Online Appendix
"Good and Bad News about the (S, T) Policy"

Fang Liu
Nanyang Business School, Nanyang Technological University, Singapore, 639798, Liu_Fang@ntu.edu.sg

Jing-Sheng Song
Fuqua School of Business, Duke University, Durham, NC 27708,
and School of Management, Fudan University, Shanghai 200433, China, jssong@duke.edu

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Proof of Lemma 1 We only provide the proofs for Brownian Motion and Poisson demand processes. Other cases can be shown similarly.

1) Suppose $X(t)$ is a Brownian motion. Then the c.d.f. of $X(t)$ is

$$F(x,t) = N\left(\frac{x-\lambda t}{\sigma\sqrt{t}}\right),$$

where N is the c.d.f. of the standard Normal distribution.

For any $x > 0$,

$$V(x) = \lim_{t\to 0^+} F^0(x,t)/t \le \lim_{t\to 0^+} \frac{1}{t}\int_{-\infty}^{\frac{-x}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}}(-y)e^{-\frac{y^2}{2}}dy = \lim_{t\to 0^+} \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\Big|_{-\infty}^{\frac{-x}{\sigma\sqrt{t}}} = 0.$$

The second inequality follows because $\frac{-x}{\sigma\sqrt{t}} < -1$ for sufficiently small t. Note that $V(x) \ge 0$, thus we have $V(x) = 0$.

Similarly, when $x < 0$,

$$V(x) = \lim_{t\to 0^+} -(1 - F^0(x,t))/t = \lim_{t\to 0^+} -N\left(\frac{x-\lambda t}{\sigma\sqrt{t}}\right)/t = 0.$$

When $x = 0$,

$$V(x) = \lim_{t\to 0^+} F(x,t)/t = \lim_{t\to 0^+} N\left(\frac{-\lambda\sqrt{t}}{\sigma}\right)/t = \infty.$$

Note that

$$\int_{-\infty}^{\infty} V(x)dx = \lim_{t\to 0^+} E[X(t)]/t = \lambda,$$

we have $V(x) = \lambda\delta(x)$.

2) Suppose $X(t)$ is a Poisson process. The c.d.f. of $X(t)$ is $F(x,t) = \sum_{i=0}^{x} e^{-\lambda t}(\lambda t)^i/i!$. For any $x > 0$,

$$V(x) = \lim_{t \to 0^+} F^0(x,t)/t = \lim_{t \to 0^+} \sum_{i=x}^{\infty} e^{-\lambda t}\lambda^i t^{i-1}/i! = 0.$$

When $x = 0$,

$$V(x) = \lim_{t \to 0^+} F^0(x,t)/t = \lim_{t \to 0^+} \sum_{i=0}^{\infty} e^{-\lambda t}\lambda^i t^{i-1}/i! = \infty.$$

Note that

$$\int_{-\infty}^{\infty} V(x)dx = \lim_{t \to 0^+} E[X(t)]/t = \lambda,$$

we have $V(x) = \lambda\delta(x)$. □

Proof of Proposition 1 Note that

$$C(S,T) = \frac{K}{T}P(X(T) > 0) + hE[S - Y(T)] + (h+b)E[Y(T) - S]^+$$

consists of three terms. The results of 1) and 2) for the first two terms follow from Rao (2003). Then it is sufficient to show that those results hold for $H(S,T) = E[Y(T) - S]^+$.

1) Since $X(T)$ is SI in T, according to Shaked and Shanthikumar (2006) Theorem 1.A.6, $Y(T_1) \geq_{st} Y(T_2)$ for any $T_1 > T_2 > 0$. Therefore, $Y(T)$ is SI in T. When demand is continuous,

$$\frac{\partial^2}{\partial T \partial S}H(S,T) = \frac{\partial}{\partial T}\Psi(S,T) \leq 0;$$

when demand is discrete,

$$H(S+1,T_1) + H(S,T_2) - H(S+1,T_2) - H(S,T_1) = Pr(Y(T_2) > S) - Pr(Y(T_1) > S) \leq 0.$$

Therefore, $H(S,T)$ is submodular.

