Technical Appendix for "Transshipment of Inventories: Dual Allocations vs. Transshipment Prices"

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Proof of Lemma 1: Note that the objective function for the dual is given by $\min \pi = \sum_k \lambda_k \bar{H}_k + \sum_k \mu_k \bar{E}_k$, while the constraints are given by $\lambda_j + \mu_k \geq p_{jk}$ and $\lambda_j, \mu_k \geq 0$. When X_k increases, $\bar{H}_k$ (weakly) increases and $\bar{E}_k$ (weakly) decreases, hence in an optimal solution λ_i is (weakly) decreasing, and μ_i is (weakly) increasing. □

Proof of Lemma 2: Suppose $Y_{ij} > 0$. Then, by complementary slackness, we have $\lambda_i + \mu_j = p_{ij}$. Because $\mu_j \geq 0$, $\lambda_{ij} \leq p_{ij}$. □

Proof of Lemma 3: When $p_{ij} = p$ for all $i \neq j$, we have only one discontinuity point. If the retailer has residual supply, the function has a downward jump at the discontinuity. The profit is increasing while the retailer has no leftover inventory, and starts decreasing afterwards, so the function is unimodal. Similarly, if the retailer has unsatisfied demand, the function has an upward jump at the discontinuity. The graph is strictly increasing before this point, and concave decreasing after it, hence the function is unimodal. When $p_{ij} \leq \min\{c_i - v_i, r_j - c_j\}$ for all $i \neq j$, there can be multiple discontinuities. Consider, for instance, the case $\sum_{j \neq i} \bar{E}_j > \sum_{j \neq i} \bar{H}_j$. It follows from equation (4) that the profit function is decreasing on every discontinuity segment, because $\lambda_i \leq p_{ij} \leq c_i - v_i$. In addition, Lemma 1 implies that the function has a downward jump at every discontinuation point. A similar analysis holds for the case $\sum_{j \neq i} \bar{E}_j < \sum_{j \neq i} \bar{H}_j$. □

Proof of Proposition 1: As can be seen from Lemma 3, the profit functions in our model are discontinuous (with at most $n-1$ discontinuities) and piecewise linear, and the set of discontinuities is of measure zero. Now, each retailer's profit function is given by (2), and it can also be written as

$$P_i^d(\mathbf{X}, \mathbf{D}) = r_i \min\{X_i, D_i\} + v_i \bar{H}_i - c_i X_i + \lambda_i \bar{H}_i + \mu_i \bar{E}_i = (r_i - c_i) X_i + \varpi(X_i - D_i, \mathbf{X}_{-i}, \mathbf{D}_{-i}),$$

where

$$\varpi(X_i - D_i, \mathbf{X}_{-i}, \mathbf{D}_{-i}) = \begin{cases} -\mu_i (X_i - D_i), & \text{if } X_i < D_i, \\ (\lambda_i - r_i + v_i)(X_i - D_i), & \text{if } X_i \geq D_i. \end{cases}$$

Let $J_i^d(x) = \mathsf{E}[P_i^d(x | D_i, \mathbf{X}_{-i}, \mathbf{D}_{-i})]$. Suppose that $p_{ij} = p$ for all $i \neq j$, or $p_{ij} \leq \min\{c_i - v_i, r_j - c_j\}$ for all $i \neq j$, and denote the set of discontinuities of $P_i^d(X_i | D_i, \mathbf{X}_{-i}, \mathbf{D}_{-i})$ by A_i. Then, for every

$x \notin A_i$, $\frac{dP_i^d(x|D_i,\mathbf{X}_{-i},\mathbf{D}_{-i})}{dx} = r_i - c_i + \varpi'(X_i - D_i, \mathbf{X}_{-i}, \mathbf{D}_{-i})$. In addition, $\frac{dP_i^d(x|D_i,\mathbf{X}_{-i},\mathbf{D}_{-i})}{dx} \geq 0$ for $X_i < D_i$ and $\frac{dP_i^d(x|D_i,\mathbf{X}_{-i},\mathbf{D}_{-i})}{dx} \leq 0$ for $X_i \geq D_i$. This implies that we can use an approach similar to that used in the proof of Theorem 1.10 in Dharmadhikari and Joag-dev (1988) to show that for $z > x$, $(J_i^d)'(z) = \int_{-\infty}^{\infty}[r_i - c_i + \varpi'(y, \mathbf{X}_{-i}, \mathbf{D}_{-i})]f_i(z-y)dy \leq \frac{f_i(z)}{f_i(x)}\int_{-\infty}^{\infty}[r_i - c_i + \varpi'(y, \mathbf{X}_{-i}, \mathbf{D}_{-i})]f_i(x-y)dy = \frac{f_i(z)}{f_i(x)}(J_i^d)'(x)$. Thus, $J_i^d(x)' \leq 0 \Rightarrow J_i^d(z)' \leq 0$ for $z > x$, and J_i^d is unimodal. □

