Appendix to "Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover"

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1 Introduction

This paper contains the proofs for "Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover," by Bradley and Robinson [2005], which is concerned with order crossover in a periodic-review inventory system. In Section 2 we (i) define the family of quasi-uniform distributions, (ii) prove that a distribution from this family provides an upper bound on the variance of the number of orders outstanding σ_N^2, and (iii) show that $\sigma_N^2 \leq \sigma_L/\sqrt{3}$, where σ_L is the standard deviation of lead time. In Section 3, we compute a lower bound on the variance of the number of orders outstanding.

2 An Upper Bound on σ_N^2

In this section we first define and analyze the class of quasi-uniform discrete probability distributions. We then show in Section 2.2 that there exists a quasi-uniform distribution with the same first two moments, μ and σ^2 respectively, as any arbitrary lead time distribution. Finally, we prove in Section 2.3 that a quasi-uniform distribution maximizes the variance of the number of orders outstanding.

2.1 The Quasi-Uniform Discrete Probability Distribution

In the context of the periodic-review inventory system analyzed in "Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover," by Bradley and Robinson [2005], we define a probability mass function (p.m.f.) as $f_i \doteq \Pr\{L = i\}$ on the integer variables $0 \le i \le m$ with the corresponding cumulative mass function (c.m.f.) as $F_i = \sum_{j=0}^{i} f_j$. This distribution defines the behavior of the random variable of lead time L, which is then restricted by a (possibly arbitrarily large) lead time m. We refer to the p.m.f. in vector form as $f = \{f_i\}$.

In particular, we are interested in a generalization of discrete uniform distributions that Bradley and Robinson [2005] refer to as "quasi-uniform" distributions, which have uniform mass at all points with nonzero mass, except possibly the lower and upper mass points (denoted by ℓ and u respectively):

$$f_i = \begin{cases} \frac{1-p}{n-p-q} & i = \ell \\ \frac{1}{n-p-q} & \ell + 1 \le i \le u - 1 \\ \frac{1-q}{n-p-q} & i = u \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $i \ge 0$, where

$$p \le 1, \tag{2}$$

$$0 \le q \le 1. \tag{3}$$

We define the number of nonnegative mass points to be $n \doteq u - \ell + 1$. Thus, the mass on the upper point may be less than the mass on intermediate points,

$$f_u \leq f_i \quad \text{for } \ell + 1 \leq i \leq u - 1,$$

and the mass on the lower-most point f_ℓ may be less than, equal to, or greater than the mass on the intermediate points. We sometimes restrict $p \geq 0$ in which case

$$f_\ell \leq f_i \quad \text{for } \ell + 1 \leq i \leq u - 1.$$

When a distribution has a large variance relative to its mean, then the quasi-uniform distribution that matches the first two moments will have a mass at zero and $p < 0$ so that the mass at zero exceeds that of the intermediate points, $f_0 > f_i$ for $\ell + 1 \leq i \leq u - 1$.

Definition: We define the collection of all distributions that satisfy (1), (2), and (3) for integer $n \geq 2$ as the family of quasi-uniform distributions $\mathcal{F}$.

We will not be concerned with deterministic lead times where $n = 1$ and no crossover is possible. Moreover, we will ultimately not be concerned with lead time distributions where $n = 2$.

Definition: Let a_i denote the period in which an order placed in period i is received. *Strict order crossover* occurs when two orders are placed in periods j and k, with $j < k$, and $a_k < a_j$.

When $n = 2$, then no strict order crossover is possible and the variance of the number of orders outstanding σ_N^2 equals the variance of the lead time σ_L^2.

Each of the following expressions for the mean of a distribution $f \in \mathcal{F}$ is useful:

$$\begin{aligned}
\mu &= \ell + \frac{(n-1)(n-2q)}{2(n-p-q)} && (4)\\
&= \ell + \frac{(u-\ell)(u-\ell+1-2q)}{2(u-\ell+1-p-q)} && (5)\\
&= \ell + \frac{a}{2}(u-\ell)(u-\ell+1-2q). && (6)
\end{aligned}$$

The variance of lead time when lead time is distributed according to a quasi-uniform distribution is

$$\sigma^2 = \frac{(n-1)}{12(n-p-q)^2}\left[n^2(n+1) - 2n(2n-1)(p+q) + 12pq(n-1)\right], \tag{7}$$

and the variance of the number of orders outstanding in that situation is

$$\sigma_N^2 = \frac{(n-1)}{6(n-p-q)^2}\left[n^2 + n - 3(p+q)n + 6pq\right]. \tag{8}$$

We will be interested in varying p as a function of q while holding μ and n constant when $p \leq 0$ and $\ell = 0$, which we do with this expression for p derived from (4):

$$p_1(q) = \frac{(n-q)}{\mu} - \frac{(n-1)(n-2q)}{2\mu}. \tag{9}$$

When $p \geq 0$ (ℓ might not be zero) we will also be interested in holding μ, ℓ, and n constant while varying p or q, and so we write the following expressions using (4) so that p or q can be varied while preserving the mean by adjusting the dependent variable:

$$q(p) = \frac{n(n-1) + 2(\mu-\ell)(p-n)}{2(n-1-\mu+\ell)} \tag{10}$$

$$p_2(q) = n - \frac{n(n-1) - 2q(n-1-\mu+\ell)}{2(\mu-\ell)}. \tag{11}$$

2.2 Matching μ and σ^2 with Quasi-Uniform Distributions

We show in this section that a quasi-uniform distribution exists for any attainable mean and variance.

2.2.1 Matching Moments When $p \geq 0$

Lemma 1 *Holding μ, ℓ, and n constant, $q(p)$ is increasing in p.*

Proof: The result is immediate from (10). □

Lemma 2 *Holding μ, ℓ, and n constant, $p_2(q)$ is increasing in q.*

Proof: The result is immediate from (11). □

Lemma 3 *For $n \geq 3$, σ^2 is decreasing in p with $q = q(p)$, and decreasing in q with $p = p_2(q)$, while μ, ℓ, and n are held constant. For $n = 2$, σ^2 is invariant for $q = q(p)$ as p decreases and invariant for $p = p_2(q)$ as q decreases while μ, ℓ, and n are held constant.*

Proof: We first compute the variance of f in an alternative form to (7):

$$\begin{aligned} \sigma^2 &= \sum_{i=\ell}^{u} i^2 f_i - \mu^2 \\ &= \frac{n(n-1)(2n-1) - 6q(n-1)^2}{6(n-p-q)} - (\mu-\ell)^2 . \end{aligned} \tag{12}$$

Then we compute the derivatives of σ^2 with respect to p and q as follows:

$$\begin{aligned} \frac{d\sigma^2(p, q(p))}{dp} &= \frac{d\sigma^2}{dp} + \frac{\partial\sigma^2}{\partial q}\frac{dq(p)}{dp} \\ \frac{d\sigma^2(p_2(q), q)}{dq} &= \frac{d\sigma^2}{dq} + \frac{\partial\sigma^2}{\partial q}\frac{dp_2(q)}{dq}. \end{aligned}$$

First, we find the following after a number of steps:

$$\frac{d\sigma^2(p, q(p))}{dp} = \frac{(n-1)^2\left[n(2n-1) - 6q(n-1) - 6(\mu-\ell)(n-p-q)\right]}{6(n-1-\mu+\ell)(n-p-q)^2}.$$

Using (4) to replace $(\mu - \ell)$ in the numerator yields

$$\begin{aligned} \frac{d\sigma^2(p, q(p))}{dp} &= \frac{(n-1)^2\left[n(2n-1) - 6q(n-1) - 3(n-1)(n-2q)\right]}{6(n-1-\mu+\ell)(n-p-q)^2} \\ &= \frac{2(n-1)^2(2-n)}{6(n-1-\mu+\ell)(n-p-q)^2} \\ &< 0 \end{aligned}$$

for $n \geq 3$ and $d\sigma^2(p, q(p))/dp = 0$ for $n = 2$. Similarly, we find after a number of computational steps

$$\frac{d\sigma^2(p_2(q), q)}{dq} = \frac{(n-1)^2}{(n-p-q)}\left[-1 + \frac{n(2n-1) - 6q(n-1)}{6(n-p-q)(\mu-\ell)}\right],$$

and replacing $(\mu - \ell)$ using (4) yields

$$\begin{aligned} \frac{d\sigma^2(p_2(q), q)}{dq} &= \frac{n(n-1)^2(2-n)}{3(n-p-q)(n-1)(n-2q)} \\ &< 0 \end{aligned}$$

for $n \geq 3$ and $d\sigma^2(p_2(q), q)/dq = 0$ for $n = 2$. □

Lemma 4 *The variance σ^2 is increasing in n.*

Proof: We start by re-writing the expression for σ^2 in (12) as

$$\sigma^2 = \frac{2n^3 - 3\left(1+2q\right)n^2 + \left(1+12q\right)n - 6q}{6\left(n-p-q\right)} - \left(\mu - \ell\right)^2.$$

Treating n as a continuous variable for the moment, we take the derivative of σ^2 w.r.t. n:

$$\frac{d\sigma^2}{dn} = \frac{4n^3 + 3\left(1+2p\right)n^2 + 6\left(1+2q\right)\left(p+q\right)n - p + 5q - 12q\left(p+q\right)}{6\left(n-p-q\right)^2}.$$

The numerator evaluated at $n = 2$ is positive,

$$20 - 13p + 17q + 12pq + 12q^2 > 0,$$

because $p \leq 1$. Thus, the derivative is positive at $n = 2$ because the denominator is positive, and remains so as n increases because it can be shown that the numerator is convex. $\square$

As in Bradley and Robinson [2005], define the fractional part of the mean to be $\delta \doteq \mu - \lfloor \mu \rfloor$.

Lemma 5 *The distributions $f \in \mathcal{F}$ with mean μ have a minimum possible variance $\sigma^2 = \delta\left(1-\delta\right)$ and at most two positive mass points, i.e., $n \leq 2$.*

Proof: As shown above, the variance is increasing in n for any p and q, and so the variance is minimized by minimizing n. If μ is noninteger, then the minimum possible variance is attained by $n = 2$. If μ is integer, then the minimum variance is zero and is attained by a distribution with a single mass point. Using equations (1) and (11), it can be shown that $f_\ell = 1 - \mu + \ell$ and $f_u = \mu - \ell$ for any p and q with $n = 2$ for a given μ — *i.e.*, the masses of a two-point distribution for a given μ are invariant with p and q because a unique two-point distribution attains a specific mean. Thus the minimum variance for distributions in $\mathcal{F}$ with noninteger means is attained by a two-point distribution, $\sigma^2 = \left(\mu - \ell\right)\left(1 - \mu + \ell\right) = \delta\left(1-\delta\right)$. $\square$

Denote the mean and the variance of a distribution $f \in \mathcal{F}$ with parameters n, ℓ, p, and q, as $\mu_{\mathcal{F}}\left(n, \ell, p, q\right)$ and $\sigma^2_{\mathcal{F}}\left(n, \ell, p, q\right)$ respectively. Also, denote the p.m.f. with parameters

n, ℓ, p, and q, as $f(n,\ell,p,q) = \{f_i(n,\ell,p,q) : 0 \le i \le m\}$. The following two lemmas show that although two $f \in \mathcal{F}$ have different parameters, they may in fact designate the same distribution. This is rather obvious when one thinks about setting either p or q to one, which reduces the mass at an end point to zero, in which case the effective number of nonzero mass points is less than the formally defined quantity n.

Lemma 6 *The probability mass functions $f^1(n,\ell,p,0)$ and $f^2(n+1,\ell,p,1)$ are equivalent.*

Proof: Let a^1 correspond with f^1, and a^2 with f^2. First we see that $a^1 = a^2$:

$$a^1 = \frac{1}{n-p-0} = \frac{1}{(n+1)-p-1} = a^2.$$

Thus $f^1_\ell(n,\ell,p,0) = (1-p)\,a^1 = (1-p)\,a^2 = f^2_\ell(n+1,\ell,p,1)$, and $f^1_i(n,\ell,p,0) = a^1 = a^2 = f^2_i(n+1,\ell,p,1)$ for $\ell+1 \le i \le u-1$. Also, $f^1_u(n,\ell,p,0) = (1-q)\,a^1 = a^1 = a^2 = f^2_u(n+1,\ell,p,1)$, and $f^1_{u+1}(n,\ell,p,0) = 0 = (1-1)\,a^2 = f^2_{u+1}(n+1,\ell,p,1)$. Finally, $f^1_i(n,\ell,p,0) = 0 = f^2_i(n+1,\ell,p,1)$ for $0 \le i \le \ell-1$, and $u+2 \le i \le m$. $\square$

Lemma 7 *The probability mass functions $f^1(n,\ell,0,q)$ and $f^2(n+1,\ell-1,1,q)$ are equivalent.*

Proof: Again, let a^1 correspond with f^1, and a^2 with f^2. First we see that $a^1 = a^2$:

$$a^1 = \frac{1}{n-0-q} = \frac{1}{(n+1)-1-q} = a^2.$$

Thus $f^1_{\ell-1}(n,\ell,0,q) = 0 = (1-1)\,a^2 = (1-p)\,a^2 = f^2_{\ell-1}(n+1,\ell-1,1,q)$, and $f^1_\ell(n,\ell,0,q) = (1-p)\,a^1 = (1-0)\,a^1 = a^1 = a^2 = f^2_\ell(n+1,\ell-1,1,q)$. Then $f^1_i(n,\ell,p,0) = a^1 = a^2 = f^2_i(n+1,\ell,p,1)$ for $\ell+1 \le i \le u-1$. Finally, $f^1_u(n,\ell,0,q) = (1-q)\,a^1 = (1-q)\,a^2 = f^2_u(n+1,\ell-1,1,q)$. Finally, $f^1_i(n,\ell,p,0) = 0 = f^2_i(n+1,\ell,p,1)$ for $0 \le i \le \ell-2$, and $u+1 \le i \le m$. $\square$

An immediate consequence of Lemmas 6 and 7 is that

$$\begin{aligned}
\mu_{\mathcal{F}}(n,\ell,p,0) &= \mu_{\mathcal{F}}(n+1,\ell,p,1) \\
\sigma^2_{\mathcal{F}}(n,\ell,p,0) &= \sigma^2_{\mathcal{F}}(n+1,\ell,p,1) \\
\mu_{\mathcal{F}}(n,\ell,0,q) &= \mu_{\mathcal{F}}(n+1,\ell-1,1,q) \\
\sigma^2_{\mathcal{F}}(n,\ell,0,q) &= \sigma^2_{\mathcal{F}}(n+1,\ell-1,1,q)\,.
\end{aligned}$$

Lemma 8 *The variance σ^2 for a given mean μ and $0 \le p \le 1$ is maximized over n and p by $n = \lceil 1+2\mu \rceil$ and $p = 0$. Moreover the lower mass point is $\ell = 0$ for this n and p, and the corresponding value of q is*

$$q = \frac{\lceil 1+2\mu \rceil \left(\lceil 2\mu \rceil - 2\mu\right)}{2\left(\lceil 2\mu \rceil - \mu\right)}.$$

Proof: To maximize the variance over n we must, according to Lemma 4, increase n to the greatest extent possible. With a quasi-uniform distribution, the only constraint we face in increasing n is $\ell \ge 0$. Thus, the n that maximizes the variance will induce $\ell = 0$. We also know from Lemma 3 that $p = 0$ and $q = 0$ maximize the variance with this n given the constraints $0 \le p \le 1$ and $q \ge 0$.

