Online Appendix to
Capacity Allocation over a Long Horizon: The Return on Turn-and-Earn

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I. Proofs

Proposition 1 Proof. We develop the proof through three lemmas.

Lemma 1 *Suppose that* $\mathbf{q}(\mathbf{s},\mathbf{x})$ *is weakly increasing in* $\mathbf{x}$. *For any* $m \in \{0,...,M\}$, *it is optimal for retailer* i *to order* $m\delta$ *if and only if* $\theta_i \geq \underline{\theta}_{im}(\mathbf{s},\alpha)$ *and* $\theta_i \leq \bar{\theta}_{im}(\mathbf{s},\alpha)$, *where*

$$\underline{\theta}_{im}(\mathbf{s},\alpha) = \begin{cases} \underline{\Theta} & \text{if } m = 0 \\ \max\left\{\frac{bm\delta-\alpha}{\varepsilon},\ \max\limits_{m'\in\{0,...,m-1\}} g(m,m',\mathbf{s},\alpha)\right\} & \text{if } m = 1,...,M \end{cases}$$

and

$$\bar{\theta}_{im}(\mathbf{s},\alpha) = \begin{cases} \bar{\Theta} & \text{if } m = M \\ \min\limits_{m'\in\{m+1,...,M\}}\left\{\max\left\{\frac{bm'\delta-\alpha}{\varepsilon},\ g(m,m',\mathbf{s},\alpha)\right\}\right\} & \text{if } m = 0,...,M-1 \end{cases}$$

The expression for $g(\cdot)$ *is given by Eq.(7) and (8) below. Accordingly his optimal order distribution* $\boldsymbol{\sigma}_i(\mathbf{s},\alpha)$ *is given by*

$$\sigma_{im}(\mathbf{s},\alpha) = \int \mathbf{1}[\underline{\theta}_{im}(\mathbf{s},\alpha) \leq \theta_i \leq \bar{\theta}_{im}(\mathbf{s},\alpha)]dF(\theta_i),\ m = 0,...,M.$$

Proof. Consider the optimization problem of retailer i:

$$V_i(\mathbf{s},\alpha,\theta_i) = \max_{x_i(s,\alpha,\theta_i)\in\{0\delta,...,\bar{m}\delta\}} \sum_{l=0}^{M} \sigma_{jl}(\mathbf{s},\alpha)\left\{\pi_i(\mathbf{s},\alpha,\theta_i,x_i,l\delta) + \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},x_i,l\delta),\alpha'\right)\right\}$$

where

$$\bar{m} \equiv \max\{m \in \mathbb{Z}|\alpha + \varepsilon\theta_i - bm\delta \geq 0\}.$$

For $m = 1, ..., M-1$, $x_i^* = m\delta$ if and only if $m \leq \bar{m}$ and

$$\begin{aligned} & \sum_{l=0}^{M} \sigma_{jl}(\mathbf{s},\alpha)\left\{\pi_i(\mathbf{s},\alpha,\theta_i,m\delta,l\delta) + \beta \sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i(\mathbf{q}(\mathbf{s},m\delta,l\delta),\alpha')\right\} \\ \geq & \max_{x_i\in\{0\delta,...,M\delta\}} \sum_{l=0}^{M} \sigma_{jl}(\mathbf{s},\alpha)\left\{\pi_i(\mathbf{s},\alpha,\theta_i,x_i,l\delta) + \beta \sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i(\mathbf{q}(\mathbf{s},x_i,l\delta),\alpha')\right\}. \end{aligned}$$

The boundary cases, i.e., $m = 0$ and $m = M$ can be easily derived following the same logic and is omitted for brevity. The second inequality is equivalent to for any $m' \neq m$, $m' \in \{0, ..., m-1, m+1, ..., M\}$,

$$\theta_i \varepsilon B(m, m', \mathbf{s}, \alpha) \geq -\Delta(m, m', \mathbf{s}, \alpha). \tag{6}$$

where

$$\begin{aligned} \Delta(m,m',\mathbf{s},\alpha) &= \sum_{l=0}^{M} \sigma_{jl}(\mathbf{s},\alpha)\left\{\pi_i(\mathbf{s},\alpha,0,m\delta,l\delta) + \beta \sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},m\delta,l\delta),\alpha'\right)\right. \\ &\quad \left. -\pi_i(\mathbf{s},\alpha,0,m'\delta,l\delta) - \beta \sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},m'\delta,l\delta),\alpha'\right)\right\}, \\ B(m,m',\mathbf{s},\alpha) &= \sum_{l=0}^{M} \sigma_{jl}(\mathbf{s},\alpha)(q_i(\mathbf{s},m\delta,l\delta) - q_i(\mathbf{s},m'\delta,l\delta)). \end{aligned}$$

