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USING IMPERFECT ADVANCE DEMAND INFORMATION IN PRODUCTION-INVENTORY SYSTEMS WITH MULTIPLE CUSTOMER CLASSES

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Online Appendix

Proof of Property 2

Let $v \in \mathcal{U}$, $i \in \mathcal{A}$ and $y_i < m_i$. From C.1 and (3), we have

$$\Delta_0 v(s(\mathbf{y}+\mathbf{e}_i),\mathbf{y}) \geq \Delta_0 v(s(\mathbf{y}+\mathbf{e}_i),\mathbf{y}+\mathbf{e}_i) > 0. \tag{1}$$

Thus $\Delta_0 v(s(\mathbf{y}+\mathbf{e}_i),\mathbf{y}) > 0$, from which we deduce using (3) that $s(\mathbf{y}+\mathbf{e}_i) \geq s(\mathbf{y})$. Furthermore, from C.2 and (3), we have

$$\Delta_0 v(s(\mathbf{y})+1,\mathbf{y}+\mathbf{e}_i) \geq \Delta_0 v(s(\mathbf{y}),\mathbf{y}) > 0. \tag{2}$$

Hence, $\Delta_0 v(s(\mathbf{y})+1,\mathbf{y}+\mathbf{e}_i) \geq 0$, from which we deduce using again (3) that $s(\mathbf{y}+\mathbf{e}_i) \leq s(\mathbf{y})+1$.

Let $j \in \mathcal{W}$. Then from C.1 and using (5) leads to

$$\Delta_0 v(r_j(\mathbf{y}+\mathbf{e}_i)-1,\mathbf{y}) \geq \Delta_0 v(r_j(\mathbf{y}+\mathbf{e}_i)-1,\mathbf{y}+\mathbf{e}_i) > -c_j. \tag{3}$$

It follows that $\Delta_0 v(r_j(\mathbf{y}+\mathbf{e}_i)-1,\mathbf{y}) > -c_j$ and we deduce from (5) that $r_j(\mathbf{y}+\mathbf{e}_i) \geq r_j(\mathbf{y})$. In addition, we have from C.2 and (5)

$$\Delta_0 v(r_j(\mathbf{y}),\mathbf{y}+\mathbf{e}_i) \geq \Delta_0 v(r_j(\mathbf{y})-1,\mathbf{y}) > -c_j. \tag{4}$$

Consequently, using (5) we have $r_j(\mathbf{y}+\mathbf{e}_i) \leq r_j(\mathbf{y})+1$. For $j \in \mathcal{A}$, we can prove similarly that $r_j(\mathbf{y}) \leq r_j(\mathbf{y}+\mathbf{e}_i) \leq r_j(\mathbf{y})+1$, which completes the proof. □

Proof of Property 3

Assume in all the proof that $v \in \mathcal{U}$, $c_i \geq c_j$, $\mathbf{y} \in \mathcal{Y}$ and $x \geq 1$.

Case 1: $(i,j) \in \mathcal{A}^2$

From Condition 1 of $\mathcal{U}$ and $c_i \geq c_j$, we have:

$$\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_i)+c_i \geq \Delta_0 v(x-1,\mathbf{y})+c_j \tag{5}$$

which implies that $r_i(\mathbf{y}) \leq r_j(\mathbf{y}+\mathbf{e}_j)$. From Property 2, we also have $r_j(\mathbf{y}+\mathbf{e}_j) \leq r_j(\mathbf{y})+1$ and we finally obtain that $r_i(\mathbf{y}) \leq r_j(\mathbf{y})+1$.

Case 2: $(i, j) \in \mathcal{W}^2$

We have:

$$\Delta_0 v(x-1, \mathbf{y}) + c_i \geq \Delta_0 v(x-1, \mathbf{y}) + c_j$$

which immediately implies that $r_i(\mathbf{y}) \leq r_j(\mathbf{y})$.

Case 3: $i \in \mathcal{A}$ **and** $j \in \mathcal{W}$

We have:

$$\Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_i) + c_i \geq \Delta_0 v(x-1, \mathbf{y}) + c_j$$

which implies that $r_i(\mathbf{y}) \leq r_j(\mathbf{y})$.

Case 4: $i \in \mathcal{W}$ **and** $j \in \mathcal{A}$

We have, from Condition 3 of $\mathcal{U}$:

$$\Delta_0 v(x, \mathbf{y}) + c_i \geq \Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_j) + c_j$$

which implies that $r_i(\mathbf{y}) \leq r_j(\mathbf{y}) + 1$.

Proof of Lemma 1

To simplify the proof, we introduce three new operators as follows:

$$\tilde{A}_k^2 v(x, \mathbf{y}) = p_k y_k A_k^2 v(x, \mathbf{y}), \tag{6}$$

$$\tilde{A}_k^3 v(x, \mathbf{y}) = (1 - p_k) y_k A_k^3 v(x, \mathbf{y}) + (m_k - y_k) v(x, \mathbf{y}), \text{ and} \tag{7}$$

$$\tilde{A}_k^{23} v(x, \mathbf{y}) = \tilde{A}_k^2 v(x, \mathbf{y}) + \tilde{A}_k^3 v(x, \mathbf{y}). \tag{8}$$

We assume throughout the proof that $v \in \mathcal{U}$ and that $(x, \mathbf{y}) \in \mathbb{N} \times \mathcal{Y}$.

