Online Appendix to
"Optimal Partition for a Multi-Type Queueing System":
Supplementary Derivations and Proofs

Proof of Theorem 1. First we consider $\alpha = 1$ ($\alpha = 0$). In this case, $\boldsymbol{x} = \mathbf{1}$ ($\boldsymbol{x} = \mathbf{0}$) is the only feasible solution, and the statement is true.

When $0 < \alpha < 1$, we first show that $\boldsymbol{x} = \mathbf{0}$ or $\boldsymbol{x} = \mathbf{1}$, i.e., assigning all customers to one queue, cannot be optimal. To proceed, we first introduce some notations. For queue with arrival rate λ and service time distribution S, we denote the zeroth order load as $C = \lambda\mathbb{E}[S^0]$, the first order load as $B = \lambda\mathbb{E}[S^1]$, and the second order load as $A = \lambda\mathbb{E}[S^2]$. We denote the zeroth, first, second order load for the first (second, resp) sub-queue under a unit server as C_x, B_x, A_x (C_y, B_y, A_y, resp.). Therefore, we have

$$
\begin{aligned}
A_x = \textstyle\sum_{i=1}^n \lambda_i v_i x_i, B_x = \sum_{i=1}^n \frac{\lambda_i}{\mu_i} x_i, C_x = \sum_{i=1}^n \lambda_i x_i,\\
A_y = \textstyle\sum_{i=1}^n \lambda_i v_i (1-x_i), B_y = \sum_{i=1}^n \frac{\lambda_i}{\mu_i}(1-x_i), C_y = \sum_{i=1}^n \lambda_i (1-x_i).
\end{aligned} \tag{A.1}
$$

We denote $s = \frac{B_x}{\alpha - B_x}, t = \frac{B_y}{1-\alpha-B_y}$. We further denote the Lagrangian function $\mathcal{L}(\boldsymbol{x}, \alpha, \boldsymbol{p}, \boldsymbol{q}) = \frac{A_x \cdot C_x}{\alpha(\alpha - B_x)} + \frac{A_y \cdot C_y}{(1-\alpha)(1-\alpha-B_y)} - \sum_i x_i p_i - \sum_i (1-x_i) q_i$, where $p_i \ge 0, q_i \ge 0$ (we remove the constant factor $\frac{1}{2\sum_{i=1}^n \lambda_i}$ for the ease of expression). From the KKT condition $\frac{\partial \mathcal{L}}{\partial x_i} = 0$, we have

$$
\begin{aligned}
p_i - q_i = \; & \left(\frac{C_x}{\alpha(\alpha - B_x)} - \frac{C_y}{(1-\alpha)(1-\alpha-B_y)} \right) \lambda_i v_i \\
& + \left(\frac{A_x}{\alpha(\alpha - B_x)} - \frac{A_y}{(1-\alpha)(1-\alpha-B_y)} \right) \lambda_i \\
& + \left(\frac{A_x C_x}{\alpha(\alpha - B_x)^2} - \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2} \right) \frac{\lambda_i}{\mu_i}.
\end{aligned} \tag{A.2}
$$

If $\boldsymbol{x} = \mathbf{0}$ is feasible, then for $\boldsymbol{x} = \mathbf{0}$,

$$
p_i - q_i = -\left(\frac{C_y}{(1-\alpha)(1-\alpha-B_y)} \lambda_i v_i + \frac{A_y}{(1-\alpha)(1-\alpha-B_y)} \lambda_i + \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2} \frac{\lambda_i}{\mu_i} \right) < 0.
$$

From non-negativity of p_i, q_i we know $q_i > 0$. However, complementary slackness condition states that $(1-x_i)q_i = 0$, thus $\boldsymbol{x} = \mathbf{0}$ is not optimal.

Similarly, if $\boldsymbol{x} = \mathbf{1}$ is feasible, then for $\boldsymbol{x} = \mathbf{1}$,

$$
p_i - q_i = \frac{C_x}{\alpha(\alpha - B_x)} \lambda_i v_i + \frac{A_x}{\alpha(\alpha - B_x)} \lambda_i + \frac{A_x C_x}{\alpha(\alpha - B_x)^2} \frac{\lambda_i}{\mu_i} > 0.
$$

From non-negativity of p_i, q_i we know $p_i > 0$. However, complementary slackness condition states that $x_i p_i = 0$, thus $\boldsymbol{x} = \mathbf{1}$ is not optimal.

Now we can restrict our consideration to $\boldsymbol{x} \neq \mathbf{0}, \mathbf{1}$. We first formulate a representative function which maps each customer type to a scalar value. In particular, for any feasible assignment $\boldsymbol{x}$, we define function $g_{\boldsymbol{x}}$ as

$$g_{\boldsymbol{x}}(u) = \left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y} v_{\mathcal{I}}(u) + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right) u. \tag{A.3}$$

It is readily shown that representative function only depends linearly on the first two moment of the customers' service time, i.e., u and $v_{\mathcal{I}}(u)$.

We complete the proof by showing that, under the optimal assignment policy, customer of type i will be allocated to the first server only when the corresponding scalar value $g_{\boldsymbol{x}}(\cdot)$ is non-positive. Consequently, due to linearity, the optimal assignment then corresponds to a polytope partition in a two-dimensional space of the first two moments. In particular, when there are two queues, the two-dimensional space are partitioned by a line into two subspaces.

Specifically, since $\boldsymbol{x} \neq \mathbf{0}, \mathbf{1}$, we have $A_x, A_y, B_x, B_y, C_x, C_y \neq 0$. Multiplying both sides of (A.2) by $\frac{B_y^2}{A_y C_y}\frac{1}{\lambda_i}$ yields

$$\begin{aligned}
\frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i} = & \left(\frac{\frac{C_x B_y}{C_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{A_y} v_i \\
& + \left(\frac{\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{C_y} \\
& + \left(\frac{\frac{C_x B_y}{C_y B_x}\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)^2/B_x^2} - \frac{1}{(1-\alpha)(1-\alpha-B_y)^2/B_y^2}\right)\frac{1}{\mu_i}.
\end{aligned}$$

Denote $\gamma = \frac{C_x}{B_x}\frac{B_y}{C_y}, \beta = \frac{A_x}{B_x}\frac{B_y}{A_y}, u = \sqrt{\gamma\beta}$. Recall that $s = \frac{B_x}{\alpha - B_x}, t = \frac{B_y}{1-\alpha-B_y}$, the above equation can then be simplified as

$$\left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y} v_i + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right)\frac{1}{\mu_i} = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}. \tag{A.4}$$

Recall definition (A.3), we know that (A.4) can be rewritten as $g_{\boldsymbol{x}}(\frac{1}{\mu_i}) = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}$. By complementary slackness conditions, for any optimal assignment $\boldsymbol{x}^*$, it holds that

$$g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) > 0 \Rightarrow x_i^* = 0, \quad g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) < 0 \Rightarrow x_i^* = 1.$$

We then finish the proof by considering two cases.

Case 1. $\frac{\gamma s}{\alpha}=\frac{t}{1-\alpha}$. Then $g_{\boldsymbol{x}^*}(u)$ reduces to a linear function. When $g_{\boldsymbol{x}^*}(u)\not\equiv 0$, $g_{\boldsymbol{x}^*}(u)$ intersects with the positive half of X-axis for at most once. By KKT condition we know there exists $i_0\in\{1,2,\dots,n\}$ such that either $\begin{cases} x_i^*=0 & i<i_0\\ x_i^*=1 & i>i_0\\ x_i^*\in[0,1] & i=i_0.\end{cases}$ or $\begin{cases} x_i^*=1 & i<i_0\\ x_i^*=0 & i>i_0\\ x_i^*\in[0,1] & i=i_0.\end{cases}$. Construct $U=\{i:i>i_0\}, E=\{i:i=i_0\}, L=\{i:i<i_0\}$. By considering a linear function passing through $(\frac{1}{\mu_{i_0}},\nu_0)$ with large enough slope, we have $(U,E,L)\in\mathcal{C}(\mathcal{I})$. When $g_{\boldsymbol{x}^*}(u)\equiv 0$, we have

$$\frac{\gamma s}{\alpha}=\frac{t}{1-\alpha},\frac{\beta s}{\alpha}=\frac{t}{1-\alpha},\frac{u^2s^2}{\alpha}=\frac{t^2}{1-\alpha},$$

