Speed Quality Trade-offs in a Dynamic Model: Online Appendix

Proof of Proposition 1

The profit maximization problem is: $Max : R = \sum_{t=1}^{N}[p_t\lambda_t - \frac{\gamma\lambda_t}{\mu_t-\lambda_t}] + \theta\Lambda_{N+1}$

$$\text{subject to } \Lambda_t = \Lambda_{t-1} - \delta\lambda_{t-1}(\mu_{t-1} - \hat{\mu})$$

$$\lambda_t = \Lambda_t - \alpha p_t.$$

The problem can be restated as a N-stage dynamic program with Λ_t as the state variable:

$$\begin{aligned} Max \quad &: \quad R_t(\Lambda_t) = (\Lambda_t - \alpha p_t)p_t - \frac{\gamma(\Lambda_t - \alpha p_t)}{\mu_t - (\Lambda_t - \alpha p_t)} + R_{t+1}(\Lambda_{t+1}) \ \forall t = 1, \ldots, N \\ \text{where } \Lambda_{t+1} &= \Lambda_t - \delta\lambda_t(\mu_t - \hat{\mu}) \text{ and } R_{N+1} = \theta\Lambda_{N+1} \text{ where } \theta > 0. \end{aligned}$$

Now we show that at each stage t, the optimization problem is concave with respect to the two variables λ_t and μ_t taking into account their impact on future decisions. First, we show that $\frac{\partial^2 R_t}{\partial\Lambda_t^2} \leq 0$. Note that $\frac{\partial^2 R_{N+1}}{\partial\Lambda_{N+1}^2} = 0$ since $R_{N+1} = \theta\Lambda_{N+1}$. Next,

$$\frac{\partial^2 R_N}{\partial\Lambda_N^2} = \frac{\partial^2\pi_N}{\partial\Lambda_N^2} + \frac{\partial^2 R_{N+1}}{\partial\Lambda_N^2} = \frac{\partial^2\pi_N}{\partial\Lambda_N^2} + \frac{\partial^2 R_{N+1}}{\partial\Lambda_{N+1}^2}\left(\frac{\partial\Lambda_{N+1}}{\partial\Lambda_N}\right)^2$$

where $\pi_t = (\Lambda_t - \alpha p_t)p_t - \frac{\gamma(\Lambda_t-\alpha p_t)}{\mu_t-(\Lambda_t-\alpha p_t)}, t = 1, .., N$ is the single-period profit in period t and so after some simplification, $\frac{\partial^2\pi_t}{\partial\Lambda_t^2} = \frac{-2\gamma\mu_t}{(\mu_t-\lambda_t)^3} < 0$. So, $\frac{\partial^2 R_N}{\partial\Lambda_N^2} < 0$ and it then follows recursively that $\frac{\partial^2 R_t}{\partial\Lambda_t^2} < 0$ for all t.

Now, we turn our attention to the first and second order derivatives:

$$\frac{\partial R_t}{\partial\mu_t} = \frac{\gamma\lambda_t}{(\mu_t - \lambda_t)^2} - \delta\lambda_t\frac{\partial R_{t+1}}{\partial\Lambda_{t+1}}$$

$$\frac{\partial^2 R_t}{\partial\mu_t^2} = -\frac{2\gamma\lambda_t}{(\mu_t - \lambda_t)^3} + \delta^2\lambda_t^2\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2}$$

$$\frac{\partial R_t}{\partial\lambda_t} = \frac{\Lambda_t}{\alpha} - \frac{2\lambda_t}{\alpha} - \frac{\gamma\mu_t}{(\mu_t - \lambda_t)^2} - \delta(\mu_t - \hat{\mu})\frac{\partial R_{t+1}}{\partial\Lambda_{t+1}}$$

$$\frac{\partial^2 R_t}{\partial\lambda_t^2} = -\frac{2}{\alpha} - \frac{2\gamma\mu_t}{(\mu_t - \lambda_t)^3} + \delta^2(\mu_t - \hat{\mu})^2\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2}$$

$$\frac{\partial^2 R_t}{\partial\lambda_t\partial\mu_t} = \gamma\frac{(\mu_t + \lambda_t)}{(\mu_t - \lambda_t)^3} - \delta\frac{\partial R_{t+1}}{\partial\Lambda_{t+1}} + \delta^2\lambda_t(\mu_t - \hat{\mu})\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2}$$

$$\text{and, } \quad \frac{\partial^2 R_t}{\partial\mu_t^2}\frac{\partial^2 R_t}{\partial\lambda_t^2} = \frac{2\gamma\lambda_t}{(\mu_t - \lambda_t)^3}\left(\frac{2}{\alpha} + \frac{2\gamma\mu_t}{(\mu_t - \lambda_t)^3}\right) + Z^+$$

where Z^+ consists of only positive terms because $\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2} < 0$. Also,

$$\left(\frac{\partial^2 R_t}{\partial\lambda_t\partial\mu_t}\right)^2 = \left[\gamma\frac{(\mu_t+\lambda_t)}{(\mu_t-\lambda_t)^3} - \delta\frac{\partial R_{t+1}}{\partial\Lambda_{t+1}} + \delta^2\lambda_t(\mu_t-\hat{\mu})\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2}\right]^2.$$

From the single-period analysis, we have $\frac{\partial R_t}{\partial\Lambda_t} > 0$. Given our definition of δ (recall that δ is divided by $\hat{\mu}$), we assume that δ is small enough so that $\gamma\frac{(\mu_t+\lambda_t)}{(\mu_t-\lambda_t)^3} > \delta\frac{\partial R_{t+1}}{\partial\Lambda_{t+1}} - \delta^2\lambda_t(\mu_t-\hat{\mu})\frac{\partial^2 R_{t+1}}{\partial\Lambda_{t+1}^2}$ and so $\frac{\partial^2 R_t}{\partial\lambda_t\partial\mu_t} > 0$.