With equations (2) to (3) and the assumption that $v(x, \mathbf{y} - \mathbf{e}_i) = 0$ if $y_i = 0$ (this assumption holds for the rest of the proof), we can rewrite the different operators as follows:

$$Pv(x,\mathbf{y}) = \begin{cases} v(x+1,\mathbf{y}) & \text{if } x < s(\mathbf{y}) \\ v(x,\mathbf{y}) & \text{if } x \geq s(\mathbf{y}), \end{cases} \tag{9}$$

$$W_k v(x,\mathbf{y}) = \begin{cases} v(x,\mathbf{y}) + c_k & \text{if } x < r_k(\mathbf{y}) \\ v(x-1,\mathbf{y}) & \text{if } x \geq r_k(\mathbf{y}), \end{cases} \tag{10}$$

$$A_k^1 v(x,\mathbf{y}) = \begin{cases} v(x,\mathbf{y}+\mathbf{e}_k) & \text{if } y_k < m_k \\ v(x,\mathbf{y}) & \text{if } y_k = m_k, \end{cases} \tag{11}$$

$$\tilde{A}_k^2 v(x,\mathbf{y}) = \begin{cases} p_k y_k [v(x,\mathbf{y}-\mathbf{e}_k) + c_k] & \text{if } x < r_k(\mathbf{y}) \\ p_k y_k v(x-1,\mathbf{y}-\mathbf{e}_k) & \text{if } x \geq r_k(\mathbf{y}), \text{ and} \end{cases} \tag{12}$$

$$\tilde{A}_k^3 v(x,\mathbf{y}) = (1-p_k) y_k v(x,\mathbf{y}-\mathbf{e}_k) + (m_k - y_k) v(x,\mathbf{y}). \tag{13}$$

Using Equations (9) to (13), we obtain

$$\Delta_0 Pv(x,\mathbf{y}) = \begin{cases} \Delta_0 v(x+1,\mathbf{y}) \leq 0 & \text{if } x < s(\mathbf{y}) - 1 \\ 0 & \text{if } x = s(\mathbf{y}) - 1 \\ \Delta_0 v(x,\mathbf{y}) > 0 & \text{if } x \geq s(\mathbf{y}), \end{cases} \tag{14}$$

$$\Delta_0 W_k v(x,\mathbf{y}) = \begin{cases} \Delta_0 v(x,\mathbf{y}) \leq -c_k & \text{if } x < r_k(\mathbf{y}) - 1 \\ -c_k & \text{if } x = r_k(\mathbf{y}) - 1 \\ \Delta_0 v(x-1,\mathbf{y}) > -c_k & \text{if } x \geq r_k(\mathbf{y}), \end{cases} \tag{15}$$

$$\Delta_0 A_k^1 v(x,\mathbf{y}) = \begin{cases} \Delta_0 v(x,\mathbf{y}+\mathbf{e}_k) & \text{if } y_k < m_k \\ \Delta_0 v(x,\mathbf{y}) & \text{if } y_k = m_k, \end{cases} \tag{16}$$

$$\Delta_0 \tilde{A}_k^2 v(x,\mathbf{y}) = \begin{cases} p_k y_k \Delta_0 v(x,\mathbf{y}-\mathbf{e}_k) \leq -p_k y_k c_k & \text{if } x < r_k(\mathbf{y}) - 1 \\ -p_k y_k c_k & \text{if } x = r_k(\mathbf{y}) - 1 \\ p_k y_k \Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_k) > -p_k y_k c_k & \text{if } x \geq r_k(\mathbf{y}), \text{ and} \end{cases} \tag{17}$$

$$\Delta_0 \tilde{A}_k^3 v(x,\mathbf{y}) = (1-p_k) y_k \Delta_0 v(x,\mathbf{y}-\mathbf{e}_k) + (m_k - y_k) \Delta_0 v(x,\mathbf{y}). \tag{18}$$

The inequalities (≤ 0) and (> 0) in (21) follow from (3). The inequalities ($\leq -c_k$) and ($> -c_k$) in (22) follow from (5). The inequalities ($\leq p_k y_k c_k$) and ($> p_k y_k c_k$) in (24) follow from (4).

Based on these preliminary results, we will prove now that Tv satisfies conditions C.1, C.2, C.3 and C.4 and we will conclude that Tv also belongs to $\mathcal{U}$.

Condition C.1

Here we assume that $i \in \mathcal{A}$ and $y_i < m_i$. Equation (21) and Property 2 imply

$$\Delta_i \Delta_0 Pv(x, \mathbf{y}) = \begin{cases} \Delta_i \Delta_0 v(x+1, \mathbf{y}) \leq 0 & \text{if } x < s(\mathbf{y}) - 1 \\ \Delta_0 v(x+1, \mathbf{y} + \mathbf{e}_i) \leq 0 & \text{if } x = s(\mathbf{y}) - 1 = s(\mathbf{y} + \mathbf{e}_i) - 2 \\ 0 & \text{if } x = s(\mathbf{y}) - 1 = s(\mathbf{y} + \mathbf{e}_i) - 1 \\ -\Delta_0 v(x, \mathbf{y}) \leq 0 & \text{if } x = s(\mathbf{y}) = s(\mathbf{y} + \mathbf{e}_i) - 1 \\ \Delta_i \Delta_0 v(x, \mathbf{y}) \leq 0 & \text{if } x \geq s(\mathbf{y} + \mathbf{e}_i). \end{cases} \tag{19}$$

To establish the four inequalities (≤ 0) in (19), we use C.1 and (3). Thus $\Delta_i \Delta_0 Pv(x, \mathbf{y}) \leq 0$ and Pv satisfies C.1.

Let $k \in \mathcal{W}$. Equation (15) and Property 2 imply

$$\Delta_i \Delta_0 W_k v(x, \mathbf{y}) = \begin{cases} \Delta_i \Delta_0 v(x, \mathbf{y}) \leq 0 & \text{if } x < r_k(\mathbf{y}) - 1 \\ \Delta_0 v(x, \mathbf{y} + \mathbf{e}_i) + c_k \leq 0 & \text{if } x = r_k(\mathbf{y}) - 1 = r_k(\mathbf{y} + \mathbf{e}_i) - 2 \\ 0 & \text{if } x = r_k(\mathbf{y}) - 1 = r_k(\mathbf{y} + \mathbf{e}_i) - 1 \\ -[\Delta_0 v(x-1, \mathbf{y}) + c_k] \leq 0 & \text{if } x = r_k(\mathbf{y}) = r_k(\mathbf{y} + \mathbf{e}_i) - 1 \\ \Delta_i \Delta_0 v(x-1, \mathbf{y}) \leq 0 & \text{if } x \geq r_k(\mathbf{y} + \mathbf{e}_i). \end{cases} \tag{20}$$

To establish the four inequalities (≤ 0) in (20), we use C.1, and (5). Thus $\Delta_i \Delta_0 W_k v(x, \mathbf{y}) \leq 0$ and $W_k v$ satisfies C.1.

