Appendix

Variation Diminishing Property (VDP) of PF_2

Let $M(u)$ be a real function on $(-\infty,\infty)$, f be PF_2 on $[0,\infty)$ and zero on $(-\infty,0]$. Suppose $M(u)$ changes sign at most once on $(-\infty,\infty)$. Then the transformation

$$g(y)=\int_{-\infty}^{\infty} M(u)\,f(y-u)du$$

also changes sign at most once over $(-\infty,\infty)$. Moreover, if both g and M change sign once, their sign changes occur in the same order (see Karlin 1968).

Proof of Lemma 1. 1) Note that $G_{\tau+L,T}(y_{\tau+L})$ is the expected newsvendor type cost with a beginning inventory level $y_{\tau+L}$, composed of the expected cost for overage in each remaining period and for underage and disposal at the end of period T. It follows that $G_{\tau+L,T}(y_{\tau+L})$ is convex in $y_{\tau+L}$.

2) Note that $B_t^x(x_t)$ is the expected sales in period t multiplied by a constant b_t, minus the expected leftovers in period t multiplied by a constant h_t, in the newsvendor model with a beginning inventory level x_t. Thus $B_t^x(x_t)$ is concave in x_t. Similarly, $B_t^y(y_t)$ is concave in y_t.

3) Expanding the recursive equation (2), we get for $t=\tau,\cdots,\tau+l-1$,

$$g_t(x_t,y_t)=\sum_{t'=t}^{\tau+L-1} E\left[B_{t'}\left(x_t-\xi_{t,t'},y_t-\xi_{t,t'}\right)\right]+\int_0^{\infty} G_{\tau+L,T}\left(y_t-\xi_{t,\tau+L-1}\right)f_{t,\tau+L-1}\left(\xi_{t,\tau+L-1}\right)d\xi_{t,\tau+L-1} \quad (12)$$

where $E\left[B_{t'}\left(x_t-\xi_{t,t'},y_t-\xi_{t,t'}\right)\right]$ is the expectation of $B_{t'}\left(x_t-\xi_{t,t'},y_t-\xi_{t,t'}\right)$ with respect to $\xi_{t,t'}$.

Then it follows, from the fact that $B_{t'}(x_{t'},y_{t'})$ is separable in $x_{t'}$ and $y_{t'}$, that $g_t(x_t,y_t)$ is separable in x_t and y_t. Similarly, it follows, from the fact that $B_{t'}(x_{t'},y_{t'})$ is convex in $x_{t'}$, that $g_t(x_t,y_t)$ is convex in x_t. ■

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Proof of Theorem 1.

1) It is sufficient for us to consider only the case $L > 0, b_\tau = \cdots = b_{\tau+L-1} = 0$, for which $H_\tau\left(x_\tau, IP_\tau\right)$ can be written as

$$H_\tau\left(x_\tau, IP_\tau\right) = c_\tau\left(IP_\tau - x_\tau\right) + \sum_{t=\tau}^{\tau+L-1} h_t \int_0^{x_\tau}\left(x_\tau - \xi_{\tau,t}\right) f_{\tau,t}\left(\xi_{\tau,t}\right) d\xi_{\tau,t}$$
$$+ \int_0^\infty G_{\tau+L,T}\left(IP_\tau - \xi_{\tau,\tau+L-1}\right) f_{\tau,\tau+L-1}\left(\xi_{\tau,\tau+L-1}\right) d\xi_{\tau,\tau+L-1}$$

The convexity of $H_\tau\left(x_\tau, IP_\tau\right)$ with respect to IP_τ follows directly by Lemma 1 part 1). Thus, a base-stock policy $Q_\tau = \max\left\{IP_\tau^* - x_\tau, 0\right\}$ is the optimal ordering rule at the beginning of period τ, where $IP_\tau^* \geq 0$ is the value of IP_τ that attains the minimum of $H_\tau\left(x_\tau, IP_\tau\right)$ over $\left[0, \infty\right)$, leading to an expression for $\rho_\tau\left(x_\tau\right)$ below:

$$\rho_\tau\left(x_\tau\right) = \begin{cases} H_\tau\left(x_\tau, IP_\tau^*\right) & \text{if } x_\tau < IP_\tau^* \\ H_\tau\left(x_\tau, x_\tau\right) & \text{if } x_\tau \geq IP_\tau^* \end{cases}$$

The convexity of $\rho_\tau\left(x_\tau\right)$ is implied by the convexity of $H_\tau\left(x_\tau, IP_\tau^*\right)$ and $H_\tau\left(x_\tau, x_\tau\right)$ with respect to x_τ, and by the inequality $\dfrac{\partial H_\tau\left(x_\tau, IP_\tau\right)}{\partial IP_\tau} \geq 0$ for $IP_\tau > IP_\tau^*$.