It then follows that $\frac{\partial^2 R_t}{\partial\mu_t^2}\frac{\partial^2 R_t}{\partial\lambda_t^2} > \left(\frac{\partial^2 R_t}{\partial\lambda_t\partial\mu_t}\right)^2$ if,

$$\frac{2\gamma\lambda_t}{(\mu_t-\lambda_t)^3}\left(\frac{2}{\alpha} + \frac{2\gamma\mu_t}{(\mu_t-\lambda_t)^3}\right) > \left(\gamma\frac{(\mu_t+\lambda_t)}{(\mu_t-\lambda_t)^3}\right)^2.$$

Suppressing the subscript t, this is equivalent to $\frac{4\gamma\lambda}{\alpha(\mu-\lambda)^6}((\mu-\lambda)^3+\alpha\gamma\mu) > (\gamma\frac{(\mu+\lambda)}{(\mu-\lambda)^3})^2$ or $\frac{4\lambda}{\alpha}((\mu-\lambda)^3+\alpha\gamma\mu) > \gamma(\mu+\lambda)^2$. Since the service rate should be higher than the arrival rate, let $\mu = k\lambda$, where $k > 1$. Then we need $\frac{4}{\alpha}((k-1)^3\lambda^2+\alpha\gamma k) > \gamma(k+1)^2$ or $\lambda^2 > \alpha\gamma\frac{(\frac{(k+1)^2}{4}-k)}{(k-1)^3} = \frac{\alpha\gamma}{4(k-1)}$. From the single-period analysis (see proof of Theorem 1), we have: $(\mu_S^* - \lambda_S^*) = \sqrt{\frac{\gamma}{\delta\theta}}$ where the subscript S is used to distinguish the single-period from the multi-period optima. We can show that $\mu - \lambda \geq \mu_S^* - \lambda_S^*$ using Theorem 5(i), the fact that $\frac{\partial(\mu_S^*-\lambda_S^*)}{\partial\Lambda} = 0$ and the fact that the Nth period optimal solution coincides with the single period optimal solution when the starting demand potential is Λ_N. Therefore, $(\mu-\lambda) \geq \sqrt{\frac{\gamma}{\delta\theta}}$ and so $\frac{1}{(k-1)} \leq \frac{\lambda}{\sqrt{\frac{\gamma}{\delta\theta}}}$ and $\frac{\lambda\alpha\sqrt{\gamma\delta\theta}}{4} \geq \frac{\alpha\gamma}{4(k-1)}$. Now, if $\lambda > \frac{\alpha\sqrt{\gamma\delta\theta}}{4}$, it then follows that $\lambda^2 > \frac{\lambda\alpha\sqrt{\gamma\delta\theta}}{4} = \frac{\alpha\gamma}{4(k-1)}$. We assume λ is large enough to satisfy the condition $\lambda > \frac{\alpha\sqrt{\gamma\delta\theta}}{4}$. Hence, $\frac{\partial^2 R_t}{\partial\mu_t^2}\frac{\partial^2 R_t}{\partial\lambda_t^2} \geq \left(\frac{\partial^2 R_t}{\partial\lambda_t\partial\mu_t}\right)^2$ and the concavity conditions are satisfied at stage t. □

Proof of Theorem 1

We start by identifying the optimal price and service speed decisions. Recall that the expected profit is

$$\mathcal{R}(\mu,p) = p(\Lambda-\alpha p) - \gamma\frac{\Lambda-\alpha p}{\mu-\Lambda+\alpha p} - \theta\delta(\Lambda-\alpha p)(\mu-\hat{\mu}) + \theta\Lambda.$$

The optimal pricing and service speed decisions will satisfy the first and second order conditions. Solving the first order conditions $\frac{\partial\mathcal{R}(\mu,p)}{\partial p} = 0$ and $\frac{\partial\mathcal{R}(\mu,p)}{\partial\mu} = 0$ gives

$$\Lambda - 2\alpha p^* + \frac{\gamma\mu^*\alpha}{(\mu^*-\Lambda+\alpha p^*)^2} + \alpha\delta\theta(\mu^*-\hat{\mu}) = 0$$
$$(\mu^*-\Lambda+\alpha p^*)^2 = \frac{\gamma}{\delta\theta}$$

and so (6) follows. Moreover since $\lambda^* > 0$, we always **require** that $\lambda^* = \frac{\Lambda+\alpha\theta\delta\hat{\mu}-2\alpha\sqrt{\gamma\delta\theta}}{2(1+\alpha\delta\theta)} > 0$. □

	speed	price	demand	profit
with respect to δ	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow/\uparrow$
with respect to γ	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$
with respect to α	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$
with respect to Λ	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
with respect to $\hat{\mu}$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$

Table 2: Sensitivity Analysis for the parameters in the Single Period Model.

Sensitivity Analysis

In this subsection we provide the derivatives of the optimal price, speed, demand and profit with respect to several parameters. To determine the sign of each expression, we have assumed that $\Lambda > \hat{\mu}$. This condition may be reasonable because we have noted earlier that $\Lambda > 2\lambda$ and since "normal" speed is not expected to be larger than twice the demand (otherwise utilization will be very low). We summarize the results in Table 2.

with respect to δ

$$\frac{\partial \mu^*}{\partial \delta} = \frac{\alpha\theta\hat{\mu} - \sqrt{\frac{\gamma}{\delta^3\theta}} - 3\alpha\sqrt{\frac{\gamma\theta}{\delta}} - \alpha\theta\Lambda}{2(1+\alpha\delta\theta)^2} < 0$$

$$\frac{\partial p^*}{\partial \delta} = \frac{\Lambda\theta - \theta\hat{\mu} - \alpha\theta\sqrt{\gamma\theta\delta} + \sqrt{\frac{\gamma\theta}{\delta}}}{2(1+\alpha\delta\theta)^2} > 0$$

$$\frac{\partial \lambda^*}{\partial \delta} = -\alpha\frac{\Lambda\theta - \theta\hat{\mu} - \alpha\theta\sqrt{\gamma\theta\delta} + \sqrt{\frac{\gamma\theta}{\delta}}}{2(1+\alpha\delta\theta)^2} < 0$$

$$\frac{\partial \mathcal{R}^*}{\partial \delta} = \theta\frac{\Lambda + \alpha\delta\theta\hat{\mu} - 2\alpha\sqrt{\gamma\theta\delta}}{4(1+\alpha\delta\theta)^2}\left(-\Lambda + \alpha\delta\theta\hat{\mu} + 2\hat{\mu} - 2\sqrt{\frac{\gamma}{\delta\theta}}\right) \begin{array}{l} < 0 \text{ if } \hat{\mu} < \mu^* \\ > 0 \text{ if } \hat{\mu} > \mu^* \end{array}$$