$\Delta(\cdot)$ represents the payoff difference between two arbitrary order quantities, say $m\delta$ and $m'\delta$, when retailer i's private demand shock is set to zero. $B(\cdot)$ represents the difference in the expected allocations of ordering m and m' batches. Next, we characterize the threshold function denoted by $g(\cdot)$. Consider two cases:

Case 1. If $B(m, m', \mathbf{s}, \alpha) \neq 0$, then

$$g(m, m', \mathbf{s}, \alpha) = -\frac{\Delta(m, m', \mathbf{s}, \alpha)}{\varepsilon B(m, m', \mathbf{s}, \alpha)}. \tag{7}$$

Because $q_i(\mathbf{s},x_i,x_j)$ is weakly increasing in x_i. Therefore when $m > m'$, we have $q_i(m\delta, l\delta, \mathbf{s}) \geq q_i(m'\delta, l\delta, \mathbf{s})$. Thus, Eq.(6) is equivalent to $\theta_i \geq g(m, m', \mathbf{s}, \alpha)$. When $m < m'$, we have $q_i(\mathbf{s},m\delta, x_j) \leq q_i(\mathbf{s},m'\delta, x_j)$. Thus, Eq.(6) is equivalent to $\theta_i \leq g(m, m', \mathbf{s}, \alpha)$. Therefore, $g(\cdot)$ represents the threshold of local demand state above (below) which ordering m batches is better than ordering m' batches when m is larger (smaller) than m'.

Case 2. If $B(m, m', \mathbf{s}, \alpha) = 0$, then

$$g(m, m', \mathbf{s}, \alpha) = \begin{cases} \underline{\Theta} & \text{if } m > m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) > 0, \\ \bar{\Theta} & \text{if } m < m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) > 0, \\ \bar{\Theta} & \text{if } m > m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) < 0, \\ \underline{\Theta} & \text{if } m < m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) < 0, \\ \bar{\Theta} & \text{if } m > m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) = 0, \\ \bar{\Theta} & \text{if } m < m' \text{ and } \Delta(m, m', \mathbf{s}, \alpha) = 0. \end{cases} \tag{8}$$

When $\Delta(m, m', \mathbf{s}, \alpha) > 0$, it is easy to see that ordering m batches is always better than ordering m' batches over the entire support of θ_i. Therefore, when $m > m'$(or $m < m'$), we set $g(m, m', \mathbf{s}, \alpha) = \underline{\Theta}$ (or $g(m, m', \mathbf{s}, \alpha) = \bar{\Theta}$) so that above (or below) this threshold, ordering m batches always dominates ordering m' batches. Conversely, if $\Delta(m, m', \mathbf{s}, \alpha) < 0$, ordering m' batches is always better than ordering m batches over the entire support of θ_i and we set $g(m, m', \mathbf{s}, \alpha)$ accordingly. If however $\Delta(m, m', \mathbf{s}, \alpha) = 0$, the two order quantities give identical expected payoff. Here we need to apply the assumption that retailer i chooses the lower order quantity.

Combining the thresholds with the constraint $m \leq \bar{m}$, we reach the conditions in Lemma 1. ■

The threshold policy says that there exists a range of θ_i over which it is optimal for retailer i to order m batches. Notice that the thresholds are state dependent. Let

$$\begin{aligned} \underline{\boldsymbol{\theta}}_i(\mathbf{s}, \alpha) &= \{\underline{\theta}_{i0}(\mathbf{s}, \alpha), ..., \underline{\theta}_{iM}(\mathbf{s}, \alpha)\}, \\ \bar{\boldsymbol{\theta}}_i(\mathbf{s}, \alpha) &= \{\bar{\theta}_{i0}(\mathbf{s}, \alpha), ..., \bar{\theta}_{iM}(\mathbf{s}, \alpha)\}. \end{aligned}$$

These two $M + 1$ dimensional vectors completely determine retailer i's optimal order distribution. We now characterize the properties of the thresholds.