To show that $TC_I\left(Q_0\right)$ is convex in Q_0, it is sufficient to show that $\rho_1\left(x_1\right)$ is convex in x_1. In view of the recursive equation (2), it is then sufficient to show that $\rho_2\left(x_2\right)$ is convex in x_2 since $b_1\int_{x_1}^\infty\left(\xi_1 - x_1\right) f_1\left(\xi_1\right) d\xi_1$ and $h_1\int_0^{x_1}\left(x_1 - \xi_1\right) f_1\left(\xi_1\right) d\xi_1$ are convex in x_1, where $x_2 = x_1 - \xi_1$. The rationale for this is that a convolution transformation of a convex function is a convex function and a summation of two convex functions is a convex function. Following the same logic, it is sufficient to show that $\rho_\tau\left(x_\tau\right)$ is convex in x_τ, which we have proved above. This concludes our proof for part 1).

2) Notice the condition $L = T - \tau + 1, b_\tau = \cdots = b_{T-1} = 0, b_T > 0$ implies that the backorder cost in periods τ through $\tau + L - 1$ is effectively charged independent of when the backorders occur and that $b_T \le c_u$ since otherwise the decision maker will not accept backorders. Thus, $H_\tau\left(x_\tau, IP_\tau\right)$ can be written as below

$$H_\tau\left(x_\tau, IP_\tau\right) = c_\tau\left(IP_\tau - x_\tau\right) + b_T \int_{x_\tau}^{IP_\tau} \left(\xi_{\tau,T} - x_\tau\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T} + b_T \int_{IP_\tau}^{\infty} \left(IP_\tau - x_\tau\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T}$$
$$+ \sum_{t=\tau}^{T} h_t \int_0^{x_\tau} \left(x_\tau - \xi_{\tau,t}\right) f_{\tau,t}\left(\xi_{\tau,t}\right) d\xi_{\tau,t} + c_u \int_{IP_\tau}^{\infty} \left(\xi_{\tau,T} - IP_\tau\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T} + c_d \int_0^{IP_\tau} \left(IP_\tau - \xi_{\tau,T}\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T}$$

With a little algebra, it can be shown that $\dfrac{\partial^2 H_\tau\left(x_\tau, IP_\tau\right)}{\partial IP_\tau^2} = \left(c_d + c_u - b_T\right) f_{\tau,T}\left(IP_\tau\right)$. Since $c_u \ge b_T$, it follows that $\dfrac{\partial^2 H_\tau\left(x_\tau, IP_\tau\right)}{\partial IP_\tau^{\,2}} > 0$. Therefore $H_\tau\left(x_\tau, IP_\tau\right)$ is convex in IP_τ. The proof for the convexity of $TC_I\left(Q_0\right)$ in Q_0 is similar to the proof of part 1) of Theorem 1 above.

3) The backorder cost is effectively charged independent of when the backorders occur and that $b_{\tau+L-1} \le c_u$. With a little algebra, $H_\tau\left(x_\tau, IP_\tau\right)$ can be expressed as

$$H_\tau\left(x_\tau, IP_\tau\right) = c_\tau\left(IP_\tau - x_\tau\right) + b_{\tau+L-1} \int_{x_\tau}^{IP_\tau} \left(\xi_{\tau,\tau+L-1} - x_\tau\right) f_{\tau,\tau+L-1}\left(\xi_{\tau,\tau+L-1}\right) d\xi_{\tau,\tau+L-1}$$
$$+ b_{\tau+L-1}\left(IP_\tau - x_\tau\right) \int_{IP_\tau}^{\infty} f_{\tau,\tau+L-1}\left(\xi_{\tau,\tau+L-1}\right) d\xi_{\tau,\tau+L-1} + \sum_{t=\tau}^{\tau+L-1} h_t \int_0^{x_\tau} \left(x_\tau - \xi_{\tau,t}\right) f_{\tau,t}\left(\xi_{\tau,t}\right) d\xi_{\tau,t}$$
$$+ c_u \int_{IP_\tau}^{\infty} \left(\xi_{\tau,T} - IP_\tau\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T} + \left(c_d + h_T\right) \int_0^{IP_\tau} \left(IP_\tau - \xi_{\tau,T}\right) f_{\tau,T}\left(\xi_{\tau,T}\right) d\xi_{\tau,T}$$

With some more analysis, we have $\dfrac{\partial^2 H_\tau\left(x_\tau, IP_\tau\right)}{\partial IP_\tau^{\,2}} = -b_{\tau+L-1} f_{\tau,\tau+L-1}\left(IP_\tau\right) + \left(c_u + c_d + h_T\right) f_{\tau,T}\left(IP_\tau\right)$.

