Online Appendix

This online appendix is an e-companion for paper "Information Design and Sharing in Supply Chains".

EC.1. Technical Proofs

EC.1.1. Proof of Lemma 1: Equivalence of the best forecast and the best linear forecast

We first establish the equivalence between the supplier's best linear forecast and best possible forecast and prove Lemma 1. This equivalence is a direct consequence of the Gaussianity of the orders $\{O_t\}$. Intuitively, for any finite-dimensional Gaussian random vector (X,Y), the conditional expectation $\mathbb{E}[X|Y]$ can be written as a linear form in Y.

We now prove Lemma 1 under no information sharing. The proof under full information sharing is similar. Since $\{O_t\}$ is a purely non-deterministic stationary Gaussian process, by Wold's Theorem (Brockwell and Davis 2006, Section 5.7), it can be written as $O_t = \sum_{n=0}^{\infty} a_n \epsilon^*_{t-n}$, where $a_0 = 1$ for model identifiability and $\{\epsilon^*_t\}$ is an uncorrelated sequence such that the linear past (see Definition 1) $\mathcal{M}_{\mathrm{O}}(t) = \mathcal{M}_{\epsilon^*}(t)$. Since $\{O_t\}$ is a Gaussian process, it follows that $\{\epsilon^*_t\}$ is Gaussian, and as a consequence, $\{\epsilon^*_t\}$ is independent, and due to invertibility, $\sigma(O_s : s \le t) = \sigma(\epsilon^*_s : s \le t)$. Therefore, the best possible forecast is indeed $\mathbb{E}[O_{t+1}|\mathcal{F}^{\mathrm{S}}(t)] = \sum_{n=1}^{\infty} a_n \epsilon^*_{t+1-n}$, which coincides with the orthogonal projection of O_{t+1} on $\mathcal{M}_{\mathrm{O}}(t)$. It follows that the MSFEs of the two forecasts are identical.

EC.1.2. Proof of Proposition 1: Full Information Sharing

Note that $O_{t+1} = d + \sum_{n=0}^{\infty} \tilde{\psi}_n \,\epsilon_{t+1-n}$. By Lemma 1, the supplier's best forecast under full information sharing is

$$m_{\mathrm{S|FI}}(t) = \mathcal{P}_{\mathcal{M}_{\mathrm{D}}(t)}(O_{t+1}) = \mathcal{P}_{\{1,\epsilon_t,\epsilon_{t-1},\epsilon_{t-2},\dots\}}(O_{t+1}) = d + \sum_{n=1}^{\infty} \tilde{\psi}_n \,\epsilon_{t+1-n}.$$

Therefore, the corresponding forecast error is $O_{t+1} - m_{\mathrm{S|FI}}(t) = \tilde{\psi}_0 \epsilon_{t+1}$, and $\sigma^2_{\mathrm{S|FI}} = \mathbb{V}\mathrm{ar}(\tilde{\psi}_0\epsilon_{t+1}) = \sigma^2_\epsilon\, \tilde{\psi}_0^2$ By (11), the supplier's average expected cost is $\mathcal{C}^{\mathrm{S}}_{\mathrm{FI}}(\tilde{\psi}) = K_{\mathrm{S}}\, \sigma_\epsilon\, |\tilde{\psi}_0|$.

EC.1.3. Real-valued process

We first present a technical lemma regarding the fact that the order process $\{O_t\}$ is real-valued. The following lemma shows that, $\tilde{\psi}(z)$ satisfies an important property $\overline{\tilde{\psi}(z)} = \tilde{\psi}(\overline{z})$, which we refer to as the *conjugate-symmetry* property. We further show that the conjugate-symmetry property and Assumption 1 are carried over to the three components after the factorization.

LEMMA 6. *For any function $\tilde{\psi}(z) \in H^2$, $\tilde{\psi}$ is conjugate-symmetric if and only if $\{\tilde{\psi}_k\}$ is real-valued. In addition, if $\tilde{\psi}(z)$ is conjugate-symmetric, its factors $B(z)$, $G(z)$, $Q(z)$ defined in Lemma 2 are conjugate-symmetric. Moreover, given Assumptions 1–2, we have*

$$Q(1) := \lim_{r \to 1^-} Q(r) = 1,$$

and $Q(e^{i\lambda})$ is continuous at $\lambda = 0$. The inner function $M(z) = B(z)G(z)$ satisfies $B(1) = G(1) = 1$.

EC.1.4. Non-tangential Convergence

We introduce the definition of a Stolz region and Abel's limit Theorem as an important technical lemma to help characterize the geometric implications of the stationary inventory assumption (Assumption 1).

DEFINITION 6. *A Stolz region in the unit disk $\mathbb{D}$ is defined as $R(K) = \{z \in \mathbb{D}, |1-z| \le K(1-|z|)\}$.*

We now present the following lemma, a proof of which can be found in (Ahlfors 1966, Page 41).

LEMMA 7 (**Abel's limit theorem**). *For every power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients,*

- *There exist $0 \le R \le +\infty$ such that, $f(z)$ converges for $|z| < R$ and diverges for $|z| > R$.*
- *If $\sum_{n=0}^{\infty} a_n$ converges then $f(z)$ converges to $f(1)$ as long as z approaches 1 along any curve in a Stolz region $R(K)$ with some fixed constant K.*

The second bullet in Lemma 7 defines a convergence called a *non-tangential limit*, and indicates that if a power series converges at a point on the boundary, the convergence extends to the sense of radial limit, as well as the nontangential limit. It is worth noting that the radial limit is a special case of the nontangential limit.

EC.1.5. Proof of Proposition 2: No Information Sharing The construction of the best linear forecast is described in (35) of Section 5.3. Since the order $\{O_t\}$ has a unique Wold representation $O_t = d + \sum_{n=0}^{\infty} q_n \epsilon_{t-n}^*$, where the full-information shocks $\{\epsilon_t^*\}$ are a white noise sequence. Since $\{O_t\}$ is invertible with respect to $\{\epsilon_t^*\}$, the orthogonal projection of O_{t+1} on $\{O_n, n \le t\}$ is the same as the orthogonal projection of $\{\epsilon_n^*, n \le t\}$, which is $m_{\text{S|NS}} = d + \sum_{n=1}^{\infty} q_n \epsilon_{t-n}^*$. Therefore, the MSFE of the best linear forecast is $\sigma_{\text{S|NS}}^2 = \mathbb{V}\text{ar}(q_0 \epsilon_{t+1}^*)$, and the supplier's average expected cost $\mathcal{C}_{\text{NS}}^{\text{S}}$ can be calculated accordingly based on (11). The MSFE $\sigma_{\text{S|NS}}^2$ can also be derived as a direct corollary of Proposition 6 using Kolmogorov's formula.

