Supplementary Material for "Decoupled Functional Central Limit Theorems for Two-Time-Scale Stochastic Approximation"

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7 Recapitulation of the Assumptions and Results

7.1 Assumptions

Assumption 3.1 (Lipschitz conditions)**.** *There exists an operator $H\colon \mathbb{R}^{d_y} \to \mathbb{R}^{d_x}$ such that for any fixed $y \in \mathbb{R}^{d_y}$, $x = H(y)$ is the unique solution of $F(x,y) = 0$. The operators H, F and G satisfy the following Lipschitz conditions: $\forall\, x \in \mathbb{R}^{d_x}, y_1, y_2 \in \mathbb{R}^{d_y}$,*

$$\begin{aligned} \|H(y_1) - H(y_2)\| &\le L_H \|y_1 - y_2\|, \\ \|F(x,y_1) - F(H(y_1),y_1)\| &\le L_F \|x - H(y_1)\|, \\ \|G(x,y_1) - G(H(y_1),y_1)\| &\le L_{G,x} \|x - H(y_1)\|, \\ \|G(H(y_1),y_1) - G(H(y^\star),y^\star)\| &\le L_{G,y} \|y - y^\star\|. \end{aligned} \tag{7.1}$$

Assumption 3.2 (Uniform local linearity of H up to order $1+\delta_H$)**.** *Assume that H is differentiable and there exist constants $S_H \ge 0$ and $\delta_H \in [0.5, 1]$ such that $\forall\, y_1, y_2 \in \mathbb{R}^{d_y}$,*

$$\|H(y_1) - H(y_2) - \nabla H(y_2)(y_1 - y_2)\| \le S_H \|y_1 - y_2\|^{1+\delta_H}.$$

Assumption 3.3 (Nested local linearity of F and G up to order $(1+\delta_F, 1+\delta_G)$)**.** *There exist matrices B_1, B_2, B_3 with compatible dimensions, constants $S_{B,F}, S_{B,G} \ge 0$ and $\delta_F, \delta_G \in (0,1]$ such that*

$$\|F(x,y) - B_1(x - H(y))\| \le S_{B,F}\left(\|x - H(y)\|^{1+\delta_F} + \|y - y^\star\|^{1+\delta_F}\right), \tag{7.2}$$

$$\|G(x,y) - B_2(x - H(y)) - B_3(y - y^\star))\| \le S_{B,G}\left(\|x - H(y)\|^{1+\delta_G} + \|y - y^\star\|^{1+\delta_G}\right). \tag{7.3}$$

Furthermore, $\|B_1\| \le L_F$, $\|B_2\| \le L_{G,x}$, and $\|B_3\| \le L_{G,y}$.

Assumption 3.4 (Conditions on step sizes)**.** *The step sizes $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ together with their ratios $\{\kappa_n = \beta_n/\alpha_n\}_{n=0}^\infty$ satisfy the following conditions.*

(i) As $n \to \infty$, $\alpha_n, \beta_n, \kappa_n \to 0$ and α_n, β_n are decreasing in n.

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(ii) $\frac{\beta_{n-1}}{\beta_n} = 1 + \mathcal{O}(\beta_n)$ *and* $\tilde{\beta} := \lim_{n\to\infty}(\beta_{n+1}^{-1} - \beta_n^{-1}) \geq 0$.

(iii) $\frac{\alpha_{n-1}}{\alpha_n} = 1 + \mathcal{O}(\beta_n)$ *and* $\frac{\kappa_{n-1}}{\kappa_n} = 1 + \mathcal{O}\big(\beta_n\big)$.

(iv) $n\alpha_n$ *and* $n\beta_n$ *are increasing in* n.

(v) $\beta_n^{(1+\delta_H)/2} = \mathcal{O}(\alpha_n)$ *and* $\alpha_n^{1+\delta_G}, \alpha_n^{1+\delta_F} = o(\beta_n)$, *with* δ_H, δ_F *and* δ_G *defined in Assumptions 3.2 and 3.3.*

Assumption 3.5 (Hurwitz matrices)**.** *The matrices* B_1 *and* B_3 *defined in Assumption 3.3 and the parameter* $\tilde{\beta}$ *defined in Assumption 3.4 satisfy that both* $-B_1$ *and* $-\left(B_3 - \frac{\tilde{\beta}}{2}I\right)$ *are Hurwitz matrices, i.e, the real parts of their eigenvalues are negative.*

Define the filtration $(\mathcal{F}_n)_{n\geq 1}$ as

$$\mathcal{F}_n = \sigma(x_0, y_0, \xi_0, \psi_0, \xi_1, \psi_1, \ldots, \xi_{n-1}, \psi_{n-1}).$$

Assumption 3.6 (Conditions on the noise)**.** *The noise sequences* $\{\xi_n\}_{n=0}^{\infty}$ *and* $\{\psi_n\}_{n=0}^{\infty}$ *are martingale difference sequences satisfying* $\mathbb{E}[\xi_n \,|\, \mathcal{F}_n] = 0$ *and* $\mathbb{E}[\psi_n \,|\, \mathcal{F}_n] = 0$. *Their fourth moments are uniformly bounded, i.e.,*

$$\sup_{n\geq 0} \mathbb{E}\|\xi_n\|^4 + \sup_{n\geq 0} \mathbb{E}\|\psi_n\|^4 < \infty.$$

Moreover, as $n \to \infty$,

$$\mathbb{E}\left[\begin{pmatrix} \xi_n\xi_n^\top & \xi_n\psi_n^\top \\ \psi_n\xi_n^\top & \psi_n\psi_n^\top \end{pmatrix} \middle| \mathcal{F}_n\right] \xrightarrow{p} \begin{pmatrix} \Sigma_\xi & \Sigma_{\xi,\psi} \\ \Sigma_{\xi,\psi}^\top & \Sigma_\psi \end{pmatrix}.$$

Assumption 4.1 (Non-asymptotic decoupled convergence)**.** *The convergence rates in Proposition 4.1 hold.*

7.2 Construction

With the errors defined by

$$\hat{x}_n = x_n - H(y_n), \quad \hat{y}_n = y_n - y^\star, \tag{7.4}$$

we rescale them as follows

$$\check{x}_n := \frac{\hat{x}_n}{\sqrt{\alpha_{n-1}}} = \frac{x_n - H(y_n)}{\sqrt{\alpha_{n-1}}}, \quad \check{y}_n := \frac{\hat{y}_n}{\sqrt{\beta_{n-1}}} = \frac{y_n - y^\star}{\sqrt{\beta_{n-1}}}. \tag{7.5}$$

Then the continuous-time trajectories are defined as

$$\bar{\mathrm{X}}_n(t) = \begin{cases} \check{x}_n, & t = 0; \\ \check{x}_m + \frac{t - \sum_{i=n}^{m-1}\alpha_i}{\alpha_m}(\check{x}_{m+1} - \check{x}_m), & t \in \left(\sum_{i=n}^{m-1}\alpha_i, \sum_{i=n}^{m}\alpha_i\right] \text{ for } m \geq n; \end{cases} \tag{7.6}$$

$$\bar{\mathrm{Y}}_n(t) = \begin{cases} \check{y}_n, & t = 0; \\ \check{y}_m + \frac{t - \sum_{i=n}^{m-1}\beta_i}{\beta_m}(\check{y}_{m+1} - \check{y}_m), & t \in \left(\sum_{i=n}^{m-1}\beta_i, \sum_{i=n}^{m}\beta_i\right] \text{ for } m \geq n. \end{cases} \tag{7.7}$$

The auxiliary sequence is defined as

$$\check{z}_n = \check{y}_n - \sqrt{\kappa_{n-1}} B_2 B_1^{-1} \check{x}_n = \frac{\hat{y}_n - \kappa_{n-1} B_2 B_1^{-1} \hat{x}_n}{\sqrt{\beta_{n-1}}}, \ n \geq 1, \tag{7.8}$$

along with the corresponding continuous-time trajectories

$$\bar{\mathrm{Z}}_n(t) = \begin{cases} \check{z}_n, & t = 0; \\ \check{z}_m + \frac{t - \sum_{i=n}^{m-1} \beta_i}{\beta_m} (\check{z}_{m+1} - \check{z}_m), & t \in \left(\sum_{i=n}^{m-1} \beta_i, \sum_{i=n}^{m} \beta_i \right] \text{ for } m \geq n. \end{cases} \tag{7.9}$$

7.3 Main Results

Theorem 4.1 (Decoupled functional central limit theorems)**.** *Suppose that Assumptions 3.1 – 3.6 and 4.1 hold.*

(i) The stochastic process $\bar{\mathrm{X}}_n(\cdot)$ defined in (7.6) converges weakly to the stationary solution of the following SDE

$$d\mathrm{X}(t) = -B_1 \mathrm{X}(t) dt + \Sigma_{\xi}^{1/2} d\mathrm{W}^{d_x}(t). \tag{7.10}$$

The rescaled iterate $\check{x}_n$ defined in (7.5) converges weakly to the invariant distribution of (7.10), i.e., $\mathcal{N}(0, \Sigma_x)$, where Σ_x satisfies the following Lyapunov equation

$$B_1 \Sigma_x + \Sigma_x B_1 = \Sigma_{\xi}. \tag{7.11}$$

(ii) The stochastic process $\bar{\mathrm{Y}}_n(\cdot)$ defined in (7.7) converges weakly to the stationary solution of the following SDE

$$d\mathrm{Y}(t) = -\left(B_3 - \frac{\tilde{\beta} I}{2} \right) \mathrm{Y}(t) dt + \widetilde{\Sigma}_{\psi}^{1/2} d\mathrm{W}^{d_y}(t), \tag{7.12}$$

where the matrix $\widetilde{\Sigma}_{\psi}$ is defined as

$$\widetilde{\Sigma}_{\psi} := \Sigma_{\psi} - B_2 B_1^{-1} \Sigma_{\xi,\psi} - \Sigma_{\xi,\psi}^{\top} B_1^{-\top} B_2^{\top} + B_2 B_1^{-1} \Sigma_{\xi} B_1^{-\top} B_2^{\top}. \tag{7.13}$$

The rescaled iterate $\check{y}_n$ defined in (7.5) converges weakly to the invariant distribution of (7.12), i.e., $\mathcal{N}(0, \Sigma_y)$, where Σ_y satisfies the following Lyapunov equation

$$\left(B_3 - \frac{\tilde{\beta} I}{2} \right) \Sigma_y + \Sigma_y \left(B_3 - \frac{\tilde{\beta} I}{2} \right) = \widetilde{\Sigma}_{\psi}. \tag{7.14}$$

Corollary 4.1 (Central limit theorems)**.** *Under the conditions of Theorem 4.1, we have $\alpha_n^{-1/2}(x_n - H(y_n)) \Rightarrow \mathcal{N}(0, \Sigma_x)$, $\beta_n^{-1/2}(y_n - y^\star) \Rightarrow \mathcal{N}(0, \Sigma_y)$ and $\alpha_n^{-1/2}(x_n - x^\star) \Rightarrow \mathcal{N}(0, \Sigma_x)$.*

7.4 Examples

In the following two examples, we consider the problem of minimizing a strongly convex function f with a unique minimizer $x_o^\star := \arg\min_x f(x)$, using noisy gradient observations.

Example 4.1 (SGD with Polyak-Ruppert averaging)**.** *Stochastic gradient descent (SGD), introduced by [26], iteratively updates the iterate x_t by (7.15a). To improve the convergence of SGD, an additional averaging step (7.15b) is often used [24, 27]*

$$x_{n+1} = x_n - \alpha_n(\nabla f(x_n) + \xi_n) \text{ with } \alpha_n = \frac{\alpha_0}{(n+1)^a} \left(\frac{1}{2} < a < 1\right), \tag{7.15a}$$

$$y_{n+1} = \frac{1}{n+1}\sum_{\tau=0}^{n} x_\tau = y_n - \beta_n(y_n - x_n) \text{ with } \beta_n = \frac{1}{n+1}. \tag{7.15b}$$

Example 4.2 (SGD with momentum)**.** *SGD with momentum is ubiquitous in machine learning. We consider a special case of quasi-hyperbolic momentum [13]*

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n(x_n - \nabla f(y_n) - \xi_n), \\ y_{n+1} &= y_n - \beta_n[\nu x_n + (1-\nu)(\nabla f(y_n) + \xi_n)]. \end{aligned} \tag{7.16}$$

Here $\nu \in [0,1]$ is an interpolation parameter: when $\nu = 1$, the method becomes a "normalized" version of stochastic heavy ball (SHB) [12, 14]; when $\nu = 0$, it reduces to SGD.

Example 4.3 (Gradient-based temporal-difference learning)**.** *Consider GTD2 and TDC with linear function approximation [29].The goal is to estimate the value function $V(s)$ for a given policy using a linear parameterization $V(s;y) = \phi(s)^\top y$, where $\phi(s)$ represents the feature vector of state s, and y denotes the parameter vector. Given i.i.d. samples $\{(s_n, r_n, s'_n)\}_{n=0}^\infty$ [28], where s_n is the current state, r_n is the reward and s'_n is the next state, the optimal parameter y can be estimated iteratively via the following update rules*

$$x_{n+1} = x_n - \alpha_n\big(\phi(s_n)^\top x_n - \rho_n\delta_n\big)\phi(s_n), \tag{7.17}$$

$$y_{n+1} = y_n - \beta_n\rho_n\big(\gamma\phi(s'_n) - \phi(s_n)\big)\phi(s_n)^\top x_n, \qquad \text{(GTD2)} \tag{7.18}$$

$$y_{n+1} = y_n - \beta_n\rho_n\big(\gamma\phi(s'_n)\phi(s_n)^\top x_n - \delta_n\phi(s_n)\big), \qquad \text{(TDC)} \tag{7.19}$$

where γ is the discount factor, ρ_n is the importance weight with its explicit form deferred to Appendix B.4, and δ_n is the TD error defined as $\delta_n = r_n + (\gamma\phi(s'_n) - \phi(s_n))^\top y_n$.

7.5 Framework of Proof

7.5.1 Derivation of One-Step Recursions

Lemma 5.1 (One-step recursions for $\check{x}_n$ and $\check{y}_n$)**.** *Suppose that Assumptions 3.1 – 3.4 and 4.1 hold. Define $\tilde{\xi}_n := \xi_n - \kappa_n H^\star \psi_n$. The rescaled sequences $\{\check{x}_n\}_{n=0}^\infty$ and $\{\check{y}_n\}_{n=0}^\infty$ defined in (7.5) satisfy the following relationships*

$$\check{x}_{n+1} = (I - \alpha_n B_1)\check{x}_n - \sqrt{\alpha_n}\tilde{\xi}_n + R_n^x, \tag{7.20}$$

$$\check{y}_{n+1} = \left(I - \beta_n B_3 + \frac{\tilde{\beta}\beta_n}{2} I\right)\check{y}_n - \sqrt{\beta_n\alpha_n} B_2 \check{x}_n - \sqrt{\beta_n}\psi_n + R_n^y. \tag{7.21}$$

For any $2 < p \le \frac{4}{1+(\delta_H \vee \delta_F \vee \delta_G)/2}$, *the residual terms* R_n^x *and* R_n^y *have the following properties: (i)* $\mathbb{E}\|\mathbb{E}[R_n^x|\mathcal{F}_n]\| = o(\sqrt{\beta_n\alpha_n})$, $\mathbb{E}\|R_n^x\|^2 = \mathcal{O}(\beta_n\alpha_n)$ *and* $\mathbb{E}\|R_n^x\|^p = o(\alpha_n)$; *(ii)* $\mathbb{E}\|R_n^y\|^2 = o(\beta_n^2)$ *and* $\mathbb{E}\|R_n^y\|^p = o(\beta_n)$.

Lemma 5.2 (One-step recursion for $\check{z}_n$)**.** *Suppose that Assumptions 3.1 – 3.4 and 4.1 hold. Define* $\tilde{\psi}_n := \psi_n - B_2 B_1^{-1}\tilde{\xi}_n$ *with* $\tilde{\xi}_n$ *defined in Lemma 5.1. The auxiliary sequences* $\{\check{z}_n\}_{n=1}^{\infty}$ *defined in* (7.8) *satisfy the following relationships*

$$\check{z}_{n+1} = \left(I - \beta_n B_3 + \frac{\tilde{\beta}\beta_n}{2} I\right)\check{z}_n - \sqrt{\beta_n}\tilde{\psi}_n + R_n^z. \tag{7.22}$$

For any $2 < p \le \frac{4}{1+(\delta_H \vee \delta_F \vee \delta_G)/2}$, *the residual terms* R_n^z *have the following properties:* $\mathbb{E}\|\mathbb{E}[R_n^z|\mathcal{F}_n]\| = o(\beta_n)$, $\mathbb{E}\|R_n^z\|^2 = \mathcal{O}(\beta_n^2)$ *and* $\mathbb{E}\|R_n^z\|^p = o(\beta_n)$.

7.5.2 Establishment of Tightness

Proposition 5.1 (Sufficient conditions for tightness)**.** *Suppose that* $\{\mathrm{U}_n(\cdot)\}_{n=0}^{\infty}$ *is a sequence of random functions in* $C([0,\infty);\mathbb{R}^d)$. *Then* $\{\mathrm{U}_n(\cdot)\}_{n=0}^{\infty}$ *is tight if the following two conditions hold:*

(i) For each positive η, *there exist a positive* a *and an integer* n_0 *such that*

$$\mathbb{P}\left(\|\mathrm{U}_n(0)\| \ge a\right) \le \eta, \quad \forall\, n \ge n_0.$$

(ii) For each positive T, *there exist positive numbers* $\gamma_1, \gamma_2, c_1, c_2, K$ *such that*

$$\mathbb{P}\left(\|\mathrm{U}_n(s) - \mathrm{U}_n(t)\| \ge \varepsilon\right) \le K\left(\frac{|s-t|^{1+\gamma_1}}{\varepsilon^{c_1}} + \frac{|s-t|^{1+\gamma_2}}{\varepsilon^{c_2}}\right), \ \forall\, s,t \in [0,T], \forall\, n, \forall\, \varepsilon > 0. \tag{7.23}$$

Lemma 5.3 (Tightness)**.** *Suppose that Assumptions 3.1 – 3.4, 3.6 and 4.1 hold. The sequences* $\{\bar{\mathrm{X}}_n(\cdot)\}_{n=1}^{\infty}$, $\{\bar{\mathrm{Y}}_n(\cdot)\}_{n=1}^{\infty}$ *and* $\{\bar{\mathrm{Z}}_n(\cdot)\}_{n=1}^{\infty}$ *constructed in* (7.6), (7.7) *and* (7.9) *are tight.*

Proposition 2.1 (Properties of tightness)**.** *Suppose that* $\{\mathrm{U}_n(\cdot)\}_{n=0}^{\infty}$ *is a tight sequence of random functions in* $C([0,\infty);\mathbb{R}^d)$ *and* $\mathrm{U}(\cdot)$ *is another random function in* $C([0,\infty);\mathbb{R}^d)$.

(i) If the finite-dimensional distributions of $\mathrm{U}_n(\cdot)$ *converge weakly to the finite-dimensional distributions of* $\mathrm{U}(\cdot)$, *i.e., for each positive integer* k *and* $t_1, t_2, \ldots, t_k > 0$, *it holds that* $(\mathrm{U}_n(t_1), \mathrm{U}_n(t_2), \ldots, \mathrm{U}_n(t_k)) \Rightarrow (\mathrm{U}(t_1), \mathrm{U}(t_2), \ldots, \mathrm{U}(t_k))$, *then* $\mathrm{U}_n(\cdot) \Rightarrow \mathrm{U}(\cdot)$ *[33, Theorem 3].*

(ii) Prokhorov's theorem [4, Theorem 5.1]: Each subsequence of $\{\mathrm{U}_n(\cdot)\}_{n=0}^{\infty}$ *contains a further subsequence that converges weakly to some random function in* $C([0,\infty);\mathbb{R}^d)$.

(iii) If each convergent subsequence of $\{\mathrm{U}_n(\cdot)\}_{n=0}^{\infty}$ *converges weakly to* $\mathrm{U}(\cdot)$, *then* $\mathrm{U}_n(\cdot) \Rightarrow \mathrm{U}(\cdot)$ *[4, Corollary on page 59].*

7.5.3 Approximately Solving of Martingale Problems

Lemma 5.4 (Approximately solving of martingale problems)**.** *Suppose that Assumptions 3.1 – 3.4, 3.6 and 4.1 hold. For each n, define the following filtrations*

$$\begin{aligned} \mathcal{F}^{\bar{\mathrm{X}}}_{n,t} &= \begin{cases} \mathcal{F}_n, & t = 0; \\ \mathcal{F}_{m+1}, & t \in \left(\sum_{i=n}^{m-1} \alpha_i, \sum_{i=n}^{m} \alpha_i\right] \text{ for } m \geq n; \end{cases} \\ \mathcal{F}^{\bar{\mathrm{Z}}}_{n,t} &= \begin{cases} \mathcal{F}_n, & t = 0; \\ \mathcal{F}_{m+1}, & t \in \left(\sum_{i=n}^{m-1} \beta_i, \sum_{i=n}^{m} \beta_i\right] \text{ for } m \geq n. \end{cases} \end{aligned} \tag{7.24}$$

(i) For any $f \in C_c^2(\mathbb{R}^{d_x})$ and $t \geq 0$, with $\mathcal{A}^x$ defined as

$$\mathcal{A}^x f(x) = \langle -B_1 x, \nabla f(x) \rangle + \frac{1}{2}\mathrm{tr}(\nabla^2 f(x)\, \Sigma_\xi), \ \forall f \in C_c^2(\mathbb{R}^{d_x}), \tag{7.25}$$

we have the following decomposition

$$f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(0)) - \int_0^t \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\, \mathrm{d}s = \mathcal{M}_n^f(t) + \mathcal{R}_n^f(t),$$

where for each n, $\mathcal{M}_n^f(t)$ is a martingale w.r.t. $(\mathcal{F}^{\bar{\mathrm{X}}}_{n,t})_{t\geq 0}$ and for each t, $\mathbb{E}|\mathcal{R}_n^f(t)| \to 0$ as $n \to \infty$.

(ii) For any $g \in C_c^2(\mathbb{R}^{d_y})$ and $t \geq 0$, with $\mathcal{A}^y$ defined as

$$\mathcal{A}^y g(y) = \left\langle -\left(B_3 - \frac{\tilde{\beta} I}{2}\right) y, \nabla g(y) \right\rangle + \frac{1}{2}\mathrm{tr}(\nabla^2 g(y)\, \tilde{\Sigma}_\psi), \ \forall g \in C_c^2(\mathbb{R}^{d_y}), \tag{7.26}$$

we have the following decomposition

$$g(\bar{\mathrm{Z}}_n(t)) - g(\bar{\mathrm{Z}}_n(0)) - \int_0^t \mathcal{A}^y f(\bar{\mathrm{Z}}_n(s))\, \mathrm{d}s = \mathcal{M}_n^g(t) + \mathcal{R}_n^g(t),$$

where for each n, $\mathcal{M}_n^g(t)$ is a martingale w.r.t. $(\mathcal{F}^{\bar{\mathrm{Z}}}_{n,t})_{t\geq 0}$ and for each t, $\mathbb{E}|\mathcal{R}_n^g(t)| \to 0$ as $n \to \infty$.

7.5.4 Integration of Pieces

Proposition 5.2 (10, Chapter 4, Theorem 8.10)**.** *Suppose that the generator $\mathcal{A}$ is defined as*

$$\mathcal{A}f(x) = -\langle Bx, \nabla f(x) \rangle + \frac{1}{2}\mathrm{tr}(\nabla^2 f(x) \Sigma), \ \forall f \in C_c^2(\mathbb{R}^d). \tag{7.27}$$

and the following conditions are met.

