Online Supplements for Achieving a Long-Term Service Target with Periodic Demand Signals: A Newsvendor Framework

A. Bensoussan

International Center for Decision and Risk Analysis, School of Management, The University of Texas at Dallas, Richardson, TX 75083. alain.bensoussan@utdallas.edu

Q. Feng

McCombs School of Business, The University of Texas at Austin, Austin, TX 78712. annabelle.feng@mccombs.utexas.edu

S. P. Sethi

Center for Intelligent Supply Networks, School of Management, The University of Texas at Dallas, Richardson, TX 75083. sethi@utdallas.edu

A. Derivation of The Two-Stage Model

Consider the problem defined in (2)-(4). With a finite T, we can define $n^\varepsilon \equiv \lceil T(1-\varepsilon)\rceil$. In other words, to meet the service target, the vendor should satisfy the demand for at least n^ε periods in the planning cycle. For this model, a speculative behavior is inevitable. Specifically, the long-term constraint would lead to an end-of-horizon effect. Such a policy is not desirable from the practical viewpoint, because it is a non-stationary policy that depends on the current demand signal X_t, the time to the end of the planning cycle $T-t$, and the service performance to date, i.e., $n_t \equiv \sum_{s=1}^{t-1} \mathbb{1}_{\{Q_s \geqslant D(x_s)\}}$. For example, if the target service level is met at some time s before T, i.e., $n_s \geqslant n^\varepsilon$, then the model would call for ordering the unconstrained newsvendor quantity for the remaining days. If, however, $n^\varepsilon - (T+1-t) = n_t < n^\varepsilon$, then the optimal policy calls for a very large order (at the maximum demand) to achieve the target.

Furthermore, the difficulty of solving the problem comes from the service constraint that is evaluated on each sample path. Also, this constraint results in the feasible set $\mathbf{Q}$ to be not convex. A rigorous analysis of this problem is analytically intractable and is beyond the scope of the paper. In what follows, we demonstrate that how this problem can be approximated by the long-term problem (5)-(7), which in turn reduces to our two-stage model. For that we need the following well-known result (see, e.g., Ross 1995).

Lemma A.1 *Suppose the Markov process $\{X_t\}$ is positive recurrent. Then there exists a limiting distribution of this process, i.e.,* $\lim_{t\to\infty} \Pr\{X_t \geqslant x | X_0 = x_0\} = \Phi(x)$ *is well-defined.*

We denote $\phi(\cdot)$ to be the density corresponding to $\Phi(x)$. Under a stationary policy $Q_t = Q(X)$, we can show that the almost sure constraint (6) reduces to the probability constraint (12).

LEMMA A.2 *Under a stationary policy $Q_t(x) = Q(x)$ for each x, we have*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{Q(X_t)\geqslant D(X_t)\}} = \Pr\{Q(X) > D(X)\}. \tag{39}$$

PROOF. To evaluate the right hand side, we consider a sample path $\{x_1, \cdots, x_t\}$. On the sample path, we can compute the probability measure $p_T(x)$ of $x_s = x$. Thus, we have

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{Q(x_t)\geqslant D(x_t)\}} = \frac{1}{T} \int_x \mathbb{1}_{\{Q(x)\geqslant D(x)\}} p_T(x) \mathrm{d}x.$$

By Lemma A.1, $p_T(x) \to \phi(x)$ as $T \to \infty$. Thus, we must have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{Q(x_t)\geqslant D(x_t)\}} = \int_x \mathbb{E}\mathbb{1}_{\{Q(x)\geqslant D(x)\}} \phi(x) \mathrm{d}x = \mathbb{E}\mathbb{1}_{\{Q(X)\geqslant D(X)\}}.$$

Hence, the result follows. □

Then, we can approximate the problem (2)-(4) by the long-run average profit maximization problem (5)-(7), which can be solved as the two-stage problem defined in (10)-(12). The above result ensures that the service constraint in the two-stage model represents the long-term service requirement.

