} f(x) = f(x_0) = \left(2Cx_0^2 + C\right)\operatorname{erf}(x_0) + \frac{2Ce^{-x_0^2}x_0}{\sqrt{\pi}} + 2x_0^2 + 1 = \frac{2Ce^{-x_0^2}(x_0^2 - x_0 + 1)}{-x_0}$. Observe that $x_0^2 - x_0 + 1 > 0$, and that $\frac{C}{-x_0} > 0$ from the above argument. The result follows.

Next we prove 9. We can divide both sides by $e^{q^2}q$ and show the following inequality instead: $F(q) \triangleq (a^2-2)\left(\frac{e^{-q^2}}{q} + \sqrt{\pi}\left(\operatorname{erf}(q) - \operatorname{erf}\left(a - \frac{1}{a}\right)\right)\right) - 2ae^{-\left(a-\frac{1}{a}\right)^2} < 0$. In fact, if $a^2 - 2 \leq 0$, then since $\operatorname{erf}(\cdot)$ is an increasing function and that $q > a > a - \frac{1}{a}$, $F(q) < 0$ is obviously true. Now consider when $a^2 - 2 > 0$. Compute $F'(q) = -\frac{(a^2-2)e^{-q^2}}{q^2} < 0$ for any $q \in \mathbb{R}$, which implies that for $q > a$ (note that $a > a - \frac{1}{a}$), $F(q) < F(a - \frac{1}{a}) = -\frac{a^3 e^{-\left(\frac{1}{a}-a\right)^2}}{a^2-1} < 0$.

Next we prove 10. Denote $F(\theta) \triangleq e^{q^2}q - e^{\theta^2}\theta - e^{\theta^2+q^2}\theta q\sqrt{\pi}\left(\operatorname{erfc}(\theta) - \operatorname{erfc}(q)\right)$. If $\theta \leq 0$, then the result easily holds since each term is non-negative. Now consider $\theta \in (0,q)$. To show $F(\theta) \geq 0$, it is equivalent to show $\frac{F(\theta)}{e^{q^2+\theta^2}q\theta} \geq 0$. In fact, $\frac{F(\theta)}{e^{q^2+\theta^2}q\theta} = f(\theta) - f(q)$, where $f(\theta) \triangleq \frac{1-\sqrt{\pi}e^{\theta^2}\theta\operatorname{erfc}(\theta)}{e^{\theta^2}\theta}$. Clearly, $e^{\theta^2}\theta$ is positive and increasing in θ for $\theta \in (0,q)$. Also, $1 - \sqrt{\pi}e^{\theta^2}\theta\operatorname{erfc}(\theta)$ is decreasing in θ by applying property 5 (i). Therefore $f(\theta)$ is decreasing in θ for $\theta \in (0,q)$, and the result follows.

Next we prove 11. Denote $F(\theta) \triangleq 2e^{q^2-\theta^2}q\theta - \left(1 + 2\theta^2\right)\left(1 + e^{q^2}q\sqrt{\pi}\left(\operatorname{erfc}(\theta) - \operatorname{erfc}(q)\right)\right)$. One can verify that $F(q) = F'(q) = 0$. Since $q > \theta$, we have $\operatorname{erf}(q) > \operatorname{erf}(\theta)$ and hence $F''(\theta) = -4 - 4e^{q^2}q\sqrt{\pi}\left(\operatorname{erf}(q) - \operatorname{erf}(\theta)\right) < 0$. This implies $F'(\theta) > F'(q) = 0$ on $(-\infty, q)$. Therefore $F(\theta) < F(q) = 0$ for $\theta < q$.

Next we prove 12. Denote $F(\theta) \triangleq \theta(1+2q^2) - e^{q^2-\theta^2}q(1+2q^2) + 2e^{q^2}q(1+q^2)\theta\sqrt{\pi}\left(\operatorname{erf}(q) - \operatorname{erf}(\theta)\right)$. One can verify that $F(q) = 0$ and $F'(q) = 1$. Since $q > \theta$, we have $\operatorname{erf}(q) > \operatorname{erf}(\theta)$ and hence $F(\theta) < 0$ on $(-\infty, 0]$. Now consider $\theta \in (0,q)$. It's not hard to see that $F''(\theta) = -2e^{q^2-\theta^2}q\left(3 + 2q^2 - 2\theta^2\right) < 0$, which implies $F'(\theta) > F'(q) > 0$ for $\theta < q$, and hence $F(\theta) < F(q) = 0$ for $\theta \in (0,q)$ as well.

Next we prove 13 – 15. Define

$$F_1(x,a) \triangleq 2 - 2ax + 2x^2 + \left(3x - a - 2ax^2 + 2x^3\right)e^{x^2}\sqrt{\pi}\operatorname{erf}(x),$$

$$F_2(x,a) \triangleq 2 - 2ax + 2x^2 - \left(3x - a - 2ax^2 + 2x^3\right)e^{x^2}\sqrt{\pi}\operatorname{erfc}(x),$$

$$F_3(x,a) \triangleq 2 - 2ax + 2x^2 + \left(3x - a - 2ax^2 + 2x^3\right)e^{x^2}\sqrt{\pi}(1 + \operatorname{erf}(x)).$$

If $a > 0$ and $x \in (a - \frac{1}{a}, a)$, then equivalently $a \in \left(\max\{x, 0\}, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)$. Therefore it is sufficient to prove $F_1(x,a) > 0$, $F_2(x,a) > 0$, and $F_3(x,a) > 0$ for $a \in \left(x, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}\right)$. Obvious that all three functions are linear in a. Consider the partial derivatives:

$$\frac{\partial F_1}{\partial a}(x,a) = -\sqrt{\pi}e^{x^2}\left(2x^2+1\right)\operatorname{erf}(x) - 2x, \qquad \frac{\partial F_2}{\partial a}(x,a) = \sqrt{\pi}e^{x^2}\left(2x^2+1\right)\operatorname{erfc}(x) - 2x,$$

$$\frac{\partial F_3}{\partial a}(x,a) = -\sqrt{\pi}e^{x^2}\left(2x^2+1\right)\operatorname{erfc}(-x) - 2x.$$

Property 1 implies that $\frac{\partial F_2}{\partial a}(x,a) \geq 0$ and $\frac{\partial F_3}{\partial a}(x,a) \leq 0$. To prove the results, it then suffices to show $F_2(x,x) > 0$, $F_3(x, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}) > 0$, $F_1(x,x) > 0$ and $F_1(x, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}) > 0$ for all $x \in \mathbb{R}$. Property 4 implies $F_1(x,x) = 2\sqrt{\pi}e^{x^2}x\operatorname{erf}(x) + 2 > 0$. Property 5(iii) implies $F_2(x,x) =$

$2 - 2\sqrt{\pi}e^{x^2}x\text{erfc}(x) > 0$. Define

$$f_1(x) \triangleq F_1(x, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}) = -\frac{1}{2}\sqrt{\pi}e^{x^2}\left(-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x\right)\text{erf}(x) + x^2 - \sqrt{x^2+4}x + 2$$

and

$$f_3(x) \triangleq F_3(x, \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}}) = -\frac{1}{2}\sqrt{\pi}e^{x^2}\left(-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x\right)\text{erfc}(-x) + x^2 - \sqrt{x^2+4}x + 2.$$

It remains to show $f_1(x) > 0$ and $f_3(x) > 0$ on $\mathbb{R}$. In fact, one can check that $-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x$ has a single root at $\frac{1}{\sqrt{2}}$, and that $-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x > 0$ on $(-\infty, \frac{1}{\sqrt{2}})$ and $-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x < 0$ on $(\frac{1}{\sqrt{2}}, \infty)$. Now we consider the cases separately.

First check that $f_1(\frac{1}{\sqrt{2}}) = 1 > 0$ and $f_3(\frac{1}{\sqrt{2}}) \approx 2 > 0$. Consider $x \neq \frac{1}{\sqrt{2}}$. Define $g_1(x) \triangleq \frac{e^{-x^2}f_1(x)}{-2x^3+2\sqrt{x^2+4}x^2+\sqrt{x^2+4}-5x}$ and $g_3(x) \triangleq \frac{e^{-x^2}f_3(x)}{-2x^3+2\sqrt{x^2+4}x^2+\sqrt{x^2+4}-5x}$. Compute their derivatives:

$$g_1'(x) = g_3'(x) = \frac{2e^{-x^2}\left(\sqrt{x^2+4} - x\right)}{\sqrt{x^2+4}\left(-2x^3 + 2\sqrt{x^2+4}x^2 + \sqrt{x^2+4} - 5x\right)^2} > 0.$$

Then $g_1(x) > \lim_{x\to-\infty} g_1(x) = \frac{\sqrt{\pi}}{2}$ for $x < \frac{1}{\sqrt{2}}$, $g_1(x) < \lim_{x\to\infty} g_1(x) = -\frac{\sqrt{\pi}}{2}$ for $x > \frac{1}{\sqrt{2}}$, $g_3(x) > \lim_{x\to-\infty} g_3(x) = 0$, and $g_3(x) < \lim_{x\to\infty} g_3(x) = -\sqrt{\pi}$. This means $f_1(x) > 0$ and $f_3(x) > 0$ everywhere.

Finally we prove 16. Denote the LHS of the inequality-to-show by $F(x)$. Observe that $F(x) + 2 - 2a^2 + 2a\theta$ is quadratic in a. In fact, as we will show next, it is convex in a. Consider the second order partial derivative of $G(a, \theta, x) \triangleq F(x) + 2 - 2a^2 + 2a\theta$ with respect to a:

$$\frac{\partial^2 G(a,\theta,x)}{\partial a^2} = 4\sqrt{\pi}\left(2\theta^2 + 1\right)e^{x^2}x(\text{erf}(x) - \text{erf}(\theta)) + 8\theta\left(\theta - xe^{x^2-\theta^2}\right).$$

We will show next that $g(\theta) \triangleq \frac{\partial^2 G(a,\theta,z)}{\partial a^2} \geq 0$ for $\theta \leq x$. One can easily verify that $g(x) = g'(x) = 0$. Compute $g''(\theta) = 16\sqrt{\pi}e^{x^2}x(\text{erf}(x) - \text{erf}(\theta)) + 16$. When $x \geq 0$, it is obvious that $g''(\theta) > 0$ for all $\theta \leq x$. When $x < 0$, since $-\text{erf}(\cdot) < 1$, property 5(iii) implies that $g''(\theta) > 16\sqrt{\pi}\left(\frac{1}{\sqrt{\pi}} - xe^{x^2}\text{erfc}(x)\right) \geq 0$, which then implies that $g'(\theta) \leq g'(x) = 0$ for $\theta \leq x$. Therefore, $g(\theta) \geq g(x) = 0$ for $\theta \leq x$, and G is convex in a. Recall that we want to show $G(a, \theta, x) \leq 0$ for $a \in \left[\max\{0, x\}, \frac{\theta}{2} + \sqrt{1 + \frac{\theta^2}{4}}\right]$, $x \in \left(\theta, \frac{\theta}{2} + \sqrt{1 + \frac{\theta^2}{4}}\right]$, and $\theta \in \mathbb{R}$. Since G is convex in a and $x \leq \max\{0, x\}$, it is sufficient to show $G(x, \theta, x) \leq 0$ and $G(\frac{\theta}{2} + \sqrt{1 + \frac{\theta^2}{4}}, \theta, x) \leq 0$ for all $\theta < x$.

First consider $f_1(\theta, x) \triangleq G(x, \theta, x)$. One can verify that $f_1(x, x) = 0$, $\left.\frac{\partial f_1(\theta,x)}{\partial\theta}\right|_{\theta=x} = 0$ and $\left.\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2}\right|_{\theta=x} = -12$. Compute $\frac{\partial^3 f_1(\theta,x)}{\partial\theta^3} = 4e^{x^2}\left(3\sqrt{\pi}(\text{erf}(x) - \text{erf}(\theta)) + 2e^{-\theta^2}x\right)$. If $x \geq 0$, then obviously $\frac{\partial^3 f_1(\theta,x)}{\partial\theta^3} > 0$ for all $\theta < x$, and hence $\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2} < \left.\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2}\right|_{\theta=x} < 0$. On the other hand, if $x < 0$, then for all $\theta < x < 0$, $\frac{\partial^4 f_1(\theta,x)}{\partial\theta^4} = -8e^{x^2-\theta^2}(2\theta x + 3) < 0$. Also $\lim_{\theta\to-\infty} \frac{\partial^3 f_1(\theta,x)}{\partial\theta^3} = 12\sqrt{\pi}e^{x^2}\text{erfc}(-x) > 0$, $\frac{\partial^3 f_1(x,x)}{\partial\theta^3} = 8x < 0$, hence there is a unique root $\theta = \theta_0$ of $\frac{\partial^3 f_1(\theta,x)}{\partial\theta^3}$ on $(-\infty, x)$, which is also the maximizer of $\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2}$ on $\theta \in (-\infty, x)$. θ_0 satisfies $\left.\frac{\partial^3 f_1(\theta,x)}{\partial\theta^3}\right|_{\theta=\theta_0} = 0$, i.e., $\text{erf}(\theta_0) - \text{erf}(x) = \frac{2e^{-\theta_0^2}x}{3\sqrt{\pi}}$. Then for $\theta < x < 0$, $\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2} \leq \left.\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2}\right|_{\theta=\theta_0} = 4e^{x^2-\theta_0^2}\left(\frac{2x(x-3\theta_0)}{3} - 3\right) < 4e^{x^2-\theta_0^2}\left(\frac{-4x\theta_0}{3} - 3\right) < 0$. We have just showed that $\frac{\partial^2 f_1(\theta,x)}{\partial\theta^2} < 0$ for all $\theta < x$ and $x \in \mathbb{R}$. This implies that for $\theta < x$, $\frac{\partial f_1(\theta,x)}{\partial \theta} > \left.\frac{\partial f_1(\theta,x)}{\partial \theta}\right|_{\theta=x} = 0$, and hence $f_1(\theta,x) < f_1(x,x) = 0$.

We have shown $f_1(\theta,x) = G(x,\theta,x) < 0$ for $\theta < x$. It only remains to show $G(\frac{\theta}{2}+\sqrt{1+\frac{\theta^2}{4}},\theta,x) \leq 0$ for $\theta < x$. This is equivalent to showing $f_2(a,x) \triangleq G(a, a-\frac{1}{a}, x)a^3 e^{-x^2} \leq 0$ for all $x \in (a-\frac{1}{a}, a]$ and $a > 0$. One can verify that $f_2(a, a-\frac{1}{a}) = \left.\frac{\partial f_2(a,x)}{\partial x}\right|_{x=a-\frac{1}{a}} = 0$. Compute $\frac{\partial^3 f_2(a,x)}{\partial x^3} = 8\left(a^2-2\right)e^{-x^2}(ax+1)$. Observe that since $x > a - \frac{1}{a}$ and $a > 0$, we have $ax+1 > a^2 > 0$, hence $\frac{\partial^3 f_2(a,x)}{\partial x^3} \leq 0$ if $a \in (0,\sqrt{2}]$, $\frac{\partial^3 f_2(a,x)}{\partial x^3} > 0$ if $a > \sqrt{2}$. Compute $\frac{\partial^2 f_2(a,x)}{\partial x^2} = -4\sqrt{\pi}\left(a^2-2\right)\left(\mathrm{erf}\left(a-\frac{1}{a}\right)-\mathrm{erf}(x)\right) - 4\left(a^2-2\right)ae^{-x^2} - 8e^{2-\frac{a^4+1}{a^2}}a$. If $a \in (0,\sqrt{2}]$, $\frac{\partial^2 f_2(a,x)}{\partial x^2} \leq \left.\frac{\partial^2 f_2(a,x)}{\partial x^2}\right|_{x=a-\frac{1}{a}} = -4a^3 e^{-(a-\frac{1}{a})^2} < 0$ for all $x > a - \frac{1}{a}$. If $a > \sqrt{2}$, $\frac{\partial^2 f_2(a,x)}{\partial x^2} < \hat{f}_2(a) \triangleq \lim_{x\to\infty} \frac{\partial^2 f_2(a,x)}{\partial x^2} = 4\sqrt{\pi}(a^2-2)\mathrm{erfc}(a-\frac{1}{a}) - 8ae^{-(a-\frac{1}{a})^2} < 0$, where the last inequality follows from $\hat{f}_2'(a) = 8a\left(e^{2-\frac{a^4+1}{a^2}}a + \sqrt{\pi}\left(\mathrm{erf}\left(\frac{1}{a}-a\right)+1\right)\right) > 0$ on $(\sqrt{2},\infty)$ and $\lim_{a\to\infty}\hat{f}_2(a) = 0$. Since $\frac{\partial^2 f_2(a,x)}{\partial x^2} < 0$ for all $a > 0$ and $x \in (a-\frac{1}{a}, a]$, we have $\frac{\partial f_2(a,x)}{\partial x} < \left.\frac{\partial f_2(a,x)}{\partial x}\right|_{x=a-\frac{1}{a}} = 0$, and thus $f_2(a,x) < f_2(a, a-\frac{1}{a}) = 0$. Proof complete. □

B Remaining proofs for the Base model

This section provides the remaining proofs for Proposition 3 as well as the proof for Corollary 2.

We first complete the proof for Proposition 3, including Statements 1, 5–7, 10–12.