We rewrite ℓ as a function of μ, n, and q:

$$\ell = \mu - \frac{(n-1)(n-2q)}{2(n-p-q)}.$$

It is easy to show by taking first differences that ℓ is decreasing in n. It is also easy to show that ℓ is increasing in p and q.

So, for the moment we set $p = 0$ and $q = 0$ (note that $p = 0$ maximizes the variance) for which the corresponding ℓ from the above expression is

$$\ell^* = \mu - \frac{(n-1)}{2}.$$

Then we find the minimum n such that $\ell^* \le 0$:

$$n^* = \arg\min_n \left\{ \ell^* = \mu - \frac{(n-1)}{2} \le 0 \right\},$$

resulting in

$$\begin{aligned} n^* &= \arg\min_n \{n - 1 \geq 2\mu\} \\ &= \arg\min_n \{n \geq 1 + 2\mu\} \\ n^* &= \lceil 1 + 2\mu \rceil . \end{aligned}$$

Then we increase q until $\ell = 0$. That value of q such that $\ell = 0$ and $p = 0$ from (10) is

$$q = \frac{n^* (n^* - 1 - 2\mu)}{2 (n^* - 1 - \mu)}.$$

Substituting $\lceil 1 + 2\mu \rceil$ for n^* yields the expression shown above in the lemma statement. It is easily verified that $0 \leq q \leq 1$.

If we were to have selected n greater by one for n^* then

$$\begin{aligned} \ell &= \mu - \frac{(\lceil 2 + 2\mu \rceil - 1)(\lceil 1 + 2\mu \rceil - 2q)}{2 (\lceil 1 + 2\mu \rceil - p - q)} \\ &= \mu - \frac{\lceil 1 + 2\mu \rceil (\lceil 1 + 2\mu \rceil - 2q)}{2 (\lceil 1 + 2\mu \rceil - p - q)}, \end{aligned}$$

which setting $p = 1$ and $q = 1$ to find the greatest ℓ possible with this choice of n, we find:

$$\begin{aligned} \ell &= \mu - \frac{\lceil 1 + 2\mu \rceil}{2} \\ &< 0, \end{aligned}$$

so that the n^* we selected is the maximum possible to have $p = 0$. $\square$

Lemma 9 *The maximum variance for a quasi-uniform distribution with* $0 \leq p \leq 1$ *is*

$$\widetilde{\sigma}^2 \doteq \frac{\lceil 2\mu \rceil \left[\lceil 1 + 2\mu \rceil^2 (\lceil 2\mu \rceil + 2) - 2 \lceil 1 + 2\mu \rceil (2 \lceil 1 + 2\mu \rceil - 1)(p + q) + 12pq \lceil 2\mu \rceil \right]}{12 (\lceil 1 + 2\mu \rceil - p - q)^2}. \quad (13)$$

Proof: This result follows directly from the parameters defined in Lemma 8. $\square$

Lemma 10 *If* $q > p$ *and* $n \geq 2$*, then*

$$0 < \frac{dq(p)}{dp} < 1,$$

and thus $q > p$ *continues to hold as* p *is reduced to zero.*

Proof: Assume p is set to some arbitrary quantity $\hat{p}$ such that $0 < \widehat{p} < q \leq 1$. From equation (10):

$$\frac{dq\,(p)}{dp} = \frac{\mu - \ell}{n - 1 - \mu + \ell}, \tag{14}$$

which, because both the numerator and denominator are positive, is less than one if

$$\begin{aligned} \mu - \ell &< n - 1 - \mu + \ell \\ \mu - \ell &< \frac{n-1}{2}. \end{aligned} \tag{15}$$

From (4)

$$\mu - \ell = \frac{(n-1)}{2} \frac{(n-2q)}{(n-p-q)},$$

and so the following condition is equivalent to (15):

$$\begin{aligned} \frac{(n-1)}{2} \frac{(n-2q)}{(n-p-q)} &< \frac{n-1}{2} \\ \frac{(n-2q)}{(n-p-q)} &< 1 \\ q &> p, \end{aligned}$$

which holds because we set $p = \widehat{p} < q$. Thus, $\frac{dq(p)}{dp} < \frac{dp}{dp} = 1$ at $p = \widehat{p} < q$, and so $p < q$ persists as p is decreased from any arbitrary $\widehat{p}$. □

Lemma 11 *If $p > q$ and $n \geq 2$, then*

$$\frac{dp_2\,(q)}{dq} < 1.$$

Proof: This proof is similar to that of Lemma 10 except that the necessary and sufficient condition for $dp_2\,(q)\,/dq < 1$ is $q < p$, which is a condition that is maintained as q decreases toward zero. □

An algorithmic proof follows that shows that a distribution $f \in \mathcal{F}$ exists that matches any mean and variance less than or equal to $\widetilde{\sigma}^2$ as shown in (13). We limit $p \geq 0$ in this algorithm. While one might devise a more expedient method of computing the parameters of a quasi-uniform distribution that match two arbitrary first moments, this algorithm suffices for our purposes of showing the existence of a quasi-uniform distribution for any two moments. Briefly, the algorithm starts with $n = 2$ and quasi-uniform parameters ℓ, u, p, and q to attain the specified mean μ. By Lemma 5, this two-point distribution has the minimum possible variance for a quasi-uniform distribution, $\sigma^2 = \delta(1-\delta)$. We hold μ constant throughout the algorithm while generating quasi-uniform distributions with a continuously increasing variances σ^2 by alternately decreasing p and q, and periodically increasing n. When p is decreased, μ is held constant by setting q according to (10). When q is decreased, μ is held constant by setting p according to (11). When either "control variable" p or q is decreased, it reaches zero at a point where the remaining control variable, q or p respectively, remains positive; at that point the control variable p or q is reset from 0 to 1 and the values for n and ℓ are updated so that μ and σ^2 are preserved. Then the alternate control variable, q or p is invoked and decreased, and so on. The algorithm maintains the n, ℓ, u, p, and q consistent with quasi-uniform distributions throughout and eventually attains the variance σ^2 specified in (13).

Lemma 12 *A distribution* $f \in \mathcal{F}$ *with* $p \geq 0$ *exists for each arbitrary combination of mean* $0 < \mu < \infty$ *and variance* $\delta(1-\delta) \leq \sigma^2 \leq \widetilde{\sigma}^2$.

Proof: An algorithm by which the parameters n, p, q and ℓ can be determined to match the first two moments, μ and σ^2, of a discrete probability distribution follows below. Denote the starting values of the algorithm as n_0, μ_0, p_0, q_0 and ℓ_0. Denote the parameter values that we seek as n^*, μ^*, p^*, q^* and ℓ^*. Denote the value of the mean and variance given the current parameters in the algorithm as $\widetilde{\mu}_{\mathcal{F}}(n, \ell, p, q)$ and $\widetilde{\sigma}^2_{\mathcal{F}}(n, \ell, p, q)$ respectively. The target moments are denoted by $\mu_\bigstar$ and $\sigma^2_\bigstar$. The algorithm generates quasi-uniform distributions with mean $\mu_\bigstar$ and continuously increasing variances σ^2. The algorithm terminates when $\widetilde{\sigma}^2_{\mathcal{F}}(n, \ell, p, q) = \sigma^2_\bigstar$.

Algorithm

1. Set $n_0 = 2, \ell_0 = \lfloor \mu \rfloor, p_0 = \delta$, and $q_0 = 1 - \delta$. Then $\widetilde{\mu}_{\mathcal{F}}(n_0, \ell_0, p_0, q_0) = \mu_{\bigstar}$, and $\widetilde{\sigma}^2_{\mathcal{F}}(n_0, \ell_0, p_0, q_0) = \delta(1-\delta)$.

2. Decrease p, and simultaneously set q according to $q(p)$, until either $p = 0$ or $q = 0$. If $q = 0$, then go to Step 3. If $p = 0$, then go to Step 4.

3. Set $q = 1$, and $n = n + 1$. Decrease p until $p = 0$, and simultaneously set q according to $q(p)$, at which point the terminal value of q is

$$\widehat{q} = \frac{2n(\mu - \ell) - n(n-1)}{2(\mu - \ell - n + 1)}. \tag{16}$$

4. Set $p = 1$, $n = n + 1$, $\ell = \ell - 1$. Decrease q until $q = 0$, and simultaneously set p according to $p_2(q)$, at which point the terminal value of p is

$$\widehat{p} = n - \frac{n(n-1)}{2\mu}. \tag{17}$$

5. Go to Step 3.

The algorithm will eventually identify the target parameters n^*, μ^*, p^*, q^* and ℓ^* for a quasi-uniform distribution with target mean $\mu_{\bigstar}$ and target variance $\delta(1-\delta) \leq \sigma^2_{\bigstar} \leq \widetilde{\sigma}^2$ because (i) the parameters n, ℓ, p, and q are maintained consistent with the quasi-uniform family, (ii) the algorithm maintains the target mean $\widetilde{\mu}_{\mathcal{F}}(n, \ell, p, q) = \mu_{\bigstar}$ throughout, and (iii) the algorithm generates a continuously increasing variance $\widetilde{\sigma}^2_{\mathcal{F}}(n, \ell, p, q)$ that starts at $\delta(1-\delta)$ and eventually attains $\widetilde{\sigma}^2$.

It is clear that n and ℓ retain valid integer values. At the start, $n = 2$ and increases in increments of one thereafter. The parameter ℓ starts as an integer and is decreased in increments of one at various points in the algorithm and, moreover, it is never allowed to be less than zero.

The parameters p and q are maintained at feasible values $0 \leq p \leq 1$ throughout the algorithm. By Lemmas 10 and 11 the control variable, p in Step 3 and q in Step 4, decrease more quickly then do $q(p)$ and $p_2(q)$. Thus the control variable attains zero while the remaining parameter is still positive, and less than one because $q(p)$ and $p_2(q)$ are decreasing in p and q respectively. It is easy to verify that $\widehat{p} > 0$ in (17) and $\widehat{q} > 0$ in (16) using (4). Then whenever p or q attains zero, it is reset to 1.

Lemmas 6 and 7 ensure that the variance is maintained whenever p or q are reset in Steps 3 and 4. Lemma 3 shows that $\widetilde{\sigma}^2_{\mathcal{F}}(n, \ell, p, q)$ decreases in p and q so that $\widetilde{\sigma}^2_{\mathcal{F}}(n, \ell, p, q)$ increases as p and q decrease throughout the algorithm. Lemma 8 shows that the variance increases to its maximum value $\widetilde{\sigma}^2$ in Lemma 9 once n increases to $n = \lceil 1 + 2\mu \rceil$ and subsequently p is decreased to $p = 0$. $\square$

2.2.2 Matching Moments When $p \leq 0$

We have shown in the previous subsection that given an arbitrary mean μ we can generate a sequence of quasi-uniform distributions with continuously increasing variances starting at $\sigma^2 = \delta(1 - \delta)$ and ending with the value identified in Lemma 9, which depends on the mean μ, while restricting $0 \leq p \leq 1$ and adhering to the constraint $\ell \geq 0$. In this subsection we will show that a sequence of quasi-uniform distributions with continuously increasing variance, starting with the value identified in Lemma 9, can be generated that increases without bound. We generate this sequence while maintaining the lower mass point at zero, $\ell = 0$, and allowing p to be negative, *i.e.*, $p \leq 0$.

Lemma 13 *The minimum variance σ^2 occurs for $p \leq 0$ when $p = 0$.*

Proof: This follows trivially from Lemma 3. $\square$

The next proof is essentially redundant with Lemmas 8 and 9, which identify the greatest variance possible with $p \geq 0$ and, in particular, for $p = 0$ and $\ell = 0$. These parameters define the distribution with the least variance under the condition $p \leq 0$, which is the circumstance we consider in this section. We show this proof for completeness and for an alternate method of proof for the lemmas in the previous section.

Lemma 14 *The minimum value of n given μ such that $\ell = 0$ and $p = 0$ is $n = \lceil 1 + 2\mu \rceil$. Moreover, the variance with these parameters is*

$$\sigma^2 = \frac{\lceil 2\mu \rceil \left[\lceil 1 + 2\mu \rceil^2 \left(\lceil 2\mu \rceil + 2 \right) - 2 \lceil 1 + 2\mu \rceil \left(2 \lceil 1 + 2\mu \rceil - 1 \right) (p + q) + 12pq \lceil 2\mu \rceil \right]}{12 \left(\lceil 2\mu \rceil + 1 - p - q \right)^2} \tag{18}$$

and the corresponding value of q is

$$q = \frac{\lceil 1 + 2\mu \rceil \left(\lceil 2\mu \rceil - 2\mu \right)}{2 \left(\lceil 2\mu \rceil - \mu \right)}. \tag{19}$$

Proof: From (4):

$$\ell = \mu - \frac{(n-1)(n-2q)}{2(n-p-q)}.$$

The first difference of the second term in the right-hand side is

$$\frac{-n^2 - (1 - 2p - 2q) n + 2q - 2pq - 2q^2}{2(n-p-q)(n+1-p-q)}.$$

The numerator can be rewritten

$$-(n + 1 - 2p - 2q) n + 2q - 2pq - 2q^2,$$

which for $p = 0$ is

$$-(n + 1 - 2q) n + 2q(1-q),$$

which, in turn is maximized w.r.t. q at $q = 1$ for all n so that the maximum numerator is

$$-(n-1) n < 0$$

for all $n = 2$. The denominator is positive, and so the first difference is negative.

So, we must solve the following to find the minimum n for $\ell = 0$:

$$\arg\min_n \left\{ \mu - \frac{(n-1)(n-2q)}{2(n-p-q)} \leq 0 \right\},$$

subject to $0 \leq q \leq 1$ and $p = 0$.

The derivative of the objective function w.r.t. q is

$$\frac{2(n-1)}{2(n-p-q)} - \frac{(n-1)(n-2q)}{2(n-p-q)^2} = \frac{n^2 - (1+2p) n + 2p}{2(n-p-q)^2},$$

which is greater than zero for $n \geq 2$ and $p = 0$ because the numerator is greater than zero:

$$\begin{aligned} n^2 - (1+2p)\,n + 2p &= n^2 - n \\ &\geq 0. \end{aligned}$$

So we need to set q as small as possible to find the smallest possible n. Thus, for the moment, set $q = 0$, which we will later compute to match the desired mean μ.