Next, we demonstrate the performance of our two-stage approach compared to the dynamic programming approach. For illustration, we consider a problem involving two signals, i.e., $X_t \in \{1, 2\}$. The transition of the Markov state is given by

$$p_{i,j} = \begin{cases} 0.2 & \text{if } j = i, \\ 0.8 & \text{if } j = 3 - i. \end{cases}$$

One can easily compute the limiting probability of $\{X_t, t = 1, 2, \cdots\}$ to be given by $\Pr\{X = 1\} = \Pr\{X = 2\} = 0.5$.

For the multi-period problem with a given T, the task is to find a sequence of optimal

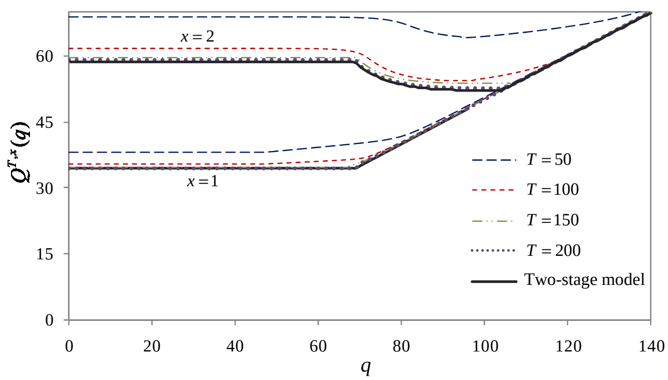
$$\{Q_t(1, n_t, q), Q_t(2, n_t, q)\}_{t\geqslant 1},$$

where q is the daily order commitment and n_t is the number of days that the demand is met during periods $1, \cdots, t$. The computation of this policy is very time consuming since it involves three states (x, n_t, t) for each given q. Moreover, the state space increases with the planning cycle. We assume that the problem starts with $n_1 = 0$, and the initial signal can be 1 or 2 with equal probabilities. Define

$$Q^{T,x}(q) = \mathbb{E} \sum_{t=1}^{T} Q_t(x, n_t, q)/T.$$

Then $Q^{T,x}(q)$ represents the expected daily ordering quantity for signal x over the planning cycle. In Figure 4, we compare $Q^{T,x}(q)$, $x = 1, 2$, for different values of T. For consistency, we use $Q^{\infty,x}(q) \equiv \max\{q, \bar{Q}(x, \lambda(q))\}$ to denote the expected daily ordering quantity for signal x in the problem (5)-(7) or the two-stage model (10)-(12).

We first observe that the trend of the curve $Q^{T,x}(q)$ is fairly consistent with that of $Q^{\infty,x}(q)$. Under the optimal solution, $Q^{T,1}(q)$ and $Q^{T,2}(q)$ tends to offset each other in response to the change of q. For a large enough T, $Q^{T,1}(q)$ is constant initially and then increasing in q, and $Q^{T,2}$ is constant initially and then decreasing in q. The discrepancy between $Q^{T,x}(q)$ and $Q^{\infty,x}(q)$ becomes smaller as T increases. These observations suggests that our two-stage model captures the key features of the multi-period model.



$c_1 = 2$, $c_2 = 3$, $h = 1$, $p = 6$, $\varepsilon = 0.2$. $F(\cdot|x)$ is normal with mean μ_x and standard deviation σ_x. $\mu_1 = 50$, $\sigma_1 = 15$, $\mu_2 = 100$, $\sigma_2 = 30$, and $\Pr\{X = 1\} = \Pr\{X = 2\} = 0.5$.

Figure 4: The expected order quantities versus the daily order commitment.

We should remark that although we provide an example involving a single-dimensional process $\{X_t\}$. In general, the signal process $\{X_t\}$ can also be multi-dimensional to incorporate a variety of information available for demand updating. In particular, X_t can include the weather forecast as well as the developing news on the previous day in our vending rack application.

A.1 The Fill Rate Targets

In this section, we derive the long-term fill rate constraints. Lemma 3 derives the aggregate fill rate constraint (19) and Lemma 4 shows the average fill rate constraint (23).