Proof of Statement 6. The case $\mu_R = \mu_S$ is trivial, since $\theta_b = \infty$ by Lemma 2. Suppose $\mu_R > \mu_S$. By the definitions of $B_S(\cdot,\cdot)$, $V_4(\cdot,\cdot)$, and C_5^* (see Eqs. (34), (21) and (38)), and the definition of q in Eq. (4), for any $x > \max\{q, \theta_{\mathrm{b}}\}$, $B_S(x, V_4(x, C_5^*))$ becomes

$$B_S(x, V_4^*) = \frac{\mu_R\sqrt{\pi}(\mu_S - x)}{\sigma_R} e^{\frac{(x-\mu_R)^2}{\sigma_R^2}} \mathrm{erfc}\left(\frac{x-\mu_R}{\sigma_R}\right) + \mu_S.$$

Observe that the RHS of the above equation is the same as the LHS of the equation in Lemma 2. Therefore $B_S(\theta_b, V_4^*) = 0$. By property 5(ii) in Lemma 5, we get $B_S(x, V_4^*) < 0$ on (θ_b, ∞). □

Proof of Statement 5. By the definitions of $\delta(\cdot,\cdot)$, $V_3(\cdot,\cdot)$, and C_4^* (see Eqs. (35), (20), and (37)), for any $x \in (q, \max\{q, \theta_b\})$, $\delta(x, V_3(x, C_4^*))$ becomes

$$\delta(x, V_3^*) = \frac{2(\mu_R - \mu_S)x^2 - 2\mu_S(\mu_R-\mu_S)x - \mu_S\sigma_R^2}{2(\mu_S - x)^2}.$$

If $\mu_R = \mu_S$, then $\delta(x, V_3^*) \leq 0$ easily follows. Suppose $\mu_R > \mu_S$. Solving $\delta(x, V_3^*) \leq 0$, we get

$$\frac{\mu_S}{2} - \sqrt{\frac{\mu_S^2}{4} + \frac{\mu_S\sigma_R^2}{2(\mu_R-\mu_S)}} \leq x \leq \frac{\mu_S}{2} + \sqrt{\frac{\mu_S^2}{4} + \frac{\mu_S\sigma_R^2}{2(\mu_R-\mu_S)}}.$$

Since $\mu_S \leq q$, we have $\frac{\mu_S}{2} - \sqrt{\frac{\mu_S^2}{4} + \frac{\mu_S\sigma_R^2}{2(\mu_R-\mu_S)}} \leq \mu_S \leq q$. Hence if we can show $\theta_{\mathrm{b}} \leq \frac{\mu_S}{2} + \sqrt{\frac{\mu_S^2}{4} + \frac{\mu_S\sigma_R^2}{2(\mu_R-\mu_S)}}$, then this means $\delta(x, V_3^*) \leq 0$ for any $x \in (q, \max\{q, \theta_b\})$. Let $\alpha = \frac{\mu_R-\mu_S}{\sigma_R}$, $\beta = \frac{\mu_S}{\mu_R}$, and $\hat{\theta}_b = \frac{\theta_b - \mu_R}{\sigma_R}$. We can equivalently show

$$\hat{\theta}_b \leq \frac{\alpha(\beta-2) + \sqrt{\alpha^2\beta^2 + 2\beta(1-\beta)}}{2(1-\beta)}. \tag{44}$$

By Lemma 2, we have $\hat{\theta}_b > -\alpha$ and $\hat{\theta}_b$ satisfies $(\hat{\theta}_b + \alpha)e^{\hat{\theta}_b^2}\mathrm{erfc}(\hat{\theta}_b) - \frac{\beta}{\sqrt{\pi}} = 0$. Using property 6 in Lemma 5 we deduce Eq. (44). □

Proof of Statement 1. By construction (see Eq. (19)), $V_2(x, C_2^*, C_2^*) > 0$ if $C_2^* > 0$. By Claim 1, we have $C_2^* > 0$, which implies that $W^*(x) > 0$ for all $x \leq q$. In particular, $W^*(q) > 0$. Also by definition of $V_3(\cdot,\cdot)$ (see Eq. (20)), $V_3(x, C_4^*)$ increases in x on $[q, \infty)$. Therefore by continuity of W^* at q, we have $W^*(x) > 0$ for all $x \leq \max\{q, \theta_b\}$. Similarly $V_4(x, C_5^*)$ increases in x as well (see Eq. (21) in Proposition 3), hence by continuity of W^* at θ_b, we have $W^*(x) > 0$ for all $x \in \mathbb{R}$. $\square$

Proof of Statement 10. By the definitions of $B_R(\cdot,\cdot)$, $V_1(\cdot,\cdot)$, and C_1^G (see Eqs. (34), (18) and (39)), for any $x < \theta_G < q$, $B_R(x, V_1^G)$ becomes

$$B_R(x, V_1^G) = \begin{cases} -b & \text{if } \theta_G \geq \mu_S; \\ \frac{\sigma_R^3 b f(\frac{x-\mu_R}{\sigma_R})}{(1-a^2+a\hat{\theta}_G)(\mu_S - x)^3} & \text{if } \theta_G < \mu_S, \end{cases}$$

where $f(z) = (a - \hat{\theta}_G)^3(1 - a^2 + az) - (a - z)^3(1 - a^2 + a\hat{\theta}_G)$. Note that $a > 0$ and $\hat{\theta}_G > a - \frac{1}{a}$ imply $1 - a^2 + a\hat{\theta}_G > 0$. Also $x < \theta_G < \mu_S$, hence to show $B_R(x, V_1^G) \leq 0$ we only need to show $f(\frac{x-\mu_R}{\sigma_R}) \leq 0$ for $x < \theta_G$, or equivalently, $f(z) \leq 0$ for $z < \hat{\theta}_G$, where $\hat{\theta}_G \in (a - \frac{1}{a}, a)$. Observe that $f(\hat{\theta}_G) = 0$ and $f'(z) = a(a - \hat{\theta}_G)^3 + 3(1 - a^2 + a\hat{\theta}_G)(a - z)^2$. In fact, $f'(z) > 0$ since $\hat{\theta}_G < a$ and $1 - a^2 + a\hat{\theta}_G > 0$. Therefore $f(z) \leq f(\hat{\theta}_G) = 0$ for $z < \hat{\theta}_G$. $\square$

Proof of Statement 7. If we can show $W^{G\prime}(x) \geq 0$ for all $x \in \mathbb{R}$, then Statement 7 easily follows since $\lim_{x\to-\infty} W^G(x) = \mu_S > 0$. Now we show $W^{G\prime}(x) \geq 0$ for all $x \in \mathbb{R}$ is true.

Lemma 3 implies that $\hat{\theta}_G > a - \frac{1}{a}$ (recall that $a = \frac{\mu_S - \mu_R}{\sigma_R}$), i.e., $1 - a^2 + a\hat{\theta}_G > 0$. Then one can check to see that $V_1^{G\prime}(\cdot) \geq 0$ on $(-\infty, \theta_G]$ (see Proposition 3 for V_1 and Eq. (39) for C_1^G). Also $V_4^{G\prime}(\cdot) \geq 0$ on $[q, \infty)$. Therefore it only remains to show $V_2^{G\prime}(\cdot) \geq 0$ on (θ_G, q), i.e., $\frac{C_3^G}{\sqrt{\pi}} + ze^{z^2}(C_2^G + C_3^G \mathrm{erf}(z)) \geq 0$ for $z \in (\hat{\theta}_G, \hat{q})$. Recall the definition of C_2^G in Eq. (40) and apply properties 4, 13 – 15 in Lemma 5, we can get $C_2^G > 0$. Therefore we can equivalently show $\frac{C}{\sqrt{\pi}} + ze^{z^2}(1 + C\mathrm{erf}(z)) \geq 0$ for $z \in (\hat{\theta}_G, \hat{q})$, where $C \triangleq \frac{C_3^G}{C_2^G}$. Compute

$$C_3^G + C_2^G = \begin{cases} b\left(e^{-\hat{\theta}_G^2} - \sqrt{\pi}\hat{\theta}_G \mathrm{erfc}(\hat{\theta}_G)\right) & \text{if } \theta_G \geq \mu_S; \\ \frac{be^{-\hat{\theta}_G^2}\left(2 - 2a\hat{\theta}_G + 2\hat{\theta}_G^2 + e^{\hat{\theta}_G^2}\sqrt{\pi}\left(a - 3\hat{\theta}_G + 2a\hat{\theta}_G^2 - 2\hat{\theta}_G^3\right)\mathrm{erfc}(\hat{\theta}_G)\right)}{2(1-a^2+a\hat{\theta}_G)} & \text{if } \theta_G < \mu_S, \end{cases}$$

and

$$C_3^G - C_2^G = \begin{cases} b\left(-e^{-\hat{\theta}_G^2} - \sqrt{\pi}\hat{\theta}_G \mathrm{erfc}(-\hat{\theta}_G)\right) & \text{if } \theta_G \geq \mu_S; \\ \frac{be^{-\hat{\theta}_G^2}\left(-2 + 2a\hat{\theta}_G - 2\hat{\theta}_G^2 + e^{\hat{\theta}_G^2}\sqrt{\pi}\left(a - 3\hat{\theta}_G + 2a\hat{\theta}_G^2 - 2\hat{\theta}_G^3\right)(1 + \mathrm{erf}(\hat{\theta}_G))\right)}{2(1-a^2+a\hat{\theta}_G)} & \text{if } \theta_G < \mu_S. \end{cases}$$

By property 5(iii) in Lemma 5, if $\theta_G \geq \mu_S$, then $C_3^G + C_2^G > 0$ and $C_3^G - C_2^G < 0$, hence $C = \frac{C_3^G}{C_2^G} \in (-1, 1)$. This is also true if $\theta_G < \mu_S$, since $1 - a^2 + a\hat{\theta}_G > 0$ and properties 13 – 15 in Lemma 5 together imply $C_3^G + C_2^G > 0$ and $C_3^G - C_2^G < 0$. Since $C \in (-1, 1)$, we can apply property 7 in Lemma 5 to get $\frac{C}{\sqrt{\pi}} + ze^{z^2}(1 + C\mathrm{erf}(z))$ is increasing in z. Therefore to show $\frac{C}{\sqrt{\pi}} + ze^{z^2}(1 + C\mathrm{erf}(z)) \geq 0$ for $z \in (\hat{\theta}_G, \hat{q})$, we only need to show $\frac{C}{\sqrt{\pi}} + \hat{\theta}_G e^{\hat{\theta}_G^2}(1 + C\mathrm{erf}(\hat{\theta}_G)) \geq 0$. Since $C = \frac{C_3^G}{C_2^G}$ is explicitly characterized by a, b and $\hat{\theta}_G$, we denote

$$\begin{aligned} F(\hat{\theta}_G) &\triangleq \frac{C}{\sqrt{\pi}} + \hat{\theta}_G e^{\hat{\theta}_G^2}(1 + C\mathrm{erf}(\hat{\theta}_G)) \\ &= \begin{cases} 0 & \text{if } \hat{\theta}_G \geq a; \\ \frac{e^{\hat{\theta}_G^2}(a - \hat{\theta}_G)}{2 - 2a\hat{\theta}_G + 2\hat{\theta}_G^2 + e^{\hat{\theta}_G^2}\sqrt{\pi}(-a + 3\hat{\theta}_G - 2a\hat{\theta}_G^2 + 2\hat{\theta}_G^3)\mathrm{erf}(\hat{\theta}_G)} & \text{if } \hat{\theta}_G \in (a - \frac{1}{a}, a). \end{cases} \end{aligned}$$

It remains to show $F(\hat{\theta}_G) \geq 0$. If $\hat{\theta}_G \geq a$ this is trivial. Suppose $\hat{\theta}_G \in (a - \frac{1}{a}, a)$. By properties 13 – 15 in Lemma 5, the above denominator is nonnegative. The statement follows. □

Proof of Statement 11. By definitions of $B_S(\cdot,\cdot)$, $V_2(\cdot,\cdot,\cdot)$, C_2^G and C_3^G (see Eqs. (34), (19), and (40) – (41)), for any $x \in (\theta_G, q)$, $B_S(x, V_2(x, C_2^G, C_3^G))$ becomes

$$B_S(x, V_2^G) = -V_2^G(x) + (\mu_S - x){V_2^G}'(x) + \mu_S.$$

Since $\lim_{x \to -\infty} W^G(x) = \mu_S$ and ${W^G}'(\cdot) \geq 0$ on $\mathbb{R}$, it follows that if $\theta_G \geq \mu_S$, then $B_S(x, V_2^G) \leq 0$ for all $x \in (\theta_G, q)$. It only remains to consider the case where $\theta_G < \mu_S$. Similarly, for all $x \in (\mu_S, q)$, the desired result $B_S(x, V_2^G) \leq 0$ easily holds. Consider the remaining region $x \in (\theta_G, \mu_S]$. In this case ($\theta_G < \mu_S$), we have

$$B_S(x, V_2^G) = \frac{b}{2\left(1 - a^2 + a\hat{\theta}_G\right)} \cdot \left(2 - 2a^2 + 2a\hat{\theta}_G + F\left(\frac{x - \mu_R}{\sigma_R}\right)\right),$$

where

$$F(z) = 2(a - z)\left(a - 3\hat{\theta}_G + 2a\hat{\theta}_G^2 - 2\hat{\theta}_G^3\right) - 2e^{z^2 - \hat{\theta}_G^2}\left(1 - a\hat{\theta}_G + \hat{\theta}_G^2\right)(1 - 2az + 2z^2)$$
$$+ e^{z^2}\sqrt{\pi}\left(a - 3\hat{\theta}_G + 2a\hat{\theta}_G^2 - 2\hat{\theta}_G^3\right)(1 - 2az + 2z^2)\left(\mathrm{erf}(\hat{\theta}_G) - \mathrm{erf}(z)\right).$$

Since $\theta_G < \mu_S$, then $\hat{\theta}_G \in (a - \frac{1}{a}, a)$ where $a > 0$, and hence $1 - a^2 + a\hat{\theta}_G > 0$. Also we know $b = \mu_S - \mu_R > 0$. Therefore to complete the proof, we only need to show $F(z) \leq -2 + 2a^2 - 2a\hat{\theta}_G$ for all $z \in (\hat{\theta}_G, a]$. Applying property 16 in Lemma 5, we obtain the desired result. □

Proof of Statement 12. For any $x > q$, we have $Q(x) = 0$ and hence (see Eq. (34) for definition of B_R, Proposition 3 for V_3 and Eq. (42) for C_4^G)

$$B_R(x, V_3^G) = (\mu_R - x){V_3^G}'(x) + \frac{1}{2}\sigma_R^2 {V_3^G}''(x) + \mu_R$$
$$= -\frac{b}{(\mu_S - x)^2}\left(x(x - \mu_S) + \frac{\mu_S \sigma_R^2}{2b}\right).$$

Since $x > q > \mu_S > 0$ and $b > 0$, it is clear that $B_R(x, V_3^G) \leq 0$. The result hence follows. □

We now prove Corollary 2.

Proof of Corollary 2. First consider the CLV under the myopic policy. It is intuitive to see that $V(q, \text{myopic}) = \mu_S < q$, since under the Safe mode, the customer's satisfaction state will stay in the unsatisfied zone until he churns (which happens according to a hazard rate of one). Now consider the optimal policy. From the proof of Proposition 3, we know that when $\mu_S > \mu_R$,

$$V^*(q) = V_3(x, C_4^G)$$

where V_3 is defined in Eq. (20) and C_4^* defined in Eq. (42). One can therefore verify that

$$V^*(q) = C_2^G e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} + C_3^G \mathrm{erf}\left(\frac{q - \mu_R}{\sigma_R}\right) e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} + \mu_R \tag{45}$$

where C_2^G and C_3^G are defined in Eqs. (40) – (41), respectively. We now show that as $\mu_S \to q$, it must be the case $\theta_G < \mu_S$, where θ_G is defined in Lemma 3. Suppose this is not true, then by Lemma 3, θ_G must be the root of Eq. (11) and satisfies $\theta_G \in [\mu_S, q)$. This indicates that the LHS of Eq. (11) must be zero as $\theta \to q$. Compute the LHS, we have

$$\lim_{\theta \to q}\left(\exp\left(\frac{(q - \mu_R)^2}{\sigma_R^2}\right)\frac{\sqrt{\pi}(\theta - \mu_R)(q - \mu_R)}{\sigma_R^2}\left(\mathrm{erf}\left(\frac{q - \mu_R}{\sigma_R}\right) - \mathrm{erf}\left(\frac{\theta - \mu_R}{\sigma_R}\right)\right)\right.$$
$$\left. + \exp\left(\frac{(q - \mu_R)^2 - (\theta - \mu_R)^2}{\sigma_R^2}\right)\frac{q - \mu_R}{\sigma_R} - \frac{\theta - \mu_R}{\sigma_R} - \frac{\mu_S \sigma_R}{2(\mu_S - \mu_R)(q - \mu_S)}\right)$$

$$= -\frac{\mu_S \sigma_R}{2(\mu_S - \mu_R)(q - \mu_S)} < 0,$$

hence a contradiction. Therefore, as $\mu_S \to q$, it must be the case $\theta_G < \mu_S$, and by Eqs. (40) – (41),

$$C_2^G = \frac{be^{-\hat{\theta}_G^2}\left(2 - 2a\hat{\theta}_G + 2\hat{\theta}_G^2 + e^{\hat{\theta}_G^2}\sqrt{\pi}\left(-a + 3\hat{\theta}_G - 2a\hat{\theta}_G^2 + 2\hat{\theta}_G^3\right)\operatorname{erf}(\hat{\theta}_G)\right)}{2(1 - a^2 + a\hat{\theta}_G)}$$

and

$$C_3^G = \frac{b\sqrt{\pi}\left(a - 3\hat{\theta}_G + 2a\hat{\theta}_G^2 - 2\theta_G^3\right)}{2(1 - a^2 + a\hat{\theta}_G)},$$

where

$$a \triangleq \frac{\mu_S - \mu_R}{\sigma_R}, \quad b \triangleq \mu_S - \mu_R, \quad \hat{\theta}_G \triangleq \frac{\theta_G - \mu_R}{\sigma_R}.$$

From Eq. (45), to show $\lim_{\mu_S \to q} V^*(q) \to \infty$, it suffices to show $\lim_{\mu_S \to q} C_3^G \to \infty$, i.e., $\lim_{\mu_S \to q} \hat{\theta}_G \to a - \frac{1}{a}$. Recall in the proof of Lemma 3, $\hat{\theta}_G$ must be the root of the following equation:

$$F_{\text{big}}(\hat{\theta}) \triangleq \left(a - 3\hat{\theta} + 2a\hat{\theta}^2 - 2\hat{\theta}^3\right) + 2e^{\hat{q}^2 - \hat{\theta}^2}\hat{q}\left(1 - a\hat{\theta} + \hat{\theta}^2\right)$$
$$+ e^{\hat{q}^2}\sqrt{\pi}\hat{q}\left(a - 3\hat{\theta} + 2a\hat{\theta}^2 - 2\hat{\theta}^3\right)\left(\operatorname{erf}(\hat{q}) - \operatorname{erf}(\hat{\theta})\right) - \frac{\mu_S\left(1 - a^2 + a\hat{\theta}\right)}{b(\hat{q} - a)} = 0$$

on interval $\in (a - 1/a, a)$, where $\hat{q} = \frac{q - \mu_R}{\sigma_R}$. Note that as $\mu_S \to q$, or equivalently, as $a \to \hat{q}$, $\hat{\theta}_G \in (a - 1/a, a)$ is bounded. Thus the first three terms in the above equation have a finite limit. At the same time, the denominator of the last term goes to zero. Therefore, to satify $F_{\text{big}}(\hat{\theta}_G) = 0$, we must have $\lim_{a \to \hat{q}} \hat{\theta}_G \to a - \frac{1}{a}$, or equivalently, $\lim_{\mu_S \to q} \hat{\theta}_G \to a - \frac{1}{a}$. □

C Robustness check: mixed policies

In this online appendix, we consider a variant of the original model where the firm is allowed to mix between the two service modes. We first setup the model. Then we give comparative statics regarding the optimal sandwich policy (see Theorem 4) and numerical evidence about the optimal sandwich policy outperforming the myopic one. In the last part of this online appendix, we provide proofs of the optimal policy structure in Theorem 3 and the monotonicity results in Theorem 4.

