10.1287/MSOM.1070.0191ec

Online Appendix

Proof of Lemma 1

We shall prove each claim in Lemma 1 by checking that the specified operations lead to membership in the correct set.

Part (a):

Let $t^* = \max\{t : g(t) \geq g(s),\ \forall\ s\}$ be the largest unconstrained maximizer of $g(\cdot)$. Then,

$$g_1(s) = \begin{cases} g(t^*) & \text{if } s \leq t^*, \\ g(s) & \text{otherwise.} \end{cases}$$

Clearly, $g_1(s) < \infty$ since $g(\cdot)$ is bounded. Next, we consider three mutually exclusive and collectively exhaustive cases representing different values of s relative to t^* to verify that properties (2) and (3) in Definition 1 hold.

Since $g_1(s) = g(t^*)$, for any $s \leq t^*$, it follows that when $s < t^*$,

$$g_1^+(s) = g(t^*) - g(t^*) = 0 = g_1^+(s-1).$$

Clearly, $g_1(s)$ has properties (2) and (3) since $g_1^+(s) \leq g_1^+(s-1)$ and $g_1^+(s) = 0$ is bounded by any strictly positive η.

If $s = t^*$, then $g_1^+(s) = g(s+1) - g(t^*) \leq 0$, whereas $g_1^+(s-1) = 0$. The inequality in the previous sentence comes from the definition of t^*. Therefore, $g_1^+(s) \leq g_1^+(s-1)$ and $g_1^+(s) \leq 0$, and once again it is shown to possess the desired properties.

Finally, suppose $s > t^*$. Here,

$$\begin{aligned} g_1^+(s) &= g(s+1) - g(s) = g^+(s) \\ &\leq g^+(s-1) = g(s) - g(s+1) \\ &= g_1^+(s-1) \end{aligned}$$

The inequality above follows from the fact that $g \in \mathcal{G}$. It also implies that there exists a $\eta < \infty$ such that $g^+(s) < \eta$. Therefore, from above, $g_1^+(s) < \eta$, and this completes the proof of part (a).

Part (b):

Since $f(\cdot) \in \mathcal{F}$, the first property of functions in $\mathcal{F}$ (see Definition 2) implies $f_1(s) < \infty$ for any s. Let x be a realization of X and $f_{1,x}(s) = f(s-x)^+$ be the corresponding function of interest. Next, we shall prove the existence of properties (2) and (3) for each s and each x, which implies the desired result.

Consider first the case when $s < x$. Here,

$$\begin{aligned} f_{1,x}^+(s) &= f(s+1-x)^+ - f(s-x)^+ = f(0) - f(0) = 0 \\ &= f(s-x)^+ - f(s-1-x)^+ = f_{1,x}^+(s-1) \end{aligned}$$

Clearly $f_{1,x}^+(s) \le f_{1,x}^+(s-1)$ and $f_{1,x}^+(s) \le 0$ for all $s < x$.

In the second instance, $s = x$. Similar arguments now yield

$$\begin{aligned} f_{1,x}^+(s) &= f(1) - f(0) \\ &\le 0 = f(0) - f(0) = f_{1,x}^+(s-1) \end{aligned}$$

Again, $f_{1,x}^+(s) \le f_{1,x}^+(s-1)$ and $f_{1,x}^+(s) \le 0$ as desired.

The final instance has $s > x$ resulting in the following comparisons.

$$\begin{aligned} f_{1,x}^+(s) &= f(s+1-x) - f(s-x) \\ &\le f(s-x) - f(s-1-x) = f_{1,x}^+(s-1) \end{aligned}$$

Clearly, $f_{1,x}^+(s) \le f_{1,x}^+(s-1)$. Moreover, because $f_{1,x}^+(s) = f^+(s-x) \le f^+(0) \le 0$, the function $f_{1,x}$ is decreasing in its argument.

We have shown that f_1 is bounded, and $f_{1,x}$ is decreasing and has monotone decreasing differences for each realization x of X. This implies that $f_1 \in \mathcal{F}$.

Part (c):

From its definition, $g_2(s) = (p+\pi)E(s \wedge X) + E[g(s-X)]$, where $g(s-x)$ is in $\mathcal{G}$ for each realization x of X. Therefore, $E[g(s-X)]$ also belongs to $\mathcal{G}$. We know that the sum of any two functions in

$\mathcal{G}$ is also in $\mathcal{G}$. Thus, $g_2 \in \mathcal{G}$ will follow if $\hat{g}(s) = (p+\pi)E(s \wedge X)$ is in $\mathcal{G}$. We show this to be the case by considering all three defining properties of functions in $\mathcal{G}$.

