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Online Supplement: Dynamic Pricing of Substitutable Products in the Presence of Capacity Flexibility

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A. Proofs of Main Results:

Proof of Theorem 1: As described in detail in the proof of Lemma 1 part(b) (provided subsequently in the Proofs of Supplementary Results), for $i = \{1, 2\}$, the KKT conditions for the problem may be represented as:

$$\frac{\partial J^t}{\partial p_i^t} = a_{1i}\lambda_1^t + a_{2i}\lambda_2^t - (a_{1i} + a_{2i})\mu^t \tag{9a}$$

$$\frac{\partial J^t}{\partial z_i} = \mu^t - \lambda_i^t \tag{9b}$$

We start by solving the KKT conditions given in (9a) for p_1^t and p_2^t and find $p_i^t = p_{i,L}^t - \frac{1}{2}(\lambda_i^t - \mu^t)$ where we define $p_{i,L}^t$ as the list-price in period t for product i, $i = \{1, 2\}$, given by $p_{1,L}^t = \frac{a_{22}b_1 - a_{12}b_2}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_1}{2}$ and $p_{2,L}^t = \frac{a_{11}b_2 - a_{12}b_1}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_2}{2}$. Further, (9b) imply implicit functions ϕ_1^t and ϕ_2^t such that $z_1^t = \phi_1^t(\lambda_1^t, \lambda_2^t, \mu^t)$ and $z_2^t = \phi_2^t(\lambda_1^t, \lambda_2^t, \mu^t)$ as stated in Lemma A.1 below.

LEMMA A.1. *There exists implicit functions ϕ_1 and ϕ_2 such that $z_1^t = \phi_1(\lambda_1^t, \lambda_2^t, \mu^t)$ and $z_2^t = \phi_2(\lambda_1^t, \lambda_2^t, \mu^t)$. Furthermore, $\phi_1(\lambda_1^t, \lambda_2^t, \mu^t)$ is increasing in λ_1^t, and decreasing in λ_2^t, μ^t whereas $\phi_2(\lambda_1^t, \lambda_2^t, \mu^t)$ is increasing in λ_2^t, and decreasing in λ_1^t, μ^t.*

Proof: Provided in the Proofs of Supplementary Results

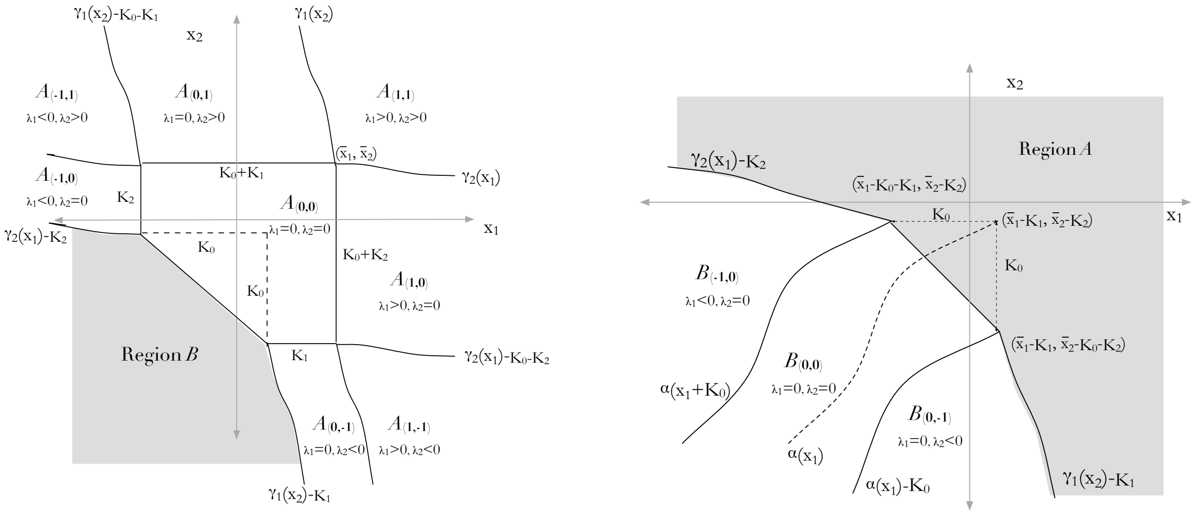
By Lemma A.1, we can rewrite the capacity constraints as follows.

$$x_1^t \le \phi_1^t(\lambda_1^t, \lambda_2^t, \mu^t) - a_{11}p_1^t(\lambda_1^t, \mu^t) - a_{12}p_2^t(\lambda_2^t, \mu^t) + b_1 \le x_1^t + K_0 + K_1 \tag{10a}$$

$$x_2^t \le \phi_2^t(\lambda_1^t, \lambda_2^t, \mu^t) - a_{21}p_1^t(\lambda_1^t, \mu^t) - a_{22}p_2^t(\lambda_2^t, \mu^t) + b_2 \le x_2^t + K_0 + K_2 \tag{10b}$$

$$\begin{aligned}\phi_1^t(\lambda_1^t, \lambda_2^t, \mu^t) + \phi_2^t(\lambda_1^t, \lambda_2^t, \mu^t) - (a_{11} + a_{21})p_1^t(\lambda_1^t, \mu^t)\\ - (a_{12} + a_{22})p_2^t(\lambda_2^t, \mu^t) + b_1 + b_2 \le x_1^t + x_2^t + K_0 + K_1 + K_2\end{aligned} \tag{10c}$$

The inventory state space may be partitioned into several regions based on the signs of λ_1^t, λ_2^t, and μ^t. (Formal partition is provided in the subsequent Lemmas A.2 and A.3.) In order to clarify the portrayal of state space segmentation, we define two broad regions, region A and region



(a)Partitioning of region A (b)Partitioning of region B (enlarged for clarification)

Figure I Segmentation of the state space

B (illustrated in Figure I), corresponding to initial inventory levels for which $\mu^t = 0$ and $\mu^t > 0$, respectively. In words, region A represents the initial inventory levels for which there remains some resource, either dedicated or flexible, that is not fully utilized. Region B, on the other hand, corresponds to inventory levels for which all resources are fully utilized. A specific point is of certain interest in our partitioning of the state space. When none of the constraints are binding, we have $\lambda_1^t = \lambda_2^t = \mu^t = 0$. Hence, $(\boldsymbol{\phi}^t(0,0,0), \mathbf{p_L^t})$ is the optimal solution to the unconstrained problem of $\max J^t(\mathbf{z}^t, \mathbf{p}^t)$. If we define $\bar{\mathbf{x}}^t$ such that $\bar{x}_1^t = \phi_1^t(0,0,0) - a_{11}p_{1L}^t - a_{12}p_{2L}^t + b_1$ and $\bar{x}_2^t = \phi_2^t(0,0,0) - a_{21}p_{1L}^t - a_{22}p_{2L}^t + b_2$, then $(\bar{\mathbf{x}}^t, \mathbf{p}_L^t)$ is the optimal solution for the unconstrained original problem $\max G^t(\mathbf{y}^t, \mathbf{p}^t)$.

Lemma A.2. *The boundaries of the state space Region A are defined by two monotone functions:*

i. $\gamma_1^t(x_2^t) : \Re \to \Re$ *with* $\gamma_1^t(x_2^t) = \bar{x}_1^t$ *for* $x_2^t \in [\bar{x}_2^t - K_0 - K_2,\, \bar{x}_2^t]$ *and* $\gamma_1^t(x_2^t)$ *strictly decreasing with respect to* x_2^t *for* $x_2^t \in \Re \setminus [\bar{x}_2^t - K_0 - K_2,\, \bar{x}_2^t]$

ii. $\gamma_2^t(x_1^t) : \Re \to \Re$ *with* $\gamma_2^t(x_1^t) = \bar{x}_2^t$ *for* $x_1^t \in [\bar{x}_1^t - K_0 - K_1,\, \bar{x}_1^t]$ *and* $\gamma_2^t(x_1^t)$ *strictly decreasing with respect to* x_1^t *for* $x_1^t \in \Re \setminus [\bar{x}_1^t - K_0 - K_1,\, \bar{x}_1^t]$

that further partitions Region A into the following eight subregions:

- $A_{(0,0)} := \{(x_1^t, x_2^t) : \bar{x}_i^t - K_0 - K_i \le x_i^t < \bar{x}_i^t \;\forall i = 1,2 \;\textit{and}\;\; x_1^t + x_2^t > \bar{x}_1^t + \bar{x}_2^t - K_0 - K_1 - K_2\}$
- $A_{(0,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} \gamma_1^t(x_2^t) - K_0 - K_1 \le x_1^t < \gamma_1^t(x_2^t) & \textit{and} \quad\; x_2^t \ge \bar{x}_2^t & \textit{if} \;\; j = 1 \\ \gamma_1^t(x_2^t) - K_1 \le x_1^t < \gamma_1^t(x_2^t) & \textit{and} \;\; x_2^t < \bar{x}_2^t - K_0 - K_2 & \textit{if} \;\; j = -1 \end{array} \right\}$