(i) Given the initial distribution, the martingale problem for $(\mathcal{A}, C_c^2(\mathbb{R}^d))$ has a unique solution.

(ii) $\{\mathrm{U}_n(\cdot)\}_{n=0}^\infty$ is a tight sequence in $C([0,\infty); \mathbb{R}^d)$.

(iii) $\mathrm{U}_n(0) \Rightarrow \mathrm{U}(0)$.

(iv) For any $k \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_k \leq t \leq t+s$, and $f, h_1, h_2, \ldots, h_k \in C_c^2(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(f(\mathrm{U}_n(t+s)) - f(\mathrm{U}_n(t)) - \int_t^{t+s} \mathcal{A}f(\mathrm{U}_n(r))\,\mathrm{d}r\right)\prod_{i=1}^k h_i(\mathrm{U}_n(t_i))\right] = 0. \tag{7.28}$$

Then we have $\mathrm{U}_n(\cdot) \Rightarrow \mathrm{U}(\cdot)$.

Corollary 5.1 (FCLT for the auxiliary sequence)**.** *Under the conditions of Theorem 4.1, the stochastic process $\bar{\mathrm{Z}}_n(\cdot)$ defined in (7.9) converges weakly to the stationary solution of the SDE in (7.12).*

A Intermediate Results about the Assumptions

This section presents some intermediate results about the Assumptions, with their proofs deferred to Appendix C.1.

Assumption 3.2 is equivalent to the Hölder continuity of ∇H.

Assumption 3.2† (δ_H-Hölder continuity of ∇H)**.** *Assume that $H(\cdot)$ is differentiable and there exist constants $\widetilde{S}_H \geq 0$ and $\delta_H \in [0,1]$ such that $\forall\, y_1, y_2 \in \mathbb{R}^{d_y}$,*

$$\|\nabla H(y_1) - \nabla H(y_2)\| \leq \widetilde{S}_H \|y_1 - y_2\|^{\delta_H}. \tag{A.1}$$

Proposition A.1 ([15, Proposition 1])**.** *Suppose that $\delta_H \in [0,1]$. Under Assumption 3.2†, Assumption 3.2 holds with $S_H = \frac{\widetilde{S}_H}{1+\delta_H}$; under Assumption 3.2, Assumption 3.2† holds with $\widetilde{S}_H = 2^{1-\delta_H}\sqrt{1+\delta_H}\left(\frac{1+\delta_H}{\delta_H}\right)^{\frac{\delta_H}{2}} S_H$ (we make the contention that $\left(\frac{1+\delta_H}{\delta_H}\right)^{\frac{\delta_H}{2}} = 1$ if $\delta_H = 0$).*

Under Assumption 3.1, Assumption 3.3 can be replaced by the following weaker condition, which requires local linearity only in a neighborhood of $(x^\star, y^\star)$.

Assumption 3.3† (Nested local linearity of F and G up to order $(1+\delta_F, 1+\delta_G)$)**.** *There exist matrices B_1, B_2, B_3 with compatible dimensions, constants $S_{B,F}, S_{B,G} \geq 0$, $\varepsilon > 0$, and $\delta_F, \delta_G \in (0,1]$ such that for any (x,y) satisfying $\max\{\|x - H(y)\|, \|y - y^*\|\} \leq \varepsilon$,*

$$\|F(x,y) - B_1(x-H(y))\| \leq S_{B,F}\left(\|x-H(y)\|^{1+\delta_F} + \|y-y^\star\|^{1+\delta_F}\right), \tag{A.2}$$

$$\|G(x,y) - B_2(x-H(y)) - B_3(y-y^\star)\| \leq S_{B,G}\left(\|x-H(y)\|^{1+\delta_G} + \|y-y^\star\|^{1+\delta_G}\right). \tag{A.3}$$

Proposition A.2. *Suppose that Assumptions 3.1 and 3.3† hold. Then Assumption 3.3 hold with $S_{B,F}$ replaced by $\max\left\{S_{B,F}, \frac{L_F+\|B_1\|}{\varepsilon^{\delta_F}}\right\}$ and $S_{B,G}$ replaced by $\max\left\{S_{B,G}, \frac{L_{G,x}+\|B_2\|+L_{G,y}+\|B_3\|}{\varepsilon^{\delta_G}}\right\}$.*

Moreover, the nested local linearity can be derived from the following standard local linearity.

Assumption 3.3‡ (Local linearity up to order $(1+\delta_F, 1+\delta_G)$)**.** *There exists matrices $A_{11}, A_{12}, A_{21}, A_{22}$ with compatible dimensions, constants $S_{A,F}, S_{A,G} \geq 0$, $\varepsilon > 0$ and $\delta_F, \delta_G \in (0,1]$ such that for any (x,y) satisfying $\max\{\|x - x^*\|, \|y - y^*\|\} \leq \varepsilon$,*

$$\|F(x,y) - A_{11}(x-x^\star) - A_{12}(y-y^\star)\| \leq S_{A,F}\left(\|x-x^\star\|^{1+\delta_F} + \|y-y^\star\|^{1+\delta_F}\right), \tag{A.4}$$

$$\|G(x,y) - A_{21}(x-x^\star) - A_{22}(y-y^\star)\| \leq S_{A,G}\left(\|x-x^\star\|^{1+\delta_G} + \|y-y^\star\|^{1+\delta_G}\right). \tag{A.5}$$

Proposition A.3. *Suppose that Assumptions 3.1, 3.2, and 3.3‡ hold and A_{11} is invertible. If we further assume $\delta_H \geq \delta_F \vee \delta_G$, then Assumption 3.3 holds with parameters*

$$
\begin{aligned}
&B_1 = A_{11}, \quad B_2 = A_{21}, \quad B_3 = A_{22} - A_{21}A_{11}^{-1}A_{12}.,\\
&S_{B,F} = S_{A,F} + \frac{L_F + \|A_{11}\| + L_F L_H + \|A_{12}\|}{\|\varepsilon\|^{\delta_F}} + L_F(S_H \vee 2L_H),\\
&S_{B,G} = S_{A,G} + \frac{L_{G,x} + \|A_{21}\| + L_{G,x}L_H + L_{G,y} + \|A_{22}\|}{\|\varepsilon\|^{\delta_G}} + L_{G,x}(S_H \vee 2L_H).
\end{aligned}
$$

The following propositions suggest that with the Lipschitz conditions, we can always select a lower order without compromising the validity of Assumptions 3.2 and 3.3.

Proposition A.4. *Suppose that (7.1) holds. If Assumption 3.2 holds up to order $1 + \delta_H$, then it also holds up to order $1 + \delta'_H$ for any $\delta'_H \in [0, \delta_H]$.*

Proposition A.5. *Suppose that Assumption 3.1 holds. If Assumption 3.3 holds up to order $(1 + \delta_F, 1 + \delta_G)$, then it also holds up to order $(1 + \delta'_F, 1 + \delta'_G)$ for any $\delta'_F \in (0, \delta_F)$ and $\delta'_G \in (0, \delta_G)$.*

With Assumptions 3.2 and 3.3, we can decompose these operators as linear parts and higher-order residual terms.

Proposition A.6 (Operator decomposition)**.** *Suppose that Assumptions 3.1 – 3.3 hold. With $\hat{x}_n$ and $\hat{y}_n$ defined in (7.4), for any $\delta'_F \in (0, \delta_F]$, $\delta'_G \in (0, \delta_G]$ and $\delta'_H \in (0, \delta_H]$, we have the following results.*

(i) $F(x_n, y_n) = B_1\hat{x}_n + R_n^F$ *with* $\|R_n^F\| = \mathcal{O}(\|\hat{x}_n\|^{1+\delta'_F} + \|\hat{y}_n\|^{1+\delta'_F})$.

(ii) $G(x_n, y_n) = B_2\hat{x}_n + B_3\hat{y}_n + R_n^G$ *with* $\|R_n^G\| = \mathcal{O}(\|\hat{x}_n\|^{1+\delta'_G} + \|\hat{y}_n\|^{1+\delta'_G})$.

(iii) $H(y_{n+1}) - H(y_n) = H^\star(y_{n+1} - y_n) + R_n^{\nabla H} + R_n^H$ *with* $H^\star := \nabla H(y^\star)$, $\|R_n^H\| = \mathcal{O}(\|y_{n+1} - y_n\|^{1+\delta'_H})$ *and* $R_n^{\nabla H} = (\nabla H(y_n) - \nabla H(y^\star))(y_{n+1} - y_n)$.

Remark A.1 (Implications of Assumption 3.4)**.** *We can derive the following results from Assumption 3.4.*

- *Assumption 3.4 implies $\beta_n = \Omega(1/n)$ and $\alpha_n/\beta_n = \omega(1)$. Then we have $\sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \beta_n = \infty$, which is a ubiquitous condition in the SA analysis to ensure convergence to the target solution [9, 11, 18].*
- *Note that $\alpha_{n+1}^{-1} - \alpha_n^{-1} = \beta_{n+1}^{-1}\kappa_{n+1} - \beta_n^{-1}\kappa_n = \beta_{n+1}^{-1}(\kappa_{n+1} - \kappa_n) + \kappa_n(\beta_{n+1}^{-1} - \beta_n^{-1}) = \beta_{n+1}^{-1}\kappa_{n+1}(1 - \kappa_n/\kappa_{n+1}) + \kappa_n(\beta_{n+1}^{-1} - \beta_n^{-1})$. If we define $\tilde{\alpha} := \lim_{n\to\infty}(\alpha_{n+1}^{-1} - \alpha_n^{-1})$ then we have $\tilde{\alpha} = 0$. This is why the choice of step sizes does not affect the SDE in (7.10). The parameters $\tilde{\beta}$ and $\tilde{\alpha}$ also appear frequently in previous analysis [11, 18].*

Remark A.2 (Linear case)**.** *If F and G are linear operators, implying H is also linear. the residual terms in Proposition A.6 vanish. Under this scenario, for Lemmas 5.1 and 5.2, as well as Theorem 4.1, to hold, Condition (v) in Assumption 3.4 is no longer necessary. Moreover, the requirement for the boundedness of fourth moments can be relaxed to p-th moments for any $p > 2$. For polynomially diminishing step sizes $\alpha_n = \alpha_0(n+1)^{-a}$ and $\beta_n = \beta_0(n+1)^{-b}$ with $a, b \in (0, 1]$, it suffices to ensure $b/a > 1$.*

B Omitted Details for Section 4

In this section, we present more details for the construction in Section 4.1 and the examples in Section 4.3.

B.1 Non-asymptotic Convergence Rates

We first give the formal statement of Proposition 4.1.

Proposition B.1 (Non-asymptotic convergence rates, [15, Theorem 3]). *Suppose that Assumptions 3.1 – 3.6 and the following strong monotonicity conditions hold:*

(i) There exists a constant $\mu_F > 0$ such that for any $y \in \mathbb{R}^{d_y}$ and $x \in \mathbb{R}^{d_x}$,

$$\langle x - H(y), F(x, y) - F(H(y), y)\rangle \geq \mu_F \|x - H(y)\|^2.$$

(ii) There exists a constant $\mu_G > 0$ such that for any $y \in \mathbb{R}^{d_y}$,

$$\langle y - y^\star, G(H(y), y) - G(H(y^\star), y^\star)\rangle \geq \mu_G \|y - y^\star\|^2.$$

Then we have $\mathbb{E}\|\hat{x}_n\|^4 + \mathbb{E}\|\hat{y}_n\|^4 = \mathcal{O}(\alpha_n^2)$. If we further assume $b/a \leq 1 + \delta_F/2$ in the step size selection of Example 3.1 and α_0 and β_0 are properly chosen, we have $\mathbb{E}\|\hat{x}_n\|^2 = \mathcal{O}(\alpha_n)$ and $\mathbb{E}\|\hat{y}_n\|^2 = \mathcal{O}(\beta_n)$.

Regarding the comparison between strong monotonicity and Assumption 3.5, strong monotonicity is a non-asymptotic condition that guarantees convergence from arbitrary initializations, whereas Assumption 3.5 is a local condition that guarantees only the asymptotic behavior. As for the additional requirement for the step sizes, we guess that this is due to proof techniques in [15] and is not inherent to the problem itself. Although [15, Assumption 7] imposes an additional condition on the step sizes, for $\alpha_n = \alpha_0(n+1)^{-a}$ and $\beta_n = \beta_0(n+1)^{-b}$ with suitably chosen α_0 and β_0, this condition holds for all sufficiently large n and therefore does not affect our asymptotic analysis.

Moreover, in the linear setting, [16] derives a sharper control of the slow-iterate error by working with the norm induced by the solution to an appropriate Lyapunov equation, rather than the Euclidean norm. We expect that extending this technique to the nonlinear case could lead to tighter non-asymptotic convergence rates.

B.2 Details for the Construction

In this subsection, we present an equivalent expression of the construction in (7.6), (7.7) and (7.9) for ease of subsequent proof. We first introduce some necessary notation inspired by [12, 22].

Definition B.1 (Time interpolation). *For the step size sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$, $n, n' \in \mathbb{N}$ with $n' > n$ and $t \geq 0$, we define*

$$\Gamma^\alpha_{n,n'} := \sum_{k=n}^{n'-1} \alpha_k,\ N^\alpha(n,t) := \max\left\{m \geq n \colon \Gamma^\alpha_{n,m} \leq t\right\}, \quad \textit{and} \quad \underline{t}^\alpha_n = \Gamma^\alpha_{n,N^\alpha(n,t)},$$

$$\Gamma^\beta_{n,n'} := \sum_{k=n}^{n'-1} \beta_k,\ N^\beta(n,t) := \max\left\{m \geq n \colon \Gamma^\beta_{n,m} \leq t\right\}, \quad \textit{and} \quad \underline{t}^\beta_n = \Gamma^\beta_{n,N^\beta(n,t)}.$$

For $n' \leq n$, we stipulate $\Gamma^\alpha_{n,n'} = \Gamma^\beta_{n,n'} := 0$.

Fact B.1. *Under Definition B.1, we have the following properties.*

(i) For $n_1 \le n_2 \le n_3$, $\Gamma^\alpha_{n_1,n_2} + \Gamma^\alpha_{n_2,n_3} = \Gamma^\alpha_{n_1,n_3}$.

(ii) For $m \ge n$, $N^\alpha(n, \Gamma^\alpha_{n,m}) = m$.

(iii) $\underline{t}^\alpha_n \le t < \underline{t}^\alpha_n + \alpha_{N^\alpha(n,t)}$. *When* $t = \Gamma^\alpha_{n,m}$ *for some* $m \ge n$, $t = \underline{t}^\alpha_n$.

(iv) As $n \to \infty$, *if* $\alpha_n \to 0$, *then* $\underline{t}^\alpha_n \uparrow t$.

Similar properties hold if the superscript is replaced by β.

In other words, $\Gamma^\alpha_{n,n'}$ denotes the accumulation of the step sizes from n-th iteration to $(n'-1)$-th iteration. $N^\alpha(n,t)$ is the maximal integer such that $\Gamma^\alpha_{n,N^\alpha(n,t)} \le t$. We define this value by $\underline{t}^\alpha_n$, which can be viewed as an approximation of t from the left.

The following lemma characterizes the magnitude of $N^\alpha(n,t)$ and $N^\beta(n,t)$.

Lemma B.1. *Suppose that Assumption 3.4 holds. There exist two positive constants* c_α *and* c_β *such that for any* $t > 0$ *and* $n \in \mathbb{N}$,

$$n \le N^\alpha(n,t) \le 2^{\lceil 2t/c_\alpha \rceil + 1} n \quad \text{and} \quad n \le N^\beta(n,t) \le 2^{\lceil 2t/c_\beta \rceil + 1} n.$$

As a corollary, $\frac{\alpha_n}{\alpha_{N^\alpha(n,t)}} \le 2^{\lceil 2t/c_\alpha \rceil + 1}$ *and* $\frac{\beta_n}{\beta_{N^\beta(n,t)}} \le 2^{\lceil 2t/c_\beta \rceil + 1}$.

Lemma B.1 shows that under our assumptions, both $N^\alpha(n,t)$ and $N^\beta(n,t)$ are of the order $\Theta(n)$. This property is useful for establishing the tightness in Section 7.5.2. For the polynomially diminishing step sizes, we can further sharpen this description, as stated in the following proposition.

Proposition B.2. *If* $\alpha_n = \frac{\alpha_0}{(n+1)^\alpha}$ *and* $\beta_n = \frac{\beta_0}{(n+1)^\beta}$ *with* $0 < a, b \le 1$, *then* $N^\alpha(n,t) - n = \Theta(n^\alpha)$ *and* $N^\beta(n,t) - n = \Theta(n^\beta)$.

The proof of Lemma B.1 and Proposition B.2 is deferred to Appendix C.2.

Fact B.2. *Under Definition B.1,* $\bar{\mathrm{X}}_n(\cdot)$ *defined in* (7.6) *satisfies the following properties.*

(i) For $m \ge n$, $\bar{\mathrm{X}}_n(\Gamma^\alpha_{n,m}) = \check{x}_m$.

(ii) $\bar{\mathrm{X}}_n(t) = \bar{\mathrm{X}}_n(\underline{t}^\alpha_n) + \frac{t - \underline{t}^\alpha_n}{\alpha_{N^\alpha(n,t)}} \big(\bar{\mathrm{X}}_n(\underline{t}^\alpha_n + \alpha_{N^\alpha(n,t)}) - \bar{\mathrm{X}}_n(\underline{t}^\alpha_n)\big)$.

(iii) For $m \ge n$ *and* $t \in [\Gamma^\alpha_{n,m}, \Gamma^\alpha_{n,m+1})$, $\bar{\mathrm{X}}_n(\underline{t}^\alpha_n) = \check{x}_m$.

Similar results also hold for $\bar{\mathrm{Y}}_n(\cdot)$ *and* $\bar{\mathrm{Z}}_n(\cdot)$ *defined in* (7.7) *and* (7.9).

Remark B.1 (Discussion on the interpolation)**.** *For the way of interpolation in our works, [11] also adopts the linear interpolation while [5, 12, 22] uses two different interpolations for different parts. Since both* $\alpha_{N^\alpha(n,t)}$ *and* $\beta_{N^\beta(n,t)}$ *converge to* 0 *as* $n \to \infty$, *different interpolation methods make no difference in the asymptotic sense in our setup. We choose simple linear interpolation for analytical simplicity such that each trajectory is continuous in time.*

B.3 An Upper Bound for the Exponential of a Hurwitz Matrix

Proposition B.3. *Suppose that $-A$ is a Hurwitz matrix, i.e., the real parts of the eigenvalues of $-A$ are negative. Then there exist $M, b > 0$ such that $\|e^{-At}\| \leq Me^{-bt}, \forall t \geq 0$. In particular, define $P := \int_0^\infty e^{-A^\top t} e^{-At} dt$. Then M and b can be set as $M := \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$ and $b := 1/(2\lambda_{\max}(P))$, where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest and smallest eigenvalues of P, respectively.*

Proposition B.3 shows that the decay of $\|e^{-At}\|$ is governed by the condition number and the largest eigenvalue of the Lyapunov matrix P. By definition, P is positive definite. In the special case where A is symmetric positive definite, we have $\lambda_{\max}(P) = \frac{1}{2\lambda_{\min}(A)}$ and $\lambda_{\min}(P) = \frac{1}{2\lambda_{\max}(A)}$. Consequently, $M = \sqrt{\lambda_{\max}(A)/\lambda_{\min}(A)}$ and $b = \lambda_{\min}(A)$. Therefore, for any $\varepsilon > 0$, if $t \geq \lambda_{\min}(A)^{-1} \log\left(\frac{\sqrt{\lambda_{\max}(A)/\lambda_{\min}(A)}}{\varepsilon}\right)$, then $\|e^{-At}\| \leq \varepsilon$. Thus, larger $\lambda_{\min}(A)$ (and a smaller condition number $\lambda_{\max}(A)/\lambda_{\min}(A)$) corresponds to faster convergence to stationarity. The proof of Proposition B.3 is deferred to Appendix C.2.

B.4 Details for Examples

For Assumption 3.6, the dominant source of stochasticity in the optimization/RL examples is mini-batch (or sample) noise, which generally depends on the current iterate (x_n, y_n). The moment conditions in Assumption 3.6 are ensured when the iterates remain in a bounded region; this is standard in stochastic approximation and can be enforced, when needed, by a projected (or truncated) variant of the algorithm. Such a modification does not change the local asymptotic behavior as long as $(x^\star, y^\star)$ lies in the interior of the projection set, so the projection is inactive in a neighborhood of the limit. Moreover, mean-square convergence implies $(x_n, y_n) \to (x^\star, y^\star)$ in probability; combined with standard continuity/uniform integrability conditions on the noise moments, this yields convergence in probability of the corresponding conditional covariance quantities.

For Assumption 4.1, the verification for Example 4.1 is provided in the main text. For the Examples 4.2 and 4.3, the strong monotonicity conditions required in Proposition B.1 are satisfied as well. Moreover, as discussed after Proposition B.1, the conclusion continues to hold with appropriately chosen step sizes.

B.4.1 Details for Example 4.3

Consider an infinite-horizon MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, R, \gamma)$ with discounted rewards, where $\mathcal{S}$ and $\mathcal{A}$ denote the (finite) state space and action space, respectively, and $\gamma \in [0, 1)$ is the discount factor. The transition kernel $\mathcal{P} \colon \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$ specifies the probability distribution over the next states for each state-action pair $(s, a) \in \mathcal{S} \times \mathcal{A}$, where $\mathcal{P}(\cdot|s, a) \in \Delta(\mathcal{S})$ denotes the distribution of transitions from state s when action a is executed. The (possibly random) reward function $R \colon \mathcal{S} \times \mathcal{A} \to [0, 1]$ represents the immediate reward received from state s when action a is taken.

A policy $\pi \colon \mathcal{S} \to \Delta(\mathcal{A})$ maps each state to a probability distribution over actions. Its quality is assessed via the value function

$$V^\pi(s) := \mathbb{E}\left[\sum_{t=0}^\infty \gamma^t R(s_t, a_t) \,|\, s_0 = s,\ a_t \sim \pi(\cdot|a_t),\ s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t)\right],$$

which gives the expected discounted cumulative reward for following policy π from initial state s. The process of estimating V^π for a fixed policy π is known as *policy evaluation*. When the data is generated under a different policy π_b to establish the value function V^π of π, the task is referred to as *off-policy evaluation* [30]. When $|\mathcal{S}|$ is large, directly estimating V^π becomes computationally infeasible. Instead, V^π can be approximated using a parameterized function. A straightforward approach is linear function approximation, where $V^\pi(s) \approx V(s;y) = \phi(s)^\top y$, with $\phi(s)$ representing the feature vector of state s [8].

To find the optimal parameter y, both GTD2 are TDC are derived by minimizing the *mean squared projected Bellman error* through SGD [29]. In Example 4.3, given i.i.d. samples $\{(s_n, a_n, R(s_n, a_n), s'_n)\}_{n=0}^\infty$ with $a_n \sim \pi_b(\cdot|s_n)$ and $s'_n \sim \mathcal{P}(\cdot|s_n, a_n)$, we define $r_n = R(s_n, a_n)$, and $\rho_n = \frac{\pi(a_n|s_n)}{\pi_b(a_n|s_n)}$. Note in this setting s'_n does not necessarily equal s_{n+1}. The importance weight ρ_n adjusts for the mismatch between π and π_b. This weight does not appear in the original formulations of GTD2 and TDC in [29] due to the use of a sub-sampling procedure [23]. We incorporate this weight in update rules (7.17)–(7.19) to align with subsequent works, e.g., [19, 34].