Lemma 2 *For any* $m < m'$, $m, m' \in \{0, ..., M\}$,

$$\bar{\theta}_{im}(\mathbf{s}, \alpha) \leq \underline{\theta}_{im'}(\mathbf{s}, \alpha).$$

Proof. By the definition in Lemma 1, $\bar{\theta}_{im}(\mathbf{s}, \alpha) \leq \max\left\{\frac{bm'\delta - \alpha}{\varepsilon}, g(m, m', \mathbf{s}, \alpha)\right\}$ and $\underline{\theta}_{im'}(\mathbf{s}, \alpha) \geq \max\left\{\frac{bm'\delta - \alpha}{\varepsilon}, g(m', m, \mathbf{s}, \alpha)\right\}$. Notice that $g(m, m', \mathbf{s}, \alpha) = g(m', m, \mathbf{s}, \alpha)$ by the definition in Eq. (7) and (8). Therefore, $\bar{\theta}_{im}(\mathbf{s}, \alpha) \leq \underline{\theta}_{im'}(\mathbf{s}, \alpha)$. ■

This monotonicity property of the thresholds implies that when the local demand state is higher, the optimal order quantity increases. The property also ensures that there is no overlapping θ_i regions whenever ordering m and m' batches have positive probability of being optimal (i.e., when $\underline{\Theta} \leq \underline{\theta}_{im} < \bar{\theta}_{im} \leq \bar{\Theta}$ and $\underline{\Theta} \leq \underline{\theta}_{im'} < \bar{\theta}_{im'} \leq \bar{\Theta}$).

Lemma 3 *For any $m < m'$, $m, m' \in \{0, ..., M\}$, if $\underline{\theta}_{im}(\mathbf{s}, \alpha) \leq \bar{\theta}_{im}(\mathbf{s}, \alpha)$ and $\underline{\theta}_{im'}(\mathbf{s}, \alpha) \leq \bar{\theta}_{im'}(\mathbf{s}, \alpha)$, then $\bar{\theta}_{im}(\mathbf{s}, \alpha) = \underline{\theta}_{im'}(\mathbf{s}, \alpha)$ when either of the following conditions is satisfied:*

i) $m' = m + 1$;

ii) $m' > m + 1$ and $\underline{\theta}_{i,m''}(\mathbf{s}, \alpha) > \bar{\theta}_{i,m''}(\mathbf{s}, \alpha)$ for all $m < m'' < m'$ and $m'' \in \{0, ..., M\}$.

Proof. By Lemma 2, we have $\bar{\theta}_{im}(\mathbf{s}, \alpha) \leq \underline{\theta}_{im'}(\mathbf{s}, \alpha)$. Thus we only need to show $\bar{\theta}_{im}(\mathbf{s}, \alpha) < \underline{\theta}_{im'}(\mathbf{s}, \alpha)$ does not hold. We will prove this by contradiction. Suppose the strict inequality holds, then there exists a $\tilde{\theta} \in (\bar{\theta}_{im}(\mathbf{s}, \alpha), \underline{\theta}_{im'}(\mathbf{s}, \alpha))$. For any such $\tilde{\theta}$, the following two statements hold: 1) ordering m batches is better than ordering z batches for any $z = 0, ..., m-1$ (because $\tilde{\theta} > \bar{\theta}_{im}(\mathbf{s}, \alpha) \geq \underline{\theta}_{im}(\mathbf{s}, \alpha)$); 2) ordering m' batches is better than ordering z batches for any $z = m+1, ..., M$ (because $\tilde{\theta} < \underline{\theta}_{im'}(\mathbf{s}, \alpha) \leq \bar{\theta}_{im'}(\mathbf{s}, \alpha)$). Moreover, because i) $m' = m + 1$ or ii) $\underline{\theta}_{im''}(\mathbf{s}, \alpha) > \bar{\theta}_{im''}(\mathbf{s}, \alpha)$ for all $m < m'' < m'$, the optimal order quantity for any $\tilde{\theta} \in (\bar{\theta}_{im}(\mathbf{s}, \alpha), \underline{\theta}_{im'}(\mathbf{s}, \alpha))$ should be from set $\{m\delta, m'\delta\}$. This contradicts the fact that $\tilde{\theta} \notin \left[\underline{\theta}_{im}(\mathbf{s}, \alpha), \bar{\theta}_{im}(\mathbf{s}, \alpha)\right]$ and $\tilde{\theta} \notin \left[\underline{\theta}_{im'}(\mathbf{s}, \alpha), \bar{\theta}_{im'}(\mathbf{s}, \alpha)\right]$, which are the necessary conditions for $m\delta$ and $m'\delta$ to be optimal, respectively, according to Lemma 1. ■