For the last inequalities observe that $\hat{\mu} < \mu^*$ means $-\Lambda + \alpha\delta\theta\hat{\mu} + 2\hat{\mu} - 2\sqrt{\frac{\gamma}{\delta\theta}} < 0$. Similarly, we can show that

$$\text{with respect to } \gamma : \frac{\partial \mu^*}{\partial \gamma} > 0, \frac{\partial p^*}{\partial \gamma} > 0, \frac{\partial \lambda^*}{\partial \gamma} < 0, \frac{\partial \mathcal{R}^*}{\partial \gamma} < 0$$

$$\text{with respect to } \hat{\mu} : \frac{\partial \mu^*}{\partial \hat{\mu}} > 0, \frac{\partial p^*}{\partial \hat{\mu}} < 0, \frac{\partial \lambda^*}{\partial \hat{\mu}} > 0, \frac{\partial \mathcal{R}^*}{\partial \hat{\mu}} > 0$$

$$\text{with respect to } \Lambda : \frac{\partial \mu^*}{\partial \Lambda} > 0, \frac{\partial p^*}{\partial \Lambda} > 0, \frac{\partial \lambda^*}{\partial \Lambda} > 0, \frac{\partial \mathcal{R}^*}{\partial \Lambda} > 0$$

$$\text{with respect to } \alpha : \frac{\partial \mu^*}{\partial \alpha} < 0, \frac{\partial p^*}{\partial \alpha} < 0, \frac{\partial \lambda^*}{\partial \alpha} < 0, \frac{\partial \mathcal{R}^*}{\partial \alpha} < 0.$$

Proof of Theorem 2

The demand potential can also be written as

$$\Lambda_i = \Lambda_{i-1} - \delta(\mu - \hat{\mu})\lambda_{i-1} = \Lambda_1 - \delta(\mu - \hat{\mu}) \sum_{j=1}^{i-1} \lambda_j$$

for $2 \le i \le N+1$, and its derivative with respect to λ_j will be

$$\frac{\partial \Lambda_i}{\partial \lambda_j} = \begin{cases} -\delta(\mu - \hat{\mu}) & \text{for } j < i \\ \\ 0 & \text{otherwise} \end{cases}.$$

Eliminating the prices p_i by using the fact that $p_i = \frac{\Lambda_i - \lambda_i}{\alpha}$ the expected profit can be written as

$$\mathcal{R}(\mu, \lambda_i) = \sum_{i=1}^{N} \left[\frac{\Lambda_i - \lambda_i}{\alpha} \lambda_i - \gamma \frac{\lambda_i}{\mu - \lambda_i} \right] + \theta \Lambda_{N+1}.$$

The optimal demand rate λ_i^* will satisfy $\frac{\partial \mathcal{R}}{\partial \lambda_i} = 0$, or equivalently,

$$\frac{\Lambda_i - 2\lambda_i^*}{\alpha} - \frac{\gamma\mu}{(\mu - \lambda_i^*)^2} - \delta\theta(\mu - \hat{\mu}) - \frac{\delta(\mu - \hat{\mu})}{\alpha} \sum_{j=i+1}^{N} \lambda_j^* = 0. \tag{24}$$

We first consider the case where $\mu > \hat{\mu}$. A similar argument holds for $\mu < \hat{\mu}$. For now, we assume that $\mu < \frac{2}{\delta}$ and in the proof of Theorem 3, we identify conditions on the model parameters such that this constraint will be satisfied. (24) for i and $(i-1)$ is,

$$\begin{aligned} \frac{\Lambda_i - 2\lambda_i^*}{\alpha} - \frac{\gamma\mu}{(\mu - \lambda_i^*)^2} - \delta\theta(\mu - \hat{\mu}) - \frac{\delta(\mu - \hat{\mu})}{\alpha} \sum_{j=i+1}^{N} \lambda_j^* &= 0 \\ \frac{\Lambda_{i-1} - 2\lambda_{i-1}^*}{\alpha} - \frac{\gamma\mu}{(\mu - \lambda_{i-1}^*)^2} - \delta\theta(\mu - \hat{\mu}) - \frac{\delta(\mu - \hat{\mu})\lambda_i^*}{\alpha} - \frac{\delta(\mu - \hat{\mu})}{\alpha} \sum_{j=i}^{N} \lambda_j^* &= 0 \end{aligned}$$

and combining both, we have

$$\frac{\Lambda_{i-1} - \delta\lambda_{i-1}^*(\mu - \hat{\mu})}{\alpha} - \frac{2\lambda_i^*}{\alpha} - \frac{\gamma\mu}{(\mu - \lambda_i^*)^2} = \frac{\Lambda_{i-1} - 2\lambda_{i-1}^*}{\alpha} - \frac{\gamma\mu}{(\mu - \lambda_{i-1}^*)^2} - \frac{\delta(\mu - \hat{\mu})\lambda_i^*}{\alpha}.$$

By rearranging the terms, we end up with

$$\frac{\delta\lambda_{i-1}^*(\mu - \hat{\mu})}{\alpha} + \frac{2\lambda_i^*}{\alpha} + \frac{\gamma\mu}{(\mu - \lambda_i^*)^2} = \frac{\delta(\mu - \hat{\mu})\lambda_i^*}{\alpha} + \frac{2\lambda_{i-1}^*}{\alpha} + \frac{\gamma\mu}{(\mu - \lambda_{i-1}^*)^2}.$$

In the above equation, both the right and the left hand side are positive with the same form. Therefore, there exists an optimal solution wherein $\lambda_i^* = \lambda_{i-1}^*$. The same is true for every i. Therefore, $\lambda_N^* = \lambda_{N-1}^* = \ldots = \lambda_1^* = \lambda^*$ and the result follows. □

Proof of Theorem 3

(a) Writing (24) for i and using Theorem 2, we have that

$$\frac{\Lambda_i - 2\lambda^*}{\alpha} - \frac{\gamma\mu}{(\mu-\lambda^*)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}(N-i)\lambda^* = 0.$$

Now we can use the fact that $\lambda^* = \Lambda_i - \alpha p_i$ and so

$$p_i - \frac{\lambda^*}{\alpha} - \frac{\gamma\mu}{(\mu-\lambda^*)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}(N-i)\lambda^* = 0.$$

Therefore, the optimal pricing policy will be

$$p_i^* = \frac{\mu^*\gamma}{(\mu^*-\lambda^*)^2} + \delta\theta(\mu^*-\hat{\mu}) + \frac{\lambda^*}{\alpha} + \frac{(N-i)\lambda^*\delta(\mu^*-\hat{\mu})}{\alpha}.$$

When speed is constant and equal to $\hat{\mu}$, the optimal price reduces to:

$$p^* = \frac{\gamma\hat{\mu}}{(\hat{\mu}-\lambda^*)^2} + \frac{\lambda^*}{\alpha}.$$