2) For any fixed T, $H(S,T)$ is clearly convex in S. The closed form solution follows directly from convexity property.

3) If $X(T)$ is SIL in T, from Theorem 8.A.17. (p. 363) of Shaked and Shanthikumar (2006), $Y(T)$ is SIL in T. Thus, for any fixed S, $H(S,T)$ is convex in T.

Since L only appears in $Y(T)$, the proof is the same for $L > 0$. □

Proof of Proposition 2 For any local minimum (x_0, y_0) of $h(x,y)$, there exists a neighborhood $\mathcal{N}$ such that for any $(x,y) \in \mathcal{N}$ we have $h(x_0,y_0) \leq h(x,y)$. (Here, if $h(x,y)$ is defined on $\mathbb{R}^2$, $\mathcal{N} = B((x_0,y_0),\ \varepsilon)$ is an ε-ball. If $h(x,y)$ is defined on $\mathbb{Z}^2$, $\mathcal{N} = \{(x_0+x, y_0+y),\ x = -1,\ 0,\ 1,\ y = -1,\ 0,\ 1\}$.) Fix $y = y_0$, $h(x,y_0)$ is unimodal in x. Therefore, x_0 is the minimizer of $h(x,y_0)$ and hence $x^*(y_0) \supseteq x_0$. Note that $h(x^*(y_0), y_0)$ is also a local minimum of $h(x^*(y), y)$, and $h(x^*(y), y)$ is unimodal in y. Thus, y_0 is the global minimum of $h(x^*(y), y)$. Note that $\min_{(x,y)}\{h(x, y)\}$ is equivalent to $\min_x\{\min_y\{h(x, y)\}\}$, therefore y_0 being the global minimum is equivalent to (x_0, y_0) being the global minimum. □

Before proving Proposition 3, we first need to establish the following lemma.

Lemma A *For $i = 0, 1$,*

1) $\Psi^i(S,T) = \frac{1}{T}\int_0^T F^i(S,x)dx$;

2) $\psi(S,T) = \frac{1}{T}\int_0^T f(S,x)dx$;

3) $\frac{\partial}{\partial T}F^0(S,T) = \int_{-\infty}^{\infty} f(x,T)V(S-x)dx \equiv (f * V)(S,T) \geq 0$, where $(f * V)$ means convolution of f and V;

4) $\frac{\partial}{\partial T}F^1(S,T) = \lambda F^0(S,T) + \int_{-\infty}^{\infty} f(x,T)\int_{-\infty}^{\infty} |V(S-x+y)|dydx \equiv \lambda F^0(S,T) + (f * \int|V|)(S,T) \geq 0$;

5) $\frac{\partial}{\partial T}\Psi^i(S,T) = -\frac{1}{T}[\Psi^i(S,T) - F^i(S,T)]$.

If $L > 0$, let $\Psi^i(S,T;L)$ be the counterpart of $\Psi^i(S,T)$ with positive leadtime, then we modify parts 1) 2) and 5) to:

1*) $\Psi^i(S,T;L) = \frac{1}{T}\int_L^{T+L} F^i(S,x)dx$;

2*) $\psi(S,T;L) = \frac{1}{T}\int_L^{T+L} f(S,x)dx$;

5*) $\frac{\partial}{\partial T}\Psi^i(S,T;L) = -\frac{1}{T}[\Psi^i(S,T;L) - F^i(S,T+L)]$,

All the results hold for discrete $X(T)$ by substituting integration with summation.