Recall that the monotone likelihood ratio property (MLRP) holds for $\dfrac{f_{\tau,T}\left(IP_\tau\right)}{f_{\tau,\tau+L-1}\left(IP_\tau\right)}$ over $\left[0,\infty\right)$. We see that $\dfrac{f_{\tau,T}\left(IP_\tau\right)}{f_{\tau,\tau+L-1}\left(IP_\tau\right)}$ is increasing in IP_τ. If $\dfrac{f_{\tau,T}\left(0\right)}{f_{\tau,\tau+L-1}\left(0\right)} \le \dfrac{b_{\tau+L-1}}{c_u + c_d + h_T}$ and $\dfrac{f_{\tau,T}\left(\infty\right)}{f_{\tau,\tau+L-1}\left(\infty\right)} \ge \dfrac{b_{\tau+L-1}}{c_u + c_d + h_T}$, then there exists an $\widetilde{IP}_\tau$ such that $\frac{f_{\tau,T}\left(\widetilde{IP}_\tau\right)}{f_{\tau,\tau+L-1}\left(\widetilde{IP}_\tau\right)} = \frac{b_{\tau+L-1}}{c_u+c_d+h_T}$. In case $\frac{f_{\tau,T}(0)}{f_{\tau,\tau+L-1}(0)} > \frac{b_{\tau+L-1}}{c_u+c_d+h_T}$, $\widetilde{IP}_\tau$ is defined as 0; in case $\frac{f_{\tau,T}(\infty)}{f_{\tau,\tau+L-1}(\infty)} < \frac{b_{\tau+L-1}}{c_u+c_d+h_T}$, $\widetilde{IP}_\tau$ is defined as ∞. Therefore we have $\frac{\partial^2 H_\tau(x_\tau, IP_\tau)}{\partial IP_\tau^2} \le 0$ for $0 \le IP_\tau \le \widetilde{IP}_\tau$ and $\frac{\partial^2 H_\tau(x_\tau, IP_\tau)}{\partial IP_\tau^2} \ge 0$ for $IP_\tau \ge \widetilde{IP}_\tau$; that is, $H_\tau(x_\tau, IP_\tau)$ is concave-convex in IP_τ: concave on $\left[0, \widetilde{IP}_\tau\right]$ and convex on $\left[\widetilde{IP}_\tau, \infty\right)$. This leads to our conclusion.

4) To prove that $H_\tau(x_\tau, IP_\tau)$ is unimodal in IP_τ, it is sufficient to show that $c_\tau + \frac{\partial g_t(x_t, y_t)}{\partial y_t}$ changes sign at most once over $(-\infty, \infty)$ for $\tau \le t \le \tau+L-1$. We do it recursively starting from $t = \tau+L-1$. With a little algebra, we can see that

$$\begin{aligned}
&c_\tau + \frac{\partial g_{\tau+L-1}(x_{\tau+L-1}, y_{\tau+L-1})}{\partial y_{\tau+L-1}} \\
&= c_\tau + b_{\tau+L-1}\int_{y_{\tau+L-1}}^{\infty} f_{\tau+L-1}(\xi_{\tau+L-1})\, d\xi_{\tau+L-1} + \int_0^\infty \frac{dG_{\tau+L,T}(y_{\tau+L-1}-\xi_{\tau+L-1})}{dy_{\tau+L-1}} f_{\tau+L-1}(\xi_{\tau+L-1})\, d\xi_{\tau+L-1} \\
&= c_\tau + (b_{\tau+L-1}-c_u)\int_{y_{\tau+L-1}}^{\infty} f_{\tau+L-1}(\xi_{\tau+L-1})\, d\xi_{\tau+L-1} + \int_0^{y_{\tau+L-1}} \frac{dG_{\tau+L,T}(y_{\tau+L-1}-\xi_{\tau+L-1})}{dy_{\tau+L-1}} f_{\tau+L-1}(\xi_{\tau+L-1})\, d\xi_{\tau+L-1}
\end{aligned}$$

By introducing a variable $u = y_{\tau+L-1} - \xi_{\tau+L-1}$, we have

$$c_\tau + \frac{\partial g_{\tau+L-1}(x_{\tau+L-1}, y_{\tau+L-1})}{\partial y_{\tau+L-1}} = \int_{-\infty}^{\infty} M(u, \tau+L-1)\, f_{\tau+L-1}(y_{\tau+L-1}-u)\, du$$

where $M(u, \tau+L-1) = \begin{cases} c_\tau + b_{\tau+L-1} - c_u & \text{if } u < 0 \\ c_\tau + \frac{\partial G_{\tau+L,T}(u)}{\partial u} & \text{if } u \ge 0 \end{cases}$. It is easy to see that $M(u, \tau+L-1)$ changes sign at most once over $(-\infty, \infty)$ in view that $G_{\tau+L,T}$ is convex, $\left.\frac{\partial G_{\tau+L,T}(u)}{\partial u}\right|_{u=0} = -c_u$, $\lim_{u\to\infty} \frac{\partial G_{\tau+L,T}(u)}{\partial u} = \sum_{t=\tau+L}^{T} h_t + c_d$ and $c_\tau + b_{\tau+L-1} - c_u \le 0$. As a result, $c_\tau + \frac{\partial g_{\tau+L-1}(x_{\tau+L-1}, y_{\tau+L-1})}{\partial y_{\tau+L-1}}$ changes

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sign at most once over $\left(-\infty,\infty\right)$ with respect to $y_{\tau+L-1}$, and

$$c_\tau+\left.\frac{\partial g_{\tau+L-1}\left(x_{\tau+L-1},y_{\tau+L-1}\right)}{\partial y_{\tau+L-1}}\right|_{y_{\tau+L-1}=0}=c_\tau+b_{\tau+L-1}-c_u\le 0.$$

