

# Online Appendix for “Efficient Distribution of Water Between Head-Reach and Tail-End Farms in Developing Countries”

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## Appendix A: Proofs of Technical Results

**Proof of Theorem 1:** The Lagrangian for the objective function is:

$$L = \sum_{\theta=1}^2 q_{\theta} \left( A_p f \left( \frac{W^L - \sum_{t=1}^v y_{st}}{A_p} + R_{\theta} \right) + \sum_{t=1}^v A_{st} f \left( \frac{y_{st}}{A_{st}} + R_{\theta} \right) \right) + \lambda_1 (W^L - \sum_{t=1}^v y_{st}) + \lambda_2 \sum_{t=1}^v y_{st} + \lambda_3 (A_p (w_{\max} - R_1) - W^L - \sum_{t=1}^v y_{st}) + \sum_{t=1}^v \mu_t (A_{st} (w_{\max} - R_1) - y_{st}) + \sum_{t=1}^v \nu_t y_{st} + \sum_{t=1}^v \omega_t (W^L - y_{st}).$$

Let  $f'_t = \frac{\partial f}{\partial y_{st}}$ . The Kuhn-Tucker (K-T) conditions are:

$$-\sum_{\theta=1}^2 q_{\theta} f'_t \left( \frac{W^L - \sum_{t=1}^v y_{st}}{A_p} + R_{\theta} \right) + \sum_{\theta=1}^2 q_{\theta} f'_t \left( \frac{y_{st}}{A_{st}} + R_{\theta} \right) - \lambda_1 + \lambda_2 - \lambda_3 - \mu_t + \nu_t - \omega_t = 0, \quad t = 1, 2, \dots, v,$$

$$\lambda_1 (W^L - \sum_{t=1}^v y_{st}) = 0, \quad \lambda_2 \sum_{t=1}^v y_{st} = 0, \quad \lambda_3 (A_p (w_{\max} - R_1) - W^L - \sum_{t=1}^v y_{st}) = 0, \quad \mu_t (A_{st} (w_{\max} - R_1) - y_{st}) = 0, \quad \nu_t y_{st} = 0, \quad \omega_t (W^L - y_{st}), \quad \lambda_1, \lambda_2, \lambda_3 \geq 0, \mu_t, \nu_t, \omega_t \geq 0, \quad t = 1, 2, \dots, v.$$

The solution of the above set of equations is:

$$y_p^* = \frac{W^L A_p}{A_p + \sum_{t=1}^v A_{st}}, \quad y_{st}^* = \frac{W^L A_{st}}{A_p + \sum_{t=1}^v A_{st}}, \quad t = 1, 2, \dots, v,$$

$$\lambda_1^* = \lambda_2^* = \lambda_3^* = \mu_t^* = \nu_t^* = \omega_t^* = 0, \quad t = 1, 2, \dots, v.$$

Since the objective function is a concave function and the constraints are linear, the K-T conditions are both necessary and sufficient. Q.E.D.

**Proof of Theorem 2:** Let  $f'_t = \frac{\partial f}{\partial y_{s_t}}$ . The first-order derivative of the primary farmer's objective under the rainfall-contingent payment scheme is:

$$\sum_{\theta=1}^2 q_{\theta} \left( \frac{-f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t} + R_{\theta}}{A_p} \right) + \beta_{\theta}}{\Pi_p + A_p f \left( \frac{W^L - \sum_{t=1}^v y_{s_t} + R_{\theta}}{A_p} \right) + \beta_{\theta} \sum_{t=1}^v (y_{s_t} - y_{s_t}^{\circ})} \right) \quad \forall t.$$

Substituting  $y_{s_t} = y_{s_t}^* = \frac{W^L A_{s_t}}{A_p + \sum_{t=1}^v A_{s_t}}$  in the above expression, we have:

$$\sum_{\theta=1}^2 q_{\theta} \left( \frac{-f'_t \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) + \beta_{\theta}}{\Pi_p + A_p f \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) + \beta_{\theta} \sum_{t=1}^v (y_{s_t}^* - y_{s_t}^{\circ})} \right) \quad \forall t.$$

Note that  $f(\cdot)$  (as defined in Section 2) is the same revenue function for each secondary farmer. Therefore,

$$f'_1 \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) = f'_2 \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) = \dots = f'_v \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right), \theta = 1, 2.$$

Accordingly, let us denote  $f'_t \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right)$  by  $f' \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right)$ ,  $\forall t$ . Consequently, for  $\beta_{\theta} = f' \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right)$ , we have

$$\sum_{\theta=1}^2 q_{\theta} \left( \frac{-f' \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) + \beta_{\theta}}{\Pi_p + A_p f \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_{\theta} \right) + \beta_{\theta} \sum_{t=1}^v (y_{s_t}^* - y_{s_t}^{\circ})} \right) = 0 \quad \forall t.$$

Thus, the primary farmer shares the socially-optimal allocation  $y_{s_t}^*$  with secondary farmer  $S_t$  under the rainfall-contingent payment scheme. The proof that secondary farmer  $S_t$  self-selects the socially-optimal solution is similar. Q.E.D.