Assume now that $k \in \mathcal{A}$ and $k \neq i$. Then

$$\Delta_i \Delta_0 A_k^1 v(x, \mathbf{y}) = \begin{cases} \Delta_i \Delta_0 v(x, \mathbf{y} + \mathbf{e}_k) \leq 0 & \text{if } y_k < m_k \\ \Delta_i \Delta_0 v(x, \mathbf{y}) \leq 0 & \text{if } y_k = m_k, \end{cases} \tag{21}$$

$$\Delta_i \Delta_0 \tilde{A}_k^2 v(x, \mathbf{y})$$
$$= \begin{cases} p_k y_k \Delta_i \Delta_0 v(x, \mathbf{y} - \mathbf{e}_k) \leq 0 & \text{if } x < r_k(\mathbf{y}) - 1 \\ p_k y_k [\Delta_0 v(x, \mathbf{y} + \mathbf{e}_i - \mathbf{e}_k) + c_k] \leq 0 & \text{if } x = r_k(\mathbf{y}) - 1 = r_k(\mathbf{y} + \mathbf{e}_i) - 2 \\ 0 & \text{if } x = r_k(\mathbf{y}) - 1 = r_k(\mathbf{y} + \mathbf{e}_i) - 1 \\ -p_k y_k [\Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_k) + c_k] \leq 0 & \text{if } x = r_k(\mathbf{y}) = r_k(\mathbf{y} + \mathbf{e}_i) - 1 \\ p_k y_k \Delta_i \Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_k) \leq 0 & \text{if } x \geq r_k(\mathbf{y} + \mathbf{e}_i), \text{ and} \end{cases} \tag{22}$$

$$\Delta_i \Delta_0 \tilde{A}_k^3 v(x, \mathbf{y}) = (m_k - y_k) \Delta_i \Delta_0 v(x, \mathbf{y}) + (1 - p_k) y_k \Delta_i \Delta_0 v(x, \mathbf{y} - \mathbf{e}_k) \leq 0. \tag{23}$$

Inequalities (≤ 0) in (21) follows from C.1. To establish the 5 inequalities (≤ 0) in (22), we use C.1 and (**??**). The inequality (≤ 0) in (23) follows from C.1. The expressions in (21)-(23) imply that $\Delta_0 A_k^1$, $\Delta_0 \tilde{A}_k^2$ and $\Delta_0 \tilde{A}_k^3$ are non-increasing in y_i.

Assume now that $k = i$.

$$\Delta_i \Delta_0 A_i^1 v(x, \mathbf{y}) = \begin{cases} \Delta_i \Delta_0 v(x, \mathbf{y} + \mathbf{e}_i) \leq 0 & \text{if } y_i < m_i - 1 \\ 0 & \text{if } y_i = m_i - 1, \end{cases} \tag{24}$$

$$\Delta_i \Delta_0 \tilde{A}_i^2 v(x, \mathbf{y}) \tag{25}$$
$$= \begin{cases} p_i y_i \Delta_i \Delta_0 v(x, \mathbf{y} - \mathbf{e}_i) + p_i \Delta_0 v(x, \mathbf{y}) & \text{if } x < r_i(\mathbf{y}) - 1 \\ p_i y_i [\Delta_0 v(x, \mathbf{y}) + c_i] + p_i \Delta_0 v(x, \mathbf{y}) & \text{if } x = r_i(\mathbf{y}) - 1 = r_i(\mathbf{y} + \mathbf{e}_i) - 2 \\ -p_i c_i & \text{if } x = r_i(\mathbf{y}) - 1 = r_i(\mathbf{y} + \mathbf{e}_i) - 1 \\ -p_i y_i [\Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_i) + c_i] - p_i c_i & \text{if } x = r_i(\mathbf{y}) = r_i(\mathbf{y} + \mathbf{e}_i) - 1 \\ p_i y_i \Delta_i \Delta_0 v(x-1, \mathbf{y} - \mathbf{e}_i) + p_i \Delta_0 v(x-1, \mathbf{y}) & \text{if } x \geq r_i(\mathbf{y} + \mathbf{e}_i), \text{ and} \end{cases} \tag{26}$$

$$\Delta_i \Delta_0 \tilde{A}_i^3 v(x, \mathbf{y}) = (m_i - y_i - 1) \Delta_i \Delta_0 v(x, \mathbf{y}) + (1 - p_i) y_i \Delta_i \Delta_0 v(x, \mathbf{y} - \mathbf{e}_i) - p_i \Delta_0 v(x, \mathbf{y}). \tag{27}$$

Inequality (≤ 0) in (24) follows from C.1. Separately $\Delta_i \Delta_0 \tilde{A}_i^2$ and $\Delta_i \Delta_0 \tilde{A}_i^3$ are not always non-positive.

However we have from (27)

$$\begin{aligned}\Delta_i\Delta_0\tilde{A}_k^3 v(x,\mathbf{y}) + p_i\Delta_0 v(x,\mathbf{y}) &= (m_i - y_i - 1)\Delta_i\Delta_0 v(x,\mathbf{y}) + (1-p_i)y_i\Delta_i\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i)\\ &\leq 0. \end{aligned} \tag{28}$$

Inequality (28) follows from C.1. On the other hand, we have

$$\begin{aligned}&\Delta_i\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) - p_i\Delta_0 v(x,\mathbf{y})\\ &= \begin{cases} p_i y_i \Delta_i\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i) \leq 0 & \text{if } x < r_i(\mathbf{y}) - 1\\ p_i y_i[\Delta_0 v(x,\mathbf{y}) + c_i] \leq 0 & \text{if } x = r_i(\mathbf{y}) - 1 = r_i(\mathbf{y}+\mathbf{e}_i) - 2\\ -p_i[c_i + \Delta_0 v(x,\mathbf{y})] \leq 0 & \text{if } x = r_i(\mathbf{y}) - 1 = r_i(\mathbf{y}+\mathbf{e}_i) - 1\\ -p_i y_i[\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_i) + c_i] - p_i[c_i + \Delta_0 v(x,\mathbf{y})] \leq 0 & \text{if } x = r_i(\mathbf{y}) = r_i(\mathbf{y}+\mathbf{e}_i) - 1\\ p_i y_i \Delta_i\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_i) - p_i\Delta_0\Delta_0 v(x-1,\mathbf{y}) \leq 0 & \text{if } x \geq r_i(\mathbf{y}+\mathbf{e}_i).\end{cases}\end{aligned} \tag{29}$$