- $A_{(1,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} x_1^t \geq \bar{x}_1^t \; \textit{and} \; \gamma_2^t(x_1^t) - K_0 - K_2 \leq x_2^t < \gamma_2^t(x_1^t) & \textit{if} & j = 0 \\ x_1^t \geq \gamma_1^t(x_2^t) \; \textit{and} \; x_2^t \geq \gamma_2^t(x_1^t) & \textit{if} & j = 1 \\ x_1^t \geq \gamma_1^t(x_2^t) \; \textit{and} \; x_2^t < \gamma_2^t(x_1^t) - K_0 - K_2 & \textit{if} & j = -1 \end{array} \right\}$
- $A_{(-1,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} x_1^t \leq \bar{x}_1^t - K_0 - K_1 \; \textit{and} \; \gamma_2^t(x_1^t) - K_2 \leq x_2^t < \gamma_2^t(x_1^t) & \textit{if} & j = 0 \\ x_1^t \leq \gamma_1^t(x_2^t) - K_0 - K_1 \; \textit{and} \; x_2^t \geq \gamma_2^t(x_1^t) & \textit{if} & j = 1 \end{array} \right\}$

Proof: The subscripts of A reflect the sign of the Lagrange variables and imply which, if any, of the constraints are binding. As an example, consider the region defined by $A_{(k_1,k_2)}$. Then, we have the index $k_i = 1$ if $\lambda_i^t > 0$, $k_i = 0$ if $\lambda_i^t = 0$, and $k_i = -1$ if $\lambda_i^t < 0$. For brevity, we only provide the results associated with regions $A_{(0,0)}$, and $A_{(0,1)}$ as the analysis for other regions are similar. We first consider region $A_{(0,0)}$ that corresponds to $\lambda_1^t = \lambda_2^t = 0$. Following (10a) - (10c), in this region we have

$$x_1^t < \phi_1^t(0,0,0) - a_{11}p_{1L}^t - a_{12}p_{2L}^t + b_1 < x_1^t + K_0 + K_1 \tag{11a}$$

$$x_2^t < \phi_2^t(0,0,0) - a_{21}p_{1L}^t - a_{22}p_{2L}^t + b_2 < x_2^t + K_0 + K_2 \tag{11b}$$

$$\phi_1^t(0,0,0) + \phi_2^t(0,0,0) - (a_{11}+a_{21})p_{1L}^t - (a_{12}+a_{22})p_{2L}^t + b_1 + b_2 < x_1^t + x_2^t + K_0 + K_1 + K_2 \tag{11c}$$

Thus, by substituting the expressions for $\bar{x}_1^t$ and $\bar{x}_2^t$ into (11a)-(11c), we can define this region as $\{(x_1^t, x_2^t) : \bar{x}_i^t - K_0 - K_i \leq x_i^t < \bar{x}_i^t \;\forall i = 1,2 \text{ and } \; x_1^t + x_2^t > \bar{x}_1^t + \bar{x}_2^t - K_0 - K_1 - K_2\}$. Next, we consider region $A_{(0,1)}$ that corresponds to $\lambda_1^t = 0$ and $\lambda_2^t > 0$. Since $\lambda_2^t > 0$, after substituting in the expressions for p_i^t, $\bar{x}_1^t$ and $\bar{x}_2^t$, the constraints (10a) - (10c) reduce to the following:

$$x_1^t < \phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t < x_1^t + K_0 + K_1 \tag{12a}$$

$$x_2^t = \phi_2^t(0,\lambda_2^t,0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t \qquad \text{(equality due to } \lambda_2^t > 0) \tag{12b}$$

We first consider (12b) which defines one boundary for this region resulting in $x_2^t = \phi_2^t(0,\lambda_2^t,0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t > \bar{x}_2^t$ (since $\phi_2^t \uparrow \lambda_2^t$ by Lemma A.1 and $a_{22} > 0$) and $\lim_{\lambda_2^t \to 0} x_2^t = \phi_2^t(0,0,0) + \bar{x}_2^t - \phi_2^t(0,0,0) = \bar{x}_2^t$. Furthermore, as $\phi_2^t \uparrow \lambda_2^t$ by Lemma A.1 and $a_{22} > 0$, x_2^t is strictly increasing with respect to λ_2^t in this region (equivalently, λ_2^t is strictly increasing with respect to x_2^t), there is a one-to-one function defining λ_2^t in terms of x_2^t, that is $\lambda_2^t = \lambda_2^t(x_2^t)$. The remaining boundaries are given by the inequalities in (12a). Since $\lambda_1^t = 0$, the constraints are not binding. We have

$$\phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t - K_0 - K_1 < x_1^t < \phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t \tag{13}$$

Temporarily defining a function $\delta_1(\lambda_2^t) := \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t$, we can rewrite (13) as $\delta_1^t(\lambda_2^t) + \bar{x}_1^t - K_0 - K_1 < x_1^t < \delta_1^t(\lambda_2^t) + \bar{x}_1^t$. Lemma A.1 and $a_{12} < 0$ yields $\delta_1^t(\lambda_2^t) < 0$ and that $\delta_1^t(\lambda_2^t)$ is strictly decreasing with respect to λ_2^t. If we now define $\gamma_1^t(x_2^t) := \bar{x}_1^t + \delta_1^t\left(\lambda_2^t(x_2^t)\right)$, we can write the boundaries for this region as $\gamma_1^t(x_2^t) - K_0 - K_1 < x_1^t < \gamma_1^t(x_2^t)$. The fact that $\gamma_1^t(x_2^t)$ strictly decreasing with respect to x_2^t follows immediately from $\delta_1^t(\lambda_2^t)$ strictly decreasing with respect to λ_2^t and λ_2^t strictly increasing with respect to x_2^t. $\square$

LEMMA A.3. *Together with $\gamma_1^t(x_2^t)$ and $\gamma_2^t(x_1^t)$, a monotone function $\alpha^t(x_1^t): [-\infty, \bar{x}_1^t - K_1] \rightarrow [-\infty, \bar{x}_2^t - K_2]$ with $\alpha^t(\bar{x}_1^t - K_1) = \bar{x}_2^t - K_2$ and $\alpha^t(x_1^t)$ strictly increasing with respect to x_1^t divides Region B into the three subregions:*

- $B_{(0,-1)} := B'_{(0,-1)} \cup B''_{(0,-1)}$ *where*
  $B'_{(0,-1)} := \{(x_1^t, x_2^t) : \bar{x}_1^t - K_1 < x_1^t \leq \gamma_1^t(x_2^t) - K_1 \text{ and } x_2^t \leq \bar{x}_2^t - K_0 - K_2\}$
  $B''_{(0,-1)} := \{(x_1^t, x_2^t) : x_1^t \leq \bar{x}_1^t - K_1 \text{ and } x_2^t \leq \alpha^t(x_1^t) - K_0\}$
- $B_{(-1,0)} := B'_{(-1,0)} \cup B''_{(-1,0)}$ *where*
  $B'_{(-1,0)} := \{(x_1^t, x_2^t) : x_1^t \leq \bar{x}_1^t - K_0 - K_1 \text{ and } \bar{x}_2^t - K_2 \leq x_2^t \leq \gamma_2^t(x_1^t) - K_2\}$
  $B''_{(-1,0)} := \{(x_1^t, x_2^t) : x_1^t \leq \bar{x}_1^t - K_0 - K_1 \text{ and } \alpha^t(x_1^t + K_0) < x_2^t \leq \bar{x}_2^t - K_2\}$
- $B_{(0,0)} := \Re^2 \backslash \{A \cup (B_{(0,-1)} \cup B_{(-1,0)})\}$.