**Proof of Theorem 3:** We need to derive the value of  $\gamma$  such that the utility of the primary farmer is maximized at  $y_p = y_p^*$ . Note that if  $y_p^* = y_p^{\circ}$ , the primary farm receives zero reward (i.e.,  $\Lambda(y_p^{\circ}) = 0$ ). Therefore, without loss of generality, we assume henceforth that  $y_p^* < y_p^{\circ}$ . Let  $f' = \frac{\partial f}{\partial y_p}$ . The first-order condition is:

$$\sum_{\theta=1}^2 q_{\theta} \left( \frac{f' \left( \frac{y_p}{A_p} + R_{\theta} \right) - \gamma}{\Pi_p + A_p f \left( \frac{y_p}{A_p} + R_{\theta} \right) + \gamma (y_p^{\circ} - y_p)} \right) = 0.$$

We need to obtain a value of  $\gamma$  such that  $y_p^*$  satisfies the condition above. That is,

$$\sum_{\theta=1}^2 q_{\theta} \left( \frac{f' \left( \frac{y_p^*}{A_p} + R_{\theta} \right) - \gamma}{\Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_{\theta} \right) + \gamma (y_p^{\circ} - y_p^*)} \right) = 0.$$

Simplifying the above equation, we have

$$\begin{aligned} \gamma^2 (y_p^{\circ} - y_p^*) + \gamma \left( q_1 (\Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_2 \right)) - (y_p^{\circ} - y_p^*) f' \left( \frac{y_p^*}{A_p} + R_1 \right) + q_2 (\Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_1 \right)) - (y_p^{\circ} - y_p^*) f' \left( \frac{y_p^*}{A_p} + R_2 \right) \right) \\ - q_1 f' \left( \frac{y_p^*}{A_p} + R_1 \right) (\Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_2 \right)) - q_2 f' \left( \frac{y_p^*}{A_p} + R_2 \right) (\Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_1 \right)) = 0. \end{aligned}$$

We need to show that the above equation has a real positive root. Let  $X_1 = \Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_1 \right)$ ,  $X_2 = \Pi_p + A_p f \left( \frac{y_p^*}{A_p} + R_2 \right)$ ,  $Y_1 = (y_p^{\circ} - y_p^*) f' \left( \frac{y_p^*}{A_p} + R_1 \right)$  and  $Y_2 = (y_p^{\circ} - y_p^*) f' \left( \frac{y_p^*}{A_p} + R_2 \right)$ . Let  $\Delta$  denote the discriminant of above equation. Then, we have  $\Delta = (q_1 (X_2 - Y_1) + q_2 (X_1 - Y_2))^2 + 4q_1 Y_1 X_2 + 4q_2 Y_2 X_1 \geq 0$ . Hence, a real root exists and the two roots are provided by the expression

$$\frac{-(q_1 (X_2 - Y_1) + q_2 (X_1 - Y_2)) \pm \sqrt{\Delta}}{2(y_p^{\circ} - y_p^*)}.$$

We select the positive root as the value of  $\gamma$ . Thus,  $\gamma = \frac{-(q_1 (X_2 - Y_1) + q_2 (X_1 - Y_2)) + \sqrt{\Delta}}{2(y_p^{\circ} - y_p^*)}$ . Hence, the payment  $\Lambda(y_p)$  induces the primary farmer's optimum to be at  $y_p^*$ . This completes the proof. Q.E.D.

**Proof of Theorem 4:** Revenue of secondary farmer  $S_t$  is  $q_1 A_{s_t} f(\frac{M_t}{A_{s_t}}) + q_2 A_{s_t} f(R_2 + \frac{\eta_t}{A_{s_t}})$ . Let  $M_t^*$  and  $\eta_t^*$  be such that

$$q_1 A_{s_t} f(\frac{M_t^*}{A_{s_t}}) + q_2 A_{s_t} f(R_2 + \frac{\eta_t^*}{A_{s_t}}) = q_1 A_{s_t} f(R_1 + \frac{y_{s_t}^*}{A_{s_t}}) + q_2 A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}).$$

It is easy to observe that  $\eta_t^* = y_{s_t}^*$  and  $M_t^* = y_{s_t}^* + R_1 A_{s_t}$  satisfy the above equation. Hence, there exists a pair  $(M_t^*, \eta_t^*)$  that achieves socially-optimal behavior from the secondary farmer. We, henceforth, consider allocation of  $y_{s_t}^*$  under the high-rainfall scenario. We next show that there exists a value of  $\alpha_t$  such that secondary farmer  $S_t$  maximizes its objective at  $M_t^*$ .

The problem of secondary farmer  $S_t$  is to find a value of  $M_t$  that maximizes its utility. Hence, we desire to obtain value of  $\alpha_t$  such that

$$M_t^* = \underset{y_{s_t}^* + R_1 A_{s_t} \leq M_t \leq y_{s_t}^* + R_1 A_{s_t}}{\operatorname{argmax}} \left[ q_1 \ln \left( \Pi_t + A_{s_t} f\left(\frac{M_t}{A_{s_t}}\right) - \alpha_t (M_t - y_{s_t}^\circ - R_1 A_{s_t}) \right) + q_2 \ln \left( \Pi_t + A_{s_t} f\left(R_2 + \frac{y_{s_t}^*}{A_{s_t}}\right) - \alpha_t (M_t - y_{s_t}^\circ - R_1 A_{s_t}) \right) \right].$$

