Online Supplement to
Continuous Review Inventory Model with Dynamic Choice of Two Freight Modes with Fixed Costs

Aditya Jain
Indian School of Business, aditya_jain@isb.edu,

Harry Groenevelt
Simon School of Business, groenevelt@simon.rochester.edu,

Nils Rudi
INSEAD, nils.rudi@insead.edu,

This document contains the supplementary material for the paper, "Continuous Review Inventory Model with Dynamic Choice of Two Freight Modes with Fixed Costs", and follows the notation introduced there.

A. Numerical Investigations

In this section we discuss the numerical investigations carried out to supplement the analysis in the paper. We consider the following set of parameters as the base case example: $h=1$, $p=9$, $L_1=0.3$, $L_2=0.7$, $L_2=0.2$, $c_2=0.25$, $K_1=250$, $K_2=50$, $k_2=25$ and the demand being a Poisson process with demand rate 50. This section is organized as follows: In §A.1 we examine the reasonability of (Q,r) policy assumption by investigating the likelihood of order-crossing in our model. §A.2 considers the sensitivityof the average cost $C(Q)$ to order quantity Q. §A.3 illustrates the sensitivity of the solution and the cost of our model to freight costs K_2, k_2 and c_2. Finally, §A.4 discusses an instance of our model at which we observe multiple solutions to the first-order condition of r (discussed in §5).

A.1. Probability of Order-crossing

A (Q,r) policy is the optimal policy for our model under the assumption that orders do not cross. As discussed in §3, order-crossings are more likely when $\mathbb{P}\left(D_{(0,L_2-l_2]}>Q\right)$ takes a large value and both freight modes are used widely. Of these, the latter is equivalent to the dual-freight mode being of value in terms of cost reduction. In order to elucidate the conditions under which a (Q,r) policy is a reasonable assumption for our model, we provide a close examination of the upper bound on the probability of order-crossing $\mathbb{P}\left(D_{(0,L_2-l_2]}>Q^*\right)$, the percent usage of the express freight $\mathbb{E}q^*/Q^*\%$, and the value of the dual-freight model measured by cost savings over the cheaper of the two single-freight models $\min\{C^s,C^f\}-C^*$.

Figures A.1 (and Figure 4 in the paper) provide the sensitivity of $\mathbb{P}(D_{(0,L_2-l_2]}>Q^*)$, $\mathbb{E}q^*/Q^*\%$ and $\min\{C^s,C^f\}-C^*$ to K_1 for different sets of parameter values. In each of these figures: In the plots on the left-hand side (i.e., Figures 4(a), A.1(a), A.1(c) and A.1(e)), the upper bound on the probability of order-crossing $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ is plotted on a logarithmic scale as a function of $K_1 \in [0,50]$. Whereas, in the plots on the right-hand side (i.e., Figures 4(b), A.1(b), A.1(d) and A.1(f)), the value of two freight modes $\min\{C^s, C^f\} - C^*$, and the percent usage of express freight $\mathbb{E}q^*/Q\%$ are plotted as functions of $K_1 \in [0,250]$, with solid and dotted lines, respectively; and the thick solid and dotted arrows indicate the direction in which h increases while keeping $h+p=10$. The demand is a Poisson process with $\mu \in \{5,25,50\}$.

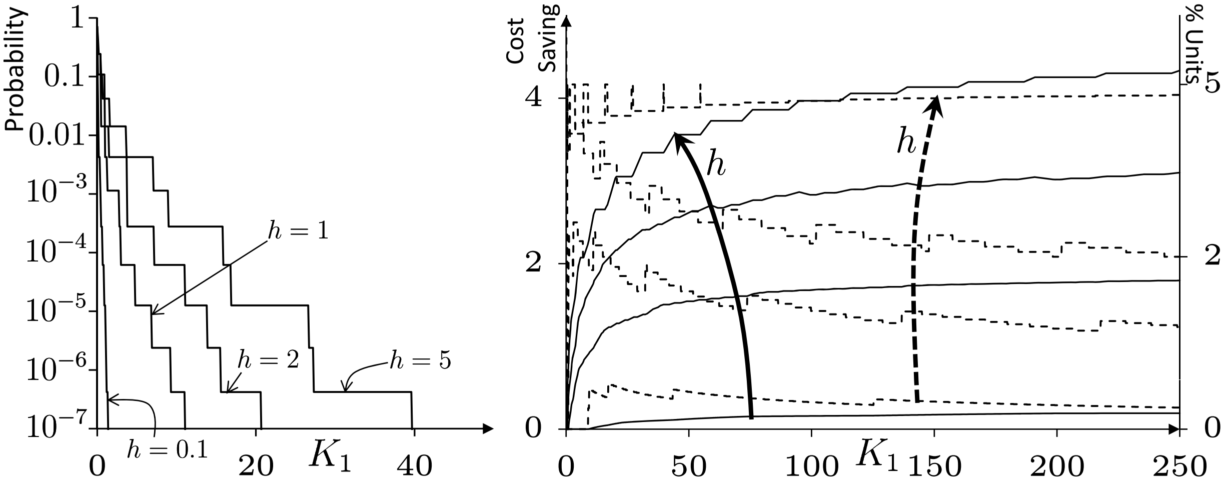
First observe that for all cases as K_1 increases, the upper bound on the probability of order-crossing $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ drops sharply, with its value becoming practically insignificant as K_1 approaches 20. Moreover, its value is larger when $h/(h+p)$ is closer to 0.5, which corresponds to practical situations with low fill-rate requirements.