Model setup. At each point in time $t \geq 0$, the firm chooses the proportion of the Risky mode $p_t \in [0, 1]$ in its service, so that the reward is generated according to

$$dY_t = ((1 - p_t)\mu_S + p_t\mu_R)dt + p_t\sigma_R dB_t. \tag{46}$$

We restrict attention to $\mu_R > \mu_S > 0$. This is motivated by the real world application of an investment manager, where he can mix between assets and where the riskier asset usually provides better rewards. We also require $\mu_S < q$ as in the original model to ensure finite lifetime.

Note that over time, the investor may dynamically adjust the fraction of assets invested in the stock market. The investment manager, who is paid proportionally to returns, would prefer the higher-return option — to be fully invested in the stock market at all times — but is aware that a period of poor returns could cause the customer to leave.

Analogous to the original model, a policy λ is admissible if the firm's action process $(p_t)_{t\geq 0}$ (by following this policy) is adapted to the filtration $\mathbb{F}$, takes value in $[0, 1]$, and is such that the corresponding satisfaction processes is an $\mathbb{F}$-adapted semimartingale specified uniquely in law. We denote the set of admissible policies by Λ.

The optimal value function under the new policy space Λ is given by

$$V^I(x) = \sup_{\lambda \in \Lambda} \mathbb{E}\left[\int_0^\infty (\mu_S + p_t(\mu_R - \mu_S))\mathbb{1}\{t < T\}dt + \int_0^\infty p_t \sigma_R \mathbb{1}\{t < T\}dB_t \,\middle|\, H_0 = x\right]. \tag{47}$$

We call this model under policy space Λ the *Investor model*.

As in the original model, we expect that there is a stationary Markov optimal policy $\lambda : \mathbb{R} \to [0, 1]$. Next we show that interval policies, suitably generalized, are a subclass of stationary Markov policies that are admissible. First we extend the definition of interval policies to allow mixed policies.

Definition 4 (Interval policy in Investor model)**.** *In the Investor model, a policy λ is an interval policy if:*

- *it is stationary Markov, that is, the corresponding action process is given by a mapping from current satisfaction to the proportion of the Risky mode, which we denote by $p_t = \lambda(H_t)$.*
- *there is a partition of the satisfaction real line into a countable number of intervals, such that $\lambda(\cdot)$ is Lipschitz continuous within each interval, and that there exists some $c > 0$ such that $\lambda(x) \in \{0\} \cup [c, 1]$ for all $x \in \mathbb{R}$.*

Interval policies for the Investor model are admissible. The argument is similar to the proof of Lemma 1. Here, the policy may choose an arbitrary blend of the service modes at different points in the "Risky" pieces as long as the fraction of the Risky mode $\lambda(x)$ is Lipschitz continuous in the satisfaction level x within each piece, and $\lambda(x)$ is uniformly bounded below everywhere on the union of all "Risky" pieces (so that Salins and Spiliopoulos [29] still applies on the closure of each "Risky" piece). It turns out (see Theorem 3 below) that the optimal policy for the Investor model belongs to the class of interval policies.

Theorem 3. *Suppose $\mu_S < \mu_R$ and $\mu_S < q$. Consider the firm's problem as presented in Eq. (47). Let θ_I be as defined in Lemma C.1. If $\theta_I \leq q$, then the myopic (pure Risky-everywhere) policy is optimal. If $\theta_I > q$, then a sandwich policy is optimal, where the proportion of the Risky mode is $\lambda^*(x)$ as defined in Lemma C.2 for satisfaction levels $x \in [q, \theta_I]$, and $\lambda^*(x) = 1$ for $x \notin [q, \theta_I]$.*

Comparative statics and numerics. Recall in Theorem 3 the structure of the optimal sandwich policy in the Investor model: the firm still uses the Risky mode for low and high levels of satisfaction, but instead of using purely the Safe mode for an intermediate satisfaction interval, the firm mixes the Risky mode with the Safe mode in $[q, \theta_I]$, where θ_I is defined in Lemma C.1 later in this online appendix. We call the interval $[q, \theta_I]$ the risk-averse region. Moreover, inside the risk-averse region, the proportion of the Risky mode which the firm employs at satisfaction level x is specified by $\lambda^*(x)$ in Lemma C.2. The next theorem provides monotonicity results regarding θ_I and $\lambda^*(x)$.

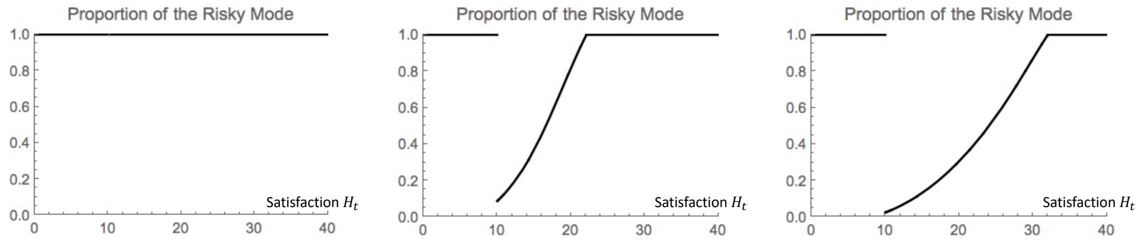
Theorem 4. *Let θ_I be as defined in Lemma C.1 and $\lambda^*(\cdot)$ be as defined in Lemma C.2. Then, the following properties hold:*

1. *The threshold θ_I increases in μ_S and σ_R.*
2. *Assume the optimal policy for a given set of parameters (see Theorem 3) is a sandwich policy. Then, the proportion of the Risky mode $\lambda^*(\cdot)$ in the risk-averse region $[q, \theta_I]$ is strictly increasing with satisfaction, and the firm strictly mixes at the satisfaction threshold q but not at θ_I; that is, we have $\lambda^*(q) > 0$ and $\lambda^*(\theta_I) = 1$.*

The first part of this theorem simply states that the size of the risk-averse region in the Investor model possesses the same monotonicities with regard to μ_S and σ_R as in the original model. The second part of Theorem 4 states that when the customer satisfaction level is in the unsatisfied zone, the firm is more risk-averse closer to the satisfaction threshold q. That is, the firm prefers a lower risk profile closer to q even though this generates a lower current reward rate. Note that $\lambda^*(q-) = 1$, and $\lambda^*(q+)$ has a value such that $(V^I)'$ is continuous at q despite the step in $Q(\cdot)$.

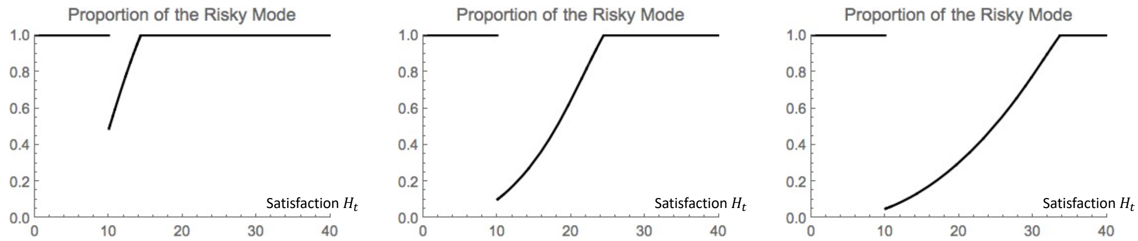
An interesting implication of part 2 of Theorem 4 is that the optimal policy never uses the Safe service mode alone. It always mixes the Risky mode with the Safe mode in the risk-averse region. The intuition is that the Risky mode has a higher drift ($\mu_R > \mu_S$), and as specified in Eq. (46), the variance is only quadratic in $\lambda(\cdot)$ while the drift is the $\lambda(\cdot)$-weighted convex combination of μ_R and μ_S, so it is always beneficial to include at least a small proportion of Risky.

We give a few numerical examples of the optimal policy in the following graphs (Figures 8–10). Notice that the model parameters considered in Figure 8 (c) here are the same as those in the original model in Figure 5. In comparison, the size of the risk-averse region here are larger than in the original model (which is $[10, 22.1]$).



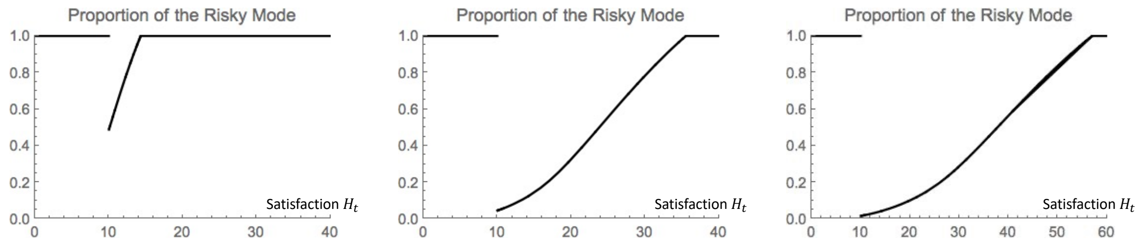
(a) $\mu_S = 3$ (b) $\mu_S = 7$ (c) $\mu_S = 8$

Figure 8: Optimal policies for the Investor model: $\mu_R = 9$, $\sigma_R = 10$, $q = 10$.



(a) $\mu_R = 9$ (b) $\mu_R = 6$ (c) $\mu_R = 5.5$

Figure 9: Optimal policies for the Investor model: $\mu_S = 5$, $\sigma_R = 10$, $q = 10$.



(a) $\sigma_R = 10$ (b) $\sigma_R = 30$ (c) $\sigma_R = 50$

Figure 10: Optimal policies for the Investor model: $\mu_S = 5$, $\mu_R = 9$, $q = 10$.

We omit to provide the improvement in CLV under the optimal policy compared with the myopic policy (pure Risky everywhere). Observe that the Investor model expands the policy space, hence increasing optimal CLV. That is, the gap for the Investor model exceeds that under the original model (Figure 6).

Proof of Theorems 3 and 4. To prove Theorem 3, we first establish two lemmas that characterize the optimal sandwich threshold θ_I and the optimal Risky proportion $\lambda^*(x)$ as a function of the satisfaction value x inside the risk-averse region $[q, \theta_I]$. First we present and prove the lemma on θ_I.

Lemma C.1. *Let Θ_I be the set of values of θ that satisfy*

$$\frac{\mu_R\sqrt{\pi}}{\sigma_R}e^{\frac{(\theta-\mu_R)^2}{\sigma_R^2}}\left(1-\operatorname{erf}\left(\frac{\theta-\mu_R}{\sigma_R}\right)\right)(2\theta-\mu_R-\mu_S)-(\mu_R+\mu_S)=0.$$

Then, the set $\Theta_I \cap (\frac{\mu_S+\mu_R}{2}, \infty)$ contains a single element, which we label θ_I.

Proof of Lemma C.1. Define

$$\alpha \triangleq \frac{\mu_R-\mu_S}{\sigma_R}, \quad \beta \triangleq \frac{\mu_S}{\mu_R} \quad \text{and} \quad z \triangleq \frac{\theta-\mu_R}{\sigma_R}.$$

We have $\alpha > 0$ and $0 < \beta < 1$. Define also

$$r(z) \triangleq (z+\frac{\alpha}{2})e^{z^2}\operatorname{erfc}(z) - \frac{1+\beta}{2\sqrt{\pi}}.$$

The lemma is equivalent to showing that $r(z)$ has a unique root on $(-\frac{\alpha}{2}, \infty)$. Applying property 5(ii) in Lemma 5, we get that $r(z)$ has a unique root on $(-\frac{\alpha}{2}, \infty)$. □

Next we define the proportion $\lambda^*(x)$ of the Risky mode which the firm employs at satisfaction level x inside the risk-averse region $[q, \theta_I]$.

Lemma C.2. *Let $X(\cdot)$ be the following function:*

$$X(y) = e^{-2a\left(\log(y+1)+\frac{1}{y+1}\right)}K_0 - (-2a)^{1+2a}be^{-2a\left(\log(y+1)+\frac{1}{y+1}\right)}\Gamma\left(-2a, -\frac{2a}{y+1}\right),$$

where

$$K_0 = \theta_I e^{2a\left(\log(g)+\frac{1}{g}\right)} + b(-2a)^{1+2a}\Gamma\left(-2a, -\frac{2a}{g}\right),$$

$$a = \frac{\sigma_R^2}{(\mu_R-\mu_S)^2}, \quad b = \mu_S, \quad g = \frac{2\theta_I}{2\theta_I-\mu_S-\mu_R},$$

and $\Gamma(s,z)$ is the upper incomplete gamma function (see Chaudry and Zubair [9]). Then, the inverse function of function $X(\cdot)$ is properly defined on $(\mu_S, \theta_I]$ and is represented by $G(\cdot) \triangleq X^{-1}(\cdot)$. Then, on $(\mu_S, \theta_I]$, the function $G(\cdot)$ is positive, strictly decreasing, differentiable, and satisfies the following ODE:

$$(G(x)+1)^2 + 2a(x-b)G(x)G'(x) - 2abG'(x) = 0.$$

Finally, define the function

$$\lambda^*(x) \triangleq \frac{(\mu_S-\mu_R)(G(x)+1)}{\sigma_R^2 G'(x)}. \tag{48}$$

Then, λ^* is strictly increasing on $[q, \theta_I]$ with $\lambda^*(q) > 0$ and $\lambda^*(\theta_I) = 1$.

The function $G(x)$ in Lemma C.2 captures the marginal benefit of satisfaction at x, when the firm uses the conjectured optimal proportion $\lambda^*(\cdot)$ of the Risky mode. The function $X(\cdot)$ is the inverse function of $G(\cdot)$.

Lemmas C.3 and C.4 are key to proving Lemma C.2.

Lemma C.3. *Let $X(y)$ be as defined in Lemma C.2. Then $X(y)$ is strictly decreasing on $\left[\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}, \infty\right)$. Moreover, $X(\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}) = \theta_I$ and $\lim_{y\to\infty} X(y) = \mu_S$.*

Proof of Lemma C.3. First we restate $X(y)$ here:

$$X(y) = e^{-2a\left(\log(y+1)+\frac{1}{y+1}\right)}K_0 - (-2a)^{1+2a}be^{-2a\left(\log(y+1)+\frac{1}{y+1}\right)}\Gamma\left(-2a, -\frac{2a}{y+1}\right). \tag{49}$$

Then one can verify that $X\left(\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}\right) = \theta_I$ and $\lim_{y\to\infty} X(y) = b$. Recall by definition of θ_I in Lemma C.1 that $\theta_I > \frac{\mu_S+\mu_R}{2} > \mu_S$. Next we want to show that $X'(y) < 0$ on $\left[\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}, \infty\right)$. Compute this derivative:

$$X'(y) = 2aye^{-\frac{2a}{y+1}}(y+1)^{-2(a+1)}F(y)$$

where

$$F(y) = \frac{be^{\frac{2a}{y+1}}(y+1)^{2a+1}}{y} + 2^{2a+1}(-a)^{2a+1}b\Gamma\left(-2a, -\frac{2a}{y+1}\right) - K_0.$$

Observe that for $y \in \left[\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}, \infty\right)$, we have $2aye^{-\frac{2a}{y+1}}(y+1)^{-2(a+1)} > 0$. Therefore we want to show $F(y) < 0$ on $\left[\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}, \infty\right)$. In fact, this is true since

$$F'(y) = -be^{\frac{2a}{y+1}}\frac{(1+y)^{2a}}{y^2} < 0$$

for $y \geq \frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R} > 0$ and

$$F\left(\frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}\right) = \frac{e^{\frac{2a}{g}}g^{2a}(bg - g\theta_I + \theta_I)}{g-1} < 0,$$

where the last step follows from $g = \frac{2\theta_I}{2\theta_I-\mu_S-\mu_R} > 1$ and $b = \mu_S < \frac{\mu_S+\mu_R}{2}$. We have thus completed the proof. □

Lemma C.4. *The function $\lambda^*(x)$ as defined in Lemma C.2 is strictly increasing on $[q, \theta_I]$, and $\lambda^*(q) > 0$, $\lambda^*(\theta_I) = 1$.*

Proof of Lemma C.4. Let $p^*(\cdot)$ be defined as Eq. (48), but on the domain $(\mu_S, \theta_I]$. Then on $(\mu_S, \theta_I]$, we have

$$p^*(x) = \frac{(\mu_S-\mu_R)(G(x)+1)}{\sigma_R^2 G'(x)}.$$

Also from Lemma C.2, $G(x)$ satisfies

$$(G(x)+1)^2 + 2a(x-b)G(x)G'(x) - 2abG'(x) = 0.$$

on this interval. Therefore we have

$$G'(x) = \frac{(G(x)+1)^2}{2a(b-(x-b)G(x))}$$

on $(\mu_S, \theta_I]$, and thus we can rewrite $p^*(x)$ as

$$p^*(x) = \frac{2a(\mu_S-\mu_R)(b-(x-b)G(x))}{\sigma_R^2(G(x)+1)}. \tag{50}$$

Since $\mu_S < \mu_R$, to show $p^*(x)$ is strictly increasing on $(\mu_S, \theta_I]$ is equivalent to showing $\frac{b-(x-b)G(x)}{G(x)+1}$ is strictly decreasing on $(\mu_S, \theta_I]$. Also $G(\cdot)$ is the inverse function of $X(\cdot)$, and by Lemma C.3, X is strictly decreasing on $[g-1, \infty)$, therefore it is equivalent if we can show $\frac{b-y(X(y)-b)}{y+1}$ is strictly increasing on $[g-1, \infty)$.