As discussed in Section 4, deriving $\widetilde{\Sigma}_\psi$ in (7.12) requires analyzing the asymptotic covariance of $\psi_n - B_2 B_1^{-1}\xi_n$. Suppose (x_n, y_n) converges to $(x^\star, y^\star)$ in probability, as guaranteed by the results in [23]. For GTD2, we have $B_1 = C$ and $B_2 = -A^\top$, leading to

$$\begin{aligned}\psi_n^{\mathrm{GTD2}} - B_2 B_1^{-1}\xi_n &= -(A_n^\top - A^\top)x_n + A^\top C^{-1}[(C_n - C)x_n + (A_n - A)y_n - (b_n - b)] \\ &= A^\top C^{-1}(A_n A^{-1} b - b_n) + o_p(1)\end{aligned}$$

For TDC, $B_1 = C$ and $B_2 = D^\top = C - A^\top$. Then

$$\begin{aligned}\psi_n^{\mathrm{TDC}} - B_2 B_1^{-1}\xi_n &= (D_n^\top - D^\top)x_n + (A_n - A)y_n - (b_n - b) \\ &\quad - (C - A^\top)C^{-1}[(C_n - C)x_n + (A_n - A)y_n - (b_n - b)] \\ &= A^\top C^{-1}(A_n A^{-1} b - b_n) + o_p(1).\end{aligned}$$

Thus, both GTD2 and TDC share the same $\widetilde{\Sigma}_\psi$.

B.5 Q-learning with Polyak-Ruppert averaging

We continue with the notation in Appendix B.4 and restrict our attention on finite state space and action space. For a fixed policy π, its Q-function is defined as

$$Q^\pi(s,a) := \mathbb{E}\Big[\sum_{t=0}^\infty \gamma^t R(s_t, a_t) \,|\, s_0 = s,\ a_0 = a,\ s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t),\ a_{t+1} \sim \pi(\cdot|a_{t+1})\Big],$$

which is the expected discounted cumulative reward for following policy π from initial state s and initial action a. For fixed s and a, define the expectation of reward as $r(s,a) = \mathbb{E}[R(s,a)]$. The optimal Q-function is defined as $Q^\star = \max_\pi Q^\pi(s,a)$ and is the unique fixed-point of the Bellman optimality equation

$$Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim \mathcal{P}(\cdot|s,a)} \max_{a' \in \mathcal{A}} Q(s', a') =: (\mathcal{T}Q)(s,a).$$

Viewing Q as a vector in $\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, we define the Bellman optimality operator $\mathcal{T}\colon \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \to \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ as the right-hand side, then $Q^\star$ is the unique fixed point of $\mathcal{T}$, i.e., $Q^\star = \mathcal{T}Q^\star$.

Q-learning is a model-free algorithm to seek the optimal $Q^\star$ [32]. Similar to Example 4.1, Polyak-Ruppert averaging could improve the convergence rate [21]. We focus on the synchronous setting where a generative model produces independent samples for all state-action pairs in every iteration [17]. Then tabular Q-learning with PR averaging can be written as

$$
\begin{aligned}
x_{n+1} &= x_n - \alpha_n\big(x_n - \mathcal{T}x_n + \xi_n\big), \qquad \alpha_n = \alpha_0(n+1)^{-a}, \quad a \in (1/2, 1), \\
y_{n+1} &= \frac{1}{n+1}\sum_{\tau=0}^{n} x_\tau \;=\; y_n - \beta_n(y_n - x_n), \qquad \beta_n = \frac{1}{n+1}.
\end{aligned}
\tag{B.1}
$$

The recursion (B.1) is therefore an instance of two-time-scale SA with $F(x, y) = x - \mathcal{T}x$ and $G(x, y) = y - x$. Consequently, $x^\star = y^\star = Q^\star$, and the unique solution of $F(x, y) = 0$ is given by $H(y) \equiv Q^\star$, where $Q^\star$ is the unique fixed point of $\mathcal{T}$. Since G is linear, it suffices to verify the local linearity condition for F.

To this end, we introduce the greedy policy induced by Q: for any Q, define $\pi_Q(\cdot \mid s)$ by $\pi_Q(a \mid s) = \mathbf{1}\{a \in \arg\max_{a' \in \mathcal{A}} Q(s, a')\}$. When the argmax is not unique, we break ties arbitrarily (e.g., at random) to obtain a deterministic policy. Let $\pi^\star := \pi_{Q^\star}$ denote the (deterministic) optimal policy. Viewing r as a vector in $\mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, we can write

$$
\mathcal{T}Q = r + \gamma P^{\pi_Q} Q, \qquad P^{\pi_Q}_{(s,a),(s',a')} := \mathcal{P}(s' \mid s, a)\, \pi_Q(a' \mid s').
$$

If the optimal policy $\pi^\star$ is unique, then there exists $L > 0$ such that, for any Q,

$$
\big\|(P^{\pi_Q} - P^{\pi^\star})(Q - Q^\star)\big\|_\infty \le L\|Q - Q^\star\|_\infty^2 \quad \text{[21, Lemma B.1]}.
$$

Intuitively, the uniqueness of $\pi^\star$ implies that, whenever Q is sufficiently close to $Q^\star$, the greedy policy π_Q coincides with $\pi^\star$; since the set of deterministic policies is finite, the Bellman optimality operator becomes locally affine around $Q^\star$.

For $F(x, y) = x - \mathcal{T}x$, we have $H(y) \equiv Q^\star$ and hence

$$
F(x, y) = x - \mathcal{T}x = (x - Q^\star) + (\mathcal{T}Q^\star - \mathcal{T}x) = (x - Q^\star) + \gamma P^{\pi^\star} Q^\star - \gamma P^{\pi_x} x,
$$

where π_x denotes the greedy policy induced by x (with a fixed tie-breaking rule), and we use $\mathcal{T}Q = r + \gamma P^{\pi_Q} Q$. Let $B_1 := I - \gamma P^{\pi^\star}$. We claim that

$$
\|F(x, y) - B_1(x - Q^\star)\|_\infty \le \gamma L \|x - Q^\star\|_\infty^2. \tag{B.2}
$$

Since all norms are equivalent in finite dimensions, the same bound also holds for the Euclidean norm up to a dimension-dependent constant. We defer the verification of (B.2) later.

With (B.2), Assumptions 3.1 – 3.5 holds with $\delta_F = \delta_G = \delta_H = 1$, $B_1 = 1 - \gamma P^{\pi^\star}$, $B_2 = -I$ and $B_3 = I$. Since $\gamma \in [0, 1)$ and $P^{\pi^\star}$ is a transition matrix, the eigenvalues of B_1 have positive real parts and $-B_1$ is Hurwitz. By [20, Theorem 2], we can obtain $\mathbb{E}\|x_n - Q^\star\|^4 = \mathcal{O}(\alpha_n^2)$ and conquently $\mathbb{E}\|x_n - Q^\star\|^2 = \mathcal{O}(\alpha_n)$. A remaining issue is whether one can further show $\mathbb{E}|y_n - Q^\star|^2 = \mathcal{O}(1/n)$. Since [21] establishes the asymptotic distribution of $\sqrt{n}, (y_n - Q^\star)$ and the rate $\mathbb{E}|y_n - Q^\star|_\infty = \mathcal{O}(n^{-1/2})$, we believe this gap can be filled.

This example is similar to Example 4.1, except that $F(x, y)$ is not the gradient of an objective function and B_1 is not symmetric. Moreover, unlike [21], which establishes the FCLT for the partial-sum process via a uniform rescaling, our results yield an alternative characterization based on an iteration-dependent scaling. Following the analysis in Example 4.1, one can also obtain that, within the family of weighted averages with $\beta_n = \frac{\beta_0}{n+1}$, the vanilla average ($\beta_0 = 1$) is asymptotically optimal in the sense of minimizing the estimator's asymptotic covariance.

Now we verify (B.2). Note that

$$F(x,y) - B_1(x - Q^\star) = \big[(x - \mathcal{T}x) - (I - \gamma P^{\pi^\star})(x - Q^\star)\big] = \gamma\big(P^{\pi^\star} - P^{\pi_x}\big)x.$$

For each (s,a), we have

$$\begin{aligned}
\big[(P^{\pi^\star} - P^{\pi_x})x\big](s,a) &= \sum_{s'\in\mathcal{S}}\sum_{a'\in\mathcal{A}} \mathcal{P}(s' \mid s,a)\, x(s',a')\big(\pi^\star(a' \mid s') - \pi_x(a' \mid s')\big) \\
&= \sum_{s'\in\mathcal{S}} \mathcal{P}(s' \mid s,a)\Big(\sum_{a'\in\mathcal{A}} \pi^\star(a' \mid s')x(s',a') - \max_{a'\in\mathcal{A}} x(s',a')\Big) \le 0,
\end{aligned}$$

by the definition of the greedy policy. Therefore, $F(x,y) - B_1(x - Q^\star) \le 0$ elementwise. Moreover, we can decompose

$$(P^{\pi^\star} - P^{\pi_x})x = (P^{\pi^\star} - P^{\pi_x})(x - Q^\star) + (P^{\pi^\star} - P^{\pi_x})Q^\star.$$

By the same argument as above, $(P^{\pi^\star} - P^{\pi_x})Q^\star \ge 0$ elementwise. Combining the two elementwise inequalities yields

$$\big|(P^{\pi^\star} - P^{\pi_x})x\big| \le (P^{\pi_x} - P^{\pi^\star})(x - Q^\star) \quad \text{elementwise},$$

and hence

$$\|F(x,y) - B_1(x - Q^\star)\|_\infty \le \gamma\big\|(P^{\pi_x} - P^{\pi^\star})(x - Q^\star)\big\|_\infty \le \gamma L\|x - Q^\star\|_\infty^2,$$

where the last inequality follows from [21, Lemma B.1].

Finally, we note that a closely related example is SSP Q-learning, which is designed for average-reward MDPs [1]. SSP Q-learning can also be interpreted as a two-time-scale SA scheme, where the fast iterate corresponds to the Q-function and the slow iterate tracks a reference (baseline) variable. This example has been analyzed within the two-time-scale SA framework in [7]. Verifying whether SSP Q-learning satisfies our local linearity condition is left for future work.

C Omitted Proofs for Auxiliary Results in Appendices

C.1 Proofs for Appendix A

Proof of Proposition A.2. We first prove the inequality for F.
Suppose that (A.2) hold for $\max\{\|x - H(y)\|, \|y - y^*\|\} \le \varepsilon$. If $\max\{\|x - H(y)\|, \|y - y^*\|\} > \varepsilon$, we can divide into the following two cases.

(i) $\|x - H(y)\| > \varepsilon$: From the definition of H, we have $F(H(y), y) = 0$. Then we have

$$\begin{aligned}
\|F(x,y) - B_1(x - H(y))\| &\le \|F(x,y) - F(H(y),y)\| + \|B_1\|\|x - H(y)\| \\
&\overset{(a)}{\le} (L_F + \|B_1\|)\|x - H(y)\| \le \frac{L_F + \|B_1\|}{\varepsilon^{\delta_F}}\|x - H(y)\|^{1+\delta_F},
\end{aligned}$$

where (a) applies the Lipschitz condition of F in Assumption 3.1.

(ii) $\|x - H(y)\| < \varepsilon$ and $\|y - y^*\| \geq \varepsilon$: Similar to the former case, we have

$$\begin{aligned}\|F(x,y) - B_1(x - H(y))\| &\leq (L_F + \|B_1\|)\|x - H(y)\| \leq (L_F + \|B_1\|)\|y - y^*\| \\ &\leq \frac{L_F + \|B_1\|}{\varepsilon^{\delta_F}}\|y - y^\star\|^{1+\delta_F}.\end{aligned}$$

In sum, if (A.2) hold for $\max\{\|x - H(y)\|, \|y - y^*\|\} \leq \varepsilon$, we can remove this requirement by replace $S_{B,F}$ with $\max\left\{S_{B,F}, \frac{L_F+\|B_1\|}{\varepsilon^{\delta_F}}\right\}$.
Next, we focus the inequality for G. Suppose that (A.3) hold for $\max\{\|x - H(y)\|, \|y - y^*\|\} \leq \varepsilon$. Since $x^* = H(y^*)$, we have $G(H(y^*), y^* = 0)$. Then we have

$$\begin{aligned}&\|G(x,y) - B_2(x - H(y)) - B_3(y - y^\star))\| \\ &\leq \|G(x,y) - G(H(y),y)\| + \|G(H(y),y) - G(H(y^*),y^*)\| \\ &\quad + \|B_2\|\|x - H(y)\|\| + \|B_3\|\|y - y^*\| \\ &\overset{(b)}{\leq} (L_{G,x} + \|B_2\|)\|x - H(y)\| + (L_{G,y} + \|B_3\|)\|y - y^*\|,\end{aligned}$$

where (b) applies the Lipschitz condition of F in Assumption 3.1. If $\max\{\|x - H(y)\|, \|y - y^*\|\} > \varepsilon$, we can divide into the following three cases.

(i) $\|x - H(y)\| > \varepsilon$ and $\|y - y^*\| \leq \varepsilon$: We have

$$\begin{aligned}\|G(x,y) - B_2(x - H(y)) - B_3(y - y^\star))\| &\leq (L_{G,x} + \|B_2\| + L_{G,y} + \|B_3\|)\|x - H(y)\| \\ &\leq \frac{L_{G,x} + \|B_2\| + L_{G,y} + \|B_3\|}{\varepsilon^{\delta_G}}\|x - H(y)\|^{1+\delta_G}.\end{aligned}$$

(ii) $\|x - H(y)\| \leq \varepsilon$ and $\|y - y^*\| > \varepsilon$: Similar to the former case, we can obtain

$$\|G(x,y) - B_2(x - H(y)) - B_3(y - y^\star))\| \leq \frac{L_{G,x} + \|B_2\| + L_{G,y} + \|B_3\|}{\varepsilon^{\delta_G}}\|y - y^*\|^{1+\delta_G}.$$

(iii) $\|x - H(y)\| > \varepsilon$ and $\|y - y^*\| > \varepsilon$: we have

$$\begin{aligned}&\|G(x,y) - B_2(x - H(y)) - B_3(y - y^\star))\| \\ &\leq \frac{L_{G,x} + \|B_2\|}{\varepsilon^{\delta_G}}\|x - H(y)\|^{1+\delta_G} + \frac{L_{G,y} + \|B_3\|}{\varepsilon^{\delta_G}}\|y - y^*\|^{1+\delta_G}.\end{aligned}$$

In sum, if (A.3) hold for $\max\{\|x - H(y)\|, \|y - y^*\|\} \leq \varepsilon$, we can remove this requirement by replace $S_{B,G}$ with $\max\left\{S_{B,G}, \frac{L_{G,x}+\|B_2\|+L_{G,y}+\|B_3\|}{\varepsilon^{\delta_G}}\right\}$. □

Proof of Proposition A.3. For F, Since $\|x - H(y)\| \leq \|x - x^*\| + \|H(y^*) - H(y)\| \leq \|x - x^*\| + L_H\|y - y^*\|$ and $F(H(y), y) = 0$, we have

$$\begin{aligned}&\|F(x,y) - A_{11}(x - x^\star) - A_{12}(y - y^\star)\| \\ &\leq \|F(x,y) - F(H(y),y)\| + \|A_{11}\|\|x - x^*\| + \|A_{12}\|\|y - y^*\| \\ &\leq (L_F + \|A_{11}\|)\|x - x^*\| + (L_F L_H + \|A_{12}\|)\|y - y^*\|.\end{aligned}$$

For G, since $G(H(y^*), y^*) = 0$, we can also obtain

$$
\begin{aligned}
&\|G(x,y) - A_{21}(x - x^\star) - A_{22}(y - y^\star)\| \\
&\le \|G(x,y) - G(H(y),y)\| + \|G(H(y),y) - G(H(y^*),y^*)\| \\
&\quad + \|A_{11}\|\|x - x^*\| + \|A_{12}\|\|y - y^*\| \\
&\le (L_{G,x} + \|A_{21}\|)\|x - x^*\| + (L_{G,x}L_H + L_{G,y} + \|A_{22}\|)\|y - y^*\|.
\end{aligned}
$$

Then similar to the proof of Proposition A.2, we can remove the requirement $\max\{\|x-x^*\|, \|y-y^*\|\} \le \varepsilon$ by replacing $S_{A,F}$ with $\tilde{S}_{A,F} := \max\left\{S_{A,F}, \frac{L_F+\|A_{11}\|+L_F L_H+\|A_{12}\|}{\|\varepsilon\|^{\delta_F}}\right\}$ and replacing $S_{A,G}$ with $\tilde{S}_{A,G} := \max\left\{S_{A,G}, \frac{L_{G,x}+\|A_{21}\|+L_{G,x}L_H+L_{G,y}+\|A_{22}\|}{\|\varepsilon\|^{\delta_G}}\right\}$. For the remaining proof, please refer to [15, Proposition 2] and the proof therein. □

Proof of Proposition A.4. Combining (7.1) and Assumption 3.2, we have

$$
\|H(y_1) - H(y_2) - \nabla H(y_2)(y_1 - y_2)\| \le S_H\|y_1 - y_2\| \cdot \min\left\{\|y_1 - y_2\|^{\delta_H}, R_H\right\}
$$

with $R_H = \frac{2L_H}{S_H}$. For any $\delta'_H \in [0, \delta_H]$, if $\|y_1 - y_2\|^{\delta_H} \le R_H$, then we have

$$
\|H(y_1) - H(y_2) - \nabla H(y_2)(y_1 - y_2)\| \le S_H R_H^{\delta_H - \delta'_H}\|y_1 - y_2\|^{1+\delta'_H};
$$

otherwise, we have

$$
\|H(y_1) - H(y_2) - \nabla H(y_2)(y_1 - y_2)\| \le S_H R_H\|y_1 - y_2\| \le S_H R_H / R_H^{\delta'_H}\|y_1 - y_2\|^{1+\delta'_H}.
$$

Thus, by replacing S_H with $\max\left\{S_H R_H^{\delta_H - \delta'_H}, S_H R_H / R_H^{\delta'_H}\right\}$, Assumption 3.2 holds up to order $1 + \delta'_H$. □

Proof of Proposition A.5. For ease of exposition, we let $\hat{x} = x - H(y)$ and $\hat{y} = y - y^\star$. First, these assumptions imply

$$
\begin{aligned}
\|F(x,y) - B_1\hat{x}\| &= \|F(x,y) - F(H(y),y) - B_1\hat{x}\| \le 2L_F\|\hat{x}\| + \varepsilon \cdot \|\hat{y}\|, \\
\|G(x,y) - B_2\hat{x} - B_3\hat{y})\| &= \|G(x,y) - G(H(y),y) + G(H(y),y) - G(H(y^\star),y^\star) - B_2\hat{x} - B_3\hat{y}\| \\
&\le 2L_{G,x}\|\hat{x}\| + 2L_{G,y}\|\hat{y}\|,
\end{aligned}
$$

where ε is an arbitrary positive constant. In the following, we only prove that (7.3) holds after replacing δ_G with $\delta'_G \in (0, \delta_G)$. The proof for (7.2) is similar.

If (i) $S_{B,G}\|\hat{x}\|^{1+\delta_G} \le 2L_{G,x}\|\hat{x}\|$ and $S_{B,G}\|\hat{y}\|^{1+\delta_G} \le 2L_{G,y}\|\hat{y}\|$ or (ii) $S_{B,G}\|\hat{x}\|^{1+\delta_G} > 2L_{G,x}\|\hat{x}\|$ and $S_{B,G}\|\hat{y}\|^{1+\delta_G} > 2L_{G,y}\|\hat{y}\|$, we have

$$
\begin{aligned}
\|G(x,y) - B_2\hat{x} - B_3\hat{y})\| &\le \min\left\{S_{B,G}\|\hat{x}\|^{1+\delta_G} + S_{B,G}\|\hat{y}\|^{1+\delta_G}, 2L_{G,x}\|\hat{x}\| + 2L_{G,y}\|\hat{y}\|\right\} \\
&= \min\left\{S_{B,G}\|\hat{x}\|^{1+\delta_G}, 2L_{G,x}\|\hat{x}\|\right\} + \min\left\{S_{B,G}\|\hat{y}\|^{1+\delta_G}, 2L_{G,y}\|\hat{y}\|\right\} \\
&= S_{B,G}\|\hat{x}\| \cdot \min\left\{\|\hat{x}\|^{\delta_G}, \frac{2L_{G,x}}{S_{B,G}}\right\} + S_{B,G}\|\hat{y}\| \cdot \min\left\{\|\hat{y}\|^{\delta_G}, \frac{2L_{G,y}}{S_{B,G}}\right\}.
\end{aligned}
$$

Then the remaining proof is similar to that of Proposition A.4.

If neither of (i) and (ii) holds, without loss of generality we assume $S_{B,G}\|\hat{x}\|^{1+\delta_G} \le 2L_{G,x}\|\hat{x}\|$ and $S_{B,G}\|\hat{y}\|^{1+\delta_G} > 2L_{G,y}\|\hat{y}\|$. This implies $\|\hat{x}\| \le c_x := (2L_{G,x}/S_{B,G})^{1/\delta_G}$ and $\|\hat{y}\| > c_y := (2L_{G,y}/S_{B,G})^{1/\delta_G}$. Then we have

$$\begin{aligned}
\|G(x,y) - B_2\hat{x} - B_3\hat{y})\| &\le \min\left\{S_{B,G}\|\hat{x}\|^{1+\delta_G} + S_{B,G}\|\hat{y}\|^{1+\delta_G}, 2L_{G,x}\|\hat{x}\| + 2L_{G,y}\|\hat{y}\|\right\} \\
&\le 2L_{G,x}\|\hat{x}\| + 2L_{G,y}\|\hat{y}\| \le 2L_{G,x}\, c_x + 2L_{G,y}\|\hat{y}\| \\
&\le \left(\frac{2L_{G,x}c_x}{c_y^{1+\delta'_G}} + \frac{2L_{G,y}}{c_y^{\delta'_G}}\right)\|\hat{y}\|^{1+\delta'_G}.
\end{aligned}$$

Thus (7.3) holds with δ_G and $S_{B,G}$ replaced with δ'_G and $\frac{2L_{G,x}c_x}{c_y^{1+\delta'_G}} + \frac{2L_{G,y}}{c_y^{\delta'_G}}$ respectively. □

Proof of Proposition A.6 . We first prove Part (i). By Assumption 3.3 and the definition of $\hat{x}_n$ and $\hat{y}_n$ in (7.4), we have $\|F(x_n,y_n) - B_1\hat{x}_n\| \le S_{B,G}(\|\hat{x}_n\|^{1+\delta_F} + \|\hat{y}_n\|^{1+\delta_F})$. Setting $R_n^F = F(x_n,y_n) - B_1\hat{x}_n$ yields that $\|R_n^F\| = \mathcal{O}(\|\hat{x}_n\|^{1+\delta_F} + \|\hat{y}_n\|^{1+\delta_F})$. By Proposition A.5, for any $\delta'_F \in (0,\delta_F]$, we also have $\|R_n^F\| = \mathcal{O}(\|\hat{x}_n\|^{1+\delta'_F} + \|\hat{y}_n\|^{1+\delta'_F})$. The proof of Part (ii) is similar to that of Part (i) by setting $R_n^G = G(x_n,y_n) - B_2\hat{x}_n - B_3\hat{y}_n$.

For Part (iii), we first set $R_n^H = H(y_{n+1}) - H(y_n) - \nabla H(y_n)(y_{n+1} - y_n)$. Then by Assumption 3.2 and Proposition A.4, for any $\delta'_H \in (0,\delta_H]$, we have $\|R_n^H\| = \mathcal{O}(\|y_{n+1} - y_n\|^{1+\delta'_H})$. With $R_n^{\nabla H} = (\nabla H(y_n) - \nabla H(y^\star))(y_{n+1} - y_n)$, we have $H(y_{n+1}) - H(y_n) = H^\star(y_{n+1} - y_n) + R_n^{\nabla H} + R_n^H$. □

C.2 Proofs for Appendix B.2

Proof of Lemma B.1. We first prove the result for the sequence $\{\beta_n\}$. From the definition of $N^\beta(n,t)$ in Definition B.1, we must have $N^\beta(n,t) \ge n$. Then the monotonicity of $n\beta_n$ in Assumption 3.4 implies $\frac{\beta_n}{\beta_{N^\beta(n,t)}} \le \frac{N^\beta(n,t)}{n}$. It remains to derive the upper bound of $\frac{N^\beta(n,t)}{n}$.