Then the optimal n is the minimum one that just allows the objective function to become zero or negative with $q = 0$ and $p = 0$,

$$\begin{aligned} \mu - \frac{(n-1)\,n}{2n} &\leq 0 \\ \mu - \frac{(n-1)}{2} &\leq 0 \\ n &\geq 1 + 2\mu \\ n &= \lceil 1 + 2\mu \rceil\,. \end{aligned}$$

Having identified the minimum n, we now set (increase) q using (4) so that desired mean is attained. Starting with

$$\mu = \ell + \frac{(n-1)\,(n-2q)}{2\,(n-p-q)},$$

we solve for q,

$$q = \frac{n\,(n-1-2\mu)}{2\,(n-1-\mu)}, \tag{20}$$

which is easily verified to satisfy $0 \leq q \leq 1$. To demonstrate this, we know that

$$\mu \leq \frac{n-1}{2}$$

from Lemma 8, and so $(n - 1 - 2\mu) \geq 0$, and thus $q \geq 0$. Then, we also know from Lemma 8 that

$$\mu \geq \frac{n-2}{2},$$

from which we find

$$\begin{aligned}
n-2-2\mu &\leq 0 \\
(n-1)(n-2-2\mu) &\leq 0 \\
(n-1)(n-2)-2\mu(n-1) &\leq 0 \\
n^2-3n-2\mu(n-1)+2 &\leq 0 \\
n(n-1-2\mu) &\leq 2(n-1-\mu) \\
\frac{n(n-1-2\mu)}{2(n-1-\mu)} &\leq 1 \\
q &\leq 1.
\end{aligned}$$

Substituting $n = \lceil 1+2\mu \rceil$ into (20) yields the expression for q in the lemma statement. $\square$

Lemma 15 *If $\ell = 0$ and $p \leq 0$, then $\lim_{n\to\infty} \sigma_{\mathcal{F}}^2 = \infty$.*

Proof: We first rewrite equation (7) as

$$\sigma^2 = \frac{1}{12}\left[\frac{(n^2-1)}{\left(1-\frac{p}{n}-\frac{q}{n}\right)^2} - \frac{2\left(1-\frac{1}{n}\right)(2n-1)(p+q)}{\left(1-\frac{p}{n}-\frac{q}{n}\right)^2} + \frac{12pq\left(1-\frac{1}{n}\right)^2}{\left(1-\frac{p}{n}-\frac{q}{n}\right)^2}\right].$$

Taking the limit as $n \to \infty$ for any arbitrary values of $0 \leq q \leq 1$ and $p \leq 0$:

$$\begin{aligned}
\lim_{n\to\infty} \sigma^2 &= \frac{1}{12}\left[\left(n^2-1\right) - 2(2n-1)(p+q) + 12pq\right] \\
&= \frac{1}{12}\left[n^2 - 4(p+q)n + 2(p+q) + 12pq - 1\right] \\
&= \infty. \quad \square
\end{aligned}$$

Lemma 16 *For $p \leq 0$, $\ell = 0$, and μ and n held constant, $p_1(q)$ is increasing in q for $n \geq 3$.*

Proof: From (9) we find that

$$\frac{dp_1(q)}{dq} = \frac{n-2}{\mu},$$

which is nondecreasing for $n = 2$ and increasing for $n \geq 3$. $\square$

Lemma 17 *The two probability mass functions* $f^1(n, 0, p, 0)$ *and* $f^2(n+1, 0, p, 1)$ *are equivalent.*

Proof: This is easily verified. $\square$

Based on the foregoing lemmas, we know that the smallest variance σ^2 that can be matched with a quasi-uniform distribution with mean μ when $\ell = 0$ and $p \le 0$ is the value shown in (18) with $n = \lceil 1 + 2\mu \rceil$ and q set according to (19) as identified in Lemma 14. By Lemmas 3 and 17 we can generate a continuous sequence of quasi-uniform distributions with increasing variance σ^2 by iteratively (i) decreasing q until it attains $q = 0$, and (ii) at that point incrementing n by one and resetting q to 1, which by Lemma 17 maintains the same distribution and hence variance, and, finally, (iii) repeating steps (i) and (ii) until the target variance is matched. Lemma 15 assures that an arbitrarily large variance can be attained. The following algorithm formalizes these steps.

Lemma 18 *Given any mean* $0 < \mu < \infty$*, the following algorithm identifies a distribution* $f \in \mathcal{F}$ *with any desired finite variance* σ^2 *greater than or equal to the quantity in (18).*

Proof: Denote the target moments by $\mu_\bigstar$ and $\sigma^2_\bigstar$. The algorithm generates a sequence of quasi-uniform distributions with a continuously increasing variance σ^2 that maintains a mean of $\mu_\bigstar$ with $0 \le q \le 1$ and $p \le 0$. The algorithm terminates at any arbitrary variance $\sigma^2_\bigstar$ greater than or equal to (18).

Algorithm

1. Set n equal to $\lceil 1 + 2\mu \rceil$, $p = 0$, and q equal to the quantity in (19) to match the target mean $\mu_\bigstar$.
2. Decrease q while setting $p = p_1(q)$ until $\sigma^2_\bigstar$ is matched or until $q = 0$.
3. Stop if $\sigma^2_\bigstar$ has been attained. Otherwise set $n = n + 1$ and $q = 1$, and go to step 2.

The algorithm always maintains $n, p,$ and q for a valid quasi-uniform distribution with a constant mean $\mu_\bigstar$. Specifically, (i) n is increasing and remains a nonzero integer, (ii) q is maintained in $0 \leq q \leq 1$, and (iii) p starts at a value of zero and then decreases throughout the algorithm so that by Lemma 16 $p \leq 0$ is always satisfied. The variance continuously increases without bound as the algorithm progresses by Lemma 15, and so any finite target mean $\mu_\bigstar < \infty$ can be attained. $\square$

2.3 The Optimality of a Quasi-Uniform Distribution

First, we calculate the variance of the number of orders outstanding.

Lemma 19 *If lead times are independent, then* $\sigma_N^2 = \sum_{l=0}^{\infty} F_l(1 - F_l)$.

Proof: Define the indicator function

$$1\left(A\right) = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases} .$$

An order remains outstanding after l periods if the random variable of lead time L for that order is greater than l, *i.e.*, if $L > l$. An order is outstanding and $1\left(L > l\right) = 1$ with probability $1 - F_l$, and the converse, $1\left(L > l\right) = 0$, is true with probability F_l. Thus the random number of orders outstanding N in an inventory system in steady state is

$$N = \sum_{l=0}^{\infty} 1\left(L > l\right). \tag{21}$$

Let $\mathbf{E}$ denote expectation. Then the variance of each of the terms in the infinite sum (21) is

$$\begin{aligned} \operatorname{Var}\left[1\left(L > l\right)\right] &= \mathbf{E}\left[1^2\left(L > l\right)\right] - \mathbf{E}^2\left[1\left(L > l\right)\right] \\ &= \left(1 - F_l\right) - \left(1 - F_l\right)^2 \\ &= F_l(1 - F_l), \end{aligned} \tag{22}$$

Then, provided lead times are independent over time, the variance of the number of orders outstanding can be computed from (21) and (22):

$$\begin{aligned} \sigma_N^2 &= \operatorname{Var} N \\ &= \sum_{l=0}^{\infty} F_l(1 - F_l) . \quad \square \end{aligned}$$

This result was shown in Zalkind (1976, 1978) and (implicitly) in Robinson *et al.* (2001).

Next, we restate this variance in a form that allows us to state Problem P1 in Bradley and Robinson [2005] as a quadratic program.

Proposition 20 *The variance of the number of orders outstanding is equal to*

$$\sigma_N^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tfrac{1}{2} \left| j-k \right| f_j f_k \ .$$

Proof: Starting with the result of Lemma 19, we compute

$$\begin{aligned} \sigma_N^2 &= \sum_{l=0}^{\infty} F_l \left(1-F_l\right) \\ &= \sum_{l=0}^{\infty} \left[\sum_{k=0}^{l} f_k \right] \left[\sum_{j=l+1}^{\infty} f_j \right] \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{j=l+1}^{\infty} f_j f_k . \end{aligned}$$

Twice interchanging the order of summation to bring the summation over l into the innermost summation yields

$$\sigma_N^2 = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \left(j-k\right) f_j f_k \ .$$

Splitting this into two equal terms, and exchanging the notation for the indices of summation in the second term gives us

$$\sigma_N^2 = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \left(j-k\right) f_j f_k + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \left(k-j\right) f_j f_k \ .$$

Adding a zero term $(k = j)$ to the second sum, and interchanging its order of summation now yields

$$\begin{aligned} \sigma_N^2 &= \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \tfrac{1}{2} \left(j-k\right) f_j f_k + \sum_{k=0}^{\infty} \sum_{j=0}^{k} \tfrac{1}{2} \left(k-j\right) f_j f_k \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tfrac{1}{2} \left| j-k \right| f_j f_k \ . \ \square \end{aligned}$$

With this result we can rewrite the objective function in Problem P1 from Bradley and Robinson [2005] as

$$\begin{array}{rrcl} \max\limits_{f} & \sigma_N^2 & = & \sum\limits_{i=0}^{m}\sum\limits_{j=0}^{m} \frac{1}{2}\left|j-i\right| f_i f_j \\ & \sum\limits_{i=0}^{m} f_i & = & 1 \\ & \sum\limits_{i=0}^{m} i f_i & = & \mu \\ & \sum\limits_{i=0}^{m} i^2 f_i & = & \sigma^2+\mu^2 \\ & f_i & \geq & 0 \qquad\qquad 0 \leq i \leq m. \end{array}$$

The value of m should be set sufficiently large so that it does not constrain the solution; *i.e.*, m should be set so that the same solution is obtained as in the limit as $m \to \infty$. Restated in vector-matrix notation, where vectors are denoted by lower case letters and matrices by upper case letters, Problem P1 is equivalently stated as:

$$\begin{array}{lrrcl} (\mathrm{P1}') & \max\limits_{f} & \sigma_N^2 & = & \frac{1}{2} f^{\mathrm{T}} Q f \\ & \text{s.t.} & Af & = & b \\ & & -If & \leq & 0 \end{array}$$

where the matrix Q is $(m+1)\times(m+1)$ with the element $q_{ij} = |i-j|$ in the i-th row and j-th column for $0 \leq i, j \leq m$. Moreover, $f \in \Re^{m+1}$, $b \in \Re^3$, and A is a $3\times(m+1)$ matrix. Specifically,

$$\begin{array}{rcl} f & = & \left[\begin{array}{cccc} f_0 & \cdots & f_{m-1} & f_m \end{array}\right]^{\mathrm{T}} \\ A & = & \left[\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & (m-1) & m \\ 0 & 1 & 4 & \cdots & (m-1)^2 & m^2 \end{array}\right] \\ b & = & \left[\begin{array}{c} 1 \\ \mu \\ \sigma^2+\mu^2 \end{array}\right] \\ Q & = & \left[\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & m \\ 1 & 0 & 1 & 2 & \cdots & m-1 \\ 2 & 1 & 0 & 1 & \cdots & m-2 \\ 3 & 2 & 1 & 0 & \cdots & m-3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m & m-1 & m-2 & m-3 & \cdots & 0 \end{array}\right]. \end{array}$$

Having shown in the previous section that a quasi-uniform distribution can be found for any mean μ and variance σ^2 provided that $\sigma^2 \geq \delta\left(1-\delta\right)$, we now present a series of lemmas that leads to Proposition 1, which states that a quasi-uniform distribution solves P1$'$. First, we show in Lemma 21 that a quasi-uniform distribution $f \in \mathcal{F}$ satisfies the KKT necessary conditions for P1$'$. Then, Lemmas 22 through 26 provide the foundation for Lemma 27, which shows that Q is negative definite for feasible search directions of f. This result renders the KKT conditions for P1$'$ in Lemma 21 sufficient as well as necessary. All these results make the proof of Proposition 1 possible, which proves the optimality of a quasi-uniform distribution $f \in \mathcal{F}$ for P1$'$.

Lemma 21 *If a p.m.f. $f \in \mathcal{F}$ exists that is feasible for P1$'$, then it satisfies the KKT necessary conditions.*

Proof: As a preliminary step to this proof, we note that if either $p = 0$ or $q = 0$, then two sets of parameters identify identical quasi-uniform distributions as shown in Lemmas 6 and 7. The alternate specification in this case has either $p = 1$ or $q = 1$. We will henceforth, without loss of generality, use the latter specification so that p and q are strictly positive.

The main KKT necessary condition for P1$'$,

$$-\nabla\left(\frac{1}{2}f^{\mathrm{T}}Qf\right) + w^{T}\nabla\left(-If\right) + v^{T}\nabla\left(Af\right) = 0,$$

simplified and shown with the remaining KKT conditions is

$$\begin{array}{rcll} -f^{T}Q - w^{T}I + v^{T}A & = & 0 & \\ Af & = & b & \\ f_i & \geq & 0 & i = 0,\ldots,m \\ w_i f_i & = & 0 & i = 0,\ldots,m \\ w_i & \geq & 0 & i = 0,\ldots,m. \end{array} \tag{23}$$

Any feasible $f \in \mathcal{F}$ for P1$'$ will satisfy $Af = b$ and $f \geq 0$. So, we need to find multipliers $v \in \Re^{3}$ and $w \in \Re^{m+1}$ that satisfy the remaining conditions in (23).