To establish the 5 inequalities (≤ 0) in (29), we use C.1, C.3 and (4). Consequently

$$\Delta_i\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) - p_i\Delta_0 v(x,\mathbf{y}) \leq 0. \tag{30}$$

If we add inequalities (28) and (30), we obtain

$$\Delta_i\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) + \Delta_i\Delta_0\tilde{A}_k^3 v(x,\mathbf{y}) = \Delta_i\Delta_0\tilde{A}_i^{23} v(x,\mathbf{y}) \leq 0. \tag{31}$$

Finally $\Delta_i\Delta_0 A_k^1 v(x,\mathbf{y}) \leq 0$ and $\Delta_i\Delta_0\tilde{A}_k^{23} v(x,\mathbf{y}) \leq 0$ for all $k \in \mathcal{A}$. Thus $A_k^1 v$ and $\tilde{A}_k^{23} v$ satisfy C.1.

Furthermore, the definition of operator T given in (2) yields

$$\Delta_0 Tv = \Delta_0 h + \mu\Delta_0 Pv + \sum_{k\in\mathcal{W}} \lambda_k\Delta_0 W_k v + \sum_{k\in\mathcal{A}} \Big\{\lambda_k\Delta_0 A_k^1 v + \nu_k(\Delta_0\tilde{A}_k^2 v + \Delta_0\tilde{A}_k^3 v)\Big\}, \tag{32}$$

which implies that $\Delta_0 Tv$ is non-increasing in y_i as a positive linear combination of non-increasing functions in y_i. Therefore Tv satisfies C.1.

Condition C.2

Here again we assume that $i \in \mathcal{A}$ and $y_i < m_i$. Equation (14) and Property 2 imply

$$\Delta_{0+i}\Delta_0 Pv(x,\mathbf{y}) = \begin{cases} \Delta_{0+i}\Delta_0 v(x+1,\mathbf{y}) \geq 0 & \text{if } x < s(\mathbf{y}+\mathbf{e}_i)-2 \\ -\Delta_0 v(x+1,\mathbf{y}) \geq 0 & \text{if } x = s(\mathbf{y}+\mathbf{e}_i)-2 = s(\mathbf{y})-2 \\ 0 & \text{if } x = s(\mathbf{y}+\mathbf{e}_i)-2 = s(\mathbf{y})-1 \\ \Delta_0 v(x+1,\mathbf{y}+\mathbf{e}_i) \geq 0 & \text{if } x = s(\mathbf{y}+\mathbf{e}_i)-1 = s(\mathbf{y})-1 \\ \Delta_{0+i}\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } x \geq s(\mathbf{y}). \end{cases} \quad (33)$$

The four inequalities (≥ 0) in (33) follows from C.2 and (3). Thus $\Delta_{0+i}\Delta_0 Pv(x,\mathbf{y}) \geq 0$ and Pv satisfies C.2.

Let $k \in \mathcal{W}$. Equation (15) and Property 2 imply

$$\Delta_{0+i}\Delta_0 W_k v(x,\mathbf{y}) = \begin{cases} \Delta_{0+i}\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } x < r_k(\mathbf{y}+\mathbf{e}_i)-2 \\ -[\Delta_0 v(x,\mathbf{y}) + c_k] \geq 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-2 = r_k(\mathbf{y})-2 \\ 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-2 = r_i(\mathbf{y})-1 \\ \Delta_0 v(x,\mathbf{y}+\mathbf{e}_i) + c_k \geq 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-1 = r_k(\mathbf{y})-1 \\ \Delta_{0+i}\Delta_0 v(x-1,\mathbf{y}) \geq 0 & \text{if } x \geq r_k(\mathbf{y}). \end{cases} \quad (34)$$

To establish the four inequalities (≥ 0) in (34), we use C.2 and (5). Thus $\Delta_{0+i}\Delta_0 W_k v(x,\mathbf{y}) \geq 0$ and $W_k v$ satisfies C.2.

Assume now that $k \in \mathcal{A}$ and $k \neq i$.

$$\Delta_{0+i}\Delta_0 A_k^1 v(x,\mathbf{y}) = \begin{cases} \Delta_{0+i}\Delta_0 v(x,\mathbf{y}+\mathbf{e}_k) \geq 0 & \text{if } y_k < m_k \\ \Delta_{0+i}\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } y_k = m_k, \end{cases} \tag{35}$$

$$\Delta_{0+i}\Delta_0 \tilde{A}_k^2 v(x,\mathbf{y})$$
$$= \begin{cases} p_k y_k \Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_k) \geq 0 & \text{if } x < r_k(\mathbf{y}+\mathbf{e}_i)-2 \\ -p_k y_k[\Delta_0 v(x,\mathbf{y}-\mathbf{e}_k)+c_k] \geq 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-2 = r_k(\mathbf{y})-2 \\ 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-2 = r_k(\mathbf{y})-1 \\ p_k y_k[\Delta_0 v(x,\mathbf{y}-\mathbf{e}_k+\mathbf{e}_i)+c_k] \geq 0 & \text{if } x = r_k(\mathbf{y}+\mathbf{e}_i)-1 = r_k(\mathbf{y})-1 \\ p_k y_k \Delta_{0+i}\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_k) \geq 0 & \text{if } x \geq r_k(\mathbf{y}), \text{ and} \end{cases} \tag{36}$$

$$\Delta_{0+i}\Delta_0 \tilde{A}_k^3 v(x,\mathbf{y}) = (m_k - y_k)\Delta_{0+i}\Delta_0 v(x,\mathbf{y}) + (1-p_k)y_k\Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_k) \geq 0. \tag{37}$$

The two inequalities (≥ 0) in (35) follow from C.2. To establish the 5 inequalities (≤ 0) in (36), we use C.2 and (4). The inequality (≥ 0) in (37) follows from C.2. Equations (35)-(37) imply that $\Delta_{0+i}\Delta_0 A_k^1 v(x,\mathbf{y})$, $\Delta_{0+i}\Delta_0 \tilde{A}_k^2 v(x,\mathbf{y})$ and $\Delta_{0+i}\Delta_0 \tilde{A}_k^3 v(x,\mathbf{y})$ are non-negative.