Next consider Figure A.1: In these plots fixed costs are $K_2 = k_2 = 0$, and lead times are $L_1 = 0$, $L_2 = 1$ and $l_2 = 0.5$. This combination of fixed costs and lead times ensures that orders are more likely to be split and the amount shipped by the express freight q^* is deterministic. In such cases when $0 < q^* < Q^*$, then the probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ is the actual probability of order-crossing (rather than an upper bound). These cases therefore represent the extreme cases of order-crossing in our model. A careful observation of Figures A.1(c), A.1(d), A.1(e) and A.1(f), where the demand rate is not too small ($\mu = 25$ and $\mu = 50$), reveals the following trend: For large values of K_1, the probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ is negligible, and the value of dual-freight model $\min\{C^s, C^f\} - C^*$ is significant. In these cases, $0 < \mathbb{E}q^*/Q\% < 100$ suggest a predominant use of both freight modes. For very small values of K_1, the probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ is large, while the value of dual-freight mode $\min\{C^s, C^f\} - C^*$ is negligible. For these cases $\mathbb{E}q^*/Q\% \approx 0$, suggests use of only regular freight. And for intermediate values of K_1, the probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*) \approx 0$ as well as the value of dual-freight model $\min\{C^s, C^f\} - C^*$ and the percent usage of express freight $\mathbb{E}q^*/Q\%$ are negligible. Together, these observations suggest that when K_1 is not too small, orders are sufficiently large to ensure that orders rarely cross and a (Q,r) policy is a reasonable assumption. For the cases where probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ is large, the value of the dual-freight model is negligible and the optimal policy is likely to be one that uses one of the freight modes (in these cases the regular freight) almost all the time, in which case the optimal policy is a (Q,r) policy. Only in Figures A.1(a) and A.1(b) where $\mu = 5$ do we observe instances where $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$, $\min\{C^s, C^f\} - C^*$ and $\mathbb{E}q^*/Q\%$ take significant values simultaneously. All such instances are observed for $K_1 < 10$, and they represent cases for which the (Q,r) policy assumption may not be reasonable.

The observations made above become stronger (in favor of a (Q,r) policy) when the fixed costs of freight modes are introduced into the model. This is illustrated in Figure 4, where K_2, k_2 and L_1 are non-zero. In this figure, when K_1 is small, $\mathbb{E}q^*/Q\% \approx 100\%$ for $h \in \{1,2,5\}$, and $\mathbb{E}q^*/Q\% \approx 0\%$

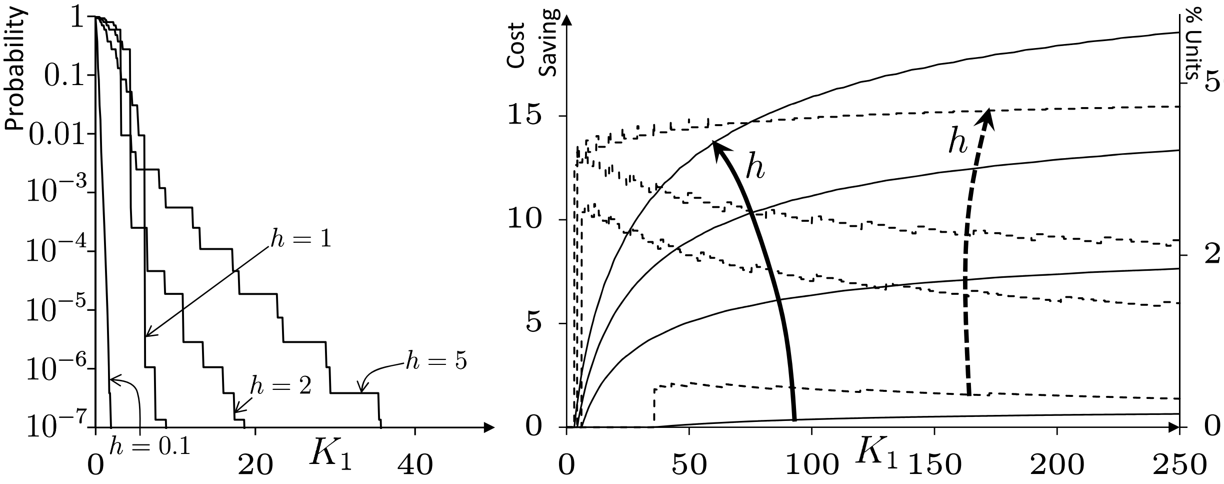
Figure A.1 **Probability $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$, $\mathbb{E}q^*/Q\%$ and $\min\{C^s, C^f\} - C^*$ as functions of K_1**