From the finiteness of $E(X)$, it follows that $\hat{g}(\infty) = (p+\pi)\lim_{s\to\infty} E(s\wedge X) \to (p+\pi)E(X) < \infty$. Moreover $\hat{g}^+(s) = (p+\pi)P(X > s) \geq 0$ implies that $\hat{g}$ is nondecreasing in s. Therefore $\hat{g}(\infty) < \infty$ implies that $\hat{g}(s) < \infty$ for all s. Next, $\hat{g}^+(s) = (p+\pi)P(X > s) \leq (p+\pi)P(X > s-1) = \hat{g}^+(s-1)$. That is, $\hat{g}$ has monotone decreasing differences. Finally, $\hat{g}^+(s) = (p+\pi)P(X > s) \leq (p+\pi)$, which implies that there exists a $\eta > (p+\pi)$ such that $\hat{g}^+(s) < \eta$ for all s.

Part (d):

The claim in part (d) is a direct consequence of the fact that $\mathcal{F} \subset \mathcal{G}$ and that the sum of any two functions in $\mathcal{G}$ is also in $\mathcal{G}$.

Part (e):

It is straightforward to verify that multiplying by a positive fraction does not affect any of the three defining properties of the functions in $\mathcal{F}$ and $\mathcal{G}$.

Proof of Proposition 2

First, we simplify and rearrange certain terms on the right hand side of (12) as shown below:

$$\begin{aligned}
-w(\sum_{m=1}^{\kappa} s_{m,n}) \quad &+ \quad (p+w)[\sum_{j=1}^{\mu-1} q_{n-j} \wedge b_n] - w\sum_{j=1}^{\mu-1} q_{n-j} \\
&= \quad -w\theta_n + w[\sum_{j=1}^{\mu-1} q_{n-j} - b_n] + (p+w)b_n - (p+w)[b_n - \sum_{j=1}^{\mu-1} q_{n-j}]^+ - w\sum_{j=1}^{\mu-1} q_{n-j} \\
&= \quad -w\theta_n - (p+w)[b_n - \sum_{j=1}^{\mu-1} q_{n-j}]^+ + pb_n \\
&= \quad -w\theta_n - (p+w)[(-\theta_n)^+] + p[b_n - \sum_{m=1}^{\kappa} s_{m,n}]^+ \\
&= \quad -w\theta_n^+ - p[(-\theta_n)^+] + p[-(\theta_n - \sum_{j=1}^{\mu-1} q_{n-j})]^+ \qquad (40)
\end{aligned}$$

The last three equalities use the fact that $b_n \cdot \sum_{m=1}^{\kappa} s_{m,n} = 0$. Therefore, $b_n = [b_n - \sum_{m=1}^{\kappa} s_{m,n}]^+$ and this allows us to conclude that $[b_n - \sum_{j=1}^{\mu-1} q_{n-j}]^+ = (-\theta_n)^+$. The last equality above is a direct consequence of the fact that $a = a^+ - (-a)^+$ for any a. With these simplifications, the terminal value function becomes

$$
\begin{aligned}
v_n(\tilde{s}_n, \tilde{q}_{n-1}, b_n) \quad = \quad & u_1(\theta_n)^+ - w\theta_n^+ - u_2[(-\theta_n)^+] - p[(-\theta_n)^+] + p[-(\theta_n - \sum_{j=1}^{\mu-1} q_{n-j})]^+ \\
& -w \sum_{j=1}^{\kappa} \hat{\alpha}_j \left(\sum_{m=j}^{\kappa} s_{m,n} \right) \qquad (41)
\end{aligned}
$$

The proof of Proposition 2 is by induction. Starting with $t = n$, we define

$$
\begin{aligned}
f_\kappa^{(n)}(s) \quad & \doteq \quad -w\hat{\alpha}_1 s \\
f_{\kappa-j}^{(n)}(s) \quad & \doteq \quad -w\hat{\alpha}_{j+1} s, \qquad \text{where } 1 \leq j < \kappa.
\end{aligned}
$$

For each non-negative integer m, let

$$
\hat{g}^{(n)}(s) \doteq u_1[(s)^+] - ws^+ - u_2[(-s)^+] - p[(-s)^+] + p(m-s)^+,
$$

where $s \in \mathcal{Z}$ is an arbitrary integer. [We abuse notation slightly here and treat $\sum_{j=1}^{\mu-1} q_{n-j}$, which are determined before period n, as a constant, although this term is also a part of θ_n.] We shall argue next that $\hat{g}^{(n)}(s) \in \mathcal{G}$. A proof of this assertion involves considering three mutually exclusive and collective exhaustive cases, as shown below.

