Online Appendix to Facility Location Decisions with Random Disruptions and Imperfect Estimation

Appendix A: Validation of the tiling scheme

The derivation of (5) is based on a diamond-shaped tiling scheme such that the facilities can be evenly spread while locating one reliable facility surrounded by unreliable facilities. Here, we test how sensitive the results are when this particular tiling scheme is relaxed. Denoting the configuration ratio between the total number of facilities and the number of reliable facilities to β (i.e., $\beta = \frac{n_t}{n_r} > 0$), the expected total cost becomes

$$\mathbb{E}[TC(\beta)] = 3\Big(\frac{\gamma}{2}\Big)^{\frac{2}{3}} f_u^{\frac{1}{3}} \Big(r+\beta-1\Big)^{\frac{1}{3}} \Big(q + \frac{1-q}{\sqrt{\beta}}\Big)^{\frac{2}{3}} \tag{11}$$

where the optimal configuration ratio is $\beta^* = \big(\frac{(1-q)(r-1)}{q}\big)^{\frac{2}{3}}$ for $q \geq q_{th}$ and $\beta^* = 1$ otherwise. The proposed tiling scheme provides an "exact" solution only for $\beta = n^2, \forall n \in \mathbb{N}$. For other configuration ratios, $\beta \neq n^2$, finding the *actual* optimal tiling scheme may be difficult. This resulting configuration may not result in diamond-shape tiles. We assume such an actual tiling scheme exists.

Based on an infinite service region (to avoid integrality issue on the number of facilities), we investigate how large the error will be for using the proposed tiling scheme instead of actual tiling scheme. We refer to the expected total cost based on an actual optimal tiling scheme to *true expected total cost*. Denoting the true expected total cost by $\overline{TC}(\beta)$ and its actual configuration ratio by $\overline{\beta}$, we provide the following proposition.

Proposition 7. *$\mathbb{E}[\overline{TC}(\beta)]$ is bounded below by $\mathbb{E}[TC(\beta)]$. Further, $\mathbb{E}[\overline{TC}(\overline{\beta})]$ is bounded below by $\mathbb{E}[TC(\beta^*)]$.*

Proof. Since all unreliable facilities have the same disruption probability, it is sufficient for the optimal configuration to satisfy the optimal tiling scheme (non-overlapping diamond-shaped tiles) for the following two extreme cases: (a) when all unreliable facilities are working, (b) when none of the unreliable facilities are working. For a contiguous service region, the only configuration that satisfies this condition is when $\beta = n^2, n \in \mathbb{N}$ (one reliable facility in the center surrounded by unreliable facilities). For $\beta \neq n^2, n \in \mathbb{N}$, an actual tiling scheme does not satisfy the optimal tiling scheme and incurs higher total cost. Hence, $\mathbb{E}[TC(\beta)] \leq \mathbb{E}[\overline{TC}(\beta)]$ for all $\beta > 0$.

Taking the infimum on each side from Proposition 7, we have $\inf_\beta \mathbb{E}[TC(\beta)] \leq \inf_\beta \mathbb{E}[\overline{TC}(\beta)]$. Since $\mathbb{E}[\overline{TC}(\overline{\beta})] = \inf_\beta \mathbb{E}[\overline{TC}(\beta)]$ and $\mathbb{E}[TC(\beta^*)] = \inf_\beta \mathbb{E}[TC(\beta)]$, we conclude that $\mathbb{E}[TC(\beta^*)] \leq \mathbb{E}[\overline{TC}(\overline{\beta})]$. □

We also obtain the upper bound on the true expected total cost. By restricting the configuration ratio to $\beta = n^2$, $\forall n \in \mathbb{N}$ and using the proposed diamond tiling scheme, we can find the configuration ratio $\tilde{\beta}$ that minimizes the expected total cost; that is, $\tilde{\beta} = \operatorname{argmin}_{\beta \in \{\beta = n^2, \forall n \in \mathbb{N}\}} \mathbb{E}[TC(\beta)]$.