Suppose $c_\tau+\frac{\partial g_{t+1}\left(x_{t+1},y_{t+1}\right)}{\partial y_{t+1}}$ changes sign at most once and has a non-positive value of $c_\tau+\sum_{j=t+1}^{\tau+L-1}b_j-c_u$ at $y_{t+1}=0$. For $c_\tau+\frac{\partial g_t\left(x_t,y_t\right)}{\partial y_t}$, we have

$$c_\tau+\frac{\partial g_t\left(x_t,y_t\right)}{\partial y_t}=c_\tau+\left(\sum_{j=t}^{\tau+L-1}b_j-c_u\right)\int_{y_t}^{\infty}f_t\left(\xi_t\right)d\xi_t+\int_0^{y_t}\frac{\partial g_{t+1}\left(x_t-\xi_t,y_t-\xi_t\right)}{\partial y_t}f_t\left(\xi_t\right)d\xi_t$$

$$=\int_{-\infty}^{\infty}M\left(u,t\right)f_t\left(y_t-u\right)du \qquad \text{where}$$

$$M\left(u,t\right)=\begin{cases} c_\tau+\sum_{j=t}^{\tau+L-1}b_j-c_u & \text{if } u<0\\ c_\tau+\left.\frac{\partial g_{t+1}\left(x_t+u-y_t,v\right)}{\partial v}\right|_{v=u} & \text{if } u\ge 0\end{cases}$$

It can be seen that $M\left(u,t\right)$ changes sign at most once over $\left(-\infty,\infty\right)$ since $M\left(u,t\right)\le 0$ for $u<0$ and $c_\tau+\left.\frac{\partial g_{t+1}\left(x_t+u-y_t,v\right)}{\partial v}\right|_{v=u}$ changes sign at most once over $\left[0,\infty\right)$ from "-" to "+". Therefore $c_\tau+\frac{\partial g_t\left(x_t,y_t\right)}{\partial y_t}$ changes sign at most once and $c_\tau+\left.\frac{\partial g_t\left(x_t,y_t\right)}{\partial y_t}\right|_{y_t=0}=c_\tau+\sum_{j=t}^{\tau+L-1}b_j-c_u\le 0$. By induction, the conclusion is established. The proof for the convexity of $TC_I\left(Q_0\right)$ in Q_0 is similar to that for part 1) of Theorem 1 above.

5) To prove that $H_\tau\left(x_\tau,IP_\tau\right)$ has at most one local minimum and one local maximum with respect to IP_τ, it is sufficient to show that $c_\tau+\frac{\partial g_t\left(x_t,y_t\right)}{\partial y_t}$ changes sign at most twice over $\left(-\infty,\infty\right)$ for

$\tau \le t \le \tau + L - 1$. If $c_\tau + \sum_{j=\tau}^{\tau+L-1} b_j - c_u \le 0$, then the result is evident by part 4) since PF_2 includes PF_3.

Next we assume $c_\tau + \sum_{j=\tau}^{\tau+L-1} b_j - c_u > 0$. Define $t^* = \max\left\{ t \middle| c_\tau + \sum_{j=t}^{\tau+L-1} b_j - c_u > 0, \tau \le t \le \tau + L - 1 \right\}$. By part 4), $c_\tau + \frac{\partial g_t\left(x_t, y_t\right)}{\partial y_t}$ changes sign at most once over $\left(-\infty, \infty\right)$ for $t > t^*$. We next show that, for any $t \le t^*$, $c_\tau + \frac{\partial g_t\left(x_t, y_t\right)}{\partial y_t}$ changes sign at most twice over $\left(-\infty, \infty\right)$. We do so recursively starting from $t = t^*$. With a little algebra, we can obtain $c_\tau + \frac{\partial g_{t^*}\left(x_{t^*}, y_{t^*}\right)}{\partial y_{t^*}} = \int_{-\infty}^{\infty} M\left(u, t^*\right) f_{t^*}\left(y_{t^*} - u\right) du$, where

$$M\left(u, t^*\right) = \begin{cases} c_\tau + \sum_{j=t^*}^{\tau+L-1} b_j - c_u & \text{if } u < 0 \\ c_\tau + \left.\frac{\partial g_{t^*+1}\left(x_{t^*} + u - y_{t^*}, v\right)}{\partial v}\right|_{v=u} & \text{if } u \ge 0 \end{cases}$$

It can be seen that $M\left(u, t^*\right)$ changes sign at most twice over $\left(-\infty, \infty\right)$ since $M\left(u, t^*\right) > 0$ for $u < 0$, and $c_\tau + \left.\frac{\partial g_{t^*+1}\left(x_{t^*} + u - y_{t^*}, v\right)}{\partial v}\right|_{v=u}$ changes sign at most once over $\left[0, \infty\right)$ starting from non-positive. Therefore $c_\tau + \frac{\partial g_{t^*}\left(x_{t^*}, y_{t^*}\right)}{\partial y_{t^*}}$ changes sign at most twice and has a positive value of $c_\tau + \sum_{j=t^*}^{\tau+L-1} b_j - c_u$ at $y_{t^*} = 0$. For $t^* - 1$, we have:

$$c_\tau + \frac{\partial g_{t^*-1}\left(x_{t^*-1}, y_{t^*-1}\right)}{\partial y_{t^*-1}} = \int_{-\infty}^{\infty} M\left(u, t^*-1\right) f_{t^*-1}\left(y_{t^*-1}-u\right) du \quad \text{where}$$

$$M\left(u, t^*-1\right) = \begin{cases} c_\tau + \sum_{j=t^*-1}^{\tau+L-1} b_j - c_u & \text{if } u < 0 \\ c_\tau + \left.\frac{\partial g_{t^*}\left(x_{t^*-1}+u-y_{t^*-1}, v\right)}{\partial v}\right|_{v=u} & \text{if } u \ge 0 \end{cases}$$