From Assumption 3.4, we have $\frac{\beta_{n-1}}{\beta_n} = 1 + \mathcal{O}(\beta_n) = 1 + \mathcal{O}(\beta_{n-1})$. It follows that $\frac{1}{\beta_n} = \frac{1}{\beta_{n-1}} + \mathcal{O}(1) = \cdots = \mathcal{O}(n)$, implying $\beta_n = \Omega\big(\frac{1}{n}\big)$. Consequently, there exists c_β such that $\beta_n \ge \frac{c_\beta}{n}$ for any $n \in \mathbb{N}$. Meanwhile, for a fixed n, there exists a unique integer $k(n)$ such that $2^{k(n)-1} \le n < 2^{k(n)}$. Then we have that for any $k' \ge k(n)$,

$$\beta_{2^{k'}+1} + \cdots + \beta_{2^{k'+1}} \ge c_\beta\left(\frac{1}{2^{k'}+1} + \frac{1}{2^{k'}+2} + \cdots \frac{1}{2^{k'+1}}\right) \ge \frac{c_\beta}{2}.$$

It follows that with $m(n) := 2^{k(n)+\lceil 2t/c_\beta\rceil}$,

$$\Gamma^\beta_{n,m(n)+1} = \sum_{l=n}^{m(n)} \beta_l > \sum_{l=2^{k(n)}+1}^{2^{k(n)+\lceil 2t/c_\beta\rceil}} \beta_l \ge \left\lceil\frac{2t}{c_\beta}\right\rceil \frac{c_\beta}{2} \ge t.$$

From Definition B.1, we have $N^\beta(n,t) \le m(n) = 2^{k(n)+\lceil 2t/c_\beta\rceil}$. Recall that $n \ge 2^{k(n)-1}$. It follows that $\frac{N^\beta(n,t)}{n} \le 2^{\lceil 2t/c_\beta\rceil+1}$.

For the sequence $\{\alpha_n\}$, Assumption 3.4 guarantees that $n\alpha_n$ is non-decreasing and $\frac{\alpha_{n-1}}{\alpha_n} = 1 + o(\beta_n) = 1 + \mathcal{O}(\alpha_n)$. Then following similar procedure, we have that there exists $c_\alpha > 0$ such that $\frac{\alpha_n}{\alpha_{N^\alpha(n,t)}} \le \frac{N^\alpha(n,t)}{n} \le 2^{\lceil 2t/c_\alpha\rceil+1}$. □

Proof of Proposition B.2. The proof for $N^{\alpha}(n,t)$ and $N^{\beta}(n,t)$ is identical and without loss of generality, we focus on the latter. Recall the definition of $\Gamma^{\beta}_{n,m}$ in Definition B.1, for $\beta_n = \frac{\beta_0}{(n+1)^{\beta}}$, we have $\Gamma^{\beta}_{n,m} = \sum_{k=n+1}^{m} \frac{\beta_0}{k^{\beta}}$.

If $\beta = 1$, then $\Gamma^{\beta}_{n,m} = \sum_{k=n+1}^{m} \frac{\beta_0}{k} \in \left(\beta_0 \log \frac{m+1}{n+1}, \beta_0 \log \frac{m}{n}\right)$. From the definition of $N^{\beta}(n,t)$ in Definition B.1, we have $\Gamma^{\beta}_{n,N^{\beta}(n,t)} \leq t$ and $\Gamma^{\beta}_{n,N^{\beta}(n,t)+1} \geq t$. Then we have $\log \frac{N^{\beta}(n,t)+1}{n+1} \leq t/\beta_0 \leq \log \frac{N^{\beta}(n,t)+1}{n}$ and consequently

$$n(e^{t/\beta_0} - 1) - 1 \leq N^{\beta}(n,t) - n \leq n(e^{t/\beta_0} - 1) + e^{t/\beta_0} - 1.$$

Thus, $N^{\beta}(n,t) - n = (e^{t/\beta_0} - 1)n + o(n)$.

If $\beta < 1$, then $\Gamma^{\beta}_{n,m} = \sum_{k=n+1}^{m} \frac{\beta_0}{k^{\beta}} \in \left(\frac{\beta_0}{1-\beta}((m+1)^{1-\beta} - (n+1)^{1-\beta}), \frac{\beta_0}{1-\beta}(m^{1-\beta} - n^{1-\beta})\right)$. From $\Gamma^{\beta}_{n,N^{\beta}(n,t)} \leq t$ and $\Gamma^{\beta}_{n,N^{\beta}(n,t)+1} \geq t$, we can obtain

$$\left[(1-\beta)t/\beta_0 + n^{1-\beta}\right]^{1/(1-\beta)} - 1 \leq N^{\beta}(n,t) \leq \left[(1-\beta)t/\beta_0 + (n+1)^{1-\beta}\right]^{1/(1-\beta)} - 1.$$

For the left-hand side, we have $\left[(1-\beta)t/\beta_0 + n^{1-\beta}\right]^{1/(1-\beta)} - 1 = n\left[1 + (1-\beta)tn^{-(1-\beta)}/\beta_0\right]^{1/(1-\beta)} - 1 = n + tn^{\beta}/\beta_0 + o(n^{\beta})$. For the right-hand side, similarly, we have $\left[(1-\beta)t/\beta_0 + (n+1)^{1-\beta}\right]^{1/(1-\beta)} - 1 = n + tn^{\beta}/\beta_0 + o(n^{\beta})$. In sum, we have $N^{\beta}(n,t) - n = \Theta(n^{\beta})$. $\square$

Proof of Proposition B.3. Since $-A$ is a Hurwitz matrix, the Lyapunov equation $A^{\top}P + PA = I_d$ has a unique symmetric positive definite solution $P \in \mathbb{R}^{d\times d}$ [3, p. 475]. Define $p_{\max} := \lambda_{\max}(P), p_{\min} := \lambda_{\min}(P), \kappa(P) := p_{\max}/p_{\min}$. For any $x \in \mathbb{R}^d$ set $y(t) := e^{-At}x$. Differentiate

$$\frac{d}{dt}\left(y(t)^{\top}Py(t)\right) = -y(t)^{\top}(A^{\top}P + PA)y(t) = -\|y(t)\|^2.$$

Using $\|y\|^2 \geq \frac{1}{p_{\max}} y^{\top}Py$ gives $\frac{d}{dt}\left(y(t)^{\top}Py(t)\right) \leq -\frac{1}{p_{\max}} y(t)^{\top}Py(t)$. Then Gronwall's inequality yields $y(t)^{\top}Py(t) \leq e^{-t/p_{\max}} y(0)^{\top}Py(0) \leq e^{-t/p_{\max}} p_{\max}\|y(0)\|^2$. Hence, with $y(0) = x$,

$$\|e^{-At}\|^2 = \sup_{\|x\|=1} \|e^{-At}x\|^2 \leq \sup_{\|x\|=1} \frac{1}{p_{\min}} (e^{-At}x)^{\top}P(e^{-At}x) \leq \frac{p_{\max}}{p_{\min}} e^{-t/p_{\max}} = \kappa(P)e^{-t/p_{\max}},$$

where the first inequality is because $P - p_{\min}I$ is positive semidefinite. Then one can check the solution to $A^{\top}P + PA = I_d$ is $P = \int_0^{\infty} e^{-A^{\top}t}e^{-At}dt$, which is well-defined [3, Theorem 7.5]. Therefore we have

$$\|e^{-At}\| \leq Me^{-bt}, \forall t \geq 0, \text{ with } M := \sqrt{\kappa(P)}, b := \frac{1}{2p_{\max}}.$$

$\square$

Derivation of $\mathbb{P}(\sup_{0\leq t\leq T} \|\mathrm{Y}_{\infty}(t)\| \geq M) \leq \delta$

For ease of notation, we omit the subscipt of Y_{∞} and assume $\mathrm{Y}(\cdot) = (\mathrm{Y}^1(\cdot), \mathrm{Y}^2(\cdot), \ldots, \mathrm{Y}^{d_y}(\cdot))^{\top}$ is a stationary d_y-dimensional OU process. Then each component $\mathrm{Y}^i(\cdot)$ is a mean-zero Gaussian process. For a fixed T, by Borell–TIS inequality [2, Theorem 2.1.1], we have that for all $u \geq 0$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T} \mathrm{Y}^i(t) \geq \mathbb{E}\sup_{0\leq t\leq T} \mathrm{Y}^i(t) + u\right) \leq \exp\left(-\frac{u^2}{2\sup_{0\leq t\leq T} E(\mathrm{Y}^i(t))^2}\right).$$

Similarly, $\mathbb{P}\left(-\inf_{0\le t\le T}\mathrm{Y}^i(t)\ge -\mathbb{E}\inf_{0\le t\le T}\mathrm{Y}^i(t)+u\right)\le \exp\left(-\frac{u^2}{2\sup_{0\le t\le T}E(\mathrm{Y}^i(t))^2}\right)$. Note that $\sup_{0\le t\le T}|\mathrm{Y}^i(t)|=\max\{\sup_{0\le t\le T}\mathrm{Y}^i(t),-\inf_{0\le t\le T}\mathrm{Y}^i(t)\}$. By Dudley integral inequality [31, Theorem 8.1.3], we can obtain an upper bound for $\mathbb{E}\sup_{0\le t\le T}\mathrm{Y}^i(t)$ and $-\mathbb{E}\inf_{0\le t\le T}\mathrm{Y}^i(t)$ and denote this bound as A_T. Then we have

$$\mathbb{P}\left(\sup_{0\le t\le T}|\mathrm{Y}^i(t)|\ge A_T+u\right)\le 2\exp\left(-\frac{u^2}{2\sup_{0\le t\le T}\mathbb{E}(\mathrm{Y}^i(t))^2}\right).$$

Let $M=\sqrt{d_y}(A_T+u)$. Then we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T}\|\mathrm{Y}(t)\|\ge M\right)&=\mathbb{P}\left(\sup_{0\le t\le T}\sum_{i=1}^{d_y}(\mathrm{Y}^{(i)}(t))^2\ge M^2\right)\le\mathbb{P}\left(\sum_{i=1}^{d_y}\sup_{0\le t\le T}(\mathrm{Y}^{(i)}(t))^2\ge M^2\right)\\&\le\sum_{i=1}^{d_y}\mathbb{P}\left(\sup_{0\le t\le T}(\mathrm{Y}^{(i)}(t))^2\ge M^2/d_y^2\right)=\sum_{i=1}^{d_y}\mathbb{P}\left(\sup_{0\le t\le T}|\mathrm{Y}^{(i)}(t)|\ge M/\sqrt{d_y}\right)\\&\le 2d_y\exp\left(-\frac{u^2}{2\sigma_T^2}\right),\end{aligned}$$

where $\sigma_T^2:=\sup_{1\le i\le d_y}\sup_{0\le t\le T}\mathbb{E}(\mathrm{Y}^i(t))^2$. It suffices to set $\delta=2d_y\exp\left(-\frac{u^2}{2\sigma_T^2}\right)$ and we obtain $M=\sqrt{d_y}(A_T+\sigma_T\sqrt{2\log(2d_y/\delta)})$.

D Omitted Proofs for the Main Theorem

D.1 Proofs in Section 7.5.1

In this subsection, we give the proof of the one-step recursions in Section 7.5.1. Since the proofs of both Lemmas 5.1 and 5.2 are closely related, we combine their proof together.

Proof of Lemmas 5.1 and 5.2. The proof is structured into three parts: we derive the one-step recursion first for $\check{y}_n$, followed by $\check{x}_n$, and finally $\check{z}_n$. To begin, we take the update rules for $\hat{x}_n$ and $\hat{y}_n$, as outlined below.

$$\hat{x}_{n+1}=\hat{x}_n-\alpha_nF(x_n,y_n)-\alpha_n\xi_n+H(y_n)-H(y_{n+1}),\tag{D.1}$$
$$\hat{y}_{n+1}=\hat{y}_n-\beta_nG(H(y_n),y_n)+\beta_n(G(H(y_n),y_n)-G(x_n,y_n))-\beta_n\psi_n.\tag{D.2}$$

Part 1: One-step recursion for $\check{y}_n$. By Proposition A.6, we can rewrite the update rule of $\hat{y}_n$ in (D.2) as

$$\hat{y}_{n+1}=(I-\beta_nB_3)\hat{y}_n-\beta_nB_2\hat{x}_n-\beta_n\psi_n-\beta_nR_n^G.\tag{D.3}$$

Then from the definition of $\check{y}_n$ and (D.3), we have

$$\begin{aligned}\check{y}_{n+1}&=\sqrt{\frac{\beta_{n-1}}{\beta_n}}(I-\beta_nB_3)\check{y}_n-\sqrt{\beta_n\alpha_{n-1}}B_2\check{x}_n-\sqrt{\beta_n}\psi_n-\sqrt{\beta_n}R_n^G\\&=\left(I-\beta_nB_3+\frac{\tilde{\beta}\beta_n}{2}I\right)\check{y}_n-\sqrt{\beta_n\alpha_n}B_2\check{x}_n-\sqrt{\beta_n}\psi_n+R_n^y,\end{aligned}\tag{D.4}$$

where

$$R_n^y = \underbrace{\left(\sqrt{\frac{\beta_{n-1}}{\beta_n}} - 1 - \frac{\tilde{\beta}\beta_n}{2}\right)\check{y}_n}_{=:R_{n,1}^y} + \underbrace{(\beta_n - \sqrt{\beta_n\beta_{n-1}})B_3\check{y}_n}_{=:R_{n,2}^y}$$
$$+ \underbrace{(\sqrt{\beta_n\alpha_n} - \sqrt{\beta_n\alpha_{n-1}})B_2\check{x}_n}_{=:R_{n,3}^y} - \underbrace{\sqrt{\beta_n}R_n^G}_{=:R_{n,4}^y}. \tag{D.5}$$

By Assumption 3.4, we have

$$\sqrt{\frac{\beta_{n-1}}{\beta_n}} = \sqrt{1 + \beta_{n-1}(\beta_n^{-1} - \beta_{n-1}^{-1})} = \sqrt{1 + \tilde{\beta}\beta_{n-1} + o(\beta_{n-1})}$$
$$= \sqrt{1 + \tilde{\beta}\beta_n + o(\beta_n)} = 1 + \frac{\tilde{\beta}\beta_n}{2} + o(\beta_n),$$
$$\sqrt{\frac{\alpha_{n-1}}{\alpha_n}} = 1 + \mathcal{O}(\beta_n), \qquad \sqrt{\frac{\kappa_{n-1}}{\kappa_n}} = 1 + \mathcal{O}(\beta_n). \tag{D.6}$$

It follows that $\|R_{n,1}^y\| + \|R_{n,2}^y\| = o(\beta_n)\|\check{y}_n\| = o(\beta_n^{1/2})\|\hat{y}_n\|$ and $\|R_{n,3}^y\| = \mathcal{O}(\beta_n^{3/2}\alpha_n^{1/2})\|\check{x}_n\| = o(\beta_n^{3/2})\|\hat{x}_n\|$. Meanwhile, by Proposition A.6, we have $\|R_{n,4}^y\| = \mathcal{O}(\beta_n^{1/2})(\|\hat{x}_n\|^{1+\delta_G} + \|\hat{y}_n\|^{1+\delta_G})$. Note that in Assumption 3.4, we require $\alpha_n^{1+\delta_G} = o(\beta_n)$. Then we have

$$\|R_n^y\| = o(\beta_n^{1/2})\|\hat{y}_n\| + o(\beta_n^{3/2})\|\hat{x}_n\| + \mathcal{O}(\beta_n)\,\frac{\alpha_n^{(1+\delta_G)/2}}{\beta_n^{1/2}} \cdot \frac{\|\hat{x}_n\|^{1+\delta_G} + \|\hat{y}_n\|^{1+\delta_G}}{\alpha_n^{(1+\delta_G)/2}}. \tag{D.7}$$

By Assumption 4.1, it follows that $\mathbb{E}\|\mathbb{E}[R_n^y|\mathcal{F}_n]\| \le \mathbb{E}\|R_n^y\| = o(\beta_n)$ and $\mathbb{E}\|R_n^y\|^2 = o(\beta_n^2)$. Moreover, by Proposition A.5, we have that (D.7) holds with δ_G replaced by $\delta_G/2$. For $2 \le p \le \frac{4}{1+\delta_G/2}$, it follows that

$$\mathbb{E}\|R_n^y\|^p = o(\beta_n) + o(\beta_n \cdot \beta_n^{p-1}/\alpha_n^{p\delta_G/2}) = o(\beta_n),$$

where the last step is due to $p - 1 \ge p/2 \ge p\delta_G/2$.

Part 2: One-step recursion for $\check{x}_n$. By Proposition A.6, the update rule of $\hat{x}_n$ in (D.1) can be rewritten as

$$\begin{aligned}\hat{x}_{n+1} &= (I - \alpha_n B_1)\hat{x}_n + H^\star(y_n - y_{n+1}) - \alpha_n\xi_n - R_n^{\nabla H} - R_n^H - \alpha_n R_n^F \\ &\overset{(D.3)}{=} (I - \alpha_n B_1 + \beta_n H^\star B_2)\hat{x}_n + \beta_n H^\star B_3\hat{y}_n - \alpha_n\xi_n + \beta_n H^\star\psi_n \\ &\quad - R_n^{\nabla H} - R_n^H - \alpha_n R_n^F + \beta_n H^\star R_n^G.\end{aligned} \tag{D.8}$$

Then from the definition of $\check{x}_n$, we have

$$\begin{aligned}\check{x}_{n+1} = &\sqrt{\frac{\alpha_{n-1}}{\alpha_n}}(I - \alpha_n B_1 + \beta_n H^\star B_2)\check{x}_n + \sqrt{\frac{\beta_{n-1}}{\alpha_n}}\beta_n H^\star B_3\check{y}_n - \sqrt{\alpha_n}\xi_n + \frac{\beta_n}{\sqrt{\alpha_n}}H^\star\psi_n \\ &- \frac{1}{\sqrt{\alpha_n}}R_n^{\nabla H} - \frac{1}{\sqrt{\alpha_n}}R_n^H - \sqrt{\alpha_n}R_n^F + \frac{\beta_n}{\sqrt{\alpha_n}}H^\star R_n^G\end{aligned}$$

$$= (I - \alpha_n B_1)\check{x}_n - \sqrt{\alpha_n}\tilde{\xi}_n + R_n^x, \tag{D.9}$$

where $\tilde{\xi}_n := \xi_n - \kappa_n H^\star \psi_n = \xi_n - \frac{\beta_n}{\alpha_n} H^\star \psi_n$ and

$$\begin{aligned} R_n^x = &\underbrace{\left(\sqrt{\frac{\alpha_{n-1}}{\alpha_n}} - 1\right)\check{x}_n}_{=:R_{n,1}^x} + \underbrace{(\alpha_n - \sqrt{\alpha_n \alpha_{n-1}})B_1 \check{x}_n}_{=:R_{n,2}^x} + \underbrace{\sqrt{\frac{\alpha_{n-1}}{\alpha_n}}\beta_n H^\star B_2 \check{x}_n}_{=:R_{n,3}^x} \\ &+ \underbrace{\sqrt{\frac{\beta_{n-1}}{\alpha_n}}\beta_n H^\star B_3 \check{y}_n}_{=:R_{n,4}^x} - \underbrace{\frac{1}{\sqrt{\alpha_n}} R_n^{\nabla H}}_{=:R_{n,5}^x} - \underbrace{\frac{1}{\sqrt{\alpha_n}} R_n^H}_{=:R_{n,6}^x} - \underbrace{\sqrt{\alpha_n} R_n^F}_{=:R_{n,7}^x} + \underbrace{\frac{\beta_n}{\sqrt{\alpha_n}} H^\star R_n^G}_{=:R_{n,8}^x}. \end{aligned} \tag{D.10}$$

By (D.6), we have $\|R_{n,1}^x\| + \|R_{n,2}^x\| = \mathcal{O}(\beta_n)\|\check{x}_n\| = \mathcal{O}(\beta_n \alpha_n^{-1/2})\|\hat{x}_n\|$. By Assumption 3.4, we have $\|R_{n,3}^x\| + \|R_{n,4}^x\| = \mathcal{O}(\beta_n)\|\check{x}_n\| + \mathcal{O}(\beta_n^{3/2}\alpha_n^{-1/2})\|\check{y}_n\| = \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\| + \|\hat{y}_n\|)$. For $R_{n,5}^x$ and $R_{n,6}^x$, from the definition of $\mathcal{F}_n$, Assumption 3.1 and Proposition A.6, we have

$$\begin{aligned} \|\mathbb{E}[R_n^{\nabla H}|\mathcal{F}_n]\| &= \|(\nabla H(y_n) - \nabla H(y^\star))\mathbb{E}[y_{n+1} - y_n|\mathcal{F}_n]\| \\ &= \beta_n \|(\nabla H(y_n) - \nabla H(y^\star))G(x_n, y_n)\| \le \beta_n \widetilde{S}_H \|\hat{y}_n\|^{\delta_H}\|G(x_n, y_n)\| \\ \|R_n^{\nabla H}\| &\le \widetilde{S}_H \|\hat{y}_n\|^{\delta_H}\|y_{n+1} - y_n\|, \\ \|R_n^H\| &= \mathcal{O}(\|y_{n+1} - y_n\|^{1+\delta_H}), \\ \|G(x_n, y_n)\| &= \|G(x_n, y_n) - G(H(y_n), y_n) + G(H(y_n), y_n) - G(H(y^\star), y^\star)\| \\ &\le L_{G,x}\|\hat{x}_n\| + L_{G,y}\|\hat{y}_n\|, \\ \|y_{n+1} - y_n\| &\le \beta_n\|G(x_n, y_n)\| + \beta_n\|\psi_n\|. \end{aligned}$$

Then by Young's inequality and Jensen's inequality, we have

$$\begin{aligned} \|\mathbb{E}[R_{n,5}^x|\mathcal{F}_n]\| &= \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_H} + \|\hat{y}_n\|^{1+\delta_H}), \\ \|R_{n,5}^x\| &= \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_H} + \|\hat{y}_n\|^{1+\delta_H} + \|\hat{y}_n\|^{\delta_H}\|\psi_n\|), \\ \|R_{n,6}^x\| &= \mathcal{O}(\beta_n^{1+\delta_H}\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_H} + \|\hat{y}_n\|^{1+\delta_H} + \|\psi_n\|^{1+\delta_H}). \end{aligned}$$

Moreover, Proposition A.6 also implies $\|R_{n,7}^x\| = \mathcal{O}(\alpha_n^{1/2})(\|\hat{x}_n\|^{1+\delta_F} + \|\hat{y}_n\|^{1+\delta_F})$ and $\|R_{n,8}^x\| = \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_G} + \|\hat{y}_n\|^{1+\delta_G})$. Summarizing the above results, we obtain

$$\begin{aligned} \|\mathbb{E}[R_n^x|\mathcal{F}_n]\| = &\ \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\| + \|\hat{y}_n\|) + \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_H} + \|\hat{y}_n\|^{1+\delta_H}) \\ &+ \mathcal{O}(\beta_n^{1+\delta_H}\alpha_n^{-1/2})\|\psi_n\|^{1+\delta_H} + \mathcal{O}(\alpha_n^{1/2})(\|\hat{x}_n\|^{1+\delta_F} + \|\hat{y}_n\|^{1+\delta_F}) \\ &+ \mathcal{O}(\beta_n\alpha_n^{-1/2})(\|\hat{x}_n\|^{1+\delta_G} + \|\hat{y}_n\|^{1+\delta_G}), \\ \|R_n^x\| = &\ \mathcal{O}(\|\mathbb{E}[R_n^x|\mathcal{F}_n]\|) + \mathcal{O}(\beta_n\alpha_n^{-1/2})\|\hat{y}_n\|^{\delta_H}\|\psi_n\|. \end{aligned} \tag{D.11}$$

Note that Assumption 3.4 requires $\alpha_n^{1+\delta_F} = o(\beta_n)$, $\beta_n^{(1+\delta_H)/2} = \mathcal{O}(\alpha_n)$. By Assumption 4.1, we have $\mathbb{E}\|\mathbb{E}[R_n^x|\mathcal{F}_n]\| = \mathcal{O}(\beta_n + \beta_n^{1+\delta_H}\alpha_n^{-1/2} + \alpha_n^{1+\delta_F/2}) = o(\sqrt{\beta_n\alpha_n})$. Meanwhile, since $\hat{y}_n \in \mathcal{F}_n$, Assumption 3.6 implies $\mathbb{E}\|\hat{y}_n\|^{2\delta_H}\|\psi_n\|^2 = \mathcal{O}(\mathbb{E}\|\hat{y}_n\|^{2\delta_H})$. By Assumption 4.1, we have $\mathbb{E}\|R_n^x\|^2 = \mathcal{O}(\beta_n^2 + \beta_n^{2+\delta_H}\alpha_n^{-1} + \alpha_n^{2+\delta_F}) = \mathcal{O}(\beta_n\alpha_n)$. Moreover, by Propositions A.4 and A.5, we have that (D.11) holds after replacing $\delta_H, \delta_F, \delta_G$ with $\delta_H/2, \delta_F/2, \delta_G/2$ respectively. For $2 \le p \le \frac{4}{1+(\delta_H \vee \delta_F \vee \delta_G)/2}$, one can check $\mathbb{E}\|R_n^x\|^p = o(\alpha_n)$.