Proposition 8. *The expected total cost with configuration ratio $\tilde{\beta}$, $\mathbb{E}[TC(\tilde{\beta})]$, can be derived as*

$$\mathbb{E}[TC(\tilde{\beta})] = \min\big\{\, \mathbb{E}[TC(\lfloor\sqrt{\beta^*}\rfloor^2)],\, \mathbb{E}[TC(\lceil\sqrt{\beta^*}\rceil^2)] \,\big\}.$$

Then, $\mathbb{E}[\overline{TC}(\overline{\beta})]$ is bounded above by $\mathbb{E}[TC(\tilde{\beta})]$.

Proof. We know that $\mathbb{E}[TC(\beta)]$ is strictly convex in β since $\frac{\partial^2 TC}{\partial \beta^2} = \frac{3\gamma}{4\sqrt{n_r}}(1-q)\beta^{-\frac{5}{2}} > 0$. Hence, $\mathbb{E}[TC(\beta)]$ is a unimodal function with the minimum at β^* and thus $\tilde{\beta}$ is the minimum between $\lfloor\sqrt{\beta^*}\rfloor^2$ and $\lceil\sqrt{\beta^*}\rceil^2$.

Further, we know that $\inf_\beta \mathbb{E}[\overline{TC}(\beta)] \leq \mathbb{E}[\overline{TC}(\tilde{\beta})]$ holds. Therefore, $\mathbb{E}[\overline{TC}(\overline{\beta})] = \inf_\beta \mathbb{E}[\overline{TC}(\beta)] \leq \mathbb{E}[\overline{TC}(\tilde{\beta})] = \mathbb{E}[TC(\tilde{\beta})]$. □

We next conduct a numerical experiment to study the performance of the current solution method. For the case example, the percentage optimality gap ($\Big[\frac{\mathbb{E}[TC(\tilde{\beta})]}{\mathbb{E}[TC(\beta^*)]} - 1\Big] \times 100\,(\%)$) in the expected total cost by restricting to a diamond-shaped tiling scheme is computed in Table 2. This result shows that the optimality gap is typically very small implying that our current solution method provides a very good approximation in terms of the expected total cost.

Table 2 **Percentage optimality gap using the diamond-shaped tiling scheme**

r	$q=0.01$	0.05	0.10	0.20	0.30	0.40	0.50
1.25	0.002%	0.216%	0.873%	0%	0%	0%	0%
1.5	0.025%	0.030%	0.462%	1.015%	0.057%	0%	0%
2	0.025%	0.144%	0.024%	1.053%	1.893%	0.451%	0%
5	0.015%	0.049%	0.183%	0.712%	0.066%	0.228%	1.288%

Appendix B: Line analysis

Denote l to be length of the line (assumed large) and other notation remain the same (such as ρ, c, n). With n facilities on the line, the expected distance for any demand to its nearest facility is $\mathbb{E}[D] = \frac{l\rho c}{4n}$. When there is only one reliable facility (that is located at the center of the line), the expected distance to the reliable facility for the demand outside of the central region is $\mathbb{E}[D_U] = \left(\frac{n+1}{4n}\right) l\rho c$.

Given that the number of each type of facilities are n_r and n_u, the expected distance to the nearest facility is $\mathbb{E}[D] = \frac{l\rho c}{4(n_r+n_u)}$. Also, the expected distance to the reliable facility when its primary unreliable facility fails is $\mathbb{E}[D_U] = \frac{l\rho c}{4}\left(\frac{1}{n_r} + \frac{1}{n_r+n_u}\right)$. Hence, the expected total cost is $\mathbb{E}[TC] = f_r n_r + f_u n_u + q\bar{\gamma}\frac{1}{n_r} + (1-q)\bar{\gamma}\frac{1}{n_r+n_u}$ where $\bar{\gamma} = \frac{l\rho c}{4}$. Note that this has the same functional form as equation (5) for our analysis on plane. The square root term is due to the difference between a line and a plane. Using this, we obtain the optimal number of facilities as:

$$\begin{cases} n_r^* = \left(\frac{\bar{\gamma}q}{f_r - f_u}\right)^{\frac{1}{2}} \text{ and } n_r^* + n_u^* = \left(\frac{\bar{\gamma}(1-q)}{f_u}\right)^{\frac{1}{2}} & \text{where } q \le q_{th} = \frac{f_r - f_u}{f_r}, \\ n_r^* = \left(\frac{\bar{\gamma}}{f_r}\right)^{\frac{1}{2}} \text{ and } n_u^* = 0 & \text{otherwise} \end{cases}.$$

Appendix C: Robustness test on the discrete model

To verify the robustness of the proposed model in this paper, we test one of our main results (underestimation hurts more than overestimation) in this paper under a more general setting. We employ a discrete version counterpart model introduced by Lim et al. (2010) briefly explained below.