Lemma 3 *(Aggregate Fill Rate)*

$$\mathrm{FR}^{\mathrm{AG}}(Q(X)) = \frac{\mathbb{E}\min\{Q(X), D(X)\}}{\mathbb{E}D(X)}. \tag{40}$$

Proof. We note that the Markov chain $\mathfrak{X}$ has the limiting density $\phi(\cdot)$. Consider a sequence of independent and identically distributed random variables $\{X_1', X_2', \cdots\}$ with X_t', $t \geqslant 1$, each having the density $\phi(\cdot)$. Then, it is easy to see that

$$\begin{aligned}\mathrm{FR}^{\mathrm{AG}}(Q(X)) = \quad & \lim_{T\to\infty} \mathbb{E}\frac{\sum_{t=1}^{T}\min\{Q(X_t), D(X_t)\}}{\sum_{t=1}^{T} D(X_t)} = \lim_{T\to\infty} \mathbb{E}\frac{\sum_{t=1}^{T}\min\{Q(X_t'), D(X_t')\}}{\sum_{t=1}^{T} D(X_t')} \\ &= \lim_{T\to\infty} \mathbb{E}\frac{\frac{1}{T}\sum_{t=1}^{T}\min\{Q(X_t'), D(X_t')\}}{\frac{1}{T}\sum_{t=1}^{T} D(X_t')} = \frac{\mathbb{E}\min\{Q(X), D(X)\}}{\mathbb{E}D(X)}.\end{aligned}$$

To see the first equation, we note that $\Pr\{X_T < x\} = \Pr\{X_T' < x\}$ for each x as $T \to \infty$. The third equation follows from the application of the renewal theory in the same way as in Sobel (2004).

□

Lemma 4 *(Average Fill Rate)*

$$\mathrm{FR}^{\mathrm{s}}(x, Q) = \frac{\mathbb{E}\min\{Q, D(x)\}}{\mathbb{E}D(x)}$$

and

$$\mathrm{FR}^{\mathrm{AV}}(Q(X)) = \mathbb{E}_X\frac{\mathbb{E}\min\{Q, D(X)\}}{\mathbb{E}D(X)}.$$

Proof. Since the Markov Chain $\mathbf{X}$ is positive recurrent, there is, on each sample path, a subsequence of periods $\{T_1, T_2, \cdots\}$ with $T_i < \infty$ *a.e.*, for which the signal equals some state x_0. Note that $D(X_{T_k})$, $k = 1, 2, \cdots$, is a sequence of i.i.d. random variables. Based on the result of Sobel (2004), we have

$$\mathrm{FR}^{\mathrm{s}}(x_0, Q) = \lim_{T\to\infty} \mathbb{E}\frac{\sum_{t\in[1,T]:X_t=x_0}\min\{Q, D(X_t)\}}{\sum_{t\in[1,T]:X_t=x_0}\mathbb{E}D(X_t)} = \frac{\mathbb{E}\min\{Q, D(x_0)\}}{\mathbb{E}D(x_0)}.$$

Taking the expectation on both sides with respect to x_0 gives the desired result. □

B. Remarks

Remark B.1 For the problem defined in (11)-(12), the feasible set defined by the service constraint may not be convex, in which case an optimal solution may not exist. A standard approach is to discretize the action space and the state space, and use a randomized strategy (e.g., Beutler and Ross 1985 and Feinberg 1994). Upon discretization, let $\mathcal{A} = \{q^1, ..., q^n\}$ be the action space and $\mathcal{X} = \{x^1, ..., x^m\}$ be the sample space of X. Then the problem becomes one of finding the probabilities $a^{i,j} \geqslant 0$, $i = 1, ..., n$ and $j = 1, ..., m$, by solving the following linear programming problem:

$$\min_{a^{i,j}} \quad \sum_{i=1}^{n}\left[c_2 q^i \sum_{j=1}^{m} a^{i,j} + \sum_{j=1}^{m}\left[\int_0^\infty L(q^i - y) f(y|x^j)dy\right]\phi(x^j)a^{i,j}\right],$$

$$\text{s.t.} \quad \sum_{i=1}^{n}\sum_{j=1}^{m} F(q^i|x^j)\phi(x^j)a^{i,j} = 1-\varepsilon,$$
$$\sum_{i=1}^{n} a^{i,j} = 1, \quad j = 1, ..., m,$$

where $L(x) = p\max(-x,0) + h(x,0)$. We have transformed the maximization problem into the above minimization problem in view of the relation (1). Since there are $(m+1)$ constraints, the optimal basic solution of the linear program can have at most $(m+1)$ nonzero solutions. Note also that for each state x^j, $j = 1, ..., m$, there should be at least one positive $a^{i,j}$. Thus, a *limited randomization* property can be established. That is, under an optimal randomized strategy, there is no more than one state $x^r \in \mathcal{X}$ at which actions are randomized, and each of the remaining states will be assigned a single action having probability 1. Moreover, the randomization at the state x^r takes no more than two actions in $\mathcal{A}$. We are indebted to Pete Veinott for his comment in this connection.