EC.1.6. Proof of Theorem 1 Under full information sharing, recall the optimization problem in (25),

$$\Sigma_{\text{FI}}^2(\sigma) := \inf_{\{\tilde\psi_n\} \in \tilde\Psi} \sigma_\epsilon^2 \left(1 + \sum_{k=1}^{\infty} \left(\sum_{n=0}^{k-1} \tilde\psi_n - 1\right)^2\right) \qquad \text{subject to} \qquad \sigma_\epsilon^2 \tilde\psi_0^2 \le \sigma^2. \tag{25}$$

Note that for any feasible policy $\{\tilde\psi_n\} \in \tilde\Psi$ we have

$$\sigma_\epsilon^2 \left(1 + \sum_{k=1}^{\infty} \left(\sum_{n=0}^{k-1} \tilde\psi_n - 1\right)^2\right) \ge \sigma_\epsilon^2 \left(1 + (\tilde\psi_0 - 1)^2\right).$$

Furthermore, it is not hard to check that equality above can only happen if $\{\tilde\psi_n\}$ satisfies $\tilde\psi_1 = 1 - \tilde\psi_0$ and $\tilde\psi_n = 0$ for $n \ge 2$. From this observation, it follows that the optimization in (25) reduces to

$$\Sigma_{\text{FI}}^2(\sigma) := \min_{\tilde\psi_0} \sigma_\epsilon^2 \left(1 + (\tilde\psi_0 - 1)^2\right) \qquad \text{subject to} \qquad \sigma_\epsilon^2 \tilde\psi_0^2 \le \sigma^2.$$

When $\sigma \ge \sigma_\epsilon$ the optimal solution is trivially $\tilde\psi_0 = 1$. On the other hand, when $\sigma < \sigma_\epsilon$, the optimal solution is $\tilde\psi_0 = \sigma/\sigma_\epsilon$. We conclude that the unique optimal solution $\{\tilde\psi_n\} \in \tilde\Psi$ to (25) is given by $\tilde\psi_0 = \min\{1, \sigma/\sigma_\epsilon\}$, $\tilde\psi_1 = 1 - \tilde\psi_0$ and $\tilde\psi_n = 0$ for $n \ge 2$.

EC.1.7. Proof of Theorem 2 Without loss of generality, we assume $\sigma_\epsilon = 1$. Theorem 1 shows that $\Sigma_{\text{FI}}^2(\sigma) = 1 + ((1-\sigma)^+)^2$ for any $\sigma > 0$, where the optimality takes place when $\tilde\psi = \sigma + (1-\sigma)z$ for

$\sigma \in [1/2, 1]$ and $\tilde{\psi} = 1$ for $\sigma > 1$. When $\sigma \geq 1/2$, it is easy to verify that the optimal order obtained is invertible, and therefore $\sigma_{\mathrm{S|FI}} = \sigma_{\mathrm{S|NS}}$. Therefore, we have,

$$\Sigma^2_{\mathrm{NS}}(\sigma) = 1 + ((1-\sigma)^+)^2, \quad \sigma \geq \frac{1}{2}.$$

In the following, we consider $\sigma \in (0, 1/2)$ only. By Lemma 2, we can factorize $\tilde{\psi}(z) = Q(z)B(z)G(z)$, where $Q(z), B(z), G(z)$ are the outer function, Blaschke product, and singular inner function factors defined in (30)–(32), respectively. We prove the theorem in the following two steps.

1. By Lemma 3, any zero(s) of $\tilde{\psi}$ in $\mathbb{D}$ described by the Blaschke product $B(z)$ would lead to a sub-optimal $\tilde{\psi}(z)$. Therefore, we assume $B(z) \equiv 1$, and we will simplify the optimization problem to the consideration only of the outer function $Q(z) = \exp(\Theta(z))$ and the singular inner function $G(z)$. Additionally, due to Proposition 6, we consider a relaxed problem (39), where we show that the optimality of the relaxed problem occurs when the singular measure μ concentrates on $\lambda = \pi$, i.e., $z = -1$. Therefore, the relaxed problem (39) is lower bounded by (40).
2. When $G(z)$ is fixed, the relaxed problem (40), as a function of $\{\theta_n\}$ where $\Theta(z) = \sum_{n=0}^{\infty} \theta_n z^n$ and $Q(z) = \exp(\Theta(z))$, is strictly convex with respect to the parameters $\{\theta_n\}$, and therefore the optimal $Q(z)$ can be obtained from the first order condition.

Step 1. By Lemma 3, we consider only $B(z) \equiv 1$, i.e., there is no zero of $\tilde{\psi}$ in $\mathbb{D}$, and therefore $\tilde{\psi}(z) = Q(z)G(z)$, where Q is the outer function in (32) and G is the singular inner function in (31).

By (23), have

$$\begin{aligned}
\sigma_{\mathrm{I}}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_{\mathrm{I}}(e^{-i\lambda})\psi_{\mathrm{I}}(e^{i\lambda})\mathrm{d}\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{zQ(z)G(z)-1}{1-z}\right)\left(\frac{\bar{z}Q(\bar{z})G(\bar{z})-1}{1-\bar{z}}\right)\mathrm{d}\lambda, \quad \text{where } z = e^{-i\lambda}, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{Q(z)G(z)Q(\bar{z})G(\bar{z}) - zQ(z)G(z) - \bar{z}Q(\bar{z})G(\bar{z}) + 1}{(1-z)(1-\bar{z})}\right)\mathrm{d}\lambda. \qquad \text{(EC.1)}
\end{aligned}$$

In the sequel, we set $z = e^{-i\lambda}$ when we use z in the integrals with respect to λ. Since

$$\int_{-\pi}^{\pi} \frac{zQ(z)G(z)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda = \int_{-\pi}^{\pi} \frac{\bar{z}Q(\bar{z})G(\bar{z})}{(1-\bar{z})(1-z)}\mathrm{d}\lambda$$

and by Lemma 2, we have $|M(z)| = |G(z)| = 1$, a.e. on $\mathbb{T}$. Therefore, we have

$$\sigma_{\mathrm{I}}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{Q(z)\overline{Q(z)} - 2zQ(z)G(z) + 1}{(1-z)(1-\bar{z})}\right)\mathrm{d}\lambda.$$

Note that

$$Q(z)\overline{Q(z)} - 2zQ(z)G(z) + 1 = (zQ(z)-1)(\overline{zQ(z)}-1) + zQ(z) + \overline{zQ(z)} - 2zQ(z)G(z).$$

We can write $\sigma_{\mathrm{I}}^2 = S_1 + S_2$, where

$$S_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(zQ(z)-1)(\overline{zQ(z)}-1)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda, \quad S_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} -\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda.$$

By Lemma 6, we have Q and G are conjugate symmetric, and $Q(1) = G(1) = 1$.

We first evaluate S_2. A standard approach is to translate the integral as a contour integral along the unit circle $\mathbb{T}$, and use Cauchy's residue theorem. However, the residue theorem requires the integrand to be holomorphic on an open set containing the curve $\mathbb{T}$, but the integrand here is not holomorphic at $z=1$. Therefore, we provide an alternative approach. We define $c_g = \tilde{G}(1)$ where

$$\tilde{G}(z) = G(z)\int_{-\pi}^{\pi} \frac{\mathrm{d}}{\mathrm{d}z}\frac{z+e^{i\lambda}}{z-e^{i\lambda}}\mathrm{d}\mu(\lambda).$$