Consider a set of demand points I where customers resides in and a set of candidate sites J where the facilities can be located. We assume that each node is a demand node and a candidate facility site, i.e., $I = J$, for this study. At each node, $j \in N$ where N is the set of all nodes, we can locate either an unreliable facility at a cost of f_j^U or a reliable facility at a cost of $f_j^R > f_j^U$. The cost of traveling to a facility j which has not failed from demand node i is given by d_{ij}^P and the cost of traveling to a backup facility (if the primary facility has failed) is given by $d_{ij}^B \ge d_{ij}^P$. We also denote d_{ij}^S $(= d_{ij}^B - d_{ij}^P \ge 0)$ as the unit savings that has to be subtracted from the objective function when demand node i is assigned to a reliable facility at j as both the primary facility at a unit cost of d_{ij}^P and the backup facility at a unit cost of d_{ij}^B since the true assignment cost of node i to the facility at j should be only d_{ij}^P, since the facility is completely reliable. Finally, we define the following decision variables:

X_j^U = 1 if an unreliable facility is located at candidate site j; 0 if not
X_j^R = 1 if a reliable facility is located at candidate site j; 0 if not
Y_{ij}^P = 1 if demands at i are assigned to a facility at j as the primary site; 0 if not
Y_{ij}^B = 1 if demands at i are assigned to a facility at j as the backup site; 0 if not
Y_{ij}^S = 1 if demands at i are assigned to a facility at j as the primary and backup site; 0 if not

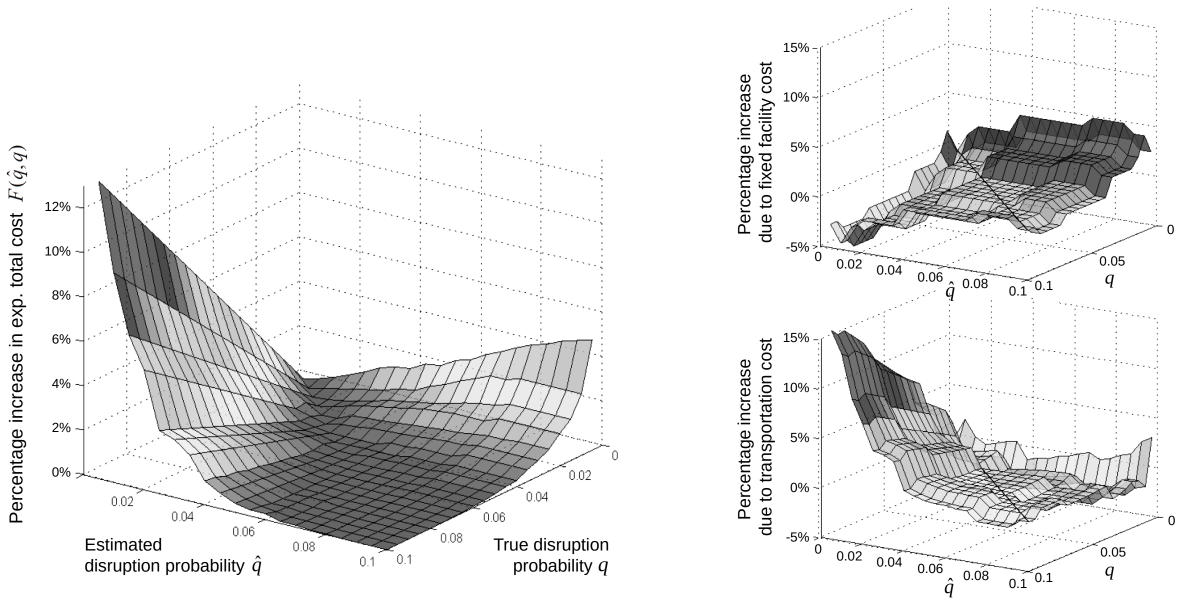
With this notation, the discrete version of the model can be formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j\in N} f_j^U X_j^U + \sum_{j\in N} f_j^R X_j^R \\ & + \sum_{i\in N}\sum_{j\in N}(1-q_j) h_i d_{ij}^P Y_{ij}^P + \sum_{i\in N}\sum_{j\in N} q_j h_i d_{ij}^B Y_{ij}^B - \sum_{i\in N}\sum_{j\in N} q_j h_i d_{ij}^S Y_{ij}^S \\ \text{subject to} \quad & \\ & \sum_{j\in N} Y_{ij}^P = 1, \qquad \sum_{j\in N} Y_{ij}^B = 1 \qquad \forall i \in N \\ & Y_{ij}^P \le X_j^R + X_j^U, \qquad Y_{ij}^B \le X_j^R \qquad \forall i, j \in N \end{aligned}$$