Note that  $M_t^* = y_{s_t}^* + R_1 A_{s_t}$ . Hence, we have  $y_{s_t}^\circ + R_1 A_{s_t} \leq M_t^* = y_{s_t}^* + R_1 A_{s_t}$ . Furthermore, if  $y_{s_t}^* = y_{s_t}^\circ$  (i.e.,  $M_t^* = y_{s_t}^\circ + R_1 A_{s_t}$ ), the secondary farmer pays zero premium (i.e.,  $\Omega_t(M_t^*) = 0$ ). Therefore, without loss of generality, we assume henceforth that  $M_t^* > y_{s_t}^\circ + R_1 A_{s_t}$ . Thus, we derive the value of  $\alpha_t$  such that the utility is maximized at  $M_t^*$ . The first-order condition is

$$q_1 \frac{f'(\frac{M_t}{A_{s_t}}) - \alpha_t}{\Pi_t + A_{s_t} f(\frac{M_t}{A_{s_t}}) - \alpha_t (M_t - y_{s_t}^\circ - R_1 A_{s_t})} + q_2 \frac{-\alpha_t}{\Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}) - \alpha_t (M_t - y_{s_t}^\circ - R_1 A_{s_t})} = 0.$$

We need to identify a value of  $\alpha_t$  such that  $M_t^*$  satisfies the condition above. Thus, we have

$$q_1 \frac{f'(\frac{M_t^*}{A_{s_t}}) - \alpha_t}{\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t (M_t^* - y_{s_t}^\circ - R_1 A_{s_t})} + q_2 \frac{-\alpha_t}{\Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}) - \alpha_t (M_t^* - y_{s_t}^\circ - R_1 A_{s_t})} = 0. \quad (1)$$

For now, we assume that the denominators  $[\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t (M_t^* - y_{s_t}^\circ - R_1 A_{s_t})]$  and  $[\Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}) - \alpha_t (M_t^* - y_{s_t}^\circ - R_1 A_{s_t})]$  are strictly positive. Later, we show that this assumption is consistent with the solution we obtain. Equation (1) can be simplified as:

$$\alpha_t^2 (M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) - \left( q_1 [f'(\frac{M_t^*}{A_{s_t}})(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) + \Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}})] + q_2 [\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}})] \right) \alpha_t + q_1 f'(\frac{M_t^*}{A_{s_t}}) [\Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}})] = 0. \quad (2)$$

To complete the proof under this case, we need to show that a real positive root exists for (2). Since (2) is of the form  $A\alpha_t^2 - B\alpha_t + C$ , where  $A > 0, B > 0$  and  $C > 0$ , the real roots (if they exist) are both positive. Thus, it suffices to show that the discriminant of (2) is positive. Let  $T_{1t} = \Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}), T_{2t} = \Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}), \Gamma_t = f'(\frac{M_t^*}{A_{s_t}})(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})$ . Let  $\Delta_t$  denote the discriminant of (2). Then, we show that

$$\Delta_t = (q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}))^2 - 4q_1\Gamma_t T_{2t} \geq 0.$$

Consider the following arguments:

- Since  $R_2 + \frac{y_{s_t}^*}{A_{s_t}} \geq M_t^*/A_{s_t}$  we have  $T_{2t} \geq T_{1t}$ . Next, we show that  $T_{1t} > \Gamma_t$ . We have
  - ◊  $T_{1t} - \Gamma_t = \Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - f'(\frac{M_t^*}{A_{s_t}})(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) > A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - f'(\frac{M_t^*}{A_{s_t}})M_t^*$ .

◇ Recall from Section 2 that the revenue function for a crop is  $f(u) = b(1 - \exp(-cu))$ , where  $u$  is the amount of surface water. Thus, we have

$$T_{1t} - \Gamma_t > A_{s_t} b(1 - \exp(-c \frac{M_t^*}{A_{s_t}})) - bc \exp(-c \frac{M_t^*}{A_{s_t}}) M_t^* = (bA_{s_t} + bcM_t^*)(1 - \exp(-c \frac{M_t^*}{A_{s_t}})) - bcM_t^*.$$

◇ We know that  $\exp(c \frac{M_t^*}{A_{s_t}}) \geq 1 + c \frac{M_t^*}{A_{s_t}}$ . Thus, we have  $1 - \exp(-c \frac{M_t^*}{A_{s_t}}) \geq \frac{cM_t^*}{A_{s_t} + cM_t^*}$ . Hence  $T_{1t} - \Gamma_t > 0$ .

• Consider the discriminant:  $\Delta_t = (q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}))^2 - 4q_1\Gamma_t T_{2t} > (q_1 T_{2t} - \Gamma_t)^2 \geq 0$ . Thus, real roots exist for (2) and the two roots are provided by the expression

$$\frac{q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}) \pm \sqrt{\Delta_t}}{2(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})}.$$

Next, we show that for  $\alpha_t = \frac{q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}) - \sqrt{\Delta_t}}{2(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})}$ , we have  $\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) > 0$  and  $\Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}) - \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) > 0$ . Since  $T_{2t} = \Pi_t + A_{s_t} f(R_2 + \frac{y_{s_t}^*}{A_{s_t}}) \geq T_{1t} = \Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}})$ , it is sufficient to show that  $\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) > 0$ . Thus, consider

$$\begin{aligned} \Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) &= T_{1t} - \frac{q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}) - \sqrt{\Delta_t}}{2} \\ &> \frac{2T_{1t} - q_1(T_{2t} + \Gamma_t) - q_2(T_{1t}) + (q_1 T_{2t} - \Gamma_t)}{2} > \frac{2T_{1t} - q_1 T_{1t} - q_2 T_{1t} - T_{1t}}{2} = 0. \end{aligned}$$

Thus, we have  $\Pi_t + A_{s_t} f(\frac{M_t^*}{A_{s_t}}) - \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}) > 0$ . Consequently, the premium  $\alpha_t = \frac{q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}) - \sqrt{\Delta_t}}{2(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})}$  results in secondary farmer  $S_t$  choosing  $M_t^*$ . Q.E.D.