Remark B.2 It is easy to show that equation (17) reduces to

$$p - c_1 \quad = \quad (h+p)\left[(1-\varepsilon) \vee \frac{p-c_1}{h+p}\right] - \lambda(\hat{q})\int f\left(x, \hat{Q}_{\hat{q}}(x)\right)\mathrm{d}x. \tag{41}$$

Thus, we can compute the optimal order quantities using the following simple procedure:

0. Check whether $f(y|x)/F(y|x)$ is decreasing in y. If yes, go to step 1. Otherwise, use the randomized approach described in Remark B.1.

1. If $(p-c_1)/(h+p) \geqslant 1-\varepsilon$, then $\lambda = 0$ and $\bar{Q}(x,\lambda) = \bar{Q}(x,0)$. Solving (41) with $\lambda = 0$ gives q^0. The optimal first-stage order is $\hat{q} = q^0$.

2. If $(p-c_1)/(h+p) < 1-\varepsilon$, then $\bar{Q}(x,\lambda)$ is solved from (13) and $\lambda(q)$ is obtained from (14).

2.1 Compute $\bar{q}$ from (16).

2.2 Find the solutions of (41) over $(q^0, \bar{q})$.

2.3 If $f(\cdot|x)$ is decreasing, $\hat{q}$ is the unique solution in step 2.2. Otherwise, if multiple solutions are obtained in step 2.2, then evaluate $J(q)$ for each solution and choose the one that gives the largest value of $J(q)$.

C. Additional Proofs

Lemma C.1 *Under the assumption that $\partial f(y|x)/\partial y \leqslant 0$, the profit function $J(q)$ is concave.*

PROOF. Assume $\theta \in [0,1]$ and let $\bar{\theta} = 1-\theta$. For the initial orders q^1 and q^2, the corresponding optimal total order quantities for a given x are $\hat{Q}_{q^1}(x)$ and $\hat{Q}_{q^2}(x)$, respectively. Then,

$$\int F(\hat{Q}_{q^i}(x)|x)\phi(x)\mathrm{d}x \geqslant 1-\varepsilon, \quad i=1,2.$$

Note that $\partial^2 F(Q|x)/\partial Q^2 = \partial f(Q|x)/\partial Q \leqslant 0$, so that $F(Q|x)$ is concave in Q. Thus,

$$\int F\left(\theta\hat{Q}_{q^1}(x)+\bar{\theta}\hat{Q}_{q^2}(x)\middle| x\right)\phi(x)dx \geqslant \int\left(\theta F(\hat{Q}_{q^1}(x)|x)+\bar{\theta}F(\hat{Q}_{q^2}(x)|x)\right)\phi(x)\mathrm{d}x \geqslant 1-\varepsilon.$$

Furthermore, $\hat{Q}_{q^i}(x) \geqslant q^i$, $i=1,2$. Then, $\theta\hat{Q}_{q^1}(x)+\bar{\theta}\hat{Q}_{q^2}(x) \geqslant \theta q^1+\bar{\theta}q^2$. So, if the first-stage order quantity is $\theta q^1+\bar{\theta}q^2$, then clearly $\theta\hat{Q}_{q^1}(x)+\bar{\theta}\hat{Q}_{q^2}(x)$ is a feasible total order quantity. Thus, we have