As we proved in Theorem 2, demand rate remains constant and thus $\Lambda_1 - \alpha p_1^* = \lambda^*$. Therefore, we have that λ^* will satisfy

$$\Lambda_1 - 2\lambda^* - \frac{\alpha\mu^*\gamma}{(\mu^*-\lambda^*)^2} - \alpha\delta\theta(\mu^*-\hat{\mu}) - (N-1)\lambda^*\delta(\mu^*-\hat{\mu}) = 0. \tag{25}$$

optimal speed: The optimal speed of the system will satisfy the first order condition, $\frac{\partial\mathcal{R}(\mu,\lambda_i)}{\partial\mu} = 0$, that is

$$\sum_{i=1}^{N}\left\{\frac{\lambda}{\alpha}\frac{\partial\Lambda_i}{\partial\mu} + \gamma\frac{\lambda}{(\mu-\lambda)^2}\right\} + \theta\frac{\partial\Lambda_{N+1}}{\partial\mu} = 0$$

given that $\lambda_i = \lambda$. Furthermore, for $i > 0$, $\frac{\partial\Lambda_i}{\partial\mu} = -(i-1)\delta\lambda$ and hence,

$$-\frac{\delta(N-1)\lambda}{2\alpha} + \frac{\gamma}{(\mu-\lambda)^2} - \theta\delta = 0. \tag{26}$$

Combining (25) and (26) suggests that λ^* and μ^* should be a solution of

$$\mu^* = \frac{\Lambda_1 - 2\lambda^* + \hat{\mu}\delta\left(\alpha\theta + (N-1)\lambda^*\right)}{\delta\left(2\alpha\theta + \frac{3}{2}(N-1)\lambda^*\right)}$$

and

$$-\frac{\delta\lambda^*(N-1)}{2} + \frac{\alpha\gamma}{(\mu^*-\lambda^*)^2} - \alpha\theta\delta = 0.$$

(b) Using the implicit function theorem and the second equation for the optimal solution of (λ, μ), we have: $\frac{\partial\mu}{\partial\theta} = -\frac{F_\theta}{F_\mu} = \frac{+\alpha\delta}{-\alpha\gamma 2(\mu-\lambda)/(\mu-\lambda)^4} = -\frac{\delta(\mu-\lambda)^3}{2\gamma} < 0.$

So μ decreases with θ. Taking derivatives of (24) with respect to θ and λ, we have:

$\frac{\partial \lambda}{\partial \theta} = -\frac{F_\theta}{F_\lambda} = \frac{-\delta(\mu-\hat{\mu})}{-(\frac{2}{\alpha}+\frac{2\gamma}{(\mu-\lambda)^3}+\frac{\delta(\mu-\hat{\mu})}{\alpha})}$. So $\frac{\partial \lambda}{\partial \theta} < 0$ if $\mu > \hat{\mu}$.

Now consider the case where $\mu < \hat{\mu}$. The expression $\frac{2}{\alpha} + \frac{2\gamma}{(\mu-\lambda)^3} + \frac{\delta(\mu-\hat{\mu})}{\alpha}$ is positive if $\frac{2}{\alpha} + \frac{2\gamma\mu}{(\mu-\lambda)^3} + \frac{\delta\mu}{\alpha} > \frac{\delta\hat{\mu}}{\alpha}$ or this is equivalent to $\frac{2}{\delta} + \frac{2\alpha\mu\gamma/\delta}{(\mu-\lambda)^3} + \mu > \hat{\mu}$. Substituting for $\frac{\gamma}{(\mu-\lambda)^2}$ from (26), we have: $\frac{2}{\delta\mu} + \frac{2\alpha\theta}{(\mu-\lambda)} + \frac{(N-1)\lambda}{(\mu-\lambda)} + 1 > \frac{\hat{\mu}}{\mu}$ and this condition is always satisfied since $\hat{\mu} < \frac{2}{\delta}$.

The analysis for the impact of θ on price p_i is similar to that for λ and is therefore omitted. The analysis for showing the effect of N on μ, λ and p_i is identical to that for θ and is omitted.

(c) Suppose that $\mu^* = \hat{\mu}$, then (25) becomes

$$\Lambda_1 - 2\lambda^* - \frac{\alpha\mu^*\gamma}{(\mu^* - \lambda^*)^2} = 0.$$

Let $\hat{\hat{\mu}}$ be defined as the value of $\hat{\mu} = \mu^*$ that satisfies the above equation and so, if $\hat{\hat{\lambda}}$ is the corresponding demand rate when service rate is $\hat{\hat{\mu}}$, then $\hat{\hat{\lambda}}$ and $\hat{\hat{\mu}}$ will also solve (26) for $\mu^* = \hat{\mu} = \hat{\hat{\mu}}$ and $\lambda^* = \hat{\hat{\lambda}}$, i.e.

$$-\frac{\delta(N-1)\hat{\hat{\lambda}}}{2\alpha} + \frac{\gamma}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \theta\delta = 0.$$

Therefore, after some algebra, $\hat{\hat{\lambda}}$ and $\hat{\hat{\mu}}$ will satisfy,

$$\Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\alpha\hat{\hat{\mu}}\gamma}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0 \quad \text{and} \quad \hat{\hat{\mu}} = \frac{2(\Lambda_1 - 2\hat{\hat{\lambda}})}{\delta\left(\hat{\hat{\lambda}}(N-1) + 2\alpha\theta\right)}.$$

Now let us discuss the case when $\hat{\mu} > \hat{\hat{\mu}}$ and observe the following equations that come from (25),

$$\frac{\Lambda_1 - 2\hat{\hat{\lambda}}}{\alpha} - \frac{\gamma\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0 \tag{27}$$

$$\frac{\Lambda_1 - 2\lambda^*}{\alpha} - \frac{\gamma\mu^*}{(\mu^* - \lambda^*)^2} - \delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\frac{\delta(\mu^* - \hat{\mu})}{\alpha} = 0 \tag{28}$$

where the second one is the optimality condition when $\hat{\mu} > \hat{\hat{\mu}}$. Now suppose that $\hat{\mu} > \hat{\hat{\mu}}$ but $\mu^* < \hat{\mu}$ is not true in the optimality equation, i.e. (28) is not satisfied when $\mu^* < \hat{\mu}$. Consider the following solution to the scenario $\hat{\mu} > \hat{\hat{\mu}}$. Let $\mu^* > \hat{\hat{\mu}}$ be the optimal service speed and let λ^* be such that

$$\frac{\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = \frac{\mu^*}{(\mu^* - \lambda^*)^2},$$

$$\text{or equivalently,} \qquad \lambda^* = \hat{\hat{\lambda}}\sqrt{\frac{\mu^*}{\hat{\hat{\mu}}}} + \mu^* - \sqrt{\mu^*\hat{\hat{\mu}}}.$$