For $L_1 = 0$, $L_2 = 1$, $l_2 = 0.5$, $K_2 = k_2 = 0$, $c_2 = 0.5$, $\mu = 5$, $h \in \{0.1, 1, 2, 5\}$ with $h + p = 10$, and $\mu = 5$



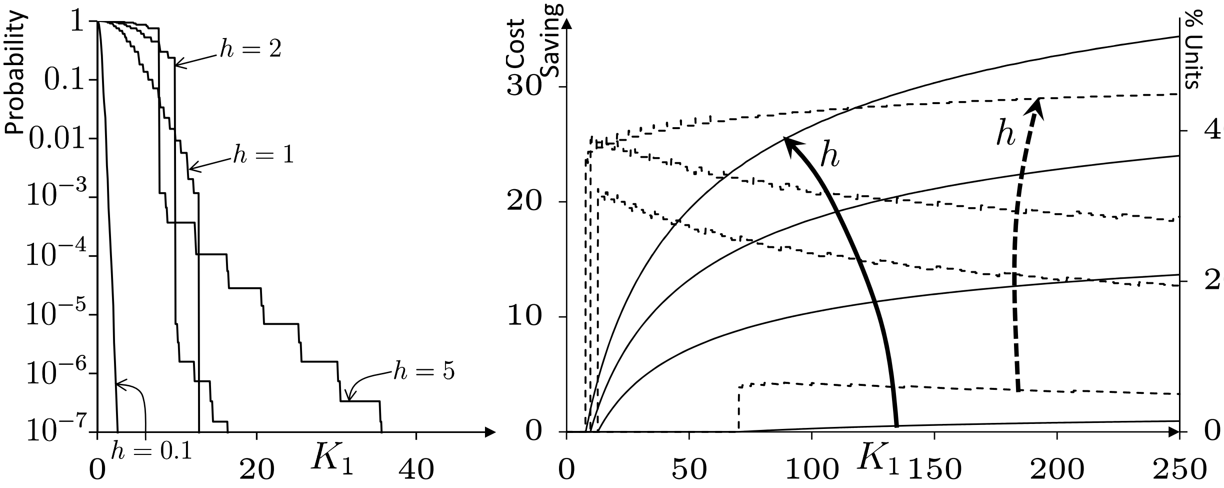
(a) $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ on log scale (b) $\min\{C^s, C^f\} - C^*$ and $\mathbb{E}q^*/Q\%$

For $L_1 = 0$, $L_2 = 1$, $l_2 = 0.5$, $K_2 = k_2 = 0$, $c_2 = 0.5$, $h \in \{0.1, 1, 2, 5\}$ with $h + p = 10$, and $\mu = 25$



(c) $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ on log scale (d) $\min\{C^s, C^f\} - C^*$ and $\mathbb{E}q^*/Q\%$

For $L_1 = 0$, $L_2 = 1$, $l_2 = 0.5$, $K_2 = k_2 = 0$, $c_2 = 0.5$, $h \in \{0.1, 1, 2, 5\}$ with $h + p = 10$, and $\mu = 50$



(e) $\mathbb{P}(D_{(0,L_2-l_2]} > Q^*)$ on log scale (f) $\min\{C^s, C^f\} - C^*$ and $\mathbb{E}q^*/Q\%$

for $h = 0.1$. In other words, when K_1 is small, our model becomes the single-freight model with the smaller optimal cost. Finally, the findings from Figures A.1 and 4 are further substantiated on a large set of numerical examples solved in §A.2.