Proof of Lemma A 1) $\Psi^0(S,T) = \dfrac{1}{T}\displaystyle\int_0^T Pr(X(\mathcal{U}[0,T]) > S \mid \mathcal{U}[0,T] = x)dx = \dfrac{1}{T}\displaystyle\int_0^T F^0(S,x)dx.$

$$\Psi^1(S,T) = \int_S^{\infty} Pr(Y(T) > x)dx = \frac{1}{T}\int_0^T\int_S^{\infty} Pr(X(\mathcal{U}[0,T]) > x|\mathcal{U}[0,T] = y)dxdy = \frac{1}{T}\int_0^T F^1(S,y)dy.$$

2) $\psi^0(S,T) = -\dfrac{\partial}{\partial S}\Psi^0(S,T) = -\dfrac{\partial}{\partial S}\dfrac{1}{T}\displaystyle\int_0^T F^0(S,x)dx = \dfrac{1}{T}\displaystyle\int_0^T f(S,x)dx.$

3) Note

$$\frac{\partial}{\partial T}F^0(S,T) = \lim_{t\to 0^+} \frac{Pr(X(T+t) > S) - Pr(X(T) > S)}{t}.$$

Since $X(T)$ is stationary with independent increments, we have $X(T+t) =_{st} X(T) + X(t)$ and $X(T)$ and $X(t)$ are independent. Therefore, the above limit can be simplified by conditioning on $X(T)$, i.e.,

$$Pr(X(T+t) > S) - Pr(X(T) > S) = \int_{-\infty}^{S} f(x,T)F^0(S-x,t)dx - \int_{S}^{\infty} f(x,T)(1 - F^0(S-x,t))dx.$$

Taking the limit over t, we have $\frac{\partial}{\partial T}F^0(S,T) = \int_{-\infty}^{\infty} f(x,T)V(S-x)dx = (f * V)(S,T)$, which is nonnegative because $X(T)$ is SI.

4) We calculate the derivative of $F^1(S,T)$ by definition. Note that

$$\begin{aligned} \frac{\partial}{\partial T}F^1(S,T) &= \lim_{t\to 0^+} \frac{E[X(T+t) - S]^+ - E[X(T) - S]^+}{t} \\ &= \lim_{t\to 0^+} \frac{E[X(T) + X(t) - S]^+ - E[X(T) - S]^+}{t}. \end{aligned} \tag{1}$$

We discuss the value of equation (1) in two cases:

Case 1: When $X(T) - S = x > 0$, equation (1) can be simplified to:

$$\lim_{t\to 0^+} \frac{E[X(T+t) - S]^+ - E[X(T) - S]^+}{t} = \lambda + \int_{-\infty}^{0} |V(y+x)|dy.$$

Case 2: When $X(T) - S = x \leq 0$, equation (1) can be simplified to:

$$\lim_{t\to 0^+} \frac{E[X(T+t) - S]^+ - E[X(T) - S]^+}{t} = \int_{0}^{\infty} |V(y+x)|dy.$$

Combining both cases and taking expectation over $X(T) - S$ yields

$$\frac{\partial}{\partial T}F^1(S,T) = \lambda F^0(S,T) + \int_{-\infty}^{\infty} f(x,T) \int_{-\infty}^{\infty} |V(S-x+y)|dydx \equiv (f * \int |V|)(S,T) \geq 0.$$

5) Follows directly from 1).

The proofs for 1*), 2*) and 5*) are similar. □

Proof of Proposition 3 a) From Proposition 1 part 2) and Lemma A parts 2) and 5), we have

$$S^{*\prime}(T) = \frac{F^0(S^*(T),T) - \Psi^0(S^*(T),T)}{\int_0^T f(S^*(T),x)dx}.$$

From the Lemma A part 1) and taking integration by parts, we obtain

$$\Psi^0(S^*(T),T) = F^0(S^*(T),T) - \frac{1}{T}\int_0^T x\frac{\partial}{\partial x}F^0(S^*(T),x)dx.$$

Substituting F^0 in the first expression by that in the second expression yields the result.