Since $\mu^* > \hat{\hat{\mu}}$, the above implies that $\lambda^* > \hat{\hat{\lambda}}$. Subtracting (27) from (28) gives

$$\frac{2\hat{\hat{\lambda}}}{\alpha} - \frac{2\lambda^*}{\alpha} + \frac{\gamma\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \frac{\gamma\mu^*}{(\mu^* - \lambda^*)^2} - \delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\frac{\delta(\mu^* - \hat{\mu})}{\alpha} = 0. \tag{29}$$

But,
$$\frac{2\hat{\hat{\lambda}}}{\alpha} - \frac{2\lambda^*}{\alpha} < 0, \text{ because } \lambda^* > \hat{\hat{\lambda}}$$
$$\frac{\gamma\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \frac{\gamma\mu^*}{(\mu^* - \lambda^*)^2} = 0, \text{ due to the definition of } \lambda^*.$$

(29) can be satisfied only if $\mu^* \leq \hat{\mu}$ and so we have a contradiction. A similar argument holds for the opposite case and the result follows.

Now we show that $\mu^* > \hat{\hat{\mu}}$ when $\mu^* < \hat{\mu}$. From (28), we know that μ^* is such that,

$$\Lambda_1 - 2\lambda^* - \frac{\alpha\mu^*\gamma}{(\mu^* - \lambda^*)^2} - \alpha\delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\delta(\mu^* - \hat{\mu}) = 0$$

which implies

$$\Lambda_1 - 2\lambda^* - \frac{\alpha\mu^*\gamma}{(\mu^* - \lambda^*)^2} = \alpha\delta\theta(\mu^* - \hat{\mu}) + (N-1)\lambda^*\delta(\mu^* - \hat{\mu}) < 0.$$

But $\hat{\hat{\mu}}$ satisfies
$$\Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\alpha\hat{\hat{\mu}}\gamma}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0.$$
Combining the last two relationships, we have

$$\Lambda_1 - 2\lambda^* - \frac{\alpha\mu^*\gamma}{(\mu^* - \lambda^*)^2} < 0 = \Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\alpha\hat{\hat{\mu}}\gamma}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2}$$

which is equivalent to
$$2\lambda^* + \frac{\alpha\mu^*\gamma}{(\mu^* - \lambda^*)^2} > 2\hat{\hat{\lambda}} + \frac{\alpha\hat{\hat{\mu}}\gamma}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2}$$
and implies that $\mu^* > \hat{\hat{\mu}}$ taking into consideration that

$$\frac{\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = \frac{\mu^*}{(\mu^* - \lambda^*)^2},$$

and also, if $\hat{\hat{\lambda}} > (<)\lambda^*$, then $\hat{\hat{\mu}} > (<)\mu^*$, as shown earlier.

Recall that we assumed that the model parameters are such that $\hat{\mu} < \frac{2}{\delta}$, $\hat{\hat{\mu}} < \frac{2}{\delta}$. Since μ^* always lies between $\hat{\mu}$ and $\hat{\hat{\mu}}$, these assumptions ensure that $\mu^* < \frac{2}{\delta}$.

We can use a similar argument to show that if $\mu^* > \hat{\mu}$, then $\hat{\mu} < \mu^* < \hat{\hat{\mu}}$. The result on prices increasing if $\hat{\hat{\mu}} < \mu^* < \hat{\mu}$ follows (and similarly for prices decreasing) because demand is constant over time and when $\mu^* < \hat{\mu}$, Λ_i increases and so p_i^* has to increase to ensure demand is constant.

□

Proof of Theorem 4

(i) The profit function is

$$\mathcal{R}(p, \mu_t) = \sum_{t=1}^{N} \left\{ p(\Lambda_t - \alpha p) - \gamma \frac{\Lambda_t - \alpha p}{\mu_t - \Lambda_t + \alpha p} \right\} + \theta\Lambda_{N+1}.$$

and the optimal price and speed will be obtained by solving the following optimization problem

$$\begin{aligned} Max \quad &: \quad R_t(\Lambda_t) = \frac{\Lambda_t - \lambda_t}{\alpha}\lambda_t - \frac{\gamma\lambda_t}{\mu_t - \lambda_t} + R_{t+1}(\Lambda_{t+1}) \ \forall t = 1, \ldots, N \\ \text{where } \Lambda_{t+1} \quad &= \quad \Lambda_t - \delta\lambda_t(\mu_t - \hat{\mu}) \text{ and } R_{N+1} = \theta\Lambda_{N+1} \text{ where } \theta > 0. \end{aligned}$$

Let

$$\pi_t(\Lambda_t) = \frac{\Lambda_t - \lambda_t}{\alpha}\lambda_t - \frac{\gamma\lambda_t}{\mu_t - \lambda_t} \ \forall t = 1, \ldots, N.$$

$\pi_t(\Lambda_t)$ is effectively the single-period profit (revenue less congestion costs) in period t.

We have,

$$\frac{\partial \mathcal{R}_t}{\partial \Lambda_t} = \frac{\partial \pi_t}{\partial \Lambda_t} + \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_t} = \frac{\partial \pi_t}{\partial \Lambda_t} + \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}\frac{\partial \Lambda_{t+1}}{\partial \Lambda_t}.$$

Then, $\frac{\partial \Lambda_{t+1}}{\partial \Lambda_t} = 1 - \delta(\mu_t - \hat{\mu}) > 1$ when $\mu_t < \hat{\mu}$. From the single-period analysis, we know that $\frac{\partial \pi_t}{\partial \Lambda_t} > 0$. Note that $\frac{\partial \mathcal{R}_{N+1}}{\partial \Lambda_{N+1}} = \theta > 0$ and it follows recursively that $\frac{\partial \mathcal{R}_t}{\partial \Lambda_t} > \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}$.