Proof of Proposition 2: Consider, for instance, the symmetric model with $n = 2$, $r = 9, c = 6, v = 1$, and $t = 4$. Suppose that there is a discontinuity at point $\tilde{\mathbf{X}}$. Without loss of generality, suppose that retailer 1 has residual supply and retailer 2 has residual demand. Figure A1 depicts profit functions as functions of order quantities for given demands and order quantity of the other retailer. In this case, $\mathbf{D} = (10, 10)$ and $\tilde{\mathbf{X}} = (15, 5)$. Let $\varepsilon > 0, \varepsilon \to 0$. Note that at $\mathbf{X} = (15 - \varepsilon, 5)$ we have $\lambda_1 = 4, \mu_2 = 0$, at $\mathbf{X} = (15 + \varepsilon, 5)$ we have $\lambda_1 = 0, \mu_2 = 4$,while at $\mathbf{X} = (15, 5 + \varepsilon)$ we have $\lambda_1 = 0, \mu_2 = 4$ and at $\mathbf{X} = (15, 5 - \varepsilon)$ we have $\lambda_1 = 4, \mu_2 = 0$.

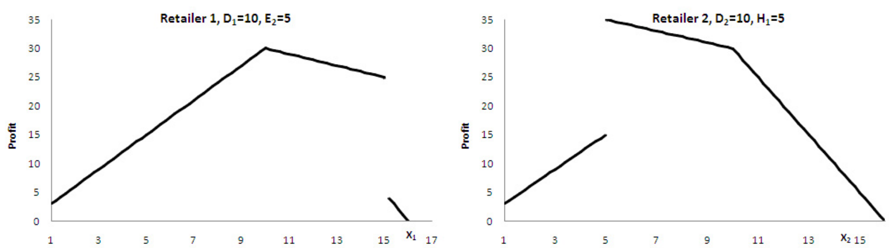

Figure A1: Profit functions for retailers 1 and 2 when $D_1 = D_2 = 10$

At point $\tilde{\mathbf{X}}$, we have multiple dual solutions, and we would need to decide what values to pick in order to completely define P_i^d. As shown in Figure A1, upper semicontinuity of the profit function for retailer 1 would require that at $\tilde{\mathbf{X}}$ we have $\lambda_1 = 4$, while at the same time upper semicontinuity of the profit function for retailer 2 would require that at $\tilde{X}$ we have $\mu_2 = 4$. As one case rules out the other, we cannot achieve upper semicontinuity of the profit functions, regardless of the rule used to deal with degenerate cases.

Now, if we consider an arbitrary case with n retailers, transshipments occur only if there is at least one retailer with residual supply and one retailer with residual demand. Let $\mathbf{X}$ be a point in which the payoff for a retailer with residual supply, say s, has a discontinuity; that is, λ_s changes to λ_s', with $\lambda_s' < \lambda_s$. Then, there is a retailer with residual demand, say d, with a discontinuity at $\mathbf{X}$, such that μ_d changes to μ_d', with $\mu_d' > \mu_d$. Similarly to the analysis shown above, in order to achieve upper semicontinuity of the profit functions we would need that at $\mathbf{X}$ dual price for retailer s attains value λ_s, and at the same time we would need that dual price for retailer d attains value μ_d', which does not give an optimal solution for the dual problem. □

Proof of Theorem 1: We first need to introduce some definitions from Reny (1999). A game,

G, is called *compact* if all pure strategy sets, S_i, are nonempty compact subsets of a topological vector space, and if all payoff functions, u_i, are bounded. Player i can *secure* a payoff of $\gamma \in I\!R$ at $\mathbf{s} \in S$ if there exists $\bar{s}_i \in S_i$ such that $u_i(\bar{s}_i, \mathbf{s}'_{-i}) \geq \gamma$ for all $\mathbf{s}'_{-i}$ in some open neighborhood of $\mathbf{s}_{-i}$. A game, G, is *better-reply secure* if whenever $(\mathbf{s}^*, \mathbf{u}^*)$ is in the closure of the graph of its payoff function and $\mathbf{s}^*$ is not a NE, some player i can secure a payoff strictly above u_i^* at $\mathbf{s}^*$. So, a game is better-reply secure if for every nonequilibrium strategy $\mathbf{s}^*$ and payoff vector $\mathbf{u}^*$ resulting from strategies approaching $\mathbf{s}^*$, a player, i, has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from $\mathbf{s}^*$. The main result from Reny (1999) states that if G is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy NE. We want to apply this result to our game.