It can be seen that $M\left(u, t^*-1\right)$ changes sign at most twice over $\left(-\infty, \infty\right)$ since $M\left(u, t^*-1\right) > 0$ for $u < 0$, and $c_\tau + \left.\frac{\partial g_{t^*}\left(x_{t^*-1}+u-y_{t^*-1}, v\right)}{\partial v}\right|_{v=u}$ changes sign at most twice over $\left[0, \infty\right)$ starting from "+".

Therefore $c_\tau + \frac{\partial g_{t^*-1}\left(x_{t^*-1}, y_{t^*-1}\right)}{\partial y_{t^*-1}}$ changes sign at most twice and has a positive value of $c_\tau + \sum_{j=t^*-1}^{\tau+L-1} b_j - c_u$ at $y_{t^*-1} = 0$. Similar logic can be applied to $t = t^*-2, \cdots, 2, 1$. Thus, part 5) follows.■

Proof of Theorem 2.

1) The outline for the proof is: we first show that for big-enough beginning inventory level x_t, $j_t\left(x_t\right) \ge k_t\left(x_t\right)$; then we show that $j_t\left(x_t\right)$ and $k_t\left(x_t\right)$ cross only once so that an $\left(s, S\right)$ policy is the optimal rule for whether to place the second order and how much when ordering at the beginning of period t. By the definition of $H_t\left(x_t, IP_t\right)$ and $L = 0$, it can be seen that

$$H_t\left(x_t, IP_t\right) = c_t\left(IP_t - x_t\right) + \sum_{t'=t}^{T} h_{t'} \int_0^{IP_t} \left(IP_t - \xi_{t,t'}\right) f_{t,t'}\left(\xi_{t,t'}\right) d\xi_{t,t'}$$
$$+ c_u \int_{IP_t}^{\infty} \left(\xi_{t,T} - IP_t\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} + c_d \int_0^{IP_t} \left(IP_t - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T}$$

It can be verified that for any t, $H_t\left(x_t, IP_t\right)$ is convex in IP_t with the limits $\lim_{IP_t \to \infty} H_t\left(x_t, IP_t\right) = \infty$ and $\lim_{IP_t \to -\infty} H_t\left(x_t, IP_t\right) = \infty$. IP_t^* is the unique minimizer of $H_t\left(x_t, IP_t\right)$ as a function of IP_t. When $x_t \ge IP_t^*$, if an order is forced to be placed then the optimal order size is zero by the convexity of $H_t\left(x_t, IP_t\right)$ with respect to IP_t. Therefore $j_t\left(x_t\right) \geq k_t\left(x_t\right)$ holds true when $x_t \geq IP_t^*$ since the $k_t\left(x_t\right)$ solution still keeps an order opportunity, while starting with the same inventory level as the $j_t\left(x_t\right)$ solution.

We next focus on the case when $x_t < IP_t^*$. From (8) and the expression for $H_t\left(x_t, IP_t\right)$ above,

$$\begin{aligned} j_t\left(x_t\right) = & -c_t x_t + c_t IP_t^* + c_u \int_{IP_t^*}^{\infty}\left(\xi_{t,T} - IP_t^*\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} + c_d \int_0^{IP_t^*}\left(IP_t^* - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} \\ & + \sum_{t'=t}^{T} h_{t'} \int_0^{IP_t^*}\left(IP_t^* - \xi_{t,t'}\right) f_{t,t'}\left(\xi_{t,t'}\right) d\xi_{t,t'} \end{aligned}$$

Thus, to compare $j_t\left(x_t\right)$ and $k_t\left(x_t\right)$, it is sufficient to compare Θ_t and $K_t\left(x_t\right) = k_t\left(x_t\right) + c_t x_t$, where

$$\begin{aligned} \Theta_t = & c_t IP_t^* + c_u \int_{IP_t^*}^{\infty}\left(\xi_{t,T} - IP_t^*\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} + c_d \int_0^{IP_t^*}\left(IP_t^* - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_{t,T} \\ & + \sum_{t'=t}^{T} h_{t'} \int_0^{IP_t^*}\left(IP_t^* - \xi_{t,t'}\right) f_{t,t'}\left(\xi_{t,t'}\right) d\xi_{t,t'} \end{aligned}$$

(Note for $x_t \geq IP_t^*$, $K_t\left(x_t\right) \leq \Theta_t$.) The question that needs to be answered next is: Given that we are at the beginning of period t and the inventory level $x_t < IP_t^*$, for what values of x_t is it optimal to place an order? We answer this question in a recursive manner by starting from $t = T$.