Assume now that $k = i$. Then,

$$\Delta_{0+i}\Delta_0 A_i^1 v(x,\mathbf{y}) = \begin{cases} \Delta_{0+i}\Delta_0 v(x,\mathbf{y}+\mathbf{e}_i) \geq 0 & \text{if } y_i < m_i - 1 \\ \Delta_0\Delta_0 v(x,\mathbf{y}+\mathbf{e}_i) \geq 0 & \text{if } y_i = m_i - 1, \end{cases} \tag{38}$$

$$\Delta_{0+i}\Delta_0 \tilde{A}_i^2 v(x,\mathbf{y})$$
$$= \begin{cases} p_i y_i \Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i) + p_i\Delta_0 v(x+1,\mathbf{y}) & \text{if } x < r_i(\mathbf{y}+\mathbf{e}_i)-2 \\ -p_i y_i[\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i)+c_i] - p_i c_i & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i)-2 = r_i(\mathbf{y})-2 \\ -p_i c_i & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i)-2 = r_i(\mathbf{y})-1 \\ p_i y_i[\Delta_0 v(x,\mathbf{y})+c_i] + p_i\Delta_0 v(x,\mathbf{y}) & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i)-1 = r_i(\mathbf{y})-1 \\ p_i y_i \Delta_{0+i}\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_i) + p_i\Delta_0 v(x,\mathbf{y}) & \text{if } x \geq r_i(\mathbf{y}), \text{ and} \end{cases} \tag{39}$$

$$\Delta_{0+i}\Delta_0 \tilde{A}_i^3 v(x,\mathbf{y}) = (m_i - y_i - 1)\Delta_{0+i}\Delta_0 v(x,\mathbf{y}) + (1-p_i)y_i\Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i)$$
$$+ \Delta_0\Delta_0 v(x,\mathbf{y}) - p_i\Delta_0 v(x+1,\mathbf{y}). \tag{40}$$

Inequalities (≥ 0) in (38) follow from C.2 and C.3 and we conclude that $\Delta_{0+i}\Delta_0 \tilde{A}_i^1 v(x,\mathbf{y}) \geq 0$. Separately

$\Delta_{0+i}\Delta_0\tilde{A}_i^2 v(x,\mathbf{y})$ and $\Delta_{0+i}\Delta_0\tilde{A}_i^3 v(x,\mathbf{y})$ are not always non-negative. However we have from (40)

$$\begin{aligned}\Delta_{0+i}\Delta_0\tilde{A}_i^3 v(x,\mathbf{y}) - \Delta_0\Delta_0 v(x,\mathbf{y}) + p_i\Delta_0 v(x+1,\mathbf{y}) \\ &= (m_i - y_i - 1)\Delta_{0+i}\Delta_0 v(x,\mathbf{y}) + (1-p_i)y_i\Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i) \\ &\geq 0. \end{aligned} \tag{41}$$

Inequality (41) follow from C.2. Also we have

$$\begin{aligned}&\Delta_{0+i}\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) + \Delta_0\Delta_0 v(x,\mathbf{y}) - p_i\Delta_0 v(x+1,\mathbf{y}) \\ &= \begin{cases} p_i y_i \Delta_{0+i}\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i) + \Delta_0\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } x < r_i(\mathbf{y}+\mathbf{e}_i) - 2 \\ -p_i y_i[\Delta_0 v(x,\mathbf{y}-\mathbf{e}_i) + c_i] + \Delta_0\Delta_0 v(x,\mathbf{y}) & \\ \qquad -p[c_i + \Delta_0 v(x+1,\mathbf{y})] \geq 0 & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i) - 2 = r_i(\mathbf{y}) - 2 \\ \Delta_0\Delta_0 v(x,\mathbf{y}) - p[c_i + \Delta_0 v(x+1,\mathbf{y})] \geq 0 & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i) - 2 = r_i(\mathbf{y}) - 1 \\ p_i y_i[\Delta_0 v(x,\mathbf{y}) + c_i] + (1-p_i)\Delta_0\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } x = r_i(\mathbf{y}+\mathbf{e}_i) - 1 = r_i(\mathbf{y}) - 1 \\ p_i y_i \Delta_{0+i}\Delta_0 v(x-1,\mathbf{y}-\mathbf{e}_i) + (1-p_i)\Delta_0\Delta_0 v(x,\mathbf{y}) \geq 0 & \text{if } x \geq r_i(\mathbf{y}). \end{cases}\end{aligned} \tag{42}$$

To establish the 5 inequalities (≥ 0) in (42), we use C.1, C.2 and (4). We can then write

$$\Delta_{0+i}\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) + \Delta_0\Delta_0 v(x,\mathbf{y}) - p_i\Delta_0 v(x+1,\mathbf{y}) \geq 0. \tag{43}$$

If we add inequalities (41) and (43) , we obtain

$$\Delta_{0+i}\Delta_0\tilde{A}_i^2 v(x,\mathbf{y}) + \Delta_{0+i}\Delta_0\tilde{A}_i^3 v(x,\mathbf{y}) = \Delta_{0+i}\Delta_0\tilde{A}_i^{23} v(x,\mathbf{y}) \geq 0. \tag{44}$$

Finally $\Delta_{0+i}\Delta_0 A_k^1 v(x,\mathbf{y}) \geq 0$ and $\Delta_{0+i}\Delta_0\tilde{A}_k^{23} v(x,\mathbf{y}) \geq 0$ for all $k \in \mathcal{A}$. Thus $A_k^1 v$ and $\tilde{A}_k^{23} v$ satisfy C.2. So does Tv by linear combination.