For S_2, we first notice from Lemma 6 that $\bar{z}Q(\bar{z})G(\bar{z}) = \overline{zQ(z)G(z)}$. Therefore

$$\begin{aligned}
S_2 =& \frac{1}{\pi}\int_{-\pi}^{0} -\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda + \frac{1}{\pi}\int_{0}^{\pi} -\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda \\
=& \frac{1}{\pi}\int_{0}^{\pi} -\frac{\overline{zQ(z)(G(z)-1)}}{(1-z)(1-\bar{z})}\mathrm{d}\lambda + \frac{1}{\pi}\int_{0}^{\pi} -\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\mathrm{d}\lambda \\
=& \frac{2}{\pi}\int_{0}^{\pi} -\mathrm{Re}\left(\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\right)\mathrm{d}\lambda \\
=& \lim_{\epsilon\to 0^+}\frac{2}{\pi}\int_{\epsilon}^{\pi} -\mathrm{Re}\left(\frac{zQ(z)(G(z)-1)}{(1-z)(1-\bar{z})}\right)\mathrm{d}\lambda \\
=& \lim_{\epsilon\to 0^+}\frac{1}{\pi}\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right) - \frac{zQ(z)(G(z)-1)}{(1-z)(1-\overline{z})}\mathrm{d}\lambda, \qquad z = e^{-i\lambda} \\
=& \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_{\gamma_\epsilon} -\frac{zQ(z)(G(z)-1)}{(1-z)(1-z^{-1})}\frac{\mathrm{d}z}{-iz} \\
=& -\frac{1}{\pi i}\lim_{\epsilon\to 0^+}\int_{\gamma_\epsilon}\frac{zQ(z)(G(z)-1)}{(1-z)^2}\mathrm{d}z,
\end{aligned}$$

a (clockwise) contour integral over the curve γ_ϵ, where $\gamma_\epsilon = \{z = e^{-i\lambda}, \lambda \in [\epsilon, 2\pi-\epsilon]\}$ is an arc on the unit circle. The integrand is holomorphic on $\mathbb{D}$ but has a pole of order 1 at $z=1$, which is on the unit circle. For some ϵ close to 0, define arc $\gamma_{r,\epsilon} = \{z = re^{-i\lambda}, \lambda \in [\epsilon, 2\pi-\epsilon]\}$ for $0<r<1$ and arc $S_\epsilon = \{z = 1+2\epsilon e^{i\theta},\ \theta \in [\pi/2, 3\pi/2]\}$. For some r close to 1, the two arcs intersect at z_0, $\bar{z}_0$ and form a clockwise closed contour, denoted by $C_{\epsilon,r} = \tilde{\gamma}_{r,\epsilon} \cup \tilde{S}_{r,\epsilon}$, where $\tilde{\gamma}_{r,\epsilon}$ is a major arc of $\gamma_{r,\epsilon}$ connecting z_0 and $\bar{z}_0$, and $\tilde{S}_{r,\epsilon}$ is a minor arc of S_ϵ connecting z_0 and $\bar{z}_0$. Thus we have

$$\begin{aligned}
S_2 =& -\frac{1}{\pi i}\int_{\gamma_\epsilon}\lim_{\epsilon\to 0^+}\frac{zQ(z)(G(z)-1)}{(1-z)^2}\mathrm{d}z \\
=& -\frac{1}{\pi i}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\int_{\tilde{\gamma}_{r,\epsilon}}\frac{zQ(z)(G(z)-1)}{(1-z)^2}\mathrm{d}z \\
=& -\frac{1}{\pi i}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\left(\int_{C_{r,\epsilon}} - \int_{\tilde{S}_{r,\epsilon}}\right)\frac{zQ(z)(G(z)-1)}{(1-z)^2}\mathrm{d}z
\end{aligned}$$

The integrand is holomorphic on $C_{r,\epsilon}$ and its interior. Therefore, by Cauchy's integral theorem, the contour integral over $C_{r,\epsilon}$ is zero, and thus

$$S_2 = \frac{1}{\pi i}\lim_{\epsilon\to 0^+}\lim_{r\to 1^-}\int_{\tilde{S}_{r,\epsilon}}\frac{zQ(z)(G(z)-1)}{(1-z)^2}\mathrm{d}z$$

Let $\theta_1, \theta_2 \in [\pi/2, 3\pi/2]$ satisfy $z_0 = 1+2\epsilon e^{i\theta_1}$ and $\bar{z}_0 = 1+2\epsilon e^{i\theta_2}$, and $\theta_1 \le \theta_2$. By Lemma 8, we have $S_2 = \lim_{\epsilon\to 0^+} c_g(\theta_2-\theta_1)/\pi$. Further, as $\epsilon \to 0^+$, we have $\theta_2 - \theta_1 \to \pi$ and thus $S_2 = c_g$. Note that S_2 depends only on $G(z)$ and S_1 depends only on $Q(z)$. Denote by $S_1 = S_1(Q)$.

By Proposition 6, we have $\sigma^2_{\mathrm{S|NS}} \geq \sigma^2_{\mathrm{S|FI}} = Q(0)^2 G(0)^2$. Note that $G(0) = \exp(-\mu(\mathbb{T}))$ and μ is symmetric. We have

$$c_g = G(1)\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{z+e^{i\lambda}}{z-e^{i\lambda}}\Big|_{z=1}\right)\mathrm{d}\mu(\lambda) = -\int_{-\pi}^{\pi}\frac{2e^{i\lambda}}{(1-e^{i\lambda})^2}\mathrm{d}\mu(\lambda) = \int_{-\pi}^{\pi}\frac{1}{1-\cos\lambda}\mathrm{d}\mu(\lambda) \geq \frac{1}{2}\mu(\mathbb{T}),$$

where equality holds when μ concentrates at $\lambda = -\pi$. Define $\gamma = \mu(\{-1\})$. Therefore, (39) is lower bounded by (40). The optimal γ in (40) is $\gamma^\star = \max(-\log(\sigma/|Q(0)|), 0)$. Therefore, we are minimizing

$$C_Q := S_1(Q) - \frac{1}{2}\min\left(\log\frac{\sigma}{|Q(0)|}, 0\right). \tag{EC.2}$$

Step 2. We further relax the objective (EC.2) to

$$\tilde{C}_Q := S_1(Q) - \frac{1}{2}\log\frac{\sigma}{|Q(0)|}, \text{ where } S_1(Q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_{\mathrm{I}}(z)Q_{\mathrm{I}}(\bar{z})\mathrm{d}\lambda,\ Q_{\mathrm{I}}(z) = \frac{zQ(z)-1}{1-z}. \tag{EC.3}$$

As $\min(a,b) \leq a$ for any $a, b \in \mathbb{R}$, we have $C_Q \geq \tilde{C}_Q$. We now consider the objective (EC.3) as a function of $\{\theta_1, \theta_2, \dots\}$. We first show that the first order condition uniquely corresponds to $Q(z) = (1+z)/2$. Then we show that $\tilde{C}_Q$ is a (strictly) convex function of $\{\theta_1, \theta_2, \dots\}$.

Since $Q(1) = 1$, we have $\theta_0 = -\sum_{j=1}^{\infty}\theta_j$. For any $z = e^{-i\lambda} \in \mathbb{T}$ and $j \geq 1$,

$$\frac{\partial}{\partial\theta_j}Q(z) = \frac{\partial}{\partial\theta_j}\exp\left(\sum_{j=0}^{\infty}\theta_j z^j\right) = (z^j - 1)Q(z), \quad \frac{\partial}{\partial\theta_j}\log|Q(0)| = \frac{\partial}{\partial\theta_j}\sum_{j=1}^{\infty} -\theta_j = -1.$$

Therefore, for any $j \geq 1$,

$$\frac{\partial}{\partial\theta_j}\tilde{C}_Q = -\frac{1}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} Q_{\mathrm{I}}(z)\frac{\overline{z(z^j-1)Q(z)}}{1-\bar{z}}\mathrm{d}\lambda.$$

In addition, we have

$$\frac{\partial}{\partial\theta_1}S_1(Q) = -\frac{1}{\pi}\int_{-\pi}^{\pi} Q_{\mathrm{I}}(z)\overline{zQ(z)}\mathrm{d}\lambda, \quad \frac{\partial}{\partial\theta_j}S_1(Q) = \frac{\partial}{\partial\theta_{j-1}}S_1(Q) - \frac{1}{\pi}\int_{-\pi}^{\pi} Q_{\mathrm{I}}(z)\overline{z^j Q(z)}\mathrm{d}\lambda.$$