We first solve for v. Denote the k-th element of the result of matrix-vector multiplication Yx as $\langle Yx\rangle_k$ and the k-th element of a vector x as x_k. Then, because $w_k = 0$ for $\ell \leq k \leq u$

we know that $\left\langle f^T Q\right\rangle_k = \left\langle v^T A\right\rangle_k$, and hence

$$\begin{aligned} k^2 v_3 + k v_2 + v_1 &= \left\langle v^T A\right\rangle_k \\ &= \left\langle f^T Q\right\rangle_k \\ &= \sum_{i=\ell}^{u} \left|k-i\right| f_i \\ &= a\left\{\sum_{i=\ell}^{u} \left|k-i\right| - p\left(k-\ell\right) - q\left(u-k\right)\right\}, \end{aligned}$$

which after a number of operations results in

$$k^2 v_3 + k v_2 + v_1 = a\left\{k^2 - \left(u+\ell+p-q\right)k + \frac{1}{2}\left[u\left(u+1\right) + \left(\ell-1\right)\ell + 2p\ell - 2qu\right]\right\},$$

where $a = 1\left/\left(n-p-q\right)\right.$. From this we can infer that

$$\begin{aligned} v_1 &= \frac{a}{2}\left[u\left(u+1\right) + \left(\ell-1\right)\ell + 2p\ell - 2qu\right] \\ v_2 &= -a\left(u+\ell+p-q\right) \\ v_3 &= a. \end{aligned}$$

It remains to be shown that $w_k \geq 0$ for $k \leq \ell - 1$ and $k \geq u+1$. First, for $k \geq u+1$, for which it is apparent that $k \geq u+1 > \mu$, we find that

$$\begin{aligned} \left\langle f^T Q\right\rangle_k &= \sum_{i=\ell}^{u} \left|k-i\right| f_i \\ &= \sum_{i=\ell}^{u} \left(k-i\right) f_i \\ &= k - \mu. \end{aligned}$$

From (23),

$$w^{\mathrm{T}} I = -f^{\mathrm{T}} Q + v^{\mathrm{T}} A,$$

and hence

$$\begin{aligned} w_k^T &= -\left\langle f^T Q\right\rangle_k + \left\langle v^T A\right\rangle_k \\ &= -k + \mu + a\left\{k^2 - \left(u+\ell+p-q\right)k + \frac{1}{2}\left[u\left(u+1\right) + \left(\ell-1\right)\ell + 2p\ell - 2qu\right]\right\}, \end{aligned}$$

which can be re-written as

$$w_k^T = a\left\{\left[k - \frac{1}{2}(1 + 2u - 2q)\right]^2 - \left(q^2 - q + \frac{1}{4}\right)\right\}. \tag{24}$$

The first term is convex and increasing in k for $k \geq u+1$ regardless of q and so it is minimized at $k = u + 1$. This implies that for all $k \geq u + 1$

$$\begin{aligned}
w_k &= a\left\{\left[k - \frac{1}{2}(1 + 2u - 2q)\right]^2 - \left(q^2 - q + \frac{1}{4}\right)\right\} \\
&\geq a\left\{\left[u + 1 - \frac{1}{2}(1 + 2u - 2q)\right]^2 - \left(q^2 - q + \frac{1}{4}\right)\right\} \\
&\geq a\left\{\left[\frac{1}{2} + q\right]^2 - \left(q^2 - q + \frac{1}{4}\right)\right\} \\
&\geq a\left\{\left(q^2 + q + \frac{1}{4}\right) - \left(q^2 - q + \frac{1}{4}\right)\right\} \\
&\geq 2aq \\
&> 0.
\end{aligned}$$

The strict inequality in the last line follows from our specification that $q = 1$ when it could otherwise be expressed as $q = 0$.

Next, where $k \leq \ell - 1$, for which it is apparent that $k \leq \ell - 1 < \mu$,

$$\begin{aligned}
\left\langle f^T Q\right\rangle_k &= \sum_{i=\ell}^{u} |k - i|\, f_i \\
&= \sum_{i=\ell}^{u} (i - k)\, f_i \\
&= \mu - k.
\end{aligned}$$

Then, with math similar to that used to derive (24) (details omitted), we find

$$\begin{aligned}
w_k &= -\left\langle f^T Q\right\rangle_k + \left\langle v^T A\right\rangle_k \\
&= k - \mu + a\left\{k^2 - (u + \ell + p - q)\, k + \frac{1}{2}\left[u\,(u + 1) + (\ell - 1)\,\ell + 2p\ell - 2qu\right]\right\} \\
&= a\left\{\left[k + \frac{1}{2}(1 - 2\ell - 2p)\right]^2 - \left(p^2 - p + \frac{1}{4}\right)\right\}.
\end{aligned}$$

The first term is convex and decreasing in k for $k \leq \ell - 1$ and thus is minimized at $k = \ell - 1$. Then for each w_k, $k \leq \ell - 1$:

$$\begin{aligned} w_k &= a\left\{\left[k + \frac{1}{2}(1 - 2\ell - 2p)\right]^2 - \left(p^2 - p + \frac{1}{4}\right)\right\} \\ &\geq a\left\{\left[-\frac{1}{2} - p\right]^2 - \left(p^2 - p + \frac{1}{4}\right)\right\} \\ &\geq a\left\{\left(p^2 + p + \frac{1}{4}\right) - \left(p^2 - p + \frac{1}{4}\right)\right\} \\ &\geq 2ap \\ &> 0. \end{aligned}$$

Again, the strict inequality in the last line follows from our specification that $p = 1$ when it could otherwise be expressed as $p = 0$. We have shown through Lemma 12 that a quasi-uniform distribution exists for any mean and variance less than that shown in (18) with $p \geq 0$ whenever $\ell \geq 0$, in which case $w_k > 0$ for $k \leq \ell - 1$ if in fact $\ell > 0$. For variances above that shown in (18), Lemma 18 shows that $p \leq 0$, although $\ell = 0$ in this case, and so no multipliers w_k exist for $k \leq \ell - 1$ because the quasi-uniform distribution is defined only on the nonnegative integers.

Through the proof of existence of v and $w \geq 0$, we have shown that a feasible p.m.f. $f \in \mathcal{F}$ satisfies the KKT necessary conditions for P1$'$. $\square$

Next we establish that Q is negative definite in the null space of A, which requires a number of preliminary lemmas.

Lemma 22 *If the lead time i is restricted to the range $0 \leq i \leq m$, then the corresponding matrix Q is of full rank, $m + 1$ for $i \geq 1$.*

Proof: Factoring $Q = LU$, where L is a lower triangular matrix and U is an upper triangular matrix, we find

$$
\begin{aligned}
L &= \left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{m} & -\frac{m-1}{m} & 1 & 0 & \cdots & 0 \\
\frac{2}{m} & -\frac{m-2}{m} & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \vdots & 0 & \ddots & 0 \\
\frac{m-1}{m} & -\frac{1}{m} & 0 & \vdots & \ddots & 1 \\
1 & 0 & 0 & 0 & \cdots & 0
\end{array}\right] \\
U &= \left[\begin{array}{cccccc}
m & m-1 & m-2 & m-3 & \cdots & 0 \\
0 & 1 & 2 & 3 & \cdots & m \\
0 & 0 & 2 & 4 & \cdots & 2\left(m-1\right) \\
0 & 0 & 0 & 2 & \cdots & 2\left(m-2\right) \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 2
\end{array}\right].
\end{aligned}
$$

The determinant of Q is

$$|Q| = |L|\,|U| = |L|\,|U| = (-1)^{m}\, m2^{m-1}.$$

The first multiplicand is equal to $|L|$ and represents the effect on the determinant of m row interchanges to move the bottom row of L to the top of L, at which point the determinant is the product of the diagonal of ones. The last two multiplicands are the product of the diagonal of U. $|Q|$ is thus non-zero for all values of m that allow for a stochastic lead-time distribution, *i.e.*, $m \geq 1$, so that the lead-time distribution may have two or more mass points. (Even if the solution of P1$'$ was a distribution with one mass point, that solution could be found by setting m arbitrarily high.) The nonzero determinant implies that Q is nonsingular and thus of full rank, $m + 1$. $\square$

Lemma 23 *A matrix P exists such that $Q = P \begin{bmatrix} A \\ \hat{Q} \end{bmatrix}$, where $\begin{bmatrix} A \\ \hat{Q} \end{bmatrix}$ is a partitioned matrix, and $\hat{Q}$ is $(m-2) \times (m+1)$.*

Proof: Because Q is of full rank by Lemma 22, a matrix $R = R_3 R_2 R_1$ can be found to form linear combinations, and exchange the rows of Q so that $RQ = \begin{bmatrix} A \\ \hat{Q} \end{bmatrix}$. Specifically,

$$R = \begin{bmatrix} \frac{1}{2m} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} \\ \frac{(1-m)}{2} & 1 & 1 & 1 & 1 & \cdots & 1 & \frac{(1-m)}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$\hat{Q} = \begin{bmatrix} Q_3 \\ Q_4 \\ \vdots \\ Q_m \\ Q_1 \end{bmatrix},$$

where Q_i denotes the i-th row of Q for $0 \leq i \leq m$. Then the following may be verified:

$$P = R^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \omega & -m & 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ m & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\omega = m(m-1)/2$. $\square$

Let M denote the row null space of A, that is, $M = \{z : Az = 0\}$. Let Y be $(m-2) \times (m+1)$, the rows of which form an orthogonal basis for M.

Lemma 24 *A matrix $\tilde{P}$ exists such that $Q = \begin{bmatrix} A^T & \hat{Q}^T \end{bmatrix} \tilde{P} \begin{bmatrix} A \\ \hat{Q} \end{bmatrix}$.*

Proof: Because

$$Q = P \begin{bmatrix} A \\ \hat{Q} \end{bmatrix}$$

we need to prove the existence of a matrix $\tilde{P}$ that satisfies $\left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right] \tilde{P} = P$. We can readily compute $\tilde{P}$ by $\tilde{P} = \left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right]^{-1} P$ because the matrix $\left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right]$ is $(m+1)\times(m+1)$ and of full rank, and thus invertible. □

We compute the following:

$$\left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right] = \left[\begin{array}{cccccccccc}
1 & 0 & 0 & 3 & 4 & 6 & \cdots & m-1 & 1 \\
1 & 1 & 1 & 2 & 3 & 5 & \cdots & m-2 & 0 \\
1 & 2 & 4 & 1 & 2 & 4 & \cdots & m-3 & 1 \\
1 & 3 & 9 & 0 & 1 & 3 & \cdots & m-4 & 2 \\
1 & 4 & 16 & 1 & 0 & 2 & \cdots & m-5 & 3 \\
1 & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
1 & m-2 & (m-2)^2 & m-1 & m-2 & m-3 & \cdots & 1 & m-3 \\
\vdots & m-1 & (m-1)^2 & m-2 & m-3 & m-4 & \cdots & 0 & m-2 \\
1 & m & m^2 & m-3 & m-4 & m-5 & \cdots & 1 & m-1
\end{array}\right]$$

$$\left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right]^{-1} = \frac{1}{4}\left[\begin{array}{ccccccccc}
2 & m(m-1) & -2m(m-1) & m(m-1) & 0 & \cdots & 0 & 2m & -2(m-1) \\
-2 & -2(m-1) & 4m & -2m & 0 & \cdots & 0 & -2 & 2 \\
0 & 2 & -4 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -2 & 6 & -6 & 2 & 0 & 0 & \cdots & 0 \\
0 & -2 & 4 & 0 & -4 & 2 & 0 & \cdots & 0 \\
0 & -2 & 4 & -2 & 2 & -4 & 2 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & -2 & 4 & -2 & \vdots & \ddots & 2 & -4 & 2 \\
2 & -6 & 6 & -2 & 0 & 0 & 0 & \cdots & 0
\end{array}\right].$$

Then, $\tilde{P}$ can be readily computed from $\tilde{P} = \left[\begin{array}{cc} A^T & \hat{Q}^T \end{array}\right]^{-1} P$.

Partition $\tilde{P}$ as follows:

$$\tilde{P} = \left[\begin{array}{cc} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{array}\right],$$

where $\tilde{P}_1$, $\tilde{P}_2$, and $\tilde{P}_3$, are of dimension 3×3, $3\times(m-2)$, and $(m-2)\times 3$, respectively. We will be concerned only with $\tilde{P}_4$, that is, the lower right $(m-2)\times(m-2)$ partition of $\tilde{P}$.

The right-most $(m-2)$ columns of $\tilde{P}$ are

$$\begin{bmatrix} \tilde{P}_2 \\ \tilde{P}_4 \end{bmatrix} = \begin{bmatrix} \frac{3m(m-1)}{4} & \frac{m(m-1)}{2} & \frac{m(m-1)}{2} & \cdots & \frac{m(m-1)}{2} & \frac{m^2}{2} & \frac{3m(m-1)}{4} \\ -\frac{3m}{2} & -m & -m & \cdots & -m & -m-\frac{1}{2} & -\frac{3m-1}{2} \\ 1.5 & 1.0 & 1.0 & \cdots & 1.0 & 1.0 & 1.5 \\ -3.0 & -1.0 & -1.5 & \cdots & -1.5 & -1.5 & -2.0 \\ -1.0 & -2.0 & -0.5 & -1.0 & \cdots & -1.0 & -1.5 \\ -1.5 & -0.5 & -2.0 & -0.5 & \ddots & -1.0 & -1.5 \\ -1.5 & -1.0 & -0.5 & -2.0 & \ddots & -1.0 & -1.5 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -0.5 & \vdots \\ -1.5 & -1.0 & -1.0 & -1.0 & -0.5 & -2.0 & -1.5 \\ -2.0 & -1.5 & -1.5 & -1.5 & \cdots & -1.5 & -3.0 \end{bmatrix},$$

such that the lower $m-2$ rows constitute $\tilde{P}_4$.

Lemma 25 *The matrix $\tilde{P}_4$ can be factored such that $\tilde{P}_4 = LU$ where L is a lower diagonal matrix with diagonal elements equal to one, and U is an upper triangular matrix with diagonal elements $u_{ii} < 0$ for $1 \leq i \leq m-2$.*

Proof: Note that in this lemma we depart from our previous convention of starting the row-column numbering at 0 and, instead, we start with 1. This makes for a clearer correspondence between the steps of the algorithm that we develop and the dimension of the matrix under consideration.

From Lemma 24,

$$\tilde{P}_4 = \begin{bmatrix} -3.0 & -1.0 & -1.5 & -1.5 & \cdots & -1.5 & -2.0 \\ -1.0 & -2.0 & -0.5 & -1.0 & \cdots & -1.0 & -1.5 \\ -1.5 & -0.5 & -2.0 & -0.5 & \ddots & -1.0 & -1.5 \\ -1.5 & -1.0 & -0.5 & -2.0 & \ddots & -1.0 & -1.5 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -0.5 & \vdots \\ -1.5 & -1.0 & \cdots & -1.0 & -0.5 & -2.0 & -1.5 \\ -2.0 & -1.5 & -1.5 & -1.5 & \cdots & -1.5 & -3.0 \end{bmatrix}.$$

We will show that $\tilde{P}_4$ can be factored $\tilde{P}_4 = LU$ by recursively applying Gaussian elimination, where the resulting diagonal elements of L and U are $l_{ii} = 1$ and $u_{ii} < 0$ respectively for $1 \leq i \leq m-2$. The proof is accomplished by first characterizing the structure of $\tilde{P}_4$, which is defined by a number of attributes that relate the elements of the (reduced) matrix to one another. We refer to these attributes collectively as the set $\mathcal{A}$. In each recursive Gaussian elimination step we will be concerned with a square matrix that is smaller in dimension than the matrix in the previous step by one. We denote the $h \times h$ matrix, $1 \leq h \leq m-2$, with elements yet to undergo additional Gaussian elimination steps by $\tilde{P}_4(h)$, and at that point say that we are in step h of the Gaussian elimination process. Thus we start the iteration counter at $h = m-2$ and decrement h by one at each step so that the iteration number and the dimension of $\tilde{P}_4(h)$ match. Then, $\tilde{P}_4(h)$ is a lower right submatrix of $\tilde{P}_4$ that has partially undergone Gaussian elimination. In iteration h the diagonal element $u_{(m-1-h)(m-1-h)}$ is determined.