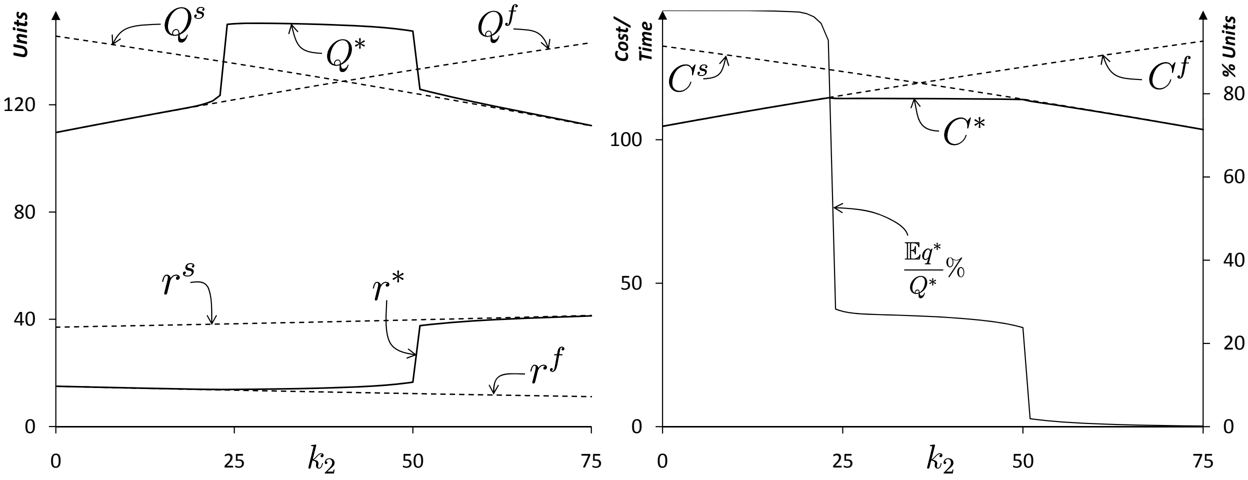
Summarizing, our numerical investigations suggest that a (Q, r) policy for placing orders is a reasonable assumption for a large set of parameter values. When $K_1 + \min\{K_2, k_2\}$ and demand rate μ are small, and $h/(h+p)$ close to 0.5, the orders cross with higher likelihood, and in such cases a (Q, r) policy may not be optimal for our model. It is worthwhile to note that in most practical cases the penalty cost rate p is much higher than the inventory holding cost rate h (firms typically keep a fill rate higher than 0.8). Furthermore, the main focus of this paper is to study the dual-freight model in the presence of significant fixed costs, i.e., when $K_1 + \min\{K_2, k_2\}$ is not very small. These parameter values are therefore not representative of the cases our paper focuses on, but are chosen to put our model in high stress to test it for order-crossing. Given the findings of our numerical investigations, practical considerations about the cost parameters h and p, and the focus of our paper, use of a (Q, r) policy is reasonable for our model. We acknowledge that there may be cases in which the above specified three conditions are met; handling such cases would require a different approach than we have taken in our paper.