The first order condition for $\tilde{C}_Q$ corresponds to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} Q_{\mathrm{I}}(e^{-i\lambda})Q(e^{i\lambda})e^{i\lambda j}\mathrm{d}\lambda = \begin{cases} -\frac{1}{2}, & j = 1 \\ 0, & j \geq 2 \end{cases}. \tag{EC.4}$$

Since $Q(1) = 1$, we have $Q_0 = 1 - \sum_{k=1}^{\infty} Q_k$, therefore for $z = e^{-i\lambda} \in \mathbb{T}$,

$$\begin{aligned} Q_{\mathrm{I}}(z) &= \frac{-1 + zQ(z)}{1-z} = \frac{-1 + zQ_0 + \sum_{k=1}^{\infty} z^{k+1}Q_k}{1-z} = \frac{-1 + z + \sum_{k=1}^{\infty} Q_k(z^{k+1} - z)}{1-z} \\ &= -1 - \sum_{k=1}^{\infty} Q_k \sum_{j=1}^{k} z^j = -1 - \sum_{j=1}^{\infty} z^j \sum_{k=j}^{\infty} Q_k. \end{aligned}$$

Define $R_j = -\sum_{k=j}^{\infty} Q_k$, $j \geq 0$. We have $Q_{\mathrm{I}}(z) = \sum_{\ell=0}^{\infty} z^{\ell} R_{\ell}$, and $Q(\bar{z})\bar{z}^j = \sum_{k=0}^{\infty} Q_k \bar{z}^{k+j}$. Since $z\bar{z} = 1$ for $z \in \mathbb{T}$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} z^k \bar{z}^{\ell} \mathrm{d}\lambda = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{i(k-\ell)\lambda} \mathrm{d}\lambda = \begin{cases} 2, & k = \ell \\ 0, & k \neq \ell \end{cases}.$$

Therefore by (EC.4),

$$\sum_{k=0}^{\infty} R_{k+j} Q_k = \begin{cases} -\frac{1}{4}, & j = 1 \\ 0, & j \geq 2 \end{cases}.$$

Since $R_{k+j+1} - R_{k+j} = Q_{k+j}$, we have

$$\sum_{k=0}^{\infty} Q_{k+j} Q_k = \sum_{k=0}^{\infty} (R_{k+j+1} - R_{k+j}) Q_k = \sum_{k=0}^{\infty} R_{k+j+1} Q_k - \sum_{k=0}^{\infty} R_{k+j} Q_k = \begin{cases} \frac{1}{4}, & j = 1 \\ 0, & j \geq 2 \end{cases}.$$

Furthermore, by $Q(1) = 1$ we have $\sum_{k=0}^{\infty} Q_k = 1$ and combined with the above equation, we have

$$\sum_{k=0}^{\infty} Q_k^2 = 1 - 2\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} Q_{k+j} Q_k = \frac{1}{2}.$$

Therefore, $|Q(e^{i\lambda})|^2 = Q(e^{i\lambda})Q(e^{-i\lambda}) = \frac{2+e^{i\lambda}+e^{-i\lambda}}{4}$. According to (29), define

$$\mathcal{Q}_Q(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z} \log|Q(e^{i\lambda})| \mathrm{d}\lambda\right) = \frac{1+z}{2}.$$

Since the quantity $|Q(e^{i\lambda})|^2$ matches that for the z-transform $\frac{1+z}{2}$, we can easily see that $\mathcal{Q}_Q(z) = c(1+z)/2$ for some $c \in \mathbb{T}$. Since Q is outer, by Proposition 4, we have $Q(z) = c'(1+z)/2$ for some $c' \in \mathbb{T}$. Since $Q(1) = 1$, we have $Q_0 = Q_1 = \frac{1}{2}$.

Lastly by Lemma 5 we have the strict convexity of $\tilde{C}_Q$ in (EC.3) with respect to $\{\theta_1, \theta_2, \dots\}$. Therefore we have

$$\tilde{C}_Q = \frac{1}{2}\log|Q_0| + S_1(Q) - \frac{1}{2}\log\sigma \geq \frac{5 - 2\log 2\sigma}{4},$$

where the inequality in (40) becomes an equality when $\sigma \in (0, \frac{1}{2}]$ and

$$Q^{\star}(z) = \frac{1+z}{2}, \quad G^{\star}(z) = \exp\left(\gamma^{\star}\frac{z-1}{z+1}\right), \text{ and } \gamma^{\star} = -\log(2\sigma).$$

EC.1.8. Proof of Proposition 3: ϵ-Optimal Solutions It is easy to see that $\tilde{\psi}_{\delta}(z)$ is holomorphic and nonzero in the disk $\mathbb{D}_{1+\delta} = \{z : |z| < 1 + \delta\}$ with radius $(1+\delta)$. By Proposition 6,

$$\sigma_{\mathrm{S}}^2(\tilde{\psi}_{\delta}) = \frac{\sigma_{\epsilon}^2}{4}\exp\left(-\frac{2\gamma_{\sigma}}{1+\delta}\right).$$

Therefore as $\delta \to 0$, we have $\sigma_{\mathrm{S}}^2(\tilde{\psi}_{\delta}) \to \frac{1}{4}\exp(-2\gamma_{\sigma})\sigma_{\epsilon}^2 = \sigma^2$.

On the other hand, the z-transform corresponding to the initialized inventory process is

$$\psi_{\mathrm{I},\delta}(z) = \frac{\frac{z(1+z)}{2}\exp\left(\gamma_{\sigma}\frac{z-1}{1+\delta+z}\right) - 1}{1-z}.$$

For any z and $\delta > 0$, we have $\mathrm{Re}(\gamma_\sigma \frac{z-1}{1+\delta+z}) = \gamma_\sigma \frac{|z|^2-1+\delta \mathrm{Re}(z-1)}{|z+1+\delta|^2} < 0$, and thus $|\exp(\gamma_\sigma \frac{z-1}{1+\delta+z})| \le 1$. Let $z_1 = \gamma_\sigma \frac{z-1}{1+\delta+z}$ and $z_2 = \gamma_\sigma \frac{z-1}{1+z}$. By the complex Rolle's theorem (Evard and Jafari 1992),

$$\frac{|\exp(z_1) - \exp(z_2)|}{|z_1 - z_2|} \le 2 \max_{\theta \in [0,1]} |\exp(z')|, \quad z' = \theta z_1 + (1-\theta) z_2. \tag{EC.5}$$

Then for any $\theta \in (0,1)$, $|\exp(z')| = |\exp(\theta z_1 + (1-\theta)z_2)| = |\exp(z_1)|^\theta |\exp(z_2)|^{1-\theta} \le 1$, and therefore we have $|\exp(z_1) - \exp(z_2)| \le 2|z_1 - z_2|$. Thus for any $z \in \mathbb{T} \backslash \{-1\}$,

$$\left|\exp\left(\gamma_\sigma \frac{z-1}{1+\delta+z}\right) - \exp\left(\gamma_\sigma \frac{z-1}{1+z}\right)\right| \le 2\gamma_\sigma \left|\frac{z-1}{1+\delta+z} - \frac{z-1}{1+z}\right| \le 2\gamma_\sigma \left|\delta \frac{z-1}{(1+z)(1+\delta+z)}\right|,$$

where the first inequality comes from the mean-value theorem in complex variables. We have

$$|\sigma_{\mathrm{I},\delta} - \sigma_{\mathrm{I},0}|^2 \le \frac{\sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} |\psi_{\mathrm{I},\delta}(z) - \psi_{\mathrm{I},0}(z)|^2 \,\mathrm{d}\lambda \le \frac{\sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{8\pi} \int_{-\pi}^{\pi} \left|\frac{z^2}{(1+\delta+z)^2}\right| \mathrm{d}\lambda.$$