Proof: For brevity, we only provide the proof for region $B_{(0,-1)}$ as the analysis of $B_{(-1,0)}$ is similar and region $B_{(0,0)}$ is defined by the remaining area in Region B. Region $B_{(0,-1)}$ corresponds to $\lambda_1^t = 0$, $\lambda_2^t < 0$, and $\mu^t > 0$ for which constraints (10a) - (10c) reduce to

$$x_1^t = \phi_1^t(0, \lambda_2^t, \mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 \tag{14a}$$

$$x_2^t = \phi_2^t(0, \lambda_2^t, \mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t + \frac{a_{22}}{2}\lambda_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2 \tag{14b}$$

The analysis of this region is simpler if we consider the cases where $x_1^t > \bar{x}_1^t - K_1$ and $x_1^t \leq \bar{x}_1^t - K_1$ separately corresponding to subregions $B'_{(0,-1)}$ and $B''_{(0,-1)}$, respectively. For subregion $B'_{(0,-1)}$, we first find the feasible values for x_2^t and then show that $\gamma_1^t(x_2^t) - K_1$ defines the remaining boundary for the possible values for x_1^t. For subregion $B''_{(0,-1)}$, we show that a function $\alpha^t(x_1^t)$ that is defined on the domain $x_1^t \leq \bar{x}_1^t - K_1$ establishes the boundary for the subregion. We first show that in the subregion $B'_{(0,-1)}$, we have $x_2^t \leq \bar{x}_2^t - K_0 - K_2$. For arbitrary $\lambda_2^t < 0$ and $\mu^t > 0$, by (14b), we have $x_2^t = \phi_2^t(0, \lambda_2^t, \mu^t) - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t + \bar{x}_2^t - K_0 - K_2 < \phi_2^t(0, \lambda_2^t, \mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - K_0 - K_2 < \bar{x}_2^t - K_0 - K_2$ where the first inequality follows from $\lambda_2^t < 0, \mu^t > 0$, and $a_{22} > 0$, $a_{21} + a_{22} > 0$ and the second inequality follows from $\phi_2^t \uparrow \lambda_2^t, \downarrow \mu^t$ and $\lambda_2^t < 0, \mu^t > 0$. We further have $\lim_{\lambda_2^t, \mu^t \to 0} x_2^t = \phi_2^t(0,0,0) - \phi_2^t(0,0,0) + \bar{x}_2^t - K_0 - K_2 = \bar{x}_2^t - K_0 - K_2$. Next, examining the expression for x_1^t given in (14a), we get $x_1^t = \phi_1^t(0, \lambda_2^t, \mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 = \phi_1^t(0, \lambda_2^t, \mu^t) - \phi_1^t(0, \lambda_2^t, 0) - \frac{(a_{11}+a_{12})}{2}\mu^t + \gamma_1^t(x_2^t) - K_1 < \gamma_1^t(x_2^t) - K_1$ where the inequality is due to $\mu^t > 0, \phi_1^t \downarrow \mu^t$ and $a_{11} + a_{12} > 0$. We also have $\lim_{\mu^t \to 0} x_1^t = \phi_1^t(0, \lambda_2^t, 0) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - K_1 = \gamma_1^t(x_2^t) - K_1$, the left-hand-side boundary for Region $A_{(0,-1)}$. We note that the increasing property of $\gamma_1^t(x_2^t)$ established in the proof of Lemma A.2 ensures that $\bar{x}_1^t - K_1 \leq \gamma_1^t(x_2^t) - K_1$. Thus the expressions $\bar{x}_1^t - K_1 \leq x_1^t \leq \gamma_1^t(x_2^t) - K_1$ and $x_2^t \leq \bar{x}_2^t - K_0 - K_2$ defines the states corresponding to $B'_{(0,-1)}$.

For subregion $B''_{(0,-1)}$, we first note that $\lim_{\lambda_2^t \to 0} x_1^t$ defines the boundary between regions $B''_{(0,-1)}$ and $B_{(0,0)}$. Along this boundary, by (14a), we have $x_1^t = \phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1$. Using Lemma A.1, we find that x_1^t is strictly decreasing with respect to μ^t and hence $x_1^t \leq \bar{x}_1^t - K_1$. Further x_1^t strictly decreasing with respect to μ^t implies that there is a one-to-one function along the boundary defining x_1^t and μ^t, i.e., $\mu^t(x_1^t)$ where μ^t is strictly decreasing with respect to x_1^t. By (14b) along the boundary, we have $x_2^t = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2$. Now, consider a point in subregion $B_{(0,0)}$, with $\lambda_1^t = 0$, $\lambda_2^t = 0$, and $\mu^t > 0$ for which the constraints (10a) - (10c) result in $\phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 = x_1^t + l$, and $\phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_2 = x_2^t + K_0 - l$ for some $0 < l < K_0$. By Lemma A.1, both $x_1^t + l$ and $x_2^t + K_0 - l$ are strictly decreasing with μ^t. There is a one-to-one function that defines $x_1^t + l$ and $x_2^t + K_0 - l$. Consequently, let $\alpha^t(x_1^t + l) = x_2^t + K_0 - l$. Approaching from a point in $B_{(0,0)}$, $\lim_{l \to 0} x_2^t$ defines the boundary between regions $B_{(0,0)}$ and $B''_{(0,-1)}$ which is equivalent to the previously defined boundary $x_2^t = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2$. Hence, the boundary can also be expressed as $x_2^t = \alpha^t(x_1^t) - K_0$, for which $x_2^t < \alpha^t(x_1^t) - K_0$ falls in region $B''_{(0,-1)}$. Let us also temporarily define $\sigma_2(\mu^t) = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t$. Then, by Lemma A.1, $\sigma_2(\mu^t)$ is strictly decreasing with respect to μ^t. Consequently, as $\mu^t(x_1^t)$ is strictly decreasing with respect to x_1^t, we have $\sigma_2(\mu^t(x_1^t))$ strictly increasing with respect to x_1^t. Since $\sigma_2(\mu^t(x_1^t)) \uparrow x_1^t$, we have $\alpha(x_1^t) \uparrow x_1^t$. Lastly, by (14a), $x_1^t = \bar{x}_1^t - K_1$ implies $\phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) - \frac{(a_{21}+a_{22})}{2}\mu^t = 0$ for which the only solution is $\mu^t = 0$. (Note $\phi_1^t \downarrow \mu^t$ and $\mu^t \geq 0$). Hence by (14b), we have $x_2^t = \bar{x}_2^t - K_0 - K_2$. which yields $\alpha(\bar{x}_1^t - K_1) = \bar{x}_2^t - K_2$. $\square$

To complete the Proof of Theorem 1, we note that part 1(a) follows directly from the definitions of the monotone functions $\gamma_1^t(x_2^t)$ and $\gamma_2^t(x_1^t)$ in Lemma A.2 and the complementary slackness conditions. For example, in region $A_{(-1,1)}$, the binding constraints yield $y_1^t = x_1^t + K_0 + K_1$ and $y_2^t = x_2^t$. In region $A_{(0,1)}$, we have $y_1^t = \gamma_1^t(x_2^t)$ since $y_1^t = z_1^t + \bar{d}_1^t = \phi_1^t(0,\lambda_2^t,0) + b_1 - a_{11}p_1^t - a_{12}p_2^t = \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \phi_1^t(0,0,0) + b_1 - a_{11}p_{1L}^t - a_{12}p_{2L}^t + \frac{a_{12}}{2}\lambda_2^t = \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t = \gamma_1^t(x_2^t)$. For part 1(b), in Regions $A_{(0,j)}$, the optimal order-up-to level for product 1 is independent of its own starting inventory x_1^t and by Lemma A.2 and part (a), it is non-increasing with x_2^t. In Region $A_{(0,0)}$, it is independent of x_2^t and in regions $A_{(0,1)}$ and $A_{(0,-1)}$, it is strictly decreasing with the inventory position of x_2^t. In Regions $A_{(1,j)}$, by part (a), we have $y_1^t = x_1^t$. For regions $A_{(-1,0)}$ and $A_{(-1,1)}$, again by part (a), we have $y_1^t = x_1^t + K_0 + K_1$, thus the order-up-to level of product 1 is increasing with x_1^t and independent of x_2^t. Symmetric arguments hold for product 2.

The proofs of part 2 (a) and (b) are due to Lemma A.3. Suppose $l^t(x_1^t, x_2^t)$ denotes the optimal amount of flexible capacity allocated to product 1. Since in Region B, the complementary slackness conditions imply full utilization of each resource, $K_0 - l^t(x_1^t, x_2^t)$ will be the amount of flexible capacity allocated to product 2. After the allocation of the flexible resource and employing the dedicated resources, the optimal production policy brings inventories of products 1 and 2 to $x_1^t + K_1 + l^t(x_1^t, x_2^t)$ and $x_2^t + K_2 + K_0 - l^t(x_1^t, x_2^t)$, respectively. Specifically, in region $B_{(-1,0)}$, complementary slackness yields $y_1^t - x_1^t = K_0 + K_1$ and $y_2^t - x_2^t = K_2$, thus $l^t(x_1^t, x_2^t) = K_0$. Similarly, in region $B_{(0,-1)}$, complementary slackness conditions give $y_1^t - x_1^t = K_1$ and $y_2^t - x_2^t = K_0 + K_2$, hence $l^t(x_1^t, x_2^t) = 0$. For region $B_{(0,0)}$, the definition of $l^t(x_1^t, x_2^t)$ yields $y_2^t - x_2^t = K_0 + K_2 - l^t(x_1^t, x_2^t)$ which leads to $\phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - x_2^t = K_0 + K_2 - l^t(x_1^t, x_2^t)$ and therefore the optimal production policy satisfies $x_2^t + K_0 - l^t(x_1^t, x_2^t) = \alpha^t(x_1^t + l^t(x_1^t, x_2^t))$. Furthermore, the complementary slackness condition (10c) yields μ^t to be a function of x_1^t and x_2^t only through their sum $x_1^t + x_2^t$. Thus, the optimal modified base stock levels for products 1 and 2 are identical for starting inventory positions for which the total inventory level, $x_1^t + x_2^t$, is identical.