Now,

$$\frac{\partial \mathcal{R}_t}{\partial \mu_t} = \frac{\gamma\lambda_t}{(\mu_t - \lambda_t)^2} + \frac{\partial \mathcal{R}_{t+1}}{\partial \mu_t} = \frac{\gamma\lambda_t}{(\mu_t - \lambda_t)^2} - \delta\lambda_t\frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}. \tag{30}$$

So the optimal μ_t^* will satisfy,

$$\frac{\gamma}{(\mu_t^* - \lambda_t)^2} = \delta\frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}. \tag{31}$$

Since $\frac{\partial \mathcal{R}_t}{\partial \Lambda_t} > \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}$, it follows that $\mu_{t-1}^* - \lambda_{t-1} < \mu_t^* - \lambda_t$. Now if $\mu_{t-1}^* < \hat{\mu}$, then we have $\Lambda_t > \Lambda_{t-1}$ and since p is constant, $\lambda_t > \lambda_{t-1}$ and so $\mu_t^* > \mu_{t-1}^*$.

Now, suppose $\mu_t^* < \hat{\mu}$. Then using the same argument as above, we can show that $\mu_{t+1}^* > \mu_t^*$. We note that since the analysis above only involves the scenario $\mu_t^* < \hat{\mu}$ and $\hat{\mu} < \frac{2}{\delta}$, the condition $\mu_t^* < \frac{2}{\delta}$ is always satisfied.

(ii) Next, we have also that $\frac{\partial \mathcal{R}(p,\mu_t)}{\partial p} = 0$. The derivative of the demand potential with respect to the price is

$$\frac{\partial \Lambda_t}{\partial p} = -\alpha\prod_{k=1}^{t-1}[1 - \delta(\mu_k - \hat{\mu})] + \alpha = \alpha\left(1 - \frac{\lambda_t}{\lambda_1}\right),$$

when $t \geq 2$. The first order condition will then be

$$\sum_{\tau=1}^{t}\frac{\partial \mathcal{R}_\tau}{\partial p} + \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}\frac{\partial \Lambda_{t+1}}{\partial p} = 0$$

or equivalently

$$\sum_{\tau=1}^{t}\lambda_\tau\left[1 - \frac{\alpha p^*}{\lambda_1} + \frac{\alpha\gamma}{\lambda_1}\frac{\mu_\tau}{(\mu_\tau - \lambda_\tau)^2}\right] + \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}\alpha\left(1 - \frac{\lambda_{t+1}}{\lambda_1}\right) = 0.$$

From (31), we have that

$$\frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}} = \frac{\gamma/\delta}{(\mu_t^* - \lambda_t)^2}$$

that leads to

$$\sum_{\tau=1}^{t} \frac{\lambda_\tau^* \lambda_1^*}{\alpha(\lambda_1^* - \lambda_{t+1}^*)} \left[1 - \frac{\alpha p^*}{\lambda_1^*} + \frac{\alpha\gamma}{\lambda_1^*} \frac{\mu_\tau^*}{(\mu_\tau^* - \lambda_\tau^*)^2}\right] + \frac{\gamma/\delta}{(\mu_t^* - \lambda_t^*)^2} = 0 \tag{32}$$

and implies that the summation is negative. Assume now that μ_t^* is increasing over time. Then the summation will be increasing (we know that λ_t^* decreases over time). Using the above equation for t and $t-1$, we have

$$\frac{\lambda_t^* \lambda_1^*}{\alpha(\lambda_1^* - \lambda_{t+1}^*)} \left[1 - \frac{\alpha p^*}{\lambda_1^*} + \frac{\alpha\gamma\mu_t^*}{\lambda_1^*(\mu_t^* - \lambda_t^*)^2}\right] + \frac{\gamma/\delta}{(\mu_t^* - \lambda_t^*)^2} - \frac{\gamma/\delta}{(\mu_{t-1}^* - \lambda_{t-1}^*)^2} = 0.$$

The term $\frac{\lambda_t^*\lambda_1^*}{\alpha(\lambda_1^* - \lambda_{t+1}^*)}\left[1 - \frac{\alpha p^*}{\lambda_1} + \frac{\alpha\gamma\mu_t^*}{\lambda_1^*(\mu_t^* - \lambda_t^*)^2}\right]$ is positive for $t > 1$ but the summation in (32) is negative which implies that $\frac{\mu_t^*}{\lambda_1^*(\mu_t^* - \lambda_t^*)^2}$ must be increasing so that the term within the brackets becomes positive (from negative). But this cannot happen if μ_t^* is increasing and so we have a contradiction.

Since μ_t^* is decreasing, to ensure that $\mu_t^* < \frac{2}{\delta}$, it is sufficient if we show that $\mu_1^* < \frac{2}{\delta}$. Suppose we assume the parameters are such that $\mu^* = \frac{\Lambda + \alpha\delta\theta\hat{\mu} + 2\sqrt{\frac{\gamma}{\delta\theta}}}{2(1+\alpha\delta\theta)} < \frac{2}{\delta}$ where μ^* is the optimal solution in the single-period problem. Then, using an analysis similar to that in the constant speed case, we can show that μ^* decreases as N increases. Since μ_1^* is the first-period solution to a $N-$period model ($N > 1$), $\mu_1^* < \mu^* < \frac{2}{\delta}$ and so the condition $\mu_t^* < \frac{2}{\delta}$ will be satisfied.

(iii) Now, we explore the convergence of μ_t^* to $\hat{\mu}$. The optimal speed in period t, μ_t^*, and the optimal demand rate in period t, $\lambda_t^*(N)$, are functions of the time horizon N and we will use $\mu_t^*(N)$ and $\lambda_t^*(N)$ here to express this dependence. Suppose $\mu_i^*(N) < \hat{\mu}$ for all $i < t$. Then we know that $\mu_1^*(N) < \mu_2^*(N) < \ldots < \mu_t^*(N) < \mu_{t+1}^*(N) < \hat{\mu}$ from part (i) and so the gap between $\mu_t^*(N)$ and $\hat{\mu}$ decreases. That is,

$$|\mu_t^*(N) - \hat{\mu}| > \left|\mu_{t+1}^*(N) - \hat{\mu}\right|.$$

Suppose that $\mu_t(N)$ converges to a value $\dot{\mu} < \hat{\mu}$. We have from (i) that

$$\mu_t^*(N) - \mu_{t-1}^*(N) > \lambda_t^*(N) - \lambda_{t-1}^*(N) = \Lambda_t - \Lambda_{t-1} = \delta\Big(\hat{\mu} - \mu_{t-1}^*(N)\Big)\lambda_{t-1}^*(N).$$

Taking limits in both sides we have that

$$\lim_{t\to\infty}\Big(\mu_t^*(\infty) - \mu_{t-1}^*(\infty)\Big) > \lim_{t\to\infty}\delta\Big(\hat{\mu} - \mu_{t-1}^*(\infty)\Big)\lambda_{t-1}^*(\infty)$$

which cannot hold unless $\dot{\mu} = \hat{\mu}$ and so $\lim_{t\to\infty}\mu_t^*(\infty) = \hat{\mu}$.