Since $z = e^{-i\lambda}$ is on the unit circle, and therefore $\bar{z} = 1/z$. Therefore

$$|\sigma_{\mathrm{I},\delta} - \sigma_{\mathrm{I},0}|^2 \le \frac{\sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{8\pi} \int_{\mathbb{T}} \frac{1}{(1+\delta+z)(1+\delta+\frac{1}{z})} \frac{\mathrm{d}z}{-iz} = \frac{\sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{4\pi i (1+\delta)} \int_{\mathbb{T}} \frac{-1}{(1+\delta+z)(\frac{1}{1+\delta}+z)} \mathrm{d}z.$$

The integrand is holomorphic on $\mathbb{D} \backslash \{-\frac{1}{1+\delta}\}$, and $-\frac{1}{1+\delta}$ is a simple pole. Therefore,

$$\frac{\sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{4\pi i (1+\delta)} \int_{\mathbb{T}} \frac{-1}{(1+\delta+z)(\frac{1}{1+\delta}+z)} \mathrm{d}z = \frac{2\pi i \sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{4\pi i (1+\delta)} \frac{1}{1+\delta-\frac{1}{1+\delta}} = \frac{2\pi i \sigma_\epsilon^2 \gamma_\sigma^2 \delta^2}{4\pi i (1+\delta)} \frac{(1+\delta)}{\delta(2+\delta)} = \frac{\sigma_\epsilon^2 \gamma_\sigma^2 \delta}{2(2+\delta)}.$$

It is easy to see that as $\delta \to 0$ where $\delta > 0$, we have $\sigma_{\mathrm{I},\delta}^2 \to \sigma_{\mathrm{I},0}^2$.

EC.1.9. Proof of Proposition 4: Outer functions The equivalence of (a) and (b) can be found in Nikolski (2019, Theorem 2.6.7). We only show (a)⇔(c) here. By Definition 2, an order process $\{O_t\}$ is invertible with respect to $\{\epsilon_t\}$ if and only if $\overline{sp}\{1, O_t, O_{t-1}, \dots\} = \overline{sp}\{1, \epsilon_t, \epsilon_{t-1}, \dots\}$, which is equivalent to $\overline{sp}\{O_t, O_{t-1}, \dots\} = \overline{sp}\{\epsilon_t, \epsilon_{t-1}, \dots\}$. We define a correspondence by $z^n \leftrightarrow \epsilon_{t-n}$ for all $n \ge 0$. Extending the correspondence by linearity, we conclude that $H^2 = \overline{sp}\{z^n, n \ge 0\}$ is isomorphic to $\mathcal{M}_\epsilon(t) = \overline{sp}\{\epsilon_{t-n}, n \ge 0\}$. Since $O_{t-k} - d = \sum_{n=0}^{\infty} \tilde{\psi}_n \epsilon_{t-n-k}$ corresponds to $z^k \tilde{\psi}(z)$, we have $\overline{sp}\{O_t - d, O_{t-1} - d, \dots\}$ is isomorphic to the function space $\overline{sp}\{z^k \tilde{\psi}, k \ge 0\}$.

EC.1.10. Proof of Proposition 5: Inventory Variance The retailer's inventory for $t > 0$ is

$$I_t = I_0 + \sum_{k=0}^{t-1} (O_k - D_{k+1}) = I_0 - (D_t - d) + (D_0 - d) + \sum_{k=0}^{t-1} (O_k - D_k).$$

Note that $O_k = d + \sum_{n=0}^{\infty} \tilde{\psi}_n \epsilon_{k-n}$, and $D_k = d + \epsilon_k$. Therefore

$$I_t = I_0 - (D_t - d) + (D_0 - d) + \sum_{k=0}^{t-1} \sum_{n=0}^{\infty} (\tilde{\psi}_n - 1_{\{n=0\}}) \epsilon_{k-n}.$$

Define $g_0 = -1$ and $g_k = -1 + \sum_{j=0}^{k-1} \tilde{\psi}_j$ for $k > 0$. In the following, we use the convention that when the upper limit of a summation is less than the lower limit, then the summation is zero. Thus we write

$g_k = -1 + \sum_{j=0}^{k-1} \tilde{\psi}_j$ for any $k \geq 0$. We can rewrite (let $\ell = k - n$)

$$\begin{aligned} I_t =& I_0 - (D_t - d) + (D_0 - d) + \sum_{k=0}^{t-1} \sum_{\ell=-\infty}^{k} (\tilde{\psi}_{k-\ell} - 1_{\{\ell=k\}})\epsilon_\ell \\ =& I_0 - (D_t - d) + (D_0 - d) + \sum_{\ell=-\infty}^{t-1} \sum_{k=\max(0,\ell)}^{t-1} (\tilde{\psi}_{k-\ell} - 1_{\{\ell=k\}})\epsilon_\ell \\ =& I_0 - (D_t - d) + (D_0 - d) + \sum_{\ell=-\infty}^{-1} (g_{t-\ell} - g_{-\ell})\epsilon_\ell + \sum_{\ell=0}^{t-1} g_{t-\ell}\epsilon_\ell. \end{aligned}$$

We set $\tilde{I}_0 = I_0 + D_0 - d - \sum_{k=1}^{\infty} g_k \epsilon_{-k}$. The retailer's initialized inventory process $\{I_t - \tilde{I}_0\}$ can be written as (write $k = t - \ell$)

$$I_t - \tilde{I}_0 = -(D_t - d) + \sum_{k=1}^{\infty} g_k \epsilon_{t-k} = \sum_{k=0}^{\infty} g_k \epsilon_{t-k}.$$

Therefore, $\{I_t - \tilde{I}_0\}$ has an MA(∞) representation with square-summable coefficients by Assumption 1, and thus $\{I_t - \tilde{I}_0\}$ is weakly stationary. For all z in the closed unit disk except for $z = 1$,

$$\psi_{\mathrm{I}}(z) = \sum_{k=0}^{\infty} g_k z^k = -\frac{1}{1-z} + \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \tilde{\psi}_j z^k = \frac{z}{1-z} \sum_{j=0}^{\infty} z^j \tilde{\psi}_j - \frac{1}{1-z} \quad .$$

Therefore, (36) follows. In addition, (37) follows from Parseval's identity.

EC.1.11. Proof of Lemma 3

Recall from (30) that

$$B(z) = z^{m_0} \prod_k \frac{z - z_k}{1 - \bar{z}_k z},$$

Since $B(\bar{z}) = \overline{B(z)}$, it is easy to see that for any $z \in \mathbb{T}$, we have $|z|^2 = z\bar{z} = 1$ and

$$B(z)B(\bar{z}) = B(z)\overline{B(z)} = (z\bar{z})^{m_0} \prod_k \frac{z - z_k}{1 - \bar{z}_k z} \frac{\bar{z} - \bar{z}_k}{1 - z_k \bar{z}} = |z|^{2m_0} \prod_k \frac{|z|^2 + |z_k|^2 - z\bar{z}_k - z_k\bar{z}}{1 + |z_k|^2|z|^2 - z\bar{z}_k - z_k\bar{z}} = 1.$$

Therefore $\sigma^2_{\mathrm{S|NS}}$ in (18) remains unchanged. Regarding σ^2_{I}, we have

$$\sigma^2_{\mathrm{I}}(\tilde{\psi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Q(z)B(z)G(z)Q(\bar{z})B(\bar{z})G(\bar{z}) - 2zQ(z)B(z)G(z) + 1}{(1-z)(1-\bar{z})} \mathrm{d}\lambda, \quad z = e^{-i\lambda}.$$