This lemma ensures that the optimal regions of θ_i for the order quantities are connected and ordered increasingly. With the properties stated in Lemma 2 and 3, we can eliminate duplicated thresholds and simplify the threshold policy as: for any $m \in \{0, ..., M\}$, it is optimal for retailer i to order $m\delta$ if and only if $\eta_{im}(\mathbf{s}, \alpha) \leq \theta_i \leq \eta_{i,m+1}(\mathbf{s}, \alpha)$, where $\eta_{im}(\mathbf{s}, \alpha)$ can be found iteratively as follows:

$$\eta_{im}(\mathbf{s}, \alpha) = \begin{cases} \underline{\Theta} & \text{if } m = 0 \\ \min\left\{\bar{\Theta},\ \ \max\left\{\eta_{i,m-1}(\mathbf{s}, \alpha), \bar{\theta}_{i,m-1}(\mathbf{s}, \alpha)\right\}\right\} & \text{if } m = 1, ..., M \\ \bar{\Theta} & \text{if } m = M + 1 \end{cases}$$

Note that the thresholds derived from Lemma 1 might be outside $[\underline{\Theta}, \bar{\Theta}]$, it is thus necessary to set $\eta_{im}(\mathbf{s}, \alpha) \in [\underline{\Theta}, \bar{\Theta}]$. In addition, $\underline{\theta}_{im}(\mathbf{s}, \alpha) > \bar{\theta}_{im}(\mathbf{s}, \alpha)$ may arise and it is nonetheless without loss of information to set $\eta_{im}(\mathbf{s}, \alpha) = \eta_{i,m+1}(\mathbf{s}, \alpha)$ because $\sigma_{im}(\mathbf{s}, \alpha) = 0$. By definition, $\eta_{im}(\mathbf{s}, \alpha) \in [\underline{\Theta}, \bar{\Theta}]$ and $\eta_{im}(\mathbf{s}, \alpha) \leq \eta_{i,m+1}(\mathbf{s}, \alpha)$, $m = 0, ..., M$. This completes the proof for Proposition 1. ■

Proposition 2 Proof. Because no intertemporal dependence exists that affects the retailers' order quantities under fixed allocation, it is optimal for the retailers to maximize their per-period profits. With no capacity constraint, retailer i's per-period profit function is concave and is maximized at $q^M(\alpha,\theta)$. With capacity constraint, it is optimal for retailer i to sell as much as possible up to $q^M(\alpha,\theta)$. Hence, his optimal order quantity is the minimum of $q^M(\alpha,\theta)$ and his maximum allocation. Retailer i's maximum allocation equals the sum of his guaranteed allocation K/N and the maximum possible capacity unclaimed by all other retailers, which occurs when they experience the lowest local demand at $\underline{\Theta}$. ■