$$\begin{aligned}
&Y_{ij}^S \le Y_{ij}^B, \qquad Y_{ij}^S \le Y_{ij}^P \qquad \forall i,j \in N \\
&X_j^R + X_j^U \le 1 \qquad \forall j \in N \\
&X_j^U, X_j^R \in \{0,1\} \qquad \forall j \in N \\
&Y_{ij}^P, Y_{ij}^B, Y_{ij}^S \in \{0,1\} \qquad \forall i,j \in N.
\end{aligned}$$

To introduce heterogeneity, we relax five assumptions from the continuous model: (a) instead of demand being uniformly distributed, each demand node has its own demand level (we use the population of the city), (b) each facility is allowed to have different fixed costs (we set the facility cost proportional to the population of the city, $f_j^U = 500,000 \times (1.7h_j)$ where h_j is the population of city j), (c) each facility can have its own disruption probability (though here we use $q_j = 0.05$ to be consistent with the current work), (d) any type of distance metric is allowed (we use great circle distances), (e) transportation unit costs are allowed to differ for the backup assignments to capture the extra cost incurred when a disruption occurs (we use $d_{ij}^B/d_{ij}^P = 1.25$). The data set of the 263 largest cities in the contiguous United States was used in the numerical study. We set all other parameters identical (or as close as possible) to the numerical example in Figures 2. For the solution methods and some structural properties of this model, please see Lim et al. (2010).

The impact of misestimating the disruption probability for the discrete model is illustrated in the below figure. This is a counterpart of Figures 2 in the paper. Although the modeling framework and some key assumptions are relaxed, we observe a strong concurrence in general trends with the results of the continuous model: the impact of underestimation is much greater than the impact of overestimation for the discrete model. We believe this suggests that the insights and the analytical results from the continuous model is quite robust under a more generalized setting. Unfortunately, capacity issue in discrete facility location problems come with a huge computational challenge, we limit the analysis to the basic setting for this study.



(a)Percentage increase with misestimation

(b)Percentage increase due to fixed facility cost (top); Percentage increase due to transportation cost (bottom)

Figure 7 Impact of disruption probability misestimation in discrete model

Appendix D: Proofs of the main results

Proof of Proposition 1. Using the first order conditions (and that (5) is convex with respect to n_r and n_u), the optimal number of reliable and unreliable facilities (6) is derived. Since the number of unreliable facilities has to be non-negative, we have $n_t^* \geq n_r^*$. Hence, for (6) to be a feasible solution, we have $\left(\frac{\gamma(1-q)}{2f_u}\right)^{\frac{2}{3}} \geq \left(\frac{\gamma q}{2(f_r-f_u)}\right)^{\frac{2}{3}}$ which in turn gives us the following threshold condition on q: if $q \leq \frac{(f_r-f_u)}{f_r}$ the optimal facility configuration is given by (6). If $q > \frac{(f_r-f_u)}{f_r}$, it is optimal to locate only reliable facilities. In this case, the optimal solution becomes (7). □