When $L > 0$, following the same steps we have:

$$S^{*\prime}(T) = \frac{\int_L^{T+L} x\frac{\partial}{\partial x}F^0(S^*(T),x)dx}{\int_L^{T+L} f(S^*(T),x)dx}.$$

b) To prove unimodality, we need to show $\partial C(S^*(T),T)/\partial T$ changes sign only once. Because $X(T)$ is continuous, $F^0(0,T) = 1$ for all $T > 0$, so the cost function becomes

$$C(S,T) = \frac{K}{T} + h(S - \lambda T/2) + (h+b)\Psi^1(S,T).$$

Substitute $S^*(T)$ into $C(S,T)$ and take derivative over T, we have:

$$\frac{\partial}{\partial T}C(S^*(T),T) = \frac{1}{T^2}A(T),$$

where

$$A(T) = -K - \frac{\lambda h}{2}T^2 + (h+b)\int_{S^*(T)}^{\infty}\left(TF^0(y,T) - \int_0^T F^0(y,x)dx\right)dy.$$

Thus, it is sufficient to show that $A(T)$ is increasing in T. Takeing derivative over T, we have

$$\begin{aligned} A'(T) &= -\lambda hT + (h+b)\int_{S^*(T)}^{\infty}\left(F^0(y,T) + T\frac{\partial}{\partial T}F^0(y,T) - F^0(y,T)\right)dy \\ &\quad -(h+b)\left(TF^0(S^*(T),T) - \int_0^T F^0(S^*(T),x)dx\right)S^{*\prime}(T). \end{aligned}$$

Note that $S^*(T)$ satisfies

$$\Psi^0(S^*(T),T) = \frac{1}{T}\int_0^T F^0(S^*(T),x)dx = h/(h+b). \tag{2}$$

Therefore

$$\begin{aligned} A'(T) &= -\lambda hT + (h+b)\int_{S^*(T)}^{\infty} T\frac{\partial}{\partial T}F^0(y,T)dy - (h+b)\left(TF^0(S^*(T),T) - hT/(h+b)\right)S^{*\prime}(T) \\ &= hT(S^{*\prime}(T) - \lambda) + (h+b)T\left[\int_{S^*(T)}^{\infty}\frac{\partial}{\partial T}F^0(y,T)dy - F^0(S^*(T),T)S^{*\prime}(T)\right]. \end{aligned}$$

From Lemma A part 4)

$$\int_{S^*(T)}^{\infty} \frac{\partial}{\partial T} F^0(y,T)dy = \frac{\partial}{\partial T} E[X(T)-S]^+|_{S=S^*(T)} = \lambda F^0(S^*(T),T) + (f * \int |V|)(S^*(T),T),$$

where $(f * \int |V|)(S^*(T),T) \geq 0$. Thus,

$$A'(T) \quad = \quad (h+b)T(\lambda - S^{*\prime}(T))(F^0(S^*(T),T) - h/(h+b)) + (h+b)T(f * \int |V|)(S^*(T),T).$$

Because $F^0(S^*(T),x)$ is increasing in x and from equation (2), we have

$$F^0(S^*(T),T) - h/(h+b) \geq 0.$$

From the assumption $\lambda \geq S^{*\prime}(T)$, we have $A'(T) \geq 0$, which proves the statement.

c) In particular, when $V(x) = \lambda\delta(x)$, substitute it into $S^{*\prime}(T)$ we have:

$$S^{*\prime}(T) \quad \leq \quad \lambda \int_0^T Tf(S^*(T),x)dx / \left(T \int_0^T f(S^*(T),x)dx \right) = \lambda,$$

where the second equality follows from Lemma A part 3). Therefore, $S^{*'}(T) \leq \lambda$ holds and hence the statement is true.

Following the same steps, the statements are true for $L > 0$. □

Proof of Proposition 5 1) From Proposition 1 part 1) $C(S,T)$ is submodular in (S,T), therefore $T_m(S)$ and $S^*(T)$ are increasing functions of S and T, respectively. By the construction of $T_m(S)$, we know that $S^*(T) = S$ in $[T_m(S-1), T_m(S))$. The points that make $S^*(T)$ discontinuous are $T_m(S)$. By definition of $T_m(S)$, the increment of $S^*(T)$ at $T_m(S)$ is one.