which implies $\alpha=\frac{1}{2}$ and $\frac{\frac{1}{2}-B_x}{\frac{1}{2}-B_y}=\frac{A_x}{A_y}=\frac{C_x}{C_y}$. For any $i\neq j$, consider another assignment $\boldsymbol{x}(\epsilon)$ where $x_i(\epsilon)=x_i^*+\left(\frac{A_x+C_x}{\alpha-B_x}\frac{\lambda_j}{\mu_j}+\lambda_j+\lambda_j\nu_j\right)\epsilon, x_j(\epsilon)=x_j^*-\left(\frac{A_x+C_x}{\alpha-B_x}\frac{\lambda_i}{\mu_i}+\lambda_i+\lambda_i\nu_i\right)\epsilon$ and $x_k(\epsilon)=x_k^*,\forall k\neq i,j$. For $i=1,2,\dots,n$, we denote $A_i=\lambda_i\nu_i, B_i=\frac{\lambda_i}{\mu_i}, C_i=\lambda_i$. We define $l(\epsilon)=f(\boldsymbol{x}(\epsilon),\alpha)$. With some computation, we have

$$\begin{aligned}l''(\epsilon)=-\frac{1}{\sum_{t=1}^n\lambda_t}\Bigg(&\frac{\left(A_x(B_jC_i-B_iC_j)+C_x(A_jB_i-A_iB_j)+(\alpha-B_x)(A_jC_i-A_iC_j)\right)^2}{\alpha\left(\alpha-B_x-(A_jB_i+C_jB_i-A_iB_j-C_iB_j)\epsilon\right)^3}\\ &+\frac{\left(-A_y(B_jC_i-B_iC_j)-C_y(A_jB_i-A_iB_j)-(1-\alpha-B_y)(A_jC_i-A_iC_j)\right)^2}{(1-\alpha)\left(1-\alpha-B_y+(A_jB_i+C_jB_i-A_iB_j-C_iB_j)\epsilon\right)^3}\Bigg),\end{aligned}$$

where the second line uses the fact that

$$\begin{aligned}x_i(\epsilon)&=x_i^*+\left(\frac{A_x+C_x}{\alpha-B_x}\frac{\lambda_j}{\mu_j}+\lambda_j+\lambda_j\nu_j\right)\epsilon=x_i^*+\left(\frac{A_y+C_y}{1-\alpha-B_y}\frac{\lambda_j}{\mu_j}+\lambda_j+\lambda_j\nu_j\right)\epsilon,\\ x_j(\epsilon)&=x_j^*-\left(\frac{A_x+C_x}{\alpha-B_x}\frac{\lambda_i}{\mu_i}+\lambda_i+\lambda_i\nu_i\right)\epsilon=x_j^*-\left(\frac{A_y+C_y}{1-\alpha-B_y}\frac{\lambda_i}{\mu_i}+\lambda_i+\lambda_i\nu_i\right)\epsilon.\end{aligned}$$

Thereby, l is concave in ϵ in the neighborhood of 0. Together with $l'(0)=0$, we can claim that there exists an optimal solution with $\boldsymbol{x}^*=\mathbf{1}$ or $\boldsymbol{x}^*=\mathbf{0}$, where contradiction arises.

Case 2. $\frac{\gamma s}{\alpha}\neq\frac{t}{1-\alpha}$. In this case, consider linear function $l(u)=-\frac{\left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{C_y}+\left(\frac{u^2s^2}{\alpha}-\frac{t^2}{1-\alpha}\right)u}{\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{A_y}}$ and its induced partition (U,E,L). It is readily shown that $(U,E,L)\in\mathcal{C}(\mathcal{I})$. When $\frac{\gamma s}{\alpha}>\frac{t}{1-\alpha}$, the KKT condition implies that an optimal solution $\boldsymbol{x}^*$ satisfies

$$\begin{cases} x_i^*=0 & i\in U\\ x_i^*=1 & i\in L\\ x_i^*\in[0,1] & i\in E.\end{cases}$$

Similarly, when $\frac{\gamma s}{\alpha} < \frac{t}{1-\alpha}$, an optimal solution $\boldsymbol{x}^*$ satisfies

$$\begin{cases} x_i^* = 1 & i \in U \\ x_i^* = 0 & i \in L \\ x_i^* \in [0,1] & i \in E. \end{cases}$$

□

Proof of Theorem 2. First we note that if the feasible set of QAP is nonempty, then an optimal solution exists and is attainable. To see this, for any given α, as $\sum_{i=1}^n \frac{\lambda_i}{\mu_i} x_i \to \alpha$ or $\sum_{i=1}^n \frac{\lambda_i}{\mu_i} x_i \to \alpha - (1 - \sum_{i=1}^n \frac{\lambda_i}{\mu_i})$, $f(\boldsymbol{x}, \alpha) \to +\infty$. Also, $f(\boldsymbol{x}, \alpha)$ is continuous in $\boldsymbol{x}$. Thereby, a minimum can be obtained in the interior of set $\mathcal{F}_\alpha$. In addition, since the set $\{\boldsymbol{x} : 0 \le x_i \le 1, \forall i\}$ is closed, we can conclude that optimal solution $\boldsymbol{x}^*$ exists.

Next we prove that for any optimal solution $\boldsymbol{x}^*$, $\mathcal{M}^* = \left\{i : 0 < x_i^* < 1\right\}$ is either empty or singleton. We prove it by contradiction and show that if two different types of customers are assigned to both queues, swapping these customers between the two queues will always decrease the expected waiting time.

Specifically, we consider an optimal assignment $\boldsymbol{x}^*$ with cardinality $|\mathcal{M}^*| \ge 2$. Denote two of $\mathcal{M}^*$'s elements as $i \neq j$. Now consider another assignment $\boldsymbol{x}(\epsilon)$ where $x_i(\epsilon) = x_i^* + \frac{\mu_i}{\lambda_i}\epsilon, x_j(\epsilon) = x_j^* - \frac{\mu_j}{\lambda_j}\epsilon$, and $x_k(\epsilon) = x_k^*$, $\forall k \neq i, j$. We define

$$l(\epsilon) := f(\boldsymbol{x}(\epsilon), \alpha). \tag{A.5}$$

Therefore, we have $l''(0) = -\frac{(\mu_i - \mu_j)^2}{\mu_i \mu_j} \frac{1}{\sum_{k=1}^n \lambda_k} \left(\frac{1}{\alpha^2 - \alpha \sum_{k=1}^n \frac{\lambda_k x_k^*}{\mu_k}} + \frac{1}{(1-\alpha)^2 - (1-\alpha)\sum_{k=1}^n \frac{\lambda_k (1 - x_k^*)}{\mu_k}} \right) < 0$. Since $\epsilon = 0$ is an interior point and $l(\epsilon)$ is strictly concave at $\epsilon = 0$, there exists $\delta > 0$ such that for $\epsilon = \pm\delta$, $\boldsymbol{x}(\epsilon)$ is feasible and $l(0) > \min\{l(-\delta), l(\delta)\}$. Now we find a feasible assignment with a smaller objective function value, which contradicts with the optimality of $\boldsymbol{x}$. Therefore, $\mathcal{M}^* = \left\{i : 0 < x_i^* < 1\right\}$ must be either empty or singleton.

In the following, we prove the rest of the statement, i.e., the "three continuous segments" property, by showing that the first two moments of each customer type's service time distribution lie on a (possibly degenerated) quadratic curve. Then by Theorem 1, the partition line then interacts with this quadratic curve for at most twice.

For simplicity, we denote

$$A_x=\sum_{i=1}^n \frac{\lambda_i}{\mu_i^2}x_i, B_x=\sum_{i=1}^n \frac{\lambda_i}{\mu_i}x_i, C_x=\sum_{i=1}^n \lambda_i x_i;$$
$$A_y=\sum_{i=1}^n \frac{\lambda_i}{\mu_i^2}(1-x_i), B_y=\sum_{i=1}^n \frac{\lambda_i}{\mu_i}(1-x_i), C_y=\sum_{i=1}^n \lambda_i(1-x_i);$$
$$s=\frac{B_x}{\alpha-B_x}, t=\frac{B_y}{1-\alpha-B_y}. \tag{A.6}$$