We show that the criteria of $\mathcal{A}$ result in $u_{hh} < 0$ in each step h, for $3 \leq h \leq m-2$. We also show that if $\tilde{P}_4(h)$ satisfies $\mathcal{A}$, then the next submatrix $\tilde{P}_4(h-1)$ to undergo Gaussian elimination also retains the structure $\mathcal{A}$. Some criteria of the structure $\mathcal{A}$ require a square matrix of minimum dimension 3, and so we halt the recursion at step 2 with $\tilde{P}_4(2)$, at which point we verify "manually" that $u_{(m-1)(m-1)} < 0$ and $u_{(m-2)(m-2)} < 0$.

We denote the elements of $\tilde{P}_4(h)$ as follows:

$$\tilde{P}_4(h) = \begin{bmatrix} a_h & b_h & c_h^1 & c_h^1 & \cdots & c_h^1 & d_h^{UR} \\ b_h & v_h & x_h & y_h & \cdots & y_h & c_h^2 \\ c_h^1 & x_h & v_h & x_h & \ddots & \vdots & c_h^2 \\ c_h^1 & y_h & x_h & v_h & \ddots & y_h & c_h^2 \\ \vdots & \vdots & \ddots & \ddots & \ddots & x_h & \vdots \\ c_h^1 & y_h & \cdots & y_h & x_h & v_h & c_h^2 \\ d_h^{LL} & c_h^2 & c_h^2 & c_h^2 & \cdots & c_h^2 & z_h \end{bmatrix},$$

where $c_h^1 = c_h^2$ and $d_h^{LL} = d_h^{UR}$. In step h, we perform Gaussian elimination by left multiplying $L_{h+1} \ldots L_{m-2} \tilde{P}_4$ by

$$L_h = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{b_h}{a_h} & 1 & 0 & \cdots & 0 & 0 \\ -\frac{c_h}{a_h} & 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ -\frac{c_h}{a_h} & 0 & \cdots & 0 & 1 & 0 \\ -\frac{d_h}{a_h} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which yields the following matrix for $L_h \ldots L_{2m-2}\tilde{P}_4\left(h\right)$:

$$\begin{bmatrix} a_h & b_h & c_h & c_h & c_h & \cdots & c_h & d_h \\ 0 & v_h - \frac{b_h^2}{a_h} & x_h - \frac{b_h c_h}{a_h} & y_h - \frac{b_h c_h}{a_h} & y_h - \frac{b_h c_h}{a_h} & \cdots & y_h - \frac{b_h c_h}{a_h} & c_h - \frac{b_h d_h}{a_h} \\ 0 & x_h - \frac{b_h c_h}{a_h} & v_h - \frac{c_h^2}{a_h} & x_h - \frac{c_h^2}{a_h} & y_h - \frac{c_h^2}{a_h} & \ddots & y_h - \frac{c_h^2}{a_h} & c_h - \frac{c_h d_h}{a_h} \\ 0 & y_h - \frac{b_h c_h}{a_h} & x_h - \frac{c_h^2}{a_h} & v_h - \frac{c_h^2}{a_h} & x_h - \frac{c_h^2}{a_h} & \ddots & \vdots & c_h - \frac{c_h d_h}{a_h} \\ 0 & y_h - \frac{b_h c_h}{a_h} & y_h - \frac{c_h^2}{a_h} & x_h - \frac{c_h^2}{a_h} & v_h - \frac{c_h^2}{a_h} & \ddots & y_h - \frac{c_h^2}{a_h} & c_h - \frac{c_h d_h}{a_h} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & x_h - \frac{c_h^2}{a_h} & \vdots \\ 0 & y_h - \frac{c_h^2}{a_h} & \cdots & y_h - \frac{c_h^2}{a_h} & y_h - \frac{c_h^2}{a_h} & x_h - \frac{c_h^2}{a_h} & v_h - \frac{c_h^2}{a_h} & c_h - \frac{c_h d_h}{a_h} \\ 0 & c_h - \frac{b_h d_h}{a_h} & c_h - \frac{c_h d_h}{a_h} & c_h - \frac{c_h d_h}{a_h} & c_h - \frac{c_h d_h}{a_h} & \cdots & c_h - \frac{c_h d_h}{a_h} & z_h - \frac{d_h^2}{a_h} \end{bmatrix}. \tag{25}$$

Thus, the elements of $\tilde{P}_4\left(h-1\right)$ are

$$\begin{array}{rclrcl} a_{h-1} & = & v_h - \frac{b_h^2}{a_h} & d_{h-1}^{LL} & = & c_h - \frac{b_h d_h}{a_h} \\ b_{h-1} & = & x_h - \frac{b_h c_h}{a_h} & v_{h-1} & = & v_h - \frac{c_h^2}{a_h} \\ c_{h-1}^1 & = & y_h - \frac{b_h c_h}{a_h} & x_{h-1} & = & x_h - \frac{c_h^2}{a_h} \\ c_{h-1}^2 & = & c_h - \frac{c_h d_h}{a_h} & y_{h-1} & = & y_h - \frac{c_h^2}{a_h} \\ d_{h-1}^{UR} & = & c_h - \frac{b_h d_h}{a_h} & z_{h-1} & = & z_h - \frac{d_h^2}{a_h}. \end{array} \tag{26}$$

We take the first Gaussian elimination step manually before beginning the recursive proof, from which we find that

$$L_{m-2}\tilde{P}_4 = \begin{bmatrix} -3.0 & -1.0 & -1.5 & -1.5 & -1.5 & \cdots & -1.5 & -2.0 \\ 0 & -1.\bar{6} & 0 & 0.25 & -0.25 & \cdots & -0.25 & -0.8\bar{3} \\ 0 & 0 & -1.25 & 0.25 & -0.25 & \ddots & -0.25 & -0.5 \\ 0 & -0.5 & 0.25 & -1.25 & 0.25 & \ddots & \vdots & \vdots \\ 0 & -0.5 & -0.25 & 0.25 & -1.25 & \ddots & -0.25 & -0.5 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0.25 & -0.5 \\ 0 & -0.5 & -0.25 & \cdots & -0.25 & 0.25 & -1.25 & -0.5 \\ 0 & -0.8\bar{3} & -0.5 & -0.5 & -0.5 & \cdots & -0.5 & -1.\bar{6} \end{bmatrix}, \tag{27}$$

and that the first diagonal term is $u_{11} = -3 < 0$.

The recursive portion of the proof begins with

$$\tilde{P}_4\left(m-3\right) = \begin{bmatrix} -1.\bar{6} & 0 & 0.25 & -0.25 & \cdots & -0.25 & -0.8\bar{3} \\ 0 & -1.25 & 0.25 & -0.25 & \ddots & -0.25 & -0.5 \\ -0.5 & 0.25 & -1.25 & 0.25 & \ddots & \vdots & \vdots \\ -0.5 & -0.25 & 0.25 & -1.25 & \ddots & -0.25 & -0.5 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0.25 & -0.5 \\ -0.5 & -0.25 & \cdots & -0.25 & 0.25 & -1.25 & -0.5 \\ -0.8\bar{3} & -0.5 & -0.5 & -0.5 & \cdots & -0.5 & -1.\bar{6} \end{bmatrix}. \tag{28}$$

The properties of the structure $\mathcal{A}$, which are easily verified for $\tilde{P}_4(m-3)$, are:

$\mathcal{A}1$. $x_h = v_h + 1.5$.

$\mathcal{A}2$. $y_h = v_h + 1.0$.

$\mathcal{A}3$. $c_h = b_h - 0.5$.

$\mathcal{A}4$. $d_h = d_h^{LL} = d_h^{UR}$.

$\mathcal{A}5$. $c_h = c_h^1 = c_h^2$.

$\mathcal{A}6$. $a_h = z_h$.

$\mathcal{A}7$. a_h, b_h, d_h, v_h satisfy $d_h = a_h + b_h - a_h(v_h + 1)/c_h$.

$\mathcal{A}8$. a_h, b_h, d_h, v_h satisfy $a_h v_h = a_h^2 + b_h^2 - d_h^2$.

$\mathcal{A}9$. $v_h \leq b_h - 1$.

$\mathcal{A}10$. $a_h \leq b_h - 1$.

$\mathcal{A}11$. $0 \leq b_h \leq 0.5$.

$\mathcal{A}12$. $a_h < 0$.

$\mathcal{A}13$. $a_h < d_h$.

$\mathcal{A}14$. $d_h < 0$.

$\mathcal{A}15$. $c_h < 0$.

Now we show that if $\tilde{P}_4(h)$ satisfies the structure $\mathcal{A}$, then $\mathcal{A}$ is preserved in $\tilde{P}_4(h-1)$.

$\mathcal{A}1$ **Proof:** We need to show that $x_{h-1} = v_{h-1} + 1.5$:

$$\begin{aligned} x_{h-1} &= x_h - \frac{c_h^2}{a_h} \\ &= v_h + 1.5 - \frac{c_h^2}{a_h} \\ &= v_{h-1} + 1.5. \quad \blacksquare \end{aligned}$$

$\mathcal{A}2$ **Proof:** We need to show that $y_{h-1} = v_{h-1} + 1$:

$$\begin{aligned} y_{h-1} &= y_h - \frac{c_h^2}{a_h} \\ &= v_h + 1 - \frac{c_h^2}{a_h} \\ &= v_{h-1} + 1. \quad \blacksquare \end{aligned}$$

$\mathcal{A}3$ **Proof:** We need to show that $c_{h-1} = b_{h-1} - 0.5$:

$$\begin{aligned} c_{h-1} &= y_h - \frac{b_h c_h}{a_h} \\ &= x_h - 0.5 - \frac{b_h c_h}{a_h} \\ &= b_{h-1} - 0.5. \quad \blacksquare \end{aligned}$$

$\mathcal{A}4$ **Proof:** We need to show that $d_{h-1} = d_{h-1}^{UR} = d_{h-1}^{LL}$, which is clearly the case because $d_{h-1}^{UR} = d_{h-1}^{LL} = c_h - \frac{b_h d_h}{a_h}$. $\blacksquare$

$\mathcal{A}5$ **Proof:** We need to show that $c_{h-1}^1 = y_h - \frac{b_h c_h}{a_h} = c_h - \frac{c_h d_h}{a_h} = c_{h-1}^2$.

$$\begin{aligned} c_h - \frac{c_h d_h}{a_h} &= c_h - \frac{c_h}{a_h}\left(a_h + b_h - \frac{a_h\left(v_h + 1\right)}{c_n}\right) \\ &= c_h - c_h - \frac{b_h c_h}{a_h} + \left(v_h + 1\right) \\ &= v_h - \frac{b_h c_h}{a_h} + 1 \\ &= y_h - \frac{b_h c_h}{a_h}. \quad \blacksquare \end{aligned}$$

$\mathcal{A}6$ **Proof:** Given that $\mathcal{A}8$ holds in iteration h, and that $a_h = z_h$ in iteration h, we find

$$\begin{aligned} a_h v_h &= a_h^2 + b_h^2 - d_h^2 \\ a_h v_h - b_h^2 &= a_h^2 - d_h^2 \\ v_h - \frac{b_h^2}{a_h} &= a_h - \frac{d_h^2}{a_h} \\ v_h - \frac{b_h^2}{a_h} &= z_h - \frac{d_h^2}{a_h} \\ a_{h-1} &= z_{h-1}. \quad \blacksquare \end{aligned}$$

$\mathcal{A}7$ **Proof:** We need to show that $d_{h-1} = a_{h-1} + b_{h-1} - a_{h-1}\left(v_{h-1} + 1\right)/c_{h-1}$. First, note from the proof for condition $\mathcal{A}5$ that

$$\begin{aligned} y_h - \frac{b_h c_h}{a_h} &= c_h - \frac{c_h d_h}{a_h} \\ &= c_h\left(1 - \frac{d_h}{a_h}\right), \end{aligned}$$

and from the $\mathcal{A}6$ proof that

$$v_h - \frac{b_h^2}{a_h} = a_h - \frac{d_h^2}{a_h}.$$

Then

$$\begin{aligned}
a_{h-1} + b_{h-1} - \frac{a_{h-1}\left(v_{h-1}+1\right)}{c_{h-1}} &= \left(v_h - \frac{b_h^2}{a_h}\right) + \left(x_h - \frac{b_h c_h}{a_h}\right) - \frac{\left(v_h - \frac{b_h^2}{a_h}\right)\left(v_h - \frac{c_h^2}{a_h} + 1\right)}{\left(v_h + 1 - \frac{b_h c_h}{a_h}\right)} \\
&= \left(y_h - \frac{b_h c_h}{a_h}\right) + 0.5 + \left(v_h - \frac{b_h^2}{a_h}\right) \\
&\quad - \frac{\left(v_h - \frac{b_h^2}{a_h}\right)\left(v_h - \frac{b_h c_h}{a_h} + 1 + \frac{0.5 c_h}{a_h}\right)}{\left(v_h + 1 - \frac{b_h c_h}{a_h}\right)} \\
&= \left(y_h - \frac{b_h c_h}{a_h}\right) + 0.5 - \frac{0.5 c_h}{a_h} \frac{\left(v_h - \frac{b_h^2}{a_h}\right)}{\left(y_h - \frac{b_h c_h}{a_h}\right)} \\
&= c_h \left(1 - \frac{d_h}{a_h}\right) + 0.5 - \frac{0.5 c_h}{a_h} \frac{\left(v_h - \frac{b_h^2}{a_h}\right)}{c_h \left(1 - \frac{d_h}{a_h}\right)} \\
&= c_h \left(1 - \frac{d_h}{a_h}\right) + 0.5 - 0.5 \frac{\left(a_h - \frac{d_h^2}{a_h}\right)}{\left(a_h - d_h\right)} \\
&= c_h \left(1 - \frac{d_h}{a_h}\right) + 0.5 - \frac{0.5}{a_h}\left(a_h + d_h\right) \\
&= \left(c_h - \frac{c_h d_h}{a_h}\right) + 0.5 - 0.5\left(1 + \frac{d_h}{a_h}\right) \\
&= c_h - \frac{\left(b_h - 0.5\right) d_h}{a_h} + 0.5 - 0.5\left(1 + \frac{d_h}{a_h}\right) \\
&= c_h - \frac{b_h d_h}{a_h} \\
&= d_{h-1}. \quad \blacksquare
\end{aligned}$$