Part 3: One-step recursion for $\check{z}_n$. From the definition of $\check{z}_n$ and the update rules of $\check{y}_n$ and $\check{x}_n$ in (D.4) and (D.9) , we have

$$\begin{aligned}
\check{z}_{n+1} &= \check{y}_{n+1} - \sqrt{\kappa_n} B_2 B_1^{-1} \check{x}_{n+1} \\
&= \left(I - \beta_n B_3 + \frac{\tilde{\beta}\beta_n}{2} I\right)(\check{z}_n + \sqrt{\kappa_{n-1}} B_2 B_1^{-1} \check{x}_n) - \sqrt{\beta_n \alpha_n} B_2 \check{x}_n - \sqrt{\beta_n}\psi_n + R_n^y \\
&\quad - \sqrt{\kappa_n} B_2 B_1^{-1}\left[(I - \alpha_n B_1)\check{x}_n - \sqrt{\alpha_n}\tilde{\xi}_n + R_n^x\right] \\
&= \left(I - \beta_n B_3 + \frac{\tilde{\beta}\beta_n}{2} I\right)\check{z}_n - \sqrt{\beta_n}\tilde{\psi}_n + R_n^z,
\end{aligned}$$

where $\tilde{\psi}_n := \psi_n - B_2 B_1^{-1}\tilde{\xi}_n = (I + \kappa_n B_2 B_1^{-1} H^\star)\psi_n - B_2 B_1^{-1}\xi_n$ and

$$R_n^z = \underbrace{(\sqrt{\kappa_{n-1}} - \sqrt{\kappa_n}) B_2 B_1^{-1}\check{x}_n}_{=:R_{n,1}^z} - \underbrace{\sqrt{\kappa_{n-1}}\beta_n (B_3 - \tilde{\beta} I/2) B_2 B_1^{-1}\check{x}_n}_{=:R_{n,2}^z} + R_n^y - \sqrt{\kappa_n} B_2 B_1^{-1} R_n^x.$$

By (D.6), we have $\|R_{n,1}^z\| + \|R_{n,2}^z\| = o(\beta_n)\|\check{x}_n\|$. Moreover, from the definitions of R_n^y and R_n^x in (D.5) and (D.10) and the analysis in Parts 1 and 2, we can obtain $\mathbb{E}\|\mathbb{E}[R_n^z|\mathcal{F}_n]\| = o(\beta_n)$, $\mathbb{E}\|R_n^z\|^2 = \mathcal{O}(\beta_n^2)$ and $\mathbb{E}\|R_n^z\|^p = o(\beta_n)$ for $2 \le p \le \frac{4}{1+(\delta_H \vee \delta_F \vee \delta_G)/2}$. □

D.2 Proofs in Section 7.5.2

In this subsection, we give the proof of Proposition 5.1 and Lemma 5.3 in Section 7.5.2. The main challenge lies in establishing the tightness of $\{\bar{\mathrm{Y}}_n(\cdot)\}_{n=1}^\infty$, which is detailed in Part 3 of this proof.

First, we emphaze that

With Definition B.1, the construction in (7.6), (7.7) and (7.9) can be equivalently expressed as follows

$$\begin{aligned}
\bar{\mathrm{X}}_n(t) &\overset{(7.6)}{=} \check{x}_{N^\alpha(n,t)} + \frac{t - \underline{t}_n^\alpha}{\alpha_{N^\alpha(n,t)}}\left(\check{x}_{N^\alpha(n,t)+1} - \check{x}_{N^\alpha(n,t)}\right) \\
&\overset{(7.20)}{=} \check{x}_{N^\alpha(n,t)} + (t - \underline{t}_n^\alpha)\left(-B_1 \check{x}_{N^\alpha(n,t)} + \frac{R^x_{N^\alpha(n,t)}}{\alpha_{N^\alpha(n,t)}}\right) - \frac{t - \underline{t}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,t)}}}\tilde{\xi}_{N^\alpha(n,t)}.
\end{aligned} \tag{X}$$

$$\begin{aligned}
\bar{\mathrm{Y}}_n(t) &\overset{(7.7)}{=} \check{y}_{N^\beta(n,t)} + \frac{t - \underline{t}_n^\beta}{\beta_{N^\beta(n,t)}}\left(\check{y}_{N^\beta(n,t)+1} - \check{y}_{N^\beta(n,t)}\right) \\
&\overset{(7.21)}{=} \check{y}_{N^\beta(n,t)} + (t - \underline{t}_n^\beta)\left[\left(-B_3 + \frac{\tilde{\beta} I}{2}\right)\check{y}_{N^\beta(n,t)} + \frac{R^y_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}\right] - \frac{t - \underline{t}_n^\beta}{\sqrt{\beta_{N^\beta(n,t)}}}\psi_{N^\beta(n,t)} \\
&\quad - (t - \underline{t}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}} B_2 \check{x}_{N^\beta(n,t)}.
\end{aligned} \tag{Y}$$

$$\begin{aligned}\bar{\mathrm{Z}}_n(t) &\overset{(7.9)}{=} \check{z}_{N^\beta(n,t)} + \frac{t-\underline{t}_n^\beta}{\beta_{N^\beta(n,t)}}\big(\check{z}_{N^\beta(n,t)+1} - \check{z}_{N^\beta(n,t)}\big) \\ &\overset{(7.22)}{=} \check{z}_{N^\beta(n,t)} + (t-\underline{t}_n^\beta)\left[\Big(-B_3 + \frac{\tilde{\beta}I}{2}\Big)\check{z}_{N^\beta(n,t)} + \frac{R^z_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}\right] - \frac{t-\underline{t}_n^\beta}{\sqrt{\beta_{N^\beta(n,t)}}}\tilde{\psi}_{N^\beta(n,t)}.\end{aligned} \tag{Z}$$

Proof of Proposition 5.1. For each $U_n(\cdot)$, if we restrict the domain as $[0,T]$ for a fixed $T>0$, then $U_n(\cdot)|_{[0,T]}$ is also a random function in $C([0,T];\mathbb{R}^d)$. Equip $C([0,T];\mathbb{R}^d)$ with the norm $\rho_T(f,g) := \max_{0\le t\le T}\|f(t)-g(t)\|$, we can similarly define the tightness of $\big\{U_n(\cdot)|_{[0,T]}\big\}_{n=0}^\infty$ as Definition ??. [33, Corollary 5] guarantees that $\{U_n(\cdot)\}_{n=0}^\infty$ is tight in $C([0,\infty);\mathbb{R}^d)$ if and only if for each integer $j\ge 1$, $\big\{U_n(\cdot)|_{[0,j]}\big\}_{n=0}^\infty$ is tight in $C([0,j];\mathbb{R}^d)$. Then it suffices to establish the tightness of $\big\{U_n(\cdot)|_{[0,j]}\big\}_{n=0}^\infty$ for each $j\ge 1$.

To this end, we resort to [4, Theorem 7.3]. Although this theorem only focus on $C([0,1];\mathbb{R})$, we can generalize it effortlessly to $C([0,j];\mathbb{R}^d)$. Then it suffices to verify that for any $\varepsilon,\eta>0$, there exists $\delta\in(0,1)$ such that

$$\mathbb{P}\bigg(\sup_{s,t\in[0,j]:\,|s-t|\le\delta}\|U_n(s)-U_n(t)\|\ge\varepsilon\bigg)\le\eta,\quad \forall\, n\ge n_0.$$

By (7.23), we can set $\delta=\min\left\{\left(\frac{\eta\varepsilon^{c_1}}{2K}\right)^{1/(1+\gamma_1)},\left(\frac{\eta\varepsilon^{c_2}}{2K}\right)^{1/(1+\gamma_2)},0.5\right\}$ to ensure $K\left(\frac{\delta^{1+\gamma_1}}{\varepsilon^{c_1}}+\frac{\delta^{1+\gamma_2}}{\varepsilon^{c_2}}\right)\le\eta$. □

Proof of Lemma 5.3. We begin by proving the tightness of $\{\bar{\mathrm{X}}_n(\cdot)\}_{n=1}^\infty$. The tightness of $\{\bar{\mathrm{Z}}_n(\cdot)\}_{n=1}^\infty$ can be established using a similar approach. Finally, we address the tightness of $\{\bar{\mathrm{Y}}_n(\cdot)\}_{n=1}^\infty$, which is the primary challenge in this proof.

Part 1: Tightness of $\{\bar{\mathrm{X}}_n(\cdot)\}_{n=1}^\infty$. Note that Assumption 4.1 implies $\mathbb{E}\|\bar{\mathrm{X}}_n(0)\|^2=\mathbb{E}\|\check{x}_n\|^2=\mathcal{O}(1)$. Then by Markov's inequality, the first condition in Proposition 5.1 holds. Now we verify the second condition. Without loss of generality, assume $0\le t<s\le T$. From the definition of $\bar{\mathrm{X}}_n(t)$ in (X), we have

$$\begin{aligned}\bar{\mathrm{X}}_n(s)-\bar{\mathrm{X}}_n(t) &\overset{(X)}{=} \check{x}_{N^\alpha(n,s)} + (s-\underline{s}_n^\alpha)\left(-B_1\check{x}_{N^\alpha(n,s)}+\frac{R^x_{N^\alpha(n,s)}}{\alpha_{N^\alpha(n,s)}}\right) - \frac{s-\underline{s}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,s)}}}\tilde{\xi}_{N^\alpha(n,s)} \\ &\qquad - \check{x}_{N^\alpha(n,t)} - (t-\underline{t}_n^\alpha)\left(-B_1\check{x}_{N^\alpha(n,t)}+\frac{R^x_{N^\alpha(n,t)}}{\alpha_{N^\alpha(n,t)}}\right) + \frac{t-\underline{t}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,t)}}}\tilde{\xi}_{N^\alpha(n,t)} \\ &\overset{(7.20)}{=} \left\{\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)-1}\alpha_k\left(-B_1\check{x}_k+\frac{R^x_k}{\alpha_k}\right) - (t-\underline{t}_n^\alpha)\left(-B_1\check{x}_{N^\alpha(n,t)}+\frac{R^x_{N^\alpha(n,t)}}{\alpha_{N^\alpha(n,t)}}\right)\right. \\ &\qquad \left. + (s-\underline{s}_n^\alpha)\left(-B_1\check{x}_{N^\alpha(n,s)}+\frac{R^x_{N^\alpha(n,s)}}{\alpha_{N^\alpha(n,s)}}\right)\right\} - \left\{\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)-1}\sqrt{\alpha_k}\tilde{\xi}_k\right. \\ &\qquad \left. - \frac{t-\underline{t}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,t)}}}\tilde{\xi}_{N^\alpha(n,t)} + \frac{s-\underline{s}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,s)}}}\tilde{\xi}_{N^\alpha(n,s)}\right\} \\ &=: \mathbf{B}^{(1x,n)}_{t,s} - \mathbf{\Xi}^{(n)}_{t,s}.\end{aligned}$$

The first term can also be expressed as

$$\mathbf{B}_{t,s}^{(1x,n)} = \Bigg\{ \sum_{k=N^\alpha(n,t)+1}^{N^\alpha(n,s)-1} \alpha_k \left(-B_1 \check{x}_k + \frac{R_k^x}{\alpha_k} \right) + \left[\alpha_{N^\alpha(n,t)} - (t - \underline{t}_n^\alpha) \right] \left(-B_1 \check{x}_{N^\alpha(n,t)} + \frac{R_{N^\alpha(n,t)}^x}{\alpha_{N^\alpha(n,t)}} \right) \\ + (s - \underline{s}_n^\alpha) \left(-B_1 \check{x}_{N^\alpha(n,s)} + \frac{R_{N^\alpha(n,s)}^x}{\alpha_{N^\alpha(n,s)}} \right) \Bigg\}.$$

Note that $t' - (\underline{t'})_n^\alpha \leq \alpha_{N^\alpha(n,t')}$ for $t' = t, s$. Then we have $0 \leq \alpha_{N^\alpha(n,t)} - (t - \underline{t}_n^\alpha) \leq \alpha_{N^\alpha(n,t)}$ and $0 \leq s - \underline{s}_n^\alpha \leq \alpha_{N^\alpha(n,s)}$. It follows that

$$\begin{aligned} \|\mathbf{B}_{t,s}^{(1x,n)}\| \leq & \sum_{k=N^\alpha(n,t)+1}^{N^\alpha(n,s)-1} \alpha_k \left\| B_1 \check{x}_k - \frac{R_k^x}{\alpha_k} \right\| + \left[\alpha_{N^\alpha(n,t)} - (t - \underline{t}_n^\alpha) \right] \left(-B_1 \check{x}_{N^\alpha(n,t)} + \frac{R_{N^\alpha(n,t)}^x}{\alpha_{N^\alpha(n,t)}} \right) \\ & + (s - \underline{s}_n^\alpha) \left\| B_1 \check{x}_{N^\alpha(n,s)} - \frac{R_{N^\alpha(n,s)}^x}{\alpha_{N^\alpha(n,s)}} \right\| \\ \leq & \sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k \left\| B_1 \check{x}_k - \frac{R_k^x}{\alpha_k} \right\|. \end{aligned} \tag{D.12}$$

By Assumption 4.1 and Lemma 5.1, we have that

$$\sup_n \mathbb{E} \left\| B_1 \check{x}_n - \frac{R_n^x}{\alpha_n} \right\|^2 \leq 2\|B_1\|^2 \sup_n \mathbb{E}\|\check{x}_n\|^2 + 2 \sup_n \frac{\mathbb{E}\|R_n^x\|^2}{\alpha_n^2} \leq C_x^{ti} \tag{D.13}$$

for some $C_x^{ti} > 0$. Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mathbb{E}\|\mathbf{B}_{t,s}^{(1x,n)}\|^2 \overset{(D.12)}{\leq} & \ \mathbb{E} \left(\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k \left\| B_1 \check{x}_k - \frac{R_k^x}{\alpha_k} \right\| \right)^2 \\ \leq & \left(\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k \right) \left(\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k \mathbb{E} \left\| B_1 \check{x}_k - \frac{R_k^x}{\alpha_k} \right\|^2 \right) \\ \overset{(D.13)}{\leq} & \ C_x^{ti} \left(\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k \right)^2 \leq C_x^{ti} (s - t + 2\alpha_n)^2 \overset{(a)}{\leq} K_1 (s-t)^2, \end{aligned} \tag{D.14}$$

where there exists K_1 such that (a) holds for any n because $\alpha_n \to 0$.

To deal with the second term, we need the following Burkholder's inequality.

Proposition D.1 (Burkholder's inequality, [6])**.** *Fix any $p \geq 2$. Let $\{X_k\}_{1 \leq k \leq n}$ be a finite $\mathbb{R}^d$-valued martingale difference sequence with $\sup_{1 \leq k \leq n} \mathbb{E}\|X_k\|^p < \infty$. Then we have*

$$\mathbb{E} \left\| \sum_{k=1}^n X_k \right\|^p \leq \left[(p-1) \vee \frac{1}{p-1} \right]^p \mathbb{E} \left(\sum_{k=1}^n \|X_k\|^2 \right)^{\frac{p}{2}},$$

which together with Jensen's inequality imply

$$\mathbb{E}\left\|\sum_{k=1}^{n} X_k\right\|^p \le \left[(p-1)\vee\frac{1}{p-1}\right]^p n^{\frac{p}{2}-1}\mathbb{E}\sum_{k=1}^{n}\|X_k\|^p. \tag{D.15}$$

With $p \in \left(2, \frac{4}{1+(\delta_H\vee\delta_F\vee\delta_G)/2}\right]$, we have

$$\mathbb{E}\|\Xi_{t,s}^{(n)}\|^p \overset{(D.15)}{\le} C_p^{ti}\left[N^\alpha(n,s)-N^\alpha(n,t)+1\right]^{\frac{p}{2}-1}\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k^{\frac{p}{2}}\mathbb{E}\|\tilde{\xi}_k\|^p,$$

where we have used $t'-\underline{t}_n'^{\alpha} \le \alpha_{N^\alpha(n,t')}$ for $t'=t,s$ and the constant number is $C_p^{ti}=\left[(p-1)\vee\frac{1}{p-1}\right]^p$.

To further derive the upper bound, we control different terms separately as follows.

- For $N^\alpha(n,s)-N^\alpha(n,t)+1$, From the definition of $N^\alpha(n,t')$ in Definition B.1, we have $\sum_{k=n}^{N^\alpha(n,t')-1}\alpha_k \le t' < \sum_{k=n}^{N^\alpha(n,t')}\alpha_k$ for $t'=s,t$. Note that α_k is decreasing in k. It follows that

  $$s-t \ge \sum_{k=n}^{N^\alpha(n,s)-1}\alpha_k - \sum_{k=n}^{N^\alpha(n,t)}\alpha_k = \sum_{k=N^\alpha(n,t)+1}^{N^\alpha(n,s)-1}\alpha_k \ge (N^\alpha(n,s)-N^\alpha(n,t)-2)\alpha_{N^\alpha(n,s)}.$$

  Thus,
  $$N^\alpha(n,s)-N^\alpha(n,t)+1 \le \frac{s-t}{\alpha_{N^\alpha(n,s)}}+3 \le \frac{s-t+3\alpha_n}{\alpha_{N^\alpha(n,s)}}.$$

- For $\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k^{\frac{p}{2}}$, since $N^\alpha(n,t)\ge n$ and α_k is decreasing in k, we have $\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k^{\frac{p}{2}} \le \alpha_n^{\frac{p}{2}-1}\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k$. From the analysis in the first bullet, we have

  $$\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k = \sum_{k=n}^{N^\alpha(n,s)-1}\alpha_k + \alpha_{N^\alpha(n,s)} - \sum_{k=n}^{N^\alpha(n,t)}\alpha_k + \alpha_{N^\alpha(n,t)} \le s-t+2\alpha_n.$$

  Thus,
  $$\sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)}\alpha_k^{\frac{p}{2}} \le \alpha_n^{\frac{p}{2}-1}(s-t+2\alpha_n).$$

- For $\mathbb{E}\|\tilde{\xi}_k\|^p$, from the definition of $\tilde{\xi}_k=\xi_k-\kappa_k H^\star\psi_k$ and Assumption 3.6, there exists $C_\xi^{ti}>0$ such that
  $$\sup_k \mathbb{E}\|\tilde{\xi}_k\|^p \le C_\xi^{ti}.$$

Combining the above bounds yields

$$\begin{aligned}
\mathbb{E}\|\boldsymbol{\Xi}_{t,s}^{(n)}\|^p &\le C_p^{ti} C_\xi^{ti} \left[N^\alpha(n,s) - N^\alpha(n,t) + 1\right]^{\frac{p}{2}-1} \sum_{k=N^\alpha(n,t)}^{N^\alpha(n,s)} \alpha_k^{\frac{p}{2}} \\
&\le C_p^{ti} C_\xi^{ti} \left(\frac{s-t+3\alpha_n}{\alpha_{N^\alpha(n,s)}}\right)^{\frac{p}{2}-1} \alpha_n^{\frac{p}{2}-1}(s-t+2\alpha_n) \\
&\le C_p^{ti} C_\xi^{ti} (s-t+3\alpha_n)^{\frac{p}{2}} \left(\frac{\alpha_n}{\alpha_{N^\alpha(n,s)}}\right)^{\frac{p}{2}-1} \\
&\overset{(a)}{\le} C_p^{ti} C_\xi^{ti} C_\alpha^{ti} (s-t+3\alpha_n)^{\frac{p}{2}} \overset{(b)}{\le} K_2 (s-t)^{\frac{p}{2}}, \qquad \text{(D.16)}
\end{aligned}$$

where (a) uses Lemma B.1 with $C_\alpha^{ti} = 2^{(\lceil 2T/c_\alpha \rceil + 1)(p/2-1)}$ with c_α specified therein, and there exists K_2 such that (b) holds for any n.

Combining (D.14) and (D.16) with Markov's inequality, we have

$$\begin{aligned}
\mathbb{P}\left(\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(t)\| \ge \varepsilon\right) &\le \mathbb{P}\left(\|\mathbf{B}_{t,s}^{(1x,n)}\| \ge \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\|\boldsymbol{\Xi}_{t,s}^{(n)}\| \ge \frac{\varepsilon}{2}\right) \\
&\le \frac{4}{\varepsilon^2}\mathbb{E}\|\mathbf{B}_{t,s}^{(1x,n)}\|^2 + \frac{2^p}{\varepsilon^p}\mathbb{E}\|\boldsymbol{\Xi}_{t,s}^{(n)}\|^p \\
&\le 8(K_1 \vee K_2)\left[\frac{(s-t)^2}{\varepsilon^2} + \frac{(s-t)^{\frac{p}{2}}}{\varepsilon^p}\right].
\end{aligned}$$

This upper bound verifies the second condition in Proposition 5.1. Consequently, $\{\bar{\mathrm{X}}_n(\cdot)\}_{n=1}^\infty$ is tight.

Part 2: Tightness of $\{\bar{\mathrm{Z}}_n(\cdot)\}_{n=1}^\infty$. Note that in the proof of Part 1, the only place we employed Lemma 5.1 is (D.13) and in fact, we only require $\mathbb{E}\|R_n^x\|^2 = \mathcal{O}(\alpha_n^2)$. Moreover, the dimension d_x would not affect the proof. The tightness of $\{\bar{\mathrm{Z}}_n(\cdot)\}_{n=1}^\infty$ follows from a similar procedure by applying the residual bounds in Lemma 5.2.