Proposition 1 implies that each retailer's expected profit function in our inventory-sharing game is unimodal when $p = p_{ij}, \forall i, j$ or $p_{ij} \leq \min\{c_i - v_i, r_j - c_j\}$ for all $i \neq j$. To show that a game is better-reply secure, it is enough to show (Proposition 3.2 in Reny) that the game is (i) reciprocally upper semicontinuous and (ii) payoff secure.

(i) Reciprocal upper semicontinuity requires that some players payoff jump up whenever some other players payoff jumps down. This is a generalization of the condition from Dasgupta and Maskin (1986), which requires that the sum of all expected profit functions is upper semicontinuous, and which is satisfied for our game.

(ii) Payoff security requires that for every strategy $\mathbf{s}$ and every $\varepsilon > 0$, each player i can secure a payoff of $u_i(\mathbf{s}) - \varepsilon$ at $\mathbf{s}$. In other words, payoff security requires that for every strategy $\mathbf{s}$, each player has a strategy that virtually guarantees the payoff he receives at $\mathbf{s}$, even if the others deviate slightly from $\mathbf{s}$. Consider a non-equilibrium point, $\mathbf{X}$, for our inventory-sharing game, and select a retailer, i. If i's expected profit is continuous in $\mathbf{X}$, it is easy to see that the condition for payoff security is satisfied. Now, suppose that i's expected payoff has a discontinuity at $\mathbf{X}$. Then, if i's expected payoff has a jump up (resp., down) at $\mathbf{X}$, he can secure a payoff that is at worst just below the status quo by increasing (resp., decreasing) her order quantity slightly. Consequently, inventory-sharing game is payoff secure. As (i) also shows its reciprocal upper semicontinuity, the inventory-sharing game is better-reply secure, and it possesses a pure strategy NE. □

Proof of Theorem 2: Theorem 5 in Dasgupta and Maskin (1986) states that our game will have a NE in mixed strategies if the sum of all expected profit functions is upper semicontinuous, and if the expected profit for a retailer, i, is bounded and weakly lower semicontinuous. It is easy to verify that these conditions are satisfied in our case: the sum of all expected profits is continuous, and the conditions for weak lower semicontinuity (Definition 6 in Dasgupta and Maskin) is satisfied by letting $\lambda = 0$ or $\lambda = 1$. □

Proof or Proposition 3: If the retailers are symmetric, then there is an equilibrium in which all retailers order the same quantity, $X_i^d = X^d, \forall i$, and $J_i^d(\mathbf{X}^d) = J^d(\mathbf{X}^d), \forall i$. Thus, $\sum_i J_i^d(\mathbf{X}^d) =$

$nJ^d(\mathbf{X}^d)$, and (6) implies that $nJ^d(\mathbf{X}^d) = J^n(\mathbf{X}^d)$.

If we consider the centralized system, there is an equilibrium in which all retailers order the same quantity, $X_i^n = X^n, \forall i$. Because centralized model maximizes the expected profit, $J^n(\mathbf{X}^n) \geq J^n(\mathbf{X}^\mathbf{d}) = nJ^d(\mathbf{X}^d)$, and (6) implies that it is optimal for symmetric retailers to order at the first-best level, $\mathbf{X}^d = \mathbf{X}^n$. □

Proof of Proposition 4: Without loss of generality, assume $\Delta > 0$ such that $c_1 = c + \Delta > c_2 = c - \Delta$ for (i) and $t_{12} = t + \Delta > t_{21} = t - \Delta$ for (ii). Let $\hat{F}(\cdot)$ be the c.d.f. of the total demand, $D = D_1 + D_2$, with the corresponding density $\hat{f}(\cdot)$, and let X be the total order quantity, $X_1 + X_2$. For our proof, we need the following result, which for brevity we state without the proof (it can be obtained from the authors).

Lemma A1. *X_1^n decreases with Δ, and X_2^n increases with Δ.*

We now continue the proof of the proposition.