A.2. Sensitivity of Average Cost $C(Q)$ to Order Quantity Q

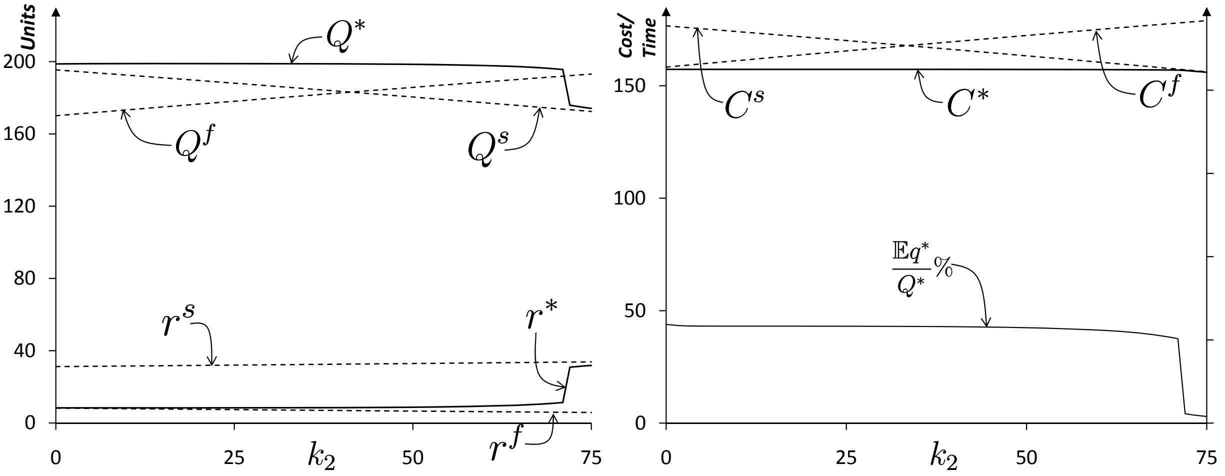
This part of our computational study is aimed towards evaluating the sensitivity of the dual-freight model cost $C(Q)$ to the order quantity Q. To this end, we solve numerical examples for a Poisson demand process with $\mu \in \{5,\ 25,\ 50\}$, lead times $L_1 = 0.3$, $L_2 = 0.7$ and $l_2 = 0.2$, holding cost $h \in \{0.1,\ 0.5,\ 1,\ 2,\ 5\}$, penalty cost p such that $h + p = 10$, $K_2, k_2 \in \{10,\ 25,\ 50,\ 100\}$, $K_1 \in \{0,\ 10,\ 50,\ 100,\ 500,\ 1000\}$ and $c_2 \in \{0.05,\ 0.10,\ 0.25,\ 0.50,\ 0.75\}$. Following Zheng (1992) for each of the 7200 problems solved, we calculate the relative cost increase $R = (C(\hat{Q}) - C^*)/C^*$ incurred by using the optimal order quantity determined assuming deterministic demand $\hat{Q}$ instead of Q^*. Of these 7200 instances of our model, we find that the value of R exceeds 0.1% in 1190 (16.53%) instances and exceeds 1% in only 192 (2.67%) instances, with the highest value at 5.85%. We also calculate R^s, the counterpart of R, for a regular-freight model for 1650 unique regular-freight models obtained from the same set of parameters. The values of R^s are also of a similar order of magnitude, exceeding 0.1% in 255 (17.71%) instances and exceeding 1% in 40 (2.78%) instances, with the highest value at 4.71%. We do not find any specific ordering of R and R^s. Our numerical findings thus suggest that the model with the optimal dual-freight policy is similarly insensitive to the order quantity as the single-freight model.

Figure A.2 Q^*, r^*, C^* **and** $\frac{\mathbb{E}q^*}{Q^*}\%$ **as functions of** k_2 **with** $K_2 + k_2 = 100$ **and** $c_2 = 0.05$**.**

(a) Plots for $K_1 = 100$



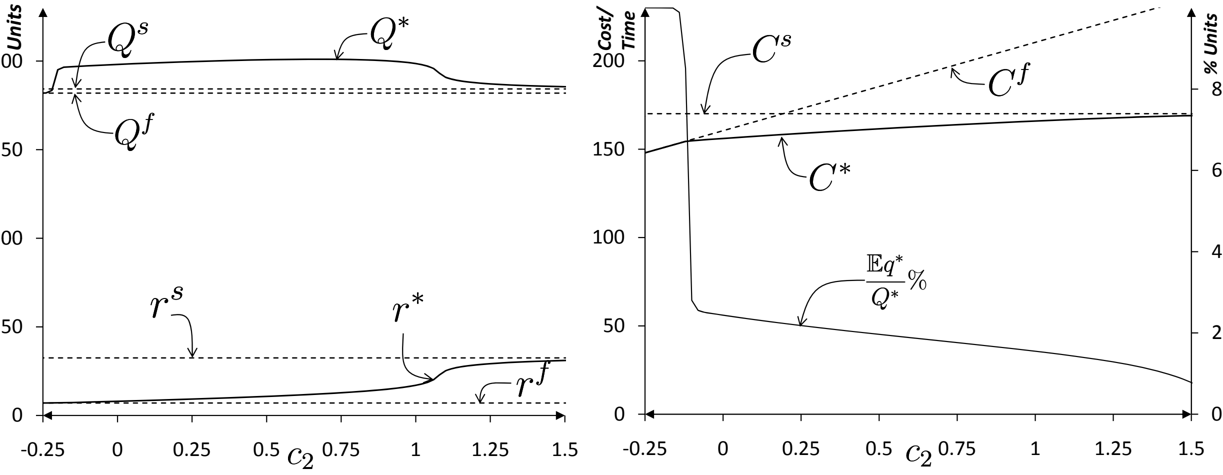
(b) Plots for $K_1 = 250$



A.3. Sensitivity of the Solution to K_2, k_2 and c_2

The results of numerically solved examples to examine the sensitivity of the model to K_2, k_2 and c_2 are displayed in Figures A.2 and A.3. The plots in these figures are presented in two columns: In the left-hand column the solution for order quantity Q^* and reorder point $r^* \left(\overset{\text{def}}{=} r(Q^*)\right)$ are plotted, whereas in the right-hand column the corresponding values of the average cost C^* and the fraction of units shipped by express freight $\mathbb{E}q^*/Q^*$ are plotted. To enable a comparison, in each plot the optimal values of order quantity, reorder point and cost for the two single-freight models are plotted using dotted lines. To obtain these plots, computations are performed by keeping the model parameters the same as the base case, i.e., lead times $L_1 = 0.3$, $L_2 = 0.7$ and $l_2 = 0.2$; and cost parameters $h = 1$ and $p = 9$. To enable continuous plots, the lead-time demands are approximated by Normally distributed random variables with infinitesimal mean 50 and standard deviation 15.