Then we can write $f(\boldsymbol{x},\alpha)=\frac{1}{\sum_{i=1}^n \lambda_i}\left(\frac{A_x\cdot C_x}{\alpha(\alpha-B_x)}+\frac{A_y\cdot C_y}{(1-\alpha)(1-\alpha-B_y)}\right)$ and

$$\sum_{i=1}^n \frac{\lambda_i x_i}{\mu_i}<\alpha<\sum_{i=1}^n \frac{\lambda_i x_i}{\mu_i}+1-\sum_{i=1}^n \frac{\lambda_i}{\mu_i}$$

can be reformulated as

$$B_x<\alpha<1-B_y.$$

First we consider $\alpha=1$. In this case, $\boldsymbol{x}=\mathbf{1}$ is the only feasible solution, and the statement is true. Now we consider $\frac{1}{2}\le\alpha<1$. We first show that $\boldsymbol{x}=\mathbf{0}$ or $\boldsymbol{x}=\mathbf{1}$, i.e., assigning all customers to one queue, cannot be optimal. Denote the Lagrangian function $\mathcal{L}(\boldsymbol{x},\alpha,\boldsymbol{p},\boldsymbol{q})=\frac{A_x\cdot C_x}{\alpha(\alpha-B_x)}+\frac{A_y\cdot C_y}{(1-\alpha)(1-\alpha-B_y)}-\sum_i x_i p_i-\sum_i(1-x_i)q_i$, where $p_i\ge 0, q_i\ge 0$ (we remove the constant factor $\frac{1}{\sum_{i=1}^n \lambda_i}$ for the ease of expression). From the KKT condition $\frac{\partial\mathcal{L}}{\partial x_i}=0$, we have

$$\begin{aligned} p_i-q_i= &\left(\frac{C_x}{\alpha(\alpha-B_x)}-\frac{C_y}{(1-\alpha)(1-\alpha-B_y)}\right)\frac{\lambda_i}{\mu_i^2}\\ &+\left(\frac{A_x}{\alpha(\alpha-B_x)}-\frac{A_y}{(1-\alpha)(1-\alpha-B_y)}\right)\lambda_i\\ &+\left(\frac{A_xC_x}{\alpha(\alpha-B_x)^2}-\frac{A_yC_y}{(1-\alpha)(1-\alpha-B_y)^2}\right)\frac{\lambda_i}{\mu_i}. \end{aligned} \tag{A.7}$$

If $\boldsymbol{x}=\mathbf{0}$ is feasible, then for $\boldsymbol{x}=\mathbf{0}$,

$$p_i-q_i=-\left(\frac{C_y}{(1-\alpha)(1-\alpha-B_y)}\frac{\lambda_i}{\mu_i^2}+\frac{A_y}{(1-\alpha)(1-\alpha-B_y)}\lambda_i+\frac{A_yC_y}{(1-\alpha)(1-\alpha-B_y)^2}\frac{\lambda_i}{\mu_i}\right)<0.$$

From non-negativity of p_i, q_i we know $q_i>0$. However, complementary slackness condition states that $(1-x_i)q_i=0$, thus $\boldsymbol{x}=\mathbf{0}$ is not optimal.

Similarly, if $\boldsymbol{x}=\mathbf{1}$ is feasible, then for $\boldsymbol{x}=\mathbf{1}$,

$$p_i-q_i=\frac{C_x}{\alpha(\alpha-B_x)}\frac{\lambda_i}{\mu_i^2}+\frac{A_x}{\alpha(\alpha-B_x)}\lambda_i+\frac{A_xC_x}{\alpha(\alpha-B_x)^2}\frac{\lambda_i}{\mu_i}>0.$$

From non-negativity of p_i, q_i we know $p_i>0$. However, complementary slackness condition states that $x_ip_i=0$, thus $\boldsymbol{x}=\mathbf{1}$ is not optimal.

Now we can restrict our consideration to $\boldsymbol{x} \neq \mathbf{0}, \mathbf{1}$. In this case, $A_x, A_y, B_x, B_y, C_x, C_y \neq 0$. Multiplying both sides of (A.7) by $\frac{B_y^2}{A_y C_y}\frac{1}{\lambda_i}$ yields

$$\begin{aligned}\frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i} = & \left(\frac{\frac{C_x B_y}{C_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{A_y}\frac{1}{\mu_i^2} \\ & + \left(\frac{\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{C_y} \\ & + \left(\frac{\frac{C_x B_y}{C_y B_x}\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)^2/B_x^2} - \frac{1}{(1-\alpha)(1-\alpha-B_y)^2/B_y^2}\right)\frac{1}{\mu_i}.\end{aligned}$$

Denote $\gamma = \frac{C_x}{B_x}\frac{B_y}{C_y}, \beta = \frac{A_x}{B_x}\frac{B_y}{A_y}, u = \sqrt{\gamma\beta}$. Recall that $s = \frac{B_x}{\alpha - B_x}, t = \frac{B_y}{1-\alpha-B_y}$, the above equation can then be simplified as

$$\left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y}\frac{1}{\mu_i^2} + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right)\frac{1}{\mu_i} = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}. \tag{A.8}$$

For a feasible assignment $\boldsymbol{x}$, define a function $g_{\boldsymbol{x}}$ as

$$g_{\boldsymbol{x}}(v) = \left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y}v^2 + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right)v. \tag{A.9}$$

Then (A.8) can be rewritten as $g_{\boldsymbol{x}}(\frac{1}{\mu_i}) = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}$. By complementary slackness conditions, for any optimal assignment $\boldsymbol{x}^*$, it holds that

$$g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) > 0 \Rightarrow x_i^* = 0, \quad g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) < 0 \Rightarrow x_i^* = 1.$$

Note that $g_{\boldsymbol{x}^*}(v)$ is a polynomial function with degree being at most two. We finish the proof by considering two cases.

In the first case, $g_{\boldsymbol{x}^*}(v) \equiv 0$. Moreover, if there exist $i \neq j$ such that $x_i^* \neq x_j^*$, then we define function $l(\epsilon)$ as in (A.5) and denote $\boldsymbol{x}(\epsilon)$ as the corresponding assignment. By earlier analysis, we have $l''(0) < 0$. Also since $g_{\boldsymbol{x}^*}(v) \equiv 0$, we have $l'(0) = 0$. Moreover, since $x_i^* \neq x_j^*$, there exists $\delta \neq 0$ such that when $\epsilon = \delta$, $\boldsymbol{x}(\epsilon)$ is feasible and $l(\delta) < l(0)$, which contradicts with the optimality of $\boldsymbol{x}^*$. Therefore in this case, $\boldsymbol{x}$ must all take the same value. Since $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{x} \neq \mathbf{1}$, we know $0 < x_i < 1$, $i = 1, \ldots, n$, which can only happen when there is only one type of customers. Thus the conclusion is justified. When $g_{\boldsymbol{x}^*}(v) \not\equiv 0$, it is either a quadratic function or an affine function, and thus intersects with the positive half of X-axis for at most twice. As a result, we can partition

$(0,+\infty)$ into intervals $(0,\underline{\mu}^*]\cup[\underline{\mu}^*,\bar{\mu}^*]\cup[\bar{\mu}^*,+\infty)$ for some $\underline{\mu}^*\le\bar{\mu}^*$. Within each interval, signs of $g_{\boldsymbol{x}^*}(v)$ are the same, while in adjacent intervals they are different. In other words, there exist $0\le l<h\le n$ such that either $x_i^*=\begin{cases}0 & \text{if } l<i<h\\ 1 & \text{if } i>h \text{ or } i<l\end{cases}$ or $x_i^*=\begin{cases}1 & \text{if } l<i<h\\ 0 & \text{if } i>h \text{ or } i<l\end{cases}$.

Now we prove that if $\alpha\ge\frac{1}{2}$, then any optimal solution can be represented by the former case. Again, we prove by contradiction. Suppose there exist $i<j<k$ such that $g_{\boldsymbol{x}^*}(\frac{1}{\mu_i})\ge 0, g_{\boldsymbol{x}^*}(\frac{1}{\mu_j})\le 0, g_{\boldsymbol{x}^*}(\frac{1}{\mu_k})\ge 0$, then it implies that

$$\frac{\gamma s}{\alpha}>\frac{t}{1-\alpha}\quad\text{(the quadratic term of } g_{\boldsymbol{x}^*}(v) \text{ is positive);}\tag{A.10}$$

$$\frac{\beta s}{\alpha}>\frac{t}{1-\alpha}\quad\text{(the constant term of } g_{\boldsymbol{x}^*}(v) \text{ is positive);}\tag{A.11}$$

$$\frac{u^2s^2}{\alpha}<\frac{t^2}{1-\alpha}\quad\text{(the linear term of } g_{\boldsymbol{x}^*}(v) \text{ is negative).}\tag{A.12}$$

Multiplying (A.10) with (A.11) and then dividing it by (A.12), we have $\alpha<\frac{1}{2}$, which contradicts with the assumption $\alpha\ge\frac{1}{2}$. Thus, there exists $0\le l<h\le n$ such that $x_i^*=\begin{cases}0 & \text{if } l<i<h\\ 1 & \text{if } i>h \text{ or } i<l\end{cases}$. □

THEOREM A.1. *Given $\lambda_i, i=1,2,\dots,n$, $\alpha\ge\frac{1}{2}$ and $\mathcal{I}$ with $\mu_1>\cdots>\mu_n, v_1,\dots,v_n$ satisfying $\frac{v_n-v_{n-1}}{1/\mu_n-1/\mu_{n-1}}\ge\cdots\ge\frac{v_2-v_1}{1/\mu_2-1/\mu_1}\ge\frac{v_1}{1/\mu_1}$. For any optimal solution $\boldsymbol{x}^*$ to the QAP, there exist $0\le l<h\le n$ such that:*

- *When $i>h$ or $i<l$, $x_i^*=1$;*
- *When $l<i<h$, $x_i^*=0$;*
- *When $i\in\{l,h\}$, $x_i^*\in[0,1]$. Moreover, the set $\mathcal{M}^*=\left\{i:0<x_i^*<1\right\}$ is either empty or singleton.*