Case $t = T$

It is easy to see that $k_T\left(x_T\right) = H_T\left(x_T, x_T\right)$, thus $K_T\left(x_T\right)$ is convex and minimized at IP_T^*, and $\Theta_T \leq K_T\left(x_T\right)$ for $x_T \leq IP_T^*$. Therefore the optimal policy at the beginning of period T is: If $x_T \leq IP_T^*$ then it is optimal to raise the inventory level to IP_T^*. As a result, $\rho_T\left(x_T\right)$ is convex in x_T.

Before further discussion we state a lemma which will be used later.

Lemma 3. $K_t'\left(x_t\right) < 0$ for $x_t \leq 0$ and $t \leq T-1$.

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Proof "$x_t \le 0$" implies that there must be an order placement in one of the remaining $T-t$ periods by the backordering assumption and the assumption $c_u > \max\{c_1, c_2, \cdots, c_T\}$. Suppose it is placed in period t_1 ($t_1 > t$) with a unit price of c_{t_1}. Then we have

$$\begin{aligned} K_t^{'}(x_t) &= c_t - b_t \int_{x_t}^{\infty} f_t(\xi_t)\, d\xi_t + h_t \int_0^{x_t} f_t(\xi_t)\, d\xi_t + \int_0^{\infty} \rho_{t+1}^{'}(x_t - \xi_t)\, f_t(\xi_t)\, d\xi_t \\ &= c_t - b_t - \cdots - b_{t_1 - 1} - c_{t_1} < 0 \end{aligned}$$

■

Knowing that $K_t(IP_t^*) \le \Theta_t$, Lemma 3 suggests that Θ_t and $K_t(x_t)$ cross at least once over $\left(-\infty, IP_t^*\right]$ (note here Θ_t is a constant function with respect to x_t). In what follows we show that they cross only once.

Case $t = T-1$

To show that Θ_{T-1} and $K_{T-1}(x_{T-1})$ cross once over $\left(-\infty, IP_{T-1}^{\ *}\right]$, it is sufficient to show that $K_{T-1}(x_{T-1})$ is convex in x_{T-1}. Notice that

$$\begin{aligned} K_{T-1}(x_{T-1}) &= c_{T-1}x_{T-1} + b_{T-1}\int_{x_{T-1}}^{\infty} (\xi_{T-1} - x_{T-1})\, f_{T-1}(\xi_{T-1})\, d\xi_{T-1} + h_{T-1}\int_0^{x_{T-1}} (x_{T-1} - \xi_{T-1})\, f_{T-1}(\xi_{T-1})\, d\xi_{T-1} \\ &\quad + \int_0^{\infty} \rho_T(x_{T-1} - \xi_{T-1})\, f_{T-1}(\xi_{T-1})\, d\xi_{T-1} \end{aligned}.$$

The convexity of $K_{T-1}(x_{T-1})$ follows because each term in the R.H.S. is convex in x_{T-1}.

In what follows to show that Θ_t and $K_t(x_t)$ cross once for all $t \le T-2$, we show that $K_t(x_t)$ is unimodal for all $t \le T-2$. Notice that

$$\begin{aligned} K_t^{'}(x_t) &= c_t - b_t \int_{x_t}^{\infty} f_t(\xi_t)\, d\xi_t + h_t \int_0^{x_t} f_t(\xi_t)\, d\xi_t + \int_0^{\infty} \rho_{t+1}^{'}(x_t - \xi_t)\, f_t(\xi_t)\, d\xi_t \\ &= \int_{-\infty}^{\infty} M_t(u)\, f_t(x_t - u)\, du \end{aligned} \quad \ldots. (13)$$

where

$$M_t(u) = \begin{cases} c_t - b_t + \rho_{t+1}^{'}(u) & \text{if } u \le 0 \\ c_t + h_t + \rho_{t+1}^{'}(u) & \text{if } u > 0 \end{cases}$$

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To prove that $K_t(x_t)$ is unimodal, it is sufficient to show that $K_t^{'}(x_t)$ changes sign once over $(-\infty,\infty)$.

It is then sufficient to show that $M_t(u)$ changes sign once over $(-\infty,\infty)$ since f_t is a PF_2 density. We first discuss the special case $t=T-2$; Then we discuss the general case by induction.

Case $t=T-2$

We follow two steps to prove that $M_{T-2}(y)$ changes sign once over $(-\infty,\infty)$.

Step 1: Claim $M_{T-2}(y)$ is negative when $y\le 0$. In fact, if $y\le 0$, we have

$$\begin{aligned} M_{T-2}(y) &= \begin{cases} c_{T-2}-b_{T-2}+j_{T-1}^{'}(y) & \text{if } s_{T-1}>0 \\ c_{T-2}-b_{T-2}+j_{T-1}^{'}(y) & \text{if } s_{T-1}\le 0 \;\&\; y<s_{T-1} \\ c_{T-2}-b_{T-2}+k_{T-1}^{'}(y) & \text{if } s_{T-1}\le 0 \;\&\; y\ge s_{T-1} \end{cases} \\ &= \begin{cases} c_{T-2}-b_{T-2}-c_{T-1} & \text{if } s_{T-1}>0 \\ c_{T-2}-b_{T-2}-c_{T-1} & \text{if } s_{T-1}\le 0 \;\&\; y<s_{T-1} \\ c_{T-2}-b_{T-2}-c_{T-1}+K_{T-1}^{'}(y) & \text{if } s_{T-1}\le 0 \;\&\; y\ge s_{T-1} \end{cases} \\ &< 0 \end{aligned}$$

where s_{T-1} is the order-triggering point in period $T-1$.