Proof of Proposition 2. To show $\frac{\partial F(q,q(1\pm\delta))}{\partial q} = \frac{TC(q,q)\frac{\partial TC(q,q(1\pm\delta))}{\partial q} - TC(q,q(1\pm\delta))\frac{\partial TC(q,q)}{\partial q}}{TC(q,q)^2} > 0$, it is sufficient to show that

$$TC(q,q)\frac{\partial TC(q,q(1\pm\delta))}{\partial q} - TC(q,q(1\pm\delta))\frac{\partial TC(q,q)}{\partial q} \tag{12}$$

is strictly positive. We start with the overestimation case, $\hat{q} = q(1+\delta)$. We know $TC(q,q) = 3f^{\frac{1}{3}}(\frac{\gamma}{2})^{\frac{2}{3}}[q^{\frac{2}{3}}(r-1)^{\frac{1}{3}} + (1-q)^{\frac{2}{3}}]$ from (8). Further, we have $TC(q,q(1+\delta)) = f^{\frac{1}{3}}(\frac{\gamma}{2})^{\frac{2}{3}}\big[q^{\frac{2}{3}}(3+\delta)(\frac{r-1}{1+\delta})^{\frac{1}{3}} + (1-q(1+\delta))^{\frac{2}{3}}\big(1+\frac{2(1-q)}{1-q(1+\delta)}\big)\big]$, $\frac{\partial TC(q,q)}{\partial q} = 2f^{\frac{1}{3}}(\frac{\gamma}{2})^{\frac{2}{3}}\big[(\frac{r-1}{q})^{\frac{1}{3}} - (\frac{1}{1-q})^{\frac{1}{3}}\big]$, and $\frac{\partial TC(q,q(1+\delta))}{\partial q} = \frac{2}{3}f^{\frac{1}{3}}(\frac{\gamma}{2})^{\frac{2}{3}}(3+\delta)\big[(\frac{r-1}{q(1+\delta)})^{\frac{1}{3}} - (\frac{1}{1-q(1+\delta)})^{\frac{4}{3}}\big(1-q(1+\delta)-\frac{\delta}{3+\delta}\big)\big]$. Using this (with some algebra), (12) can be expressed as the sum of three terms as follows:

$$\begin{aligned}2f^{\frac{2}{3}}\Big(\frac{\gamma}{2}\Big)^{\frac{4}{3}}\Big\{&\Big[q\delta^2\Big(\frac{1}{1-q}\Big)^{\frac{1}{3}}\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{4}{3}}\Big]\\ &+q^{\frac{2}{3}}(r-1)^{\frac{1}{3}}(3+\delta)\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{1}{3}}\Big[-1+\frac{1}{(1-q(1+\delta))(3+\delta)}+\Big(\frac{1}{(1-q)(1+\delta)}\Big)^{\frac{1}{3}}\Big]\\ &+\Big(\frac{r-1}{q}\Big)^{\frac{1}{3}}(1-q)^{\frac{2}{3}}\Big[(3+\delta)\Big(\frac{1}{1+\delta}\Big)^{\frac{1}{3}}-\Big(\frac{1-q(1+\delta)}{1-q}\Big)^{\frac{2}{3}}-2\Big(\frac{1-q}{1-q(1+\delta)}\Big)\Big]\Big\}.\end{aligned}$$

It follows that for $0 < q \leq \frac{q_{th}}{2} = \frac{r-1}{2r}$, $r > 1$, and $0 < \delta < 1$, all the terms within each bracket are positive. More specifically, it is straightforward to see that the first term is strictly positive. For the second term, one can show that this is strictly positive by investigating: (a) $(1-q)(1+\delta) > 1$ and (b) $(1-q)(1+\delta) \leq 1$. The last term can be shown to be strictly positive by using $0 < \delta < 1$, $0 < q \leq \frac{r-1}{2r} < \frac{1}{2}$. Thus the sign of (12) is also strictly positive. The same result can be shown for the underestimation case, $\hat{q} = q(1-\delta)$, through a very similar exercise. Therefore, we conclude $\frac{\partial F(q,q(1\pm\delta))}{\partial q} > 0$ for $0 < \delta < 1$. □