Part 3: Tightness of $\{\bar{\mathrm{Y}}_n(\cdot)\}_{n=1}^\infty$. Note that Assumption 4.1 implies $\mathbb{E}\|\bar{\mathrm{Y}}_n(0)\|^2 = \mathbb{E}\|\check{y}_n\|^2 = \mathcal{O}(1)$. Then by Markov's inequality, the first condition in Proposition 5.1 holds. Now we verify the second condition. Without loss of generality, we assume $0 \le t < s \le T$. For ease of notation, we define $\widetilde{B}_3 := B_3 - \frac{\tilde{\beta} I}{2}$. From the definition of $\bar{\mathrm{Y}}_n(t)$ in (Y), we have

$$\begin{aligned}
\bar{\mathrm{Y}}_n(s) - \bar{\mathrm{Y}}_n(t) &\overset{(Y)}{=} \check{y}_{N^\beta(n,s)} + (s - \underline{s}_n^\beta)\left(-\widetilde{B}_3 \check{y}_{N^\beta(n,s)} + \frac{R^y_{N^\beta(n,s)}}{\beta_{N^\beta(n,s)}}\right) - \frac{s - \underline{s}_n^\beta}{\sqrt{\beta_{N^\beta(n,s)}}}\psi_{N^\beta(n,s)} \\
&\quad - \check{y}_{N^\beta(n,t)} - (t - \underline{t}_n^\beta)\left(-\widetilde{B}_3 \check{y}_{N^\beta(n,t)} + \frac{R^y_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}\right) + \frac{t - \underline{t}_n^\beta}{\sqrt{\beta_{N^\beta(n,t)}}}\psi_{N^\beta(n,t)} \\
&\quad - (s - \underline{s}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,s)}}{\beta_{N^\beta(n,s)}}} B_2 \check{x}_{N^\beta(n,s)} + (t - \underline{t}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}} B_2 \check{x}_{N^\beta(n,t)} \\
&\overset{(7.21)}{=} \left\{ \sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1} \beta_k \left(-\widetilde{B}_3 \check{y}_k + \frac{R_k^y}{\beta_k}\right) - (t - \underline{t}_n^\beta)\left(-\widetilde{B}_3 \check{y}_{N^\beta(n,t)} + \frac{R^y_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ (s - \underline{s}_n^\beta)\left(-\widetilde{B}_3 \check{y}_{N^\beta(n,s)} + \frac{R^y_{N^\beta(n,s)}}{\beta_{N^\beta(n,s)}}\right)\Bigg\} - \Bigg\{\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1} \sqrt{\beta_k}\psi_k \\
&- \frac{t - \underline{t}_n^\beta}{\sqrt{\beta_{N^\beta(n,t)}}}\psi_{N^\beta(n,t)} + \frac{s - \underline{s}_n^\beta}{\sqrt{\beta_{N^\beta(n,s)}}}\psi_{N^\beta(n,s)}\Bigg\} - \Bigg\{\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1} \beta_k \sqrt{\frac{\alpha_k}{\beta_k}} B_2 \check{x}_k \\
&- (t - \underline{t}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}} B_2 \check{x}_{N^\beta(n,t)} + (s - \underline{s}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,s)}}{\beta_{N^\beta(n,s)}}} B_2 \check{x}_{N^\beta(n,s)}\Bigg\} \\
=:\ & \mathbf{B}_{t,s}^{(3y,n)} - \mathbf{\Psi}_{t,s}^{(n)} - \mathbf{B}_{t,s}^{(2x,n)}.
\end{aligned}
$$

For the first and second terms, similar to the derivation of (D.14) and (D.16), with $p \in \left(2, \frac{4}{1+(\delta_H \vee \delta_F \vee \delta_G)/2}\right]$, there exist K_3, K_4 such that for any n,

$$
\mathbb{E}\|\mathbf{B}_{t,s}^{(3y,n)}\|^2 \le K_3(s-t)^2 \quad \text{and} \quad \mathbb{E}\|\mathbf{\Psi}_{t,s}^{(n)}\|^p \le K_4(s-t)^{\frac{p}{2}}. \tag{D.17}
$$

Main difficulty. The main difficulty lies to how to deal with the last term $\mathbf{B}_{t,s}^{(2x,n)}$. If we adopt a similar procedure of deriving (D.14) by the Cauchy-Schwarz inequality, we could only obtain

$$
\mathbb{E}\|\mathbf{B}_{t,s}^{(2x,n)}\|^2 \lesssim (s-t)\left(\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)} \beta_k \frac{\alpha_k}{\beta_k}\right) \lesssim (s-t)^2 \max_{k \ge N^\beta(n,s)} \frac{\alpha_k}{\beta_k}.
$$

Since Assumption 3.4 (i) implies $\max_{k\ge N^\beta(n,s)} \alpha_k/\beta_k \to \infty$ as $n \to \infty$, this upper bound becomes ineffective.

Solution approach. To address this issue, we need to provide a more precise characterization of $\mathbf{B}_{t,s}^{(2x,n)}$. A natural approach is to further expand each $\check{x}_k$. While this idea is feasible, it leads to more complex calculations, as each $\check{x}_k$ depends on both $\check{x}_{k-1}$ and $\check{y}_{k-1}$. Below, we outline the detailed process.

To begin, we define another term $\mathbf{B}_{n_0,n_1}^{(2x)} := \sum_{k=n_0}^{n_1} \sqrt{\beta_k} B_2 \hat{x}_k$. Here, we use the unrescaled iterate to simplify the subsequent calculations. Given that $\check{x}_k = \hat{x}_k/\sqrt{\alpha_{k-1}}$, it follows that $\mathbf{B}_{n_0,n_1}^{(2x)} = \sum_{k=n_0}^{n_1} \sqrt{\beta_k \alpha_{k-1}} B_2 \check{x}_k$, which demonstrates that this term is closely related to $\mathbf{B}_{t,s}^{(2x,n)}$, as shown in the following equation

$$
\begin{aligned}
\mathbf{B}_{t,s}^{(2x,n)} = \mathbf{B}^{(2x)}_{N^\beta(n,t),N^\beta(n,s)-1} &- (t - \underline{t}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,t)}}{\beta_{N^\beta(n,t)}}} B_2 \check{x}_{N^\beta(n,t)} \\
&+ (s - \underline{s}_n^\beta)\sqrt{\frac{\alpha_{N^\beta(n,s)}}{\beta_{N^\beta(n,s)}}} B_2 \check{x}_{N^\beta(n,s)} + \sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1} \sqrt{\beta_k}\left(\sqrt{\frac{\alpha_k}{\alpha_{k-1}}} - 1\right) B_2 \hat{x}_k.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\mathbb{E}\left\|\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1} \sqrt{\beta_k}\left(\sqrt{\frac{\alpha_k}{\alpha_{k-1}}} - 1\right) B_2 \hat{x}_k\right\|^2
$$

$$\leq \left(\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1}\beta_k\right)\left[\sum_{k=N^\beta(n,t)}^{N^\beta(n,s)-1}\left(\sqrt{\frac{\alpha_k}{\alpha_{k-1}}}-1\right)^2\mathbb{E}\|B_2\hat{x}_k\|^2\right].$$

In (D.6), we have shown that $\sqrt{\frac{\alpha_k}{\alpha_{k-1}}}-1=o(\beta_k)$. Assumption 4.1 implies $\mathbb{E}\|B_2\hat{x}_k\|^2=o(1)$. Moreover, $t'-(\underline{t'})_n^\beta \leq \beta_{N^\beta(n,t')}$ for $t'=t,s$. Summarizing the above results and following a similar procedure as (D.14), we can obtain that there exists K_5 such that

$$\mathbb{E}\|\mathbf{B}_{t,s}^{(2x,n)}-\mathbf{B}_{N^\beta(n,t),N^\beta(n,s)-1}^{(2x)}\|^2 \leq K_5(s-t)^2 \tag{D.18}$$

for any n. This inequality controls the difference between $\mathbf{B}_{t,s}^{(2x,n)}$ and $\mathbf{B}_{N^\beta(n,t),N^\beta(n,s)-1}^{(2x)}$.

From now on, we always assume $n_0=N^\beta(n,t)$ and $n_1=N^\beta(n,s)-1$ and It remains to deal with the term $\mathbf{B}_{n_0,n_1}^{(2x)}$. Recall that the update rule of $\hat{x}_n$ is shown in (D.8). For ease of notation, we define $A_n:=I-\alpha_nB_1+\beta_nH^\star B_2$, $\tilde{\xi}_n:=\xi_n-\beta_nH^\star\psi_n/\alpha_n$ and $\tilde{R}_n^x:=R_n^{\nabla H}+R_n^H+\alpha_nR_n^F-\beta_nH^\star R_n^G$. Then the update rule of $\hat{x}_n$ can be rewritten as

$$\hat{x}_{n+1}=A_n\hat{x}_n+\beta_nH^\star B_3\hat{y}_n-\alpha_n\tilde{\xi}_n-\tilde{R}_n^x.$$

Applying this equation multiple times yields

$$\hat{x}_{n+m}=\left(\prod_{i=1}^m A_{n+m-i}\right)\hat{x}_n+\sum_{l=n}^{n+m-1}\left(\prod_{i=1}^{n+m-l-1}A_{n+m-i}\right)\left(\beta_lH^\star B_3\hat{y}_l-\alpha_l\tilde{\xi}_l-\tilde{R}_l^x\right).$$

It follows that

$$\begin{aligned}
\mathbf{B}_{n_0,n_1}^{(2x)} &= \sum_{k=n_0}^{n_1}\sqrt{\beta_k}B_2\left[\left(\prod_{i=1}^{k-n_0}A_{k-i}\right)\hat{x}_{n_0}+\sum_{l=n_0}^{k-1}\left(\prod_{i=1}^{k-l-1}A_{k-i}\right)\left(\beta_lH^\star B_3\hat{y}_l-\alpha_l\tilde{\xi}_l-\tilde{R}_l^x\right)\right]\\
&= \sum_{k=n_0}^{n_1}\sqrt{\beta_k}B_2\left(\prod_{i=1}^{k-n_0}A_{k-i}\right)\hat{x}_{n_0}+\sum_{l=n_0}^{n_1-1}\sum_{k=l+1}^{n_1}\sqrt{\beta_k}B_2\left(\prod_{i=1}^{k-l-1}A_{k-i}\right)\left(\beta_lH^\star B_3\hat{y}_l-\alpha_l\tilde{\xi}_l-\tilde{R}_l^x\right)\\
&= \underbrace{B_2\widetilde{\Gamma}_{n_0,n_1}\hat{x}_{n_0}+B_2\sum_{l=n_0}^{n_1-1}\widetilde{\Gamma}_{l+1,n_1}\left(\beta_lH^\star B_3\hat{y}_l-\tilde{R}_l^x\right)}_{:=\mathbf{B}_{n_0,n_1}^{(2x,1)}}\underbrace{-B_2\sum_{l=n_0}^{n_1-1}\widetilde{\Gamma}_{l+1,n_1}\alpha_l\tilde{\xi}_l}_{:=\mathbf{B}_{n_0,n_1}^{(2x,2)}},
\end{aligned}$$

where we define $\widetilde{\Gamma}_{m,j}:=\sum_{k=m}^{j}\sqrt{\beta_k}\left(\prod_{i=1}^{k-m}A_{k-i}\right)$ in the last step. The magnitude of $\widetilde{\Gamma}_{m,j}$ can be controlled by the following auxiliary lemma, whose proof is deferred to the end of this subsection.

Lemma D.1. *Suppose that Assumption 3.4 holds and the real parts of the eigenvalues of B_1 are no less than μ_1. Define $A_k:=I-\alpha_kB_1+\beta_kH^\star B_2$. If t and s are fixed positive numbers satisfying $s>t$ and n is sufficiently large, then for any $N^\beta(n,t)\leq l<N^\beta(n,s)$, we have*

$$\left\|\sum_{k=l+1}^{N^\beta(n,s)-1}\sqrt{\beta_k}\prod_{i=1}^{k-l-1}A_{k-i}\right\|\leq\frac{2^{\lceil 2s/c_\alpha\rceil+3}}{\mu_1}\cdot\frac{\sqrt{\beta_l}}{\alpha_l}.$$

For the first term $\mathbf{B}^{(2x,1)}_{n_0,n_1}$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
&\mathbb{E}\|\mathbf{B}^{(2x,1)}_{n_0,n_1}\|^2 \\
&\leq 2\mathbb{E}\|B_2\widetilde{\Gamma}_{n_0,n_1}\hat{x}_{n_0}\|^2 + 2\mathbb{E}\left\|B_2\sum_{l=n_0}^{n_1-1}\widetilde{\Gamma}_{l+1,n_1}(\beta_l H^\star B_3\hat{y}_l - \tilde{R}^x_l)\right\|^2 \\
&\leq 2(\|B_2\|\|\widetilde{\Gamma}_{n_0,n_1}\|)^2\mathbb{E}\|\hat{x}_{n_0}\|^2 + 2\|B_2\|^2\left(\sum_{l=n_0}^{n_1-1}\beta_l\right)\left(\sum_{l=n_0}^{n_1-1}\beta_l\|\widetilde{\Gamma}_{l+1,n_1}\|^2\mathbb{E}\left\|H^\star B_3\hat{y}_l - \frac{\tilde{R}^x_l}{\beta_l}\right\|^2\right).
\end{aligned}$$

By Lemma D.1, when n is sufficiently large, we $\|\widetilde{\Gamma}_{l+1,n_1}\| \leq \frac{2^{\lceil 2T/c_\alpha\rceil+3}}{\mu_1}\frac{\sqrt{\beta_l}}{\alpha_l}$ for any $n_0 \leq l < n_1$ and $\|\widetilde{\Gamma}_{n_0,n_1}\| \leq \frac{2^{\lceil 2T/c_\alpha\rceil+3}}{\mu_1}\frac{\sqrt{\beta_{n_0}}}{\alpha_{n_0}} + \sqrt{\beta_{n_0}}$, with μ_1 and c_α specified in Lemma D.1; Assumption 4.1 implies $\mathbb{E}\|\hat{x}_{n_0}\|^2 = \mathcal{O}(\alpha_{n_0})$ and $\mathbb{E}\|\hat{y}_l\|^2 = \mathcal{O}(\beta_l)$; the upper bounds in (D.11) and the subsequent analysis therein imply that under Assumption 3.4, $\mathbb{E}\|\tilde{R}^x_l\|^2 = \mathcal{O}(\beta_l^2\alpha_l^{1+\delta_H} + \beta_l^{2+\delta_H} + \alpha_l^{3+\delta_F} + \beta_l^2\alpha_l^{1+\delta_G}) = \mathcal{O}(\alpha_l^2\beta_l)$. Therefore, we have $\|\widetilde{\Gamma}_{n_0,n_1}\|^2\mathbb{E}\|\hat{x}_{n_0}\|^2 = \mathcal{O}\left(\frac{\beta_{n_0}}{\alpha_{n_0}}\right) = o(1)$ and

$$\|\widetilde{\Gamma}_{l+1,n_1}\|^2\mathbb{E}\left\|H^\star B_3\hat{y}_l - \frac{\tilde{R}^x_l}{\beta_l}\right\|^2 = \mathcal{O}\left(\frac{\beta_l}{\alpha_l^2}\right)\left(\mathbb{E}\|\hat{y}_l\|^2 + \frac{\mathbb{E}\|\tilde{R}^x_l\|^2}{\beta_l^2}\right) = \mathcal{O}(1).$$

Then similar to the derive of (D.14), there exist K_6 and m_1 such that

$$\mathbb{E}\|\mathbf{B}^{(2x,1)}_{n_0,n_1}\|^2 \leq K_6(s-t)^2 \tag{D.19}$$

holds for any $n \geq m_1$. Modifying K_6, we could make (D.19) hold for any n.

For the second term $\mathbf{B}^{(2x,2)}_{n_0,n_1}$, applying $\|\widetilde{\Gamma}_{l+1,n_1}\| \leq C_T\frac{\sqrt{\beta_l}}{\alpha_l}$ with $C_T := \frac{2^{\lceil 2T/c_\alpha\rceil+3}}{\mu_1}$, we obtain that with $p \in \left(2, \frac{4}{1+(\delta_H\vee\delta_F\vee\delta_G)/2}\right]$ and sufficiently large n,

$$\begin{aligned}
\mathbb{E}\|\mathbf{B}^{(2x,2)}_{n_0,n_1}\|^p &\overset{(D.15)}{\leq} C_p^{ti}(n_1-n_0)^{\frac{p}{2}-1}\|B_2\|^p\sum_{l=n_0}^{n_1-1}\|\widetilde{\Gamma}_{l+1,n_1}\|^p\alpha_l^p\mathbb{E}\|\tilde{\xi}_l\|^p \\
&\leq C_p^{ti}C_T^p(n_1-n_0)^{\frac{p}{2}-1}\|B_2\|^p\sum_{l=n_0}^{n_1-1}\beta_l^{\frac{p}{2}}\mathbb{E}\|\tilde{\xi}_l\|^p,
\end{aligned}$$

where we have used Burkholder's inequality in Proposition D.1 with $C_p^{ti} = \left[(p-1)\vee\frac{1}{p-1}\right]^p$. Then following a similar procedure to (D.16), we have that there exist K_7 and m_2 such that

$$\mathbb{E}\|\mathbf{B}^{(2x,2)}_{n_0,n_1}\|^p \leq K_7(s-t)^{\frac{p}{2}} \tag{D.20}$$

for any $n \geq m_2$. Modifying K_7, we could make (D.20) hold for any n.

The upper bounds in (D.18), (D.19) and (D.20) collectively control the term $\mathbf{B}^{(2x,n)}_{t,s}$. Combining them with (D.17) and Markov's inequality yields the following upper bound

$$\mathbb{P}\left(\|\bar{\mathrm{Y}}_n(s) - \bar{\mathrm{Y}}_n(t)\| \geq \varepsilon\right) \leq \mathbb{P}\left(\|\mathbf{B}^{(3y,n)}_{t,s}\| \geq \frac{\varepsilon}{5}\right) + \mathbb{P}\left(\|\mathbf{\Psi}^{(n)}_{t,s}\| \geq \frac{\varepsilon}{5}\right) + \mathbb{P}\left(\|\mathbf{B}^{(2x,1)}_{n_0,n_1}\| \geq \frac{\varepsilon}{5}\right)$$

$$
\begin{aligned}
&+\mathbb{P}\left(\|\mathbf{B}_{n_0,n_1}^{(2x,2)}\| \geq \frac{\varepsilon}{5}\right) + \mathbb{P}\left(\|\mathbf{B}_{t,s}^{(2x,n)} - \mathbf{B}_{n_0,n_1}^{(2x)}\| \geq \frac{\varepsilon}{5}\right) \\
&\leq \frac{25}{\varepsilon^2}\left(\mathbb{E}\|\mathbf{B}_{t,s}^{(3y,n)}\|^2 + \mathbb{E}\|\mathbf{B}_{n_0,n_1}^{(2x,1)}\|^2 + \mathbb{E}\|\mathbf{B}_{t,s}^{(2x,n)} - \mathbf{B}_{n_0,n_1}^{(2x)}\|^2\right) \\
&\quad + \frac{5^p}{\varepsilon^p}\left(\mathbb{E}\|\mathbf{\Psi}_{t,s}^{(n)}\|^p + \mathbb{E}\|\mathbf{B}_{n_0,n_1}^{(2x,2)}\|^p\right) \\
&\leq 5^4\left(\max_{3\leq i\leq 7} K_i\right)\left[\frac{(s-t)^2}{\varepsilon^2} + \frac{(s-t)^{\frac{p}{2}}}{\varepsilon^p}\right].
\end{aligned}
$$

This upper bound verifies the second condition in Proposition 5.1, and hence $\{\bar{\mathrm{Y}}_n(\cdot)\}_{n=1}^{\infty}$ is tight. $\square$

Proof of Lemma D.1. For ease of notation, we define $n_0 := N^\beta(n,t)$ and $n_1 := N^\beta(n,s) - 1$. Since the real parts of the eigenvalues of B_1 are no less than μ_1, $\beta_j = o(\alpha_j)$ and n is sufficiently large, we have $\|A_j\| \leq \|I - \alpha_j B_1\| + \beta_j\|H^\star B_2\| \leq 1 - \mu_1\alpha_j/2$ for any $j > l \geq n$. It follows that

$$
\left\|\sum_{k=l+1}^{n_1}\sqrt{\beta_k}\prod_{i=1}^{k-l-1}A_{k-i}\right\| \leq \sum_{k=l+1}^{n_1}\sqrt{\beta_k}\prod_{i=l+1}^{k-1}\left(1-\frac{\mu_1\alpha_i}{2}\right) \leq \sum_{k=l+1}^{n_1}\sqrt{\beta_k}\exp\left(-\frac{\mu_1}{2}\sum_{i=l+1}^{k-1}\alpha_i\right).
$$

Moreover, by Lemma B.1 and the monotonicity of the step sizes, we have $\alpha_k \geq c_1\alpha_l$ and $\beta_k \leq \beta_l$ for any $l \leq k \leq n_1$. Here $c_1 := 2^{-\lceil 2s/c_\alpha\rceil - 1}$ with c_α specified in Lemma B.1. This implies

$$
\begin{aligned}
\sum_{k=l+1}^{n_1}\sqrt{\beta_k}\exp\left(-\frac{\mu_1}{2}\sum_{i=l+1}^{k-1}\alpha_i\right) &\leq \sqrt{\beta_l}\sum_{k=l+1}^{n_1}\exp\left[-(k-1-l)\frac{c_1\mu_1\alpha_l}{2}\right] \\
&\leq \frac{\sqrt{\beta_l}}{1-\exp(-c_1\mu_1\alpha_l/2)}.
\end{aligned}
$$

For $x \in (0,1)$, we have $\exp(-x) \leq 1 - x/2$. Therefore, for sufficiently large n, we have

$$
\left\|\sum_{k=l+1}^{n_1}\sqrt{\beta_k}\prod_{i=1}^{k-l-1}A_{k-i}\right\| \leq \frac{4}{c_1\mu_1}\cdot\frac{\sqrt{\beta_l}}{\alpha_l}
$$

for any $n_0 \leq l < n_1$. With $c_1 = 2^{-\lceil 2s/c_\alpha\rceil - 1}$, the coefficient is $\frac{4}{c_1\mu_1} = \frac{2^{\lceil 2s/c_\alpha\rceil + 3}}{\mu_1}$. $\square$

D.3 Proofs in Section 7.5.3

In this subsection, we present the proof of Lemma 5.4 in Section 7.5.3. We begin by presenting a discrete version of Lemma 5.4, which serves as an intermediate result leading to the full proof.

Lemma D.2. *Suppose that Assumptions 3.1 – 3.4, 3.6 and 4.1 hold. Then*

(i) For any $f \in C_c^\infty(\mathbb{R}^{d_x})$, with $\mathcal{A}^x$ defined in (7.25), we have

$$
\mathbb{E}[f(\check{x}_{n+1}) - f(\check{x}_n)|\mathcal{F}_n] = \alpha_n\mathcal{A}^x f(\check{x}_n) + R_n^f, \tag{D.21}
$$

where $\frac{1}{\alpha_n}\mathbb{E}|R_n^f| \to 0$ as $n \to \infty$.

(ii) For any $g \in C_c^\infty(\mathbb{R}^{d_y})$, with $\mathcal{A}^y$ defined in (7.26), we have

$$\mathbb{E}[g(\check{z}_{n+1}) - g(\check{z}_n)|\mathcal{F}_n] = \beta_n \mathcal{A}^y g(\check{z}_n) + R_n^g, \tag{D.22}$$

where $\frac{1}{\beta_n}\mathbb{E}|R_n^g| \to 0$ as $n \to \infty$.

Proof of Lemma D.2. We first prove the part (i). Throughout the proof, C will represent a universal constant whose value may change from line to line, for the sake of convenience. Because $f \in C_c^\infty(\mathbb{R}^{d_x})$, $\|\nabla f(\cdot)\|$ and $\|\nabla^2 f(\cdot)\|$ are both bounded functions.