Step 2: $M_{T-2}(y)$ changes sign at most once over $(0,\infty)$ from “-” to “+”.

Without loss of generality, we can only discuss the case $s_{T-1}>0$ since the opposite case can be discussed similarly. Notice that

$$c_{T-2}+h_{T-2}+\rho_{T-1}^{'}(y)=\begin{cases} c_{T-2}+h_{T-2}-c_{T-1} & \text{if } \; y<s_{T-1} \\ c_{T-2}+h_{T-2}-c_{T-1}+K_{T-1}^{'}(y) & \text{if } \; y\ge s_{T-1} \end{cases}$$

$c_{T-2}+h_{T-2}+\rho_{T-1}^{'}(y)$ is negative if $y<s_{T-1}$; For $y\ge s_{T-1}$, $c_{T-2}+h_{T-2}+\rho_{T-1}^{'}(y)$ increases to a positive value of $c_{T-2}+h_{T-2}+c_d$ as y approaches to ∞ by the convexity of $K_{T-1}(y)$, which follows from the convexity of $\rho_T(x_T)$. Therefore $c_{T-2}+h_{T-2}+\rho_{T-1}^{'}(y)$ changes sign at most once as y traverses from 0 to ∞.

Similarly, we can show that $c_{T-3}+h_{T-3}-c_{T-2}+M_{T-2}(y)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

Case General t

Analogous to the case $t=T-2$, we follow two steps to show that $M_t(y)$ changes signs once.

Step 1: Claim $M_t(y)$ is negative when $y\le 0$. A logic similar to that for case $\tau=T-2$ can be applied.

Step 2: $M_t(y)$ changes sign at most once over $(0,\infty)$ from "-" to "+".

Without loss of generality, we can only discuss the case $s_{t+1}>0$, since the opposite case can be discussed similarly. We will discuss the sign of $M_t(y)$ for different categories of y. For $0<y<s_{t+1}$, $M_t(y)=c_t+h_t+\rho_{t+1}{}'(y)=c_t+h_t+j_{t+1}{}'(y)=c_t+h_t-c_{t+1}\le 0$. For $s_{t+1}\le y$

$$\begin{aligned} M_t(y) &= c_t+h_t+\rho_{t+1}{}'(y)=c_t+h_t+k_{t+1}{}'(y) \\ &= c_t+h_t-c_{t+1}+K_{t+1}{}'(y) \\ &= \int_{-\infty}^{\infty}\left(c_t+h_t-c_{t+1}+M_{t+1}(u)\right) f_{t+1}(y-u)\,du \end{aligned}$$

Since $c_t+h_t-c_{t+1}+M_{t+1}(u)$ changes sign once over $(-\infty,\infty)$, $M_t(y)$ changes sign at most once over $[s_{t+1},\infty)$. In particular, if it does change sign once then the sign changes from negative to positive; Otherwise the sign remains positive. For either case, $M_t(y)$ changes sign at most once over $(0,\infty)$.

To summarize, for general t, we show that $M_t(y)$ changes sign once from negative to positive as y traverses from $-\infty$ to ∞. Similarly, we can show that $c_{t-1}+h_{t-1}-c_t+M_t(u)$ changes sign once over $(-\infty,\infty)$ from "-" to "+".

By induction, we have shown that $K_t(x_t)$ is unimodal for $t\le T-2$. Therefore there exists a unique $x_t=s_t<IP_t^*$ such that $K_t(x_t)$ intersects with Θ_t, i.e., an (s,S) policy with $s=s_t, S=IP_t^*$ is optimal.

2) Since $L > 0, b_1 = \cdots = b_{T-1} = 0, b_T \geq 0$, period $T-L+1$ is the last period when an order can be placed. Moreover, we can see that $H_t\left(x_t, IP_t\right)$ has an expression below

$$H_t\left(x_t, IP_t\right) = \begin{cases} c_t\left(IP_t - x_t\right) + b_T\int_{x_t}^{IP_t}\left(\xi_{t,T} - x_t\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_t + b_T\int_{IP_t}^{\infty} IP_t f_{t,T}\left(\xi_{t,T}\right) d\xi_t & \\ \quad + c_u\int_{IP_t}^{\infty}\left(\xi_{t,T} - IP_t\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_t + c_d\int_0^{IP_t}\left(IP_t - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_t & t = T-L+1 \\ \quad + \sum_{t'=\tau}^{\tau+L-1} h_{t'}\int_0^{x_\tau}\left(x_\tau - \xi_{\tau,t'}\right) f_{\tau,t'}\left(\xi_{\tau,t'}\right) d\xi_{\tau,t'} & \\ c_t\left(IP_t - x_t\right) + c_u\int_{IP_t}^{\infty}\left(\xi_{t,T} - IP_t\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_t + c_d\int_0^{IP_t}\left(IP_t - \xi_{t,T}\right) f_{t,T}\left(\xi_{t,T}\right) d\xi_t & \\ \quad + \sum_{t'=\tau}^{\tau+L-1} h_{t'}\int_0^{x_\tau}\left(x_\tau - \xi_{\tau,t'}\right) f_{\tau,t'}\left(\xi_{\tau,t'}\right) d\xi_{\tau,t'} & t < T-L+1 \end{cases}$$