For part 2(c), first let $\alpha'^t(x_1^t + l^t)$ denote the derivative of $\alpha^t(x_1^t + l^t)$ with respect to its argument. By Lemma A.3, α^t is increasing, thus $\alpha'^t(x_1^t + l^t) > 0$. Next, differentiating both sides of $l^t(x_1^t, x_2^t) + \alpha^t(x_1^t + l^t(x_1^t, x_2^t)) = x_2^t + K_0$ with respect to x_1^t, we get $\frac{\partial l^t}{\partial x_1^t} = -\frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)} < 0$. Thus, l^t is decreasing with respect to x_1^t. Similarly, differentiating both sides of $l^t(x_1^t, x_2^t) + \alpha^t(x_1^t + l^t(x_1^t, x_2^t)) = x_2^t + K_0$ with respect to x_2^t, we get $\frac{\partial l^t}{\partial x_2^t} = \frac{1}{1+\alpha'^t(x_1^t+l^t)} > 0$. Hence, l^t is increasing with respect to x_2^t. Finally, for part 2(d), the order-up-to level for product 1 is $x_1^t + l^t(x_1^t, x_2^t) + K_1$. Differentiating it with respect to x_1^t and with respect to x_2^t and using the expressions for $\frac{\partial l^t}{\partial x_1^t}$ and $\frac{\partial l^t}{\partial x_2^t}$, we get $\frac{\partial x_1^t+l^t(x_1^t,x_2^t)+K_1}{\partial x_1^t} = \frac{1}{1+\alpha'^t(x_1^t+l^t)} > 0$ and $\frac{\partial x_1^t+l^t(x_1^t,x_2^t)+K_1}{\partial x_2^t} = \frac{1}{1+\alpha'^t(x_1^t+l^t)} > 0$. The order-up-to level for product 2 is $x_2^t + K_0 - l^t(x_1^t, x_2^t) + K_2$. Again, differentiating it both with respect to x_1^t and x_2^t, we get $\frac{\partial x_2^t+K_0-l^t(x_1^t,x_2^t)+K_2}{\partial x_1^t} = \frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)} > 0$ and $\frac{\partial x_2^t+K_0-l^t(x_1^t,x_2^t)+K_2}{\partial x_2^t} = \frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)} > 0$. Hence, the order-up-to level for both products is increasing with respect to either starting inventory position.□

Proof of Theorem 2: The proof follows from the expressions for p_1^t and p_2^t given in the Proof of Theorem 1, i.e. $p_i^t = p_{i,L}^t - \frac{1}{2}(\lambda_i^t - \mu^t)$ where $p_{i,L}^t$ is defined as the list-price in period t for product i, $i = \{1,2\}$, given by $p_{1,L}^t = \frac{a_{22}b_1 - a_{12}b_2}{2(a_{11}a_{22}-a_{12}^2)} + \frac{c_1}{2}$ and $p_{2,L}^t = \frac{a_{11}b_2 - a_{12}b_1}{2(a_{11}a_{22}-a_{12}^2)} + \frac{c_2}{2}$. For part (a), corresponding to Region A, $\mu^t = 0$, therefore using (5) we have $m_1^t(x_1^t, x_2^t) = -\frac{1}{2}\lambda_1^t(x_1^t, x_2^t)$ and $m_2^t(x_1^t, x_2^t) = -\frac{1}{2}\lambda_2^t(x_1^t, x_2^t)$. Thus, we have $m_1^t(x_1^t, x_2^t) > 0$ for $\lambda_1^t < 0$, $m_1^t(x_1^t, x_2^t) = 0$ for $\lambda_1^t = 0$, and $m_1^t(x_1^t, x_2^t) < 0$ for $\lambda_1^t > 0$. Following the state space segmentation set forth in Lemma A.2, $\lambda_1^t < 0$, $\lambda_1^t = 0$, and $\lambda_1^t > 0$ correspond to item 1 being critically understocked, moderately understocked, and overstocked, respectively. Hence, a price surcharge is applied if the item is critically understocked, list price is charged if the item is moderately understocked and a discount is given if the item is overstocked. Similar arguments yield the results corresponding to product 2.

For part (b), corresponding to Region B, we have $m_1^t(x_1^t, x_2^t) = -\frac{1}{2}\left(\lambda_1^t(x_1^t, x_2^t) - \mu^t(x_1^t, x_2^t)\right)$ and $m_2^t(x_1^t, x_2^t) = -\frac{1}{2}\left(\lambda_2^t(x_1^t, x_2^t) - \mu^t(x_1^t, x_2^t)\right)$. Since region B is defined as the states corresponding to $\mu^t > 0$ and non-positive λ_1^t and λ_2^t, we have $m_1^t(x_1^t, x_2^t) > 0$ and $m_2^t(x_1^t, x_2^t) > 0$ indicating markups for both items. In the states that correspond to $B_{(0,0)}$, we have $m_1^t(x_1^t, x_2^t) = \frac{1}{2}\mu^t(x_1^t, x_2^t)$ and $m_2^t(x_1^t, x_2^t) = \frac{1}{2}\mu^t(x_1^t, x_2^t)$. Therefore, $m_1^t(x_1^t, x_2^t) = m_2^t(x_1^t, x_2^t)$. Further, (5) then yields $p_2^t(x_1^t, x_2^t) = p_1^t(x_1^t, x_2^t) + C^t$ where $C^t = p_{2L}^t - p_{1L}^t$. The fact that $m_i^t(x_1^t, x_2^t)$ is a function of x_1^t and x_2^t through their sum follows from (10c) which for this region implies that μ^t is a function of $x_1^t + x_2^t$.

For part (c), we only show the proof for product 1, as similar arguments yield the desired monotonicity results for product 2. In regions $A_{(0,0)}$, $A_{(0,1)}$, and $A_{(0,-1)}$, we have $p_1^t = p_{1L}^t$ and hence p_1^t is independent of both x_1^t and x_2^t. In region $A_{(1,0)}$, we have $p_1^t = p_{1L}^t - \frac{1}{2}\lambda_1^t$. Based on Lemma A.2, in this region λ_1^t increases with x_1^t and is independent of x_2^t, hence p_1^t decreases with x_1^t and is independent of x_2^t. With a similar analysis, we also find that p_1^t decreases with x_1^t and is independent of x_2^t in region $A_{(-1,0)}$ as well. In region $A_{(1,1)}$, we have $x_1^t = \phi_1^t(\lambda_1^t, \lambda_2^t, 0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{11}}{2}\lambda_1^t + \frac{a_{12}}{2}\lambda_2^t$ and $x_2^t = \phi_2^t(\lambda_1^t, \lambda_2^t, 0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{21}}{2}\lambda_1^t + \frac{a_{22}}{2}\lambda_2^t$. By differentiating these two expressions with respect to x_1^t, we find that both λ_1^t and λ_2^t are increasing with respect to x_1^t. Similarly, λ_1^t and λ_2^t are increasing with respect to x_2^t. Since, in this region p_1^t is given by $p_1^t = p_{1L}^t - \frac{1}{2}\lambda_1^t$, p_1^t decreases with respect to both x_1^t and x_2^t. Similar analysis yield λ_1^t to be independent of x_1^t and x_2^t in Regions $B_{(0,-1)}$ and $B_{(0,0)}$ and be increasing with respect to x_1^t and x_2^t in Region $B_{(-1,0)}$. Likewise, we find λ_2^t to be independent of x_1^t and x_2^t in Regions $B_{(-1,0)}$ and $B_{(0,0)}$ and be increasing with respect to x_1^t and x_2^t in Region $B_{(0,-1)}$. Lastly, we find μ^t to be increasing with respect to x_1^t and x_2^t in regions $B_{(-1,0)}, B_{(0,-1)}$ and $B_{(0,0)}$. Therefore, the desired monotonicity results follow immediately form the definitions of p_1^t in these regions. □

Proof of Theorem 3: The result follows from a similar methodology given in the proofs of Theorems 1 and 2 and hence omitted for brevity.