2) From Proposition 1 part 2) we know that $S^*(T)$ is piecewise constant in T. Therefore, in any interval that $S^*(T)$ is constant following Proposition 1 part 3), $C(S^*(T),T)$ is convex in T .

3) Suppose this is not true. If $C(S,T_m(S)) < C(S+1,T_m(S))$, then because both $C(S,T)$ and $C(S+1,T)$ are continuous in T, for sufficiently small $\Delta > 0$, we must have $C(S,T_m(S)+\Delta) < C(S+1,T_m(S)+\Delta)$, contradicting to $S^*(T_m(S)+\Delta) = S+1$. If $C(S,T_m(S)) > C(S+1,T_m(S))$, then it contradicts $S^*(T_m(S)) = S$. Therefore, we must have $C(S,T_m(S)) = C(S+1,T_m(S))$.

4) Using property 3), the right directive satisfies:

$$\frac{\partial}{\partial T} C(S^*(T),T)\bigg|_{T_m(S)-} = \frac{\partial}{\partial T} C(S,T)\bigg|_{T_m(S)}.$$

$$\begin{aligned}\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)+} &= \lim_{\Delta\to 0}\frac{C(S^*(T_m(S)+\Delta),T_m(S)+\Delta)-C(S^*(T_m(S)),T_m(S))}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{C(S+1,T_m(S)+\Delta)-C(S+1,T_m(S))}{\Delta}\\ &= \left.\frac{\partial}{\partial T}C(S+1,T)\right|_{T_m(S)}.\end{aligned}$$

The left directive satisfies:

$$\begin{aligned}\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)-} &= \lim_{\Delta\to 0}\frac{C(S^*(T_m(S)),T_m(S))-C(S^*(T_m(S)-\Delta),T_m(S)-\Delta)}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{C(S,T_m(S))-C(S,T_m(S)-\Delta)}{\Delta}\\ &= \left.\frac{\partial}{\partial T}C(S,T)\right|_{T_m(S)}.\end{aligned}$$

The last equality in both equations follow because fix S, $C(S,T)$ is differentiable at T.

5)

$$\begin{aligned}&\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)-}-\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)+}=\left.\frac{\partial}{\partial T}C(S+1,T)\right|_{T_m(S)}-\left.\frac{\partial}{\partial T}C(S,T)\right|_{T_m(S)}\\ &= \frac{h+b}{T_m(S)}[F^1(S-1,T_m(S))-\Psi^1(S-1,T_m(S))-F^1(S,T_m(S))+\Psi^1(S,T_m(S))].\end{aligned}$$

Note that $\Psi^1(S-1,T_m(S))-\Psi^1(S,T_m(S))=\Psi^0(S,T_m(S))=h/(h+b)$ and $F^1(S-1,T_m(S))-F^1(S,T_m(S))=F^0(S,T_m(S))$. The above equation can be further simplified to

$$\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)-}-\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)+} = \frac{h+b}{T_m(S)}\left[F^0(S,T_m(S))-\frac{h}{h+b}\right].$$

Since $0\geq F^0(S,T_m(S))\leq 1$, $F^0(S,T_m(S))-\frac{h}{h+b}$ is bounded for all $T_m(S)$. Therefore,

$$\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)-}-\left.\frac{\partial}{\partial T}C(S^*(T),T)\right|_{T_m(S)+}=O\left(\frac{1}{T_m(S)}\right).$$

□

The following definitions and lemma are useful for proving Proposition 6. Assume $\{X(t),t\geq 0\}$ is Poisson with rate λ. Thus, $Y(T)=X[\mathcal{U}[0,T]]$ and $X(T)\sim Poisson(\lambda T)$. Let $g(\cdot,T)$ be the pmf of $X(T)$ and $p(\cdot,T)$ the pmf of $Y(T)$. Define

$$G^0(S,T)=\sum_{u=S+1}^{\infty}g(u,T),\qquad G^i(S,T)=\sum_{u=S+1}^{\infty}G^{i-1}(u,T),\ i=1,2.$$

$$P^0(S,T)=\sum_{u=S+1}^{\infty}p(u,T),\qquad P^i(S,T)=\sum_{u=S+1}^{\infty}P^{i-1}(u,T),\ i=1,2.$$