Condition C.3

Tv satisfies conditions C.3 as a direct consequence of satisfying conditions C.1 and C.2. If $y_i < m_i$

$$\Delta_0 v(x,\mathbf{y}) \leq \Delta_0 v(x+1,\mathbf{y}+\mathbf{e}_i) \tag{45}$$

$$\leq \Delta_0 v(x+1,\mathbf{y}) \tag{46}$$

where (46) follows from C.2. If $y_i = m_i$

$$\Delta_0 v(x, \mathbf{y}) \leq \Delta_0 v(x, \mathbf{y} - \mathbf{e}_i) \quad (47)$$

$$\leq \Delta_0 v(x + 1, \mathbf{y}) \quad (48)$$

where (48) follows from C.3.

Condition C.4

By assumption h is increasing in x thus $\Delta_0 h(x, \mathbf{y}) > 0$. Applying C.4 to (14)-(18) leads to $\Delta_0 P v(x, \mathbf{y}) \geq -c_1$, $\Delta_0 W_k v(x, \mathbf{y}) \geq -c_1$, $\Delta_0 A_k^1 v(x, \mathbf{y}) \geq -c_1$ and $\Delta_0 \tilde{A}_k^{23} v(x, \mathbf{y}) \geq -m_k c_1$. From (32), we obtain

$$\Delta_0 T v(x, \mathbf{y}) \geq -\left(\mu + \sum_{k \in \mathcal{A} \cup \mathcal{W}} \lambda_k + \sum_{k \in \mathcal{A}} m_k \nu_k\right) c_1 \quad (49)$$

$$\geq -c_1. \quad (50)$$

Inequality (50) follows from the rescaling condition $\alpha + \mu + \sum_{k \in \mathcal{A} \cup \mathcal{W}} \lambda_k + \sum_{k \in \mathcal{A}} m_k \nu_k = 1$. We conclude that Tv satisfies C.4. This completes the proof of lemma 1.

Proof of Theorem 1

Our problem verifies Assumption P in Bertsekas (2001, Section 3.1). From Proposition 3.1.5 and 3.1.6 of Bertsekas (2001), $v^* = \lim_{n \to \infty} T^{(n)} v$ for any v in $\mathcal{U}$, where $T^{(n)}$ refers to n compositions of operator T. Since $\mathcal{U}$ is complete, v^* belongs to $\mathcal{U}$ from Lemma 1. Define $s^*(\mathbf{y}) = \min[x \geq 0 | \Delta_0 v^*(x, \mathbf{y}) > 0]$, $r_i^*(\mathbf{y}) = \min[x \geq 1 | \Delta_0 v^*(x - 1, \mathbf{y} - \mathbf{e}_i) + c_i > 0]$ if $i \in \mathcal{A}$, and $r_i(\mathbf{y}) = \min[x \geq 1 | \Delta_0 v^*(x - 1, \mathbf{y}) + c_i > 0]$ if $i \in \mathcal{W}$. Let π^* be the policy described by the base-stock level $s^*(\mathbf{y})$ and rationing levels $r_j^*(\mathbf{y})$ such that the facility produces if $x < s^*(\mathbf{y})$ and idles otherwise, and an order from class j that becomes due is satisfied if $x \geq r_j^*(\mathbf{y})$ and is rejected otherwise. Then π^* is optimal since for each state $(x, \mathbf{y})$ it specifies an action that attains the minimum in $Tv^*(x)$ (see Propositions 3.1.1 and 3.1.3 of Bertsekas (2001)). Results P.1-P.6 follow from the fact that v^* satisfies conditions C.1-C.4 and the base-stock and rationing levels satisfy Properties 2 and 3.

Proof of Theorem 2

The proof is in two parts.We first show that the optimal cost of the unbounded problem satisfies certain optimality equations and that the problem admits an optimal stationary policy. We then show that the value functions of certain bounded problems converge to a limit as the bound on the state-space goes to infinity. It turns out that this limit satisfies the optimality equations of the unbounded problem and is therefore optimal.

For the purpose of this proof, we describe the state of the system with an $n+1$-dimensional vector $\mathbf{y} = (y_0, \ldots, y_n)$ such that y_0 represents the on-hand inventory and y_i the number of announced orders of class i; $\mathbf{e}_k$ now denotes the $k+1$-th unit vector of dimension $n+1$ (e.g., $\mathbf{e}_0 = (1, 0, \ldots, 0)$, $\mathbf{e}_1 = (0, 1, \ldots, 0)$ etc.).

Consider first the problem for which $m_1 = \cdots = m_n = +\infty$. In the following, we refer to this problem as the ∞-problem. Denote by $q(\mathbf{z}|\mathbf{y}, \mathbf{a})$ the transition rate from state $\mathbf{y}$ to state $\mathbf{z}$ given the decision $\mathbf{a} \in A := \{0, 1\}^{n+1}$. For example, for all $\mathbf{a}$, $q(\mathbf{y} - \mathbf{e}_k|\mathbf{y}, \mathbf{a}) = \nu_k y_k$ and we will sometimes use $q(\mathbf{y} - \mathbf{e}_k|\mathbf{y}, \mathbf{a})$ in place of $\nu_k y_k$ when it simplifies the presentation. Without loss of generality, we also introduce dummy transactions with rate μ when the policy states not to produce. In this case, $\sum_{\mathbf{z}\neq\mathbf{y}} q(\mathbf{z}|\mathbf{y}, \mathbf{a}) = \beta(\mathbf{y})$ with

$$\beta(\mathbf{y}) = \sum_{k\in\mathcal{W}\cup\mathcal{A}} \lambda_k + \mu + \sum_{k\in\mathcal{A}} \nu_k y_k$$

which does not depend on decision $\mathbf{a}$ and hence also simplifies the presentation. It follows that the transition probability from state $\mathbf{y}$ to state $\mathbf{z}$ given $\mathbf{a}$ is equal to $q(\mathbf{z}|\mathbf{y}, \mathbf{a})/\beta(\mathbf{y})$.