Figure A.3 Sensitivity of optimal Q^*, r^*, C^* and $\mathbb{E}q^*/Q\%$ to c_2 for $K_1=250$, $K_2=50$, $k_2=25$.



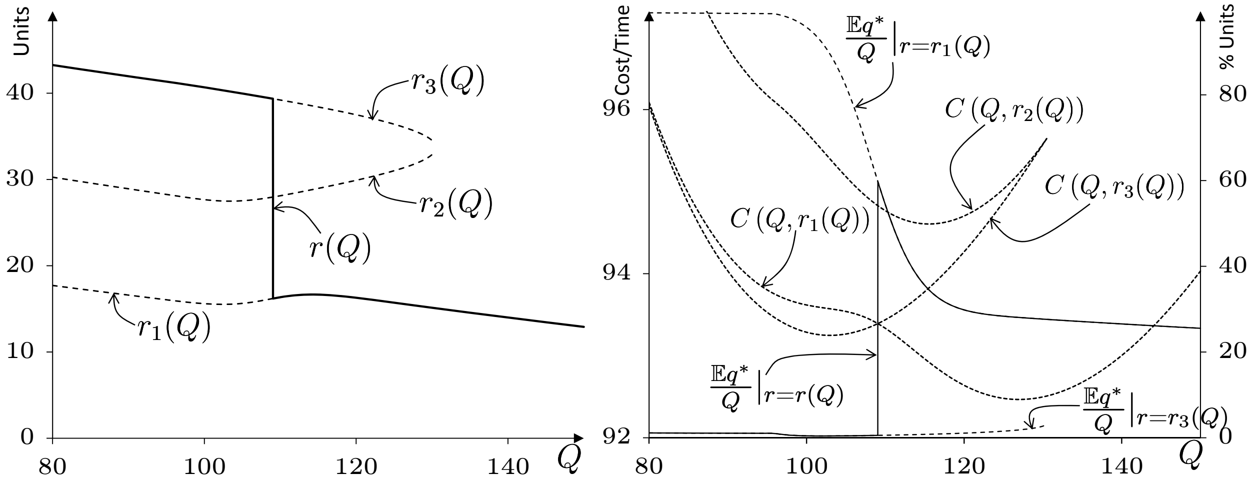
In order to illustrate the sensitivity of our model to fixed costs K_2 and k_2, in Figure A.2, the solution of our model is plotted for $k_2 \in [0,75]$ with $k_2+K_2=75$. A smaller value of $c_2=0.05$ is chosen to highlight the effects of K_2 and k_2 for three different values of K_1. In these plots observe that at the left and right extremes (i.e., near $K_2=75$, $k_2=0$ and $K_2=0$, $k_2=75$) the solution to our model resembles the optimal policy for the single-freight model with smaller fixed cost. On the other hand, for intermediate values of K_2 and k_2, both freight modes are used for shipping a significant fraction of ordered units, and the cost with dual-freight model C^* is much smaller than optimal costs with both single-freight models, C^s and C^f. Also note that the intermediate region is larger when K_1 is larger, implying that the value of dual-freight modes is greater for wider combinations of K_2 and k_2, and is less sensitive to the values of K_2 and k_2.

Figure A.3, where $K_1=250$, $K_2=50$, $k_2=25$ and c_2 is varied from -0.25 to 1.5, indicates that the performance of the model with the optimal dual-freight policy is very sensitive to the value of c_2, and can typically be characterized by three phases: For very small values of c_2, $\mathbb{E}q^*/Q^* \approx 1$ and the use of express freight dominates; for very large values of c_2, $\mathbb{E}q^*/Q^* \approx 0$ and the use of regular freight dominates; and for intermediate values of c_2, significant fractions of the order quantity are shipped by each freight mode. For larger values of k_2, the transitions between these phases are sharper and occur at smaller values of c_2. The phase with dominant use of express freight does not occur if k_2 is of comparable magnitude to or larger than K_2.

A.4. Multiple Solutions to First-Order Condition of r

For the solution to the reorder-point decision discussed in §5, we describe an example in which there exist multiple solutions to the first-order condition of r in equation (17). The values of parameters in this instance of our model are $L_1=0.3$, $L_2=0.7$ and $l_2=0.2$, $h=1$, $p=9$, $c_2=.05$, $K_1=50$, $K_2=37.5$ and $k_2=37.5$. The lead-time demands are approximated by Normal distribution with

Figure A.4 Example of multiple solutions to equation (17).