(i) For DA, the sufficient and necessary condition for a first-best outcome is $G_1 = G_2$ at $\mathbf{X}^n$. As $p_{12} = p_{21} = r-v-t$, the condition is equivalent to $(g_1 = g_2)\,|_{\mathbf{X}^n}$. However, Lemma A1 indicates that $X_1^n < X_2^n$, hence $(g_1 \neq g_2)\,|_{\mathbf{X}^n}$ and DA cannot coordinate the system when $|\Delta| > 0$. For TP, with the given r and v, we can find a $\bar{c}$ such that $(r, \bar{c}, v)$ is in the $\bar{a}$-set. Let $\bar{\mathbf{X}}^n$ be the system-optimal order quantity for parameters $(r, \bar{c}, v)$. Recall that $a_i(\mathbf{X}) = Pr\{D_i > X_i, D \leq X\} = Pr\{D \leq X\} - Pr\{D_i \leq X_i, D \leq X\}$, and $b_i(\mathbf{X}) = Pr\{D_i \leq X_i, D > X\}$, hence $a_i(\mathbf{X}) - b_i(\mathbf{X}) = \hat{F}(X) - F(X_i)$. Then, if $c_1 = c_2 = \bar{c}$, we have $a_i(\bar{\mathbf{X}}^n) - b_i(\bar{\mathbf{X}}^n) = \hat{F}(\bar{X}^n) - F(\bar{X}_i^n) = 0$. It can also be verified that if $c_1 = c_2 > \bar{c}$, then the first-best order quantity, X_i^n, satisfies $X_i^n < \bar{X}_i^n$, and $a_i(\mathbf{X}^n) - b_i(\mathbf{X}^n) = \hat{F}(X^n) - F(X_i^n) < 0$; otherwise, for $c_1 = c_2 < \bar{c}$, $a_i(\mathbf{X}^n) - b_i(\mathbf{X}^n) = \hat{F}(X^n) - F(X_i^n) > 0$.

Now, denote by $\bar{\mathbf{X}}^n$ and $\bar{\mathbf{X}}^n_\Delta$ the system-optimal order quantity for parameters $(r, \bar{c}, v)$ and $(r; \bar{c} + \Delta, \bar{c} - \Delta; v)$, respectively. Suppose that $c > \bar{c}$ such that $X_i^n < \bar{X}_i^n$ and $a_i(\mathbf{X}^n) - b_i(\mathbf{X}^n) < 0$ when $\Delta = 0$. Because $c_1 = c + \Delta > c_2 = c - \Delta$ and $a_i(\mathbf{X}^n) - b_i(\mathbf{X}^n) = \hat{F}(X^n) - F(X_i^n)$, Lemma A1 implies that $a_1(\mathbf{X}^n_\Delta) - b_1(\mathbf{X}^n_\Delta) > a_2(\mathbf{X}^n_\Delta) - b_2(\mathbf{X}^n_\Delta)$. Let $\Delta_t = \sup\{\Delta : a_1(\mathbf{X}^n_\Delta) - b_1(\mathbf{X}^n_\Delta) \leq 0\}$. Then, the TP coordinate the system whenever $\Delta \leq \Delta^t$ (which implies $(A_1 - B_1)(A_2 - B_2) \geq 0$). A similar analysis holds for $c < \bar{c}$.

(ii) $t_{12} = t + \Delta > t_{21} = t - \Delta$ implies $p_{12} < p_{21}$, and Lemma A1 implies $X_2^n > X_1^n$. For DA, this means

$$g_1(\mathbf{X}^n) = \int_0^{X_2^n} (X_2^n - u) f(X^n - u) f(u) du > g_2(\mathbf{X}^n) = \int_0^{X_1^n} (X_1^n - u) f(X^n - u) f(u) du.$$

Consequently, $G_1 = p_{21} g_1(\mathbf{X}^n) > G_2 = p_{12} g_2(\mathbf{X}^n)$, and DA cannot coordinate the system. The proof for TP follows similar steps as described in (i).

(iii) When (r, c, v) is not in the $\bar{a}$-set and $\Delta = 0$, we have $|A_i - B_i| > 0$ for $t \in (0, r - v)$. Thus, there exists some $\epsilon > 0$ such that the signs of $A_i - B_i$ remains unchanged when $\Delta \leq \epsilon$. Now, suppose that (r, c, v) is in the $\bar{a}$-set. Then, at $\Delta = 0$,we have $a_1(\mathbf{X}^n) - b_1(\mathbf{X}^n) = a_2(\mathbf{X}^n) - b_2(\mathbf{X}^n) = 0$ and $f_1(X_1^n) = f_2(X_2^n)$. Suppose that $c_1 = c + \Delta > c_2 = c - \Delta$; similarly as before, this implies that

$$a_1(\mathbf{X}_\Delta^n) - b_1(\mathbf{X}_\Delta^n) > a_1(\mathbf{X}^n) - b_1(\mathbf{X}^n) = 0 = a_2(\mathbf{X}^n) - b_2(\mathbf{X}^n) > a_2(\mathbf{X}_\Delta^n) - b_2(\mathbf{X}_\Delta^n).$$