Similarly, if $\mu_t^*(N) > \hat{\mu}$, we can use the same argument (with the opposite inequality signs) to show that $\lim_{t\to\infty}\mu_t^*(\infty) = \hat{\mu}$.

Next, we show that there exists $\theta_N > 0$ such that $\mu_N^* = \hat{\mu}$ if $\Lambda_N > 2\hat{\mu}$. The first order conditions for period N are:

$$\frac{\partial \mathcal{R}}{\partial \mu_N} = \frac{\gamma \lambda_N^*}{(\mu_N^* - \lambda_N^*)^2} - \theta\delta\lambda_N^* = 0$$

$$\frac{\partial \mathcal{R}}{\partial \lambda_N} = \frac{\Lambda_N - 2\lambda_N^*}{\alpha} - \frac{\gamma}{\mu_N^* - \lambda_N^*} - \theta\delta\mu_N^* - \lambda_N^* - 2\theta\delta\lambda_N^* + \theta\delta\hat{\mu} = 0.$$

Combining them together and replacing μ_N with $\hat{\mu}$, we need to satisfy:

$$\Lambda_N - 2\hat{\mu} + 2\sqrt{\frac{\gamma}{\theta\delta}} - \alpha\theta\delta\hat{\mu} = 0.$$

This is equivalent to a cubic equation in θ which will have at least one real solution. Further, it is easy to show that if $\Lambda_N > 2\hat{\mu}$, this root will be positive since otherwise (if the root is negative), the equation will not be satisfied.

(iv) The optimal price should satisfy the first order condition $\frac{\partial \mathcal{R}}{\partial p} = 0$, i.e.

$$\sum_{i=1}^{N} \left\{ \lambda_i - \alpha \frac{\lambda_i}{\lambda_1} \left(p^* - \frac{\gamma \mu_i}{(\mu_i - \lambda_i)^2} \right) \right\} + \alpha\theta \left(1 - \frac{\lambda_N \left[1 - \delta(\mu_N - \hat{\mu})\right]}{\lambda_1} \right) = 0.$$

If we further use the condition $\frac{\partial \mathcal{R}}{\partial \mu_i} = 0$, that gives

$$-\delta \left\{ \sum_{j=i+1}^{N} \left(p^* - \frac{\gamma \mu_j^*}{(\mu_j^* - \lambda_j^*)^2} \right) \lambda_j^* + \theta\lambda_N^* \left[1 - \delta(\mu_N^* - \hat{\mu})\right] \right\} + \frac{\gamma \lambda_{i+1}^*}{(\mu_i^* - \lambda_i^*)^2} = 0,$$

then it becomes

$$\sum_{i=1}^{N} \lambda_i^* - \alpha \left\{ p^* - \frac{\gamma \left[1 - \delta(2\mu_1^* - \hat{\mu})\right]}{\delta(\mu_1^* - \lambda_1^*)^2} \right\} + \theta\alpha = 0. \tag{33}$$

Therefore, the optimal price will satisfy

$$\frac{\Lambda_1}{\alpha} \left\{ 1 + \sum_{i=2}^{N} \prod_{k=1}^{i-1} \left[1 - \delta(\mu_k^* - \hat{\mu})\right] \right\} + p^* \sum_{i=2}^{N} \prod_{k=1}^{i-1} \left[1 - \delta(\mu_k^* - \hat{\mu})\right] - \frac{\gamma \left[1 - \delta(2\mu_1^* - \hat{\mu})\right]}{\delta(\mu_1^* - \Lambda_1 + \alpha p^*)^2} = 0.$$

When speed is constant and equal to $\hat{\mu}$, market potential and demand are also constant over time, it follows from the first order condition $\frac{\partial \mathcal{R}}{\partial p} = 0$ above that:

$$\sum_{i=1}^{N} \left\{ \lambda^* - \alpha \frac{\lambda^*}{\lambda^*} \left(p - \frac{\gamma \mu^*}{(\mu^* - \lambda^*)^2} \right) \right\} + \alpha\theta \left(1 - \frac{\lambda^* \left[1 - \delta(\mu^* - \hat{\mu})\right]}{\lambda^*} \right) = 0$$

which leads to $p^* = \frac{\gamma\hat{\mu}}{(\hat{\mu} - \lambda^*)^2} + \frac{\lambda^*}{\alpha}$.

(v) p^* increases with θ follows directly from the equation (33). One can implicitly differentiate this equation to get the result.

Since $\frac{\partial p}{\partial \theta} > 0$, given that $\lambda_1 = \Lambda_1 - \alpha p$, λ_1 decreases with θ.

Since $\frac{\partial \mathcal{R}_{N+1}}{\partial \Lambda_{N+1}} = \theta$ and $\frac{\partial \mathcal{R}_t}{\partial \Lambda_t}$ for all $t \leq N$ can be expressed as a function of $\frac{\partial \mathcal{R}_{N+1}}{\partial \Lambda_{N+1}}$, it follows that $\frac{\partial \mathcal{R}_t}{\partial \Lambda_t}$ increases in θ (so long as $\frac{\partial \Lambda_{t+1}}{\partial \Lambda_t} > 0$).

It then follows from (31) in the proof of Theorem 4(i) that $(\mu_t - \lambda_t)$ decreases with θ.

