Online Companion to
"A Closed-Form Approximation for Serial Inventory Systems and Its Application to System Design"

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Appendix: Proofs

Proposition 6

Without loss of generality, we assume $L_{[1,N]} = 1$ and $h_{[1,N]} = 1$. Thus equal leadtimes and holding costs implies $L_j = 1/N$ and $h_j = 1/N$ for all j. In this case, (12) becomes

$$\xi_j = \frac{2(j\beta + (N-j))}{(1-\beta)(j+1)}$$

and (13) becomes

$$s_j^{'a} \;=\; \frac{\lambda}{N} + 0.617\sqrt{\frac{\lambda}{N}}\left[\sqrt{j}ln\left(\frac{2(j\beta+(N-j))}{(1-\beta)(j+1)}\right) - \sqrt{j-1}ln\left(\frac{2(j-1)\beta+(N-j+1))}{j(1-\beta)}\right)\right].$$

To show $s_{j+1}^{'a} < s_j^{'a}$, it is equivalent to show $f(j) \stackrel{def}{=} \sqrt{j}ln(\xi_j)$ is concave in j.

Define the continuous extension of $f(j)$ on the real interval $[1, N]$ as

$$\begin{aligned} f(x) &= \sqrt{x}ln\left(\frac{2(\beta x + (N-x))}{(1-\beta)(1+x)}\right) \\ &= \sqrt{x}\left(ln\left(\frac{2}{1-\beta}\right) + ln((\beta-1)x+N) - ln(1+x)\right). \end{aligned}$$

We now take the second derivative of each term in $f(x)$ with respect to the continuous variable x. Let $f_1(x) \stackrel{def}{=} \sqrt{x}ln\left(\frac{2}{1-\beta}\right)$. Then, $f_1^{''}(j) = -\frac{1}{4}x^{-3/2}ln\left(\frac{2}{1-\beta}\right) < 0$. Next, let $f_2(x) \stackrel{def}{=} \sqrt{x}ln(\beta x + N - x)$. Then,

$$\begin{aligned} f_2^{''}(x) &= -\frac{1}{4}x^{-3/2}ln(\beta x + N - x) + \frac{1}{2}x^{-1/2}\frac{\beta-1}{\beta x + N - x} \\ &\quad +(\beta-1)\left(\frac{\frac{1}{2}x^{-1/2}(\beta x + N - x) - (\beta-1)\sqrt{x}}{(\beta x + N - x)^2}\right) < 0. \end{aligned}$$

Further, let $f_3(x) \stackrel{def}{=} \sqrt{x} ln(x+1)$. Then

$$f_3^{''}(x) \;=\; \frac{1}{\sqrt{x}}\left(\frac{4x-(1+x)^2 ln(1+x)}{4x(1+x)^2}\right).$$

Thus, $f_3^{''}(x) < 0$ when $x \geq 2$. Therefore, we have shown that when $x \geq 2$, $f(x)$ is concave in x. This implies that $f(j) - f(j-1) > f(j+1) - f(j)$, $j = 2, ..., N-1$. Thus, we have $s_2^{'a} > s_3^{'a} > ... > s_N^{'a}$.

Finally, from the definition,

$$s_1^{'a} - s_2^{'a} \;=\; 0.617\sqrt{\frac{\lambda}{N}}\left(2ln\left(\frac{\beta+N-1}{1-\beta}\right) - \sqrt{2}ln\left(\frac{2(2\beta+N-2)}{3(1-\beta)}\right)\right).$$

Since $\frac{\beta+N-1}{1-\beta} > \frac{2(2\beta+N-2)}{3(1-\beta)} > 1$, $s_1^{'a} - s_2^{'a} > 0$. This completes the proof.