**Proof of Theorem 5:** We will consider the unconstrained problems of the primary farmer and each secondary farmer. The unconstrained optima also satisfy the respective constraints of these problems, and are therefore optimal for the constrained problems as well.

Since the objective function of the primary farmer's decision problem is concave, the first-order conditions are both necessary and sufficient to obtain the optimum solution. Thus, we have

$$-\sum_{\theta=1}^2 q_\theta f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} + R_\theta \right) + \pi'_t(y_{s_t}) = 0, \quad t = 1, 2, \dots, v,$$

$$\text{or, } -\sum_{\theta=1}^2 q_\theta f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} + R_\theta \right) + 0.5 \sum_{\theta=1}^2 q_\theta \left[ f'_t \left( \frac{W_t^L - y_{s_t}}{A_{p_t}} + R_\theta \right) + f'_t \left( \frac{y_{s_t}}{A_{s_t}} + R_\theta \right) \right] = 0,$$

$$\text{or, } 0.5 \sum_{\theta=1}^2 q_\theta \left[ f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} + R_\theta \right) - f'_t \left( \frac{W_t^L - y_{s_t}}{A_{p_t}} + R_\theta \right) + f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} + R_\theta \right) - f'_t \left( \frac{y_{s_t}}{A_{s_t}} + R_\theta \right) \right] = 0.$$

We claim that the above first-order condition holds only when,

$$\frac{W_t^L - y_{s_t}}{A_{p_t}} = \frac{y_{s_t}}{A_{s_t}} = \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p}.$$

By construction of the virtual field, we have  $\frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} = \frac{W_t^L - y_{s_t}}{A_{p_t}}$ . Thus, we only need to show that  $\frac{y_{s_t}}{A_{s_t}} = \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p}$ . Suppose, on the contrary, that  $\frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} > \frac{y_{s_t}}{A_{s_t}}$ . Since  $f'_t$  is a decreasing function, we have

$$f'_t \left( \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} + R_\theta \right) - f'_t \left( \frac{y_{s_t}}{A_{s_t}} + R_\theta \right) < 0.$$

Hence, we arrive at a contradiction. Similarly, we can disqualify  $\frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p} < \frac{y_{s_t}}{A_{s_t}}$ . Thus, we must have

$$\frac{W_t^L - y_{s_t}}{A_{p_t}} = \frac{y_{s_t}}{A_{s_t}} = \frac{W^L - \sum_{t=1}^v y_{s_t}}{A_p}.$$

Solving the above equations, we have that the decentralized optimum solution to the primary farmer's decision problem is same as that of the social planner's problem. That is,

$$y_{s_t}^* = \frac{W^L A_{s_t}}{A_p + \sum_{t=1}^v A_{s_t}}, t = 1, 2, \dots, v, \quad y_p^* = \frac{W^L A_p}{A_p + \sum_{t=1}^v A_{s_t}}.$$

Similarly, we can show that under the payment scheme, the optimum solution to each secondary farmer's decision problem is also to purchase the socially-optimum amount of water. The individual rationality of this scheme for the primary and the secondary farmers is straightforward.

Next, we discuss the distribution of surplus (over the naive decentralized solution) generated by socially-optimal allocation. Consider the value of primary farmer's objective at  $y_p^*$  under the payment scheme  $\pi_t$ :

$$\underbrace{\sum_{\theta=1}^2 q_{\theta} A_p f\left(\frac{y_p^*}{A_p} + R_{\theta}\right)}_{\text{Primary Farmer's Revenue With the Payment Scheme}} + \underbrace{0.5 \sum_{t=1}^v \sum_{\theta=1}^2 q_{\theta} \left[ \left( A_{p_t} f\left(\frac{y_{p_t}^{\circ}}{A_{p_t}} + R_{\theta}\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_{\theta}\right) \right) + \left( A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_{\theta}\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_{\theta}\right) \right) \right]}_{\text{Payment Made by Secondary Farmers}}.$$

Note that  $\frac{y_p}{A_p} = \frac{y_{p_t}}{A_{p_t}}$ . Hence, we can rewrite the above expression as

$$\begin{aligned} & \sum_{\theta=1}^2 q_{\theta} A_p f\left(\frac{y_p^{\circ}}{A_p} + R_{\theta}\right) - 0.5 \sum_{\theta=1}^2 q_{\theta} \left( A_p f\left(\frac{y_p^{\circ}}{A_p} + R_{\theta}\right) - A_p f\left(\frac{y_p^*}{A_p} + R_{\theta}\right) \right) + \\ & 0.5 \sum_{t=1}^v \sum_{\theta=1}^2 q_{\theta} \left( A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_{\theta}\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_{\theta}\right) \right) \\ & = \underbrace{\sum_{\theta=1}^2 q_{\theta} A_p f\left(\frac{y_p^{\circ}}{A_p} + R_{\theta}\right)}_{\text{Primary Farmer's Revenue Without the Payment Scheme}} + \\ & 0.5 \sum_{\theta=1}^2 q_{\theta} \left[ \sum_{t=1}^v \left( A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_{\theta}\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_{\theta}\right) \right) - \left( A_p f\left(\frac{y_p^{\circ}}{A_p} + R_{\theta}\right) - A_p f\left(\frac{y_p^*}{A_p} + R_{\theta}\right) \right) \right]. \end{aligned}$$

Surplus = Gain of the Secondary Farmers - Loss of the Primary Farmer

Thus, the primary farmer receives half of the surplus. The remaining half of the surplus is shared among the secondary farmers. Q.E.D.