Denote $L(y) \triangleq \frac{b-y(X(y)-b)}{y+1}$. Compute its derivative:

$$L'(y) = 2ae^{-\frac{2a}{y+1}}(y+1)^{-2a-3}\Bigg(2^{2a+1}a(-a)^{2a}b\left(2ay^2-y-1\right)\Gamma\left(-2a, -\frac{2a}{y+1}\right) - 2abye^{\frac{2a}{y+1}}(y+1)^{2a+1} + K_0\left(2ay^2-y-1\right)\Bigg).$$

We want to show that $L'(y) > 0$ on $[g-1,\infty)$. Consider y sufficiently large such that $2ay^2-y-1>0$. We want to show

$$f(y) = 2^{2a+1}a(-a)^{2a}b\Gamma\left(-2a,-\frac{2a}{y+1}\right) - \frac{2abye^{\frac{2a}{y+1}}(y+1)^{2a+1}}{2ay^2-y-1} + K_0 > 0.$$

One can check that $f'(y) = \frac{2ab(y+2)e^{\frac{2a}{y+1}}(y+1)^{2a+1}}{(-2ay^2+y+1)^2} > 0$. Also

$$\begin{aligned} f(g-1) &= 2^{2a+1}a(-a)^{2a}b\Gamma\left(-2a,-\frac{2a}{g}\right) - \frac{2ab(g-1)e^{\frac{2a}{g}}g^{2a+1}}{2a(g-1)^2-g} + K_0 \\ &= -\frac{e^{\frac{2a}{g}}g^{2a+1}\left(2a(g-1)^2(\mu_S-\mu_R)+g(\mu_R+\mu_S)\right)}{2(g-1)\left(2a(g-1)^2-g\right)}. \end{aligned}$$

If we can show g is such that $2a(g-1)^2-g>0$ (so that $-2ay^2+y+1>0$ for all $y \geq g-1$) and $2a(g-1)^2(\mu_S-\mu_R)+g(\mu_R+\mu_S) \leq 0$, then we are done. In fact, we can check that the second inequality implies the first, therefore we only need to show the second one is true. Now let $\alpha \triangleq \frac{1}{2}\sqrt{\frac{1}{a}}$, $\beta \triangleq \frac{\mu_S+\mu_R}{2\mu_R}$, $\hat{\theta}_I \triangleq \frac{\theta_I-\mu_R}{\sigma_R}$, and recall the definition of $g \triangleq \frac{2\theta_I}{2\theta_I-\mu_S-\mu_R}$. After solving the desired inequality on g, we can get an equivalent desired inequality on $\hat{\theta}_I$:

$$\hat{\theta}_I \leq \frac{\alpha(\beta-2)+\sqrt{\alpha^2\beta^2+2\beta(1-\beta)}}{2(1-\beta)}.$$

By Lemma C.1, $\hat{\theta}_I$ is the unique root of $F(\theta) = (x+\alpha)e^{\theta^2}\text{erfc}(\theta) - \frac{\beta}{\sqrt{\pi}}$on $(-\alpha,\infty)$. Then apply property 6 in Lemma 5, we get that the desired inequality of $\hat{\theta}_I$ is true. Therefore we have proved that $p^*(x)$ is strictly increasing on $(\mu_S,\theta_I]$.

Finally, consider $\lim_{x\to\mu_S^+} p^*(x)$ and $p^*(\theta_I)$. Since $G(\cdot)$ is the inverse function of $X(\cdot)$, then by Lemma C.3 we know that $\lim_{x\to\mu_S^+} G(x) = \infty$, and $G(\theta_I) = \frac{\mu_S+\mu_R}{2\theta_I-\mu_S-\mu_R}$. Combine with Eq. (50), we get $\lim_{x\to\mu_S^+} p^*(x) = 0$, and $p^*(\theta_I) = 1$. Therefore since $p^*(\cdot) = \lambda^*(\cdot)$ on $[q,\theta_I]$ and $q > \mu_S$, we have $\lambda^*(q) = p^*(q) > 0$ and $\lambda^*(\theta_I) = p^*(\theta_I) = 1$. □

Now we are ready to prove Lemma C.2.

Proof of Lemma C.2. By Lemma C.3, the function $X(\cdot) : [g-1,\infty) \to (\mu_S,\theta_I]$ as specified in Lemma C.2 is strictly decreasing and differentiable, and $X(g-1) = \theta_I$, and $\lim_{y\to\infty} X(y) = \mu_S$. Therefore, its inverse function $G(\cdot) : (\mu_S,\theta_I] \to [g-1,\infty)$ is well-defined, strictly decreasing and differentiable, and $G'(x) = \frac{1}{X'(G(x))}$ by the inverse function theorem. Let $G(x) = y$, $x = X(y)$, and $G'(x) = \frac{1}{X'(y)}$, one can easily check that $(y+1)^2 + 2a(X(y)-b)\frac{y}{X'(y)} - \frac{2ab}{X'(y)} = 0$. Therefore $G(\cdot)$ satiesfies the ODE $(G(x)+1)^2+2a(x-b)G(x)G'(x)-2abG'(x) = 0$ on $[\mu_S,\infty)$. Since $g-1>0$, we have that $G(\cdot)$ is strictly positive on $(\mu_S,\theta_I]$. Finally by Lemma C.4, the function $\lambda^*(\cdot)$ is strictly increasing on $[q,\theta_I]$, with $\lambda^*(q) > 0$ and $\lambda^*(\theta_I) = 1$. □

Next we establish the full proof of Theorem 3.

Proof of Theorem 3. The proof technique is similar to one we used prove Theorem 1 (see Section 5). Recall that in Section 5, we first obtain a candidate value function $W(\cdot)$ by solving the HJB equation (13), then we prove its optimality by showing that it satisfies a set of optimality conditions (Conditions 1-6 in Proposition 2). Similarly, for the Investor model, we will first provide a candidate value function $W^I(\cdot)$ by solving the HJB equation

$$\max_{p\in[0,1]}\left\{-Q(x)V(x)+(\mu_S+p(\mu_R-\mu_S)-x)V'(x)+\frac{1}{2}p^2\sigma_R^2V''(x)+\mu_S+p(\mu_R-\mu_S)\right\} = 0, \quad (51)$$

(where p denotes the proportion of the Risky mode to invest in) and then prove its optimality by showing that $W^I(\cdot)$ together with the policy stated in Theorem 3 satisfies Conditions 1-6, with Condition 4^I-5^I (see below) replacing Condition 4-5.

4^I. the following inequality[14] is true for any $x \in \mathbb{R}$ when $p = 0$ and for any $x \in \mathbb{R} \setminus \mathcal{E}$ when $p \in (0, 1]$:

$$-Q(x)\bar{V}(x) + (\mu_S + p(\mu_R - \mu_S) - x)\bar{V}'(x) + \frac{1}{2}p^2\sigma_R^2\bar{V}''(x) + \mu_S + p(\mu_R - \mu_S) \leq 0; \tag{52}$$

and

5^I. for all $x \in \mathbb{R}$ and some interval policy (see Definition 4 in Online Appendix C) $\bar{\lambda} \in \Lambda$ such that $\bar{\lambda}(y) = 0$ for all $y \in \mathcal{E}$, the process $(\bar{V}(H_t^{x,\bar{\lambda}}))_{t\geq 0}$ is an $\mathbb{F}$-adapted semimartingale, and

$$-Q(x)\bar{V}(x) + (\mu_S + \bar{\lambda}(x)(\mu_R - \mu_S) - x)\bar{V}'(x) + \frac{1}{2}\bar{\lambda}(x)^2\sigma_R^2\bar{V}''(x) + \mu_S + \bar{\lambda}(x)(\mu_R - \mu_S) = 0 \tag{53}$$

Definition of $W^I(\cdot)$. Define

$$W^I(x) = \begin{cases} V_1(x, C_1^I) \text{ if } x < q; \\ V_2^I(x, C_2^I) \text{ if } q \leq x \leq \theta_I; \\ V_3(x, C_3^I) \text{ if } x > \max\{q, \theta_I\}; \end{cases} \tag{54}$$

for some uniquely specified C_1^I, C_2^I and C_3^I, where V_1 and V_3 are as defined in Eqs. (19) and (21), restated here:

$$V_1(x, C_1) = C_1 e^{\frac{(x-\mu_R)^2}{\sigma_R^2}} \left(1 + \text{erf}\left(\frac{x - \mu_R}{\sigma_R}\right)\right) + \mu_R;$$

$$V_3(x, C_3) = C_3 + \int_0^x \frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{\frac{(z-\mu_R)^2}{\sigma_R^2}} (1 - \text{erf}(\frac{z - \mu_R}{\sigma_R}))dz.$$

V_2^I is defined as follows:

$$V_2^I(x, C_2) = C_2 + \int_q^x G(z)dz, \tag{55}$$

where G is as defined in Lemma C.2. Observe that $V_2^{I\prime}(\cdot, C_2)$ and $V_3'(\cdot, C_3)$ are independent of C_2 and C_3, hence we will use $V_2^{I\prime}(\cdot)$ and $V_3'(\cdot)$ to denote the two derivatives, respectively.

We now provide the explicit expressions of C_1^I, C_2^I and C_3^I:

$$C_1^I = \begin{cases} \dfrac{\frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} (1-\text{erf}(\frac{q-\mu_R}{\sigma_R}))}{\frac{2}{\sigma_R}\left(\frac{1}{\sqrt{\pi}} + e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} \frac{q-\mu_R}{\sigma_R}\left(1+\text{erf}\left(\frac{q-\mu_R}{\sigma_R}\right)\right)\right)} & \text{if } \theta_I < q; \\ \dfrac{G(q)}{\frac{2}{\sigma_R}\left(\frac{1}{\sqrt{\pi}} + e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} \frac{q-\mu_R}{\sigma_R}\left(1+\text{erf}\left(\frac{q-\mu_R}{\sigma_R}\right)\right)\right)} & \text{if } \theta_I \geq q, \end{cases} \tag{56}$$

$$C_2^I = C_1^I e^{\frac{(q-\mu_R)^2}{\sigma_R^2}} \left(1 + \text{erf}\left(\frac{q - \mu_R}{\sigma_R}\right)\right) + \mu_R, \tag{57}$$

and

$$C_3^I = \begin{cases} V_1(q, C_1^I) - \int_0^q \frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{\frac{(z-\mu_R)^2}{\sigma_R^2}} (1 - \text{erf}(\frac{z-\mu_R}{\sigma_R}))dz & \text{if } \theta_I < q; \\ V_2^I(\theta_I, C_2^I) - \int_0^{\theta_I} \frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{\frac{(z-\mu_R)^2}{\sigma_R^2}} (1 - \text{erf}(\frac{z-\mu_R}{\sigma_R}))dz & \text{if } \theta_I \geq q. \end{cases} \tag{58}$$

[14]When $p = 0$, we define $\frac{1}{2}p^2\sigma_R^2\bar{V}''(x)$ to be zero for any x including $x \in \mathcal{E}$.

To reduce the burden of notation, we define $V_1^I(\cdot) \triangleq V_1(\cdot, C_1^I)$, $V_2^I(\cdot) \triangleq V_2^I(\cdot, C_2^I)$ and $V_3^I(\cdot) \triangleq V_3(\cdot, C_3^I)$.

Conditions 1, 2, 3, 4^I, 5^I and 6. Now we have an explicitly defined candidate value function $W^I(\cdot)$ and an explicit stationary Markov policy $\lambda^*(\cdot)$ (see Lemma C.2). Next we want to show that $W^I(\cdot)$ and $\lambda^*(\cdot)$ together satisfy Conditions 1-3 (see Proposition 2), 4^I and 5^I (see (52) and (53)) and 6 (see Proposition 2).

We start with Condition 1. We want to show that $W^I(\cdot) \geq 0$. By construction of V_1 (see Eq. (19)), if $C_1^I > 0$ then $V_1^I(\cdot) > 0$. Indeed this is true since $G(q) > 0$ (see Lemma C.2). Then, in particular $V_1^I(q) > 0$ and $W^I(\cdot) > 0$, since $W^I(\cdot)$ is continuous everywhere (see below) including at q and increases on $[q, \infty)$, see Eq. (54).

Condition 2 requires $W^I(\cdot)$ to be continuously differentiable everywhere and twice continuously differentiable almost everywhere. By construction, $W^I(\cdot)$ is continuously differentiable and twice continuously differentiable everywhere except possibly at q and θ_I (note that by Lemma C.2, G is differentiable on $[q, \theta_I]$). Hence, we only need to show that $W^I(\cdot)$ is differentiable at q and θ_I. Equivalently, we want to show

$$V_1^I(q) = \begin{cases} V_3^I(q) & \text{if } \theta_I < q; \\ V_2^I(q) & \text{if } \theta_I \geq q, \end{cases} \tag{59}$$

$$V_1^{I\prime}(q) = \begin{cases} V_3^{I\prime}(q) & \text{if } \theta_I < q; \\ V_2^{I\prime}(q) & \text{if } \theta_I \geq q, \end{cases} \tag{60}$$

and if $\theta_I \geq q$,

$$V_2^I(\theta_I) = V_3^I(\theta_I), \tag{61}$$

$$V_2^{I\prime}(\theta_I) = V_3^{I\prime}(\theta_I). \tag{62}$$

Eq. (59) is implied by the definitions of C_2^I and C_3^I (see Eqs. (57) and (58)). Eq. (60) is implied by the definition of C_1^I (see Eq. (56)). Eq. (61) is implied by the definition of C_3^I (see Eq. (58)). Eq. (62) is implied by the fact that $G^{-1}(V_3^{I\prime}(\theta_I)) = \theta_I$ (see Lemma C.2).

Condition 3 requires that $W^{I\prime}$ be bounded. Since we have just proved that $W^{I\prime}(\cdot)$ is continuous in $\mathbb{R}$, to show that $W^{I\prime}(\cdot)$ is bounded, it suffices to show $\left|\lim_{x\to-\infty} W^{I\prime}(x)\right| < \infty$ and $\left|\lim_{x\to\infty} W^{I\prime}(x)\right| < \infty$. This is equivalent to showing $\left|\lim_{x\to-\infty} V_1^{I\prime}(x)\right| < \infty$ and $\left|\lim_{x\to\infty} V_3^{I\prime}(x)\right| < \infty$. By the definitions of V_1^I (see Eqs. (19) and (56)) and V_3^I (see Eqs. (21) and (58)), we have

$$\lim_{x\to-\infty} V_1^{I\prime}(x) = \lim_{x\to-\infty} \frac{2C_1^I}{\sigma_R}\left(\frac{1}{\sqrt{\pi}} + e^{\frac{(x-\mu_R)^2}{\sigma_R^2}} \frac{x-\mu_R}{\sigma_R}\left(1 + \operatorname{erf}\left(\frac{x-\mu_R}{\sigma_R}\right)\right)\right)$$
$$= \lim_{z\to\infty} \frac{2C_1^I}{\sigma_R}\left(\frac{1}{\sqrt{\pi}} - z e^{z^2} \operatorname{erfc}(z)\right)$$

and

$$\lim_{x\to\infty} V_3^{I\prime}(x) = \lim_{x\to\infty} \frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\left(1 - \operatorname{erf}\left(\frac{x-\mu_R}{\sigma_R}\right)\right)$$
$$= \lim_{z\to\infty} \frac{\mu_R\sqrt{\pi}}{\sigma_R} e^{z^2} \operatorname{erfc}(z)\,,$$

for $z = (x-\mu_R)/\sigma_R$. By property 2 in Lemma 5, the limits above are zero.

Conditions 5^I and 4^I respectively require $W^I(\cdot)$ and $\lambda^*(\cdot)$ to satisfy

$$-Q(x)W^I(x) + (\mu_S(1-\lambda^*(x)) + \mu_R\lambda^*(x) - x)W^{I\prime}(x) + \frac{1}{2}(\sigma_R\lambda^*(x))^2 W^{I\prime\prime}(x)$$
$$+ \mu_S(1-\lambda^*(x)) + \mu_R\lambda^*(x) = 0 \tag{63}$$

and

$$-Q(x)W^I(x) + (\mu_S(1-p) + \mu_R p - x)W^{I'}(x) + \frac{1}{2}(\sigma_R p)^2 W^{I''}(x) + \mu_S(1-p) + \mu_R p \le 0 \quad \text{for all } p \in [0,1] \tag{64}$$

for all $x \in \mathbb{R}$, except possibly at q and θ_I. Eq. (63) is true by the construction of $W^I(\cdot)$ and $\lambda^*(\cdot)$. Specifically, $V_1^I(x)$ satisfies

$$-V_1^I(x) + (\mu_R - x)V_1^{I'}(x) + \frac{1}{2}\sigma_R^2 V_1^{I''}(x) + \mu_R = 0 \text{ for all } x < q;$$

$V_3^I(x)$ satisfies

$$(\mu_R - x)V_3^{I'}(x) + \frac{1}{2}\sigma_R^2 V_3^{I''}(x) + \mu_R = 0 \text{ for all } x \ge \theta_I;$$

and $G(x)$ satisfies (see Lemma C.2)

$$(G(x)+1)^2 + 2a(x-b)G(x)G'(x) - 2abG'(x) = 0,$$

which is equivalent to (since $G'(\cdot)$ is nonzero)

$$(\mu_S(1-\lambda^*(x)) + \mu_R\lambda^*(x) - x)G(x) + \frac{1}{2}(\sigma_R\lambda^*(x))^2 G'(x) + \mu_S(1-\lambda^*(x)) + \mu_R\lambda^*(x) = 0 \text{ for all } x \in [q, \theta_I].$$

Next we will show Eq. (64) to complete Condition 4^I. Denote the LHS of Eq. (64) by $B(p,x,W^I)$. Then, we need to show $B(p,x,V_1^I) \le 0$ for $x < q$, $B(p,x,V_2^I) \le 0$ for $x \in [q,\theta_I]$, and $B(p,x,V_3^I) \le 0$ for $x > \max\{q,\theta_I\}$; each for all $p \in [0,1]$.