Proof of Theorem A.1. The proof is similar to Theorem 2. We first prove that the set $\mathcal{M}^*=\left\{i:0<x_i^*<1\right\}$ is either empty or singleton. Recall the l function defined in (A.5), we only need to show that $l''(0)<0$. We notice that

$$l''(0)=(\mu_i-\mu_j)(\mu_iv_i-\mu_jv_j)\frac{1}{2\sum_{k=1}^n\lambda_k}\left(\frac{1}{\alpha^2-\alpha\sum_{k=1}^n\frac{\lambda_kx_k^*}{\mu_k}}+\frac{1}{(1-\alpha)^2-(1-\alpha)\sum_{k=1}^n\frac{\lambda_k(1-x_k^*)}{\mu_k}}\right).$$

Since the condition $\frac{v_n-v_{n-1}}{1/\mu_n-1/\mu_{n-1}}\ge\cdots\ge\frac{v_2-v_1}{1/\mu_2-1/\mu_1}\ge\frac{v_1}{1/mu_1}$ implies $\frac{v_n}{1/\mu_n}\ge\cdots\ge\frac{v_21}{1/\mu_2}\ge\frac{v_1}{1/mu_1}$. We know that $(\mu_i-\mu_j)(\mu_iv_i-\mu_jv_j)$ is negative. Thereby $l''(0)<0$.

We next prove the rest of the statement. We follow the notation in Theorem 2, except for $A_x=\sum_{i=1}^n\lambda_i\frac{v_i}{2}x_i$ and $A_y=\sum_{i=1}^n\lambda_i\frac{v_i}{2}(1-x_i)$. Recall the definition of $g_{\boldsymbol{x}}$ function in (A.9) and we show in Theorem 2 that $g_{\boldsymbol{x}^*}$ is at most a quadratic function, thus intersecting with the positive half of X-axis for at most twice. Similarly we define

$$g_{\boldsymbol{x}}(u) = \left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y}\cdot\frac{1}{2}v_{\mathcal{I}}(u) + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right)u,$$

where $v_{\mathcal{I}}(u)$ is defined in (4). By condition $\frac{v_n - v_{n-1}}{1/\mu_n - 1/\mu_{n-1}} \geq \cdots \geq \frac{v_2 - v_1}{1/\mu_2 - 1/\mu_1} \geq \frac{v_1}{1/mu_1}$, we know that $v_{\mathcal{I}}(u)$ is a convex function. Therefore, $g_{\boldsymbol{x}^*}$ is either convex or concave. Again it intersects with the positive half of X-axis for at most twice. Following the argument in the proof of Theorem 2, there exists $0 \leq l < h \leq n$ such that $x_i^* = \begin{cases} 0 & \text{if } l < i < h \\ 1 & \text{if } i > h \text{ or } i < l \end{cases}$. □

Lemma A.1. *Given $\lambda_i, i = 1, 2, \ldots, n$ and $\mathcal{I}$ with $\mu_1 > \cdots > \mu_n, v_1, \ldots, v_n$ satisfying Assumption 1. For any optimal solution $(\boldsymbol{x}^*, \alpha^*)$ to the queue partition problem, there exists i^* such that either $x_i^* = \begin{cases} 1 & \textit{if } i < i^* \\ 0 & \textit{if } i > i^* \end{cases}$ or $x_i^* = \begin{cases} 0 & \textit{if } i < i^* \\ 1 & \textit{if } i > i^* \end{cases}$.*

Proof of Lemma A.1. We prove it by showing that otherwise the optimality condition of α and $\boldsymbol{x}$ cannot hold at the same time.

We start with the optimality condition of α. Specifically, recall the definition of s, t, we have

$$B_x = \frac{s}{1+s}\alpha, \quad B_y = \frac{t}{1+t}(1-\alpha).$$

Also recall $\gamma = \frac{C_x}{B_x}\frac{B_y}{C_y}, \beta = \frac{A_x}{B_x}\frac{B_y}{A_y}, u^2 = \beta\gamma$. Now we can simplify the optimality condition of α as

$$\begin{aligned}
&\frac{A_x C_x}{\alpha(\alpha - B_x)}\left(\frac{1}{\alpha} + \frac{1}{\alpha - B_x}\right) = \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)}\left(\frac{1}{1-\alpha} + \frac{1}{1-\alpha-B_y}\right)\\
\Leftrightarrow\ &\frac{\frac{C_x B_y}{C_y B_x}\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)/B_x^2}\left(\frac{1}{\alpha} + \frac{1}{\alpha - B_x}\right) = \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y^2}\left(\frac{1}{1-\alpha} + \frac{1}{1-\alpha-B_y}\right)\\
\Leftrightarrow\ &\frac{u^2}{\alpha(\alpha - B_x)/B_x^2}\left(\frac{1}{\alpha} + \frac{1}{\alpha - B_x}\right) = \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y^2}\left(\frac{1}{1-\alpha} + \frac{1}{1-\alpha-B_y}\right)\\
\Leftrightarrow\ &\frac{u^2}{\frac{1}{1+s}/(\frac{s}{1+s})^2}\left(\frac{1}{\alpha} + \frac{1+s}{\alpha}\right) = \frac{1}{\frac{1}{1+t}/(\frac{t}{1+t})^2}\left(\frac{1}{1-\alpha} + \frac{1+t}{1-\alpha}\right)
\end{aligned}$$

where the second to last line is by the definition of u, and the last line replaces B_x, B_y with $B_x = \frac{s}{1+s}\alpha$ and $B_y = \frac{t}{1+t}(1-\alpha)$.

By rearranging terms, we further have

$$\frac{\alpha}{1-\alpha} = \frac{u^2 s^2 \frac{s+2}{s+1}}{t^2 \frac{t+2}{t+1}}. \tag{A.13}$$

Next we show that if the argument is wrong, contradiction arises from the optimality condition of $\boldsymbol{x}$. In particular, by the proof of Theorem 1, if the argument is wrong, then $v_{\mathcal{I}}(u)$ must intersect with linear function $l(u) = -\frac{\left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{C_y}+\left(\frac{u^2s^2}{\alpha}-\frac{t^2}{1-\alpha}\right)u}{\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{A_y}}$ for at least twice. Since $v_{\mathcal{I}}(u)$ satisfies Assumption 1, we know that either $\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}<0, \frac{\beta s}{\alpha}-\frac{t}{1-\alpha}<0$, or $\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}>0, \frac{\beta s}{\alpha}-\frac{t}{1-\alpha}>0$. Define $\theta = \frac{s\frac{s+2}{s+1}}{t\frac{t+2}{t+1}}$. Substituting α by (A.13), we have either $\gamma < u^2\theta, \beta < u^2\theta$, or $\gamma > u^2\theta, \beta > u^2\theta$. Recall that $u^2 = \beta\gamma$, we then have

$$\gamma + \beta < u^2\theta + \frac{1}{\theta}. \tag{A.14}$$

We multiply both sides of (A.4) with λ_i, which yields

$$\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{A_y}\lambda_i v_i + \left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{C_y}\lambda_i + \left(\frac{u^2 s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{\lambda_i}{\mu_i} = \frac{B_y^2}{A_y C_y}(p_i - q_i). \tag{A.15}$$

By the complementary slackness conditions, we know that $x_i(p_i - q_i) = -q_i \le 0$. Thus, multiplying (A.15) by x_i and summing up yields

$$\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)A_x\frac{B_y}{A_y} + \left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)C_x\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha}-\frac{t^2}{1-\alpha}\right)B_x \le 0. \tag{A.16}$$

Here we use the fact that $A_x = \sum_i \lambda_i v_i x_i, B_x = \sum_i \frac{\lambda_i}{\mu_i}x_i, C_x = \sum_i \lambda_i x_i$. Next, we further have

$$\begin{aligned}
\text{(A.16)} &\Rightarrow \left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)\beta + \left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)\gamma + \left(\frac{u^2 s^2}{\alpha}-\frac{t^2}{1-\alpha}\right) \le 0 \\
&\Rightarrow 2\beta\gamma + u^2 s \le u^2\theta(\beta+\gamma+t) \\
&\Rightarrow 2u^2 + u^2 s < u^2\theta(u^2\theta + \frac{1}{\theta} + t) \\
&\Rightarrow u^2\theta^2 > 1 + s - t\theta \\
&\Rightarrow u^2\theta^2 > \frac{(s+1)^2 + (t+1)}{(t+2)(s+1)}
\end{aligned}$$

where the second line is by (A.13), the third line is by (A.14), and the last line is by the definition of θ.