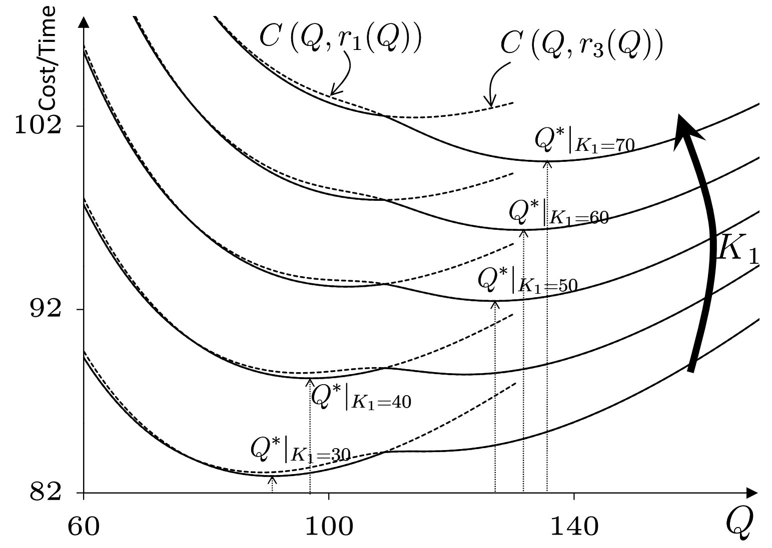
(a) The two local minima and the optimal reorder point.

(b) Costs and fractions of units shipped via express freight

$\mu = 50$, $\sigma = 10$. In this example, for $Q < 130.5$, there exist multiple solutions r to (17). These three solutions (labeled $r_1(Q)$, $r_2(Q)$ and $r_3(Q)$, in the order of their magnitudes) along with the optimal reorder point $r(Q)$ are plotted in Figure A.4(a) as a function of Q. Corresponding values of average cost $C(Q,r)$ and fraction of units shipped with express freight are plotted in Figure A.4(b). We can observe that for all values of Q, $\max\{C(Q,r_1(Q)), C(Q,r_3(Q))\} < C(Q,r_2(Q))$. Clearly, the smallest solution $r_1(Q)$ and the largest solution $r_3(Q)$ are local minima, while the intermediate solution $r_2(Q)$ is a local maximum. The optimal solution for reorder point $r(Q)$ is therefore the more economical of $r_1(Q)$ and $r_3(Q)$. Also note that for smaller values of Q, the fraction of units shipped at reorder point $r_1(Q)$ is close to 100%, while at reorder point $r_3(Q)$ it is close to 0%. In other words, the local minima $r_1(Q)$ and $r_3(Q)$ correspond to dominant use of express freight and regular freight, respectively, with the latter being more cost-effective and hence optimal. However, as Q increases, inventory-cost benefits resulting from splitting orders outweigh the fixed costs incurred in using both freights simultaneously. This results in a significant use of both freight modes at $r_1(Q)$, leading to $r_1(Q)$ becoming more economical than $r_3(Q)$ (hence optimal) for $Q > 109.05$. This switch between the local minima at $Q = 109.05$ makes $r(Q)$ discontinuous and $C(Q)$ non-differentiable at that point.

Observe that as Q increases, one of the local minima, ($r_3(Q)$ in this case) and the local maximum converge and vanish. Thus for larger values of Q ($Q > 130.5$ in this case) there exists a unique solution to the first-order condition of r. When a local minimum and a local maximum converge to a point, that point becomes an inflection point, which is neither a local maximum nor a local minimum. Note that function $C(Q)$ can be discontinuous in Q only if the optimal reorder point switches from a local minimum to the other at a point where the former vanishes. Clearly, this

Figure A.5 **Examples of multiple solutions to equation (20).**



cannot happen, as at such a point the former is not a local minimum. Thus $C(Q)$ is continuous at all points.

Finally, extending this example to multiple values of K_1, Figure A.5 illustrates the uniqueness property of order quantity that satisfies (20). In this figure, for different values of $K_1 \in \{30, 40, 50, 60\}$, the two sub-functions $C(Q, r_1(Q))$ and $C(Q, r_3(Q))$ are plotted with dotted lines, and $C(Q) = \min\{C(Q, r_1(Q)), C(Q, r_3(Q))\}$ with a solid line. For each value of K_1, the order quantity solution that minimizes $C(Q)$ is also depicted. First note that Q^* is never at a point of discontinuity, and hence always satisfies (20). When K_1 is small, two locally optimal Q satisfy (20): The first is smaller than 109.5, corresponding to the dominant use of regular freight; and the second is greater than 109.5, corresponding to greater use of both freights. As K_1 increases, the former locally optimal Q vanishes and $Q^* > 109.5$ is the unique solution to (20). Also note that when K_1 becomes very small, the latter locally optimal Q vanishes and $Q^* < 109.5$ is the unique solution to (20).