N retailers. For the case of N retailers, we modify the proof of Proposition 1. All three lemmas needed to prove the proposition still hold. The proofs of Lemma 2 and 3 are the same as in the case of two retailers. For Lemma 1, the proof is modified as follows. We start by writing the Bellman equation of retailer i:

$$\begin{aligned} V_i(\mathbf{s},\alpha,\theta_i) &= \max_{x_i(s,\alpha,\theta_i)\in\{0\delta,\ldots,\bar{m}\delta\}} \sum_{\mathbf{x}_{-i}\in\{0\delta,\ldots,M\delta\}^{N-1}} \Pr(\mathbf{x}_{-i}) \\ &\quad \cdot\left\{\pi_i(\mathbf{s},\alpha,\theta_i,x_i,\mathbf{x}_{-i}) + \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},x_i,\mathbf{x}_{-i}),\alpha'\right)\right\}. \end{aligned}$$

where $\bar{m}\equiv\max\{m\in\mathbb{Z}|\alpha+\varepsilon\theta_i-bm\delta\geq 0\}$ and

$$\Pr(\mathbf{x}_{-i}) = \prod_{j\neq i}\prod_{l=0}^{M}\sigma_{jl}(\mathbf{s},\alpha)^{\mathbf{1}[x_j=l\delta]}.$$

For $m=1,\ldots,M-1$, $x_i^*=m\delta$ if and only if $m\leq\bar{m}$ and

$$\begin{aligned} &\sum_{\mathbf{x}_{-i}\in\{0\delta,\ldots,M\delta\}^{N-1}} \Pr(\mathbf{x}_{-i})\left\{\pi_i(\mathbf{s},\alpha,\theta_i,m\delta,\mathbf{x}_{-i}) + \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i(\mathbf{q}(\mathbf{s},m\delta,\mathbf{x}_{-i}),\alpha')\right\} \\ \geq\ &\max_{x_i\in\{0\delta,\ldots,M\delta\}} \sum_{\mathbf{x}_{-i}\in\{0\delta,\ldots,M\delta\}^{N-1}} \Pr(\mathbf{x}_{-i})\left\{\pi_i(\mathbf{s},\alpha,\theta_i,x_i,\mathbf{x}_{-i}) + \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i(\mathbf{q}(\mathbf{s},x_i,\mathbf{x}_{-i}),\alpha')\right\}. \end{aligned}$$

The boundary cases, i.e., $m=0$ and $m=M$ can be easily derived following the same logic and is omitted for brevity. The second inequality is equivalent to for any $m'\neq m$, $m'\in\{0,\ldots,m-1,m+1,\ldots,M\}$,

$$\theta_i\varepsilon B(m,m',\mathbf{s},\alpha)\geq -\Delta(m,m',\mathbf{s},\alpha).$$

where

$$\begin{aligned} \Delta(m,m',\mathbf{s},\alpha) &= \sum_{\mathbf{x}_{-i}\in\{0\delta,\ldots,M\delta\}^{N-1}} \Pr(\mathbf{x}_{-i})\left\{\pi_i(\mathbf{s},\alpha,0,m\delta,\mathbf{x}_{-i}) + \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},m\delta,\mathbf{x}_{-i}),\alpha'\right)\right. \\ &\qquad \left. -\pi_i(\mathbf{s},\alpha,0,m'\delta,\mathbf{x}_{-i}) - \beta\sum_{\alpha'\in\mathcal{A}} P(\alpha'|\alpha)V_i\left(\mathbf{q}(\mathbf{s},m'\delta,\mathbf{x}_{-i}),\alpha'\right)\right\}, \\ B(m,m',\mathbf{s},\alpha) &= \sum_{\mathbf{x}_{-i}\in\{0\delta,\ldots,M\delta\}^{N-1}} \Pr(\mathbf{x}_{-i})(q_i(\mathbf{s},m\delta,\mathbf{x}_{-i}) - q_i(\mathbf{s},m'\delta,\mathbf{x}_{-i})). \end{aligned}$$

The rest of the proof is the same as in the case of two retailers. ■

II. Approximation Logic for Fractional Allocations

We use a two-retailer example to illustrate the logic for approximating fractional number of batches using nearby integer number of batches. The logic can be extended to the N-retailer setting and is omitted for brevity. Let $m_i \equiv q_i/\delta$ and let $\lceil\cdot\rceil$ ($\lfloor\cdot\rfloor$) denote the operator to take the nearest integer larger (smaller) than the argument. It is straightforward to show that fractional number of batches only occur when capacity is lower than the sum of orders. Therefore, when $m_i \neq \lceil m_i \rceil$, we must have $\sum_{i=1}^{N} \lceil m_i \rceil = M+1$ and $m_1 - \lfloor m_1 \rfloor + m_2 - \lfloor m_2 \rfloor = 1$. The allocations can be approximated using the following random variable:

$$\mathbf{q} = \begin{cases} (\lceil m_1 \rceil \delta, \lfloor m_2 \rfloor \delta) & \text{w.p. } m_1 - \lfloor m_1 \rfloor, \\ (\lfloor m_1 \rfloor \delta, \lceil m_2 \rceil \delta) & \text{w.p. } m_2 - \lfloor m_2 \rfloor. \end{cases}$$

