10.1287/msom.1070.0160

Online Appendix to Accompany Further Improvements on Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover

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January 22, 2007

This online appendix contains three technical topics in support of the paper "Further Improvements on Base-Stock Approximations for Independent Stochastic Lead Times with Order Crossover." The first section describes how to fit a beta distribution to any four moments. The second section provides the proof of Theorem 1, while the third section summarizes the cost performance of the various heuristics.

1 Fitting the Parameters of a Beta Distribution

Given the first four moments (μ_L, σ_L, α_3, and α_4) of a probability distribution, the transformation of Elderton and Johnson (1969), as cited in Johnson *et al.* (1995), can be used to compute the four parameters (α, β, ℓ, u) that define the matching beta distribution as follows:

$$\begin{aligned}
\alpha &\doteq \tfrac{1}{2}\rho\left[1-(\rho+2)\,\alpha_3 \div \sqrt{(\rho+2)^2\,\alpha_3^2+16\,(\rho+1)}\right] \\
\beta &\doteq \rho-\alpha\ , \\
\text{where } \rho &\doteq \frac{6\,(\alpha_4-\alpha_3^2-1)}{6+3\alpha_3^2-2\alpha_4}=\alpha+\beta\ , \text{ and} \\
\ell &\doteq \mu_L - w\alpha/\rho \\
u &\doteq \ell+w\ , \\
\text{where } w &\doteq \sigma_L\cdot\rho\sqrt{\frac{\rho+1}{\alpha\beta}}=u-\ell\ .
\end{aligned}$$

2 Proof of Theorem 1

Theorem 1 *For any discrete distribution of the non-negative lead time,* $\sigma_N^2 \le \left(\dfrac{c_v^2+1/9}{c_v^2+1}\right)\mu_L$.

Proof: Because Bradley and Robinson (2005) showed that the "quasi-uniform" discrete distribution for the lead time maximized the variance of the number of orders outstanding σ_N^2, it is sufficient to show that Theorem 1 holds for the family of quasi-uniform distributions.

Note that we can rewrite this upper bound as

$$\left(\frac{c_v^2+1/9}{c_v^2+1}\right)\mu_L=\left(1-\frac{8\mu_L^2}{9\,(\sigma_L^2+\mu_L^2)}\right)\mu_L\ ; \tag{1}$$

taking the partial derivative with respect to μ_L yields, after some simplification,

$$\frac{\partial}{\partial \mu_L}\left(1-\frac{8\mu_L^2}{9\left(\sigma_L^2+\mu_L^2\right)}\right)\mu_L=\left(\frac{\mu_L^2/3-\sigma_L^2}{\sigma_L^2+\mu_L^2}\right)^2\geq 0\ ,$$

showing that the bound is increasing in μ_L. This implies that we need test the bound only for those quasi-uniform distributions that have a positive mass on zero. (We can translate other distributions towards zero, which would tighten this upper bound while leaving σ_N^2 unchanged.)

It is possible to define these quasi-uniform probability distributions by two parameters p and n as follows. Define $p \in [0,1)$ to be a point-mass on 0, with the remaining mass of $1-p$ uniformly distributed over $[0,n]$ and then aggregated onto the nearest integer. (Because of this aggregation, p and n are slightly different than their counterparts p_0 and n_0 in the continuous distribution.) For any real number n, define $\langle n\rangle \doteq \lfloor n+0.5\rfloor$ to be its nearest integer. Then the discrete distribution $\{f_l\}$ defined by (p,n) will be:

$$\begin{aligned}
f_0 &= p+\frac{1-p}{2n}\\
f_l &= \frac{1-p}{n} \qquad\qquad l=1,...,\langle n\rangle-1\\
f_{\langle n\rangle} &= \frac{1-p}{n}\left[n-\left(\langle n\rangle-\tfrac{1}{2}\right)\right]\ ,
\end{aligned}$$

with $f_l=0$ outside of this range. In order to allow f_0 to assume values in the range $(0,\frac{1}{2n})$. we generalize the definition of p to allow it to take on negative values, so that $p\in\left(-\frac{1}{2n-1},1\right)$; f_l and $f_{\langle n\rangle}$ are both always well-defined when $n>1.5$. As a function of p and n,