$\mathcal{A}8$ **Proof:** We need to show that $a_{h-1} v_{h-1} = a_{h-1}^2 + b_{h-1}^2 - d_{h-1}^2$. As a preliminary step, we can find the following two relationships by manipulating the equalities from (26):

$$\begin{aligned}
c_{h-1} - a_{h-1} - 1 &= \frac{0.5 b_h}{a_h}, \\
c_{h-1} - d_{h-1} &= \frac{0.5 d_h}{a_h}.
\end{aligned}$$

Then starting with $\mathcal{A}7$,

$$\begin{aligned}
d_{h-1} &= a_{h-1} + b_{h-1} - a_{h-1}\left(v_{h-1}+1\right)/c_{h-1} \\
a_{h-1}v_{h-1} &= -a_{h-1} + a_{h-1}c_{h-1} + b_{h-1}c_{h-1} - c_{h-1}d_{h-1} \\
&= a_{h-1}\left(c_{h-1} - 1 - a_{h-1} + a_{h-1}\right) + b_{h-1}\left(b_{h-1} - 0.5\right) - d_{h-1}\left(c_{h-1} - d_{h-1} + d_{h-1}\right) \\
&= a_{h-1}^2 + b_{h-1}^2 - d_{h-1}^2 + a_{h-1}\left(c_{h-1} - a_{h-1} - 1\right) - 0.5b_{h-1} - d_{h-1}\left(c_{h-1} - d_{h-1}\right),
\end{aligned}$$

so that we obtain the desired result if the remainder $r_{h-1} = a_{h-1}\left(c_{h-1} - a_{h-1} - 1\right) - 0.5b_{h-1} - d_{h-1}\left(c_{h-1} - d_{h-1}\right)$ is zero:

$$\begin{aligned}
r_{h-1} &= \left(v_h - \frac{b_h^2}{a_h}\right)\left(\frac{0.5b_h}{a_h}\right) - 0.5\left(v_h + 1.5 - \frac{b_hc_h}{a_h}\right) - \left(v_h + 1 - \frac{b_hc_h}{a_h} - \frac{0.5d_h}{a_h}\right)\left(\frac{0.5d_h}{a_h}\right) \\
&= \frac{1}{a_h^2}\left[0.5b_h\left(a_hv_h - b_h^2\right) - 0.5a_h\left(a_hv_h + 1.5a_h - b_hc_h\right) - 0.5d_h\left(a_hv_h + a_h - b_hc_h - 0.5d_h\right)\right],
\end{aligned}$$

and then substituting $a_hv_h - b_h^2 = a_h^2 - d_h^2$ in the first instance of a_hv_h and $a_hv_h = -a_h + a_hc_h + b_hc_h - c_hd_h$ in the second and third instances,

$$\begin{aligned}
r_{h-1} &= \frac{1}{a_h^2}\left\{0.5b_h\left(a_h^2 - d_h^2\right) - 0.5a_h\left[0.5a_h + c_h\left(a_h - d_h\right)\right] - 0.5d_h\left[c_h\left(a_h - d_h\right) - 0.5d_h\right]\right\} \\
&= \frac{1}{a_h^2}\left[0.5b_h\left(a_h^2 - d_h^2\right) - 0.5c_h\left(a_h + d_h\right)\left(a_h - d_h\right) - 0.25\left(a_h^2 - d_h^2\right)\right] \\
&= \frac{0.5}{a_h^2}\left[\left(b_h - 0.5 - c_h\right)\left(a_h^2 - d_h^2\right)\right] \\
&= 0,
\end{aligned}$$

where the last step follows because $c_h = b_h - 0.5$. ■

$\mathcal{A}9$ **Proof:** We need to show that $v_{h-1} \leq b_{h-1} - 1$:

$$\begin{aligned}
b_{h-1} &= x_h - \frac{b_hc_h}{a_h} \\
&= v_h - \frac{c_h^2}{a_h} + 1.5 - 0.5\frac{\left(b_h - 0.5\right)}{a_h} \\
b_{h-1} - 1 &= v_{h-1} + 0.5 - 0.5\frac{\left(b_h - 0.5\right)}{a_h} \\
&= v_{h-1} + 0.5\left[1 - \frac{\left(b_h - 0.5\right)}{a_h}\right] \\
&\geq v_{h-1},
\end{aligned}$$

where the last line follows because $a_h \leq b_h - 1$, which with $0 \leq b_h \leq 0.5$ implies that $a_h \leq -0.5$ and -0.5 $\leq b_h - 0.5 \leq 0$, and hence $0 \leq (b_h - 0.5)/a_h \leq 1$. ■

$\mathcal{A}10$ **Proof:** We need to show that $a_{h-1} \leq b_{h-1} - 1$:

$$\begin{aligned}
b_{h-1} &= x_h - \frac{b_h c_h}{a_h} \\
b_{h-1} - 1 &= v_h + 1.5 - \frac{b_h^2}{a_h} + 0.5\frac{b_h}{a_h} - 1 \\
&= v_h - \frac{b_h^2}{a_h} + 0.5\left(1 + \frac{b_h}{a_h}\right) \\
&= a_{h-1} + 0.5\left(1 + \frac{b_h}{a_h}\right),
\end{aligned}$$

and from the proof for $\mathcal{A}9$ where we found that $a_h \leq -0.5$ and $0 \leq b_h \leq 0.5$, we can conclude that $-1 \leq b_h/a_h \leq 0$, from which the result is immediate. ■

$\mathcal{A}11$ **Proof:** We need to show that $0 \leq b_{h-1} \leq 0.5$. As preliminaries, we know that:

From $\mathcal{A}9$ and $b_h \leq 0.5$ we find that $a_h \leq -0.5$.

From $\mathcal{A}13$ and $\mathcal{A}14$ we know that $0 \leq d_h/a_h < 1$, from which we can conclude that $d_{h-1} \geq -0.5$:

$$\begin{aligned}
d_{h-1} &= c_h - \frac{b_h d_h}{a_h} \\
&\geq (b_h - 0.5) - b_h \\
&\geq -0.5.
\end{aligned}$$

Also recall from $\mathcal{A}7$ that

$$y_h - \frac{b_h c_h}{a_h} = c_h\left(1 - \frac{d_h}{a_h}\right).$$

First, we show that $b_{h-1} \geq 0$:

$$\begin{aligned}
b_{h-1} &= x_h - \frac{b_h c_h}{a_h} \\
&= y_h + 0.5 - \frac{b_h c_h}{a_h} \\
&= 0.5 + c_h\left(1 - \frac{d_h}{a_h}\right)
\end{aligned}$$

$$\begin{aligned} b_{h-1} &= 0.5 + (b_h - 0.5)\left(1 - \frac{d_h}{a_h}\right) \\ &= b_h - \frac{b_h d_h}{a_h} + 0.5\frac{d_h}{a_h} \\ &= b_h\left(1 - \frac{d_h}{a_h}\right) + 0.5\frac{d_h}{a_h} \\ &\geq 0, \end{aligned}$$

where the last step follows because $0 \leq d_h/a_h < 1$.

Now we show that $b_{h-1} \leq 0.5$:

$$\begin{aligned} b_{h-1} - 0.5 &= c_{h-1} \\ b_{h-1} &= c_h\left(1 - \frac{d_h}{a_h}\right) + 0.5 \\ &\leq 0.5, \end{aligned}$$

where the last step follows from $\mathcal{A}15$ ($c_h < 0$), and from $d_h/a_h < 1$. ■

$\mathcal{A}12$ **Proof:** We need to show that $a_{h-1} < 0$:

$$\begin{aligned} a_{h-1} &= v_h - \frac{b_h^2}{a_h} \\ &\leq b_h - 1 - \frac{b_h^2}{(b_h - 1)} \\ &\leq \frac{(b_h - 1)^2 - b_h^2}{(b_h - 1)} \\ &\leq \frac{-2(b_h - 0.5)}{(b_h - 1)} \\ &< 0, \end{aligned}$$

where the last line follows from $0 \leq b_h \leq 0.5$. ■

$\mathcal{A}13$ **Proof:** We need to show that $a_{h-1} < d_{h-1}$:

$$\begin{aligned} d_{h-1} &= c_h - \frac{b_h d_h}{a_h} \\ &= c_h\left(1 - \frac{d_h}{a_h}\right) - 0.5\frac{d_h}{a_h} \\ &= y_h - \frac{b_h c_h}{a_h} - 0.5\frac{d_h}{a_h} \end{aligned}$$

$$\begin{aligned}
d_{h-1} &= v_h + 1 - \frac{b_h^2}{a_h} + 0.5\frac{b_h}{a_h} - 0.5\frac{d_h}{a_h} \\
&= a_{h-1} + 1 + 0.5\frac{b_h}{a_h} - 0.5\frac{d_h}{a_h} \\
&= a_{h-1} + 0.5\left(1 + \frac{b_h}{a_h}\right) + 0.5\left(1 - \frac{d_h}{a_h}\right) \\
d_{h-1} &> a_{h-1},
\end{aligned}$$

where the last line follows because $(1 + b_h/a_h) \geq 0$ as shown in $\mathcal{A}10$ and $d_h/a_h < 1$ as shown in $\mathcal{A}11$. ■

$\mathcal{A}14$ **Proof:** We need to show that $d_{h-1} < 0$. The result follows immediately because $d_{h-1} = c_h - \frac{b_h d_h}{a_h}$, $c_h = b_h - 0.5 \leq 0$, and $b_h d_h / a_h > 0$. ■

$\mathcal{A}15$ **Proof:** We need to show that $c_{h-1} < 0$. Because $c_{h-1} = c_h\left(1 - \frac{d_h}{a_h}\right)$, $c_h < 0$, and $d_h/a_h < 1$, then $c_{h-1} < 0$. ■

The structure $\mathcal{A}$ is thus preserved throughout the recursive portion of our proof until $h = 2$, at which point

$$\tilde{P}_4(2) = \begin{bmatrix} a_2 & d_2 \\ d_2 & a_2 \end{bmatrix},$$

so that the Gaussian elimination results in

$$L_2\tilde{P}_4(2) = \begin{bmatrix} a_2 & d_2 \\ 0 & a_2 - \frac{d_2^2}{a_2} \end{bmatrix}.$$

Thus $u_{(m-3)(m-3)} = a_2 < 0$ by property $\mathcal{A}12$ in iteration 3, and $u_{(m-2)(m-2)} = a_2 - \frac{d_2^2}{a_2} < 0$ by properties $\mathcal{A}13$ and $\mathcal{A}14$.

The full factorization is then

$$L_1 L_2 \ldots L_{m-2}\tilde{P}_4 = U.$$

Define $L^{-1} = L_1 L_2 \ldots L_{m-2}$. Note that each L_h is lower triangular with diagonal elements equal to one, so that L^{-1} is also lower triangular with elements of one. The determinant of L^{-1} is non-zero because all diagonal elements are non-zero, which implies that L^{-1} cannot be singular Thus L^{-1} is invertible, and $\tilde{P}_4 = LU$, where we know from the structure of L^{-1} that L is also lower triangular with diagonal elements of one. The diagonal elements of U are $u_{hh} = a_h < 0$ as determined above. □

Lemma 26 *The matrix $\tilde{P}_4$ is negative definite.*

Proof: We prove this lemma by computing the determinants of the upper left submatrices of $\tilde{P}_4$. Denote the submatrix of $\tilde{P}_4$ that is constituted by the upper k rows and left k columns by $\tilde{P}_4^k$ and its determinant by $\left|\tilde{P}_4^k\right|$. $\tilde{P}_4$ is negative definite if we can show that $\left|\tilde{P}_4^k\right| = (-1)^k \eta_k$ with $\eta_k > 0$ for $1 \leq k \leq m-2$ (see Strang [1988], pg. 338). The LU factorization in Lemma 25 is convenient because the determinant of a matrix is the product of the diagonal elements of the matrices L and U.

Any upper left submatrix of $\tilde{P}_4^k$ can be factored as $\tilde{P}_4$ was factored in its entirety in Lemma 25 using the same Gaussian elimination factors in the L_h matrices, which would still have all its diagonal elements equal to one. Furthermore, the U matrix for $\tilde{P}_4^k$ has the same diagonal elements as U for the entire $\tilde{P}_4$ matrix. Thus for any upper left submatrix $\tilde{P}_4^k$ we find that

$$\left|\tilde{P}_4^k\right| = \prod_{i=1}^{k} l_{ii} \prod_{i=1}^{k} u_{ii} = 1^k (-1)^k \prod_{i=1}^{k} |u_{ii}| \,.$$

It is clear when we set $\eta_k = \prod_{i=1}^{k} |u_{ii}|$ that $\left|\tilde{P}_4^k\right|$ is negative for odd k and positive for even k. $\square$

Lemma 27 *The matrix Q is negative definite over the row null space of A. That is, $z^T Q z < 0$ for all $z \in M$.*

Proof: To establish that Q is negative definite in the null space of A, we need to show for all $z \in M$ that

$$\begin{aligned} z^T Q z &< 0 \\ z^T \begin{bmatrix} A^T & \hat{Q}^T \end{bmatrix} \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{bmatrix} \begin{bmatrix} A \\ \hat{Q} \end{bmatrix} z &< 0 \\ \begin{bmatrix} z^T A^T & z^T \hat{Q}^T \end{bmatrix} \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{bmatrix} \begin{bmatrix} Az \\ \hat{Q} z \end{bmatrix} &< 0 \\ \begin{bmatrix} 0 & z^T \hat{Q}^T \end{bmatrix} \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{Q} z \end{bmatrix} &< 0 \\ z^T \hat{Q}^T \tilde{P}_4 \hat{Q} z &< 0, \end{aligned}$$

which is true because $\tilde{P}_4$ is negative definite. $\square$

The foregoing lemmas culminate in the following proposition, which is stated without proof in Bradley and Robinson [2005].

Proposition 1: The unique globally optimal solution to Problem P1$'$, f^*, is from the family of quasi-uniform distributions $\mathcal{F}$.

Proof: The "active set" approach from Luenberger [1989] identifies a direction of improvement $d \in \Re^{m+1}$, if it exists, from a candidate solution. Our candidate solution is the quasi-uniform distribution $f_c \in \mathcal{F}$ that we have shown will exist for any two arbitrary first moments and that satisfies the KKT necessary conditions. If no direction of improvement exists, then the candidate solution is optimal. The active set formulation has constraints that correspond to (i) the equality constraints in Problem P1$'$ and (ii) the inequality constraints that are tight in P1$'$ under the candidate solution f_c:

$$\begin{array}{lll} (\text{P1}'') & \max\limits_{d} & \frac{1}{2}d^{\text{T}}Qd + f_c^{\text{T}}Qd \\ & & \begin{array}{rcl} Ad & = & 0 \\ -I_{\ell-1}^{0}d & = & 0 \\ -I_{m}^{u+1}d & = & 0 \end{array} \end{array} \tag{29}$$

where I_z^y denotes the y-th through z-th rows of the identity matrix. The candidate solution is optimal for formulation P1$''$ if (i) the optimal direction of improvement is $d^* = 0$ and (ii) the Lagrange multipliers for the constraints in P1$''$ that correspond to the tight inequality constraints in Problem P1$'$ are nonnegative.