Similarly, by the complementary slackness conditions, $(1 - x_i)(p_i - q_i) = p_i \ge 0$. Thus, by multiplying (A.15) by $1 - x_i$ and summing up yields

$$\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)A_y\frac{B_y}{A_y} + \left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)C_y\frac{B_y}{C_y} + \left(\frac{u^2 s^2}{\alpha}-\frac{t^2}{1-\alpha}\right)B_y \ge 0. \tag{A.17}$$

Here we use the fact that $A_y = \sum_i \lambda_i v_i(1-x_i)$, $B_y = \sum_i \frac{\lambda_i}{\mu_i}(1-x_i)$, $C_y = \sum_i \lambda_i(1-x_i)$. And we further have

$$\begin{aligned}
(\text{A.17}) &\Rightarrow \beta+\gamma+u^2 s \geq u^2\theta(t+2)\\
&\Rightarrow u^2\theta+\frac{1}{\theta}+u^2 s > u^2\theta(t+2)\\
&\Rightarrow \frac{1}{u^2\theta^2} > t+1-\frac{s}{\theta} = \frac{(t+1)^2+(s+1)}{(s+2)(t+1)}
\end{aligned}$$

where the first line is by (A.13), the second line is by (A.14), and the last line is by definition of θ.

Together, it implies

$$\frac{(s+1)^2+(t+1)}{(t+2)(s+1)}\cdot\frac{(t+1)^2+(s+1)}{(s+2)(t+1)} < 1,$$

which is equivalent to

$$((s+1)+(t+1))\,(s-t)^2 < 0.$$

Therefore, we reach a contradiction. Thus there can not be $i<j<k$ such that $x_i^* > x_j^*$ and $x_k^* > x_j^*$. It means that for any optimal solution $\boldsymbol{x}^*$ there exists i^* such that for $i \neq i^*$ either $x_i^* = \begin{cases} 1 & \text{if } i < i^* \\ 0 & \text{if } i > i^* \end{cases}$ or $x_i^* = \begin{cases} 0 & \text{if } i < i^* \\ 1 & \text{if } i > i^* \end{cases}$. □

Lemma A.2. *Given $\lambda_i, i=1,2,\ldots,n$ and $\mathcal{I}$ with $\mu_1 > \cdots > \mu_n, v_1,\ldots,v_n$ satisfying Assumption 1. Any solution $(1,\ldots,1,x_i,0,\ldots,0),\alpha$ with $\alpha\in(0,1)$ and $x_i\in(0,1)$ for some i cannot be optimal.*

Proof of Lemma A.2. We prove it by contradiction and show that if one type of customers are assigned to both queues, either transferring the customers from one queue to the other, or transferring the customers back from the other queue, together with the transfer of serving capacity, will always decrease the expected waiting time. Consequently, it cannot be optimal.

We assume there exist at least one 1 and one 0 in $\boldsymbol{x}$, since we can split customers with respect to x_i to construct artificial customer types. As a result, $\mu_1 \geq \cdots \geq \mu_i \geq \cdots \geq \mu_n$.

Before proceeding, we first derive some useful relations from optimality conditions of $\boldsymbol{x}$ and α. Specifically, we denote

$$\beta_x = \frac{A_x}{B_x}\frac{B_i}{A_i}, \gamma_x = \frac{C_x}{B_x}\frac{B_i}{C_i}, \beta_y = \frac{A_y}{B_y}\frac{B_i}{A_i}, \gamma_y = \frac{C_y}{B_y}\frac{B_i}{C_i}. \tag{A.18}$$

Since $\mu_1 \geq \cdots \geq \mu_i \geq \cdots \geq \mu_n$ satisfy Assumption 1, we know

$$\beta_x \leq 1 \leq \beta_y, \gamma_y \leq 1 \leq \gamma_x. \tag{A.19}$$

By first order optimality condition of α, we have

$$\begin{aligned}
&\frac{A_x C_x}{\alpha(\alpha - B_x)}\left(\frac{1}{\alpha} + \frac{1}{\alpha - B_x}\right) = \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)}\left(\frac{1}{1-\alpha} + \frac{1}{1-\alpha-B_y}\right)\\
\Leftrightarrow\ &\frac{A_x/B_x \cdot C_x/B_x}{\alpha \cdot (\alpha - B_x)/B_x}\left(\frac{B_x}{\alpha} + \frac{B_x}{\alpha - B_x}\right) = \frac{A_y/B_y \cdot C_y/B_y}{(1-\alpha)\cdot(1-\alpha-B_y)/B_y}\left(\frac{B_y}{1-\alpha} + \frac{B_y}{1-\alpha-B_y}\right)\\
\Leftrightarrow\ &\frac{A_i C_i}{B_i^2}\frac{\beta_x\gamma_x}{\alpha/s}\left(\frac{s}{s+1} + s\right) = \frac{A_i C_i}{B_i^2}\frac{\beta_y\gamma_y}{(1-\alpha)/t}\left(\frac{t}{t+1} + t\right)\\
\Leftrightarrow\ &\frac{\alpha}{1-\alpha} = \frac{\beta_x\gamma_x\frac{s^2(s+2)}{s+1}}{\beta_y\gamma_y\frac{t^2(t+2)}{t+1}}
\end{aligned} \tag{A.20}$$

where the third line is from (A.6) and (A.18).

By first order optimality condition of x_i, we have

$$\begin{aligned}
&\left(\frac{C_x}{\alpha(\alpha-B_x)} - \frac{C_y}{(1-\alpha)(1-\alpha-B_y)}\right)A_i\\
&+\left(\frac{A_x}{\alpha(\alpha-B_x)} - \frac{A_y}{(1-\alpha)(1-\alpha-B_y)}\right)C_i\\
&+\left(\frac{A_x C_x}{\alpha(\alpha-B_x)^2} - \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2}\right)B_i = 0\\
\Leftrightarrow\ &\left(\frac{C_x/B_x}{\alpha(\alpha-B_x)/B_x} - \frac{C_y/B_y}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)A_i\\
&+\left(\frac{A_x/B_x}{\alpha(\alpha-B_x)/B_x} - \frac{A_y/B_y}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)C_i\\
&+\left(\frac{A_x/B_x\cdot C_x/B_x}{\alpha(\alpha-B_x)^2/B_x^2} - \frac{A_y/B_y\cdot C_y/B_y}{(1-\alpha)(1-\alpha-B_y)^2/B_y^2}\right)B_i = 0\\
\Leftrightarrow\ &\left(\frac{\gamma_x}{\alpha/s} - \frac{\gamma_y}{(1-\alpha)/t}\right)A_iC_i/B_i\\
&+\left(\frac{\beta_x}{\alpha/s} - \frac{\beta_y}{(1-\alpha)/t}\right)A_iC_i/B_i\\
&+\left(\frac{\beta_x\gamma_x}{\alpha/s^2} - \frac{\beta_y\gamma_y}{(1-\alpha)/t^2}\right)A_iC_i/B_i = 0\\
\Leftrightarrow\ &\frac{\gamma_x}{\alpha/s} + \frac{\beta_x}{\alpha/s} + \frac{\beta_x\gamma_x}{\alpha/s^2} = \frac{\gamma_y}{(1-\alpha)/t} + \frac{\beta_y}{(1-\alpha)/t} + \frac{\beta_y\gamma_y}{(1-\alpha)/t^2}\\
\Leftrightarrow\ &\frac{\beta_x\gamma_x s}{\alpha}\left(\frac{1}{\beta_x} + \frac{1}{\gamma_x} + s\right) = \frac{\beta_y\gamma_y t}{1-\alpha}\left(\frac{1}{\beta_y} + \frac{1}{\gamma_y} + t\right)
\end{aligned}$$

where the third to last line is from (A.6) and (A.18). Substituting α by (A.20), we have

$$\frac{\frac{1}{\beta_x}+\frac{1}{\gamma_x}+s}{\frac{s(s+2)}{s+1}}=\frac{\frac{1}{\beta_y}+\frac{1}{\gamma_y}+t}{\frac{t(t+2)}{t+1}}. \tag{A.21}$$

To complete the proof, we construct a way to transfer both x_i customers and serving capacity α such that expected total waiting time must decrease. Specifically, we denote $r=\frac{\frac{A_i}{B_iA_y}+\frac{C_i}{B_iC_y}+\frac{1}{1-\alpha-B_y}}{\frac{1}{1-\alpha}+\frac{1}{1-\alpha-B_y}}$. Multiply numerator and denominator with B_y, we have $r=\frac{\frac{1}{\beta_y}+\frac{1}{\gamma_y}+t}{\frac{t(t+2)}{t+1}}$. Together with (A.21), we have

$$r=\frac{\frac{1}{\beta_x}+\frac{1}{\gamma_x}+s}{\frac{s(s+2)}{s+1}}=\frac{\frac{1}{\beta_y}+\frac{1}{\gamma_y}+t}{\frac{t(t+2)}{t+1}}.$$

Now consider partition $(1,\cdots,1,x_i+\frac{1}{B_i}\epsilon,0,\cdots,0),\alpha+r\epsilon$. Note that as long as $|\epsilon|>0$ is small, it is a feasible partition. Multiplied by $\sum_{i=1}^n\lambda_i$, the corresponding objective function is then denoted as

$$F(\epsilon)=G(\epsilon)+H(\epsilon)$$

where $G(\epsilon)=\frac{\left(A_x+\frac{A_i}{B_i}\epsilon\right)\left(C_x+\frac{C_i}{B_i}\epsilon\right)}{(\alpha+r\epsilon)(\alpha-B_x+(r-1)\epsilon)}$ and $H(\epsilon)=\frac{\left(A_y-\frac{A_i}{B_i}\epsilon\right)\left(C_y-\frac{C_i}{B_i}\epsilon\right)}{(1-\alpha-r\epsilon)(1-\alpha-B_y-(r-1)\epsilon)}$. We next show that $F'(0)=0$ and $F''(0)\le 0$.