$$\begin{aligned}
\mu_L &= \sum_{l=1}^{\langle n\rangle-1} l\left(\frac{1-p}{n}\right)+\langle n\rangle\left(\frac{1-p}{n}\right)\left[n-\left(\langle n\rangle-\tfrac{1}{2}\right)\right]\\
&= (1-p)\langle n\rangle\left[\frac{2n-\langle n\rangle}{2n}\right]\ ,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_L^2+\mu_L^2 &= \sum_{l=1}^{\langle n\rangle-1} l^2\left(\frac{1-p}{n}\right)+\langle n\rangle^2\left(\frac{1-p}{n}\right)\left[n-\left(\langle n\rangle-\tfrac{1}{2}\right)\right]\\
&= \left(\frac{1-p}{n}\right)\left[\frac{\left(\langle n\rangle-1\right)\langle n\rangle\left(2\langle n\rangle-1\right)}{6}+\langle n\rangle^2 n-\langle n\rangle^3+\tfrac{1}{2}\langle n\rangle^2\right]\\
&= \left(\frac{1-p}{6n}\right)\langle n\rangle\left[1+6\langle n\rangle n-4\langle n\rangle^2\right]\ ,
\end{aligned}$$

with

$$\begin{aligned}
\sigma_N^2 &= \sum_{l=0}^{\infty} F_l (1 - F_l) \\
&= \sum_{l=0}^{\langle n \rangle - 1} \left[p + \frac{1-p}{2n} + l \left(\frac{1-p}{n} \right) \right] \left[1 - p - \frac{1-p}{2n} - l \left(\frac{1-p}{n} \right) \right] \\
&= \left(\frac{1-p}{4n^2} \right) \sum_{l=0}^{\langle n \rangle - 1} (2np + 1 - p)(2n - 1) \\
&\quad + \left(\frac{1-p}{4n^2} \right) \sum_{l=0}^{\langle n \rangle - 1} \left[-2 (2np + 1 - p) + 2 (1-p)(2n-1) \right] l \\
&\quad - \left(\frac{1-p}{4n^2} \right) \sum_{l=0}^{\langle n \rangle - 1} 4 (1-p) l^2 \\
&= \left(\frac{1-p}{12n^2} \right) \langle n \rangle \left[12n^2 p + 1 - p + 6n \langle n \rangle (1 - 2p) - 4 \langle n \rangle^2 (1-p) \right] \\
&= \mu_L \left[1 - \left(\frac{1-p}{6n} \right) \left(\frac{12n \left[n - \langle n \rangle \right] + 4 \langle n \rangle^2 - 1}{2n - \langle n \rangle} \right) \right] . \qquad (2)
\end{aligned}$$

In order to prove Theorem 1, we need to compare (2) with (1):

$$\begin{aligned}
\mu_L \left[1 - \left(\frac{1-p}{6n} \right) \left(\frac{12n \left[n - \langle n \rangle \right] + 4 \langle n \rangle^2 - 1}{2n - \langle n \rangle} \right) \right] &\leq \left(1 - \frac{8\mu_L^2}{9 (\sigma_L^2 + \mu_L^2)} \right) \mu_L \\
\left(\frac{1-p}{2n} \right) \left(\frac{12n \left[n - \langle n \rangle \right] + 4 \langle n \rangle^2 - 1}{2n - \langle n \rangle} \right) &\geq \frac{8\mu_L^2}{3 (\sigma_L^2 + \mu_L^2)} \\
\frac{12n \left[n - \langle n \rangle \right] + 4 \langle n \rangle^2 - 1}{2n - \langle n \rangle} &\geq \frac{8 \langle n \rangle \left[2n - \langle n \rangle \right]^2}{1 + 6 \langle n \rangle n - 4 \langle n \rangle^2} .
\end{aligned}$$

Note that p has dropped out of this comparison. Cross-multiplying, expanding, and simplifying leads in a straightforward if tedious manner to the equivalent inequality:

$$8 \langle n \rangle \left[n - \langle n \rangle \right]^3 + 12 \left[n - \langle n \rangle \right]^2 + 6 \langle n \rangle \left[n - \langle n \rangle \right] + 2 \langle n \rangle^2 - 1 \geq 0 . \qquad (3)$$