$$\begin{aligned}
K(\theta q^1+\bar{\theta}q^2) &\geqslant p\mathbb{E}D - c_2\int\left(\theta\hat{Q}_{q^1}(x)+\bar{\theta}\hat{Q}_{q^2}(x)\right)\phi(x)\mathrm{d}x \\
&\quad -\iint L\left(\theta\hat{Q}_{q^1}(x)+\bar{\theta}\hat{Q}_{q^2}(x)-y\right)f(y|x)\phi(x)\mathrm{d}y\mathrm{d}x \\
&\geqslant p\mathbb{E}D-\theta\left[\int c_2\hat{Q}_{q^1}(x)\phi(x)\mathrm{d}x - \iint L\left(\hat{Q}_{q^1}(x)-y\right)f(y|x)\phi(x)\mathrm{d}xdy\right] \\
&\quad -\bar{\theta}\left[\int c_2\hat{Q}_{q^2}(x)\phi(x)\mathrm{d}x + \iint L\left(\hat{Q}_{q^2}(x)-y\right)f(y|x)\phi(x)\mathrm{d}y\mathrm{d}x\right] \\
&= \theta K(q^1)+\bar{\theta}K(q^2).
\end{aligned}$$

□

OPTIMALITY CONDITIONS UNDER AN AGGREGATE FILL RATE TARGET

We form the Lagrangian for the problem (10)-(11) and (19) as

$$\mathcal{L}(Q(X),\lambda) = \int \Pi(q,Q(x),x)\phi(x)\mathrm{d}x + \lambda\left[\int\left(\frac{\int_0^{Q(x)} yf(y|x)\mathrm{d}x}{\mu}+\frac{Q(x)}{\mu}[1-F(Q(x)|x)]\right)\phi(x)\mathrm{d}x - 1+\varepsilon\right].$$

Differentiating the Lagrangian with respect to Q, we deduce that for each x,

$$c_2 - p + (h+p)F(Q(x)|x) - \frac{\lambda}{\mu}[1-F(Q(x)|x)] = 0,$$

which gives (20). The Kuhn-Tucker necessary conditions are

$$\begin{aligned}
&Q(x)\geqslant q, \quad c_2-p+\lambda/\mu+(h+p+\lambda/\mu)F(Q(x)|x)\geqslant 0,\\
&(Q(x)-q)[c_2-p+\lambda/\mu+(h+p+\lambda/\mu)F(Q(x)|x)] = 0.
\end{aligned} \tag{42}$$

The key to the solution of the second-stage problem is to find a value of λ such that

$$\int\left(\frac{\int_0^{Q(x)} yf(y|x)\mathrm{d}x}{\mu}+\frac{Q(x)}{\mu}[1-F(Q(x)|x)]\right)\phi(x)\mathrm{d}x \geqslant 1-\varepsilon. \tag{43}$$

Note that by the complementary slackness condition for the constraint (43), we must have $\lambda=0$ when the constraint (43) is not binding and $\lambda>0$ when it is binding.

Proof of Theorem 3. The proof is straightforward and the details are omitted. □

Proof of Theorem 4. It is straightforward to show that the profit function $J(q)$ is concave. Note that $\lambda(q)$ is a solution of

$$\frac{\int \left[\hat{Q}_q(x) - \int_0^{\hat{Q}_q(x)} F(y|x)dy\right]\phi(x)\mathrm{d}x}{\mu} = 1 - \varepsilon.$$

By differentiating both sides with respect to q, we obtain

$$\frac{d\lambda(q)}{dq} = \frac{-\int_{\{\bar{Q}(x,\lambda(q))<q\}} [1 - f(q|x)]\phi(x)\mathrm{d}x}{\int_{\{q<\bar{Q}(x,\lambda(q))\}} [\phi(x) - f(\bar{Q}(x,\lambda(q))|x)\phi(x)] \frac{\partial \bar{Q}(x,\lambda(q))}{\partial \lambda}\mathrm{d}x}. \tag{44}$$

We can then compute dJ/dq to obtain the first-order condition. The details are omitted here. □

Optimality Conditions under an Average Fill Rate Target

We form the Lagrangian for the problem (10)-(11) and (23) as

$$\mathcal{L}(Q(X),\lambda) = \int \Pi(q, Q(x), x)\phi(x)\mathrm{d}x + \lambda\left[\int\left(\frac{\int_0^{Q(x)} yf(y|x)\mathrm{d}x}{\mu_x} + \frac{Q(x)}{\mu_x}[1 - F(Q(x)|x)]\right)\phi(x)\mathrm{d}x - 1 + \varepsilon\right].$$