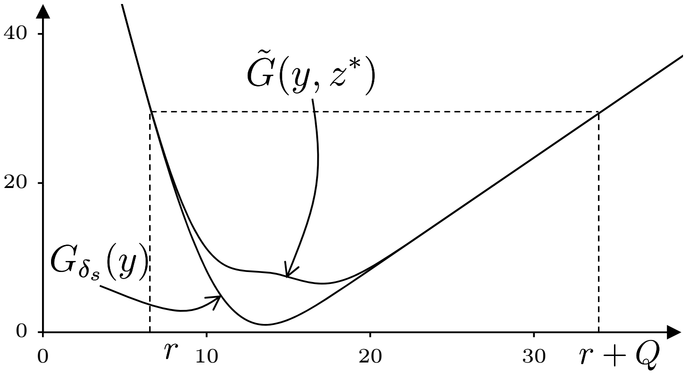
A.5. Comparing Q^* with Q^s and Q^f when $K_2 = k_2 = 0$ and $c_2 < p(L_2 - l_2)$

We provide an heuristic argument, based on the geometric representation of the optimal solution, for $Q^* \geq \max\{Q^s, Q^f\}$. Let $A_s(Q)$ ($A_f(Q)$) denote the single regular (express) freight model counterpart of the function $A(Q)$ defined in Proposition 4. First, we argue that when $K_2 = k_2 = 0$, $c_2 < p(L_2 - l_2)$ and Q is sufficiently large, $A(Q) \leq \min\{A_s(Q), A_f(Q)\}$. This inequality then implies the desired result. We present a geometric argument for $A(Q) \leq A_s(Q)$. An analogous argument can be constructed for $A(Q) \leq A_f(Q)$. Define the cost rate function

$$G_{\delta_s}(y) \overset{\text{def}}{=} G\left(y + \frac{\mu\delta_s}{h}, D_{(0,L]}\right) - \mu\delta_s,$$

and let $r_{\delta_s}(Q) = \min_r \left\{\int_r^{r+Q} G_{\delta_s}(y)dy\right\}$, the optimal r for given Q with the cost rate function $G_{\delta_s}(y)$. First, note that $G_{\delta_s}(y)$ is obtained by shifting the cost rate function for the pure regular freight policy $G\left(y, D_{(0,L]}\right)$ right by $\mu\delta_s/h$ and down by $\mu\delta_s$, so that $G_{\delta_s}(y)$ asymptotically approaches $\tilde{G}(y,z^*)$. Thus, for a sufficiently large value of Q, $G_{\delta_s}\left(r_{\delta_s}(Q)\right) \approx \tilde{G}\left(r(Q), z^*\right)$. Second, in Lemma 7 we established that $S_s(Q) - \mu\delta_s Q = \min_r \left\{\int_r^{r+Q} G_{\delta_s}(y)dy\right\} \leq \min_r \left\{\int_r^{r+Q} \tilde{G}(y,z^*)dy\right\}$. These two facts imply that for sufficiently large values of Q, with r chosen optimally for Q, $G_{\delta_s}(y)$ has a greater area enclosed between the cost rate function and the chord connecting the cost rate function at points r and $r+Q$, than $\tilde{G}(y,z^*)$ has. In other words, the counterpart of $A(Q)$ for $G_{\delta_s}(y)$, which is equal to $A_s(Q)$, is greater than $A(Q)$. Figure A.6, where typical plots of $\tilde{G}(y,z^*)$ and $G_{\delta_s}(y)$ are shown, illustrates our argument. Finally, note that $A_s(Q)$ and $A_f(Q)$ are increasing in Q, Q^s satisfies $A_s(Q) = \mu K_1$ and Q^f satisfies $A_f(Q) = \mu K_1$ (Lemma 6 of Zheng 1992). Given $A(Q) \leq \min\{A_s(Q), A_f(Q)\}$, these facts immediately imply $Q^* \geq \min\left\{Q^s, Q^f\right\}$.

Figure A.6 $G_{\delta_s}(y)$ **and** $\tilde{G}(y,z^*)$ **plotted together.**



References

Zheng, Y.S. 1992. On properties of stochastic inventory systems. *Management Science* **38** 87–103.