We want to simplify the formulation P1$''$ by eliminating the constraint $Ad = 0$, which we accomplish by constraining the search direction d to the null space of A. Let Y be $(m-2)\times(m+1)$, the rows of which form an orthogonal basis for M. Then, any column vector d in the null space of A can be generated with Y, $d = Y^{\text{T}}d_1$, and so we use the surrogate search direction $d_1 \in \Re^{m-2}$ rather than d. Replacing $Y^{\text{T}}d_1$ for d in P1$''$ yields a new formulation:

$$\begin{array}{lll} (\text{P1}''') & \max\limits_{d_1} & \frac{1}{2}d_1^{\text{T}}Y\hat{Q}^{\text{T}}\tilde{P}_4\hat{Q}Y^{\text{T}}d_1 + f_c^{\text{T}}QY^{\text{T}}d_1 \\ & & \begin{array}{rcl} -I_{\ell-1}^{0}Y^{\text{T}}d_1 & = & 0 \\ -I_{m}^{u+1}Y^{\text{T}}d_1 & = & 0. \end{array} \end{array} \tag{30}$$

The objective function of P1$'''$ is negative definite from Lemma 26, and so we can compute the optimal solution in closed form using Luenberger's approach [1989, pg. 425]. Specifically, for P1$'''$ we find that $d_1^* = 0$, and thus $d^* = Y^{\mathrm{T}} d_1^* = 0$.

We denote the Lagrangian multipliers in the KKT necessary conditions for P1$'''$ by λ and now verify that $\lambda > 0$. Given $d_1^* = 0$, the KKT conditions

$$\begin{aligned} Y\hat{Q}^{\mathrm{T}}\tilde{P}_4\hat{Q}Y^{\mathrm{T}}d_1 - Y\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda + YQf_c &= 0 \\ \begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix} Y^{\mathrm{T}} d_1 &= 0 \end{aligned}$$

simplify to

$$Y\left\{\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - Qf_c\right\} = 0.$$

(If $\ell = 0$, then we define $I_{\ell-1}^0$ to be a matrix with zero dimension.) From Lemma 21 we know that $Qf_c = A^{\mathrm{T}}v - Iw$ where $w_i = 0$ for $\ell \le i \le u$ and $w_i > 0$ otherwise so that

$$\begin{aligned} Y\left\{\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - A^{\mathrm{T}}v - Iw\right\} &= 0 \\ Y\left\{\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - Iw\right\} - YA^{\mathrm{T}}v &= 0 \\ Y\left\{\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - Iw\right\} &= 0. \end{aligned}$$

Unless we can find a solution for λ that satisfies

$$\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - Iw = A^{\mathrm{T}}\omega, \tag{31}$$

then the solution is

$$\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}} \lambda - Iw = 0. \tag{32}$$

Elements i for $\ell \le i \le u$ of the left-hand side of (31) are zero because $w_i = 0$ for those elements i and all the elements of rows ℓ through u of $\begin{bmatrix} I_{\ell-1}^0 \\ I_m^{u+1} \end{bmatrix}^{\mathrm{T}}$, which is multiplied by λ, are zeros. It can also be shown that no more than two adjacent elements of the vector $A^{\mathrm{T}}\omega$ can be zero. Thus, the only solution to (31) for $n = u - \ell + 1 > 2$ is $\omega = 0$. So, equation

(32) provides the solution for λ. From Lemma 21 we know that $w_i > 0$ for $0 \leq i \leq \ell - 1$ and for $u + 1 \leq i \leq m$, and so $\lambda > 0$ for all its $m - n$ elements. Note that (32) is satisfied also for the elements corresponding to $w_i = 0$ for $\ell \leq i \leq u$ because, again, the elements in the corresponding rows ℓ through u of the transpose, partial identity matrix are all zeros. Thus the optimal solution to the formulation in P1$''$ is $d^* = 0$ with $\lambda > 0$, from which we can conclude that $f_c \in \mathcal{F}$ exists and is the unique and globally optimal solution to P1$'$, and hence P1. □

2.4 Computing the Upper Bound

Proposition 2: *For any quasi-uniform lead time distribution with variance* σ_L^2*,*

$$\sigma_N^2 \leq \frac{\sigma_L}{\sqrt{3}} .$$

Proof: Consider the quasi-uniform distribution defined in (1), for arbitrary integer $\ell \geq 0$ and $n \geq 2$, and arbitrary $p, q \in [0, 1]$. Without loss of generality, we can redefine the origin so that $\ell = 0$, so that the p.m.f. becomes

$$f_l = \begin{cases} \dfrac{1-p}{n-p-q} & l = 0 \\[2ex] \dfrac{1}{n-p-q} & l = 1, 2, ..., n-2 \\[2ex] \dfrac{1-q}{n-p-q} & l = n-1 \end{cases} .$$

We start by calculating μ_L and σ_L^2 for this (shifted) distribution as follows:

$$\begin{aligned} \mu_L &= \sum_{l=0}^{\infty} f_l \cdot l \\ &= \left(\frac{1}{n-p-q}\right) \left[\sum_{l=0}^{n-1} l - q(n-1)\right] \\ &= \left(\frac{1}{n-p-q}\right) \left[\frac{(n-1)n}{2} - q(n-1)\right] \\ &= \left(\frac{n-1}{2}\right) \left(1 + \frac{p-q}{n-p-q}\right) , \end{aligned}$$

$$
\begin{aligned}
\sigma_L^2 &= \sum_{l=0}^{\infty} f_l \cdot l^2 - \mu_L^2 \\
&= \left(\frac{1}{n-p-q}\right)\left[\sum_{l=0}^{n-1} l^2 - q(n-1)^2\right] - \left(\frac{n-1}{2}\right)^2 \left(1+\frac{p-q}{n-p-q}\right)^2 \\
&= \left(\frac{1}{n-p-q}\right)\left[\frac{(n-1)n(2n-1)}{6} - q(n-1)^2\right] - \left(\frac{n-1}{2}\right)^2 \left(\frac{n-2q}{n-p-q}\right)^2 \\
&= \frac{1}{12}\left(\frac{n-1}{n-p-q}\right)^2 \left[\frac{2n(2n-1)(n-p-q)}{n-1} - 12q\,(n-p-q) - 3\,(n-2q)^2\right] ,
\end{aligned}
$$

and so

$$
\frac{\sigma_L}{\sqrt{3}} = \frac{1}{6}\left(\frac{n-1}{n-p-q}\right)\sqrt{\frac{2n(2n-1)(n-p-q)}{n-1} - 12q\,(n-p-q) - 3\,(n-2q)^2} \,.
$$

The next step is to calculate σ_N^2 for this shifted distribution. Its c.d.f. is

$$
F_l = \begin{cases} \dfrac{l+1-p}{n-p-q} & l = 0, ..., n-2 \\ \\ 1 & l \geq n-1 \end{cases} ,
$$

so that

$$
\begin{aligned}
\sigma_N^2 &= \sum_{l=0}^{\infty} F_l \cdot (1-F_l) = \sum_{l=0}^{n-2} F_l - \sum_{l=0}^{n-2} F_l^2 \\
&= \left(\frac{1}{n-p-q}\right)\sum_{l=0}^{n-2}(l+1-p) - \left(\frac{1}{n-p-q}\right)^2 \sum_{l=0}^{n-2}\left[l^2 + 2(1-p)l + (1-p)^2\right] \\
&= \left(\frac{1}{n-p-q}\right)\left(\frac{(n-2)(n-1)}{2} + (n-1)(1-p)\right) \\
&\quad - \left(\frac{1}{n-p-q}\right)^2 \left(\frac{(n-2)(n-1)(2n-3)}{6} + 2(1-p)\frac{(n-2)(n-1)}{2} + (1-p)^2(n-1)\right) \\
&= \frac{1}{6}\left(\frac{n-1}{n-p-q}\right)\left[3n - 6p - \frac{2n^2 - n - 6pn + 6p^2}{n-p-q}\right] \\
&= \left(\frac{\sigma_L}{\sqrt{3}}\right)\left[\frac{3n - 6p - \dfrac{2n^2 - n - 6pn + 6p^2}{n-p-q}}{\sqrt{\dfrac{2n(2n-1)(n-p-q)}{n-1} - 12q\,(n-p-q) - 3\,(n-2q)^2}}\right] .
\end{aligned}
$$

Now $\frac{\sigma_L}{\sqrt{3}}$ will be an upper bound for σ_N^2 if its multiplicand does not exceed 1; *i.e.*, if and only if

$$\left[3n - 6p - \frac{2n^2 - n - 6pn + 6p^2}{n - p - q}\right]^2 \overset{?}{\le} \left[\frac{2n(2n-1)(n-p-q)}{n-1} - 12q\,(n-p-q) - 3\,(n-2q)^2\right] .$$

Multiplying through by $(n-1)(n-p-q)^2$ yields the equivalent condition

$$\begin{aligned}
&(n-1)\left[(n-p-q)\,(3n-6p) - \left(2n^2 - n - 6pn + 6p^2\right)\right]^2 \\
&\qquad\qquad \overset{?}{\le} (n-p-q)^2\left[2n(2n-1)(n-p-q) - 12q(n-1)\,(n-p-q)\right. \\
&\qquad\qquad\qquad\qquad \left. -3(n-1)\,(n-2q)^2\right] \\
&(n-1)\left[n^2 + n - 3\,(pn + qn - 2pq)\right]^2 \\
&\qquad\qquad \overset{?}{\le} (n-p-q)^2\left[n^3 + n^2 - 4(p+q)n^2 + 2(p+q)n + 12pq(n-1)\right] .
\end{aligned}$$

Multiplying out both sides yields

$$\begin{aligned}
&n^5 + n^4 - n^3 - n^2 - 6(p+q)n^4 + 3\left[3p^2 + 10pq + 3q^2\right]n^3 \\
&+3(p+q)\left[2 - 3(p+q+4pq)\right]n^2 - 12pq\left[1 - 3(p+q+pq)\right]n - 36p^2q^2 \\
&\qquad\qquad \overset{?}{\le} n^5 + n^4 - 6(p+q)n^4 + 3\left[3p^2 + 10pq + 3q^2\right]n^3 \\
&\qquad\qquad\quad - \left[12pq(1 + 2(p+q)) + 3\,(p+q)^2 + 4\,(p+q)^3\right]n^2 \\
&\qquad\qquad\quad +2(p+q)\left[(p+q)^2 + 6pq(2+p+q)\right]n - 12pq(p+q)^2 \\
&-n^3 - n^2 + 3(p+q)\left[2 - 3(p+q+4pq)\right]n^2 - 12pq\left[1 - 3(p+q+pq)\right]n - 36p^2q^2 \\
&\qquad\qquad \overset{?}{\le} - \left[12pq(1 + 2(p+q)) + 3\,(p+q)^2 + 4\,(p+q)^3\right]n^2 \\
&\qquad\qquad\quad +2(p+q)\left[(p+q)^2 + 6pq(2+p+q)\right]n - 12pq(p+q)^2 \ ,
\end{aligned}$$

or, eventually,

$$\begin{aligned}
B(p,q|n) \ &\doteq \ 2\left[(1-p)^3 + (1-q)^3\right]n^2 + 6pq(2-p-q)n + 12(n-1)pq(p-q)^2 \\
&\quad +12(n-1)p^2q^2 - 2n(n-1)\left[p^3 + q^3\right] + n^2(n-3) \overset{?}{\ge} 0 \ .
\end{aligned}$$

We need to show that $B(p,q|n) \ge 0$ for all $p, q \in [0,1)$, for all integer $n \ge 3$. (We are not interested in the cases $n = 1$ and $n = 2$ where there can be no strict order crossover, in which case $\sigma_N^2 = \sigma_L^2$.) To do this, we will evaluate $B(p,q|n)$ for every feasible combination of p and q that satisfy either the first-order conditions or a boundary condition. One of these points will be the minimum over the region $0 \le p,q < 1$, so that if all of them (that exist) yield nonnegative values of B, we can be sure that B will always be nonnegative.

For the four pure boundary solutions $p,q \in \{0,1\}$, $B(p,q|n)$ can be easily found by direct substitution as follows:

$$\begin{aligned}
B(0,0|n) &= n^2(n+1) \\
B(0,1|n) &= B(1,0|n) = n(n-1)(n-2) \\
B(1,1|n) &= (n-3)(n-2)^2 \ .
\end{aligned}$$

The minimum value occurs at $B(1,1|n) = (n-3)(n-2)^2 \ge 0$ for all $n \ge 3$. We now consider the situation where at least one first-order condition is satisfied:

$$\begin{aligned}
\frac{\partial B(p,q|n)}{\partial p} &= -6n^2 + 12pn^2 - 12p^2n^2 + 12qn - 12pqn - 6q^2n + 36qnp^2 - 24pq^2n + 12q^3n \\
&\quad -36qp^2 + 24pq^2 - 12q^3 + 6np^2 \overset{\text{set}}{=} 0 \ .
\end{aligned}$$

Solving for p yields

$$p^*(q|n) = \frac{n(n-q) - 2q^2(n-1) \pm \sqrt{(n-1)(n-2q)b(q|n)}}{n(2n-1) - 6q(n-1)} \ ,$$

where $b(q|n)$ is defined to be the cubic function

$$b(q|n) \doteq 4(n-1)q^3 + 6nq(1-q) - n^2 \ .$$

In order for $p^*(q|n)$ to be real, the argument under the square root must be nonnegative. For $n \ge 3$ and $q \in [0,1]$, this requires $b(q|n) \ge 0$. Nickalls [1993] shows that a general cubic function $ax^3 + bx^2 + cx + d$ will have no turning points (*i.e.*, no local minimum or maximum) if $b^2 < 3ac$. For our $b(q|n)$, this condition becomes

$$\begin{aligned}
[-6n]^2 &\overset{?}{<} 3\,[4(n-1)]\,[6n] \\
36n^2 &\overset{?}{<} 72n(n-1) \\
n &\overset{?}{>} 2,
\end{aligned}$$

which is always satisfied. Because $b(q|n)$ has no turning points, and because the coefficient of the q^3 term is positive for $n \geq 3$, $b(q|n)$ is nondecreasing in q. Directly substituting $q = 1$ yields $b(1|n) = -(n-2)^2 < 0$. Thus $b(q|n) < 0$ for all $q \leq 1$, showing that for any n, no real solution exists for $p^*(q|n)$, for any q within the allowable range $[0, 1]$. Because $B(p, q|n)$ is symmetric in p and q, the analysis of the partial derivative with respect to q is identical. Thus the minimum value of $B(p, q|n)$ is $(n-3)(n-2)^2 > 0$, showing that the multiplicand of $\frac{\sigma_L}{\sqrt{3}}$ is less than one. $\square$

3 A Lower Bound on σ_N^2

The following lemma is a basis for subsequent proofs in this section for lower bounds on the variance of the number of orders outstanding.