**Proof of Theorem 6:** We first consider the upfront internal payment scheme. The optimal decision of secondary farmer  $S_t$  is to purchase  $g_{1t}^* = y_{s_t}^*$  amount of surface at a price  $\pi_t(y_{s_t}^*)$ . We have

$$\pi_t(y_{s_t}^*) = 0.5 \sum_{\theta=1}^2 q_{\theta} \left[ A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_{\theta}\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_{\theta}\right) + A_{p_t} f\left(\frac{y_{p_t}^{\circ}}{A_{p_t}} + R_{\theta}\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_{\theta}\right) \right],$$

$$\begin{aligned} \frac{\partial \pi_t(y_{s_t}^*)}{\partial q_2} &= -0.5 \left[ A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_1\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_1\right) \right] + 0.5 \left[ A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_2\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_2\right) \right] - \\ & 0.5 \left[ A_{p_t} f\left(\frac{y_{p_t}^{\circ}}{A_{p_t}} + R_1\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_1\right) \right] + 0.5 \left[ A_{p_t} f\left(\frac{y_{p_t}^{\circ}}{A_{p_t}} + R_2\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_2\right) \right]. \end{aligned}$$

To complete the proof, we next show that the gain of secondary farmer  $S_t$  in the low-rainfall scenario ( $\theta = 1$ ) is more than its gain in the high-rainfall scenario ( $\theta = 2$ ). That is,

$$A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_1\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_1\right) \geq A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_2\right) - A_{s_t} f\left(\frac{y_{s_t}^{\circ}}{A_{s_t}} + R_2\right).$$

Recall from Section 2 that the revenue function  $f = b(1 - \exp(-cu))$ , where  $u$  is the amount of water. Also, we have  $y_{s_t}^* \geq y_{s_t}^\circ$ . Thus,

$$\begin{aligned} A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_1\right) - A_{s_t} f\left(\frac{y_{s_t}^\circ}{A_{s_t}} + R_1\right) &= b \exp(-cR_1) \left( \exp\left(-c\left(\frac{y_{s_t}^\circ}{A_{s_t}}\right)\right) - \exp\left(-c\left(\frac{y_{s_t}^*}{A_{s_t}}\right)\right) \right) A_{s_t} \\ &\geq b \exp(-cR_2) \left( \exp\left(-c\left(\frac{y_{s_t}^\circ}{A_{s_t}}\right)\right) - \exp\left(-c\left(\frac{y_{s_t}^*}{A_{s_t}}\right)\right) \right) A_{s_t} = A_{s_t} f\left(\frac{y_{s_t}^*}{A_{s_t}} + R_2\right) - A_{s_t} f\left(\frac{y_{s_t}^\circ}{A_{s_t}} + R_2\right). \end{aligned}$$

Similarly, we can show that

$$A_{p_t} f\left(\frac{y_{p_t}^\circ}{A_{p_t}} + R_1\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_1\right) \geq A_{p_t} f\left(\frac{y_{p_t}^\circ}{A_{p_t}} + R_2\right) - A_{p_t} f\left(\frac{y_{p_t}^*}{A_{p_t}} + R_2\right).$$

Hence,  $\frac{\partial \pi_t(y_{s_t}^*)}{\partial q_2} \leq 0$ . Since  $y_{s_t}^*$  remains constant as probability of high-rainfall scenario changes, the price per unit of water  $\pi_t(y_{s_t}^*)/y_{s_t}^*$  increases as the probability  $q_2$  of high-rainfall scenario decreases.

We now consider the reward and water-guarantee scheme. Under this scheme, optimal decision of secondary farmer  $S_t$  is to purchase water guarantee of amount  $M_t^* = y_{s_t}^* + R_1 A_{s_t}$  at the price  $\Omega_t(M_t^*) = \alpha_t(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})$ . Hence, the secondary farmer receives  $g_{2t}^* = y_{s_t}^* = M_t^* - R_1 A_{s_t}$  amount of surface water. From Theorem 4, for  $M_t^* > y_{s_t}^\circ + R_1 A_{s_t}$ , we have

$$\begin{aligned} \alpha_t &= \frac{q_1(T_{2t} + \Gamma) + q_2(T_{1t}) - \sqrt{\Delta_t}}{2(M_t^* - y_{s_t}^\circ - R_1 A_{s_t})} \text{ where } \Delta_t = (q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}))^2 - 4q_1\Gamma_t T_{2t}, \\ T_{1t} &= \Pi_t + A_{s_t} f\left(\frac{M_t^*}{A_{s_t}}\right), T_{2t} = \Pi_t + A_{s_t} f\left(R_2 + \frac{y_{s_t}^*}{A_{s_t}}\right), \Gamma_t = f'\left(\frac{M_t^*}{A_{s_t}}\right)(M_t^* - y_{s_t}^\circ - R_1 A_{s_t}). \end{aligned}$$

Since we have  $M_t^* = y_{s_t}^* + R_1 A_{s_t}$ , rewriting the expression for  $\alpha_t$ ,

$$\begin{aligned} \alpha_t &= \frac{q_1(T_{2t} + \Gamma) + q_2(T_{1t}) - \sqrt{(q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}))^2 - 4q_1\Gamma_t T_{2t}}}{2(y_{s_t}^* - y_{s_t}^\circ)}, \\ \frac{\partial \alpha_t}{\partial q_2} &= \frac{1}{2(y_{s_t}^* - y_{s_t}^\circ)} \left( -(T_{2t} + \Gamma_t) + T_{1t} - \frac{4\Gamma_t T_{2t} + 2(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t})(-(T_{2t} + \Gamma_t) + T_{1t})}{2\sqrt{(q_1(T_{2t} + \Gamma_t) + q_2(T_{1t}))^2 - 4q_1\Gamma_t T_{2t}}} \right), \\ &= \frac{1}{2(y_{s_t}^* - y_{s_t}^\circ)} \left( (T_{2t} + \Gamma_t - T_{1t}) \frac{(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t}}{\sqrt{\Delta_t}} - \frac{2\Gamma_t T_{2t}}{\sqrt{\Delta_t}} \right). \end{aligned}$$