Let us first start with $x < q$. We already know that $B(1,x,V_1^I) = 0$. Therefore, to show $B(p,x,V_1^I) \le 0$ is equivalent to showing $B(1,x,V_1^I) - B(p,x,V_1^I) \ge 0$. Rearranging the terms, we can get

$$B(1,x,V_1^I) - B(p,x,V_1^I) = (1-p)\left((\mu_R - \mu_S)V_1^{I'}(x) + \frac{(1+p)\sigma_R^2}{2}V_1^{I''}(x) + \mu_R - \mu_S\right), \tag{65}$$

In fact, by properties 1, 5(iii) in Lemma 5, and the fact that $C_1^I > 0$, we have

$$V_1^{I'}(x) = \frac{2C_1^I}{\sigma_R}\left(\frac{1}{\sqrt{\pi}} + e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\frac{x-\mu_R}{\sigma_R}\operatorname{erfc}\left(-\frac{x-\mu_R}{\sigma_R}\right)\right) \ge 0$$

and

$$V_1^{I''}(x) = \frac{2C_1^I}{\sigma_R^2}\left(\frac{2(x-\mu_R)}{\sigma_R\sqrt{\pi}} + e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\left(1 + \frac{2(x-\mu_R)^2}{\sigma_R^2}\right)\operatorname{erfc}\left(-\frac{x-\mu_R}{\sigma_R}\right)\right) \ge 0.$$

Therefore since $p \in [0,1]$ and $\mu_R > \mu_S$, following Eq. (65), we obtain the desired inequality $B(1,x,V_1^I) - B(p,x,V_1^I) \ge 0$, and we have proved Eq. (64) for $x < q$.

Now consider the case $x > \max\{q,\theta_I\}$. We know that $B(1,x,V_3^I) = 0$. Therefore, to show $B(p,x,V_3^I) \le 0$ is equivalent to proving $B(1,x,V_3^I) - B(p,x,V_3^I) \ge 0$. Rearranging the terms, we get

$$B(1,x,V_3^I) - B(p,x,V_3^I) = (1-p)\left((\mu_R - \mu_S)V_3^{I'}(x) + \frac{(1+p)\sigma_R^2}{2}V_3^{I''}(x) + \mu_R - \mu_S\right) = (1-p)\Lambda(p), \tag{66}$$

where

$$\Lambda(p) \triangleq \frac{\mu_R\sqrt{\pi}}{\sigma_R}e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\operatorname{erfc}\left(\frac{x-\mu_R}{\sigma_R}\right)(-p\mu_R - \mu_S + (p+1)x) - p\mu_R - \mu_S.$$

By property 5(iii) in Lemma 5, we obtain

$$\Lambda'(p) = \frac{\mu_R\sqrt{\pi}}{\sigma_R}e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\operatorname{erfc}\left(\frac{x-\mu_R}{\sigma_R}\right)(x-\mu_R) - \mu_R \le 0.$$

Therefore, since $p \in [0,1]$, following Eq. (66) we have

$$\begin{aligned} B(1,x,V_3^I) - B(p,x,V_3^I) &= (1-p)\Lambda(p) \\ &\geq (1-p)\Lambda(1) \\ &= (1-p)\left(\frac{\mu_R\sqrt{\pi}}{\sigma_R}e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\operatorname{erfc}\left(\frac{x-\mu_R}{\sigma_R}\right)(2x-\mu_R-\mu_S)-\mu_R-\mu_S\right) \\ &\geq 0\ , \end{aligned}$$

where the last step follows from the definition of θ_I (see Lemma C.1) and property 5(ii) in Lemma 5.

Now we are only left with $x \in [q, \theta_I]$. Similar to the argument above, we want to show $B(\lambda^*(x), x, V_2^I) - B(p, x, V_2^I) \geq 0$. Rearranging the terms, we get

$$\begin{aligned} &B(\lambda^*(x),x,V_2^I) - B(p,x,V_2^I) \\ &\quad = (\lambda^*(x)-p)\left((\mu_R-\mu_S)G(x)+\frac{\lambda^*(x)+p}{2}\sigma_R^2G'(x)+\mu_R-\mu_S\right) \\ &\quad = -\frac{(\lambda^*(x)-p)^2\sigma_R^2}{2}G'(x) \qquad \geq 0. \end{aligned}$$

The second step follows from the definition of $\lambda^*(x)$ (see Lemma C.2). The last step follows from the fact that $G(\cdot)$ is nonincreasing (see Lemma C.2). Finally, it remains to be shown that Condition 6 holds. Condition 6 holds straightforwardly by an application[15] of Lemma 4. □

Finally we prove Theorem 4.

Proof of Theorem 4. The monotonicity of $\lambda^*(x)$ on $[q, \theta_I]$ and the boundary values follow from Lemma C.2. It remains to show the monotonicity of θ_I. Let

$$\begin{aligned} F(\mu_S,\mu_R,\sigma_R,x) &\triangleq \frac{2x-\mu_S-\mu_R}{2\sigma_R}e^{\frac{(x-\mu_R)^2}{\sigma_R^2}}\left(1-\operatorname{erf}\left(\frac{x-\mu_R}{\sigma_R}\right)\right)-\frac{\mu_S+\mu_R}{2\mu_R\sqrt{\pi}} \\ &= (y+\frac{a}{2})e^{y^2}(1-\operatorname{erf}(y))-\frac{b+1}{2\sqrt{\pi}}, \end{aligned}$$

where

$$y \triangleq \frac{x-\mu_R}{\sigma_R}, \quad a \triangleq \frac{\mu_R-\mu_S}{\sigma_R}, \quad b \triangleq \frac{\mu_S}{\mu_R}.$$

By Lemma C.1, θ_I is the only root of $F(\mu_S,\mu_R,\sigma_R,\cdot)$ on $(\frac{\mu_S+\mu_R}{2},\infty)$. From property 5(i) in Lemma 5, we obtain that $F(\mu_S,\mu_R,\sigma_R,\cdot) < 0$ for all $x \in (\frac{\mu_S+\mu_R}{2}, \theta_I)$, and $F(\mu_S,\mu_R,\sigma_R,\cdot) > 0$ for all $x \in (\theta_I, \infty)$. Hence to prove θ_I is increasing in μ_S and σ_R, it suffices to show $F(\cdot,\mu_R,\sigma_R,x)$ is decreasing and $F(\mu_S,\mu_R,\cdot,x)$ is decreasing. Equivalently, we want to show $\frac{\partial F}{\partial \mu_S}(\mu_S,\mu_R,\sigma_R,x) < 0$, and $\frac{\partial F}{\partial \sigma_R}(\mu_S,\mu_R,\sigma_R,x) < 0$ for any $\mu_R > \mu_S > 0$ and $x > \frac{\mu_S+\mu_R}{2}$. Suppose now we fix such μ_R, μ_S, σ_R and x. Then we have $y > \frac{a}{2}$. Compute the partial derivatives:

$$\begin{aligned} \frac{\partial F}{\partial \mu_S}(\mu_S,\mu_R,\sigma_R,x) &= -\frac{1}{2\sigma_R}e^{y^2}\operatorname{erfc}(y)-\frac{1}{2\mu_R\sqrt{\pi}}; \\ \frac{\partial F}{\partial \sigma_R}(\mu_S,\mu_R,\sigma_R,x) &= -\frac{1}{2\sigma_R}\left[e^{y^2}\operatorname{erfc}(y)(2y^2+1)-\frac{2y}{\sqrt{\pi}}\right](a+2y). \end{aligned}$$

The first result needed, $\frac{\partial F}{\partial \mu_S}(\mu_S,\mu_R,\sigma_R,x) < 0$, follows from the fact that $\operatorname{erfc}(\cdot) > 0$ everywhere. The second, $\frac{\partial F}{\partial \sigma_R}(\mu_S,\mu_R,\sigma_R,x) \leq 0$, follows from the fact that $x > \frac{\mu_S+\mu_R}{2}$ (so that $a + 2y > 0$) and

[15] Lemma 4 applies here as long as the customer lifetime is finite in expectation. Recall the proof of Proposition 1 for finite lifetime. Consider the satisfaction process under the policy stated in Theorem 3. If the initial satisfaction level is below q, then we are done since satisfaction process spends positive measure of time below q. On the other hand, if the initial satisfaction level is above q, then the first passage time to q must have finite expectation since both μ and σ terms are Lipschitz and bounded below, and hence we are done.

using a Chernoff-type bound of the error function (Chang et al. [8]), $\text{erfc}(y) \geq \frac{2y}{\sqrt{\pi}(2y^2+1)} e^{-y^2}$. This completes the proof. □

D Robustness check: Geometric Brownian Motion reward process

Motivated by an investment problem where a risk-free asset generates returns (interest) with percentage drift μ_S deterministically and a risky asset generates returns (capital gains and dividends, with automatic reinvestment of dividends) with percentage drift μ_R and percentage volatility σ_R, in this online appendix, we consider a model that uses geometric Brownian motion (GBM) reward process instead of arithmetic Brownian motion reward. We will first define the model. Then we will establish optimality conditions (Proposition 4), explain how we numerically solve for the optimal value function, and finish by presenting our numerical findings.

Model setup. In this model, under the firm's choice of service mode $u_t \in \{R, S\}$, the total reward $\tilde{Y}_t$ evolves according to a Geometric Brownian Motion (GBM)

$$d\tilde{Y}_t = \mu_{u_t} \tilde{Y}_t dt + \sigma_{u_t} \tilde{Y}_t dB_t \tag{67}$$

with[16] $\tilde{Y}_0 = 1$. We are interested in cases where $\mu_R > \mu_S > 0$. (However, we do not impose the restriction $\mu_R > \mu_S$ for our analytical development.) Customer satisfaction $\tilde{H}_t$ follows a stochastic differential equation:

$$d\tilde{H}_t = d\tilde{Y}_t / \tilde{Y}_t - \tilde{H}_t dt, \tag{68}$$

where $\tilde{H}_0 = x$ is the initial customer satisfaction. Comparing Eqs. (67) and (68) with Eqs. (1) and (3), we see that under the same action process u_t, customer satisfaction $\tilde{H}_t$ follows the same dynamics as H_t does in the original model. We assume that the hazard rate of customer churn is still a step function but is positive even for $\tilde{H}_t \geq q$. Specifically,

$$\tilde{Q}(\tilde{H}_t) \triangleq Q_1 \mathbb{1}\{\tilde{H}_t < q\} + Q_2 \mathbb{1}\{\tilde{H}_t \geq q\} \tag{69}$$

with $Q_1 > Q_2 > 0$. Then the customer's survival probability $\tilde{S}_t$ at time t is given by

$$\tilde{S}_t \triangleq P\left(\tilde{T} > t \mid \tilde{\mathcal{F}}_t\right) = e^{-\int_0^t \tilde{Q}(\tilde{H}_s) ds}. \tag{70}$$

Denote by $\tilde{T}$ the customer lifetime:

$$\tilde{T} \triangleq \inf\left\{t \geq 0 : e^{-\int_0^t \tilde{Q}(\tilde{H}_s) ds} = w\right\}, \tag{71}$$

where w is a uniform random variable over $[0, 1]$ independent of filtration $\mathbb{F}$.

We require the next condition on Q_1 and Q_2 to ensure that the expected CLV (which will be defined next) is finite[17].

Condition 1. $Q_1 > Q_2 > \max\{\mu_S, \mu_R\}$.

Denote by $\tilde{\Pi}$ the space of admissible policies that satisfy the usual conditions as in the original model, which is that under the policy, the corresponding action process u_t should be adapted to filtration $\mathbb{F}$, takes value in $\{S, R\}$, and the corresponding $\tilde{H}_t$ is an $\mathbb{F}$-adapted semimartingale uniquely specified in law. For a given starting satisfaction x and admissible policy π, let $\tilde{Y}_t^{x,\pi}$ denote the reward gained up to time t and $\tilde{T}^{x,\pi}$ be the customer lifetime. Then the CLV is equal to

$$\tilde{V}(x, \pi) = \mathbb{E}\left[1 + \int_0^\infty \mathbb{1}\{t < T^{x,\pi}\} d\tilde{Y}_t^{x,\pi} \;\middle|\; \tilde{H}_0 = x\right]. \tag{72}$$

[16] The solution to Eq. (67) is $\tilde{Y}_t = \exp\left(\int_0^t \left(\mu_{u_s} - \sigma_{u_s}^2/2\right) ds + \int_0^t \sigma_{u_s} dB_s\right)$.

[17] Suppose the hazard rate is Q for all satisfaction states and the firm always uses the Risky service mode, then it is not hard to show that the expected CLV is $\frac{Q}{Q-\mu_R}$ if $Q > \mu_R$, and ∞ if $Q \leq \mu_R$.

The firm's objective is to maximize the CLV it earns from interacting with the customer. The optimal CLV given a starting satisfaction x is given by

$$\tilde{V}^*(x) = \sup_{\pi\in\tilde{\Pi}} \tilde{V}(x,\pi). \tag{73}$$

Next we establish optimality conditions in order to find the optimal value function.

Optimality conditions. As in the original model, we present the optimality conditions for a function $\tilde{W}:\mathbb{R}\to\mathbb{R}$ to be the optimal value function.

Proposition 4. *Suppose a function $\tilde{W}:\mathbb{R}\to\mathbb{R}$ satisfies*

1. *the function value $\tilde{W}(x) > 1$ for any $x\in\mathbb{R}$;*
2. *the function $\tilde{W}$ is continuously differentiable everywhere on $\mathbb{R}$ and twice continuously differentiable everywhere on $\mathbb{R}\setminus\mathcal{E}$ for some countable set $\mathcal{E}$;*
3. *the function $\tilde{W}$ is bounded;*
4. *the function $\tilde{W}'$ is bounded;*
5. *for any $x\in\mathbb{R}$ for $i=S$ and for any $x\in\mathbb{R}\setminus\mathcal{E}$ for $i=R$ the following holds[18]:*
$$\tilde{Q}(x) + (\mu_i - \tilde{Q}(x))\tilde{W}(x) + \left(\mu_i + \sigma_i^2 - x\right)\tilde{W}'(x) + \frac{1}{2}\sigma_i^2\tilde{W}''(x) \le 0;$$
6. *for some interval policy $\tilde{\pi}$ (see Definition 2) such that $\tilde{\pi}(y) = S$ for all $y\in\mathcal{E}$, the process $\tilde{W}(\tilde{H}_t)$ is an $\mathbb{F}$-adapted semimartingale, and for all $x\in\mathbb{R}$ it holds that*
$$\tilde{Q}(x) + (\mu_{\tilde{\pi}(x)} - \tilde{Q}(x))\tilde{W}(x) + \left(\mu_{\tilde{\pi}(x)} + \sigma_{\tilde{\pi}(x)}^2 - x\right)\tilde{W}'(x) + \frac{1}{2}\sigma_{\tilde{\pi}(x)}^2\tilde{W}''(x) = 0. \tag{74}$$

Then, the function $\tilde{W}$ is the optimal value function $\tilde{V}^*$, and $\tilde{\pi}$ is an optimal policy.

Note that Conditions 5 and 6 together establish the HJB equation for this new setting:

$$\max_{i\in\{S,R\}}\left\{\tilde{Q}(x) + (\mu_i - \tilde{Q}(x))\tilde{V}(x) + \left(\mu_i + \sigma_i^2 - x\right)\tilde{V}'(x) + \frac{1}{2}\sigma_i^2\tilde{V}''(x)\right\} = 0 \tag{75}$$

for all $x\in\mathbb{R}$ where $\tilde{V}''$ exists.

Before we present the proof of Proposition 4, we want to give an overview of the remainder of the GBM online appendix. Recall from the analysis of the original model (Section 5), we would like to solve the HJB equation to get a candidate value function such that all optimality conditions are satisfied. A general solution to the HJB equation (75) (for a single service mode) is of the following form:

$$\tilde{W}(x,C_1,C_2) = \frac{\tilde{Q}(x)}{\tilde{Q}(x)-\mu_R} + C_1 H\left(\mu_R - \tilde{Q}(x), \frac{x-\mu_R-\sigma_R^2}{\sigma_R}\right) + C_2 M\left(\frac{\tilde{Q}(x)-\mu_R}{2}, \frac{1}{2}, \frac{\left(x-\mu_R-\sigma_R^2\right)^2}{\sigma_R^2}\right) \tag{76}$$

under the Risky service mode, and

$$\tilde{W}(x,C_3,S) = \frac{\tilde{Q}(x)}{\tilde{Q}(x)-\mu_S} + C_3(\mu_S - x)^{\mu_S - \tilde{Q}(x)}$$

under the Safe service mode, where C_1, C_2, C_3 are free parameters, and in Eq. (76) $H(\cdot,\cdot)$ is a Hermite polynomial function, and $M(\cdot,\cdot,\cdot)$ is the Kummer confluent hypergeometric function.[19] We

[18]For any $x\in\mathcal{E}$, we define the term $\frac{1}{2}\sigma_S^2\tilde{W}''(x)$ to be zero consistent with $\sigma_S = 0$.

[19]The two functions $H(\lambda,x)$ and $M(-\frac{\lambda}{2},\frac{1}{2},x^2)$ are the two linearly independent solutions to the Hermite Differential Equation $y''(x) - 2xy'(x) + 2\lambda y(x) = 0$.

then numerically find the values of C_1, C_2 and C_3 such that the optimality conditions in Proposition 4 are satisfied. Later in this online appendix, we describe how we numerically solve for the optimal policy and value function, and provide our numerical findings, which shows that our main structural results for the original model also hold in the GBM setting. We defer the proof of Proposition 4 to the end of the GBM online appendix.

How we numerically solve for the optimal policy and value function. We need to numerically find the values of C_1, C_2 and C_3 of the following functions (that solves Eq. (74) for the Risky service mode and for the Safe mode, respectively,)

$$\tilde{W}(x, C_1, C_2, R) = \frac{\tilde{Q}(x)}{\tilde{Q}(x) - \mu_R} + C_1 H\left(\mu_R - \tilde{Q}(x), \frac{x - \mu_R - \sigma_R^2}{\sigma_R}\right) + C_2 M\left(\frac{\tilde{Q}(x) - \mu_R}{2}, \frac{1}{2}, \frac{\left(x - \mu_R - \sigma_R^2\right)^2}{\sigma_R^2}\right)$$

and

$$\tilde{W}(x, C_3, S) = \frac{\tilde{Q}(x)}{\tilde{Q}(x) - \mu_S} + C_3(\mu_S - x)^{\mu_S - \tilde{Q}(x)}$$

in different satisfaction regions where the firm chooses different service modes, such that the optimality conditions in Proposition 4 are satisfied.