Proof of Theorem 4: We first note that $J^t(\mathbf{z^t}, \mathbf{p^t})$ is strictly concave. This follows from similar arguments as in the proof of Lemma 1(a) and due to the fact that the matrix $\mathbf{A}$ is positive definite, since $\mathbf{A}$ is a symmetric, strictly diagonally dominant matrix with positive diagonal elements. We construct the following KKT conditions:

$$\frac{\partial J^t}{\partial p_i^t} = \sum_{j \in \mathcal{N}} a_{ij}\lambda_j^t \quad \forall i \in \mathcal{N} \tag{15a}$$

$$\frac{\partial J^t}{\partial z_i^t} = -\lambda_i^t \quad \forall i \in \mathcal{N} \tag{15b}$$

where $\lambda_i^t > 0$ implies that product i is overstocked and $z_i^t + b_i - \sum_{j=1:N} a_{ij} p_j = x_i^t$; $\lambda_i^t = 0$ implies that product i is moderately understocked and $x_i^t + K_i > z_i^t + b_i - \sum_{j=1:N} a_{ij} p_j > x_i^t$; and $\lambda_i^t < 0$ implies that product i critically understocked with $z_i^t + b_i - \sum_{j=1:N} a_{ij} p_j = x_i^t + K_i$.

Using (8) and (15a), we can solve for $\mathbf{p}^t$ to get $p_i^t = p_{iL}^t - \lambda_i^t/2$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$. By (15b), we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial \mathbf{z}^t}\right) = -\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)$. Since $J(\mathbf{z^t}, \mathbf{p^t})$ is strictly concave, $\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)$ is invertible. By the implicit function theorem, we can write $\mathbf{z^t}$ as a function of $\boldsymbol{\lambda}^t$ as $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) = -\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)^{-1}$.

The below definition and lemmas describe a certain property on the objective function that enables us to characterize the structure of the optimal policy and its monotonicity with respect to starting inventory levels.

Definition 1. A Stieltjes matrix (symmetric M-matrix) is a real symmetric and positive definite matrix $\mathbf{M} = [m_{i,j}]$ in $R^{n,n}$ for which $m_{i,j} \leq 0$ for all $i \neq j$.

Lemma A.4. (Nabben and Varga 1994) *The inverse of a Stieltjes matrix is a real nonsingular and symmetric matrix with all of its entries nonnegative.*

Lemma A.5. $-\left(\frac{\partial J^t}{\partial z_i^t \partial z_j^t}\right)^{-1}$ *is a strictly diagonally dominant Stieltjes matrix.*

Proof: Provided in the Proofs of Supplementary Results.

By Lemmas A.4 and A.5, we also have $\frac{\partial \lambda_j^t}{\partial x_i^t} \geq 0 \, \forall i, j \in \mathcal{N}$. To see why, first consider the cases where $\lambda_j^t \neq 0 \, \forall j$. Then, we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1} e_i$. By Lemma A.4, $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$ is positive, hence $\frac{\partial \lambda_i^t}{\partial x_j^t} \geq 0$. Now consider the case where $\exists j$ s.t. $\lambda_j^t = 0$. In this case, we have $\frac{\partial \lambda_j^t}{\partial x_i^t} = 0$ and $\frac{\partial \lambda_i^t}{\partial x_j^t} = 0 \, \forall i$ where the second relationship follows from $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right)$ being symmetric. Differentiating the active constraints with respect to x_i^t and solving for $\frac{\partial \lambda_k^t}{\partial x_i^t}$, we get $\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}$ to be of the form $\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t} = \mathbf{S}^{-1} e_i$ where $\mathbf{S}$ is a principle submatrix of $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)$ and is also a strictly diagonal dominant Stieltjes matrix with a nonnegative inverse. Thus, $\frac{\partial \lambda_k^t}{\partial x_i^t} \geq 0$.

We can now show the structure of the optimal policy. For the optimal production policy described in part (a), consider a product $i \in \mathcal{P}$, i.e. $\lambda_i^t \leq 0$ and it is optimal to order product i. If $\lambda_i^t = 0$, then product i is moderately understocked and we have $x_i^t < z_i^t + b_i - \sum_{j=1:N} a_{ij} p_j < x_i^t + K_i$. The optimal base stock level is given by the expression $y_i^t = z_i^t + b_i - \sum_{k=1:N} a_{ik} p_k^t$ which can be equivalently written as $y_i^t = z_i^t + b_i - \sum_{k=1:N} a_{ik} p_{kL}^t + \sum_{k=1:N} a_{ik} \frac{\lambda_k^t}{2}$. Differentiating with respect to $x_j^t, j \neq i$, we get $\frac{\partial y_i^t}{\partial x_j^t} = \sum_{k=1:N} \frac{\partial z_i^t}{\partial \lambda_k^t} \frac{\partial \lambda_k^t}{\partial x_j^t} + \frac{a_{ik}}{2} \frac{\partial \lambda_k^t}{\partial x_j^t}$. Since $\lambda_i^t = 0$, we have $\frac{\partial y_i^t}{\partial x_j^t} = \sum_{k=1:N, k \neq i} \left(\frac{\partial z_i^t}{\partial \lambda_k^t} + \frac{a_{ik}}{2}\right) \frac{\partial \lambda_k^t}{\partial x_j^t}$. Since the term in the parenthesis is a Stieltjes matrix and $k \neq i$, it is nonpositive. Further the term $\frac{\partial \lambda_k^t}{\partial x_j^t}$ is nonnegative. Therefore, $\frac{\partial y_i^t}{\partial x_j^t} \leq 0$. Now, consider $\lambda_i^t < 0$. Then $y_i^t = x_i^t + K_i$ and hence y_i^t is independent of all x_j^t for $j \neq i$.

For the pricing policy given in part (b), the optimal price to charge for product i in period t was given by $p_i^t = p_{iL}^t - \frac{\lambda_i^t}{2}$. Consider $i \in \mathcal{P}$, i.e. $\lambda_i^t \leq 0$. If $\lambda_i^t = 0$, the product i is moderately understocked and $p_i^t = p_{iL}^t$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$ as given earlier. If $\lambda_i^t < 0$, then the product is critically understocked and it is optimal to mark up the price the of product i. If the product is overstocked, i.e., for product j, $j \in \mathcal{N} \backslash \mathcal{P}$, then $\lambda_j^t > 0$ and hence, it is optimal to give a discount on product j. Regarding the monotonocity of the optimal price policy, we have $\frac{\partial p_i^t}{\partial x_k^t} = -\frac{1}{2}\frac{\partial \lambda_i^t}{\partial x_k^t} \leq 0$. Therefore, the price of an item is decreasing with respect to the inventory positions of all other items. □

Proof of Theorem 5: As in the proof of Theorem 5, it can be verified that $J^t(\mathbf{z^t}, \mathbf{p^t})$ is strictly concave. A similar construction of KKT conditions yields:

$$\frac{\partial J^t}{\partial p_i^t} = \sum_{n=1,\ldots,N} a_{in}\lambda_n^t \quad \forall i \leq k, i \in \mathcal{N} \tag{16a}$$

$$\frac{\partial J^t}{\partial p_j^t} = \sum_{n=1,\ldots,N} a_{in}(\lambda_n^t - \mu) \quad \forall j > k,\, j \in \mathcal{N} \tag{16b}$$

$$\frac{\partial J^t}{\partial z_i^t} = -\lambda_i^t \quad \forall i \leq k,\, i \in \mathcal{N} \tag{16c}$$

$$\frac{\partial J^t}{\partial z_j^t} = -\lambda_j^t + \mu \quad \forall j > k,\, j \in \mathcal{N} \tag{16d}$$

where for product i, $i \leq k$, $\lambda_i^t > 0$ implies the product is overstocked and $z_i^t + b_i - \sum_{n=1:N} a_{in}p_n = x_i^t$, $\lambda_i^t = 0$ implies the product is moderately understocked and $x_i^t + K_i > z_i^t + b_i - \sum_{n=1:N} a_{in}p_n > x_i^t$; and $\lambda_i^t < 0$ implies that the product is critically understocked with $z_i^t + b_i - \sum_{n=1:N} a_{in}p_n = x_i^t + K_i$. Similarly, for product j, $j > k$, $\lambda_j^t > 0$ implies that the product is overstocked and $z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n = x_j^t$ while $\lambda_j^t = 0$ implies that the product is understocked and $z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n > x_j^t$. Further, $\mu = 0$ implies that the available flexible capacity is not fully utilized, i.e., $\sum_j \left(z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n\right) < \sum_j x_j^t + K_0$ whereas $\mu > 0$ implies the capacity is entirely used with the corresponding active constraint $\sum_j \left(z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n\right) = \sum_j x_j^t + K_0$.