The corresponding optimality operator is then given by

$$\begin{aligned}
\tilde{T}v(\mathbf{y}) &= \min_{\mathbf{a}\in A} \int_0^{+\infty} \left(h(y_0) + \sum_{\mathbf{z}\neq\mathbf{y}} \frac{q(\mathbf{z}|\mathbf{y}, \mathbf{a})}{\beta(\mathbf{y})} v(\mathbf{z}) + \frac{\mu}{\beta(\mathbf{y})} v(\mathbf{y}) \mathbf{1}_{a_0=0} \right) \beta(\mathbf{y}) e^{-(\alpha+\beta(\mathbf{y}))t} dt \\
&\quad + \left(\int_0^{+\infty} \beta(\mathbf{y}) e^{-(\alpha+\beta(\mathbf{y}))t} dt \right) \sum_{i\in\mathcal{W}\cup\mathcal{A}} \frac{q(\mathbf{y} - \mathbf{e}_k|\mathbf{y}, \mathbf{a})}{\beta(\mathbf{y})} \ c_i \ \mathbf{1}_{a_i=0} \qquad (51) \\
&= \min_{\mathbf{a}\in A} \frac{c(\mathbf{y}, \mathbf{a})}{\alpha + \beta(\mathbf{y})} + \frac{\beta(\mathbf{y})}{\alpha + \beta(\mathbf{y})} \sum_{\mathbf{z}\neq\mathbf{y}} \frac{q(\mathbf{z}|\mathbf{y}, \mathbf{a})}{\beta(\mathbf{y})} v(\mathbf{z}) + \frac{\mu}{\alpha + \beta(\mathbf{y})} v(\mathbf{y}) \mathbf{1}_{a_0=0}, \qquad (52)
\end{aligned}$$

with

$$c(\mathbf{y}, \mathbf{a}) = hy_0 + \sum_{i\in\mathcal{W}\cup\mathcal{A}} \nu_i y_i c_i \mathbf{1}_{a_i=0},$$

where $\mathbf{1}_{a_i=0} = 1$ if $a_i = 0$ and is null otherwise. The costs c_i in (51) are discounted over a time period between two transition epochs (which is exponentially distributed with rate $\beta(\mathbf{y})$) since the costs c_i are incurred when the system leaves the current state $\mathbf{y}$. The last term of (52) is due to the dummy transition with rate μ when the policy states not to produce.

The ∞-problem is a continuous time Markov decision process with unbounded transition and reward rates. Next, we use a result due to Guo and Hernandez-Lerma (2003) to show the following proposition; see also Benjaafar et al. (2007).

Proposition 1 *The optimal cost of the ∞-problem v^* is the unique solution of the optimality equations*

$$\tilde{T}v^* = v^* \tag{53}$$

In addition, an optimal stationary policy exists.

Proof: The proposition results from the direct application of Theorem 3.2, parts (b) and (e) respectively, of Guo and Hernandez-Lerma (2003). We show in the following that our problem satisfies the conditions of their theorem (see Assumptions A (1)(2)(3), B (1)(2) and C (1)(2) in Guo and Hernandez-Lerma (2003) for more details).

Assumption A (1) is immediate by taking S_m equal to the state space of the m-problem (i.e. such that $y_i \leq m$). Consider now the function $R(\mathbf{y}) = \max_i y_i$. Assumption A (2) holds since $\inf_{\mathbf{z} \notin S_m} R(\mathbf{z}) = m$. For Assumption A (3), note that

$$\begin{aligned}
\sum_{\mathbf{z}} q(\mathbf{z}|\mathbf{y}, \mathbf{a}) R(\mathbf{z}) &= \sum_{k \in \mathcal{W}} \lambda_k R(\mathbf{y}) + \sum_{k \in \mathcal{A}} \lambda_k R(\mathbf{y} + \mathbf{e}_k) + \sum_{k \in \mathcal{A}} \nu_k y_k R(\mathbf{y} - \mathbf{e}_k) + \\
&\quad \mu R(\mathbf{y}) \mathbf{1}_{a_0 = 0} + \mu R(\mathbf{y} + \mathbf{e}_0) \mathbf{1}_{a_0 = 1} - \beta(\mathbf{y}) R(\mathbf{y}) \\
&\leq \mu R(\mathbf{y} + \mathbf{e}_0) + \sum_{k \in \mathcal{W}} \lambda_k R(\mathbf{y}) + \sum_{k \in \mathcal{A}} \lambda_k R(\mathbf{y} + \mathbf{e}_k) + \sum_{k \in \mathcal{A}} \nu_k y_k R(\mathbf{y}) - \beta(\mathbf{y}) R(\mathbf{y}) \\
&\leq \mu (R(\mathbf{y}) + 1) + \sum_{k \in \mathcal{W}} \lambda_k R(\mathbf{y}) + \sum_{k \in \mathcal{A}} \lambda_k (R(\mathbf{y}) + 1) + \sum_{k \in \mathcal{A}} \nu_k y_k R(\mathbf{y}) - \beta(\mathbf{y}) R(\mathbf{y}) \\
&\leq \left(\mu + \sum_{k \in \mathcal{A}} \lambda_k \right).
\end{aligned}$$

Hence Assumption A (3) holds with $c = 0$, which also shows that Assumption B (1) is satisfied. Assumption B(2) is satisfied since $c(\mathbf{y}, \mathbf{a}) \leq n \max(h, \max_i \nu_i c_i) R(\mathbf{y})$. Furthermore, checking the two first parts of Assumption C is immediate from the finiteness of the action set. For Assumption C (3), take $w'(\mathbf{y}) = (\max_i y_i)^2$. Note that

$$\beta(\mathbf{y}) R(\mathbf{y}) \leq \left(\mu + \sum_{k \in \mathcal{A}} \lambda_k + \sum_{k \in \mathcal{A}} \nu_k R(\mathbf{y})) \right) R(\mathbf{y}) \leq \left(\mu + \sum_{k \in \mathcal{A}} \lambda_k + \sum_{k \in \mathcal{A}} \nu_k \right) w'(\mathbf{y})$$

and

$$\begin{aligned}
\sum_{\mathbf{z}} q(\mathbf{z}|\mathbf{y},\mathbf{a})w'(\mathbf{z}) &\le \mu(\max_i y_i+1)^2 + \sum_{k\in\mathcal{W}} \lambda_k(\max_i y_i)^2 + \sum_{k\in\mathcal{A}} \lambda_k(\max_i y_i+1)^2 + \\
&\quad \sum_{k\in\mathcal{A}} \nu_k y_k(\max_i y_i)^2 - \beta(\mathbf{y})(\max_i y_i)^2 \\
&\le \left(\mu + \sum_{k\in\mathcal{A}} \lambda_k\right)(2\max_i y_i + 1) \\
&\le c'\max_i y_i + b' \le c'w'(\mathbf{y}) + b',
\end{aligned}$$

with $b' = \mu + \sum_{k\in\mathcal{A}} \lambda_k$ and $c' = 2b'$ showing that C (3) holds for our problem. □