By Taylor's theorem, we have

$$\begin{aligned} f(\check{x}_{n+1}) - f(\check{x}_n) &= \langle \nabla f(\check{x}_n), \check{x}_{n+1} - \check{x}_n \rangle + \frac{1}{2}(\check{x}_{n+1} - \check{x}_n)^\top \nabla^2 f(\check{x}_n)(\check{x}_{n+1} - \check{x}_n) \\ &\quad + \underbrace{\frac{1}{2}(\check{x}_{n+1} - \check{x}_n)^\top \left(\nabla^2 f(\check{u}_n) - \nabla^2 f(\check{x}_n)\right)(\check{x}_{n+1} - \check{x}_n)}_{R_{n,0}^f}, \end{aligned} \tag{D.23}$$

where $\check{u}_n$ is a point between $\check{x}_n$ and $\check{x}_{n+1}$. For the first term, we have

$$\mathbb{E}[\langle \nabla f(\check{x}_n), \check{x}_{n+1} - \check{x}_n \rangle |\mathcal{F}_n] \overset{(7.20)}{=} -\alpha_n \langle \nabla f(\check{x}_n), B_1 \check{x}_n \rangle + \underbrace{\langle \nabla f(\check{x}_n), \mathbb{E}[R_n^x|\mathcal{F}_n] \rangle}_{R_{n,1}^f}. \tag{D.24}$$

By Lemma 5.1, we have

$$\mathbb{E}|R_{n,1}^f| \lesssim \mathbb{E}\|\mathbb{E}[R_n^x|\mathcal{F}_n]\| = o(\alpha_n). \tag{D.25}$$

For the second term, we have

$$\begin{aligned} &\mathbb{E}\left[(\check{x}_{n+1} - \check{x}_n)^\top \nabla^2 f(\check{x}_n)(\check{x}_{n+1} - \check{x}_n)|\mathcal{F}_n\right] \\ &\overset{(7.20)}{=} \alpha_n \mathbb{E}\left[\tilde{\xi}_n^\top \nabla^2 f(\check{x}_n)\tilde{\xi}_n|\mathcal{F}_n\right] + 2\sqrt{\alpha_n}\mathbb{E}\left[(-\alpha_n B_1 \check{x}_n + R_n^x)^\top \nabla^2 f(\check{x}_n)\tilde{\xi}_n|\mathcal{F}_n\right] \\ &\quad + \mathbb{E}[(-\alpha_n B_1 \check{x}_n + R_n^x)^\top \nabla^2 f(\check{x}_n)(-\alpha_n B_1 \check{x}_n + R_n^x)|\mathcal{F}_n] \\ &= \alpha_n \mathrm{tr}(\nabla^2 f(\check{x}_n)\Sigma_\xi) + \underbrace{\alpha_n \mathrm{tr}\left(\nabla^2 f(\check{x}_n)(\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n] - \Sigma_\xi)\right)}_{2R_{n,2}^f} + \underbrace{2\alpha_n^{3/2}\mathbb{E}\left[\left(\frac{R_n^x}{\alpha_n}\right)^\top \nabla^2 f(\check{x}_n)\tilde{\xi}_n \Big| \mathcal{F}_n\right]}_{2R_{n,3}^f} \\ &\quad + \underbrace{\alpha_n^2 \mathbb{E}\left[\left(-B_1\check{x}_n + \frac{R_n^x}{\alpha_n}\right)^\top \nabla^2 f(\check{x}_n)\left(-B_1\check{x}_n + \frac{R_n^x}{\alpha_n}\right)\Big|\mathcal{F}_n\right]}_{2R_{n,4}^f}. \end{aligned} \tag{D.26}$$

Recall that $\tilde{\xi}_n = \xi_n - \kappa_n H^\star \psi_n$ with $\kappa_n = \beta_n/\alpha_n = o(1)$. Then we have $\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n] \xrightarrow{p} \Sigma_\xi$. Moreover, Assumption 3.6 ensures each entry of $\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n]$ is uniformly integrable, which together with convergence in probability imply $\mathbb{E}\|\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n] - \Sigma_\xi\| \to 0$. From our assumptions and the results in Assumption 4.1 and Lemma 5.1, we have

$$\begin{aligned}
&\mathbb{E}|R^f_{n,2}| \lesssim \alpha_n \mathbb{E}\|\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n] - \Sigma_\xi\| = o(\alpha_n),\\
&\mathbb{E}|R^f_{n,3}| \lesssim \alpha_n^{3/2}\left(\frac{\mathbb{E}\|R^x_n\|^2}{\alpha_n^2} + \mathbb{E}\|\tilde{\xi}_n\|^2\right) = \mathcal{O}(\alpha_n^{3/2}) = o(\alpha_n),\\
&\mathbb{E}|R^f_{n,4}| \lesssim \alpha_n^2\left(\mathbb{E}\|\check{x}_n\|^2 + \frac{\mathbb{E}\|R^x_n\|^2}{\alpha_n^2}\right) = \mathcal{O}(\alpha_n^2) = o(\alpha_n).
\end{aligned} \tag{D.27}$$

Since $\nabla^2 f$ is Lipschitz continuous and compactly supported, $\nabla^2 f$ is also ε-Hölder continuous for any $\varepsilon \in (0,1]$. It follows that

$$\begin{aligned}
\mathbb{E}|R^f_{n,0}| &\lesssim \mathbb{E}\|\check{x}_{n+1} - \check{x}_n\|^{2+\varepsilon} \overset{(7.20)}{\leq} \mathbb{E}\left\|\alpha_n B_1 \check{x}_n + \sqrt{\alpha_n}\tilde{\xi}_n - R^x_n\right\|^{2+\varepsilon}\\
&\lesssim \alpha_n^{2+\varepsilon}\mathbb{E}\|\check{x}_n\|^{2+\varepsilon} + \alpha_n^{1+\frac{\varepsilon}{2}}\mathbb{E}\|\tilde{\xi}_n\|^{2+\varepsilon} + \mathbb{E}\|R^x_n\|^{2+\varepsilon} = o(\alpha_n),
\end{aligned} \tag{D.28}$$

where we employ Assumption 4.1 and Lemma 5.1 with $\varepsilon \leq \frac{4}{1+(\delta_H\vee\delta_F\vee\delta_G)/2} - 2$ for the last inequality. Plugging (D.24) to (D.28) into (D.23) yields

$$\mathbb{E}[f(\check{x}_{n+1}) - f(\check{x}_n)|\mathcal{F}_n] = \alpha_n \mathcal{A}^x f(\check{x}_n) + R^f_n,$$

where $R^f_n := \sum_{i=0}^4 R^f_{n,i}$ satisfies $\frac{1}{\alpha_n}\mathbb{E}|R^f_n| \to 0$ as $n\to\infty$. Thus we obtain Part (i).

Note that aside from the assumptions, the proof of Part (i) primarily relies on the fast-time-scale part of Lemma 5.1. Similarly, the proof of part (ii) follows an analogous procedure, leveraging the results in Lemma 5.2. A key aspect worth emphasizing is about the asymptotic covariance. In the proof of Part (i), it is established that $\mathbb{E}[\tilde{\xi}_n\tilde{\xi}_n^\top|\mathcal{F}_n] \xrightarrow{p} \Sigma_\xi$, indicating that the asymptotic covariance of $\tilde{\xi}_n$ matches that of ξ_n. In contrast, for Part (ii), based on the definition in Lemma 5.2, $\tilde{\psi}_n$ can be expressed as

$$\tilde{\psi}_n = \psi_n - B_2B_1^{-1}\tilde{\xi}_n = \psi_n - B_2B_1^{-1}\xi_n + \kappa_n B_2 B_1^{-1}H^\star\psi_n.$$

It follows that $\mathbb{E}[\tilde{\psi}_n\tilde{\psi}_n^\top|\mathcal{F}_n] \xrightarrow{p} \widetilde{\Sigma}_\psi$, where $\widetilde{\Sigma}_\psi$ is defined in (7.13). Using a similar procedure as in the proof of Part (i), we can show that for any $g \in C_c^\infty(\mathbb{R}^{d_y})$,

$$\mathbb{E}[g(\check{z}_{n+1}) - g(\check{z}_n)|\mathcal{F}_n] = \beta_n \mathcal{A}^y g(\check{z}_n) + R^g_n,$$

where R^g_n satisfies $\frac{1}{\beta_n}\mathbb{E}|R^g_n| \to 0$ as $n\to\infty$. □

With Lemma D.2, we are prepared to prove Lemma 5.4.

Proof of Lemma 5.4. We first prove Part (i). For convenience, with $\Gamma^\alpha_{n,m}$ introduced in Definition B.1, we define the following notation

$$\bar{N}^\alpha(n,t) := \min\left\{m \geq n \colon \Gamma^\alpha_{n,m} \geq t\right\}.$$

Then the filtration defined in (7.24) can be written succinctly as $\mathcal{F}^{\bar{\mathrm{X}}}_{n,t} = \mathcal{F}_{\bar{N}^\alpha(n,t)}$. We construct $\mathcal{M}^f_n(t)$ as

$$\mathcal{M}^f_n(t) := \sum_{k=n}^{\bar{N}^\alpha(n,t)-1}\left\{f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\right\}. \tag{D.29}$$

Clearly, $\mathcal{M}_n^f(t)$ is adapted to the filtration $(\mathcal{F}_{n,t}^{\bar{\mathrm{X}}})_{t\geq 0}$. Next we verify that $\mathcal{M}_n^f(t)$ is a martingale w.r.t. $(\mathcal{F}_{n,t}^{\bar{\mathrm{X}}})_{t\geq 0}$.

For any $t > s \geq 0$, if $k \leq \bar{N}^\alpha(n,s) - 1$, then $\check{x}_k, \check{x}_{k+1} \in \mathcal{F}_{n,s}^{\bar{\mathrm{X}}}$ and $\mathcal{F}_k \subset \mathcal{F}_{n,s}^{\bar{\mathrm{X}}}$, implying

$$\mathbb{E}\big\{f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\big|\mathcal{F}_{n,s}^{\bar{\mathrm{X}}}\big\} = f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k].$$

Otherwise, $k \geq \bar{N}^\alpha(n,s)$ and $\mathcal{F}_{n,s}^{\bar{\mathrm{X}}} \subseteq \mathcal{F}_k$, implying

$$\mathbb{E}\big\{f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\,\big|\,\mathcal{F}_{n,s}^{\bar{\mathrm{X}}}\big\} = 0.$$

Consequently,

$$\begin{aligned}
\mathbb{E}[\mathcal{M}_n^f(t)|\mathcal{F}_{n,s}^{\bar{\mathrm{X}}}] &= \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \mathbb{E}\big\{f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\,\big|\,\mathcal{F}_{n,s}^{\bar{\mathrm{X}}}\big\} \\
&= \sum_{k=n}^{\bar{N}^\alpha(n,s)-1} \big\{f(\check{x}_{k+1}) - f(\check{x}_k) - \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\big\} = \mathcal{M}_n^f(s),
\end{aligned}$$

which implies that $\mathcal{M}_n^f(t)$ is a martingale w.r.t. $\mathcal{F}_{n,t}^{\bar{\mathrm{X}}}$.

Now we focus on the residual term $\mathcal{R}_n^f(t)$. For notational simplicity, we define $\bar{t}_n^\alpha := \Gamma_{n,\bar{N}^\alpha(n,t)}^\alpha$. Then we have $\bar{t}_n^\alpha - \alpha_{\bar{N}^\alpha(n,t)-1} < t \leq \bar{t}_n^\alpha$. With this notation and $\underline{t}_n^\alpha$ introduced in Definition B.1,[1] $\mathcal{R}_n^f(t)$ can be expressed as

$$\begin{aligned}
\mathcal{R}_n^f(t) &= f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(0)) - \int_0^t \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\,\mathrm{d}s - \mathcal{M}_n^f(t) \\
&\overset{(\mathrm{D.29})}{=} f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(\bar{t}_n^\alpha)) - \int_0^t \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\,\mathrm{d}s + \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k] \\
&\overset{(\mathrm{D.21})}{=} f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(\bar{t}_n^\alpha)) - \int_0^t \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\,\mathrm{d}s + \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \Big(\alpha_k \mathcal{A}^x f(\check{x}_k) + R_k^f\Big) \\
&\overset{(a)}{=} f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(\bar{t}_n^\alpha)) + \int_0^{\bar{t}_n^\alpha} \big(\mathcal{A}^x f(\bar{\mathrm{X}}_n(\underline{s}_n^\alpha)) - \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\big)\,\mathrm{d}s \\
&\quad + \int_t^{\bar{t}_n^\alpha} \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\,\mathrm{d}s + \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} R_k^f,
\end{aligned} \tag{D.30}$$

where for the second equality, the telescoping component $f(\check{x}_{k+1}) - f(\check{x}_k)$ in (D.29) helps us simplify the expression of $\mathcal{M}_n^f(t)$ as

$$\mathcal{M}_n^f(t) = f(\check{x}_{\bar{N}^\alpha(n,t)}) - f(\check{x}_n) - \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \big\{\mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\big\}$$

[1] In the notation $\underline{t}_n^\alpha$, we view t as a variable.

$$= f(\bar{\mathrm{X}}_n(\bar{t}_n^\alpha)) - f(\bar{\mathrm{X}}_n(0)) - \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \left\{\mathbb{E}[f(\check{x}_{k+1}) - f(\check{x}_k)|\mathcal{F}_k]\right\};$$

for step (a), we used Fact B.2 (iii) such that $\alpha_k \mathcal{A}^x f(\check{x}_k) = \int_{\Gamma_{n,k}^\alpha}^{\Gamma_{n,k+1}^\alpha} \mathcal{A}^x f(\bar{\mathrm{X}}_n(\underline{s}_n^\alpha))\,\mathrm{d}s$.

Notice that $f \in C_c^\infty(\mathbb{R}^{d_x})$. For a fixed t, to control $\mathcal{R}_n^f(t)$ in the L_1 sense, we need to control the following L_1 distances $\mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\|$ and $\mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\bar{s}_n^\alpha)\|$ for any $s \in [0, \bar{t}_n^\alpha]$. From the definition of $\bar{\mathrm{X}}_n(\cdot)$ in (X), we have $\bar{\mathrm{X}}_n(\underline{s}_n^\alpha) = \check{x}_{N^\alpha(n,s)}$. Consequently,

$$\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha) = (s - \underline{s}_n^\alpha)\left(-B_1 \check{x}_{N^\alpha(n,s)} + \frac{R^x_{N^\alpha(n,s)}}{\alpha_{N^\alpha(n,s)}}\right) - \frac{s - \underline{s}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,s)}}}\tilde{\xi}_{N^\alpha(n,s)}.$$

It follows that for each $s \in [0, \bar{t}_n^\alpha]$, the following results hold

$$\begin{aligned}
\mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\| &\le (s - \underline{t}_n^\alpha)\mathbb{E}\left\|-B_1 \check{x}_{N^\alpha(n,s)} + \frac{R^x_{N^\alpha(n,s)}}{\alpha_{N^\alpha(n,s)}}\right\| + \frac{s - \underline{s}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,s)}}}\mathbb{E}\|\tilde{\xi}_{N^\alpha(n,s)}\| \\
&\overset{\text{(D.13)}}{\lesssim} (s - \underline{s}_n^\alpha) + \sqrt{s - \underline{s}_n^\alpha} \lesssim \sqrt{\alpha_{N^\alpha(n,s)}}, \\
\mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\bar{s}_n^\alpha)\| &\le \mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\| + \mathbb{E}\|\bar{\mathrm{X}}_n(\bar{s}_n^\alpha) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\| \\
&\le \mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\| + (\bar{s}_n^\alpha - \underline{s}_n^\alpha)\mathbb{E}\left\|-B_1 \check{x}_{N^\alpha(n,s)} + \frac{R^x_{N^\alpha(n,s)}}{\alpha_{N^\alpha(n,s)}}\right\| \\
&\quad + \frac{\bar{s}_n^\alpha - \underline{s}_n^\alpha}{\sqrt{\alpha_{N^\alpha(n,s)}}}\mathbb{E}\|\tilde{\xi}_{N^\alpha(n,s)}\| \\
&\lesssim \sqrt{\alpha_{N^\alpha(n,s)}}.
\end{aligned}$$

Since $f \in C_c^\infty(\mathbb{R}^{d_x})$, let $K := \mathrm{supp}(f)$ be its compact support. Then there exist finite constants

$$R := \sup_{x \in K}\|x\|,\ L_1 := \sup_x \|\nabla f(x)\|,\ L_2 := \sup_x \|\nabla^2 f(x)\|,\ L_3 := \sup_{x \ne y}\frac{\|\nabla^2 f(x) - \nabla^2 f(y)\|}{\|x - y\|}.$$

In particular, f is L_1-Lipschitz, ∇f is L_2-Lipschitz and $\nabla^2 f$ is L_3-Lipschitz. Recall $\mathcal{A}^x f(x) = \langle -B_1 x, \nabla f(x)\rangle + \frac{1}{2}\mathrm{tr}\big(\nabla^2 f(x)\,\Sigma_\xi\big)$. It is bounded in the sense that

$$|\mathcal{A}^x f(x)| \le \|B_1\|RL_1 + \frac{1}{2}\|\nabla^2 f(x)\|_F\|\Sigma_\xi\|_F \le \|B_1\|RL_1 + \frac{1}{2}d_x L_2\|\Sigma_\xi\|.$$

For the trace term, Lipschitz continuity follows from that of $\nabla^2 f$:

$$\left|\mathrm{tr}\big((\nabla^2 f(x) - \nabla^2 f(y))\,\Sigma_\xi\big)\right| \le \|\Sigma_\xi\|_F\,\|\nabla^2 f(x) - \nabla^2 f(y)\|_F \le dL_3\|\Sigma_\xi\|\|x - y\|.$$

For the drift term, we have

$$\begin{aligned}
\left|\langle B_1 x, \nabla f(x)\rangle - \langle B_1 y, \nabla f(y)\rangle\right| &\le \|B_1\|\Big(\|x\|\,\|\nabla f(x) - \nabla f(y)\| + \|x - y\|\,\|\nabla f(y)\|\Big) \\
&\le \|B_1\|RL_2\|x - y\| + \|B_1\|L_1\|x - y\|.
\end{aligned}$$

Combining the two parts shows that $\mathcal{A}^x f$ is Lipschitz. The above results can be summarized as

$$|f(x) - f(y)| \lesssim \|x - y\|, \quad |\mathcal{A}^x f(x)| \lesssim 1, \quad |\mathcal{A}^x f(x) - \mathcal{A}^x f(y)| \lesssim \|x - y\|.$$

Then we have

$$\begin{aligned}
\mathbb{E}|f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(\bar{t}_n^\alpha))| &\lesssim \mathbb{E}\|\bar{\mathrm{X}}_n(t) - \bar{\mathrm{X}}_n(\bar{t}_n^\alpha)\| \lesssim \sqrt{\alpha_{N^\alpha(n,t)}} = o(1),\\
\mathbb{E}\left|\int_t^{\bar{t}_n^\alpha} \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\,\mathrm{d}s\right| &\lesssim \int_t^{\bar{t}_n^\alpha} 1\,\mathrm{d}s = o(1),\\
\mathbb{E}\left|\int_0^{\bar{t}_n^\alpha} \left(\mathcal{A}^x f(\bar{\mathrm{X}}_n(\underline{s}_n^\alpha)) - \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))\right)\,\mathrm{d}s\right| &\lesssim \int_0^{\bar{t}_n^\alpha} \mathbb{E}\|\bar{\mathrm{X}}_n(s) - \bar{\mathrm{X}}_n(\underline{s}_n^\alpha)\|\,\mathrm{d}s\\
&\lesssim \int_0^{\bar{t}_n^\alpha} \sqrt{\alpha_{N^\alpha(n,s)}} ds \lesssim \sqrt{\alpha_n} = o(1).
\end{aligned} \tag{D.31}$$

Moreover, for a fixed t, Lemma D.2 implies

$$\mathbb{E}\left|\sum_{k=n}^{\bar{N}^\alpha(n,t)-1} R_k^f\right| \le \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \alpha_k \mathbb{E}\left|\frac{R_k^f}{\alpha_k}\right| \le \sup_{k' \ge n} \mathbb{E}\left|\frac{R_{k'}^f}{\alpha_{k'}}\right| \cdot \sum_{k=n}^{\bar{N}^\alpha(n,t)-1} \alpha_k \lesssim o(1). \tag{D.32}$$

Plugging (D.31) and (D.32) into (D.30) yields that for a fixe t, $\mathbb{E}|\mathcal{R}_n^f(t)| \to 0$ as $n \to \infty$.

For Part (ii), we similarly define

$$\bar{N}^\beta(n,t) := \min\left\{m \ge n \colon \Gamma_{n,m}^\beta \ge t\right\}$$

and

$$\mathcal{M}_n^g(t) = \sum_{k=n}^{\bar{N}^\beta(n,t)-1} \left\{f(\check{z}_{k+1}) - f(\check{z}_k) - \mathbb{E}[f(\check{z}_{k+1}) - f(\check{z}_k)|\mathcal{F}_k]\right\}.$$

The remaining proof is similar to that of Part (i). □

D.4 Proofs in Section 7.5.4

In this subsection, we present the detailed proof for the procedure in Section 7.5.4. We first give the formal proof of Theorem 4.1 (i) based on Proposition 5.2 and the following proposition, which guarantees the geometric ergodicity of the solutions of the SDEs in (7.10) and (7.12).

Proposition D.2 (Geoometric ergodicity for SDEs)**.** *Under Assumption 3.5, the SDE in (7.10) has a unique invariant distribution $\pi = \mathcal{N}(0, \Sigma_x)$, where Σ_x is the unique solution to (7.11). Moreover, suppose $\mathrm{X}(t)$ is a solution, for any compact set $K \subset \mathbb{R}^{d_x}$ and function $f \in C_c^\infty(\mathbb{R}^{d_x})$. Then there exist positive numbers A and b such that*

$$\sup_{x \in K} |\mathbb{E}[f(\mathrm{X}(t))|X(0) = x] - \mathbb{E}_{\mathrm{X} \sim \pi^\star} f(\mathrm{X})| \le Ae^{-bt}, \ \forall t \ge 0.$$

Similar results also hold for the SDE in (7.12).

Proof of Proposition D.2 . Without loss of generality, we focus on the following SDE

$$d\mathrm{X}(t) = -A\mathrm{X}(t)dt + Bd\mathrm{W}(t), \tag{D.33}$$

where $A, B \in \mathbb{R}^{d\times d}$, $-A$ is a Hurwitz matrix, and $\mathrm{W}(t)$ is an d-dimensional standard Brownian motion. For the SDE in (7.10), $A = B_1$ and $B = \Sigma_\xi^{1/2}$, while for the SDE in (7.12), $A = B_3 - \frac{\tilde{\beta} I}{2}$ and $B = \widetilde{\Sigma}_\psi^{1/2}$. By Proposition B.3, we have

$$\|e^{-At}\| \le Me^{-bt}, \forall t \ge 0, \text{ with } M := \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} b := \frac{1}{2\lambda_{\max}(P)}, \tag{D.34}$$

where $P := \int_0^\infty e^{-A^\top t} e^{-At} dt$.

Step 1: Uniqueness of invariant distribution.

Let

$$\Sigma_t := \int_0^t e^{-As} BB^\top e^{-A^\top s}\, ds, \qquad \Sigma := \int_0^\infty e^{-As} BB^\top e^{-A^\top s}\, ds.$$

By (D.34), Σ is well-defined. The solution $X(t)$ of (D.33) has the explicit representation

$$X(t) = e^{-At} X(0) + \int_0^t e^{-A(t-s)} B\, dW(s),$$

so conditionally on $X(0) = x$, $X(t) \sim \mathcal{N}(e^{-At}x, \Sigma_t)$.