Differentiating the Lagrangian with respect to Q, we deduce that for each x,

$$c_2 - p + (h+p)F(Q(x)|x) - \frac{\lambda}{\mu_x}[1 - F(Q(x)|x)] = 0,$$

which gives (24). The Kuhn-Tucker necessary conditions are

$$\begin{aligned} &Q(x) \geqslant q, \quad c_2 - p + \lambda/\mu_x + (h + p + \lambda/\mu_x)F(Q(x)|x) \geqslant 0, \\ &(Q(x) - q)[c_2 - p + \lambda/\mu_x + (h + p + \lambda/\mu_x)F(Q(x)|x)] = 0. \end{aligned} \tag{45}$$

The key to the solution of the second-stage problem is to find a value of λ such that

$$\int\left(\frac{\int_0^{Q(x)} yf(y|x)\mathrm{d}x}{\mu_x} + \frac{Q(x)}{\mu_x}[1 - F(Q(x)|x)]\right)\phi(x)\mathrm{d}x \geqslant 1 - \varepsilon. \tag{46}$$

Note that by the complementary slackness condition for the constraint (46), we must have $\lambda = 0$ when the constraint (46) is not binding and $\lambda > 0$ when it is binding.

Proof of Theorem 5. It can be shown that the cost function $J(q)$ is concave. Note that $\lambda(q)$ is a solution of

$$\int \frac{\hat{Q}_q(x) - \int_0^{\hat{Q}_q(x)} F(y|x)dy}{\mu_x}\phi(x)\mathrm{d}x = 1 - \varepsilon.$$

Differentiating both sides with respect to q gives

$$\frac{d\lambda(q)}{dq}=\frac{-\int_{\{\bar{Q}(x,\lambda(q))<q\}}[\phi(x)-f(q|x)\phi(x)]/\mu_x\mathrm{d}x}{\int_{\{q<\bar{Q}(x,\lambda(q))\}}[\phi(x)-f(\bar{Q}(x,\lambda(q))|x)\phi(x)]/\mu_x\frac{\partial\bar{Q}(x,\lambda(q))}{\partial\lambda}\mathrm{d}x}. \tag{47}$$

We can then compute dJ/dq to obtain the first-order condition. The details are omitted here. □

Theorem 10 *When the demand distribution is given by* (28)*, the periodic fill rates* $\mathrm{FR}^{\mathrm{s}}(x,\bar{Q}(x,\lambda(q)))$ *is decreasing in x for any given q when the following conditions are satisfied:*

(i) *Under a long-term in-stock rate target, G has a decreasing reversed hazard rate and σ_x is increasing in x and σ_x/μ_x is decreasing in x.*

(ii) *Under an aggregate fill rate target, σ_x/μ_x is decreasing in x.*

(iii) *Under an average fill rate target, μ_x is increasing in x and σ_x/μ_x is decreasing in x.*

Proof. We denote $z_x=(\bar{Q}(x,\lambda(q))-\mu_x)/\sigma_x)$, for any given q. Under a long-term in-stock rate target, $G(z_x)$ is decreasing in x since σ_x is increasing in x. The corresponding periodic in-stock rate is given by

$$\mathrm{FR}^{\mathrm{s}}(x,\bar{Q}(x,\lambda(q)))=\int_0^{\bar{Q}(x,\lambda(q))}(1-F(y|x))\mathrm{d}y=\int_0^{z_x}\frac{\sigma_x}{\mu_x}(1-G(t))\mathrm{d}t. \tag{48}$$

The result follows from the fact that σ_x/μ_x is decreasing in x.

Under an aggregate fill rate target, z_x is constant in x in (48). Hence, the result follows from the above argument.

Under an average fill rate target, $G(z_x)$ is decreasing in x since μ_x is decreasing in x. The results follow from observing that the right-hand side of (48) is decreasing in x. □

References

Beutler, F. J. and K. W. Ross. 1985. Optimal Policies for Controlled Markov Chains with a Constraint. *J. Math. Analysis & Appl.* **112** 236-252.

Feinberg, E. A. 1994. Coonstrained Semi-Markov Decision Processes with Average Rewards. *ZOR Math. Methods of Opns. Res.* **39** 257-288.

Ross, S. 1995. *Stochastic Process.* 2nd ed. John Wiley & Sons.