We use QG to denote the function $Q(z)G(z)$. Therefore

$$\begin{aligned} \sigma^2_{\mathrm{I}}(QG) - \sigma^2_{\mathrm{I}}(\tilde{\psi}) =& \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2zQ(z)G(z)\,(B(z)-1)}{(1-z)(1-\bar{z})} \mathrm{d}\lambda, \qquad z = e^{-i\lambda} \\ =& \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2zQ(z)G(z)\,(B(z)-1)}{(1-z)^2} \mathrm{d}z. \end{aligned}$$

The second equation above uses $\mathrm{d}\lambda = \frac{\mathrm{d}z}{-iz}$ and $\bar{z} = 1/z$. Using the same procedure in step 1 of the proof of Theorem 2 for $f(z) = \frac{2zQ(z)G(z)(B(z)-1)}{(1-z)^2}$, we have that

$$\sigma_{\mathrm{I}}^2(QG) - \sigma_{\mathrm{I}}^2(\tilde{\psi}) = -\lim_{z\to 1} zQ(z)G(z)\frac{B(z)-1}{z-1}.$$

We have

$$\lim_{z\to 1}\frac{B(z)-1}{z-1} = B(z)\left(m_0 z^{-1} + \sum_k \frac{1-z_k\bar{z}_k}{(z-z_k)(1-\bar{z}_k z)}\right)\Bigg|_{z=1} = m_0 + \sum_k \frac{1-|z_k|^2}{|1-z_k|^2} \geq 0.$$

Since $|z_k| < 1$, the inequality above is strict unless m_0 and $\{z_k\}$ is an empty set, i.e., $B(z) \equiv 1$.

EC.1.12. Proof of Lemma 5 For any $\ell, j \geq 1$, we first calculate the second partial derivatives of the objective $\tilde{C}_Q(\theta_1, \theta_2, \cdots) := \tilde{C}_Q(\theta)$,

$$H_{\ell,j} = \frac{\partial}{\partial\theta_\ell}\frac{\partial}{\partial\theta_j}C_Q(\theta) \ .$$

We obtain

$$\begin{aligned}
\frac{\partial}{\partial\theta_\ell}\frac{\partial}{\partial\theta_j}S_1(Q) &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta_\ell}\left[Q_{\mathrm{I}}(e^{i\lambda})Q(e^{-i\lambda})\sum_{k=1}^{j}e^{-i\lambda k}\right]\mathrm{d}\lambda \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\frac{\partial}{\partial\theta_\ell}\left[Q_{\mathrm{I}}(e^{i\lambda})Q(e^{-i\lambda})\right]\mathrm{d}\lambda \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\left[Q_{\mathrm{I}}(e^{i\lambda})\frac{\partial}{\partial\theta_\ell}Q(e^{-i\lambda}) + Q(e^{-i\lambda})\frac{\partial}{\partial\theta_\ell}Q_{\mathrm{I}}(e^{i\lambda})\right]\mathrm{d}\lambda \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\left[Q_{\mathrm{I}}(e^{i\lambda})\frac{1}{2}(e^{-i\lambda\ell}-1)Q(e^{-i\lambda}) + Q(e^{-i\lambda})\frac{1}{2}e^{i\lambda}Q(e^{i\lambda})\frac{e^{i\lambda\ell}-1}{1-e^{i\lambda}}\right]\mathrm{d}\lambda \\
&= -\frac{1}{2\pi}\frac{1}{2}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\left[\frac{e^{i\lambda}Q(e^{i\lambda})-1}{1-e^{i\lambda}}(e^{-i\lambda\ell}-1)Q(e^{-i\lambda}) + Q(e^{-i\lambda})e^{i\lambda}Q(e^{i\lambda})\frac{e^{i\lambda\ell}-1}{1-e^{i\lambda}}\right]\mathrm{d}\lambda \ .
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
H_{\ell,j} &= \frac{1}{2}\log|Q_0| + \frac{(-1)}{2\pi}\frac{1}{2}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\left[\frac{e^{-i\lambda\ell}-1}{1-e^{i\lambda}}\left(|Q(e^{i\lambda})|^2e^{i\lambda} - Q(e^{-i\lambda})\right) + \frac{e^{i\lambda\ell}-1}{1-e^{i\lambda}}e^{i\lambda}\left|Q(e^{i\lambda})\right|^2\right]\mathrm{d}\lambda \\
&= \frac{1}{2}\log|Q_0| + \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{j}e^{-i\lambda k}\left[\frac{e^{i\lambda}}{1-e^{i\lambda}}(\cos(\ell\lambda)-1)\left|Q(e^{i\lambda})\right|^2 - \frac{1}{2}\frac{e^{-i\lambda\ell}-1}{1-e^{i\lambda}}Q(e^{-i\lambda})\right]\mathrm{d}\lambda
\end{aligned} \tag{EC.6}$$

To show that $C_Q(\theta)$ is convex, it suffices to show that for any $a = (a_1, a_2, \cdots)^\top \in \ell^2$, we have

$$a^\top Ha := \sum_{\ell\geq 1}\sum_{j\geq 1}a_\ell a_j H_{\ell,j} > 0.$$

Let us consider some expressions for $a^\top Ha$. We have

$$\begin{aligned}
a^\top Ha =& \frac{(\sum_{j=1}^\infty a_j)^2}{(\sum_{j=1}^\infty \theta_j)^2} + \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=1}^\infty\sum_{j=1}^\infty a_\ell a_j \sum_{k=1}^{j} e^{-i\lambda k}(\cos(\ell\lambda)-1)\frac{e^{i\lambda}}{1-e^{i\lambda}}|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=1}^\infty\sum_{j=1}^\infty a_\ell a_j \sum_{k=1}^{j} e^{-i\lambda k}(e^{-i\lambda\ell}-1)\frac{1}{2}\frac{1}{1-e^{i\lambda}}Q(e^{-i\lambda})\mathrm{d}\lambda \\
=& \frac{(\sum_{j=1}^\infty a_j)^2}{(\sum_{j=1}^\infty \theta_j)^2} + \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=1}^\infty\sum_{j=1}^\infty a_\ell a_j \frac{1-e^{-i\lambda j}}{1-e^{-i\lambda}}(\cos(\ell\lambda)-1)\frac{1}{1-e^{i\lambda}}|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=1}^\infty\sum_{j=1}^\infty a_\ell a_j\, e^{-i\lambda}\frac{1-e^{-i\lambda j}}{1-e^{-i\lambda}}(e^{-i\lambda\ell}-1)\frac{1}{2}\frac{1}{1-e^{i\lambda}}Q(e^{-i\lambda})\mathrm{d}\lambda \\
=& \frac{(\sum_{j=1}^\infty a_j)^2}{(\sum_{j=1}^\infty \theta_j)^2} + \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\sum_{\ell=1}^\infty a_\ell\left(\frac{1}{2}e^{i\ell\lambda}+\frac{1}{2}e^{-i\ell\lambda}-1\right)\frac{1}{|1-e^{-i\lambda}|^2}|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\sum_{\ell=1}^\infty a_\ell(e^{-i\lambda\ell}-1)\, e^{-i\lambda}\frac{1}{2}\frac{1}{|1-e^{i\lambda}|^2}Q(e^{-i\lambda})\mathrm{d}\lambda \\
=& \frac{(\sum_{j=1}^\infty a_j)^2}{(\sum_{j=1}^\infty \theta_j)^2} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left|\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right|^2\frac{1}{|1-e^{i\lambda}|^2}\,|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right)^2\frac{1}{|1-e^{i\lambda}|^2}\,|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{(-1)}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right)^2\frac{1}{2}\,e^{-i\lambda}\,\frac{1}{|1-e^{i\lambda}|^2}Q(e^{-i\lambda})\mathrm{d}\lambda \qquad \text{(EC.7)}\\
=& \frac{(\sum_{j=1}^\infty a_j)^2}{(\sum_{j=1}^\infty \theta_j)^2} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left|\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right|^2\frac{1}{|1-e^{i\lambda}|^2}\,|Q(e^{i\lambda})|^2 \mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right)^2\frac{1}{|1-e^{i\lambda}|^2}\,Q(e^{-i\lambda})[Q(e^{i\lambda})-e^{-i\lambda}]\mathrm{d}\lambda \qquad \text{(EC.8)}
\end{aligned}$$