At this point we replace $n - \langle n \rangle$ with ε, representing the difference between n and its closest integer value. It is clear that $\varepsilon \in \left[-\frac{1}{2}, \frac{1}{2} \right)$. With this substitution, (3) becomes

$$8 \langle n \rangle \varepsilon^3 + 12\varepsilon^2 + 6 \langle n \rangle \varepsilon + 2 \langle n \rangle^2 - 1 \geq 0 . \qquad (4)$$

We need to show that (4) holds for all $\langle n\rangle \geq 1$ and all $\varepsilon \in \left[-\frac{1}{2}, \frac{1}{2}\right)$; we start with the latter. Taking the partial derivative of (4) with respect to ε yields

$$\begin{aligned} & 24\langle n\rangle \varepsilon^2 + 24\varepsilon + 6\langle n\rangle \\ = \; & 6\left[(\langle n\rangle - 1)\left(4\varepsilon^2 + 1\right) + (2\varepsilon + 1)^2\right] \ , \end{aligned}$$

which is non-negative since $\langle n\rangle \geq 1$. Thus the left-hand side of (4) is increasing in ε; to show that (4) holds for all ε, it is sufficient to show that it holds for the lowest feasible value of ε: $\varepsilon = -\frac{1}{2}$. Making this substitution, (4) becomes

$$2\left[\langle n\rangle - 1\right]^2 \geq 0 \ ,$$

which is always true, completing the proof. ■

3 Heuristic Cost Performance

For ease of comparison, the numerical results from Bradley and Robinson (2005) are repeated in the first six lines of Table 1. As an aside, note that many of the numbers given here are trivially different than theirs; this is because they had prematurely truncated the lead-time distribution at $l = 100$. This truncation was inconsequential for $\mu_L = 6$ and 10, but not for the highly skewed lead-time distribution with $\mu_L = 2$, where F_{100} was only 99.925% in the worst case ($\sigma_L = 8$). These minor corrections do not alter their conclusions.

Table 1

Summary of the Percentage Cost Increase

Across the Test Bed of 145,800 Parameter Combinations

(All numbers are in percent.)

Distribution	Variance[1]	Mean	Std Dev	95%-ile	99%-ile	Worst Case	Prob{Δ=0}	Prob{$\Delta\leq$1%}	Prob{$\Delta\leq$5%}
Normal	LT: σ_L^2	63.91	59.91	179.24	235.62	286.85	9.97	14.38	20.85
Normal	Old: $\hat{\sigma}_N^2$	0.32	1.33	1.44	5.65	36.62	61.08	93.21	98.82
Normal	True: σ_N^2	0.58	2.25	2.79	9.51	61.33	59.30	87.74	97.51
Neg. Bin.	LT: σ_L^2	69.06	86.69	231.51	402.52	1,080.33	10.02	14.23	21.49
Neg. Bin.	Old: $\hat{\sigma}_N^2$	0.38	1.10	1.97	5.46	23.19	57.60	89.89	98.81
Neg. Bin.	True: σ_N^2	0.08	0.30	0.41	1.47	9.15	77.35	98.16	99.98
Neg. Bin.	New: $\tilde{\sigma}_N^2$	0.18	0.55	1.00	2.74	12.20	65.06	95.01	99.80
Normal	New: $\tilde{\sigma}_N^2$	0.43	1.94	2.10	7.89	61.33	63.71	91.99	98.15
Lognormal	New: $\tilde{\sigma}_N^2$	1.12	3.04	5.72	15.84	41.92	44.31	78.43	94.18
Gamma	New: $\tilde{\sigma}_N^2$	0.36	0.93	1.96	4.75	15.39	56.64	89.14	99.12
Beta[2]	New: $\tilde{\sigma}_N^2$	0.13	0.44	0.60	1.87	13.05	64.53	97.42	99.88
Beta[3]	New: $\tilde{\sigma}_N^2$	0.05	0.14	0.25	0.66	3.96	70.93	99.61	100.00

[1] This column identifies the variance used in place of σ_N^2 when calculating σ_{SF}^2.

[2] For these beta distributions, $\alpha_3 = 0.3166$ and $\alpha_4 = 3.1199$.

[3] For these beta distributions, α_3 and α_4 are approximated from c_v^2 through the linear regressions

$$\hat{\alpha}_3 = 0.1156 + 2.1617 \cdot c_v^2 \text{ , and}$$

$$\hat{\alpha}_4 = 2.9406 + 1.9289 \cdot c_v^2 \text{ .}$$

References

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