Next, we show that  $\left( (T_{2t} + \Gamma_t - T_{1t}) \frac{(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t}}{\sqrt{\Delta_t}} - \frac{2\Gamma_t T_{2t}}{\sqrt{\Delta_t}} \right) \leq 0$ . Note that  $\sqrt{\Delta_t} \leq (q_1(T_{2t} + \Gamma_t) + q_2 T_{1t})$ ,  $T_{2t} \geq T_{1t} \geq \Gamma_t$ . Hence,  $(T_{2t} + \Gamma_t - T_{1t}) \frac{(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t}}{\sqrt{\Delta_t}} \geq 0$ . Clearly,  $(T_{2t} + \Gamma_t - T_{1t}) \leq T_{2t}$ . We next show that  $(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t} \leq 2\Gamma_t$ , or,  $(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - 2\Gamma_t \leq \sqrt{\Delta_t}$ . Note that  $((q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - 2\Gamma_t)^2 = \Delta_t + 4q_2\Gamma_t^2 - 4q_2 T_{1t}\Gamma_t \leq \Delta_t$ . Consequently,  $(T_{2t} + \Gamma_t - T_{1t})((q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t}) \leq 2\Gamma_t T_{2t}$ . Hence,

$$(T_{2t} + \Gamma_t - T_{1t}) \frac{(q_1(T_{2t} + \Gamma_t) + q_2 T_{1t}) - \sqrt{\Delta_t}}{\sqrt{\Delta_t}} - \frac{2\Gamma_t T_{2t}}{\sqrt{\Delta_t}} \leq 0, \text{ or, } \frac{\partial \alpha_t}{\partial q_2} \leq 0.$$

Thus, the premium per unit of water  $\alpha_t$  increases as the probability of high-rainfall decreases. Q.E.D.

## Appendix B: Socially-Optimal Schemes for Multiple, Sequential Secondary Farmers

In this section, we analyze the case of multiple, sequential secondary farmers. We discuss three schemes that achieve the following properties: (i) social optimality and (ii) individual rationality. We first consider the rainfall-contingent payment scheme (Section 4.2.1) for risk-averse farmers. We then briefly discuss the reward and water-guarantee scheme (Section 4.2.2), also for risk-averse farmers. Finally, we consider the linear pricing scheme described in Remark 1 (Section 5.1) for risk-neutral farmers.

To begin our discussion, we consider a single-lane single-crop system. In the case of sequential secondary farmers, the geometry of a lane is as follows: Let  $P = S_0$  denote the primary farmer in a lane and let  $S = \{S_1, S_2, \dots, S_v\}$  denote the set of secondary farmers in this lane. The secondary farmers are in a sequence, with  $S_1$  first and followed by  $S_2, S_3, \dots, S_v$ , in that order.

The rainfall-contingent payment scheme works as follows: The first secondary farmer ( $S_1$ ) purchases water (from primary farmer  $S_0$ ) for its own use and for possible sale to the other secondary farmers. Then,  $S_1$  acts as a primary farmer for the remaining set  $S \setminus \{S_1\}$  of secondary farmers. Next,  $S_2$  purchases water from  $S_1$  for its own use and for possible sale to the remaining downstream farmers in the set  $S \setminus \{S_1 \cup S_2\}$ . This sequence of purchasing and selling continues until the last secondary farmer.

For Scenario  $\theta$ , let  $\mathcal{K}_\theta(\widehat{Y}_{S_t})$  be the payment by secondary farmer  $S_t$  to purchase  $\widehat{Y}_{S_t}$  amount of surface water from its immediately-upstream counterpart (i.e.,  $S_{t-1}$ ) for its own use and for the use of farmers  $S_{t+1}, \dots, S_v$ . For  $\theta = 1, 2$ , let the constant  $\beta_\theta$  be defined as

$$\beta_\theta = f' \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_\theta \right) \text{ and let } \mathcal{K}_\theta(\widehat{Y}_{S_t}) = \beta_\theta (\widehat{Y}_{S_t} - \widehat{Y}_{S_t}^\circ), \forall t, \forall \theta,$$

where  $W^L$  is the total water available to the lane,  $A_p$  (resp.,  $A_{s_t}$ ) is the area of primary farmer  $S_0$  (resp., secondary farmer  $S_t$ ),  $R_\theta$  (resp.,  $q_\theta$ ) is the rainfall amount (resp., probability) of Scenario  $\theta$ ,  $\widehat{Y}_{S_t}^\circ$  is the sum of the surface water quantities received under naive decentralization by secondary farmers  $S_t, S_{t+1}, \dots, S_v$ . That is,  $\widehat{Y}_{S_t}^\circ = y_{s_t}^\circ + y_{s_{t+1}}^\circ + \dots + y_{s_v}^\circ$ . We now describe the problems of the primary and secondary farmers under this pricing scheme.