III. Distribution Function for Beta (3,3)

$$F(\theta) = \begin{cases} 0 & \text{if} \quad \theta < -1, \\ \frac{1}{2} + \frac{15}{16}\theta - \frac{5}{8}\theta^3 + \frac{3}{16}\theta^5 & \text{if} \quad -1 \le \theta < 1, \\ 1 & \text{if} \quad \theta \ge 1. \end{cases}$$

IV. Additional Figures

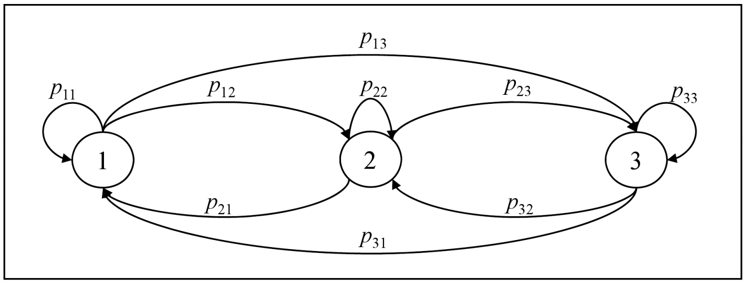

Figure 8: A Three-State Common Demand Process with Transition Probability p_{uv}

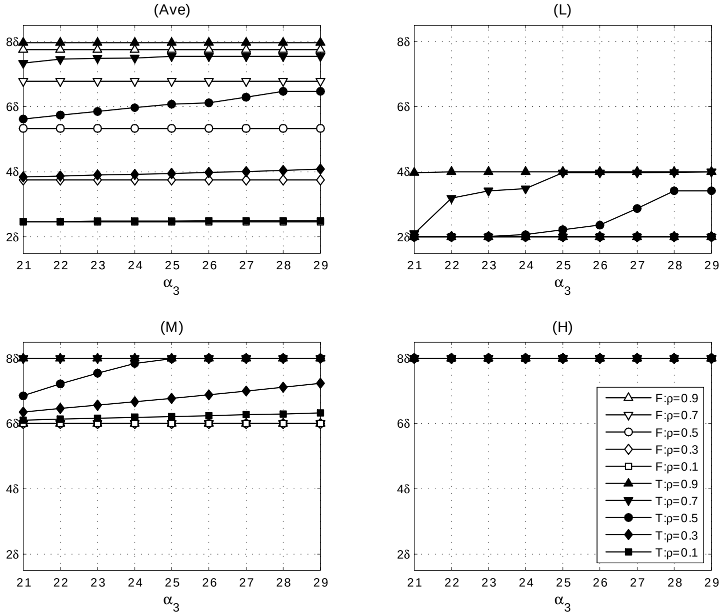

Figure 9: Expected Total Sales as a Function of α_3 Parameterized by ρ Under Fixed (F) and Turn-and-Earn (T) Allocation. $K = 8\delta$, $\mathcal{A} = \{5, 13, \alpha_3\}$, $\varepsilon = 4$.

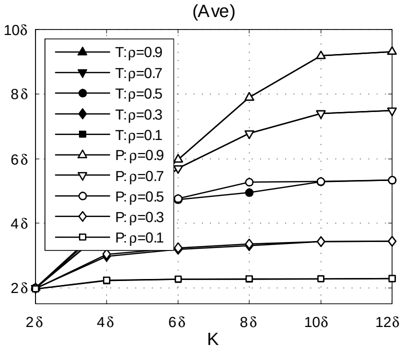

Figure 10: Expected Total Sales as a Function of K Parameterized by ρ Under Turn-and-Earn (T) and Proportional Turn-and-Earn (P) Allocation. $\mathcal{A} = \{5, 13, 21\}$, $\varepsilon = 4$.