By reorganizing its different terms, Equation (52) can be rewritten as

$$\tilde{T}v = \frac{h(y_0)}{\alpha+\beta(\mathbf{y})} + \frac{\beta(\mathbf{y})}{\alpha+\beta(\mathbf{y})}\Big(\frac{\mu}{\beta(\mathbf{y})}Pv + \sum_{k\in\mathcal{W}} \frac{\lambda_k}{\beta(\mathbf{y})}\tilde{A}_k^1 v + \sum_{k\in\mathcal{A}} \frac{\nu_k y_k}{\beta(\mathbf{y})}\left[p_k A_k^2 v + (1-p_k)A_k^3 v\right]\Big)$$

where $\tilde{A}_k^1 v(x,\mathbf{y}) = v(x,\mathbf{y}+\mathbf{e}_k)$. From Proposition (1) the optimal value function of the ∞-problem v^* satisfies

$$v^* = \tilde{T}v^*. \tag{54}$$

Similarly, consider a problem such that $m_1 = \cdots = m_n = m < \infty$ which we refer to as an m-Problem. The corresponding optimality operator is similar to the optimal operator $\tilde{T}$ except for arrivals of demands with ADI,

$$T_m v = \frac{h(y_0)}{\alpha+\beta(\mathbf{y})} + \frac{\beta(\mathbf{y})}{\alpha+\beta(\mathbf{y})}\Big(\frac{\mu}{\beta(\mathbf{y})}Pv + \sum_{k\in\mathcal{W}} \frac{\lambda_k}{\beta(\mathbf{y})}A_{k,m}^1 v + \sum_{k\in\mathcal{A}} \frac{\nu_k y_k}{\beta(\mathbf{y})}\left[p_k A_k^2 v + (1-p_k)A_k^3 v\right]\Big)$$

where $A_{k,m}^1$ is another notation for A_k^1 which explicitly refers to m (A_k^1 is actually the only operator which depends on m). The optimal cost v_m^* satisfies then the Bellman's equations

$$v_m^* = T_m v_m^*. \tag{55}$$

Recall that from Lippman (1975), v_m^* also satisfies the uniformized optimality equation $Tv_m^* = v_m^*$, where T is the operator defined in equation (2).

We show next that v_m^* converges to a limit.

Proposition 2 *v_m^* converges point-wise as $m \to +\infty$.*

Proof: We know that for all $\mathbf{y}$, $v_m^*(\mathbf{y})$ is lower-bounded by 0. Let us prove that, for all $\mathbf{y}$, $v_m^*(\mathbf{y})$ is non-increasing in m by a sample path argument. Consider the optimal policy of the m-problem and v_m^*

the associated total expected cost. Construct now π_{m+1}, a policy for the $(m+1)$-problem (not necessarily optimal) that is identical to π_m^*, except when $y_k = m+1$. In this case consider the order of class k announced at time t that brought y_k to $m+1$ and define π_{m+1} such that the order is rejected when it is requested after the demand leadtime L_k, i.e. at time instant $t + L_k$. We denote by $v^{\pi_{m+1}}$ the associated total expected discounted cost.

The only difference between the two resulting total discount costs occurs for an order of class k that brings y_k to $y_k = m_k + 1$. This order generates then an additional (discounted) cost equal to

- $\exp[-\alpha t]c_k$ for π_m^*, and
- $\exp[-\alpha(t + L_k)]c_k$ for $\pi^{\pi_{m+1}}$.

It follows that

$$v^{\pi_{m+1}} \leq v_m^*,$$

and from the definition of v_{m+1}^*,

$$v_{m+1}^* \leq v^{\pi_{m+1}}.$$

We conclude that v_m^* is decreasing in m, lower-bounded and thus point-wise converging.□

Denote by $\bar{v}^*$, the limit of v_m^* as $m \to +\infty$. The following result state that $\bar{v}^*$ is also the optimal value function of the ∞-problem.

Proposition 3 *$\bar{v}^*$ is an optimal value function of the ∞-problem ($m = +\infty$) and the corresponding optimal policy is characterized by base-stock and rationing levels that satisfy P.1-P.6 of Theorem 1.*

Proof: Fix a state $\mathbf{y}$. From the definitions of $A_{k,m}^1$ and $\tilde{A}_k^1$, we deduce that $\forall m > \max_k \mathbf{y}_k$,

$$A_{k,m}^1 v_m^*(\mathbf{y}) = v_m^*(\mathbf{y} + \mathbf{e}_k) = \tilde{A}_k^1 v_m^*(x, \mathbf{y}) \tag{56}$$

so that $\forall m > \max_k \mathbf{y}_k$,

$$T_m v_m^*(\mathbf{y}) = \tilde{T} v_m^*(\mathbf{y}) \tag{57}$$

Since $\tilde{T}$ is the finite sum of minimums of functions, we have

$$\lim_{m \to +\infty} \tilde{T} v_m^*(\mathbf{y}) = \tilde{T} \lim_{m \to +\infty} v_m^*(\mathbf{y}) = \tilde{T} \bar{v}^*(\mathbf{y}) \tag{58}$$

It follows that $T_m v_m^*(\mathbf{y})$ converges to $\tilde{T}\bar{v}^*(\mathbf{y})$. Taking the limits of both sides in (55) at $\mathbf{y}$, we have

$$\bar{v}^*(\mathbf{y}) = \tilde{T}\bar{v}^*(\mathbf{y}). \tag{59}$$

and $\bar{v}^*$ is the optimal value function of the ∞-problem from (1). The action set is finite which guarantees the existence of an optimal policy. Furthermore $\bar{v}^* \in \mathcal{U}$ from Lemma 1 and since limits preserve weak inequalities. The last part of the proposition follows then directly from conditions C.1-C.4 and Properties 1, 2 and 3. $\square$