Consequently, $(A_1 - B_1)(A_2 - B_2) = p^2(a_1(\mathbf{X}_\Delta^n) - b_1(\mathbf{X}_\Delta^n))(a_2(\mathbf{X}_\Delta^n) - b_2(\mathbf{X}_\Delta^n)) < 0$, hence $\Delta^t = 0$. A similar analysis holds for $t_{12} = t + \Delta > t_{21} = t - \Delta$. □

Proof of Proposition 5: To show there are instances in which DA coordinates while TP cannot, let us first consider an example with two retailers who have the parameters described in Table A1. Both retailers are facing Beta-distributed demand, with $\mathsf{E}[D_1] = 0.5, \mathsf{E}[D_2] = 0.5$. It can be

	r	c	v	t_{ij}	p_{ij}	D_i
Retailer 1	10	5.7848	1	0.5040	8.4960	$Beta(2,\ 2)$
Retailer 2	10	5.2152	1	2.4240	6.5760	$Beta(2,\ 2)$

Table A1: Parameter values for the two retailers

verified that at $X_1 = 0.48$ and $X_2 = 0.52$, $A_1(X_1, X_2) = B_2(X_1, X_2) = 0.9235$,, $A_2(X_1, X_2) = B_1(X_1, X_2) = 0.9384$, and $G_1(X_1, X_2) = G_2(X_1, X_2) = 0.6984$, and the FOC's are zero, $r_i - c_i - (r_i - v_i)F(X_i) + B_i(X_1, X_2) - A_i(X_1, X_2) = 0$, hence $(X_1,\ X_2)$ is the first-best order quantity, and $G_1 = G_2$ indicates that DA coordinate the system. However, note that $(B_1 - A_1)(B_2 - A_2) < 0$, thus the coordinating TP do not exist. Proposition 4 provides examples of instances in which TP coordinate the system, while DA do not. □

Proof of Theorem 4: Let $m \in \{1,\ n,\ d,\ t\}$. Denote $\pi^m = J^m(X_1^m,\ X_2^m)$, $\bar{X} = X_i^m|_{c_i=c}$, $(J_i^m)_{ii} = \frac{\partial^2 J_i^m}{\partial X_i^2}$, and $(J_i^m)_{ij} = \frac{\partial^2 J_i^m}{\partial X_1 \partial X_2}$[1]. We first prove the following lemma that is used in our proof.

Lemma A2. *Given any $\pi > 0$, $\mathcal{X}(\pi) = \{\, (x_1,\ x_2) \,|\, J^n(x_1,\ x_2) >= \pi \,\}$ is a convex set.*

Proof. To show this statement, it is enough to prove that the Hessian matrix of $J^n(X_1,\ X_2)$ is negative semi-definite. It can be verified that

$$\begin{aligned}
(J^n)_{ij} &= -p_{ij} \textstyle\int_0^{X_i} f_j(X_1 + X_2 - u)dF_i(u) - p_{ji} \int_0^{X_j} f_i(X_1 + X_2 - u)dF_j(u) < 0; \\
(J^n)_{ii} &= (J^n)_{ij} - (r_i - v_i)f_i(X_i) + p_{ij}\bar{F}_j(X_j)f_i(X_i) + p_{ji}F_j(X_j)f_i(X_i) \\
&= (J^n)_{ij} - (r_i - v_i - p_{ij}\bar{F}_j(X_j) - p_{ji}F_j(X_j))f_i(X_i) \\
&\leq -\min\{t_{ij} + r_i - r_j,\ t_{ji} + v_j - v_i\}f_i(X_i) + (J^n)_{ij}.
\end{aligned}$$

To avoid trivial cases, we assume $t_{ij} + r_i \geq r_j$ and $t_{ji} + v_j \geq v_i$ as mentioned in §2. Thus $(J^n)_{ii} \leq (J^n)_{ij} < 0$, which also implies that $(J^n)_{11}(J^n)_{22} - (J^n)_{12}(J^n)_{21} \geq 0$. Hence, the matrix is negative semi-definite and $\mathcal{X}(\pi)$ is a convex set. □

[1] When $m = n$, we let $J_1^n = J_2^n = J^n$.

We now continue the proof of the theorem.