In the above expressions, $H(\cdot, \cdot)$ is a Hermite Polynomial, and $M(\cdot, \cdot, \cdot)$ is the Kummer confluent hypergeometric function: the two functions $H(\lambda, x)$ and $M(-\frac{\lambda}{2}, \frac{1}{2}, x^2)$ are the two linearly independent solutions to the Hermite Differential Equation $y''(x) - 2xy'(x) + 2\lambda y(x) = 0$. One challenge of calculating this $\tilde{W}$ is that, as the satisfaction value x decreases, both $H()$ and $M()$ grow exponentially. Since both functions cannot be evaluated to their exact values, the error in calculation $\tilde{W}$ (which is the difference of two very large numbers) can get large for negative x with large magnitude. To control for this error, we place a reflecting boundary at $q - B$ in the unsatisfied zone for some large $B > 0$ and let the satisfaction process only evolve on $[q - B, \infty)$. Recall that the satisfaction process is an Ornstein-Uhlenbeck (O-U) process in the unsatisfied zone, if the firm always utilizes the Risky service mode there. With a reflecting boundary $q - B < q$, it becomes a reflected O-U process. To preserve our insights, we want to choose B large enough that the customer churns before hitting the reflecting boundary $q - B$ with probability close to 1.

We use the following method to choose the reflecting boundary $q - B$. Consider a reflected O-U process $\tilde{X}_t$ on $(-\infty, q]$ with infinitesimal drift $\mu_R - \tilde{X}_t$, infinitesimal volatility σ_R, initial value $\tilde{X}_0 = q$, and reflecting boundary at the satisfaction threshold q. Note that this is an approximation of the satisfaction process $\tilde{H}_t$ under the sandwich policy with the risk-averse region right above the unsatisfied zone, which we conjecture to be optimal, if not the myopic policy. Under this policy, the satisfaction process becomes a delayed reflected O-U process on $(-\infty, q]$ once it hits the unsatisfied zone. Notice the difference between process $\tilde{X}_t$ and the satisfaction process just stated — $\tilde{X}_t$ has instantaneous reflection at q while $\tilde{H}_t$ has delayed reflection at q. Nevertheless, this means that the stationary probability of $\tilde{X}_\infty < q - B$ is an overestimation of the stationary probability of $\tilde{H}_\infty < q - B$, and we can bound the latter by bounding $\Pr\{\tilde{X}_\infty < q - B\}$. From Ward and Glynn's paper [38], we know the stationary distribution of $\tilde{X}_t < q - B$ is $\Pr\left[N\left(\mu_R, \frac{\sigma_R^2}{2}\right) < q - B \middle| N\left(\mu_R, \frac{\sigma_R^2}{2}\right) \leq q\right]$. Hence we would like to choose B such that this probability is small.

Numerical findings. Next we give details of how random instances are generated to compute the optimal policies. We are interested in the regime where $q > \mu_S$, which implies that the customer is eventually not satisfied if the firm uses the Safe mode all the time. We also let $\mu_R > \mu_S$, so that the Risky asset accumulates higher rewards on average. We solve for the optimal policy for random instances and make careful use of a reflecting boundary $q - B$ for some large positive B

to ensure numerical stability while ensuring that the effect on CLV is very small. In particular, in each iteration, the parameters are randomly generated in the following sequence:

1. randomly generate $\sigma_R \sim \text{Uniform}[0.2, 1]$;
2. randomly generate $\mu_R \sim \text{Uniform}\,[0, 0.8]$;
3. randomly generate $\mu_S \sim \text{Uniform}[0, \mu_R]$;
4. randomly generate $q = \mu_S + \text{Uniform}[0, 0.8]$;
5. assign $B = q - \mu_R + 5\sigma_R$;
6. randomly generate $Q_1 \sim \text{Uniform}[\mu_R, \mu_R + 5]$;
7. randomly generate $Q_2 \sim \text{Uniform}[\mu_R, Q_1]$.

Note that the range of the specifications of μ_S, μ_R and σ_R are chosen to have a similar magnitude with the GBM drift and volatility estimations from the financial market (for example, see Schneider et al. [30]). Also note that the choice of B is made to ensure that the second argument $\frac{x-\mu_R-\sigma_R^2}{\sigma_R}$ inside the Hermite Polynomials is always bounded below on $[q-B, \infty)$, in order to ensure numerical stability. We numerically solve 1000 random instances generated by the above procedure, by solving for the free parameters C_1 and C_2 in Eq. (76) and verify that all the conditions in Proposition 4 are satisfied. In each instance, the stationary probability of a reflected O-U process $\tilde{H}_t < q - B$ is calculated. In fact, they are all less than 10^{-8}. Therefore we can say that with very high probability, the customer churns before hitting the boundary $q - B$, and that placing a boundary at $q - B$ will very likely not affect the firm's optimal policy. In fact, we also check this by perturbing the choice of B within $[q-\mu_R+4\sigma_R, q-\mu_R+5\sigma_R]$ and showing that both the value function and the sandwich structure are extremely insensitive to the choice of B. Remarkably, in all the randomly generated instances, the optimal policy is either a myopic policy (Risky mode everywhere) or a sandwich policy (Safe mode only in an interval just above q). More details are presented next.

Optimal policy is either myopic or sandwich. As in the original model (see Theorem 1 for the $\mu_R > \mu_S$ case), the optimal policy is either a myopic one or a sandwich policy. In particular, among all the randomly generated instances, the (numerically solved) optimal solutions are either a myopic policy that always uses the Risky mode everywhere, or a sandwich policy that uses the Risky mode for all satisfaction states except for some intermediate satisfaction range $[q, \tilde{\theta}_b]$ (for some numerically specified $\tilde{\theta}_b$). It is worth noting that in the GBM setting, the optimal sandwich policy once again provides substantial CLV increase over the myopic policy. For example, consider the model primitives $\mu_S = 0.12$, $\mu_R = 0.14$, $\sigma_R = 0.3$, $q = 0.13$, $Q_1 = 1.5$ and $Q_2 = 0.5$. (Here one may think of 1 time unit in the model as being a period of about 2 years.) This means the Safe asset's rate of return is 12% (with continuous compounding), and the Risky asset's expected rate of return is 14%, with volatility 30%. Also the customer is not satisfied with a rate of return below 13%, and his hazard rate of churn increases from 0.5 to 1.5 if he estimates (as quantified by his satisfaction) that the rate of return is below 13%. Under this set of model primitives, the optimal sandwich policy (see the dotted vertical line in Figure 11) is to use the Safe mode for satisfaction values on $[0.13, 0.147]$, and use the Risky mode elsewhere. The CLV increase from using the optimal sandwich policy relative to the myopic policy is 7.0%. Though we do not permit mixed strategies in this section, we briefly observe that the CLV increase relative to the myopic policy will be even larger if mixed strategies are permitted, since the myopic strategy remains unaffected.

Optimal switching threshold exhibits similar monotonicity as in the original model. The optimal switching threshold $\tilde{\theta}_b$ shows similar monotonicity in model primitives as in the original model (see Theorem 2). That is, $\tilde{\theta}_b$ decreases as we increase μ_R, and increases as we increase σ_R.

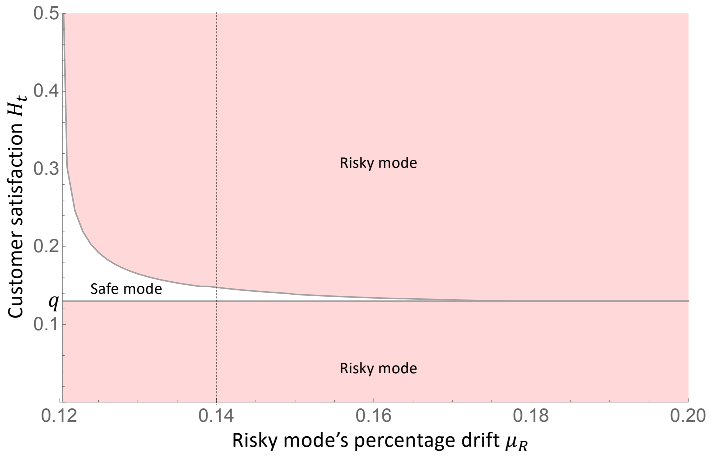


Figure 11: The optimal sandwich policies for different values of μ_R, under $\mu_S = 0.12$, $\sigma_R = 0.3$, $q = 0.13$, $Q_1 = 1.5$ and $Q_2 = 0.5$. The horizontal axis corresponds to the value of μ_R, and the vertical line marks the satisfaction value. The two curves are the switching boundaries between the Risky mode and the Safe mode.

Figure 11 plots the prescribed optimal sandwich policy for various model primitives. In particular, we fix $\mu_S = 0.12$, $\sigma_R = 0.3$, $q = 0.13$, $Q_1 = 1.5$, $Q_2 = 0.5$, and vary μ_R in $[0.12, 0.20]$. The horizontal axis is μ_R, and the vertical axis is the satisfaction value. The two curves on the plot are the two switching thresholds, separating the satisfaction regions where the firm should use different service modes. The white region represents the risk-averse region where the firm should choose the Safe service mode. Notice that $\tilde{\theta}_b$ here has the same monotonicity (i.e., decreasing in μ_R) as θ_b in the original model (see Theorem 2).

We end this online appendix with the proof of Proposition 4.

Proof of Proposition 4. In order to prove Proposition 4, we first establish the following lemma:

Lemma D.1. *For any admissible policy $\pi \in \tilde{\Pi}$, any starting satisfaction $x \in \mathbb{R}$ and any $t > 0$, the following holds:*

$$\mathbb{E}\left[\int_0^t \left\{\tilde{Y}_s^{x,\pi}\right\}^2 ds\right] < \infty,$$

where $\tilde{Y}_s$ is as defined in Eq. (67). In other words, the process $\tilde{Y}_s^{x,\pi} \in L^2[0,t]$, and the stochastic integral $\int_0^t \tilde{Y}_s^{x,\pi} dB_s$ is a martingale for any $t > 0$, which then implies that

$$\mathbb{E}\left[\int_0^t \tilde{Y}_s^{x,\pi} dB_s\right] = 0.$$

Proof. Fix an admissible policy $\pi \in \tilde{\Pi}$, a starting satisfaction $x \in \mathbb{R}$, and a time $t > 0$. Denote u_t the corresponding action process. Since the solution to Eq. (67) is

$$\tilde{Y}_t = \exp\left(\int_0^t \left(\mu_{u_s} - \sigma_{u_s}^2/2\right) ds + \int_0^t \sigma_{u_s} dB_s\right),$$

we have

$$\begin{aligned}\mathbb{E}\left[\int_0^t \left\{\tilde{Y}_s^{x,\pi}\right\}^2 ds\right] &= \int_0^t \mathbb{E}\left[\left\{\tilde{Y}_s^{x,\pi}\right\}^2\right] ds \\ &= \int_0^t \mathbb{E}\left[e^{\int_0^s \left(2\mu_{u_z} - \sigma_{u_z}^2\right) dz + \int_0^s 2\sigma_{u_z} dB_z}\right] ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \mathbb{E}\left[e^{\int_0^s (2\mu_{u_z}+\sigma_{u_z}^2)dz}\right] ds \\
&\le \int_0^t e^{\left(2\mu_R+\sigma_R^2\right)s} ds \\
&= \frac{e^{\left(2\mu_R+\sigma_R^2\right)t}-1}{2\mu_R+\sigma_R^2} < \infty,
\end{aligned}
$$

where the inequality results from the fact that $\mu_S < \mu_R$ and $\sigma_S < \sigma_R$. Hence we have showed that $\tilde{Y}_s^{x,\pi} \in L^2[0,t]$, and that by the Martingale property of stochastic integrals, the process $\int_0^t \tilde{Y}_s^{x,\pi} dB_s$ is a martingale for any $t > 0$. □

Now we are ready to prove Proposition 4.

Proof of Proposition 4. As in the proof of Proposition 2, we will show that a function $\tilde{W}$ as described in Proposition 4 is an upper bound of the optimal CLV $\tilde{V}^*$, and that the gap $\tilde{W} - \tilde{V}^*$ is zero.

To show that a function $\tilde{W}$ as described in Proposition 4 is an upper bound for $\tilde{V}^*$, it suffices to show $\tilde{W}(x) \ge \tilde{V}(x,\pi)$ for $\forall x \in \mathbb{R}$ and for any admissible policy $\pi \in \tilde{\Pi}$. Now fix any $x \in \mathbb{R}$ and any $\pi \in \tilde{\Pi}$. Let u_t denote the action process under policy π. Define a process $X_t, t \ge 0$ by

$$
X_t = \tilde{W}(\tilde{H}_t)\tilde{Y}_t\tilde{S}_t + \int_0^t \tilde{Y}_s\tilde{Q}(\tilde{H}_s)\tilde{S}_s ds, \tag{77}
$$

where $\tilde{H}_t$ is the satisfaction process under policy π and initial satisfaction x (see Eq. (68)), $\tilde{Y}_t$ the corresponding cumulative reward (conditional on no quitting) up to time t (see Eq. (67)), and $\tilde{S}_t$ is the corresponding customer survival probability at time t (see Eq. (70)). Since π is admissible, the corresponding process $\tilde{H}_t$ is a semimartingale (and hence $\tilde{Y}_t$ and $\tilde{S}_t$ are also semimartingales). Next we expand X_t in integral form.

Since $\tilde{W}$ is continuously differentiable everywhere on $\mathbb{R}$ and twice continuously differentiable everywhere on $\mathbb{R} \setminus \mathcal{E}$ for some countable set $\mathcal{E}$ (Condition 2 in Proposition 4), we can apply the Itô-Tanaka formula to conclude that $\tilde{W}(\tilde{H}_t)$ is also a semimartingale:

$$
\begin{aligned}
\tilde{W}(\tilde{H}_t) = \tilde{W}(x) + \int_0^t \tilde{W}'(\tilde{H}_s)(\mu_{u_s} - \tilde{H}_s)ds + \int_0^t \tilde{W}'(\tilde{H}_s)\sigma_{u_s} dB_s \\
+ \frac{1}{2}\int_0^t \mathbb{1}\{\tilde{H}_s \notin \mathcal{E}\}\tilde{W}''(\tilde{H}_s)\sigma_{u_s}^2 ds + \frac{1}{2}\sum_{y\in\mathcal{E}}(\tilde{W}_r'(y) - \tilde{W}_l'(y))L^{\tilde{H}}(t,y),
\end{aligned} \tag{78}
$$

where $\tilde{W}_r'$ and $\tilde{W}_l'$ are the right and left derivatives of $\tilde{W}$, and $L^{\tilde{H}}(t,y)$ is the symmetric local time of $\tilde{H}_t$ at y. In fact, we can still apply the results of Lemma 6 to the GBM setting, since the dynamics of the satisfaction process in GMB setting is the same with the satisfaction process in the original model (if under the same action process). From Lemma 6, we know that $L^{\tilde{H}}(t,y) < \infty$ almost surely, and since $\tilde{W}$ is continuously differentiable everywhere, we have $\tilde{W}_r'(y) = \tilde{W}_l'(y)$ and hence the last term involving the local time in Eq. 78 is zero almost surely.