We first prove that $\pi := \mathcal{N}(0, \Sigma)$ is an invariant distribution. Let $\mu_t^x = \mathcal{N}(e^{-At}x, \Sigma_t)$ denote the distribution of $X(t)$ when $X_0 = x$. To check the invariance of π, it suffices to check for its characteristic function $\widehat{\pi}(\theta) = \exp\big(-\frac{1}{2}\theta^\top \Sigma\theta\big)$. From the definition of Σ_t and Σ, we have

$$\Sigma - \Sigma_t = \int_t^\infty e^{-As} BB^\top e^{-A^\top s}\, ds = e^{-At}\Big(\int_0^\infty e^{-As} BB^\top e^{-A^\top s}\, ds\Big) e^{-A^\top t} = e^{-At}\Sigma e^{-A^\top t}.$$

Then the characteristic function of $X(t)$ under initial distribution π equals

$$\begin{aligned}\mathbb{E}_{X(0)\sim\pi}[e^{i\theta^\top X(t)}] &= \mathbb{E}_{X(0)\sim\pi}\big[e^{i\theta^\top e^{-At} X(0)}\big] \cdot e^{-\frac{1}{2}\theta^\top \Sigma_t \theta} = \widehat{\pi}(e^{-A^\top t}\theta)\, e^{-\frac{1}{2}\theta^\top \Sigma_t\theta} \\ &= \exp\Big(-\tfrac{1}{2}\theta^\top (e^{-At}\Sigma e^{-A^\top t} + \Sigma_t)\theta\Big) = \exp\Big(-\tfrac{1}{2}\theta^\top \Sigma\theta\Big) = \widehat{\pi}(\theta)\end{aligned}$$

Thus the characteristic function of the distribution of X_t under initial law π equals $\widehat{\pi}(\theta)$ for every t, showing that π is the invariant distribution.

Next, we verify the uniqueness. Let μ be an arbitrary invariant distribution of (D.33). Then for every bounded continuous test function f and $t > 0$,

$$\int \mathbb{E}[f(X_t) \mid X_0 = x]\, \mu(dx) = \int f\, d\mu.$$

Then the characteristic function $\widehat{\mu}$ of μ satisfies that for all $\theta \in \mathbb{R}^d$ and all $t > 0$,

$$\widehat{\mu}(\theta) = \mathbb{E}_{X(0)\sim\mu}[e^{i\theta^\top X(t)}] = \mathbb{E}_{X(0)\sim\mu}\big[e^{i\theta^\top e^{-At} X(0)}\big] \cdot e^{-\frac{1}{2}\theta^\top\Sigma_t\theta} = \widehat{\mu}(e^{-A^\top t}\theta)\, e^{-\frac{1}{2}\theta^\top \Sigma_t\theta}$$

Fix $\theta \in \mathbb{R}^d$. Let $t \to \infty$. The exponential bound in (D.34) implies $e^{-A^\top t}\theta \to 0$ and $\Sigma_t \to \Sigma$. Moreover, the characteristic function $\widehat{\mu}$ is continuous at 0 with $\widehat{\mu}(0) = 1$, leading to $\widehat{\mu}(\theta) = 1 \cdot e^{-\frac{1}{2}\theta^\top \Sigma \theta}$. Because this holds for every θ, $\widehat{\mu}$ coincides with the characteristic function of $\mathcal{N}(0,\Sigma)$, hence $\mu = \mathcal{N}(0,\Sigma) = \pi$. This proves uniqueness. Moreover, Σ is the unique solution to the Lyapunov equation $AX + XA^\top = BB^\top$ [3, Theorem 7.5].

Step 2: Exponential ergodicity.

For convenience, we denote

$$R := \sup_{x\in K} \|x\|, \quad L_1 := \sup_y |\nabla f(y)|, \quad L_2 := \sup_y \|\nabla^2 f(y)\|,$$

Then we have

$$\begin{aligned}\|\Sigma - \Sigma_t\| &= \Big\| \int_t^\infty e^{-As} BB^\top e^{-A^\top s}\, ds \Big\|_2 \le \|BB^\top\| \int_t^\infty \|e^{-As}\|^2\, ds \\ &\le \|BB^\top\| M^2 \int_t^\infty e^{-2bs}\, ds = C_\Sigma e^{-2bt},\end{aligned} \tag{D.35}$$

where $C_\Sigma := \frac{\|BB^\top\|_2 M^2}{2b}$ and b is defined in (D.34). Then we have the following decomposition

$$\begin{aligned}&\mathbb{E}[f(X(t)) \mid X(0) = x] - \mathbb{E}_{X\sim\pi} f(X) = \mathbb{E}\big[f(e^{-At}x + Y_t)\big] - \mathbb{E}\big[f(Y_\infty)\big] \\ &= \big(\mathbb{E}[f(e^{-At}x + Y_t)] - \mathbb{E}[f(Y_t)]\big) + \big(\mathbb{E}[f(Y_t)] - \mathbb{E}[f(Y_\infty)]\big),\end{aligned} \tag{D.36}$$

where $Y_t \sim \mathcal{N}(0,\Sigma_t)$ and $Y_\infty \sim \mathcal{N}(0,\Sigma)$.

To control the first term, we apply a second-order Taylor expansion around y. For any fixed y and vector $m := e^{-At}x$, there exists a point $\tilde{y}$ on the line segment between y and $y + m$ such that

$$f(y+m) - f(y) = \nabla f(y) \cdot m + \tfrac{1}{2} m^\top \nabla^2 f(\tilde{y}) m.$$

Take expectation w.r.t. Y_t and use uniform bounds L_1, L_2. For $x \in K$ with $\|x\| \le R$,

$$\begin{aligned}\big|\mathbb{E}[f(e^{-At}x + Y_t)] - \mathbb{E}[f(Y_t)]\big| &\le L_1 \|e^{-At}\| \|x\| + \tfrac{1}{2} L_2 \|e^{-At}\|^2 \|x\|^2 \\ &\le L_1 R M e^{-bt} + \tfrac{1}{2} L_2 R^2 M^2 e^{-2bt}.\end{aligned} \tag{D.37}$$

For the second term, we use Gaussian interpolation: for $s \in [0,1]$ set $\Sigma'_s := \Sigma + s(\Sigma_t - \Sigma)$ and let $Y'_s \sim \mathcal{N}(0,\Sigma'_s)$. The map $s \mapsto \mathbb{E}[f(Y'_s)]$ is differentiable and [31, Lemma 7.2.7]

$$\frac{d}{ds}\mathbb{E}[f(Y'_s)] = \tfrac{1}{2}\mathbb{E}\big[\mathrm{tr}((\Sigma_t - \Sigma)\nabla^2 f(Y'_s))\big].$$

Integrating s from 0 to 1 gives

$$\big|\mathbb{E}[f(Y_t)] - \mathbb{E}[f(Y_\infty)]\big| \le \tfrac{1}{2}\|\Sigma_t - \Sigma\|_F \mathbb{E}\|\nabla^2 f(Y'_s)\|_F \le \tfrac{1}{2} d L_2 \|\Sigma_t - \Sigma\|_2 \overset{\text{(D.35)}}{\le} \tfrac{1}{2} d L_2 C_\Sigma e^{-2bt}. \tag{D.38}$$

Substituting (D.37) and (D.38) into (D.36) yields, for all $x \in K$,

$$\sup_{x\in K} \big|\mathbb{E}[f(X_t) \mid X_0 = x] - \mathbb{E}_\pi f\big| \le L_1 R M e^{-bt} + \Big(\tfrac{1}{2} L_2 R^2 M^2 + \tfrac{1}{2} d L_2 C_\Sigma\Big) e^{-2bt} \le A e^{-bt}$$

where

$$A := L_1 R M + \tfrac{1}{2} L_2 R^2 M^2 + \tfrac{1}{2} d L_2 C_\Sigma, \tag{D.39}$$

This proves the theorem with explicit constants b, A given in (D.34) and (D.39). $\square$

Proof of Theorem 4.1 (i). The proof is divided into three steps. In the first step, we apply Proposition 5.2 to establish the convergence of convergent subsequences. In the second step, we verify Condition (iii) in Proposition 5.2. Finally, we apply Proposition 5.2 to establish the convergence of the whole sequence.

Step 1: Establishing subsequence convergence. For the conditions in Proposition 5.2, Condition (i) has been discussed in Section 2.2. Lemma 5.3 ensures that Condition (ii) holds. To verify Condition (iv), we apply Lemma 5.4, which shows that for any $f \in C_c^\infty(\mathbb{R}^{d_x})$ and $t \geq 0$, one has that

$$f(\bar{\mathrm{X}}_n(t)) - f(\bar{\mathrm{X}}_n(0)) - \int_0^t \mathcal{A}^x f(\bar{\mathrm{X}}_n(s))ds = \mathcal{M}_n^f(t) + \mathcal{R}_n^f(t),$$

where for each n, $\mathcal{M}_n^f(t)$ is a martingale w.r.t. $(\mathcal{F}_{n,t}^{\bar{\mathrm{X}}})_{t\geq 0}$ defined in (7.24) and for each t, $\mathbb{E}|\mathcal{R}_n^f(t)| \to 0$ as $n \to \infty$. Then for any $k \geq 0$, $0 \leq t_1 < t_2 < \cdots < t_k \leq t \leq t+s$, $f \in C_c^\infty(\mathbb{R}^{d_x})$ and $h_1, h_2, \ldots, h_k \in C_c(\mathbb{R}^{d_x})$

$$\begin{aligned}
&\left|\mathbb{E}\Big[\Big(f(\bar{\mathrm{X}}_n(t+s)) - f(\bar{\mathrm{X}}_n(t)) - \int_t^{t+s} \mathcal{A}^x f(\bar{\mathrm{X}}_n(r))dr\Big)\prod_{i=1}^k h_i(\bar{\mathrm{X}}_n(t_i))\Big]\right| \\
&= \left|\mathbb{E}\Big[\big(\mathcal{M}_n^f(t+s) - \mathcal{M}_n^f(t) + \mathcal{R}_n^f(t+s) - \mathcal{R}_n^f(t)\big)\prod_{i=1}^k h_i(\bar{\mathrm{X}}_n(t_i))\Big]\right| \\
&= \left|\mathbb{E}\Big\{\prod_{i=1}^k h_i(\bar{\mathrm{X}}_n(t_i))\,\mathbb{E}\big[\mathcal{M}_n^f(t+s) - \mathcal{M}_n^f(t) + \mathcal{R}_n^f(t+s) - \mathcal{R}_n^f(t)\,\big|\,\mathcal{F}_{n,t}^{\bar{\mathrm{X}}}\big]\Big\}\right| \\
&\leq \mathbb{E}\Big\{\prod_{i=1}^k |h_i(\bar{\mathrm{X}}_n(t_i))|\,\mathbb{E}\big[|\mathcal{R}_n^f(t+s)| + |\mathcal{R}_n^f(t)|\,\big|\,\mathcal{F}_{n,t}^{\bar{\mathrm{X}}}\big]\Big\} \to 0
\end{aligned}$$

as $n \to \infty$. Now we assume $\{\bar{\mathrm{X}}_{n_k}(\cdot)\}_{k=1}^\infty$ is a weakly convergent subsequence of $\{\bar{\mathrm{X}}_n(\cdot)\}_{n=1}^\infty$. Clearly, this subsequence satisfies Condition (iii) in Proposition 5.2. Suppose $\bar{\mathrm{X}}_{n_k}(0) \Rightarrow \tilde{\pi}$. Then Proposition 5.2 ensures that $\bar{\mathrm{X}}_{n_k}(\cdot) \Rightarrow \mathrm{X}^{\tilde{\pi}}(\cdot)$ as $k \to \infty$, where $\mathrm{X}^{\tilde{\pi}}(\cdot)$ is the solution of the SDE in (7.12) with the initial distribution $\tilde{\pi}$.

Step 2: Verifying initial conditions. To establish the convergence of $\bar{\mathrm{X}}_n(0)$, it suffices to prove that all the convergent subsequences have the same limit. In fact, this limit is $\pi^\star$, the unique invariant distribution of (7.10) . The uniqueness is guaranteed by Proposition D.2. In the next, we prove that if the subsequence $\bar{\mathrm{X}}_{n_k}(0) \Rightarrow \tilde{\pi}$, one must have $\tilde{\pi} = \pi^\star$. The proof is divided to the following two parts: (i) for any fixed $T > 0$, we show that there exists a solution $\mathrm{X}(t)$ to (7.10) such that $\tilde{\pi}$ has the same distribution as $\mathrm{X}(T)$; (ii) when T is sufficiently large, the geometric ergodicity implies that the distribution of $\mathrm{X}(T)$ can be arbitrarily close to $\pi^\star$.

Part (i). We first introduce some additional notation. Recall that $\Gamma_{n,n'}^\alpha = \sum_{k=n}^{n'-1} \alpha_k$. For $n \in \mathbb{N}$ and $t \geq 0$, we define

$$M^\alpha(n,t) := \min\left\{m \geq 0 \colon \Gamma_{m,n}^\alpha \leq t\right\} \quad \text{and} \quad \tilde{t}_n^\alpha := \Gamma_{M^\alpha(n,t),n}^\alpha.$$

Since $\alpha_n \to 0$, we have $\tilde{t}_n^\alpha \uparrow t$.

For a fixed $T > 0$, we consider the subsequence $\{\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\cdot)\}_{k=0}^{\infty}$, which satisfies

$$\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(0) = \check{x}_{M^\alpha(n_k,T)} \quad \text{and} \quad \bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\widetilde{T}^\alpha_{n_k}) = \check{x}_{n_k},$$

where $\widetilde{T}^\alpha_n := \Gamma^\alpha_{M^\alpha(n,T),n}$ By Lemma 5.3, $\{\bar{\mathrm{X}}_n(t)\}_{n=1}^{\infty}$ is tight. Then Prokhorov's theorem (i.e., Proposition 2.1 (ii)) implies that the subsequence $\{\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\cdot)\}_{k=0}^{\infty}$ further has a weakly convergent subsequence. Without loss of generality, we assume the subsequence $\{\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\cdot)\}_{k=0}^{\infty}$ itself weakly converges. Then its initial distribution sequence also converges weakly, i.e., $\check{x}_{M^\alpha(n_k,T)} \Rightarrow \mu_T$ for some distribution μ_T. Then by Proposition 5.2, $\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\cdot) \Rightarrow \mathrm{X}^{\mu_T}(\cdot)$, where $\mathrm{X}^{\mu_T}(\cdot)$ is a solution to (7.10) with initial distribution μ_T. Consequently, we also have $\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(T) \Rightarrow \mathrm{X}^{\mu_T}(T)$. Moreover, recall that we have verified the second condition of Proposition 5.1 in the proof of Lemma 5.3. Then $|T - \widetilde{T}^\alpha_{n_k}| \to 0$ implies that $\left\|\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\widetilde{T}^\alpha_{n_k}) - \bar{\mathrm{X}}_{M^\alpha(n_k,T)}(T)\right\| \xrightarrow{p} 0$ as $k \to \infty$. By Slutsky's theorem, $\bar{\mathrm{X}}_{M^\alpha(n_k,T)}(\widetilde{T}^\alpha_{n_k}) \Rightarrow \mathrm{X}^{\mu_T}(T)$, i.e., $\check{x}_{n_k} \Rightarrow \mathrm{X}^{\mu_T}(T)$. Since $\check{x}_{n_k} \Rightarrow \tilde{\pi}$, this implies that for any $T > 0$, $\mathrm{X}^{\mu_T}(T) \sim \tilde{\pi}$.

Part (ii). Next, we apply $\check{x}_{M^\alpha(n_k,T)} \Rightarrow \mu_T$ and $\mathrm{X}^{\mu_T}(T) \sim \tilde{\pi}$ to prove $\tilde{\pi} = \pi^\star$. Since $\{\check{x}_n\}_{n=1}^{\infty}$ is tight, for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{R}^{d_x}$ only depending on ε such that $\sup_{n\geq 0} \mathbb{P}(\check{x}_n \in K_\varepsilon) \geq 1-\varepsilon$. It follows that $\mu_T(K_\varepsilon) \geq 1-\varepsilon$ for any $T > 0$. Moreover, Proposition D.2 implies that for a fixed $g \in C_c^\infty(\mathbb{R}^{d_x})$, we can find T_ε such that

$$\sup_{x\in K_\varepsilon} |\mathcal{P}_{T_\varepsilon}\, g(x) - \mathbb{E}_{\mathrm{X}\sim\pi^\star}\, g(\mathrm{X})| \leq \varepsilon, \tag{D.40}$$

where $\mathcal{P}_t$ represents a semigroup induced by the SDE in (7.10), that is,

$$\mathcal{P}_t f(x) = \mathbb{E}[f(\mathrm{X}(t)) \,|\, \mathrm{X}(0) = x], \tag{D.41}$$

where $\mathrm{X}(\cdot)$ is the solution to the SDE in (7.10). Setting $T = T_\varepsilon$, we have $\mathrm{X}^{\mu_{T_\varepsilon}}(T_\varepsilon) \sim \tilde{\pi}$. Since $\mathrm{X}^{\mu_{T_\varepsilon}}(\cdot)$ is a solution to (7.10) with initial distribution μ_{T_ε}, we have

$$\mathbb{E}_{\mathrm{X}\sim\tilde{\pi}}\, g(\mathrm{X}) = \mathbb{E}\left[g(\mathrm{X}^{\mu_{T_\varepsilon}}(T_\varepsilon)) \,|\, \mathrm{X}(0) \sim \mu_{T_\varepsilon}\right] \overset{(D.41)}{=} \int_{\mathbb{R}^{d_x}} \mathcal{P}_{T_\varepsilon}\, g(x) d\mu_{T_\varepsilon}(x)$$

It follows that

$$\begin{aligned}
|\mathbb{E}_{\mathrm{X}\sim\tilde{\pi}}\, g(\mathrm{X}) - \mathbb{E}_{\mathrm{X}\sim\pi^\star}\, g(\mathrm{X})| &\leq \int_{\mathbb{R}^{d_x}} |\mathcal{P}_{T_\varepsilon} g(x) - \mathbb{E}_{\mathrm{X}\sim\pi^\star} g(\mathrm{X})|\, d\mu_{T_\varepsilon}(x) \\
&= \int_{K_\varepsilon} |\mathcal{P}_{T_\varepsilon} g(x) - \mathbb{E}_{\mathrm{X}\sim\pi^\star} g(\mathrm{X})|\, d\mu_{T_\varepsilon}(x) \\
&\quad\quad + \int_{K_\varepsilon^c} |\mathcal{P}_{T_\varepsilon} g(x) - \mathbb{E}_{\mathrm{X}\sim\pi^\star} g(\mathrm{X})|\, d\mu_{T_\varepsilon}(x) \\
&\leq \int_{K_\varepsilon} |\mathcal{P}_{T_\varepsilon} g(x) - \mathbb{E}_{\mathrm{X}\sim\pi^\star} g(\mathrm{X})|\, d\mu_{T_\varepsilon}(x) + 2\|g\|_\infty \mu_{T_\varepsilon}(K_\varepsilon^c) \\
&\overset{(a)}{\leq} \varepsilon + 2\|g\|_\infty \varepsilon,
\end{aligned}$$

where (a) is due to $\mu_{T_\varepsilon}(K_\varepsilon) \geq 1-\varepsilon$ and (D.40). Letting $\varepsilon \to \infty$, we obtain $\mathbb{E}_{\mathrm{X}\sim\tilde{\pi}}\, g(\mathrm{X}) = \mathbb{E}_{\mathrm{X}\sim\pi^\star}\, g(\mathrm{X})$. For any $g \in C_c^\infty(\mathbb{R}^{d_x})$, we can repeat the above procedure, implying $\tilde{\pi} = \pi^\star$.

Step 3: Concluding whole sequence convergence. This step is straightforward. In Steps 1 and 2, we have verified the all the conditions in Proposition 5.2. Moreover, $\bar{X}_n(0) \Rightarrow \pi^\star$, the invariant distribution of (7.10). Consequently, $\bar{X}_n(\cdot)$ converges weakly to the solution of (7.10) with the initial distribution $\pi^\star$, implying that this solution is a stationary process. Since B_1 is a Hurwitz matrix, the invariant distribution $\pi^\star = \mathcal{N}(0, \Sigma_x)$ with Σ_x satisfying the equation in (7.11) [25, Section 6.5]. $\square$

The proof of Corollary 5.1 is similar to that of Theorem 4.1 (i). Next, we give the proof of Theorem 4.1 (ii) based on Corollary 5.1.

Proof of Theorem 4.1 (ii). Under the tightness in Lemma 5.3, Proposition 2.1 shows that it suffices to examine the finite-dimensional distributions of $\bar{Y}_n(\cdot)$.

Recall that $\check{z}_n$ is defined as $\check{z}_n = \check{y}_n - \sqrt{\kappa_{n-1}} B_2 B_1^{-1} \check{x}_n$ with $\kappa_{n-1} = \beta_{n-1}/\alpha_{n-1} = o(1)$; Assumption 4.1 implies $\mathbb{E}\|\check{x}_n\|^2 = \mathcal{O}(1)$. It follows that $\check{y}_n - \check{z}_n \xrightarrow{p} 0$. Meanwhile, from the definition in (Y) and (Z), for each fixed t, $\bar{Y}_n(t) - \bar{Z}_n(t)$ is the linear combination of $\check{y}_{N^\beta(n,t)} - \check{z}_{N^\beta(n,t)}$ and $\check{y}_{N^\beta(n,t)+1} - \check{z}_{N^\beta(n,t)+1}$, both converging to the zero vector in probability. Consequently, $\bar{Y}_n(t) - \bar{Z}_n(t) \xrightarrow{p} 0$. Similarly, we could prove that for any positive integer k and $t_1, \ldots, t_k$, $(\bar{Y}_n(t_1) - \bar{Z}_n(t_1), \ldots, \bar{Y}_n(t_k) - \bar{Z}_n(t_k)) \xrightarrow{p} 0$. Applying Slutsky's theorem and Theorem 4.1, we obtain that the finite-dimensional distributions of $\bar{Y}_n(\cdot)$ converge weakly to the finite-dimensional distributions of the stationary distribution of (7.12). Consequently, $\bar{Y}_n(\cdot)$ converges weakly to the stationary distribution of (7.12).

Note that Assumption 3.5 implies that $-\left(B_3 - \frac{\tilde{\beta}}{2} I\right)$ is a Hurwitz matrix. The property of the SDE in (7.12) implies that the invariant distribution is $\mathcal{N}(0, \Sigma_y)$ with Σ_y satisfying the equation in (7.14) in [25, Section 6.5]. $\square$

Proof of Corollary 4.1. From the definition of $\check{x}_n$, we have $\alpha_{n-1}^{-1/2} \hat{x}_n \Rightarrow \mathcal{N}(0, \Sigma_x)$. Assumption 3.4 ensures that $\alpha_{n-1}/\alpha_n \to 1$. Then Slutsky's theorem implies $\alpha_n^{-1/2} \hat{x}_n \Rightarrow \mathcal{N}(0, \Sigma_x)$. Similarly, we could prove $\beta_n^{-1/2} \hat{y}_n \Rightarrow \mathcal{N}(0, \Sigma_y)$.

Moreover, since $x^\star = H(y^\star)$, we have $\alpha_n^{-1/2}(x_n - x^\star) = \alpha_n^{-1/2} \hat{x}_n + \alpha_n^{-1/2}(H(y_n) - H(y^\star))$. Using the delta method, we have $\beta_n^{-1/2}(H(y_n) - H(y^\star)) \Rightarrow \mathcal{N}(0, H^\star \Sigma_y (H^\star)^\top)$ with $H^\star = \nabla H(y^\star) \in \mathbb{R}^{d_x \times d_y}$. Since $\beta_n/\alpha_n \to 0$, applying Slutsky's theorem yields $\alpha_n^{-1/2}(H(y_n) - H(y^\star)) \Rightarrow 0$. Consequently, $\alpha_n^{-1/2}(x_n - x^\star) \Rightarrow \mathcal{N}(0, \Sigma_x)$. $\square$

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