Then

$$C(S,T) \;=\; \frac{K}{T}G^0(0,T) + h[S - P^1(-1,T)] + (h+b)P^1(S-1,T). \tag{3}$$

For any discrete function $\varphi(x)$, let $\Delta_x\varphi(x) = \varphi(x+1) - \varphi(x)$. The following lemma is a counterpart of Lemma A under Poisson demand. (Similar properties of $G(S,T)$ in Lemma B can be found in Hadley and Whitin 1963.)

Lemma B *For $i = 1,\ 2$,*

1) $\dfrac{\partial}{\partial T}P^i(S,T) = \dfrac{1}{T}[G^i(S,T) - P^i(S,T)]$*;*

2) $\dfrac{\partial}{\partial T}G^0(S,T) = \lambda g(S,T)$ *and* $\dfrac{\partial}{\partial T}G^1(S,T) = \lambda G^0(S-1,T)$*;*

3) $p(S,T) = \dfrac{1}{\lambda T}G^0(S,T)$ *and* $P^{i-1}(S,T) = \dfrac{1}{\lambda T}G^i(S,T)$*;*

4) $\Delta_S P^i(S,T) = P^{i-1}(S+1,T)$ *and* $\Delta_S G^i(S,T) = G^{i-1}(S+1,T)$*;*

5) $\Delta_S P^0(S,T) = p(S+1,T)$ *and* $\Delta_S G^0(S,T) = g(S+1,T)$*;*

6) both G^1 and P^1 can be expressed as summations of g:

$$\begin{aligned} G^1(S,T) &= \sum_{u=S+1}^{\infty}(u-S-1)g(u,T), \\ P^1(S,T) &= \frac{1}{2\lambda T}\sum_{u=S+3}^{\infty}(u^2 - 3u - (S-1)(S+2))g(u,T). \end{aligned}$$

When $L > 0$, let $P^i(S,T;L)$ be the corresponding $P^i(s,T)$, we modify the above results to:

1)* $\dfrac{\partial}{\partial T}P^i(S,T;L) = \dfrac{1}{T}[G^i(S,T+L) - P^i(S,T;L)]$;

3)* $p(S,T) = \dfrac{1}{\lambda T}[G^0(S,T+L) - G^0(S,L)]$ *and* $P^{i-1}(S,T;L) = \dfrac{1}{\lambda T}[G^i(S,T+L) - G^i(S,L)]$*;*

4)* $\Delta_S P^i(S,T;L) = P^{i-1}(S+1,T;L)$*;*

5)* $\Delta_S P^0(S,T;L) = p(S+1,T;L)$*;*

6)*

$$P^1(S,T;L) \;=\; \frac{1}{2\lambda T}\sum_{u=S+3}^{\infty}(u^2-3u-(S-1)(S+2))[g(u,T+L)-g(u,L)].$$

Proof of Lemma B 1) Note that

$$P^0(S,T)=\sum_{u=S+1}^{\infty}p(u,T)=\frac{1}{T}\int_0^T\sum_{u=S+1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^u}{u!}dt=\frac{1}{T}\int_0^T G^0(S,t)dt.$$

Taking derivatives over T on both sides yields

$$\frac{\partial}{\partial T}P^0(S,T)=\frac{\partial}{\partial T}\left[\frac{1}{T}\int_0^T G^0(S,t)dt\right]=\frac{1}{T^2}\left[G^0(S,T)T-\int_0^T G^0(S,t)dt\right]=\frac{1}{T}[G^0(S,T)-P^0(S,T)].$$

We can calculate $\frac{\partial}{\partial T}P^1(S,T)$ through $\frac{\partial}{\partial T}P^0(S,T)$:

$$\frac{\partial}{\partial T}P^1(S,T)=\sum_{u=S+1}^{\infty}\frac{\partial}{\partial T}P^0(u,T)=\sum_{u=S+1}^{\infty}\frac{1}{T}[G^0(S,T)-P^0(S,T)]=\frac{1}{T}[G^1(S,T)-P^1(S,T)].$$

2) Follows directly from Lemma 1 and Lemma A.