Using (8), (16a) and (16b), we can solve for prices to get $p_i^t = p_{iL}^t - \lambda_i^t/2$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b}+\mathbf{c})_i/2$ and $p_j^t = p_{jL}^t - \lambda_j^t/2 + \mu^t/2$ where $p_{jL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_j/2$. For part (a), consider product $i \leq k$, $i \in \mathcal{P}$, If $\lambda_i^t = 0$, then $p_i^t = p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$. If $\lambda_i^t < 0$, then $p_i^t = p_{iL}^t - \lambda_i^t/2$ implies it is optimal to markup the price of item i. If, on the other hand, $i \leq k$, $i \in \mathcal{N} \backslash \mathcal{P}$, then $\lambda_i^t > 0$ and it is optimal to give a price discount on product i. For part (b), if the flexible capacity is not fully utilized, we have $\mu_t = 0$. Considering a product $j > k$, if $j \in \mathcal{P}$, then $\lambda_j^t = 0$ and $p_j^t = p_{jL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_j/2$. If, however, $j \in \mathcal{N} \backslash \mathcal{P}$, then $\lambda_j^t > 0$ and $p_j^t = p_{jL}^t - \lambda_j^t$. Hence it is optimal to give a discount to product j. If the flexible capacity is fully utilized, i.e. $\mu^t > 0$. For each product $j > k$, $j \in \mathcal{P}$, we then have $p_j^t = p_{jL}^t + \mu^t/2$. Therefore it is optimal to give the same price surcharge for each item that is produced.

B. Proofs of Supplementary Results:

Proof of Lemma 1: Part (a): The proof is by induction. It can be verified that $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ is strictly concave due to the assumptions on the demand parameters (i.e., $a_{ii} > 0$ and $a_{ii} > |a_{ij}|$) and that the terms associated with holding and backorder costs are strictly concave in (z_1^1, z_2^1). Since $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ is formed by the addition of strictly concave and linear functions, itself is strictly concave. Next, note that the capacity constraints result in a convex domain over which the maximization is performed. Since concavity is preserved under maximization in a convex domain, we have $V^1(x_1^1, x_2^1)$ concave. Now, assume that $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ is strictly concave which yields $V^t(x_1^t, x_2^t)$ to be concave. Then, we have $J^{t+1}(z_1^{t+1}, z_2^{t+1}, p_1^{t+1}, p_2^{t+1})$ strictly concave since it is formed by the addition of a strictly concave term in $(z_1^{t+1}, z_2^{t+1}, p_1^{t+1}, p_2^{t+1})$ and a concave function in (z_1^{t+1}, z_2^{t+1}).

Parts (b) and (c): We first construct the KKT conditions and introduce Lagrange multipliers $\lambda_{ij}^t > 0$ for $i, j = \{1, 2\}$ and $\mu^t > 0$ where $\lambda_{i1}^t > 0$ and $\lambda_{i2}^t > 0$ are associated with constraints $z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t \geq x_i^t$ and $z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t \leq x_i^t + K_0 + K_i$, respectively and μ^t corresponds to the constraint $z_1^t + z_2^t + b_1 + b_2 - (a_{11} + a_{21})p_1^t - (a_{12} + a_{22})p_2^t \leq x_1^t + x_2^t + K_0 + K_1 + K_2$. Together with the complementary slackness conditions, we then have for i={1,2}, $\frac{\partial J^t}{\partial p_i^t} = a_{1i}(\lambda_{11}^t - \lambda_{12}^t) + a_{2i}(\lambda_{21}^t - \lambda_{22}^t) - (a_{1i} + a_{2i})\mu^t$ and $\frac{\partial J^t}{\partial z_i} = \mu^t - (\lambda_{i1}^t - \lambda_{i2}^t)$. Several pairs of constraints form "box constraints" and may not be simultaneously active for positive capacity parameters. As the following observation suggests, we can exploit this special structure of constraints to represent the first-order optimality conditions in simpler notation.

OBSERVATION 1: For $i = 1, 2$, let λ_i^t be defined such that $\lambda_i^t := \lambda_{i1}^t - \lambda_{i2}^t$. Then, λ_i^t uniquely determines λ_{ij}^t for $j = 1, 2$ where (a) $\lambda_i^t < 0$ implies $\lambda_{i1}^t = 0$ and $\lambda_{i2}^t > 0$, (b) $\lambda_i^t > 0$ implies $\lambda_{i1}^t > 0$ and $\lambda_{i2}^t = 0$; and (c) $\lambda_i^t = 0$ implies $\lambda_{i1}^t = \lambda_{i2}^t = 0$. In addition, for $i = \{1, 2\}$, the KKT conditions may be represented as:

$$\frac{\partial J^t}{\partial p_i^t} = a_{1i}\lambda_1^t + a_{2i}\lambda_2^t - (a_{1i} + a_{2i})\mu^t \tag{17a}$$

$$\frac{\partial J^t}{\partial z_i} = \mu^t - \lambda_i^t \tag{17b}$$

Proof: We first observe that having $\lambda_{11}^t > 0$ and $\lambda_{12}^t > 0$ simultaneously, implies that both $z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t - x_1^t = 0$ and $z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t - x_1^t - K_0 - K_1 = 0$. Since this is not possible for any $K_0 + K_1 > 0$ (as there must be some capacity to produce product 1), we conclude that λ_{11}^t and λ_{12}^t cannot be simultaneously positive. Thus, if we define $\lambda_1^t := \lambda_{11}^t - \lambda_{12}^t$, any value of λ_1^t uniquely determines the values of λ_{11}^t and λ_{12}^t. We note that with this definition, λ_1^t is no longer sign restricted. Specifically, we have $\lambda_1^t < 0$ for the case where $\lambda_{11}^t = 0$, $\lambda_{12}^t > 0$, and we have $\lambda_1^t > 0$ for the case where $\lambda_{11}^t > 0$ and $\lambda_{12}^t = 0$. For the case where $\lambda_{11}^t = \lambda_{12}^t = 0$, we have $\lambda_1^t = 0$. An analogous argument holds for λ_{21}^t and λ_{22}^t, hence a corresponding $\lambda_2^t := \lambda_{21}^t - \lambda_{22}^t$ can be similarly defined. □

Following the observation, λ_i^t is no longer sign restricted and is associated with two constraints where its sign - negative, positive or zero - identifies which of the corresponding constraints, if any, is binding. We can now continue with the proof of parts (b) and (c) which are by induction. To simplify the notation, recalling that J^t is separable in (z_1^t, z_2^t) and (p_1^t, p_2^t), we let $J_{ij}^t := \frac{\partial^2 J^t}{\partial z_i^t \partial z_j^t}$. Similarly, we also let $V_{ij}^t = \frac{\partial^2 V^t}{\partial x_i^t \partial x_j^t}$. For $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$, both cross partials are zero, thus $J_{12}^1 = J_{22}^1 = 0$ and part (b) follows. Part (c) results from $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ being strictly concave. We now assume that the Lemma holds for period t and show that it continues to hold for $t+1$. Due to the strictly concave and separable additional terms on holding and backorder costs, it is sufficient to show that $\mathbb{E}V^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)$ preserves these properties with weak inequalities. It can be verified recursively that the first and second derivatives of $V^t(x_1^t, x_2^t)$ are bounded. Through the interchangeability of differentiation and expectation, it is then sufficient to show that $V^t(x_1^t, x_2^t)$ has the required properties. From Envelope Theorem, we have $\frac{\partial V^t(x_1^t, x_2^t)}{\partial x_1^t} = \frac{\partial J^t}{\partial x_1^t} - \lambda_1^t + \mu^t = c_1 - \lambda_1^t + \mu^t$ and $\frac{\partial V^t(x_1^t, x_2^t)}{\partial x_2^t} = \frac{\partial J^t}{\partial x_2^t} - \lambda_2^t + \mu^t = c_2 - \lambda_2^t + \mu^t$. At this point, it is helpful to partition the state space in two broad regions: Region A where $\mu^t = 0$ and Region B where $\mu^t > 0$.