Lemma 28 *For any* μ_L and σ_L^2, *if the replenishment lead time* L *is restricted to a finite range, then the probability distribution that minimizes* σ_N^2 *will have at most three points with positive probability mass.*

Proof: Suppose that the distribution L is constrained to a finite range bounded below at l and above at u. Then we can state this problem as

$$\begin{array}{lrcll} \min & \sigma_N^2 & = & \sum_{k=l}^{u}\sum_{j=l}^{u}\frac{1}{2}\left|j-k\right| f_j f_k & \\ \text{s.t.} & \displaystyle\sum_{j=l}^{u} f_j & = & 1 & \\ & \displaystyle\sum_{j=l}^{u} j \cdot f_j & = & \mu_L & \\ & \displaystyle\sum_{j=l}^{u} j^2 \cdot f_j & = & \sigma_L^2 + \mu_L^2 & \\ & f_j & \geq & 0 & j = l, ..., u. \end{array}$$

Equivalently stated in matrix-vector notation we have

$$\begin{array}{lrcl} \min & \sigma_N^2 & = & \frac{1}{2} f^T Q f \\ \text{s.t.} & Af & = & b \\ & -If & \leq & 0. \end{array}$$

Given that we have shown in Lemma 27 that Q is negative definite over the row space of A, a boundary solution is optimal for this problem, meaning that $u-l+1$ linearly independent constraints must be tight. Because the three equality constraints contained in $Af=b$ must be tight for any feasible solution, at most three of the $u-l+1$ constraints implicit in $-If \leq 0$ are not tight in the optimal solution: *i.e.*, at most three mass points can be non-zero. $\square$

First, we establish a lower bound on the variance of the number of orders outstanding if the support for the lead-time distribution is not restricted.

Lemma 29 *For general values of μ_L and σ_L^2, the tightest lower bound on σ_N^2 is $\delta(1-\delta)$.*

Proof: Define

$$\begin{aligned} m &\doteq \lceil \mu_L \rceil - 1, \text{ with} \\ \delta &\doteq \mu_L - m \in (0,1] \ . \end{aligned}$$

Note that δ is the fractional part of μ_L except that when μ_L is an integer, δ is now one rather than zero; the product $\delta(1-\delta)$ does not change with this redefinition. (Note that the notation m in this proof is not the same m that appears elsewhere in this appendix.)

To show that $\delta(1-\delta)$ is the tightest lower bound on σ_N^2, it is sufficient to construct a specific distribution that attains this bound, at least in the limit. Consider a three-point distribution with positive masses f_m, f_{m+1}, and f_{m+k} on m, $m+1$, and a third point $m+k$, where k is large. This three-point distribution will match μ_L and σ_L^2 only if the following three linear equations

$$\begin{aligned} f_m + f_{m+1} + f_{m+k} &= 1 \\ mf_m + (m+1)f_{m+1} + (m+k)f_{m+k} &= m+\delta \\ m^2 f_m + (m+1)^2 f_{m+1} + (m+k)^2 f_k &= \sigma_L^2 + (m+\delta)^2 \end{aligned}$$

are satisfied by nonnegative values. Some tedious calculations yield the unique solution of

$$\begin{aligned} f_m &= 1-\delta+\frac{\sigma_L^2+\delta(1-\delta)}{k} \\ f_{m+1} &= \frac{\delta\,(k-2+\delta)-\sigma_L^2}{k-1} \\ f_{m+k} &= \frac{\sigma_L^2+\delta(1-\delta)}{k(k-1)}\ . \end{aligned}$$

Both f_m and f_{m+k} are clearly positive. f_{m+1} will be positive only if its numerator is positive, which we can ensure by restricting $k > 2-\delta+\sigma_L^2/\delta$; because δ is positive by construction, this inequality will hold for sufficiently large finite values of k. The final step is to compute

$$\begin{aligned} \sigma_N^2 &= \sum_{l=0}^{\infty} F_l(1-F_l) \\ &= f_m(1-f_m)+\sum_{l=m+1}^{m+k-1}(1-f_{m+k})f_{m+k} \\ &= f_m(1-f_m)+(k-1)f_{m+k}(1-f_{m+k}) \\ &= \left[1-\delta+\frac{\sigma_L^2+\delta(1-\delta)}{k}\right]\left[\delta-\frac{\sigma_L^2+\delta(1-\delta)}{k}\right] \\ &\quad+\left[\frac{\sigma_L^2+\delta(1-\delta)}{k}\right]\left[1-\frac{\sigma_L^2+\delta(1-\delta)}{k(k-1)}\right]\ . \end{aligned}$$

Because the common numerator $\sigma_L^2+\delta(1-\delta)$ is independent of k,

$$\lim_{k\to\infty}\sigma_N^2=\delta(1-\delta)\ ,$$

completing the proof. □

Put another way, Lemma 29 states that σ_N^2 can assume its lowest possible value, $\delta(1-\delta)$, regardless of the value of σ_L^2 upon which it is based. We cannot tighten the lower bound at all beyond $\delta(1-\delta)$ due to the possible nonintegrality of the mean μ_N. However, if it is reasonable to additionally assume that the lead time L is restricted to some finite range, then the proposition below shows that this lower bound, which is stated in Bradley and Robinson [2005] without proof, can be tightened substantially.

Proposition 3: For any μ_L and σ_L^2, if the replenishment lead time L falls within the compact range $[\mu_L - k_l\sigma_L, \mu_L + k_u\sigma_L]$, then

$$\sigma_N^2 \geq \frac{\sigma_L}{k + 1/k}, \tag{33}$$

where $k \doteq \max\{k_l, k_u\}$.

Proof: From Lemma 28, we know that the minimizing distribution will have positive probability masses on at most three points. Let f_l, f_m, and f_u be the masses on the lower, middle, and upper points $\mu_L - x_l\sigma_L$, $\mu_L + x_m\sigma_L$, and $\mu_L + x_u\sigma_L$, respectively, where

$$-x_l \leq x_m \leq x_u. \tag{34}$$

This three-point distribution must satisfy the three linear equations

$$\begin{aligned} f_l + f_m + f_u &= 1 \\ f_l\left[\mu_L - x_l\sigma_L\right] + f_m\left[\mu_L + x_m\sigma_L\right] + f_u\left[\mu_L + x_u\sigma_L\right] &= \mu_L \\ f_l\left[x_l\sigma_L\right]^2 + f_m\left[x_m\sigma_L\right]^2 + f_u\left[x_u\sigma_L\right]^2 &= \sigma_L^2, \end{aligned}$$

or

$$\begin{aligned} f_l + f_m + f_u &= 1 \\ -f_l x_l + f_m x_m + f_u x_u &= 0 \\ f_l x_l^2 + f_m x_m^2 + f_u x_u^2 &= 1, \end{aligned}$$

which has the solution

$$\begin{aligned} f_l &= \frac{1 + x_m x_u}{(x_l + x_u)(x_l + x_m)} \\ f_m &= \frac{x_l x_u - 1}{(x_l + x_m)(x_u - x_m)} \\ f_u &= \frac{1 - x_l x_m}{(x_l + x_u)(x_u - x_m)}. \end{aligned}$$

Because the denominators are all positive, these probabilities will be proper only if all three numerators are nonnegative:

$$\begin{aligned} x_m x_u &\geq -1 \\ x_l x_u &\geq 1 \\ x_l x_m &\leq 1. \end{aligned}$$

Taken together, these three inequalities imply that the constraints

$$-1/x_u \leq x_m \leq 1/x_l \tag{35}$$

are tighter than the initial constraints (34).

We now calculate the variance of the number of orders outstanding for this distribution:

$$\begin{aligned} \sigma_N^2 &= \sum_{k=0}^{\infty} F_k \left(1 - F_k\right) \\ &= \sum_{k=\mu_L - x_l\sigma_L}^{\mu_L + x_m\sigma_L - 1} f_l(1 - f_l) + \sum_{k=\mu_L + x_m\sigma_L}^{\mu_L + x_u\sigma_L - 1} \left(1 - f_u\right) f_u \\ &= f_l(1 - f_l)\left(x_l + x_m\right)\sigma_L + \left(1 - f_u\right) f_u \left(x_u - x_m\right)\sigma_L \\ &= \sigma_L \left[\left(\frac{1 + x_m x_u}{x_l + x_u}\right)\left(\frac{\left(x_l + x_u\right)\left(x_l + x_m\right) - 1 - x_m x_u}{\left(x_l + x_u\right)\left(x_l + x_m\right)}\right)\right. \\ &\quad \left. + \left(\frac{\left(x_l + x_u\right)\left(x_u - x_m\right) - 1 + x_l x_m}{\left(x_l + x_u\right)\left(x_u - x_m\right)}\right)\left(\frac{1 - x_l x_m}{x_l + x_u}\right)\right] \\ &= \frac{\sigma_L}{x_l + x_u}\left[x_l x_u - \frac{\left(x_l x_u - 1\right)^2}{\left(x_l + x_m\right)\left(x_u - x_m\right)}\right]. \end{aligned}$$

At this point we can specify the minimization model:

$$\begin{aligned} \min \quad \sigma_N^2 &= \frac{\sigma_L}{x_l + x_u}\left[x_l x_u - \frac{\left(x_l x_u - 1\right)^2}{\left(x_l + x_m\right)\left(x_u - x_m\right)}\right] \\ \text{s.t.} \quad 0 &< x_l \leq k_l \\ 0 &< x_u \leq k_u \\ x_l x_u &\geq 1 \\ -1/x_u &\leq x_m \leq 1/x_l \\ &\quad \mu_L - x_l\sigma_L,\ \mu_L + x_m\sigma_L,\ \mu_L + x_u\sigma_L \text{ integer.} \end{aligned}$$

In order to ensure that a lead time distribution exists that attains the given variance σ_L^2, we require this condition:

$$k_l k_u \geq 1.$$

For any given values of x_l and x_u, we examine the partial derivative of σ_N^2 with respect to x_m,

$$\frac{\partial \sigma_N^2}{\partial x_m} = \left(\frac{\sigma_L}{x_l + x_u}\right)\left(\frac{x_l x_u - 1}{(x_l + x_m)(x_u - x_m)}\right)^2 (x_u - x_l - 2x_m)\,,$$

which we set equal to zero. The first two fractions are positive, yielding an extreme point of

$$x_m^* = \frac{x_u - x_l}{2}.$$

Note that the slope of $\partial \sigma_N^2 / \partial x_m$ is positive for $x_m < x_m^*$ and negative for $x_m > x_m^*$, showing that σ_N^2 is quasiconcave in x_m for fixed x_l and x_u, and that x_m^* is a local maximizer of σ_N^2. Thus the optimal value of x_m is at one of the two boundary points, $-1/x_u$ or $1/x_l$. At this point we relax the constraint that $\mu_L + x_m \sigma_L$ is integer. Substituting in these two boundary values of x_m yields the two-point distribution solutions in Table 1.

	x_m	
	$-1/x_u$	$1/x_l$
f_l	0	$\dfrac{1}{x_l^2 + 1}$
f_m	$\dfrac{x_u^2}{x_u^2 + 1}$	$\dfrac{x_l^2}{x_l^2 + 1}$
f_u	$\dfrac{1}{x_u^2 + 1}$	0
σ_N^2	$\dfrac{\sigma_L}{x_u + 1/x_u}$	$\dfrac{\sigma_L}{x_l + 1/x_l}$

Table 1: Boundary Values for x_m

Note that one of the two endpoints x_l or x_u is effectively eliminated as its probability mass becomes zero with this new value of x_m. Setting $x_m = 1/x_l$, the minimization model becomes

$$\begin{aligned} \min \quad \sigma_N^2 &= \frac{\sigma_L}{x_l + 1/x_l} \\ \text{s.t.} \quad 1/k_u &\leq x_l \leq k_l \end{aligned}$$

where the lower bound on x_l is derived from the constraint $x_l x_u \geq 1$, by setting x_u at its least restrictive value of k_u. The objective function is quasiconcave for $x_l \geq 0$, and attains its maximum value at $x_l = 1$; its minimizing value x_l^* will be one of the two endpoints $1/k_u$ or k_l. Since the objective function takes on the same value for both x_l and $1/x_l$, we can equivalently evaluate the objective function at the two values k_l and k_u. Since $k_l k_u \geq 1$ for feasibility, it follows that $k \doteq \max\{k_l, k_u\} \geq 1$. Define $\underline{k} \doteq \min\{k_l, k_u\}$ to be the other value of k. If $\underline{k} \geq 1$, then $x_l^* = k$, since the objective function is decreasing in x_l for $x_l \geq 1$. On the other hand, if $\underline{k} < 1$, then we can equivalently evaluate the objective function at $1/\underline{k} > 1$. Since $k\underline{k} = k_l k_u \geq 1$ for feasible distributions, $k \geq 1/\underline{k}$, and again $x_l^* = k$. Thus the optimal objective function value will be $\sigma_N^2 = \dfrac{\sigma_L}{k + 1/k}$. Setting $x_m = -1/x_u$ instead yields the analogous problem in terms of x_u, and the same value for σ_N^2. $\square$

References

[2005] Bradley, J. R. and L. W. Robinson. 2005. Improved Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover. Forthcoming in *Manufacturing and Service Operations Management.*

[1989] Luenberger, D. G. 1989. *Linear and Nonlinear Programming (Second Edition).* Addison-Wesley. Menlo Park, CA.

[1993] Nickalls, R. W. D. 1993. A new approach to solving the cubic: Cardan's solution revealed. *The Mathematical Gazette* **77** 354-359.

[2001] Robinson, L. W., J. R. Bradley and L. J. Thomas. 2001. Consequences of Order Crossover Under Order-Up-To Inventory Policies. *Manufacturing and Service Operations Management*, Volume 3, Number 3, 175-188.

[1988] Strang, G. *Linear Algebra and Its Applications (Third Edition)*, 1988. Harcourt Brace Jovanovich College Publishers. New York.

[1976] Zalkind, D. 1976. Further results for order-level inventory systems with independent stochastic leadtimes. *Technical Report 76-6* Department of Health Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill.

[1978] Zalkind, D. 1978. Order-level inventory systems with independent stochastic leadtimes. *Management Sci.* **24** 1384-1392.