(i) Without loss of generality, assume $c_1 = c + \epsilon > c_2 = c$. It can be verified that this implies $X_1^n < X_1^1 < X_2^1 = \bar{X} < X_2^n$, so coordinating TP do not exists because neither $\mathbf{X}^n \leq \mathbf{X}^1$ nor $\mathbf{X}^n \geq \mathbf{X}^1$ holds. Moreover, $X_1^n + X_2^n < 2\bar{X}$. We proceed by identifying the best pair of TP, $(\tilde{\gamma_1}, \tilde{\gamma_2})$, which provides the highest efficiency among all non-negative (γ_1, γ_2). It can be shown (the proof is omitted due to space constraints) that when $c_1 = c + \epsilon > c_2 = c$, the best pair of non-negative TP is either (1, 0) or (0, 1). Assume the best pair of TP is achieved by letting $(\tilde{\gamma_1}, \tilde{\gamma_2}) = (1, 0)$; the FOCs for each retailer under the best pair of TP are

$$\frac{\partial J_i^t}{\partial X_i} = (r - c_i) - (r - v)F(X_i) + \tilde{\gamma_i} p b_i - (1 - \tilde{\gamma_j}) p a_i = (r - c_i) - (r - v)F(X_i) + p b_i - p a_i.$$

Let us further denote that $(J_i^m)_i' = \frac{\partial^2 J_i^m}{\partial X_i \partial c_1}$. We next prove that $\pi^d > \pi^t$:

(1) We first show that $X_1^n - X_2^n > X_1^d - X_2^d \geq X_1^t - X_2^t$: For any J^m, we have $\frac{\partial J_i^m}{\partial X_i} = 0$ at $\mathbf{X}^m = [X_1^m, X_2^m]'$, with $m \in \{1, n, d, t\}$. The total derivative of $\frac{\partial J_i^m}{\partial X_i} = 0$ gives

$$\begin{bmatrix} (J_1^m)_{11} & (J_1^m)_{12} \\ (J_2^m)_{21} & (J_2^m)_{22} \end{bmatrix} \frac{\partial \mathbf{X}^m}{\partial c_1} = - \begin{bmatrix} (J_1^m)_1' \\ (J_2^m)_2' \end{bmatrix}.$$

Solving the system leads to

$$\frac{\partial X_i^m}{\partial c_1} = \frac{(J_j^m)_j'(J_i^m)_{ij} - (J_i^m)_i'(J_j^m)_{jj}}{(J_1^m)_{11}(J_2^m)_{22} - (J_1^m)_{12}(J_2^m)_{21}}. \tag{A1}$$

Table A2 summarizes the values of these expressions at $c_1 = c$, with $L = tf(\bar{X})$ and $K = p\int_0^{\bar{X}} f(2\bar{X} - u)dF(u)$.

m	$(J_1^m)_{11}$	$(J_2^m)_{22}$	$(J_1^m)_{12}$	$(J_2^m)_{21}$	$(J_1^m)_1'$	$(J_2^m)_2'$
n	$-L-2K$	$-L-2K$	$-2K$	$-2K$	-1	0
d	$-L-3K$	$-L-3K$	$-K$	$-K$	-1	0
t	$-L-2K$	$-(r-v)f(\bar{X})$	$-2K$	0	-1	0

Table A2: When $c_1 = c + \epsilon$ and $(J_i^m)_i'$ is defined as $\frac{\partial^2 J_i^m}{\partial X_i \partial c_1}$

(A1) implies that

$$\begin{aligned} \left.\frac{\partial X_1^n}{\partial c_1}\right|_{c_1=c} &= -\frac{(L+2K)}{L(L+4K)}, & \left.\frac{\partial X_1^d}{\partial c_1}\right|_{c_1=c} &= \frac{-(L+3K)}{(L+2K)(L+4K)}, & \left.\frac{\partial X_1^t}{\partial c_1}\right|_{c_1=c} &= -\frac{1}{L+2K}; \\ \left.\frac{\partial X_2^n}{\partial c_1}\right|_{c_1=c} &= \frac{2K}{L(L+4K)}, & \left.\frac{\partial X_2^d}{\partial c_1}\right|_{c_1=c} &= \frac{K}{(L+2K)(L+4K)}, & \left.\frac{\partial X_2^t}{\partial c_1}\right|_{c_1=c} &= 0. \end{aligned}$$

As seen from above,

$$\left.\left(\frac{\partial X_2^t}{\partial c_1} - \frac{\partial X_1^t}{\partial c_1}\right)\right|_{c_1=c} = \left.\left(\frac{\partial X_2^d}{\partial c_1} - \frac{\partial X_1^d}{\partial c_1}\right)\right|_{c_1=c} = \frac{1}{L+2K} < \left.\left(\frac{\partial X_2^n}{\partial c_1} - \frac{\partial X_1^n}{\partial c_1}\right)\right|_{c_1=c} = \frac{1}{L}.$$

Hence expression $X_1^n - X_2^n > X_1^d - X_2^d \geq X_1^t - X_2^t$ holds.