Proposition 7

We only prove s_j^a decreases in h_j; the rest of the results are straightforward from (11) and (13) so the proof is omitted. To show the property, it is sufficient to show ξ_j decreases in h_j. But this is implied by

$$\frac{\partial \xi_j}{\partial h_j} = \frac{\beta(1-\beta)\sum_{i=1}^{j-1} h_i(L_{[1,i]}/L_{[1,j]}) - h_{[1,j-1]} - h_{[j+1,N]}}{[(1-\beta)h_j + (1-\beta)\sum_{i=1}^{j-1} h_i(L_{[1,i]}/L_{[1,j]})]^2} < 0.$$

Proposition 8

Without loss of generality, we assume $h_1 = h_2 = 1$ and $L_1 = L_2 = 1$. From (15), this is equivalent to showing

$$\beta_2^a - \beta \;=\; \left(1 + \left(\frac{1+\beta}{1-\beta}\right)\left(\frac{3(1-\beta)}{4\beta}\right)^{\sqrt{2}}\right)^{-1} - \beta < 0$$

for $0 < \beta < 1$. This can be further simplified by showing

$$\frac{1+\beta}{\beta^{\sqrt{2}-1}\cdot(1-\beta)^{2-\sqrt{2}}} - \left(\frac{4}{3}\right)^{\sqrt{2}} \tag{16}$$

is positive. Note that the first term in (16) is a convex function of β, $0 < \beta < 1$, and its minimizer is $\beta = (\sqrt{2}-1)/(3-\sqrt{2})$. At this minimizer, (16) achieves its lowest value $2.626 - 1.502 > 0$. The result follows immediately.

Proposition 9

We introduce a concept of *nominal fill rate* before we proceed. The nominal fill rate is the fill rate the stage would achieve if it had ample supply. Denote the nominal fill rate by using the base-stock level $s_j^{'a}$ at stage j to be $\beta_j^{'a}$, then

$$\beta_j^{'a} \quad = \quad \mathsf{P}(D_j < s_j^{'a}).$$

Obviously, the nominal fill rate at each stage is larger than the real internal fill rate, i.e., $\beta_j^{'a} \geq \beta_j^a$, for all j.

¿From Proposition 6, it is easy to see that $\beta_2^{'a} \geq \beta_3^{'a} \geq ... \geq \beta_N^{'a}$. Note that $\beta_j^a \leq \beta_j^{'a} \leq \beta_2^{'a}$. It is sufficient to show $\beta_2^{'a} < \beta$ for $\beta > \beta_0$. From the expression

$$\begin{aligned} \beta_2^{'a} &= \mathsf{P}(D_2 < s_2^{'a}) \\ &= \left(1 + \left(\frac{\beta + N - 1}{1 - \beta}\right)\left(\frac{3(1-\beta)}{2(2\beta + N - 2)}\right)^{\sqrt{2}}\right)^{-1}, \end{aligned}$$

this is equivalent to showing

$$\left(\frac{\beta + N - 1}{1 - \beta}\right)\left(\frac{3(1-\beta)}{2(2\beta + N - 2)}\right)^{\sqrt{2}} > \frac{1-\beta}{\beta}, \qquad \beta > \beta_0$$

or

$$w(\beta) \stackrel{def}{=} \beta(\beta + N - 1) - \left(\frac{2}{3}\right)^{\sqrt{2}} (2\beta + N - 2)^{\sqrt{2}}(1-\beta)^{2-\sqrt{2}} > 0, \qquad \beta > \beta_0. \qquad (17)$$

We now argue there indeed exists β_0 so that (17) holds. Note that $lim_{\beta \to 1} w(\beta) = N$. So, given $\epsilon = \frac{N}{2} > 0$, there exists $\delta_\epsilon > 0$ such that for all $|\beta - 1| < \delta_\epsilon$, we have $|w(\beta) - N| < \epsilon = \frac{N}{2}$. This implies that, for $\beta > \beta_0 = 1 - \delta_\epsilon$, $w(\beta) > \frac{N}{2} > 0$, proving (17).