Denote $g(\epsilon)=\log G(\epsilon), h(\epsilon)=\log H(\epsilon)$. Then $F'(0)=G(0)g'(0)+H(0)h'(0)$. By $r=\frac{\frac{A_i}{B_iA_y}+\frac{C_i}{B_iC_y}+\frac{1}{1-\alpha-B_y}}{\frac{1}{1-\alpha}+\frac{1}{1-\alpha-B_y}}$, we have $h'(0)=\frac{-A_i}{B_iA_y}+\frac{-C_i}{B_iC_y}+\frac{r}{1-\alpha}+\frac{r-1}{1-\alpha-B_y}=0$. By first order optimality condition, we know $F'(0)=0$, thus $g'(0)=h'(0)=0$. Note that

$$\begin{aligned}
g(\epsilon) &= \log\left(A_x+\frac{A_i}{B_i}\epsilon\right)+\log\left(C_x+\frac{C_i}{B_i}\epsilon\right)\log(\alpha+r\epsilon)-\log\left(\alpha-B_x+(r-1)\epsilon\right);\\
g'(\epsilon) &= \frac{A_i/B_i}{A_x+\frac{A_i}{B_i}\epsilon}+\frac{C_i/B_i}{C_x+\frac{C_i}{B_i}\epsilon}\frac{r}{\alpha+r\epsilon}\frac{r-1}{\alpha-B_x+(r-1)\epsilon};\\
g''(\epsilon) &= \frac{-A_i^2/B_i^2}{\left(A_x+\frac{A_i}{B_i}\epsilon\right)^2}+\frac{-C_i^2/B_i^2}{\left(C_x+\frac{C_i}{B_i}\epsilon\right)^2}+\frac{r^2}{(\alpha+r\epsilon)^2}+\frac{(r-1)^2}{(\alpha-B_x+(r-1)\epsilon)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
g''(0) &= \frac{-A_i^2/B_i^2}{A_x^2}+\frac{-C_i^2/B_i^2}{C_x^2}+\frac{r^2}{\alpha^2}+\frac{(r-1)^2}{(\alpha-B_x)^2}\\
\Leftrightarrow B_x^2g''(0) &= -\frac{A_i^2B_x^2}{A_x^2B_i^2}-\frac{C_i^2B_x^2}{C_x^2B_i^2}+\frac{r^2B_x^2}{\alpha^2}+\frac{(r-1)^2B_x^2}{(\alpha-B_x)^2}\\
\Leftrightarrow B_x^2g''(0) &= r^2\left(\frac{s}{s+1}\right)^2+(r-1)^2s^2-\left(\frac{1}{\beta_x^2}+\frac{1}{\gamma_x^2}\right)\\
&= \frac{-2(s+1)\frac{1}{\beta_x^2}+\frac{2}{\beta_x}\left(\frac{1}{\gamma_x}(s^2+2s+2)-s^2\right)+\left(\frac{1}{\gamma_x}+s\right)^2+\left(\frac{1}{\gamma_x}(s+1)-s\right)^2-(s+2)^2\frac{1}{\gamma_x}}{(s+2)^2}
\end{aligned}$$

where the third line is from (A.18) and (A.6). Note that it is a quadratic function of $\frac{1}{\beta_x}$ whose maximum is obtained at $\frac{\frac{1}{\gamma_x}(s^2+2s+2)-s^2}{2(s+1)}$. From (A.19), we know $\gamma_x \geq 1$, thereby $\frac{\frac{1}{\gamma_x}(s^2+2s+2)-s^2}{2(s+1)} \leq 1$. Thereby, such quadratic function is decreasing in $[1,+\infty)$. Since $\frac{1}{\beta_x} \geq 1$, we know $B_x^2 g''(0) \leq \frac{-2(s+1)}{(s+2)^2}\left(\frac{1}{\gamma_x}-1\right)^2 \leq 0$, where equality holds if and only if $\beta_x = \gamma_x = 1$.

Similarly,

$$B_y^2 h''(0) = \frac{-2(t+1)\frac{1}{\beta_y^2} + \frac{2}{\beta_y}\left(\frac{1}{\gamma_y}(t^2+2t+2)-t^2\right) + \left(\frac{1}{\gamma_y}+t\right)^2 + \left(\frac{1}{\gamma_y}(t+1)-t\right)^2 - (t+2)^2\frac{1}{\gamma_y}}{(t+2)^2}$$

which is a quadratic function of $\frac{1}{\beta_y}$ whose maximum is obtained at $\frac{\frac{1}{\gamma_y}(t^2+2t+2)-t^2}{2(t+1)} \geq 1$. Thereby, such quadratic function is increasing in $[0,1]$. Since $\frac{1}{\beta_y} \leq 1$, we know $B_y^2 h''(0) \leq \frac{-2(t+1)}{(t+2)^2}\left(\frac{1}{\gamma_y}-1\right)^2 \leq 0$, where equality holds if and only if $\beta_y = \gamma_y = 1$.

Thereby,

$$F''(0) = G(0)\left(g''(0)+(g'(0))^2\right) + H(0)\left(h''(0)+(h'(0))^2\right) = G(0)g''(0) + H(0)h''(0) \leq 0$$

where equality holds if and only if $\beta_x = \gamma_x = \beta_y = \gamma_y = 1$.

In the case of $F''(0) < 0$, due to the openness of feasible domain of ϵ, $F'(0) = 0$ and strict concavity of $F(\epsilon)$ at $\epsilon = 0$, there exists $\delta \neq 0$ such that corresponding partition and assignment are feasible and $F(0) > F(\delta)$. This contradicts with the optimality assumption.

In the case of $F''(0) = 0$, it is equivalent to the case where there is only one type of customers, and the optimization problem turns into

$$\begin{aligned} \inf_{x,\alpha} \quad & \frac{\lambda\nu}{2}\left(\frac{1}{\frac{\alpha}{x}\left(\frac{\alpha}{x}-\frac{\lambda}{\mu}\right)} + \frac{1}{\frac{1-\alpha}{1-x}\left(\frac{1-\alpha}{1-x}-\frac{\lambda}{\mu}\right)}\right) \\ \text{s.t.} \quad & \frac{\alpha}{x} > \frac{\lambda x}{\mu}, \frac{1-\alpha}{1-x} > \frac{\lambda x}{\mu}, 0 < x < 1. \end{aligned}$$

Note that if (x,α) is a feasible solution, then $(1-x, 1-\alpha)$ is also a feasible solution with the same objective function value. Thereby, without loss of generality, assume $\alpha \leq x$. Thus objective function $\geq \frac{\lambda\nu}{2}\frac{1}{1-\frac{\lambda}{\mu}}$. Note that equality holds if and only if $\alpha = x = 0$ or $\alpha = x = 1$. Thus single queue is the unique optimal solution. □

Proof of Theorem 3. In Lemma A.1, we prove the "two continuous segments" property, i.e., for any optimal solution $(\boldsymbol{x}^*, \alpha^*)$, there exists i^* such that either $x_i^* = \begin{cases} 1 & \text{if } i < i^* \\ 0 & \text{if } i > i^* \end{cases}$ or $x_i^* = \begin{cases} 0 & \text{if } i < i^* \\ 1 & \text{if } i > i^* \end{cases}$.

By plugging in Lemma A.2, we know that for any optimal solution $(\boldsymbol{x}^*, \alpha^*)$, the set $I^* := \left\{i : 0 < x_i^* < 1\right\}$ is empty. Thereby, either $x_{i^*}^* = 1$ or $x_{i^*}^* = 0$. And we can merge it into the $i < i^*$ case or the $i > i^*$ case, which completes the proof. □

THEOREM A.2. *The* deterministic assignment problem *for given* α*:*

$$\begin{aligned} \text{(DAP)} \quad & \inf_{\boldsymbol{x}} \quad f(\boldsymbol{x}, \alpha) \\ & s.t. \quad \boldsymbol{x} \in \mathcal{F}_\alpha; \quad x_i \in \{0,1\}, \ i = 1, \ldots, n, \end{aligned} \tag{A.22}$$

is NP-hard.