Region A: We first treat the cases associated with $\mu^t = 0$. For these cases, we have $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_1^t}{\partial x_2^t}$. From the KKT conditions, we further have $-\frac{\partial \lambda_1^t}{\partial x_2^t} = \frac{\partial}{\partial x_2^t}\left(\frac{\partial J^t}{\partial z_1^t}\right)$ Therefore,

$$V_{12}^t(x_1^t, x_2^t) = \frac{\partial}{\partial x_2^t}\left(\frac{\partial J^t}{\partial z_1^t}\right) = J_{11}^t \frac{\partial z_1^t}{\partial x_2^t} + J_{12}^t \frac{\partial z_2^t}{\partial x_2^t} \tag{18}$$

We implicitly assume $V_{12}^t(x_1^t, x_2^t) = V_{21}^t(x_1^t, x_2^t)$ which requires continuity of the second partial derivatives. This is fulfilled since J^t is strictly concave and twice continuously differentiable in (z_1^t, z_2^t) and (z_1^t, z_2^t) are differentiable in (x_1^t, x_2^t). There are four cases to consider: (1) $\lambda_1^t = 0$ or $\lambda_2^t = 0$, (2) $\lambda_1^t > 0$ and $\lambda_2^t > 0$, (3) $\lambda_1^t > 0$ and $\lambda_2^t < 0$, and (4) $\lambda_1^t < 0$ and $\lambda_2^t > 0$. (Note that the case $\lambda_1^t < 0$ and $\lambda_2^t < 0$ is not feasible as the flexible capacity may not be utilized in full for each product individually.)

Case 1: When $\lambda_1^t = 0$, we have $\frac{\partial \lambda_1^t}{\partial x_2^t} = 0$. Thus $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_1^t}{\partial x_2^t} = 0$. A similar argument for $\lambda_2^t = 0$ also yields $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_2^t}{\partial x_1^t} = 0$. This establishes the result for part (b), i.e., that $V^t(x_1^t, x_2^t)$ is submodular. For part (c), since $V^t(x_1^t, x_2^t)$ is concave, we have $V_{11}^t(x_1^t, x_2^t) \le 0$, hence $V_{11}^t(x_1^t, x_2^t) \le V_{12}^t(x_1^t, x_2^t)$. The result for $V_{22}^t(x_1^t, x_2^t)$ is similar.

Case 2: When $\lambda_1^t > 0$ and $\lambda_2^t > 0$, from KKT conditions we have $p_1^t = p_{1L}^t + \frac{1}{2}\frac{\partial J^t}{\partial z_1^t}$ and $p_2^t = p_{2L}^t + \frac{1}{2}\frac{\partial J^t}{\partial z_2^t}$ where $p_{1L}^t = \frac{a_{22}b_1 - a_{12}b_2}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_1}{2}$ and $p_{2L}^t = \frac{a_{11}b_2 - a_{12}b_1}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_2}{2}$. Complementary slackness yields $x_1^t = z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t$ and $x_2^t = z_2^t + b_2 - a_{21}p_1^t - a_{22}p_2^t$. Combining these we get $x_1^t = z_1^t + b_1 - a_{11}p_{1L}^t - a_{12}p_{2L}^t - \frac{a_{11}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{12}}{2}\frac{\partial J^t}{\partial z_2^t}$ and $x_2^t = z_2^t + b_2 - a_{21}p_{1L}^t - a_{22}p_{2L}^t - \frac{a_{21}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{22}}{2}\frac{\partial J^t}{\partial z_2^t}$. Taking partial derivatives with respect to x_2^t and solving for $\frac{\partial z_1^t}{\partial x_2^t}$ and $\frac{\partial z_2^t}{\partial x_2^t}$, we get $\frac{\partial z_1^t}{\partial x_2^t} = \frac{1}{\Lambda}\left(a_{11}J_{12}^t + a_{12}J_{22}^t\right)$ and $\frac{\partial z_2^t}{\partial x_2^t} = \frac{1}{\Lambda}\left(2 - a_{11}J_{11}^t - a_{12}J_{21}^t\right)$ where $\Lambda = 2 - (a_{11}J_{11}^t + 2a_{12}J_{12}^t + a_{22}J_{22}^t) + \frac{1}{2}\left[(a_{11}a_{22} - a_{12}^2)(J_{11}^t J_{22}^t - J_{12}^{t\,2})\right]$. We note that $\Lambda > 0$ by first observing that the terms in the brackets are strictly positive since $a_{11}a_{22} - a_{12}^2 > 0$ by the assumptions on demand parameters and $J^t_{11}J^t_{22} - {J^t_{12}}^2 \geq 0$. We only need to show that $a_{11}J^t_{11} + 2a_{12}J^t_{12} + a_{22}J^t_{22} \leq 0$. We have $a_{11}J^t_{11} + 2a_{12}J^t_{12} + a_{22}J^t_{22} \leq (a_{11} + 2a_{12} + a_{22})\, J^t_{12} \leq 0$ where the first inequality is due to diagonal dominance and the second is due to $a_{11} + a_{12} > 0, a_{12} + a_{22} > 0$ and $J^t_{12} \leq 0$. Substituting the expressions for $\frac{\partial z^t_1}{\partial x^t_2}$ and $\frac{\partial z^t_2}{\partial x^t_2}$ into (18) establishes submodularity as follows: $V^t_{12}(x^t_1, x^t_2) = \frac{1}{\Lambda}\left(J^t_{11}\left(a_{11}J^t_{12} + a_{12}J^t_{22}\right) + J^t_{12}\left(2 - a_{11}J^t_{11} - a_{12}J^t_{21}\right)\right) = \frac{1}{\Lambda}\left(a_{12}\left(J^t_{11}J^t_{22} - {J^t_{12}}^2\right) + 2J^t_{12}\right) \leq 0$ where the inequality is due to $J^t_{12} \leq 0$, $J^t_{11}J^t_{22} - {J^t_{12}}^2 > 0$, and $a_{12} \leq 0$. Part (c) may be shown similarly by evaluating the expressions for $V^t_{11}(x^t_1, x^t_2)$ and $V^t_{22}(x^t_1, x^t_2)$, $\frac{\partial z^t_1}{\partial x^t_1}$ and $\frac{\partial z^t_2}{\partial x^t_1}$. The analysis for Cases 3 and 4 are very similar to the analysis of Case 2 and are omitted for brevity.

Region B: We now consider the region corresponding to $\mu^t > 0$. By the definition of the multipliers and their relationships among each other, this region is subdivided into three subregions such that (1) $\mu^t > 0$, $\lambda^t_1 < 0$, $\lambda^t_2 = 0$; (2) $\mu^t > 0$, $\lambda^t_1 = 0$, $\lambda^t_2 = 0$; (3) $\mu^t > 0$, $\lambda^t_1 = 0$, $\lambda^t_2 < 0$.

Case 1 corresponds to the regions where the flexible capacity is used solely and fully to produce item 1. Once again, the Envelope Theorem yields $V^t_{12}(x^t_1, x^t_2) = \frac{\partial}{\partial x^t_2}\left(\frac{\partial J^t}{\partial z^t_1}\right) = J^t_{11}\frac{\partial z^t_1}{\partial x^t_2} + J^t_{12}\frac{\partial z^t_2}{\partial x^t_2}$. Complementary slackness conditions yield $x^t_1 = z^t_1 + b_1 - a_{11}p^t_1 - a_{12}p^t_2 - K_0 - K_1$ and $x^t_2 = z^t_2 + b_2 - a_{21}p^t_1 - a_{22}p^t_2 - K_2$. Combining these, we get $x^t_1 = z^t_1 + b_1 - a_{11}p^t_{1L} - a_{12}p^t_{2L} - K_0 - K_1 - \frac{a_{11}}{2}\frac{\partial J^t}{\partial z^t_1} - \frac{a_{12}}{2}\frac{\partial J^t}{\partial z^t_2}$ and $x^t_2 = z^t_2 + b_2 - a_{21}p^t_{1L} - a_{22}p^t_{2L} - K_2 - \frac{a_{21}}{2}\frac{\partial J^t}{\partial z^t_1} - \frac{a_{22}}{2}\frac{\partial J^t}{\partial z^t_2}$. The same arguments as presented in the analysis of the previous case yields the desired result. Further, the analysis for Case 3 is also symmetric to the analysis of Case 1 and hence omitted.