3) First we show a recursion for $p(S,T)$.

$$p(S,T) \;=\; \frac{1}{T}\int_0^T e^{-\lambda t}\frac{(\lambda t)^S}{S!}dt=-\frac{1}{\lambda T}g(S,T)+p(S-1,T).$$

and $p(0,T)=(1-e^{-\lambda T})/(\lambda T)$. Therefore, we can calculate $p(S,T)$ recursively.

$$\begin{aligned}
p(S,T) &= -\sum_{u=1}^{S}\frac{1}{\lambda T}g(u,T)+p(0,T)=\frac{1}{\lambda T}[1-\sum_{u=0}^{S}g(u,T)]=\frac{1}{\lambda T}G^0(S,T)\\
P^0(S,T) &= \sum_{u=S+1}^{\infty}p(u,T)=\sum_{u=S+1}^{\infty}\frac{1}{\lambda T}G^0(u,T)=\frac{1}{\lambda T}G^1(S,T).\\
P^1(S,T) &= \sum_{u=S+1}^{\infty}P^0(u,T)=\sum_{u=S+1}^{\infty}\frac{1}{\lambda T}G^1(u,T)=\frac{1}{\lambda T}G^2(S,T).
\end{aligned}$$

4) 5) see Hadley and Whitin 1963.

6)

$$G^1(S,T) \;=\; \sum_{u=S+1}^{\infty}G^0(u,T)=\sum_{u=S+1}^{\infty}(u-S-1)g(u,T),$$

$$\begin{aligned}
P^1(S,T) &= \frac{1}{\lambda T}\sum_{u=S+1}^{\infty}G^1(u,T)=\frac{1}{2\lambda T}\sum_{u=S+2}^{\infty}(u^2-(2S+3)u-(S+1)(S+2))g(u,T)\\
&= \frac{1}{2\lambda T}\sum_{u=S+3}^{\infty}(u^2-3u-(S-1)(S+2))g(u,T).
\end{aligned}$$

The proofs for 1*), 3*), 4*), 5*) and 6*) are similar. □

Proof of Proposition 6 For simplicity, let $\tau = T_m(S)$. With Poisson demand, we have

$$\begin{aligned}\frac{\partial}{\partial T}C(S^*(T),T)\Big|_{\tau+} &= \frac{\partial}{\partial T}C(S+1,T)\Big|_{\tau+}\\ &= \frac{1}{\tau}K\lambda e^{-\lambda\tau} - K\frac{1-e^{-\lambda\tau}}{(T_m(S))^2} - \frac{h\lambda}{2} + \frac{h+b}{\tau}[G^1(S,\tau) - P^1(S,\tau)].\end{aligned}$$

Note that $P^0(S,\tau) = \frac{1}{\lambda\tau}G^1(S,\tau) = \frac{h}{h+b}$. Substitute this into the above equation and apply Lemma B leads to

$$\begin{aligned}&\frac{\partial}{\partial T}C(S^*(T),T)\Big|_{\tau+}\\ = \;&\frac{h+b}{\tau}\left(K\frac{\tau\lambda e^{-\lambda\tau}-1+e^{-\lambda\tau}}{\tau(h+b)} - \frac{h\lambda\tau}{2(h+b)} + \frac{h(S+1)}{h+b} + \frac{(S+1)(2S+3)}{\lambda\tau}\sum_{u=S+2}^{\infty} g(u,\tau)\right).\end{aligned}$$