Proof of Proposition 3. To show $F(q,q(1+\delta)) < F(q,q(1-\delta))$, we equivalently show $TC(q,q(1+\delta)) < TC(q,q(1-\delta))$. From (5), we have

$$\begin{aligned}&TC(q,q(1+\delta)) - TC(q,q(1-\delta))\\ &= f_r\big(n_r(q(1+\delta)) - n_r(q(1-\delta))\big) + f_u\big(n_u(q(1+\delta)) - n_u(q(1-\delta))\big)\\ &\quad + q\gamma\Big(\sqrt{\tfrac{1}{n_r(q(1+\delta))}} - \sqrt{\tfrac{1}{n_r(q(1-\delta))}}\Big) + (1-q)\gamma\Big(\sqrt{\tfrac{1}{n_t(q(1+\delta))}} - \sqrt{\tfrac{1}{n_t(q(1-\delta))}}\Big)\\ &= f^{\frac{1}{3}}\Big(\frac{\gamma}{2}\Big)^{\frac{2}{3}}q^{\frac{2}{3}}(r-1)^{\frac{1}{3}}\Big((3+\delta)(1+\delta)^{-\frac{1}{3}} - (3-\delta)(1-\delta)^{-\frac{1}{3}}\Big).\end{aligned}$$

The above quantity is strictly negative for all $0 < q \leq \frac{q_{th}}{2} = \frac{r-1}{2r}$, $r > 1$, and $0 < \delta < 1$. Therefore, we conclude $F(q,q(1+\delta)) < F(q,q(1-\delta))$. □

Proof of Proposition 4. Define $G(q,\delta) := \frac{F(q,q(1-\delta))}{F(q,q(1+\delta))} = \frac{TC(q,q(1-\delta))}{TC(q,q(1+\delta))}$. Then, $G(q,0) = 1$. To show that the percentage increase due to underestimating the disruption probability by δ relative to overestimation increases with error rate δ, we show

$$\frac{\partial G(q,\delta)}{\partial \delta} = \frac{TC\big(q,q(1+\delta)\big)\frac{\partial TC(q,q(1-\delta))}{\partial \delta} - TC\big(q,q(1-\delta)\big)\frac{\partial TC(q,q(1+\delta))}{\partial \delta}}{TC\big(q,q(1+\delta)\big)^2}$$

is strictly positive for $0 < \delta < 1$. Since $\frac{\partial TC(q,q(1\pm\delta))}{\partial \delta} = \gamma\sqrt{\frac{1}{n_r(q(1\pm\delta))}} - \gamma\sqrt{\frac{1}{n_t(q(1\pm\delta))}}$, it is sufficient to examine the sign of

$$TC\big(q,q(1+\delta)\big)\Big[\sqrt{\tfrac{1}{n_r(q(1-\delta))}} - \sqrt{\tfrac{1}{n_t(q(1-\delta))}}\Big] - TC\big(q,q(1-\delta)\big)\Big[\sqrt{\tfrac{1}{n_r(q(1+\delta))}} - \sqrt{\tfrac{1}{n_t(q(1+\delta))}}\Big]. \tag{13}$$

After some algebra, (13) can be expressed as follows:

$$f^{\frac{2}{3}}\Big(\frac{\gamma}{2}\Big)^{\frac{1}{3}}\Big\{2\delta(r-1)^{\frac{2}{3}}\Big(\frac{q}{1-\delta^2}\Big)^{\frac{1}{3}} + 2(1-q)\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{1}{3}}\Big[\Big(\frac{r-1}{q(1-\delta)}\Big)^{\frac{1}{3}}-\Big(\frac{1}{1-q(1-\delta)}\Big)^{\frac{1}{3}}\Big] - 2(1-q)\Big(\frac{1}{1-q(1-\delta)}\Big)^{\frac{1}{3}}\Big[\Big(\frac{r-1}{q(1+\delta)}\Big)^{\frac{1}{3}}-\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{1}{3}}\Big]\Big\}.$$