Proof of Theorem A.2. We prove it by reduction from the *Set Partition* problem. Recall the *Set Partition* problem takes a set S of numbers and outputs whether there exists a partition A and $\bar{A}$ such that $A \cup \bar{A} = S$ and $\sum_{w \in A} w = \sum_{w \in \bar{A}} w$.

For set S in the *Set Partition* problem with elements $w_1 \le w_2 \le \cdots \le w_n$, we construct an instance of the DAP with parameters $\lambda_i = w_i$, $\mu_i = \bar{\mu} = 2\sum_{i=1}^n w_i$, for all i and $\alpha = \frac{1}{2}$. To complete the reduction, we next prove

- If equal set partition exists, then all optimal solutions to the DAP satisfy $\sum_{i=1}^n \lambda_i x_i = \frac{1}{2}\sum_{i=1}^n \lambda_i$.
- If equal set partition does not exist, then for any optimal solution to the DAP, $\sum_{i=1}^n \lambda_i x_i \neq \frac{1}{2}\sum_{i=1}^n \lambda_i$.

Note that if the above statements hold, then we can solve the *Set Partition* problem by first solving the constructed DAP and then checking whether the optimal solution satisfies the condition.

Now we consider the DAP with the constructed parameters. Multiply the objective function with $\sum_{i=1}^n \lambda_i$, the DAP can be reformulated as

$$\begin{aligned} \inf_{\boldsymbol{x}} \quad & \frac{(\sum_{i=1}^n \lambda_i x_i)^2}{\frac{\bar{\mu}}{2} - \sum_{i=1}^n \lambda_i x_i} + \frac{(\sum_{i=1}^n \lambda_i - \sum_{i=1}^n \lambda_i x_i)^2}{\frac{\bar{\mu}}{2} - (\sum_{i=1}^n \lambda_i - \sum_{i=1}^n \lambda_i x_i)} \\ \text{s.t.} \quad & -\frac{\bar{\mu}}{2} + \sum_{i=1}^n \lambda_i < \sum_{i=1}^n \lambda_i x_i < \frac{\bar{\mu}}{2}; \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned}$$

If we drop the binary constraint and denote $y = \sum_{i=1}^n \lambda_i x_i$, then it can be relaxed to

$$\begin{aligned} \inf_{y} \quad & h(y) := \frac{y^2}{\frac{\bar{\mu}}{2} - y} + \frac{\left(\sum_{i=1}^n \lambda_i - y\right)^2}{\frac{\bar{\mu}}{2} - \left(\sum_{i=1}^n \lambda_i - y\right)} \\ \text{s.t.} \quad & -\frac{\bar{\mu}}{2} + \sum_{i=1}^n \lambda_i < y < \frac{\bar{\mu}}{2}. \end{aligned}$$

Note that $h''(y) > 0, y^* = \frac{1}{2}\sum_{i=1}^n \lambda_i = \frac{\bar{\mu}}{4}$ is feasible, and $h'(y^*) = 0$. Thereby, y^* is the unique optimal solution to the relaxed problem.

Therefore, if there exists subset I such that $\sum_{i\in I}\lambda_i = \frac{1}{2}\sum_{i=1}^n \lambda_i$, then y^* is also feasible for the original problem thus optimal. If there is no such I, then $\forall I, y_I := \sum_{i\in I}\lambda_i \neq \frac{1}{2}\sum_{i=1}^n \lambda_i$, which completes the proof. □

Proof of Theorem 4. Since each element of $C(\mathcal{I}, k)$ can be represented by $\mathcal{L}$ and some $\mathcal{M}$, we prove that the optimal assignment can be represented by $\mathcal{L}$ and some $\mathcal{M}$. Start from the first queue. Following the proof of Theorem 1, we know there exist at most $k-1$ of lines to separate customers in this queue from customers in the other $k-1$ queues. And if $(\mathbb{E}[S_i], \mathbb{E}[S_i^2])$ is in the interior of the polygon, then $X_{i1} = 1$. Those in the boundary correspond to $X_{i1} \in [0, 1]$ and the others $X_{i1} = 0$. Remove points in the interior of polygon, and continue the process. Thereby the optimal assignment can be induced by these $\frac{(k-1)k}{2}$ lines which constitute the set $\mathcal{L}$. □

Proof of Theorem 5. Start from the kth queue. Following the proof of Theorem 2, we know there exist $0 \le l_k^1 \le l_k^2 \le n+1$ such that $X_{ik} = 1$ if and only if $l_k^1 < i < l_k^2$. Remove all ith type of customers such that $l_k^1 \le i \le l_k^2$. Then consider the $(k-1)$th queue. We can repeat the process for the rest of the customers until we are considering the first queue. For the first queue, let $l_1^1 = 0, l_1^2 = n+1$, obviously, the rest of customers must be assigned to the first queue. Now we have $2k$ of l's, which completes the proof. □

Proof of Theorem 6. Following the proof of Theorem 3, we know that for any two queues j_1, j_2, an optimal solution (X^*, α^*) splits customers in these two queues by service rate μ_i, such that there exists $\bar{\mu}$ and customers whose service rate $\mu_i \le \bar{\mu}$ are assigned to one queue while customers whose service rate $\mu_i > \bar{\mu}$ are assigned to the other queue. Define interval $I_j = [\underline{\mu}_j, \bar{\mu}_j]$ where $\underline{\mu}_j = \min\left\{\mu_i : X_{ik}^* > 0, k = j\right\}, \bar{\mu}_j = \max\left\{\mu_i : X_{ik}^* > 0, k = j\right\}$. Then, we know for any $1 \le j_1 < j_2 \le k$, it holds that $I_{j_1} \cap I_{j_2} = \emptyset$. Thereby, the conclusion is justified.

Proof of Theorem 8. The idea is similar to the waiting time case. We only need to prove the $k = 2$ case, since the general case proof can be extended following the same argument in the proof of Theorem 4.

First we consider $\alpha = 1$ ($\alpha = 0$). In this case, $\boldsymbol{x} = \mathbf{1}$ ($\boldsymbol{x} = \mathbf{0}$) is the only feasible solution, and the statement is true.

When $0 < \alpha < 1$, we first show that $\boldsymbol{x} = \mathbf{0}$ or $\boldsymbol{x} = \mathbf{1}$, i.e., assigning all customers to one queue, cannot be optimal. We follow the notation of $A_x, B_x, C_x, A_y, B_y, C_y, s, t$ in the proof of Theorem 1 and denote C, B, A as the zeroth, first, second order load of a pooled queue with unit capacity which serves all types of customers. Therefore,

$$A = \sum_{i=1}^{n} \lambda_i v_i, B = \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}, C = \sum_{i=1}^{n} \lambda_i.$$

Then we can write $\tilde{f}(\boldsymbol{x}, \alpha) = \frac{1}{2\sum_{i=1}^{n} \lambda_i} \left(\frac{A_x \cdot C_x}{\alpha(\alpha - B_x)} + \frac{2B_x}{\alpha} + \frac{A_y \cdot C_y}{(1-\alpha)(1-\alpha-B_y)} + \frac{2B_y}{1-\alpha} \right)$.

We denote the Lagrangian function $\mathcal{L}(\boldsymbol{x}, \alpha, \boldsymbol{p}, \boldsymbol{q}) = \frac{A_x \cdot C_x}{\alpha(\alpha - B_x)} + \frac{2B_x}{\alpha} + \frac{A_y \cdot C_y}{(1-\alpha)(1-\alpha-B_y)} + \frac{2B_y}{1-\alpha} - \sum_i x_i p_i - \sum_i (1 - x_i) q_i$, where $p_i \geq 0, q_i \geq 0$ (we remove the constant factor $\frac{1}{2\sum_{i=1}^{n} \lambda_i}$ for the ease of expression). From the KKT condition $\frac{\partial \mathcal{L}}{\partial x_i} = 0$, we have

$$\begin{aligned} p_i - q_i = & \left(\frac{C_x}{\alpha(\alpha - B_x)} - \frac{C_y}{(1-\alpha)(1-\alpha-B_y)} \right) \lambda_i v_i \\ & + \left(\frac{A_x}{\alpha(\alpha - B_x)} - \frac{A_y}{(1-\alpha)(1-\alpha-B_y)} \right) \lambda_i \\ & + \left(\frac{A_x C_x}{\alpha(\alpha - B_x)^2} + \frac{2}{\alpha} - \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2} - \frac{2}{1-\alpha} \right) \frac{\lambda_i}{\mu_i}. \end{aligned} \tag{A.23}$$

If $\boldsymbol{x} = \mathbf{0}$ is feasible, then for $\boldsymbol{x} = \mathbf{0}$,

$$\begin{aligned} p_i - q_i = & - \left(\frac{C_y}{(1-\alpha)(1-\alpha-B_y)} \lambda_i v_i + \frac{A_y}{(1-\alpha)(1-\alpha-B_y)} \lambda_i \right. \\ & \left. + \left(\frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2} + \frac{2}{1-\alpha} \right) \frac{\lambda_i}{\mu_i} \right) \\ & < 0. \end{aligned}$$

From non-negativity of p_i, q_i we know $q_i > 0$. However, complementary slackness condition states that $(1 - x_i) q_i = 0$, thus $\boldsymbol{x} = \mathbf{0}$ is not optimal.