<u>Case 1: $s \geq m$</u>

In this range of values of s, $\hat{g}^{(n)}(s) = u_1(s) - ws$. This immediately implies that (a) $\hat{g}^{(n)}(s) < \infty$, being the sum of bounded functions, (b) $\hat{g}^{(n)+}(s) = u_1^+(s) - w < 0$, by the Proposition statement, and (c) $\hat{g}^{(n)+}(s) - \hat{g}^{(n)+}(s-1) = u_1^+(s) - u_1^+(s-1) \leq 0$, since $u_2 \in \mathcal{G}$. Therefore, $\hat{g}^{(n)}(s) \in \mathcal{G}$.

<u>Case 2: $s \in [0, m)$</u>

Now $\hat{g}^{(n)}(s) = u_1(s) - (p+w)s + pm$. The following facts are easily verified: (a) $\hat{g}^{(n)}(s) < \infty$, since it is a sum of bounded functions (note $m < \infty$), (b) $\hat{g}^{(n)+}(s) = u_1^+(s) - (p+w) < 0$, by the Proposition statement and the fact that $p > 0$, and (c) $\hat{g}^{(n)+}(s) - \hat{g}^{(n)+}(s-1) = u_1^+(s) - u_1^+(s-1) \leq 0$, since $u_2 \in \mathcal{G}$. Therefore, $\hat{g}^{(n)}(s) \in \mathcal{G}$ when $s \in [0, m)$.

<u>Case 3: $s < 0$</u>

Now $\hat{g}^{(n)}(s) = -u_2(-s) + pm$. Notice that $\hat{g}^{(n)}(s)$ is the sum of functions in $\mathcal{F}$ (being a constant, pm

trivially belongs to $\mathcal{F}$). Since the set $\mathcal{F}$ is closed under summation, $\hat{g}^{(n)}(s) \in \mathcal{F}$, which immediately implies that $\hat{g}^{(n)}(s) \in \mathcal{G}$ since $\mathcal{F} \subset \mathcal{G}$.

With the above functional definitions in hand, we rewrite terms on the right hand side of (12) in terms of functions $f_j^{(n)}$ and $g^{(n)}$. The arguments of functions $f_j^{(n)}$ are $\sum_{m=\kappa-j+1}^{\kappa} s_{m,n}$ and the argument of $g^{(n)}$ is θ_n with the constant $m = \sum_{j=1}^{\mu-1} q_{n-j}$. These substitutions result in the following alternative representation of v_n.

$$v_n(\tilde{s}_n, \tilde{q}_{n-1}, b_n) = f_1^{(n)}(s_{\kappa,n}) + f_2^{(n)}(s_{\kappa,n} + s_{\kappa-1,n}) + \cdots + f_\kappa^{(n)}(\sum_{m=1}^{\kappa} s_{m,n}) + g^{(n)}(\theta_n).$$

So far, we have established that the terminal value function has the desired structure of equation (14).

Next we assume that the value function has the desired structure for time indices $t+1, t+2, \cdots, n$, and prove that the structure is preserved in v_t. After substituting equation (7) and some straightforward algebraic manipulations, equation (11) can be rewritten as

$$\begin{aligned} v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t) \quad &= \quad -w \sum_{i=1}^{\kappa} \hat{\alpha}_i \left(\sum_{m=i}^{\kappa} s_{m,t} \right) - \pi E(D_t) + \max_{q_t \geq 0} E\Big\{ (p+\pi)[(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t) \wedge D_t] \\ &+ \quad pb_t - h(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - D_t - b_t)^+ + \beta v_{t+1}(\tilde{s}_{t+1}, \tilde{q}_t, b_{t+1}) \Big\}. \end{aligned} \tag{42}$$

From the induction hypothesis, v_{t+1} can be written as the sum of functions of partial sums of $\{s_{m,t+1}\}$ and b_{t+1}. Therefore, we first express these partial sums in terms of $\{s_{m,t}\}$. From equations (2) – (10) and the definition of θ_t, we get:

$$\sum_{m=i}^{\kappa} s_{m,t+1} = (\sum_{m=i-1}^{\kappa} s_{m,t} - D_t)^+ \qquad \text{for } i = 2, \cdots, \kappa, \tag{43}$$

$$\sum_{m=1}^{\kappa} s_{m,t+1} - b_{t+1} = (q_{t-\mu+1} + \sum_{m=1}^{\kappa} s_{m,t} - D_t - b_t), \text{ and} \tag{44}$$

$$\theta_{t+1} = \theta_t + q_t - D_t. \tag{45}$$

Using the induction hypothesis, equations (42) – (45), and rearranging the resulting terms, we get:

$$
\begin{aligned}
v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t) &= -w \sum_{i=1}^{\kappa} \hat{\alpha}_i \left(\sum_{m=i}^{\kappa} s_{m,t} \right) - \pi E(D_t) + \beta E\{ f_1^{(t+1)}((\sum_{m=\kappa-1}^{\kappa} s_{m,t} - D_t)^+) + \cdots \\
&+ f_{\kappa-1}^{(t+1)}((\sum_{m=1}^{\kappa} s_{m,t} - D_t)^+)\} + E\Big\{ pb_t + (p+\pi)[(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t) \wedge D_t] \\
&- h(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - D_t - b_t)^+ + \beta f_{\kappa}^{(t+1)}((\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - D_t - b_t)^+) \Big\} \\
&+ \beta \max_{q_t \geq 0} E\{ g^{(t+1)}(\theta_t + q_t - D_t)\}. \qquad (46)
\end{aligned}
$$