The first term is strictly positive. The second and third terms are also strictly positive since $\Big[\Big(\frac{r-1}{q(1-\delta)}\Big)^{\frac{1}{3}}-\Big(\frac{1}{1-q(1-\delta)}\Big)^{\frac{1}{3}}\Big] > \Big[\Big(\frac{r-1}{q(1+\delta)}\Big)^{\frac{1}{3}}-\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{1}{3}}\Big] > 0$ and $2(1-q)\Big(\frac{1}{1-q(1+\delta)}\Big)^{\frac{1}{3}} > 2(1-q)\Big(\frac{1}{1-q(1-\delta)}\Big)^{\frac{1}{3}} > 0$. Hence, (13) is strictly positive for all $0<q\le\frac{q_{th}}{2}=\frac{r-1}{2r}$, $r>1$, and $0<\delta<1$; therefore $\frac{\partial G(q,\delta)}{\partial\delta}>0$. Since $G(q,0)=1$ and $G(q,\delta)$ is monotonically increasing in δ, we conclude that the percentage increase in expected total cost for underestimating the disruption probability by error rate δ relative to overestimation increases with error rate δ. □

Proof of Proposition 5. If $\underline{q}\ge q_{th}$, $F(q,\hat{q})$ is a constant regardless of q; that is, $F(q,\hat{q})=K$ for $\{\hat{q}>q_{th}, q\le q_{th}\}$ (region 4) and $F(q,\hat{q})=0$ for $\{\hat{q}>q_{th},\ q>q_{th}\}$ (region 1). Hence, any $\hat{q}\in(\underline{q},\bar{q})$ satisfies (9).

To prove the case for $\underline{q}<q_{th}$, we need the following two lemmas.

Lemma 1. $\sup_{q\in(\underline{q},\bar{q})} F(q,\hat{q}) = \max_{\hat{q}\le q_{th}}[F(\underline{q},\hat{q}),F(\bar{q},\hat{q})]$.

Proof. We first show that $F(q,\hat{q})$ is unimodal in q. Note that we can reduce the region of $\hat{q}$ to $(\underline{q},q_{th})$ since $F(q,\hat{q})$ is a constant for $\hat{q}>q_{th}$. We further note that $F(q,\hat{q})=0$ at $q=\hat{q}$ and $F(q,\hat{q})\ge 0$ elsewhere. Thus, to complete the proof, we shall show that $F(q,\hat{q})$ monotonically decreases in q for $q<\hat{q}$ and monotonically increases for $q\ge\hat{q}$. We examine the sign of

$$\frac{\partial F(q,\hat{q})}{\partial q}=\frac{TC(q,q)\frac{\partial TC(q,\hat{q})}{\partial q}-TC(q,\hat{q})\frac{\partial TC(q,q)}{\partial q}}{TC(q,q)^2}$$

where $\frac{\partial TC(q,q)}{\partial q}=\gamma\sqrt{\frac{1}{n_r(q)}}-\gamma\sqrt{\frac{1}{n_t(q)}}$ and $\frac{\partial TC(q,\hat{q})}{\partial q}=\gamma\sqrt{\frac{1}{n_r(\hat{q})}}-\gamma\sqrt{\frac{1}{n_t(\hat{q})}}$. To verify the unimodality of $F(q,\hat{q})$, it is sufficient to examine the sign of

$$TC(q,q)\Big[\sqrt{\frac{1}{n_r(\hat{q})}}-\sqrt{\frac{1}{n_t(\hat{q})}}\Big]-TC(q,\hat{q})\Big[\sqrt{\frac{1}{n_r(q)}}-\sqrt{\frac{1}{n_t(q)}}\Big]\ . \tag{14}$$

We examine three cases: (a) $q<\hat{q}\le q_{th}$, (b) $\hat{q}\le q\le q_{th}$, and (c) $\hat{q}\le q_{th}<q$. After algebraic work by plugging the solutions from (6)-(7) into (14), the sign of (14) can be shown to be negative, positive and positive for each case respectively.[1] Therefore, $F(q,\hat{q})$ is unimodal in q and the $\sup_{q\in(\underline{q},\bar{q})}F(q,\hat{q})$ is achieved at the maximum between $F(\underline{q},\hat{q})$ and $F(\bar{q},\hat{q})$ for $\hat{q}\le q_{th}$. □