Since $\tilde{W}(\tilde{H}_t), \tilde{Y}_t$ and $\tilde{S}_t$ are all semimartingales, we can then apply the multi-dimensional Itô's formula on semimartingales to $g(\tilde{W}(\tilde{H}_t), \tilde{Y}_t, \tilde{S}_t) = \tilde{W}(\tilde{H}_t)\tilde{Y}_t\tilde{S}_t$ and rewrite X_t as:

$$
\begin{aligned}
X_t =& \tilde{W}(x) \\
&+ \int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}'(\tilde{H}_s)\left(\mu_{u_s} - \tilde{H}_s\right) ds + \int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}'(\tilde{H}_s)\sigma_{u_s} dB_s \\
&- \int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}(\tilde{H}_s)\tilde{Q}(\tilde{H}_s)ds + \frac{1}{2}\int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}''(\tilde{H}_s)\mathbb{1}\{\tilde{H}_s \notin \mathcal{E}\}\sigma_{u_s}^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}(\tilde{H}_s)\mu_{u_s}ds+\int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}(\tilde{H}_s)\sigma_{u_s}dB_s\\
&+\int_0^t \tilde{Y}_s\tilde{S}_s\tilde{W}'(\tilde{H}_s)\sigma_{u_s}^2ds+\int_0^t \tilde{Y}_s\tilde{S}_s\tilde{Q}(\tilde{H}_s)ds. \qquad (79)
\end{aligned}
$$

Note that by the Martingale property of stochastic integrals, the two stochastic integrals above have zero expectations if $\tilde{Y}_s\tilde{S}_s\tilde{W}'(\tilde{H}_s)\sigma_{u_s}$ and $\tilde{Y}_s\tilde{S}_s\tilde{W}(\tilde{H}_s)\sigma_{u_s}$ are in the $L^2[0,t]$ space. This is indeed true since $\tilde{Y}_s^{x,\pi}\in L^2[0,t]$ (see Lemma D.1), $\tilde{S}_s\in[0,1]$, $\tilde{W}$ is bounded (see Condition 3 in Proposition 4), $\tilde{W}'$ is bounded (see Condition 4 in the proposition), and $\sigma_{u_s}\in\{0,\sigma_R\}$. Hence we can take expectation on both sides of Eq. (79) and remove the two stochastic integrals, while replacing 1 with $\mathbb{1}\{\tilde{H}_s\notin\mathcal{E}\}+\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=S\}+\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}$, and get

$$
\begin{aligned}
\mathbb{E}X_t=\tilde{W}(x)+\mathbb{E}\int_0^t \tilde{Y}_s\tilde{S}_s\Bigg[&\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\\
&+\frac{1}{2}\sigma_{u_s}^2\tilde{W}''(\tilde{H}_s)\bigg)\mathbb{1}\{\tilde{H}_s\notin\mathcal{E}\}\\
&+\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\bigg)\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=S\}\\
&+\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\bigg)\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}\Bigg]ds\\
\le \tilde{W}(x)+\mathbb{E}\int_0^t \tilde{Y}_s\tilde{S}_s&\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)\\
&+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\bigg)\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}ds\\
=\tilde{W}(x).& \qquad (80)
\end{aligned}
$$

The inequality in Eq.(80) results from applying Condition 5 of Proposition 4 and the fact that $\tilde{Y}_s\tilde{S}_s>0$. The last step in Eq. (80) follows from applying the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^t \tilde{Y}_s\tilde{S}_s\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\bigg)\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}ds\right|\\
&\le\sqrt{\left(\mathbb{E}\int_0^t \tilde{Y}_s^2\tilde{S}_s^2ds\right)}\\
&\quad\cdot\sqrt{\mathbb{E}\int_0^t\bigg(\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)\bigg)^2\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}ds}\\
&=0.
\end{aligned}
$$

In the last step above, the first squared root term is bounded since $\tilde{Y}_s\in L^2[0,t]$ by Lemma D.1 and $\tilde{S}_s\in[0,1]$. The second squared root term above is zero, since $\int_0^t\mathbb{1}\{\tilde{H}_s\in\mathcal{E}\ \&\ u_s=R\}ds=0$ almost surely by Lemma 6, and $\tilde{Q}(\tilde{H}_s)+\Big(\mu_{u_s}-\tilde{Q}(\tilde{H}_s)\Big)\tilde{W}(\tilde{H}_s)+(\mu_{u_s}+\sigma_{u_s}^2-\tilde{H}_s)\tilde{W}'(\tilde{H}_s)$ is bounded for $\tilde{H}_s$ in a countable set $\mathcal{E}$. Since Inequality (80) holds for any $t\ge0$, it also holds in the limit:

$$
\limsup_{t\to\infty}\mathbb{E}X_t\le\tilde{W}(x). \qquad (81)
$$

We will now show $\limsup_{t\to\infty}\mathbb{E}X_t\ge\tilde{V}(x,\pi)$ to complete the proof of $\tilde{W}(x)\ge\tilde{V}(x,\pi)$. Observe that since $\tilde{Y}_t>0$, $\tilde{S}_t>0$ for any $t>0$ and $\mu_R>0$, $\mu_S>0$, the integral $\int_0^t\tilde{Y}_s\tilde{S}_s\mu_{u_s}ds$ is pathwise monotone increasing in t and hence converges pathwise to $\int_0^\infty\tilde{Y}_s\tilde{S}_s\mu_{u_s}ds$ as $t\to\infty$. Therefore

$\lim_{t\to\infty} \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right] = \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right]$. Define $\tilde{V}_t(x,\pi) = \tilde{Y}_0 + \mathbb{E}\left[\int_0^t \tilde{S}_s d\tilde{Y}_s\right]$. It follows that

$$\begin{aligned}\tilde{V}_t(x,\pi) &= \tilde{Y}_0 + \mathbb{E}\left[\int_0^t \tilde{S}_s d\tilde{Y}_s\right]\\ &= 1 + \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds + \int_0^t \tilde{Y}_s \tilde{S}_s \sigma_{u_s} dB_s\right]\\ &= 1 + \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right]\end{aligned}$$

is monotone increasing in t. Note that the stochastic integral has zero expectation since $\tilde{Y}_s \tilde{S}_s \sigma_{u_s} \in L^2[0,t]$, the same reasoning as before. Then by the Monotone Convergence Theorem, we have

$$\begin{aligned}\tilde{V}(x,\pi) &= \tilde{V}_\infty(x,\pi)\\ &= \lim_{t\to\infty} \tilde{V}_t(x,\pi)\\ &= 1 + \lim_{t\to\infty} \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right]\\ &= 1 + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right], \end{aligned} \tag{82}$$

where the first equation follows from Eq. (72) and replacing $\mathbb{1}\{t < T\}$ with $\tilde{S}_t$ by the tower property of conditional expectation and the applying the definition of $\tilde{S}_t$ in Eq. (70). With Eq. 82, we are ready to take expectations on both sides of Eq. (77) and let $t \to \infty$. Apply Itô's formula on $\tilde{Y}_t \tilde{S}_t$, utilize Monotone Convergence Theorem to exchange limits with expectations and Lemma D.1 to get rid of the stochastic integral, we have

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{E}X_t &= \limsup_{t\to\infty} \mathbb{E}\left[\tilde{W}(\tilde{H}_t)\tilde{Y}_t\tilde{S}_t\right] + \limsup_{t\to\infty} \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{Q}(\tilde{H}_s)\tilde{S}_s ds\right]\\ &= \limsup_{t\to\infty} \mathbb{E}\left[\tilde{W}(\tilde{H}_t)\tilde{Y}_t\tilde{S}_t\right] + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{Q}(\tilde{H}_s)\tilde{S}_s ds\right]\\ &\geq \limsup_{t\to\infty} \mathbb{E}\left[\tilde{Y}_t\tilde{S}_t\right] + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{Q}(\tilde{H}_s)\tilde{S}_s ds\right]\\ &= 1 + \limsup_{t\to\infty} \mathbb{E}\left[\int_0^t \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds + \int_0^t \tilde{Y}_s \tilde{S}_s \sigma_{u_s} dB_s - \int_0^t \tilde{Y}_s \tilde{S}_s \tilde{Q}(\tilde{H}_s) ds\right]\\ &\quad + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{Q}(\tilde{H}_s)\tilde{S}_s ds\right]\\ &= 1 + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right] - \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{S}_s \tilde{Q}(\tilde{H}_s) ds\right] + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{Q}(\tilde{H}_s)\tilde{S}_s ds\right]\\ &= 1 + \mathbb{E}\left[\int_0^\infty \tilde{Y}_s \tilde{S}_s \mu_{u_s} ds\right]\\ &= \tilde{V}(x,\pi). \end{aligned} \tag{83}$$

The inequality results from the fact that $\tilde{W}(\cdot) > 1$ on $\mathbb{R}$ (see Condition 1 of Proposition 4) and that $\tilde{Y}_t \tilde{S}_t > 0$. Combining Eqs. (81) and (83), we obtain the desired result $W(x) \geq \tilde{V}(x,\pi)$ for $\forall x \in \mathbb{R}$ and any admissible policy π. Hence W as described in Proposition 4 is an upper bound for the optimal CLV $\tilde{V}^*$.

Now it remains to show that the gap $\tilde{W} - \tilde{V}^*$ is zero when the policy is chosen to be $\tilde{\pi}$, as described in Proposition 4. Observe in the above analysis that it suffices to show inequalities (80) and (83) are tight. Inequality (80) is tight follows from Condition 74 in Proposition 4. Hence it only remains to show Inequality (83) is tight. In fact, since $\tilde{W}$ is bounded (see Condition 3), this is true if we can show $\limsup_{t\to\infty} \mathbb{E}\left[\tilde{Y}_t\tilde{S}_t\right] = 0$. From definitions of $\tilde{Y}_t$ and $\tilde{S}_t$ (see Eqs. (67) and (70)), we know that (since $\tilde{\pi}$ is an interval policy by Condition 6 of Proposition 4, we use $\tilde{\pi}(\tilde{H}_t)$ to denote the policy's choice of service mode at time t)

$$\mathbb{E}\left[\tilde{Y}_t \tilde{S}_t\right] = \mathbb{E}\left[e^{\int_0^t \left(\mu_{\tilde{\pi}(\tilde{H}_z)} - \frac{\sigma^2_{\tilde{\pi}(\tilde{H}_z)}}{2} - \tilde{Q}(\tilde{H}_z)\right) dz + \int_0^t \sigma_{\tilde{\pi}(\tilde{H}_z)} dB_z}\right]$$
$$= \mathbb{E}\left[e^{\int_0^t \left(\mu_{\tilde{\pi}(\tilde{H}_z)} - \tilde{Q}(\tilde{H}_z)\right) dz}\right].$$

Since $Q_1 > Q_2 > \max\{\mu_S, \mu_R\}$ (see Condition 1), the exponent $\int_0^t \left(\mu_{\tilde{\pi}(\tilde{H}_z)} - \tilde{Q}(\tilde{H}_z)\right) dz$ is bounded above by $(\max\{\mu_S, \mu_R\} - Q_2)t$. Hence

$$0 \le \limsup_{t\to\infty} \mathbb{E}\left[e^{\int_0^t \left(\mu_{\tilde{\pi}(\tilde{H}_z)} - \tilde{Q}(\tilde{H}_z)\right) dz}\right] \le \limsup_{t\to\infty} \mathbb{E}\left[e^{\int_0^t (\max\{\mu_S,\mu_R\} - Q_2) dz}\right] = 0\,.$$

Thus we have proved the desired result. □

E Robustness check: switching costs

In this online appendix we first define the model with switching costs, and then (informally) derive the HJB equations required to numerically find the optimal policy and value function. Finally we present our numerical findings.

Model with switching costs. Assume that each transition from one mode to the other incurs a fixed cost, denoted by K, and keep all other assumptions in the original model unchanged. In this setting, the decision of which service mode to adopt should not only depend on the customer's current satisfaction level, but should also depend on the firm's current mode of service. In other words, the setting is stationary Markov with respect to a two-variable state which includes both the satisfaction and the current mode of service, and so, without loss of optimality, we can restrict attention to stationary Markov policies with respect to this state. Let π be such a stationary Markov policy, which maps from $\mathbb{R} \times \{S, R\}$ to $\{S, R\}$, i.e., $\pi(x, i)$ prescribes which service mode to use when the customer's satisfaction level is x and the firm's current service mode is $i \in \{S, R\}$. Denote by u_t the firm's service mode prescription at time t. Given an initial satisfaction $H_0 = x$ and initial service mode $u_{0-} = i$, under policy π, the firm's service mode prescription at each time $t \ge 0$ is $u_t = \pi(H_t, u_{t-})$. (As before, H_t and u_t are defined for all $t \ge 0$ independent of the customer lifetime T.) We restrict attention to policies such that, with probability 1, the resulting u_t process is right-continuous with left limits (cadlag).

The firm's objective is to find the policy that maximizes the difference between the expected reward earned during the customer's lifetime and the total switching cost incurred. Accordingly, the optimal expected CLV for a starting satisfaction state x and starting service mode $i \in \{S, R\}$ is

$$V_i^*(x) \triangleq \sup_{\pi\in\Pi} \mathbb{E}\left[\int_0^\infty \mathbb{1}\{t < T\} dY_t - K \sum_{k=1}^\infty \mathbb{1}\{\tau_k < T\} \,\middle|\, H_0 = x, u_{0-} = i\right],$$

where τ_k denotes the time point of the kth switch in service mode in the u_t sample path, and the customer lifetime T is defined as per Eq. (6) as before. (Since u_t is cadlag under admissible policy π, the set of time points where u_t switches is countable.)

HJB equations. The optimal CLV under switching cost should satisfy the following HJB equations (derived informally below):

$$0 = \max\left\{-Q(x)V_R^*(x) + (\mu_R - x)V_R^{*\prime}(x) + \frac{\sigma_R^2}{2} V_R^{*\prime\prime}(x) + \mu_R, V_S^*(x) - V_R^*(x) - K\right\}$$
$$\forall x \in \mathbb{R} \text{ where } V_R^{*\prime\prime}(x) \text{ exists;} \quad (84)$$

$$0 = \max\left\{-Q(x)V_S^*(x) + (\mu_S - x)V_S^{*\prime}(x) + \mu_S, V_R^*(x) - V_S^*(x) - K\right\} \quad \forall x \in \mathbb{R}. \tag{85}$$

We now summarize how we obtain these equations: We have two equations in this setting, each corresponding to one of the two possible service modes (the Risky mode or the Safe mode) being used currently. In particular, Eq. (84) comes from comparing the continuation value of staying with the Risky mode with the value of switching to the Safe mode while incurring a switching cost K. Similarly, Eq. (85) comes from comparing the continuation value of staying with the Safe mode with the value of switching to the Risky mode while incurring a switching cost K.

Next, we go over the heuristic steps to obtain the HJB equation.

Consider a starting satisfaction value at $H_0 = x$ and the firm's current service mode being the Risky mode. We answer the following question: should the firm stick with the Risky mode for a very short time t and then continue optimally, or should the firm immediately switch to the Safe mode but incur a switching cost of $K > 0$? In the first option, the total reward collected is

$$\int_0^t \mathbb{1}\{T > s\}(\mu_R ds + \sigma_R dB_s) + \mathbb{1}\{T > t\}V_R^*(H_t),$$

where $V_i^*(x)$ is the continuation value when satisfaction level starts at x and the firm's starting service mode is $i \in \{S, R\}$. Take expectation of the above expression and apply Itô's formula[20] on $e^{-\int_0^t Q(H_s)ds}V_R^*(H_t)$, we get

$$\begin{aligned}
&\mathbb{E}\int_0^t \mathbb{1}\{T > s\}(\mu_R ds + \sigma_R dB_s) + \mathbb{E}\left[\mathbb{1}\{T > t\}V_R^*(H_t)\right] \\
&= \mathbb{E}\int_0^t e^{-\int_0^s Q(H_v)dv}\mu_R ds + \mathbb{E}e^{-\int_0^t Q(H_s)ds}V_R^*(H_t) \\
&= \mathbb{E}\int_0^t e^{-\int_0^s Q(H_v)dv}\mu_R ds + \mathbb{E}\left[V_R^*(H_0) + \int_0^t e^{-\int_0^s Q(H_v)dv}(\mu_R - H_s)V_R^{*\prime}(H_s)ds\right. \\
&\qquad \left. + \int_0^t e^{-\int_0^s Q(H_v)dv}\frac{\sigma_R^2}{2}V_R^{*\prime\prime}(H_s)ds - \int_0^t Q(H_s)e^{-\int_0^s Q(H_v)dv}V_R^*(H_s)ds\right].
\end{aligned}$$

On the other hand, if the firm immediately switches to the Safe mode by incurring a cost of K, the total reward (minus cost) collected is $V_S^*(H_0) - K$. Since the firm wants to maximize total reward, we must have

$$\begin{aligned}
V_R^*(H_0) = \max\Bigg\{&\mathbb{E}\int_0^t e^{-\int_0^s Q(H_v)dv}\mu_R ds + \mathbb{E}\left[V_R^*(H_0) + \int_0^t e^{-\int_0^s Q(H_v)dv}(\mu_R - H_s)V_R^{*\prime}(H_s)ds\right. \\
&\left. + \int_0^t e^{-\int_0^s Q(H_v)dv}\frac{\sigma_R^2}{2}V_R^{*\prime\prime}(H_s)ds - \int_0^t Q(H_s)e^{-\int_0^s Q(H_v)dv}V_R^*(H_s)ds\right], V_S^*(H_0) - K\Bigg\}.
\end{aligned}$$

Consider the limit as $t \to 0$, the above equation reduces to

$$0 = \max\left\{-Q(x)V_R^*(x) + (\mu_R - x)V_R^{*\prime}(x) + \frac{\sigma_R^2}{2}V_R^{*\prime\prime}(x) + \mu_R, V_S^*(x) - V_R^*(x) - K\right\}.$$

Similarly, if the firm's current service mode is the Safe mode and the customer's satisfaction value is $H_0 = x$, we have

$$0 = \max\left\{-Q(x)V_S^*(x) + (\mu_S - x)V_S^{*\prime}(x) + \mu_S, V_R^*(x) - V_S^*(x) - K\right\}.$$

How we numerically solve for the optimal policy and value function. We focus on the case $\mu_R > \mu_S$ (the other case $\mu_R < \mu_S$ can be treated similarly with similar results, hence we omit the repetition), and numerically solve the HJB equations (84) and (85) to find the firm's optimal policy under switching cost K. For example, when K is small, we first conjecture that the optimal policy should follow a sandwich type with buffer zones. Then, we need to determine the boundaries l_R, r_R, r_S of the buffer zones (we also conjecture the fourth boundary is exactly at q)

[20] Fix $i \in \{S, R\}$. Since H_t is a semimartingale, by the Itô-Tanaka formula, if $V_i^*(\cdot)$ is sufficiently smooth, then $V_i^*(H_t)$ is also a semimartingale. Also $e^{-\int_0^t Q(H_s)ds}$ is a semimartingale. Therefore by the multidimensional Itô formula, $e^{-\int_0^t Q(H_s)ds}V_i^*(H_t)$ is also a semimartingle.

via continuity conditions of the value function, and numerically find the values of eight parameters $C_1, C_3, C_4, C_5, C_6, l_R, r_R, r_S$ of the following functions (that partially solves Eq. (84) and Eq. (85) for the Risky service mode and for the Safe mode in corresponding satisfaction regions, respectively),

$$W_R(x) = \begin{cases} C_1 e^{\frac{(x-\mu_R)^2}{\sigma_R^2}} \operatorname{erfc}\left(\frac{\mu_R - x}{\sigma_R}\right) + \mu_R & \text{if } x < q \\ C_3 + C_4 \int_q^x e^{-\frac{2\left(\mu_R u - \frac{u^2}{2}\right)}{\sigma_R^2}} du - \int_q^x \frac{e^{\frac{(u-\mu_R)^2}{\sigma_R^2}} \mu_R \sqrt{\pi} \operatorname{erf}\left(\frac{u-\mu_R}{\sigma_R}\right)}{\sigma_R} du & \text{if } q < x < l_R \\ C_5 + \mu_S \log(x - \mu_S) - K & \text{if } l_R < x < r_R \\ C_6 + \int_{r_R}^x \frac{\mu_R \sqrt{\pi}}{\sigma_R} e^{\frac{(u-\mu_R)^2}{\sigma_R^2}} \operatorname{erfc}\left(\frac{u-\mu_R}{\sigma_R}\right) du & \text{if } x > r_R \end{cases}$$

and

$$W_S(x) = \begin{cases} C_1 e^{\frac{(x-\mu_R)^2}{\sigma_R^2}} \operatorname{erfc}\left(\frac{\mu_R - x}{\sigma_R}\right) + \mu_R - K & \text{if } x < q \\ C_5 + \mu_S \log(x - \mu_S) & \text{if } q < x < r_S \\ C_6 + \int_{r_R}^x \frac{\mu_R \sqrt{\pi}}{\sigma_R} e^{\frac{(u-\mu_R)^2}{\sigma_R^2}} \operatorname{erfc}\left(\frac{u-\mu_R}{\sigma_R}\right) du - K & \text{if } x > r_S. \end{cases}$$

To be more concrete, we first determine the value of C_4 as a function of l_R via $\lim_{x \to l_R^-} W_R'(x) = \lim_{x \to l_R^+} W_R'(x)$. Then we equate $\lim_{x \to q^-} W_R'(x) = \lim_{x \to q^+} W_R'(x)$ to determine the value of C_1, also as a function of l_R. Solving $\lim_{x \to q^-} W_R(x) = \lim_{x \to q^+} W_R(x)$ then gives the value of C_3 as a function of l_R. After that, C_5 as a function of l_R can be determined by solving $\lim_{x \to l_R^-} W_R(x) = \lim_{x \to l_R^+} W_R(x)$. The value of l_R can then be solved by letting $\lim_{x \to q^-} W_S(x) = \lim_{x \to q^+} W_S(x)$. Having determined C_1, C_3, C_4, C_5, l_R, we can solve for the value of r_R by equating $\lim_{x \to r_R^-} W_R'(x) = \lim_{x \to r_R^+} W_R'(x)$, after which C_6 can be determined via $\lim_{x \to r_R^-} W_R(x) = \lim_{x \to r_R^+} W_R(x)$, and subsequently r_S can be determined by $\lim_{x \to r_S^-} W_S(x) = \lim_{x \to r_S^+} W_S(x)$.