Case 2 defines the only remaining region and it corresponds to $\mu^t > 0$, $\lambda^t_1 = 0$, $\lambda^t_2 = 0$, where the flexible capacity is used fully to produce both products simultaneously. In this region we have $x^t_1 + x^t_2 = z^t_1 + z^t_2 + b_1 + b_2 - (a_{11} + a_{21})p^t_1 - (a_{12} + a_{22})p^t_2 - K_0 - K_1 - K_2$. Differentiating with respect to x^t_2, we get:

$$1 = \left(1 - \frac{(a_{11} + a_{21})}{2}J^t_{11} - \frac{(a_{12} + a_{22})}{2}J^t_{21}\right)\frac{\partial z^t_1}{\partial x^t_2} + \left(1 - \frac{(a_{11} + a_{21})}{2}J^t_{12} - \frac{(a_{12} + a_{22})}{2}J^t_{22}\right)\frac{\partial z^t_2}{\partial x^t_2} \tag{19}$$

Through the KKT conditions, in this region we also have $J^t_1 = J^t_2$, hence differentiating with respect to x^t_2 we get

$$J^t_{11}\frac{\partial z^t_1}{\partial x^t_2} + J^t_{12}\frac{\partial z^t_2}{\partial x^t_2} = J^t_{21}\frac{\partial z^t_1}{\partial x^t_2} + J^t_{22}\frac{\partial z^t_2}{\partial x^t_2} \tag{20}$$

Combining (19) and (20), we get $\frac{\partial z^t_1}{\partial x^t_2} = \frac{1}{\Lambda'}(J^t_{12} - J^t_{22})$ and $\frac{\partial z^t_2}{\partial x^t_2} = \frac{1}{\Lambda'}(J^t_{12} - J^t_{11})$. where $\Lambda' = -J^t_{12} + 2J^t_{12} - J^t_{12} + (a_{11} + a_{12})(J^t_{11}J^t_{22} - {J^t_{12}}^2)/2 + (a_{22} + a_{12})(J^t_{11}J^t_{22} - {J^t_{12}}^2)/2$. As in the previous discussion for Λ, it can easily be shown that $\Lambda' > 0$. Substituting $\frac{\partial z^t_1}{\partial x^t_2}$ and $\frac{\partial z^t_2}{\partial x^t_2}$ into (18), we get $V^t_{12}(x^t_1, x^t_2) = \frac{1}{\Lambda}\left(J^t_{11}\left(J^t_{12} - J^t_{22}\right) + J^t_{12}\left(J^t_{12} - J^t_{11}\right)\right) = \frac{1}{\Lambda}\left({J^t_{12}}^2 - J^t_{11}J^t_{22}\right) \leq 0$. Similar steps verify part (c). □

Proof of Lemma A.1: We first introduce two functions $F_1(\mathbf{L}^t, \mathbf{z}^t)$ and $F_2(\mathbf{L}^t, \mathbf{z}^t)$ where $\mathbf{L}^t = (\lambda^t_1, \lambda^t_2, \mu^t)$ and $\mathbf{z}^t = (z^t_1, z^t_2)$. We define these functions to represent KKT conditions (17b).

$$F_1(\mathbf{L}^t, \mathbf{z}^t) = J^t_1(z^t_1, z^t_2) + \lambda^t_1 - \mu^t \tag{21a}$$

$$F_2(\mathbf{L}^t, \mathbf{z}^t) = J^t_2(z^t_1, z^t_2) + \lambda^t_2 - \mu^t \tag{21b}$$

Differentiating (21a) and (21b), and letting $D_{\mathbf{L}}F$ and $D_{\mathbf{z}}F$ to denote partial Jacobians, we have

$$D_{\mathbf{L}}F = \begin{bmatrix} \frac{\partial F_1}{\partial \lambda_1} & \frac{\partial F_1}{\partial \lambda_2} & \frac{\partial F_1}{\partial \mu} \\ \frac{\partial F_2}{\partial \lambda_1} & \frac{\partial F_2}{\partial \lambda_2} & \frac{\partial F_2}{\partial \mu} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad D_{\mathbf{z}}F = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_2} & \frac{\partial F_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} J^t_{11} & J^t_{12} \\ J^t_{21} & J^t_{22} \end{bmatrix}$$

Since $J^t(z_1^t, z_2^t)$ is strictly concave by Lemma 1, $D_{\mathbf{z}}F$ is invertible. Thus, there exists implicit functions ϕ_1 and ϕ_2 such that $z_1^t = \phi_1(\lambda_1^t, \lambda_2^t, \mu^t)$ and $z_2^t = \phi_2(\lambda_1^t, \lambda_2^t, \mu^t)$. Moreover, by the Implicit Function Theorem, we have $D\phi = -D_{\mathbf{z}}F^{-1}D_{\mathbf{L}}F$, that is,

$$\begin{bmatrix} \frac{\partial\phi_1}{\partial\lambda_1} & \frac{\partial\phi_1}{\partial\lambda_2} & \frac{\partial\phi_1}{\partial\mu} \\ \frac{\partial\phi_2}{\partial\lambda_1} & \frac{\partial\phi_2}{\partial\lambda_2} & \frac{\partial\phi_2}{\partial\mu} \end{bmatrix} = -\begin{bmatrix} J_{11}^t & J_{12}^t \\ J_{21}^t & J_{22}^t \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{J_{11}^t J_{22}^t - (J_{12}^t)^2} \begin{bmatrix} -J_{22}^t & J_{12}^t & J_{22}^t - J_{12}^t \\ J_{12}^t & -J_{11}^t & J_{11}^t - J_{12}^t \end{bmatrix}$$

The strict concavity established in Lemma 1 yields $J_{11}^t < 0, J_{22}^t < 0$, and $J_{11}^t J_{22}^t - (J_{12}^t)^2 > 0$. The submodularity and diagonal dominance properties in Lemma 1, gives $J_{12}^t \le 0, J_{11}^t - J_{12}^t < 0, J_{22}^t - J_{12}^t < 0$. Therefore, the monotonicity results follow immediately. □

Proof of Lemma A.5: The proof is by induction. For t=1, $-\left(\frac{\partial J^t}{\partial z_i^t \partial z_j^t}\right)^{-1}$ is a diagonal matrix with positive diagonal elements (due to the strict concavity of expected holding and shortage costs) and hence is a strictly diagonally dominant Stieltjes matrix. Now assume that the result holds for period t. By Envelope Theorem, we have $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = -\left(\frac{\partial \lambda_i^t}{\partial x_j^t}\right)$. Several cases arise since each λ_i^t may be greater than, less than, or equal to zero.

We first consider the case where $\lambda_i^t \neq 0 \; \forall i \in \mathcal{N}$. Similar to the steps in the proof of Lemma 1, differentiating all active constraints with respect to each x_i^t, we get $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) \cdot \left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) + \frac{\mathbf{A}}{2} \cdot \left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = e_i$. Hence, we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1} e_i$, where it can be verified by inspection that both $\mathbf{A}$ and the sum $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}$ are also strictly diagonally dominant Stieltjes matrices, thus invertible by Lemma 4. We therefore have $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = -\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$. If we let $\mathbf{D} := diag(h_1'', h_2'', ..., h_N'')$ denote a diagonal matrix with positive diagonal elements which refers to the partial derivatives of the expected holding and shortage costs, then, $\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right) = -\mathbf{D} - \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$. We note that if a matrix $\mathbf{M}$ is a strictly diagonally dominant Stieltjes matrix, and $\mathbf{D}$ is a positive diagonal matrix, then $(\mathbf{D} + \mathbf{M}^{-1})^{-1}$ is a strictly diagonally dominant Stieltjes matrix. (The proof of this claim can be found in Lemma A.3 in Ye (2008) where their definition of competitive matrix refers to a diagonally dominant Stieltjes matrix and their analysis to show that diagonal dominance holds is extendable to show that strict diagonal dominance holds as well.) Therefore, $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1} = \left(\mathbf{D} + \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}\right)^{-1}$ is a strictly diagonally dominant Stieltjes matrix.

Consider now a case for which there exists some $\lambda_k^t = 0$, $k \in \mathcal{N}$. For representation purposes and without loss of generality, assume $\lambda_N^t = 0$. Then, differentiating the active constraints for x_i^t, $i \neq N$ and noting that $\frac{\partial \lambda_N^t}{\partial x_i^t} = \frac{\partial \lambda_i^t}{\partial x_N^t} = 0$, we find $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$ where $\mathbf{S}$ is a principle submatrix of $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)$ of dimension $N-1$ x $N-1$ and is also a strictly diagonal dominant Stieltjes matrix (Varga 2009). Then, $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1} = \left(\mathbf{D} + \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}\right)^{-1} = \left(\mathbf{D}' + \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \delta_N \end{bmatrix}\right)^{-1} = \left(\mathbf{D}' + \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\delta_N} \end{bmatrix}^{-1}\right)^{-1}$ where $\mathbf{D}' = diag(h_1'', h_2'', ..., h_N'' - \delta_N)$ for $\delta_N > 0$ sufficiently small such that $h_N'' - \delta_N > 0$. It can be verified that $\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\delta_N} \end{bmatrix}$ is also a Stieltjes matrix, hence by the same argument as in the previous case, we have $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1}$ a strictly diagonally dominant Stieltjes matrix. The analysis for other cases where more than one $\lambda_k^t = 0$ are similar. □

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