**Decision Problem for Primary Farmer  $S_0$ :** Let  $y_p = W^L - \widehat{Y}_{S_1}$  be the water used by primary farmer  $S_0$  and let  $\widehat{A}_{S_t}$  be the sum of the areas of secondary farmers  $t, t+1, \dots, v$ . That is,  $\widehat{A}_{S_t} = A_{s_t} + A_{s_{t+1}} + \dots + A_{s_v}$ . The objective of the primary farmer is to maximize the utility of its wealth (initial wealth + revenue of the crop + payment received from secondary farmer  $S_1$ ). That is,

$$\max_{\widehat{Y}_{S_1}} \sum_{\theta=1}^2 q_\theta \ln \left( \underbrace{\Pi_p}_{\text{Initial Wealth}} + \underbrace{A_p f \left( \frac{W^L - \widehat{Y}_{S_1}}{A_p} + R_\theta \right)}_{\text{Revenue}} + \underbrace{\beta_\theta (\widehat{Y}_{S_1} - \widehat{Y}_{S_1}^\circ)}_{\text{Payment Received from Secondary Farmer } S_1} \right)$$

where  $0 \leq \widehat{Y}_{S_1} \leq \min \{W^L, \widehat{A}_{S_1} (w_{\max} - R_1)\}$ ,  $0 \leq W^L - \widehat{Y}_{S_1} \leq \min \{W^L, A_p (w_{\max} - R_1)\}$ . It is straightforward to show that the optimal solution of the above optimization problem is  $\widehat{Y}_{S_1}^* = \frac{W^L \widehat{A}_{S_1}}{A_p + \sum_{t=1}^v A_{s_t}}$ . Thus, the primary farmer uses the optimal amount  $y_p^* = \frac{W^L A_p}{A_p + \sum_{t=1}^v A_{s_t}}$  and sells the remaining water  $\widehat{Y}_{S_1}^* = \frac{W^L \widehat{A}_{S_1}}{A_p + \sum_{t=1}^v A_{s_t}}$  to secondary farmer  $S_1$ .

**Decision Problem for Secondary Farmer  $S_1$ :** Secondary farmer  $S_1$  receives  $\widehat{Y}_{S_1}^* = \frac{W^L \widehat{A}_{S_1}}{A_p + \sum_{t=1}^v A_{s_t}}$  amount of surface water from primary farmer  $S_0$ . Then,  $S_1$  decides the amount to use for itself and the amount to sell to secondary farmer  $S_2$ . Let  $y_{s_1} = (\widehat{Y}_{S_1}^* - \widehat{Y}_{S_2})$  be the water used by secondary farmer  $S_1$ . The objective of secondary farmer  $S_1$  is

$$\max_{\widehat{Y}_{S_2}} \sum_{\theta=1}^2 q_\theta \ln \left( \underbrace{\Pi_{s_1}}_{\text{Initial Wealth}} + \underbrace{A_{s_1} f \left( \frac{(\widehat{Y}_{S_1}^* - \widehat{Y}_{S_2})}{A_{s_1}} + R_\theta \right)}_{\text{Revenue}} + \underbrace{\beta_\theta (\widehat{Y}_{S_2} - \widehat{Y}_{S_2}^\circ)}_{\text{Payment Received from Secondary Farmer } S_2} - \underbrace{\beta_\theta (\widehat{Y}_{S_1}^* - \widehat{Y}_{S_1}^\circ)}_{\text{Payment Made to Primary Farmer } S_0} \right)$$

where  $0 \leq \widehat{Y}_{S_2} \leq \min \{ \widehat{Y}_{S_1}^*, \widehat{A}_{S_2}(w_{\max} - R_1) \}$ . The optimal solution of the above problem is  $\widehat{Y}_{S_2}^* = \frac{W^L \widehat{A}_{S_2}}{A_p + \sum_{t=1}^v A_{S_t}}$ . That is, in the optimal solution,  $S_1$  uses  $y_{S_1}^* = \frac{W^L A_{S_1}}{A_p + \sum_{t=1}^v A_{S_t}}$  and releases the amount  $\widehat{Y}_{S_2}^*$  to  $S_2$ .

Continuing the above argument, it can be verified that secondary farmer  $S_t$  uses  $y_{S_t}^* = \frac{W^L A_{S_t}}{A_p + \sum_{t=1}^v A_{S_t}}$  amount of surface water and sells an amount  $\widehat{Y}_{S_{t+1}}^* = \frac{W^L \widehat{A}_{S_{t+1}}}{A_p + \sum_{t=1}^v A_{S_t}}$  to secondary farmer  $S_{t+1}$ ,  $t = 1, 2, \dots, v-1$ .

**Two Properties of the Rainfall-Contingent Payment Scheme.** The scheme has two properties:

- It achieves social optimality: The solution  $(y_p^*, y_{S_1}^*, y_{S_2}^*, \dots, y_{S_t}^*)$  derived above is the socially-optimal solution in Theorem 1 (Section 4.1).