We note that

$$\frac{1-e^{-i\lambda}}{1-e^{i\lambda}} = \frac{e^{-i\lambda/2}2\,i\sin(\lambda/2)}{e^{i\lambda/2}2\,i\sin(-\lambda/2)} = -e^{-i\lambda}\ . \qquad \text{(EC.9)}$$

To establish our claim of positive definiteness, we note that the first term on the righthand side of Equation (EC.7) is non-negative, the second term is positive, and the third term is real with absolute value less than the second term. It therefore suffices to show that the fourth term on the righthand side of Equation (EC.7) is zero, which we now prove. Using Equation (EC.9), we have

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left(\sum_{j=1}^\infty a_j(1-e^{-i\lambda j})\right)^2 e^{-i\lambda}\,\frac{1}{|1-e^{i\lambda}|^2}Q(e^{-i\lambda})\mathrm{d}\lambda \\
=& \sum_{j_1=1}^\infty\sum_{j_2=1}^\infty a_{j_1}a_{j_2}\sum_{k=0}^\infty \beta_k\int_{-\pi}^{\pi}\frac{(1-e^{-i\lambda j_1})(1-e^{-i\lambda j_2})}{(1-e^{-i\lambda})(1-e^{i\lambda})}e^{-i\lambda}(e^{-i\lambda k}-1)\mathrm{d}\lambda
\end{aligned}$$

$$= \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1}a_{j_2}\sum_{k=0}^{\infty}\beta_k \int_{-\pi}^{\pi}\sum_{L=0}^{j_1-1} e^{-i\lambda L}\frac{(1-e^{-i\lambda j_2})(1-e^{-i\lambda})}{(1-e^{-i\lambda})(1-e^{i\lambda})}e^{-i\lambda}(e^{-i\lambda k}-1)\mathrm{d}\lambda$$

$$= -\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1}a_{j_2}\sum_{k=0}^{\infty}\beta_k \int_{-\pi}^{\pi}\sum_{L=0}^{j_1-1} e^{-i\lambda L}\sum_{M=0}^{j_2-1} e^{-i\lambda M}e^{-i\lambda}e^{-i\lambda}(e^{-i\lambda k}-1)\mathrm{d}\lambda$$

$$= -\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} a_{j_1}a_{j_2}\sum_{k=0}^{\infty}\beta_k \sum_{L=0}^{j_1-1}\sum_{M=0}^{j_2-1}\int_{-\pi}^{\pi} e^{-i\lambda(L+M+2)}(e^{-i\lambda k}-1)\mathrm{d}\lambda = 0.$$

EC.1.13. Proof of Proposition 6: Value of Information Sharing According to (32) and Kolmogorov's formula,

$$\sigma^2_{\mathrm{S|NS}} = \sigma^2_\epsilon \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\tilde\psi(e^{-i\lambda})\tilde\psi(e^{i\lambda})\right)\mathrm{d}\lambda\right) = \sigma^2_\epsilon\left(\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\tilde\psi(e^{-i\lambda})|\mathrm{d}\lambda\right)\right)^2 = \sigma^2_\epsilon Q(0)^2.$$

Equation (42) follows directly from Proposition 1 and the definition of the Blaschke products and singular inner functions in (30) and (31). Furthermore, by Lemma 2, $Q(0) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\tilde\psi(e^{-i\lambda})|\mathrm{d}\lambda\right) \neq 0$.

EC.1.14. Proof of the additional technical lemmas in the appendix

Proof of Lemma 6 We first prove the conjugate symmetry of $Q(z)$. Recall from Lemma 2, and note that we have

$$Q(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\log|\tilde\psi(e^{i\lambda})|\mathrm{d}\lambda\right) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\lambda'}+z}{e^{-i\lambda'}-z}\log|\tilde\psi(e^{-i\lambda'})|\mathrm{d}\lambda'\right).$$

where we define $\lambda' = -\lambda$. Since $\tilde\psi$ is conjugate-symmetric, we have $|\tilde\psi(e^{i\lambda'})| = |\tilde\psi(e^{-i\lambda'})|$ and

$$Q(\bar z) = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-i\lambda'}+\bar z}{e^{-i\lambda'}-\bar z}\log|\tilde\psi(e^{i\lambda'})|\mathrm{d}\lambda'\right) = \exp\left(\overline{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda'}+z}{e^{i\lambda'}-z}\log|\tilde\psi(e^{i\lambda'})|\mathrm{d}\lambda'}\right).$$

Since $\exp(\bar z) = \overline{\exp(z)}$, we have $Q(z) = \overline{Q(\bar z)}$.

Next we prove the conjugate symmetry of $B(z)$. Since $B(z)$ is the Blaschke product formed from the zero set $\{z_k\}$ of $\tilde\psi$, and $\tilde\psi(z)$ is conjugate-symmetric, we have that $\{\bar z_k\}$ is also the zero set of $\tilde\psi(z) = \overline{\tilde\psi(\bar z)}$. Using $\{\bar z_k\}$ as the zero set, by the definition of $B(z)$ in (30),

$$B(z) = z^{m_0}\prod_k \frac{z-\bar z_k}{1-z_k z},$$

Therefore, $B(z) = \overline{B(\bar z)}$ follows from replacing z in the above equation by $\bar z$, and we arrive at the conjugate symmetry of $B(z)$. Since $\tilde\psi(z) = Q(z)B(z)G(z)$, the conjugate symmetry of $G(z)$ follows from that of $\tilde\psi(z)$, $Q(z)$, and $B(z)$ proved above.

LEMMA 8. *Define $C_r(z_0,\theta_1,\theta_2) = \{z_0 + re^{-i\lambda} : \lambda\in[\theta_1,\theta_2]\}$ where $0\le\theta_2-\theta_1\le 2\pi$. Let $f(z)$ be a function such that for any $\epsilon>0$, there exists some $\delta>0$ such that for all $0<r<\delta$, $|(z-z_0)f(z)-c|<\epsilon$*

for all $z \in C_r(z_0,\theta_1,\theta_2)$. *Then*

$$\lim_{r\to 0^+}\int_{C_r(z_0,\theta_1,\theta_2)} f(z)\mathrm{d}z = -i(\theta_2-\theta_1)c.$$

Proof of Lemma 8 Denote by $h(z)=(z-z_0)f(z)-c$. Since $z=z_0+re^{-i\lambda}$, we have $\mathrm{d}z=-(z-z_0)i\mathrm{d}\lambda$, and

$$\int_{C_r(z_0,\theta_1,\theta_2)} f(z)\mathrm{d}z = \int_{C_r(z_0,\theta_1,\theta_2)} \frac{h(z)+c}{z-z_0}\mathrm{d}z = -i\int_{\theta_1}^{\theta_2}(c+h(z_0+re^{i\lambda}))\mathrm{d}\lambda.$$

Hence,

$$\left|\lim_{r\to 0^+}\int_{C_r(z_0,\theta_1,\theta_2)} f(z)\mathrm{d}z + i(\theta_2-\theta_1)c\right| \le \int_{\theta_1}^{\theta_2}\left|h(z_0+re^{i\lambda})\right|\mathrm{d}\lambda < \epsilon(\theta_2-\theta_1).$$