Note that when $S \to \infty$, by definition of $T_m(S)$, $T_m(S) \to \infty$. By the Central Limit Theorem, $S \approx \lambda\tau + z\sqrt{\lambda\tau}$, where z is the critical fractile of the standard normal distribution at $b/(h+b)$. Takeing the limit of each term, we have

$$\begin{aligned}&\lim_{S\to\infty} K\frac{\tau\lambda e^{-\lambda\tau}-1+e^{-\lambda\tau}}{\tau(h+b)} = 0,\\ &\lim_{S\to\infty} -\frac{h\lambda\tau}{2(h+b)} + \frac{h(S+1)}{h+b} = \infty,\\ &\lim_{S\to\infty} \frac{(S+1)(2S+3)}{\lambda\tau}\sum_{u=S+2}^{\infty} g(u,\tau) \geq 0.\end{aligned}$$

Observe that the only term that is not bounded is $-\frac{h\lambda\tau}{2(h+b)} + \frac{h(S+1)}{h+b}$. When S is sufficiently large, this term is positive. Therefore, we can find S large enough so that $\frac{\partial}{\partial\tau}C(S^*(\tau),\tau) > 0$. Thus, $C(S^*(T),T)$ has a finite number of local minima. Following the same steps, similar results can be shown when $L > 0$. □

Proof of Proposition 7 a) Let $X_d(T) = \lambda T$ and $Y_d(T) = X_d[\mathcal{U}[0,T]] = \mathcal{U}[0,\lambda T]$. Then $X_d(T)$ and $Y_d(T)$ are SIL in T. Further define $C_d(S,T)$ be the system cost function under $X_d(T)$ and $Y_d(T)$:

$$C_d(S,T) = \frac{K}{T} + h(S - \lambda T/2) + (h+b)E[U[0,\lambda T] - S]^+.$$

Let $S_d^*(T)$ be the optimal S that minimizes $C_d(S,T)$, then $S_d^*(T) = \frac{b}{h+b}\lambda T$. Substitute $S_d^*(T) = \frac{b}{h+b}\lambda T$ into $C_d(S,T)$, we have the cost function optimized over S:

$$\min_S\{C_d(S,T)\} = C_d(S_d^*(T),T) = \frac{hb}{2(h+b)}\lambda T + K/T.$$

Note that, for any T, $X_d(T) \leq_{cx} X(T)$. From Theorem 3.A.12 (p. 358) of Shaked and Shanthikumar (2006), this implies $Y_d(T) \leq_{cx} Y(T)$. Since $[T-S]^+$ is convex in T, using the convex order property, we have for any (S,T)

$$C_d(S,T) - \frac{K}{T} \leq C(S,T) - K\frac{Pr(X(T)>0)}{T}.$$

Therefore,

$$\min_S\{C_d(S,T) - \frac{K}{T}\} \leq \min_S\{C(S,T) - K\frac{Pr(X(T)>0)}{T}\}.$$

$K\frac{Pr(X(T)>0)}{T}$ will not affect the value of $S^*(T)$, adding it to both sides of the inequality we have

$$C(S^*(T),T) \geq \frac{hb}{2(h+b)}\lambda T + K\frac{Pr(X(T)>0)}{T}.$$

b) Note that $LB(T) - K\frac{Pr(X(T)>0)}{T}$ is linear and increasing in T. For any $T > \bar{T}$, we have

$$C(S^*(T),T) > LB(T) - K\frac{Pr(X(T)>0)}{T} > LB(\bar{T}) - K\frac{Pr(X(\bar{T})>0)}{\bar{T}} = C(S_0,T_0).$$

Note that $S_0 = S^*(T_0)$, therefore $C(S^*(T),T) > C(S^*(T_0),T_0)$, implying T is not the global minimum. The upper bound for S follows from the submodularity property of $C(S,T)$.

When $L > 0$, the corresponding lower bound

$$C_d(S_d^*(T),T,L) = \frac{3hb}{2(h+b)}\lambda T + h\lambda L + K/T$$

can be obtained following the same steps as $L = 0$. □