- The scheme is also individually rational for each farmer. That is, for farmer  $S_t$ , we have (difference in the utility of secondary farmer  $S_t$  under the scheme and that under the naive decentralization)  $\geq 0$ ,  $t = 0, 1, 2, \dots, v$ . To show the individual rationality for farmer  $S_t$ , we need to show the following:

$$\underbrace{\sum_{\theta=1}^2 q_{\theta} \ln \left( \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_1}} + R_{\theta} \right) + \beta_{\theta} (\widehat{Y}_{S_{t+1}}^* - \widehat{Y}_{S_{t+1}}^{\circ}) - \beta_{\theta} (\widehat{Y}_{S_t}^* - \widehat{Y}_{S_t}^{\circ}) \right) - \sum_{\theta=1}^2 q_{\theta} \ln \left( \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) \right)}_{\text{Change in the Utility of } S_t} \geq 0$$

Since  $\widehat{Y}_{S_t}^* = y_{S_t}^* + y_{S_{t+1}}^* + \dots + y_{S_v}^*$  and  $\widehat{Y}_{S_t}^{\circ} = y_{S_t}^{\circ} + y_{S_{t+1}}^{\circ} + \dots + y_{S_v}^{\circ}$ , we need

$$\sum_{\theta=1}^2 q_{\theta} \ln \left( \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_1}} + R_{\theta} \right) - \beta_{\theta} (y_{S_t}^* - y_{S_t}^{\circ}) \right) - \sum_{\theta=1}^2 q_{\theta} \ln \left( \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) \right) \geq 0$$

To show that the above condition holds, it is sufficient to establish that

$$\begin{aligned} \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_1}} + R_{\theta} \right) - \beta_{\theta} (y_{S_t}^* - y_{S_t}^{\circ}) &\geq \Pi_{S_t} + A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) \quad \theta = 1, 2, \\ \text{or, } \beta_{\theta} (y_{S_t}^{\circ} - y_{S_t}^*) &\geq A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) - A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) \quad \theta = 1, 2, \end{aligned}$$

Substituting the value of  $\beta_{\theta}$  and  $\frac{W^L A_{S_t}}{A_p + \sum_{t=1}^v A_{S_t}} = y_{S_t}^*$ , we need

$$f' \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) (y_{S_t}^{\circ} - y_{S_t}^*) \geq A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) - A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) \quad \theta = 1, 2. \quad (3)$$

It is easy to see that (3) holds. The concavity and the differentiability of the revenue function  $f$  implies that  $f'(u) \geq \frac{f(v) - f(u)}{v - u}$ ,  $\forall u, v$  with  $u \neq v$  (Boyd and Vandenberghe 2009). Thus, we have

$$\begin{aligned} f' \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} - \frac{y_{S_t}^*}{A_{S_t}} \right) &\geq f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) - f \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right), \quad \theta = 1, 2, \\ \text{or, } f' \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) (y_{S_t}^{\circ} - y_{S_t}^*) &\geq A_{S_t} f \left( \frac{y_{S_t}^{\circ}}{A_{S_t}} + R_{\theta} \right) - A_{S_t} f \left( \frac{y_{S_t}^*}{A_{S_t}} + R_{\theta} \right) \quad \theta = 1, 2, \end{aligned}$$

which is the same as (3). This establishes the individual rationality for secondary farmer  $S_t$ . The proof of this property for primary farmer  $S_0$  is similar, and is therefore avoided.

This completes our discussion of the rainfall-contingent payment scheme for a single-lane single-crop system. Next, consider the reward and water-guarantee scheme of Section 4.2.2. For our sequential system, observe that the secondary farmers who are earlier in the sequence receive more surface water under the naive allocation than the socially-optimal allocation. It is easy to show the existence of a non-negative integer  $\kappa \leq v$  (which can be determined a priori, given the parameters of the lane) such that secondary farmers  $S_1, S_2, \dots, S_{\kappa}$  (resp.,  $S_{\kappa+1}, \dots, S_v$ ) receive more (resp., less) surface water under the naive solution than

the socially-optimal solution. For brevity, we avoid a detailed proof of this statement. Thus,  $y_{s_t}^\circ \geq y_{s_t}^*$ ,  $t = 1, 2, \dots, \kappa$  and  $y_{s_t}^\circ < y_{s_t}^*$ ,  $t = \kappa + 1, \dots, v$ . Given the value of  $\kappa$ , the reward and water-guarantee scheme works as follows: the government purchases water from the primary farmer and secondary farmers  $S_1, S_2, \dots, S_\kappa$ , via the reward and sells it to the remaining secondary farmers via water guarantees. As in Section 4.2.2, it can be easily shown that this scheme achieves social optimality for the primary farmer and the secondary farmers, and is also individually rational for each farmer.

For risk-neutral farmers, something better can be done: A secondary farmer can directly (i.e., without third-party intervention) purchase water from its immediately-upstream counterpart, while keeping the payment independent of the rainfall. Consider the following linear pricing scheme: let  $\mathcal{C}(\widehat{Y}_{S_t})$  be the payment by secondary farmer  $S_t$  to purchase  $\widehat{Y}_{S_t}$  amount of surface water from its primary counterpart (i.e.,  $S_{t-1}$ ) for its own use and for the use of farmers  $S_{t+1}, \dots, S_v$ . Let the constant  $\tau$  be defined as

$$\tau = \sum_{\theta=1}^2 q_\theta f' \left( \frac{W^L}{A_p + \sum_{t=1}^v A_{s_t}} + R_\theta \right) \quad \text{and let } \mathcal{C}(\widehat{Y}_{s_t}) = \tau(\widehat{Y}_{S_t} - \widehat{Y}_{S_t}^\circ), \quad t = 1, 2, \dots, v,$$

An argument similar to that for the rainfall-contingent payment scheme shows that the above scheme achieves social optimality and is individually rational for each farmer (i.e., change in the revenue for farmer  $S_t$  under the scheme + payment received from farmer  $S_{t+1}$  - payment made to farmer  $S_{t-1} \geq 0$   $t = 0, 1, 2, \dots, v$ ).

Generalizations of these schemes to multiple-lanes and multiple-crops are also similar to those in Sections 4.3 and 5.2.

## References

Boyd, S., L. Vandenberghe. 2009. *Convex Optimization*. Cambridge University Press, New York.