EC.2. Some Popular Inventory Policies and their MA(∞) representation In this appendix we describe a number of inventory replenishment polices that have been proposed in the literature to study the value of information sharing and provide their MA(∞) representation. We also use this discussion to point out to some classes of inventory policies that do not admit such MA(∞) representation.

a) SIMPLE MOVING AVERAGE: (MA) Suppose the retailer's inventory strategy belongs to the family of simple moving averages:

$$O_t^{\text{MA}} = \frac{1}{N}\sum_{n=0}^{N-1} D_{t-n},$$

for some fixed integer $N\ge 1$. The use of a simple moving average ordering policy in the context of a two-tier supply chain system was studied by Balakrishnan et al. (2004) for the case in which market demand is IID, as we also do in this paper. In this case, it follows that

$$O_t^{\text{MA}} = d + \frac{1}{N}\sum_{n=0}^{N-1}\epsilon_{t-n}.$$

b) EXPONENTIAL SMOOTHING: (ES) This is another order smoothing policy proposed by Balakrishnan et al. (2004). In this case, the retailer's replenishment policy is

$$O_t^{\text{ES}} = (1-\rho)\sum_{n=0}^{\infty}\rho^n D_{t-n},$$

for some fixed $\rho\in[0,1)$. For the case of IID demand, O_t^{ES} admits the MA(∞) representation

$$O_t^{\text{ES}} = d + (1-\rho)\sum_{n=0}^{\infty}\rho^n\,\epsilon_{t-n}.$$

c) BASE-STOCK POLICIES: There is an extensive literature in supply chain that consider restricting the firms' inventory policies to base-stock strategies. For the case of an IID demand, as we consider in this paper, the retailer can trivially implement a static base-stock policy with base-stock level S by setting the initially inventory $I_0=S$ and then ordering $O_t=D_t$ for all $t\ge 1$. More generally, state-dependent base-stock policies have been consistently used in the supply chain inventory management literature (e.g., Aviv

(2001), Gaur et al. (2005), Giloni et al. (2014)). The time-dependent base-stock level S_t is determined as the sum of the best mean demand forecast plus a fixed safety-stock that is proportional to the MSFE. Lemma 1 establishes that for a demand process with an MA(∞) representation with respect to a sequence of Gaussian shocks, the best mean demand forecast coincides with the best linear forecast. From this it follows that the state-dependent base-stock policies considered in the said papers admits an MA(∞) representation.

d) STATE-DEPENDENT ORDER-UP POLICIES: In one of the earliest papers investigating the value of information sharing in supply chains, Lee et al. (2000), the authors considers the case in which the retailer's demand is AR(1), that is, $D_t = d + \rho\, D_{t-1} + \epsilon_t$, where the autoregressive coefficient satisfies $|\rho| < 1$. The authors focused on the class of state-dependent order-up to inventory policies to model the retailer's inventory replenishment policy, which takes the form (see equation (3.5) in the paper)

$$O_t = D_t + \frac{\rho\,(1-\rho^{l+1})}{1-\rho}\,(D_t - D_{t-1}),$$

where l is the supplier's replenishment leadtime. Using the fact that D_t admits the MA(∞) representation

$$D_t = \frac{d}{1-\rho} + \sum_{k=0}^{\infty} \rho^k\, \epsilon_{t-k}$$

it follows that O_t also admits an MA(∞) representation.

e) CL: Chen and Lee (2009) studied a two-tier supply chain system similar to the one we consider and proposed an order smoothing policy that minimizes the cumulative supply chain (retailer and supplier) costs. On the one hand, the mathematical framework that they consider allows for more general demand models and replenishment lead-times. On the other hand, their analysis does not explicitly address the issue of invertibility or lack thereof.

In the context of our model, the class of demand processes considered by Chen and Lee (2009) includes general MA(∞) models of the form:

$$D_t = d + \sum_{n=0}^{\infty} \psi_n\, \epsilon_{t-n}.$$

In this case, the CL ordering policy admits the MA(∞) representation:

$$O_t^{\rm CL} = d + \sum_{k=0}^{\infty} \tilde{\psi}_k^{\rm CL}\, \epsilon_{t-k}, \text{ with } \tilde{\psi}_0^{\rm CL} := (1-\gamma)(\psi_0+\psi_1),\ \tilde{\psi}_1^{\rm CL} := \psi_2 + \gamma(\psi_0+\psi_1),\ \tilde{\psi}_n^{\rm CL} := \psi_{n+1}\ \forall n \geq 2,$$

where the parameter $\gamma \in [0,1]$ is a function of the relative cost structures of the supplier and retailer as well as a relative measure of uncertainty faced by the supplier (see Chen and Lee (2009) for details).

f) GKH: In a paper by Graves, Kletter, and Hetzel (1998), a production smoothing policy is proposed with the objective of minimizing the variance of the order process subject to a constraint that upper bounds the variance of the inventory process. The demand model considered by Graves et al. (1998) is more general than ours but a special case that coincides with ours is studied in detail in Section 2 of that paper. In this case, the demand admits the MA(∞) representation

$$D_t = d + \sum_{n=0}^{\infty} \psi_n\, \epsilon_{t-n},$$

for some sequence $\{\psi_n\}_{n\geq 0}$. Note that the IID case that we consider in the paper is a special case with $\psi_0 = 1$ and $\psi_n = 0$ for $n \geq 1$.

The GKH odering policy has an MA(∞) representation and is defined by a matrix of weights $W = [w_{ij}]$ such that

$$O_t^{\text{GKH}} = d + \sum_{k=0}^{\infty} \tilde{\psi}_k^{\text{GKH}} \epsilon_{t-k}, \qquad \text{with} \quad \tilde{\psi}_k^{\text{GKH}} := \sum_{j=0}^{H} w_{kj} \psi_j,$$

where H is a non-negative integer that parametrizes how far into the future the retailer's updates her demand forecasts. The value of W are computed by solving the optimization problem

$$\min_{w_{ij}} \sum_{i=0}^{H} \sum_{j=0}^{H} (w_{ij} \psi_j \sigma_\epsilon)^2 \qquad \text{subject to} \quad \mathbb{V}\text{ar}(I_t) \le K^2 \quad \text{and} \quad \sum_{i=0}^{H} w_{ij} = 1 \; \forall j,$$

where K is a parameter that bounds the variance of the retailer's inventory. Graves et al. (1998) provide a solution to this optimization.

The aforementioned papers are standard references in the literature focusing on the effect of demand forecasting and information sharing in the operations of supply chains and manufacturing systems. They consider alternative replenishment policies motivated by different considerations. Yet, all of them admit an MA(∞) representation and, as such, fall within our proposed framework. It is worth noting that in all these cases, we can extend the IID Gaussian assumption made in our paper to consider an arbitrary invertible MA(∞) demand process. These replenishment policies will still admit an MA(∞) representation. Collectively, these examples effectively illustrate the flexibility of the MA(∞) replenishment orders assumed in our model.

At the same time, we must notice that our formulation does not capture every possible replenishment strategy, including the popular (s, S) periodic-review policy or the continuous-review (Q, R) policy. Regarding the (s, S) policy, the fact that our model does not include any fixed ordering cost suggests that there is no loss of optimality when restricting ourselves to purely order-up-to policies (i.e., policies with $s = S$). In our case, with IID demand, a fixed value of S would be optimal within this class of policies, which again can be represented by an MA(∞) ordering process. On the other hand, the continuous-review (Q, R) policy cannot be represented within our discrete-time framework.