From parts (b) and (e) of Lemma 1, we can conclude that for each $i = 1, \cdots \kappa - 1$, the functions $\beta E\{f_i^{(t+1)}((\sum_{m=\kappa-i}^{\kappa} s_{m,t} - D_t)^+)\} \in \mathcal{F}$. Moreover, for each $i = 1, \cdots \kappa$, the function $-w\hat{\alpha}_i \sum_{m=i}^{\kappa} s_{m,t}$ is a decreasing linear function of $\sum_{m=i}^{\kappa} s_{m,t}$, and hence belongs to the set $\mathcal{F}$. Next, we define the following new functions:

$$
\begin{aligned}
f_1^{(t)}(s_{\kappa,t}) &\doteq -w\hat{\alpha}_1 s_{\kappa,t} - \pi E(D_t), \qquad \text{and} \qquad (47) \\
f_i^{(t)}(\sum_{m=\kappa-i+1}^{\kappa} s_{m,t}) &\doteq -w\hat{\alpha}_{\kappa-i+1} \left(\sum_{m=\kappa-i+1}^{\kappa} s_{m,t} \right) + \beta E\{ f_{i-1}^{(t+1)}[(\sum_{m=\kappa-i+1}^{\kappa} s_{m,t} - D_t)^+]\}, \quad (48)
\end{aligned}
$$

for each $i = 2, \cdots, \kappa$. Since the set $\mathcal{F}$ is closed under summation, this implies that each $f_i^{(t)}$ also belongs to $\mathcal{F}$.

We now turn our attention to the remaining terms in equation (46). From arguments similar to those leading up to equation (40), we can rewrite the term $pb_t + pE\left[(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t) \wedge D_t\right]$ as follows:

$$
\begin{aligned}
& pb_t + pE\left[(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t) \wedge D_t \right] \\
= \; & pb_t + p(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t) - pE\left[(\sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - b_t - D_t)^+ \right] \\
= \; & p(\theta_t + b_t - \sum_{j=1}^{\mu-2} q_{t-j}) - pE\left[(\theta_t - \sum_{j=1}^{\mu-2} q_{t-j} - D_t)^+ \right].
\end{aligned}
$$

From arguments used to prove part (c) of Lemma 1, it follows that the function $\hat{g}^{(t)}(\theta_t) \doteq \pi E[(\theta_t - \sum_{j=1}^{\mu-2} q_{t-j}) \wedge D_t] + p(\theta_t + b_t - \sum_{j=1}^{\mu-2} q_{t-j})$ belongs to $\mathcal{G}$. It is also straightforward to check that

$E[-(h+p)(\theta_t - \sum_{j=1}^{\mu-2} q_{t-j} - D_t)^+]$ is decreasing and has monotone decreasing differences in θ_t. Therefore, this function belongs to the set $\mathcal{F}$. Moreover, since $\mathcal{F}$ is closed under the sum, this means $\hat{f}_\kappa^{(t)}(\theta_t) = E\{-(h+p)(\theta_t - \sum_{j=1}^{\mu-2} q_{t-j} - D_t)^+ + \beta f_\kappa^{(t+1)}((\theta_t - \sum_{j=1}^{\mu-2} q_{t-j} - D_t)^+)\}$ also belongs to the set $\mathcal{F}$. The previous statement makes use of parts (b) and (e) of Lemma 1.

We define a function $\bar{g}^{(t)}(\theta_t) \doteq \beta \max_{q_t \geq 0} E\{g^{(t+1)}(\theta_t + q_t - D_t)\}$. Then, from part (a) and (e) of Lemma 1, $\bar{g}^{(t)} \in \mathcal{G}$. Since sum of functions in $\mathcal{G}$ is also in $\mathcal{G}$, it follows that $g^{\dagger(t)}(\theta_t) = \hat{g}^{(t)}(\theta_t) + \bar{g}^{(t)}(\theta_t)$ also belongs to $\mathcal{G}$. Finally, using part (d) of Lemma 1, we conclude that $g^{(t)}(\theta_t) = g^{\dagger(t)}(\theta_t) + \hat{f}_\kappa^{(t)}(\theta_t)$ belongs to the set $\mathcal{G}$. Now, from equations (46), (47) and (48), and the arguments above, we obtain equation (14). Hence proved. #