Lemma 2. *For $\hat{q}\le q_{th}$, $F(\underline{q},\hat{q})$ is strictly increasing in $\hat{q}$ and $F(\bar{q},\hat{q})$ is strictly decreasing in $\hat{q}$.*

Proof. (a) From $F(\underline{q},\hat{q})=\frac{TC(\underline{q},\hat{q})}{TC(\underline{q},\underline{q})}-1$, we have $\frac{\partial F(\underline{q},\hat{q})}{\partial\hat{q}}=\frac{\frac{\partial TC(\underline{q},\hat{q})}{\partial\hat{q}}}{TC(\underline{q},\underline{q})}$. Hence, $\frac{\partial TC(\underline{q},\hat{q})}{\partial\hat{q}}>0$ implies $\frac{\partial F(\underline{q},\hat{q})}{\partial\hat{q}}>0$. After some algebra, we have

$$\frac{\partial TC(\underline{q},\hat{q})}{\partial\hat{q}}=\frac{2}{3}f^{\frac{1}{3}}\Big(\frac{\gamma}{2}\Big)^{\frac{2}{3}}\Big[(\hat{q}-\underline{q})\Big((r-1)^{\frac{1}{3}}\hat{q}^{-\frac{4}{3}}+(1-\hat{q})^{-\frac{4}{3}}\Big)\Big]>0.$$

Hence, $F(\underline{q},\hat{q})$ is strictly increasing in $\hat{q}$.

(b) Similar to the above case, we have

$$\frac{\partial TC(\bar{q},\hat{q})}{\partial\hat{q}}=\frac{2}{3}f^{\frac{1}{3}}\Big(\frac{\gamma}{2}\Big)^{\frac{2}{3}}\Big[(\hat{q}-\bar{q})\Big((r-1)^{\frac{1}{3}}\hat{q}^{-\frac{4}{3}}+(1-\hat{q})^{-\frac{4}{3}}\Big)\Big]<0.$$

This implies $\frac{\partial F(\bar{q},\hat{q})}{\partial\hat{q}}$ is strictly negative. Hence, $F(\bar{q},\hat{q})$ is strictly decreasing in $\hat{q}$. □

[1] Note that $\frac{\partial F(q,\hat{q})}{\partial q}$ here is for a fixed $\hat{q}$. This is different from the $\frac{\partial F(q,\hat{q})}{\partial q}$ from the Proposition 2 since the estimate $\hat{q}$ $(=q(1\pm\delta))$ is fixed for δ but dependent on q.

From Lemma 1, $\sup_{q\in(\underline{q},\bar{q})} F(q,\hat{q}) = \max_{\hat{q}\le q_{th}} [F(\underline{q},\hat{q}), F(\bar{q},\hat{q})]$. Also, since $F(\underline{q},\hat{q})$ is strictly increasing in $\hat{q}$ and $F(\bar{q},\hat{q})$ is strictly decreasing in $\hat{q}$ from Lemma 2, the infimum of $\sup_{\substack{q\in(\underline{q},\hat{q})\\ \hat{q}\le q_{th}}} F(q,\hat{q})$ can be uniquely obtained by setting $F(\underline{q},\hat{q}) = F(\bar{q},\hat{q})$. Thus, if $\underline{q} < q_{th}$, an optimal estimate of the disruption probability $\hat{q}^*$ which satisfies (9) is unique and this value is strictly less than the threshold q_{th} ($\hat{q}^* < q_{th}$). □

Proof of Proposition 6. The variance of $X\left(\frac{n_u}{n_r}, q, \Sigma\right)$ increases with d since $\sigma_{ij}(d)$ is increasing in d. Hence, $\left[\frac{A\rho}{n_t}\left(1+X\left(\frac{n_u}{n_r}, q, \Sigma\right)\right) - K\right]^+$ is stochastically increasing in d. Therefore, $\mathbb{E}[TC]$ increases with d.

Similarly, $\left[\frac{A\rho}{n_t}\left(1+X\left(\frac{n_u}{n_r}, q, \Sigma\right)\right) - K\right]^+$ is stochastically decreasing in K. Therefore, $\mathbb{E}[TC]$ decreases with K. □