(2) We now show that $\mathbf{X^t}$ is not an interior point of $\mathcal{X}(\pi^d)$. On the contour of $J^n(\mathbf{X}) = \pi^d$, we have

$$\left.\frac{\partial X_2}{\partial X_1}\right|_{\mathbf{X}=\mathbf{X}^d} = -\frac{\left.\frac{\partial J^n}{\partial X_1}\right|_{\mathbf{X}=\mathbf{X}^d}}{\left.\frac{\partial J^n}{\partial X_2}\right|_{\mathbf{X}=\mathbf{X}^d}} = -\frac{G_1-G_2}{G_2-G_1} = 1.$$

By Lemma A2 and the first inequality in (1) ($X_2^n - X_1^n > X_2^d - X_1^d$), any $\mathbf{X}$ which is an interior point of $\mathcal{X}(\pi^d)$ should satisfy $X_2 - X_1 > X_2^d - X_1^d$. However, (1) implies that this does not hold for $\mathbf{X^t}$. Thus, $\pi^d \geq \pi^t$.

(ii) When $r_2 = r+\epsilon > r_1 = r$, assume without loss of generality $t \in [0, \tau']$, with $\tau' \in (c-v, r-v)$ defined in Theorem 1 of HDK (2007). We first look at the case when $(\tilde{\gamma}_1, \tilde{\gamma}_2) = (1, 0)$; following an analysis similar to the one that we used in the proof of (i), it is then enough to show that

$$\left.\left(\frac{\partial X_2^n}{\partial r_2} - \frac{\partial X_1^n}{\partial r_2}\right)\right|_{r_2=r} > \left.\left(\frac{\partial X_2^d}{\partial r_2} - \frac{\partial X_1^d}{\partial r_2}\right)\right|_{r_2=r} \geq \left.\left(\frac{\partial X_2^t}{\partial r_2} - \frac{\partial X_1^t}{\partial r_2}\right)\right|_{r_2=r}. \tag{A2}$$

The rest of the proof follows the steps from part (2) of (i). Table A3 summarizes the values of the expressions for different derivatives.

m	$(J_1^m)_{11}$	$(J_2^m)_{22}$	$(J_1^m)_{12}$	$(J_2^m)_{21}$	$(J_1^m)_1'$	$(J_2^m)_2'$
n	$-L-2K$	$-L-2K$	$-2K$	$-2K$	b_1	$\bar{F}(\bar{X}) - a_2$
d	$-L-3K$	$-L-3K$	$-K$	$-K$	$b_1 - g_2$	$\bar{F}(\bar{X}) - a_2 + g_2$
t	$-L-2K$	$-(r-v)f(\bar{X})$	$-2K$	0	b_1	$\bar{F}(\bar{X})$

Table A3: When $r_2 = r+\epsilon > r_1 = r$ and $(J_i^m)_i'$ is defined as $\frac{\partial^2 J_i^m}{\partial X_i \partial r_2}$

It follows from (A1) that:

$$\left.\frac{\partial X_2^n}{\partial r_2}\right|_{r_2=r} = \frac{L\bar{F}(\bar{X})+(L+4K)\bar{F}(\bar{X})^2}{2L(L+4K)}, \qquad \left.\frac{\partial X_1^n}{\partial r_2}\right|_{r_2=r} = \frac{L\bar{F}(\bar{X})-(L+4K)\bar{F}(\bar{X})^2}{2L(L+4K)};$$

$$\left.\frac{\partial X_2^d}{\partial r_2}\right|_{r_2=r} = \frac{(b_1-g_2)(-K)-(\bar{F}(\bar{X})-a_2+g_2)(-L-3K)}{(L+2K)(L+4K)}, \qquad \left.\frac{\partial X_1^d}{\partial r_2}\right|_{r_2=r} = \frac{(\bar{F}(\bar{X})-a_2+g_2)(-K)-(b_1-g_2)(-L-3K)}{(L+2K)(L+4K)};$$

$$\left.\frac{\partial X_1^t}{\partial r_2}\right|_{r_2=r} = \frac{b_1(-2K)+\bar{F}(\bar{X})(r-v)f(\bar{X})}{(L+2K)(r-v)f(\bar{X})}, \qquad \left.\frac{\partial X_1^t}{\partial r_2}\right|_{r_2=r} = \frac{(L+2K)b_1}{(L+2K)(r-v)f(\bar{X})}.$$

Therefore,

$$\left.\left(\frac{\partial X_2^n}{\partial r_2} - \frac{\partial X_1^n}{\partial r_2}\right)\right|_{r_2=r} = \frac{\bar{F}(\bar{X})-a_2-b_1}{L}; \qquad \left.\left(\frac{\partial X_2^d}{\partial r_2} - \frac{\partial X_1^d}{\partial r_2}\right)\right|_{r_2=r} = \frac{\bar{F}(\bar{X})-a_2-b_1+2g_2}{L+2K};$$

$$\left.\left(\frac{\partial X_2^t}{\partial r_2} - \frac{\partial X_1^t}{\partial r_2}\right)\right|_{r_2=r} = \frac{\bar{F}(\bar{X})}{L+2K} - \frac{(L+4K)b_1}{(L+2K)(r-v)f(\bar{X})}.$$