$k_t\left(x_t\right)$ can be recursively expressed as $k_t\left(x_t\right) = h_t\int_0^{x_t}\left(x_t - \xi_t\right) f_t\left(\xi_t\right) d\xi_t + \int_0^{\infty} \rho_{t+1}\left(x_t - \xi_t\right) f_t\left(\xi_t\right) d\xi_t$ for $t < T-L+1$, and expressed as $k_t\left(x_t\right) = H_t\left(x_t, x_t\right)$ for $t = T-L+1$. Then the approach in the proof for part 1) of Theorem 2 can be applied to establish the conclusion. ■

Proof of Theorem 3. It is sufficient to show that $TC_{II}\left(Q_0\right)$ is unimodal. Since $TC_{II}\left(Q_0\right)$ has a negative slope $c_0 - c_1$ for $Q_0 < s_1$, it remains to show that $TC_{II}\left(Q_0\right)$ is unimodal over $\left(s_1, \infty\right)$. This is true since $TC_{II}\left(Q_0\right)$, with an expression below

$$TC_{II}\left(Q_0\right) = c_0 Q_0 + b_1\int_{Q_0}^{\infty}\left(\xi_1 - Q_0\right) f_1\left(\xi_1\right) d\xi_1 + h_1\int_0^{Q_0}\left(Q_0 - \xi_1\right) f_1\left(\xi_1\right) d\xi_1 + \int_0^{\infty} \rho_2\left(Q_0 - \xi_1\right) f_1\left(\xi_1\right) d\xi_1$$

has a structure similar to $K_1\left(x_1\right)$, which we have shown is unimodal. ■

Proof of Corollary 1. It can be seen that in the 3-period problem, $j_2\left(x\right)$ and $k_2\left(x\right)$ have the following expressions, respectively,

$$\begin{aligned} j_2\left(x\right) &= c_2\left(IP_2^* - x\right) + h_2\int_0^{IP_2^*}\left(IP_2^* - \xi_2\right) f_2\left(\xi_2\right) d\xi_2 \\ &+ \left(h_3 + c_d\right)\int_0^{IP_2^*}\left(IP_2^* - \xi_{23}\right) f_{23}\left(\xi_{23}\right) d\xi_{23} + c_u\int_{IP_2^*}^{\infty}\left(\xi_{23} - IP_2^*\right) f_{23}\left(\xi_{23}\right) d\xi_{23} \end{aligned}$$

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$$
\begin{aligned}
k_2(x) &= b_2\int_x^{\infty}(\xi_2-x)f_2(\xi_2)\,d\xi_2 + h_2\int_0^{x}(x-\xi_2)f_2(\xi_2)\,d\xi_2 \\
&\quad + \int_{x-IP_3^*}^{\infty}\left(c_3\left(IP_3^*-x+\xi_2\right)+c_u\int_{IP_3^*}^{\infty}\left(\xi_3-IP_3^*\right)f_3(\xi_3)\,d\xi_3+\left(h_3+c_d\right)\int_0^{IP_3^*}\left(IP_3^*-\xi_3\right)f_3(\xi_3)\,d\xi_3\right)f_2(\xi_2)\,d\xi_2 \\
&\quad + \int_0^{x-IP_3^*}\left(c_u\int_{x-\xi_2}^{\infty}\left(\xi_3-x+\xi_2\right)f_3(\xi_3)\,d\xi_3+\left(h_3+c_d\right)\int_0^{IP_3^*}\left(IP_3^*-\xi_3\right)f_3(\xi_3)\,d\xi_3\right)f_2(\xi_2)\,d\xi_2
\end{aligned}
$$

From the expressions above we can see that, $\Delta_2(x) \hat{=} j_2(x)-k_2(x)$, is convex in x. Notice that $\Delta_2(x)$ is positive at IP_2^* and is negative at $-\infty$. Putting the convexity of $\Delta_2(x)$ and its boundary conditions together allows us to claim that $\Delta_2(x)$ is monotonically increasing in x. With a little algebra, taking derivative with respect to c_2 and c_3 yields, respectively,

$$\frac{\partial \Delta_2(x)}{\partial c_2} = IP_2^* - x > 0$$

$$\frac{\partial \Delta_2(x)}{\partial c_3} = -\int_{x-IP_3^*}^{\infty}\left(IP_3^*-x+\xi_2\right) f_2(\xi_2)\,d\xi_2 < 0$$

That is, $\Delta_2(x)$ is increasing in c_2 and is decreasing in c_3. Since $\Delta_2(x)$ is monotonically increasing in x, the root of $\Delta_2(x)=0$, s_2, is decreasing in c_2 and is increasing in c_3. ■