After determining the values of $C_1, C_3, C_4, C_5, C_6, l_R, r_R, r_S$ and the corresponding conjectured value functions $W_R(x)$ and $W_S(x)$, we numerically verify that

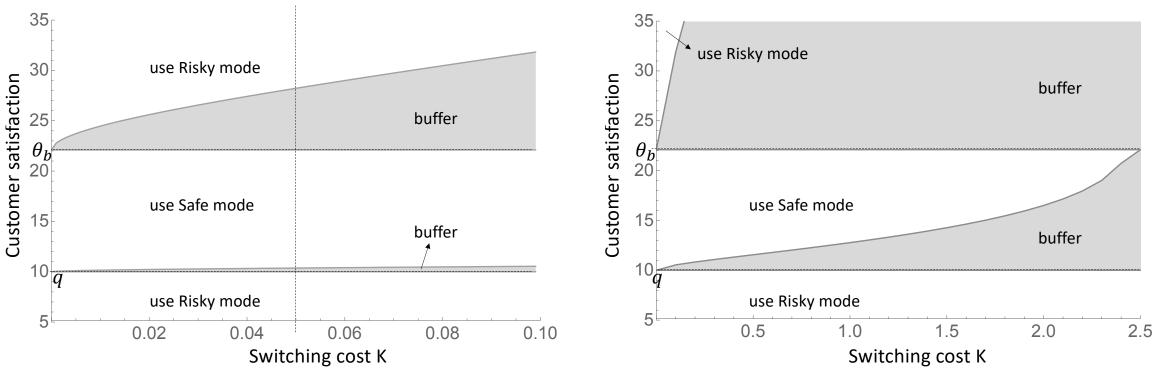
$$\begin{aligned} W_R(x) &\geq W_S - K && \text{if } x \in (-\infty, l_R) \cup (r_R, \infty) \\ -Q(x)W_R(x) + (\mu_R - x)W_R'(x) + \frac{\sigma_R^2}{2} W_R''(x) + \mu_R &\leq 0 && \text{if } x \in (l_R, r_R) \\ -Q(x)W_S(x) + (\mu_S - x)W_S'(x) + \mu_S &\leq 0 && \text{if } x \in (-\infty, q) \cup (r_S, \infty) \\ W_S(x) &\geq W_R(x) - K && \text{if } x \in (q, r_S), \end{aligned}$$

so that $W_R(x), W_S(x)$ fully solve the HJB equations (84)-(85).

Findings. We focus on the case $\mu_R > \mu_S$, and numerically solve the HJB equations (84) and (85) to find the firm's optimal policy under switching cost K. Among all the various instances we tried, we observe similar patterns of the optimal policy. In short, we find that adding a small switching cost results in an optimal policy which is very similar to the sandwich policy we find to be optimal in the original model.

Recall our result for the original model for $\mu_R > \mu_S$, namely, that in cases where a sandwich policy is optimal, θ_b and q are the thresholds separating the satisfaction values where the firm should use the Safe service mode from those where the firm should use the Risky service mode (Figure 2). With a small positive switching cost K, our numerics reveal that each switching threshold is replaced by a *buffer* interval. Specifically, above and below a buffer the firm should prefer opposite service modes (as in the original model, regardless of which service mode is currently in use), whereas inside a buffer interval the firm should not switch service modes. (Intuitively, the CLV benefit of having buffers in place of sharp switching thresholds for $K > 0$ is to reduce the number of switches between service modes.)

Figure 12 illustrates the optimal policy as a function of switching cost K. In the plot, the



(a) Small switching costs

(b) Larger switching costs

Figure 12: Firm's optimal policy as a function of switching cost K. We fix $\mu_S = 8$, $\mu_R = 9$, $\sigma_R = 10$, and $q = 10$. The shaded areas represent the buffers where the optimal policy retains the current service mode. The white areas represents the regions where the policy employs a specific service mode, even if this would incur a switching cost. For each K, the buffers of the optimal policy are given by the intervals where the (horizontal coordinate $= K$) vertical line intersects the shaded areas, e.g, the dotted line in the left subfigure gives the optimal policy for $K = 0.05$.

horizontal coordinate is the switching cost, and the vertical coordinate is the customer's current satisfaction value. All model primitives besides K are held fixed as per $\mu_S = 8$, $\mu_R = 9$, $\sigma_R = 10$, and $q = 10$. The shaded areas in Figure 12 represents the buffers of the optimal policy. Observe that the buffers grow when we increase the switching costs.[21] The special case $K = 0$ corresponds to the original model and leads to an interval optimal policy as before with sharp thresholds q and $\theta_b = 22.10$. Consider $K = 0.05$. The optimal policy has buffer zones $(q, 10.37) = (10, 10.37)$ and $(\theta_b, 28.22) = (22.10, 28.22)$, corresponding to the intersection between the $K = 0.05$ line and the shaded areas in Figure 12. Outside the buffer zones, the policy is identical to that in the original model, i.e., it uses the Risky mode for satisfaction values above 28.22 and below q, and it uses the Safe mode for satisfaction values in $(10.37, 22.10)$. To illustrate the role of the buffer zones: if the current service mode is Risky, then the policy prescribes to stay with Risky as long as the satisfaction level is on $(0, 10.37)$ or $(22.10, \infty)$, and switches to Safe once the satisfaction level leaves these regions. If the current service mode is Safe, then stick to Safe while the satisfaction level is on $(10, 28.22)$. The buffer zones $(10, 10.37)$ and $(22.10, 28.22)$ are hence overlapping bands for the two service modes, where the firm may use either the Risky or the Safe service mode depending on the current service mode in use. Despite the positive switching cost, the optimal policy produces a substantially larger expected CLV than the myopic policy. Specifically, the payoff is nearly 79% larger under the optimal policy $\frac{V_R^*(q)}{V(q,\text{Myopic})} = 1.79$, where $V(q, \text{Myopic})$ is the CLV of using the Risky mode always.

In the cases where the myopic (Risky always) policy is optimal under the original model (with $K = 0$), it is clear that the Risky always policy remains optimal even for $K > 0$, since the policy does not incur any switching cost.[22]

Implications for a setting where the firm cannot perfectly measure satisfaction. So

[21] In fact, we also find in numerical solutions that as the switching cost K increases to above $\mu_R - \mu_S$, there emerges a threshold $\theta_0 < q$, increasing with K, such that for satisfaction states under this threshold, the firm should not switch to the Risky mode, if currently using the Safe mode. Here $\mu_R - \mu_S$ is the CLV difference between the Risky-always policy and Safe-always policy in the limit of initial satisfaction $x \to -\infty$, since $Q(x') = 1 \; \forall x' < q$.

[22] There is technical caveat here: if the starting service mode is Safe, then if $K > 0$ there is a nontrivial decision regarding whether to switch to Risky and under what conditions. This case has limited practical relevance, so we avoid discussing it in the interest of space.

far we have assumed that the firm is able to perfectly estimate the customer's current satisfaction state. A reader might be concerned about the robustness of our findings to estimation errors (or delays). We now argue that the results we have obtained under switching costs suggest that small to medium-sized errors in estimating customer satisfaction would not significantly impair the CLV benefits of the optimal policy (relative to the myopic policy). In each case where the optimal policy for the case of switching costs is one where switching is postponed by a buffer (see Figure 12), by definition this policy produces higher CLV than the myopic policy. We can hence conclude this policy also produces higher CLV than the myopic policy if there were no switching costs (for $K = 0$, the CLV under the former policy is even larger whereas under the CLV under the latter policy stays the same). To give a quantitative example, for $K = 0.70$ the lower buffer interval of the optimal policy is $(10, 12.03)$ as per Figure 12. Thus the policy decisions are loosely similar to those of the optimal (sandwich) policy under the original model acting on *estimated* customer satisfaction when there are estimation errors of size similar to the size of the lower buffer interval ~ 2 (the upper buffer interval plays only a small role since the customer satisfaction only rarely rises to that level). The CLV increase from using the optimal (buffer) policy relative to the myopic policy is 30.0% for $K = 0.70$, and so the CLV increase from using the same policy in the absence of switching costs is even larger. This gives us confidence that our proposed policies still substantially increase the CLV in the face of small to medium-sized errors in estimating customer satisfaction. Along similar lines, interpreting the effect of the buffer intervals as delays in switching, one can argue that (small) delays in estimating customer satisfaction are unlikely to erode the benefits of using our proposed policies.

F Robustness check: Alternative hazard rate functions

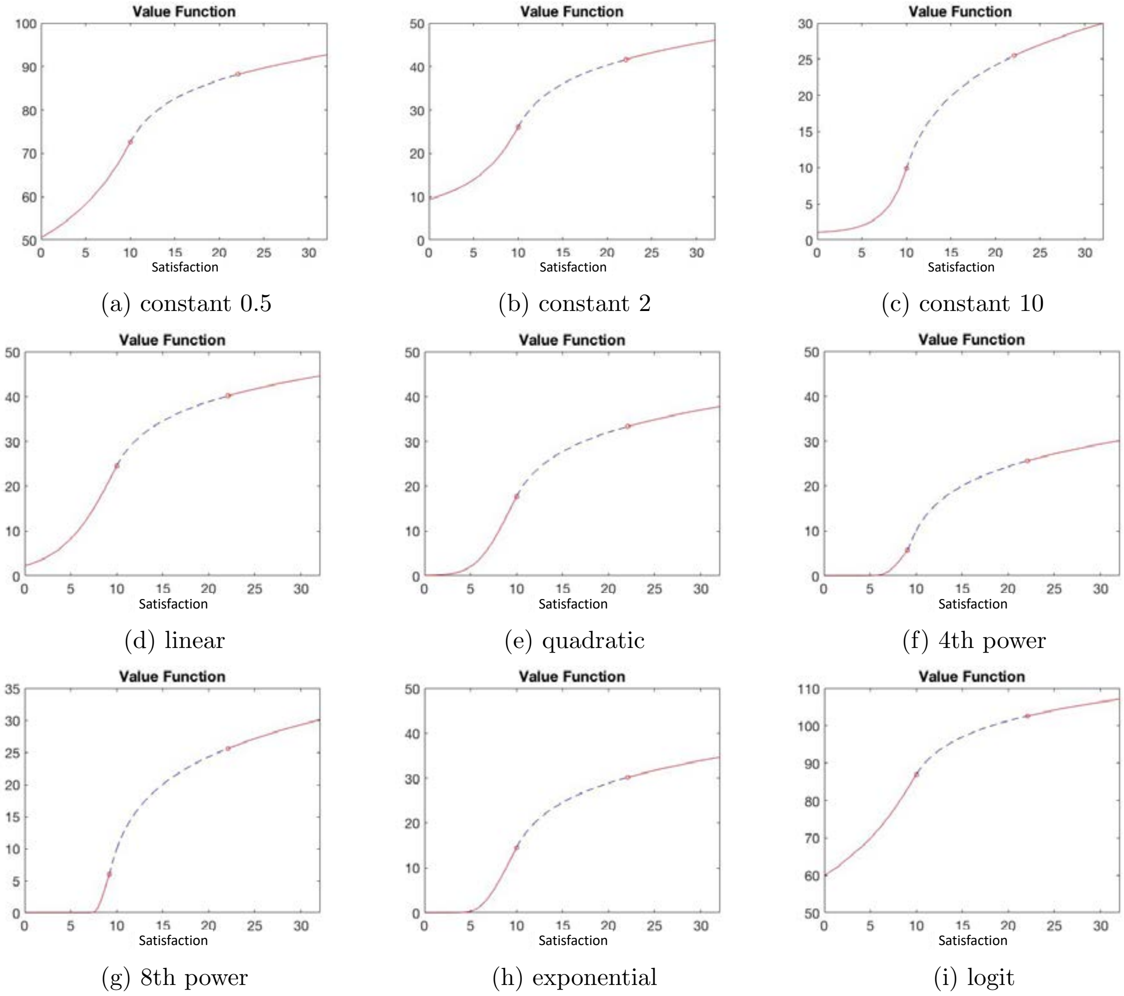
We now provide numerical evidence for our results' robustness to alternative specifications of the hazard rate function (recall the original step function in Eq. (4)) in the case $\mu_R > \mu_S > 0$. The most important takeaway is that, for a variety of hazard rate functions and different model primitives, the optimal policy is still either myopic or a sandwich policy, and moreover, the switching thresholds in the optimal sandwich policy are fairly robust to the shape of hazard rate functions.

First, we consider a variety of different hazard rate specifications in the unsatisfied zone, while keeping the original assumption of zero hazard rate in the satisfied zone. Let q be the satisfaction threshold separating the unsatisfied zone and the satisfied zone. Consider the following four types of hazard rate functions:

1. constant k: $Q(x) = k\mathbb{1}\{x < q\}$;
2. nth power: $Q(x) = (q - x)^n \mathbb{1}\{x < q\}$;
3. exponential: $Q(x) = (e^{q-x} - 1)\mathbb{1}\{x < q\}$;
4. logit: $Q(x) = \left(\frac{e^{q-x}}{1+e^{q-x}} - \frac{1}{2}\right) \mathbb{1}\{x < q\}$.

The value function and optimal policy associated with each hazard rate functions can be established by solving the HJB equation (13) and checking that the optimality conditions in Proposition 2 are still satisfied. Under all these different choices of the hazard rate function in the unsatisfied zone (including several different constants, and powers) and different model primitives with $\mu_R > \mu_S$, we find that (similar to Theorem 1) the optimal policy is either myopic or a sandwich policy. Figure 13 presents the firm's optimal policy and the associated CLV for some of the numerical instances.

One interesting observation from Figure 13 is that the size of the risk-averse region depends on how fast the hazard rate of leaving increases as the customer satisfaction level descends into the unsatisfied zone. In the original model, the firm switches from Safe to Risky as soon the customer



(a) constant 0.5 (b) constant 2 (c) constant 10
(d) linear (e) quadratic (f) 4th power
(g) 8th power (h) exponential (i) logit

Figure 13: Value function and optimal policies for various forms of $Q(\cdot)$; $\mu_R = 9$, $\mu_S = 8$, $\sigma_R = 10$, $q = 10$. Solid red lines correspond to the Risky service mode, and dashed blue lines correspond to the Safe service mode.

satisfaction crosses from the satisfied zone into the unsatisfied zone. This is not always the case with arbitrary hazard rate functions. When the hazard rate grows relatively fast as customer satisfaction goes down, the lower switching threshold remains at q. However, in the plotted cases where the hazard rate does not grow swiftly when customer satisfaction crosses into the unsatisfied zone, the switching point to Risky is strictly below q. In our numerics, this occurs for the cases of hazard rate functions growing as the fourth and eighth powers (see Figures 13(f) and 13(g)), where the lower boundary of the risk-averse region is strictly below q.

In Figure 14, we also show that the gap in CLVs between under the optimal policy and the myopic policy remains large for different hazard rate functions. In particular, the second curve from the bottom corresponds to the original step function hazard rate in Eq. (4). The other three curves correspond to the hazard rate function being linear, exponential, and logit in the unsatisfied zone (and zero in the satisfied zone) as introduced earlier in this section.

Other than the choice of hazard rate functions listed above, we also numerically examined cases where the hazard rate is strictly positive everywhere (including in the satisfied zone), again in the original model and restricting $\mu_R > \mu_S$, and find our main results remain intact. In particular,

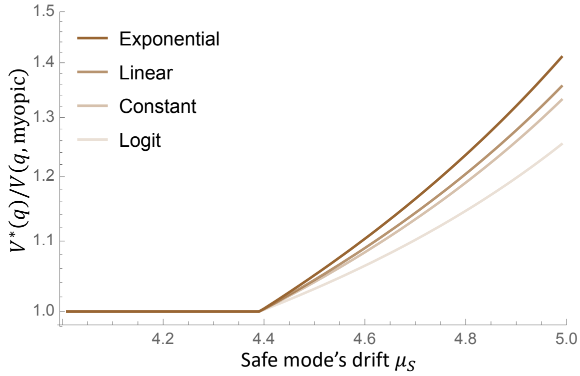


Figure 14: The ratio of CLV under the optimal policy to CLV under the myopic policy versus μ_S under different hazard rate functions), for $\mu_R = 5$, $\sigma_R = 2$, $q = 6$, and initial satisfaction q.

we considered hazard rate functions of the form $Q(x) = \mathbb{1}\{x < q\} + \epsilon\mathbb{1}\{x \geq q\}$ for $\epsilon \in (0, 1)$, and various model primitives such that $\mu_R > \mu_S$, and found that the optimal policy is still either myopic or a sandwich policy, where (in the optimal sandwich policy) the upper switching threshold decreases smoothly as we increase the value of ϵ. The CLV increase from using the optimal sandwich policy relative to the myopic policy is still large for small values of ϵ. We omit more details of this robustness check for brevity.