As $t \to 0$, we have $L \to 0$ and $K \to (r-v)\hat{f}(\bar{X})$, where $\hat{f}$ is the density function of the average demand $\hat{D} = \frac{D_1+D_2}{2}$. Under the assumption that the density function of D_i is log-concave, $\hat{D} <_{cx} D_i$ (in convex order), and the CDF of D_i and $\hat{D}$ cross only once, at $\bar{X}$. It is then straightforward that $\hat{f}(\bar{X}) > f(\bar{X})$. Therefore, as $t \to 0$,

$$\begin{aligned}\left.\left(\frac{\partial X_2^t}{\partial r_2} - \frac{\partial X_1^t}{\partial r_2}\right)\right|_{r_2=r} &\to \frac{\bar{F}(\bar{X})-4b_1\frac{\hat{f}(\bar{X})}{f(\bar{X})}}{L+2K} < \frac{\bar{F}(\bar{X})-4b_1}{L+2K} < \left.\left(\frac{\partial X_2^d}{\partial r_2} - \frac{\partial X_1^d}{\partial r_2}\right)\right|_{r_2=r} = \frac{\bar{F}(\bar{X})-a_2-b_1+2g_2}{L+2K} \\ &< \left.\left(\frac{\partial X_2^n}{\partial r_2} - \frac{\partial X_1^n}{\partial r_2}\right)\right|_{r_2=r} \to \infty,\end{aligned}$$

and there exists $0 \leq t^{'} \leq \tau^{'}$ such that (A2) holds when $t \in [0, t^{'}]$. If $(\tilde{\gamma}_1, \tilde{\gamma}_2) = (0, 1)$ for $r_2 = r + \epsilon > r_1 = r$, we instead consider $r_2 = r - \epsilon < r = r_1$ in which the best TP will be $(\tilde{\gamma}_1, \tilde{\gamma}_2) = (1, 0)$. In this case, DA are more efficient when r_i decreases by a small amount. □

Proof of Proposition 6: As has been proved in Proposition 4, for $\Delta \in (0, \Delta^t]$ DA cannot coordinate the system, while TP can. Therefore, at $\Delta = \Delta^t$, we have $J^d(\Delta) < J^t(\Delta)$. Let $\Delta^* = \sup\{\Delta : J^d(\Delta) < J^t(\Delta)\}$. Then, either $\Delta^* = t/2$ (border of the feasible area), or $J^d(\Delta^*) = J^t(\Delta^*)$. In the latter case, the profit under DA must increase faster then the profit under TP at Δ^*, hence $J^d(\Delta) > J^t(\Delta)$ when $\Delta \to \Delta^* + \epsilon$. □

Proof of Proposition 7:

1. Suppose first that the retailers are symmetric; then, $A_i = A_j = A$ and $B_i = B_j = B$. Moreover, $p\gamma_i = p\frac{(A-B)B}{A^2-B^2} = p\frac{B}{A+B} = \Lambda_i^n$, $p(1-\gamma_i) = p\left(1 - \frac{(A-B)B}{A^2-B^2}\right) = p\frac{A}{A+B} = \mathrm{M}_i^n$, thus in the symmetric case we have one-to-one correspondence between the expected value of the dual prices and the TP. In addition, because $\frac{p_{ij}\gamma_i}{\Lambda_i^n} = \frac{(A_j-B_j)B_i}{A_1A_2-B_1B_2} / \frac{B_i}{A_j+B_i} = 1 + \frac{A_j+B_i-(A_i+B_j)}{A_1A_2-B_1B_2}A_j$, the same is true when $A_1 + B_2 = A_2 + B_1$; we can use (18) to show that this condition corresponds to $p_{12}F_1(X_1^n)\bar{F}_2(X_2^n) = p_{21}\bar{F}_1(X_1^n)F_2(X_2^n)$.

2. First, note that (16) implies that $1 - \gamma_j = \frac{(A_j-B_j)A_i}{A_1A_2-B_1B_2}$, hence $\frac{1-\gamma_j}{\gamma_i} = \frac{A_i}{B_i} = \frac{1}{p_{ji}}\mathrm{M}_i^n(A_i + B_j)/\frac{1}{p_{ij}}\Lambda_i^n(A_j + B_i)$, and now (19) holds from (18). □