Similarly, if $\boldsymbol{x} = \mathbf{1}$ is feasible, then for $\boldsymbol{x} = \mathbf{1}$,

$$p_i - q_i = \frac{C_x}{\alpha(\alpha - B_x)} \frac{\lambda_i}{\mu_i^2} + \frac{A_x}{\alpha(\alpha - B_x)} \lambda_i + \left(\frac{A_x C_x}{\alpha(\alpha - B_x)^2} + \frac{2}{\alpha} \right) \frac{\lambda_i}{\mu_i} > 0.$$

From non-negativity of p_i, q_i we know $p_i > 0$. However, complementary slackness condition states that $x_i p_i = 0$, thus $\boldsymbol{x} = \mathbf{1}$ is not optimal.

Now we can restrict our consideration to $\boldsymbol{x} \neq \mathbf{0}, \mathbf{1}$. We first formulate a representative function which maps each customer type to a scalar value. In particular, for any feasible assignment $\boldsymbol{x}$, we define function $g_{\boldsymbol{x}}$ as

$$g_{\boldsymbol{x}}(u) = \left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y} v_{\mathcal{I}}(u) + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{2B_y^2}{A_y C_y}\left(\frac{1}{\alpha} - \frac{1}{1-\alpha}\right) + \frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right) u. \tag{A.24}$$

It is readily shown that representative function only depends linearly on the first two moment of the customers' service time, i.e., u and $v_{\mathcal{I}}(u)$.

We complete the proof by showing that, under the optimal assignment policy, customer of type i will be allocated to the first server only when the corresponding scalar value $g_{\boldsymbol{x}}(\cdot)$ is non-positive. Consequently, due to linearity, the optimal assignment then corresponds to a polytope partition in a two-dimensional space of the first two moments. In particular, when there are two queues, the two-dimensional space are partitioned by a line into two subspaces.

Specifically, since $\boldsymbol{x} \neq \mathbf{0}, \mathbf{1}$, we have $A_x, A_y, B_x, B_y, C_x, C_y \neq 0$. Multiplying both sides of (A.23) by $\frac{B_y^2}{A_y C_y}\frac{1}{\lambda_i}$ yields

$$\begin{aligned}\frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i} = &\left(\frac{\frac{C_x B_y}{C_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{A_y} v_i \\ &+ \left(\frac{\frac{A_x B_y}{A_y B_x}}{\alpha(\alpha - B_x)/B_x} - \frac{1}{(1-\alpha)(1-\alpha-B_y)/B_y}\right)\frac{B_y}{C_y} \\ &+ \frac{B_y^2}{A_y C_y}\left(\frac{A_x C_x}{\alpha(\alpha - B_x)^2} + \frac{2}{\alpha} - \frac{A_y C_y}{(1-\alpha)(1-\alpha-B_y)^2} - \frac{2}{1-\alpha}\right)\frac{1}{\mu_i}.\end{aligned}$$

Denote $\gamma = \frac{C_x}{B_x}\frac{B_y}{C_y}, \beta = \frac{A_x}{B_x}\frac{B_y}{A_y}$. Recall that $s = \frac{B_x}{\alpha - B_x}, t = \frac{B_y}{1-\alpha-B_y}$, the above equation can then be simplified as

$$\begin{aligned}\left(\frac{\gamma s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{A_y} v_i + \left(\frac{\beta s}{\alpha} - \frac{t}{1-\alpha}\right)\frac{B_y}{C_y} + \left(\frac{2B_y^2}{A_y C_y}\left(\frac{1}{\alpha} - \frac{1}{1-\alpha}\right) + \frac{u^2 s^2}{\alpha} - \frac{t^2}{1-\alpha}\right)\frac{1}{\mu_i} \\ = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}.\end{aligned} \tag{A.25}$$

Recall definition (A.24), we know that (A.25) can be rewritten as $g_{\boldsymbol{x}}(\frac{1}{\mu_i}) = \frac{B_y^2}{A_y C_y}\frac{p_i - q_i}{\lambda_i}$. By complementary slackness conditions, for any optimal assignment $\boldsymbol{x}^*$, it holds that

$$g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) > 0 \Rightarrow x_i^* = 0, \quad g_{\boldsymbol{x}^*}\left(\frac{1}{\mu_i}\right) < 0 \Rightarrow x_i^* = 1.$$

We then consider two cases. In the first case, $\frac{\gamma s}{\alpha} \neq \frac{t}{1-\alpha}$. Consider linear function $l(u) = -\frac{\left(\frac{\beta s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{C_y}+\left(\frac{2B_y^2}{A_yC_y}\left(\frac{1}{\alpha}-\frac{1}{1-\alpha}\right)+\frac{u^2s^2}{\alpha}-\frac{t^2}{1-\alpha}\right)u}{\left(\frac{\gamma s}{\alpha}-\frac{t}{1-\alpha}\right)\frac{B_y}{A_y}}$ and its induced partition (U, E, L). It is readily shown that $(U, E, L) \in C(\mathcal{I})$. When $\frac{\gamma s}{\alpha} > \frac{t}{1-\alpha}$, the KKT condition implies that an optimal solution $\boldsymbol{x}^*$ satisfies

$$\begin{cases} x_i^* = 0 & i \in U \\ x_i^* = 1 & i \in L \\ x_i^* \in [0,1] & i \in E. \end{cases}$$

Similarly, when $\frac{\gamma s}{\alpha} < \frac{t}{1-\alpha}$, an optimal solution $\boldsymbol{x}^*$ satisfies

$$\begin{cases} x_i^* = 1 & i \in U \\ x_i^* = 0 & i \in L \\ x_i^* \in [0,1] & i \in E. \end{cases}$$

Otherwise, $\frac{\gamma s}{\alpha} = \frac{t}{1-\alpha}$. Then $g_{\boldsymbol{x}^*}(u)$ reduces to a linear function. When $g_{\boldsymbol{x}^*}(u) \not\equiv 0$, $g_{\boldsymbol{x}^*}(u)$ intersects with the positive half of X-axis for at most once. By KKT condition, we know there exists $i_0 \in \{1, 2, \dots, n\}$ such that either $\begin{cases} x_i^* = 0 & i < i_0 \\ x_i^* = 1 & i > i_0 \\ x_i^* \in [0,1] & i = i_0. \end{cases}$ or $\begin{cases} x_i^* = 1 & i < i_0 \\ x_i^* = 0 & i > i_0 \\ x_i^* \in [0,1] & i = i_0. \end{cases}$ Construct $U = \{i : i > i_0\}, E = \{i : i = i_0\}, L = \{i : i < i_0\}$. By considering a linear function passing through $(\frac{1}{\mu_{i_0}}, v_0)$ with large enough slope, we have $(U, E, L) \in C(\mathcal{I})$. When $g_{\boldsymbol{x}^*}(u) \equiv 0$, we have

$$\frac{\gamma s}{\alpha} = \frac{t}{1-\alpha}, \frac{\beta s}{\alpha} = \frac{t}{1-\alpha}, \frac{2B_y^2}{A_yC_y}\left(\frac{1}{\alpha} - \frac{1}{1-\alpha}\right) + \frac{u^2s^2}{\alpha} - \frac{t^2}{1-\alpha} = 0.$$

Multiply the first two equations, we have

$$\frac{u^2s^2}{\alpha^2} = \frac{t^2}{(1-\alpha)^2}.$$

Without loss of generality assume $\alpha \geq \frac{1}{2}$. Thereby, $\frac{u^2s^2}{\alpha} \geq \frac{t^2}{1-\alpha}$. And we have $\frac{A_x}{A_y} = \frac{C_x}{C_y} = \frac{\alpha}{1-\alpha}, \alpha - B_x = 1 - \alpha - B_y, \frac{A_xC_x}{\alpha(\alpha - B_x)} = \frac{2}{1-\alpha}$, which implies $\frac{AC}{1-B} = \frac{1}{\alpha(1-\alpha)}$. Consider any α_0 in the neighborhood of α, $\frac{AC}{1-B} \neq \frac{1}{\alpha_0(1-\alpha_0)}$. Thereby, the conclusion holds for α_0. Let $\alpha_0 \to \alpha$, we can conclude that an optimal solution satisfying the condition exists for α, which completes the proof. □