So, in period 1, since we know that λ_1 decreases with θ, μ_1 has to decrease with θ. $\square$

Proof of Theorem 5

(i) For the demand potential, we have that

$$\Lambda_i = \Lambda_1 - \delta \sum_{j=1}^{i-1} (\mu_j - \hat{\mu})\lambda_j$$

and its derivative with respect to μ_i and p_i are,

$$\frac{\partial \Lambda_j}{\partial \mu_i} = \begin{cases} -\delta\lambda_i \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k - \hat{\mu})] & , \ j \geq i+1 \\ \\ 0 & , \text{else} \end{cases}$$

$$\frac{\partial \Lambda_j}{\partial p_i} = \begin{cases} \alpha\delta(\mu_i - \hat{\mu}) \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k - \hat{\mu})] & , \ i < j \\ \\ 0 & , \text{else} \end{cases} .$$

We start by showing that service speed increases faster that the demand rate. The optimal service speed will satisfy the first order condition $\frac{\partial \mathcal{R}}{\partial \mu_i} = 0$ that is

$$\sum_{j=i+1}^{N} \left(p_j - \frac{\gamma \mu_j^*}{(\mu_j^* - \lambda_j)^2} \right) \frac{\partial \Lambda_j}{\partial \mu_i} + \theta \frac{\partial \Lambda_{N+1}}{\partial \mu_i} + \frac{\gamma \lambda_i}{(\mu_i^* - \lambda_i)^2} = 0,$$

or equivalently

$$-\delta \left\{ \sum_{j=i+1}^{N} \left(p_j - \frac{\gamma \mu_j^*}{(\mu_j^* - \lambda_j)^2} \right) \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k^* - \hat{\mu})] + \theta \prod_{k=i+1}^{N} [1 - \delta(\mu_k^* - \hat{\mu})] \right\} + \frac{\gamma}{(\mu_i^* - \lambda_i)^2} = 0. \tag{34}$$

If we write (34) for $i-1$,

$$-\delta \left\{ \sum_{j=i}^{N} \left(p_j - \frac{\gamma \mu_j^*}{(\mu_j^* - \lambda_j)^2} \right) \prod_{k=i}^{j-1} [1 - \delta(\mu_k^* - \hat{\mu})] + \theta \prod_{k=i}^{N} [1 - \delta(\mu_k^* - \hat{\mu})] \right\} + \frac{\gamma}{(\mu_{i-1}^* - \lambda_{i-1})^2} = 0,$$

and combine the last two equations, we get

$$\begin{aligned} &- \delta \left(p_i - \frac{\gamma \mu_i^*}{(\mu_i^* - \lambda_i)^2} \right) - \delta \sum_{j=i+1}^{N} \left(p_j - \frac{\gamma \mu_j^*}{(\mu_j^* - \lambda_j)} \right) [-\delta(\mu_i^* - \hat{\mu})] \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k^* - \hat{\mu})] \\ &+ \delta^2 \theta (\mu_i^* - \hat{\mu}) \prod_{k=i+1}^{N} [1 - \delta(\mu_i^* - \hat{\mu})] + \frac{\gamma}{(\mu_{i-1}^* - \lambda_{i-1})^2} - \frac{\gamma}{(\mu_i^* - \lambda_i)^2} = 0. \end{aligned} \tag{35}$$

Let us now derive the optimal price for period i. We have already calculated the derivative of the demand potential with respect to price. The optimal price p_i will satisfy the first order condition $\frac{\partial \mathcal{R}}{\partial p_i} = 0$ that is

$$\Lambda_i - 2\alpha p_i^* + \frac{\alpha\gamma\mu_i}{(\mu_i - \lambda_i)^2} + \sum_{j=i+1}^{N} \left(p_j^* - \frac{\gamma\mu_j}{(\mu_j - \lambda_j)^2} \right) \frac{\partial \Lambda_j}{\partial p_i} + \theta \frac{\partial \Lambda_{N+1}}{\partial p_i} = 0$$

and using the derivative of the demand potential this gives

$$\Lambda_i - 2\alpha p_i^* + \frac{\alpha\gamma\mu_i}{(\mu_i - \lambda_i)^2} + \alpha\delta(\mu_i - \hat{\mu}) \left\{ \begin{array}{l} \sum_{j=i+1}^{N} \left[p_j^* - \frac{\gamma\mu_j}{(\mu_j - \lambda_j)^2} \right] \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k - \hat{\mu})] \\ + \theta \prod_{k=i+1}^{N} [1 - \delta(\mu_k - \hat{\mu})] \end{array} \right\} = 0. \quad (36)$$

Furthermore, (35) can be written as

$$p_i - \frac{\gamma\mu_i^*}{(\mu_i^* - \lambda_i)} - \frac{\gamma}{\delta(\mu_{i-1}^* - \lambda_{i-1})^2} + \frac{\gamma}{\delta(\mu_i^* - \lambda_i)^2}$$
$$+ \delta(\hat{\mu} - \mu_i^*) \left\{ \sum_{j=i+1}^{N} \left[p_j - \frac{\gamma\mu_j^*}{(\mu_j^* - \lambda_j)^2} \right] \prod_{k=i+1}^{j-1} [1 - \delta(\mu_k^* - \hat{\mu})] + \theta \prod_{k=i+1}^{N} [1 - \delta(\mu_k^* - \hat{\mu})] \right\} = 0.$$

We use the above in (36),

$$\Lambda_i - 2\alpha p_i^* + \frac{\alpha\gamma\mu_i^*}{(\mu_i^* - \lambda_i^*)^2} + \alpha p_i^* - \frac{\alpha\gamma\mu_i^*}{(\mu_i^* - \lambda_i^*)^2} - \frac{\alpha\gamma}{\delta(\mu_{i-1}^* - \lambda_{i-1}^*)^2} + \frac{\alpha\gamma}{\delta(\mu_i^* - \lambda_i^*)^2} = 0$$

and so

$$0 < \lambda_i^* = \frac{\alpha\gamma}{\delta} \left\{ \frac{1}{(\mu_{i-1}^* - \lambda_{i-1}^*)^2} - \frac{1}{(\mu_i^* - \lambda_i^*)^2} \right\}. \quad (37)$$

The last equation implies that $\mu_i^* - \lambda_i^* > \mu_{i-1}^* - \lambda_{i-1}^*$, or that the service speed increases faster than the demand rate.

(ii) We will now show that μ_i^*, λ_i^* and p_i^* are increasing over time if $\mu_i^* < \hat{\mu}$. We start by showing that μ_i^* is increasing. Suppose that $\mu_{N-1}^* < \hat{\mu}$ and hence, $\Lambda_N > \Lambda_{N-1}$. In period N, the first order condition with respect to μ_N, given Λ_N is

$$\begin{aligned} \mathcal{R}_N^{\prime\mu}(\Lambda_N) := \frac{\partial \mathcal{R}_N}{\partial \mu_N} &= \frac{\gamma\lambda_N}{(\mu_N - \lambda_N)^2} - \theta\delta\lambda_N = 0 \\ &= \lambda_N \left[\frac{\gamma}{(\mu_N - \lambda_N)^2} - \theta\delta \right] = 0 \end{aligned} \quad (38)$$

and $\mu_N^*(\Lambda_N)$ is the solution to (38) given starting state Λ_N. From single-period analysis, $\mu_N^*(\Lambda_N) > \mu_N^*(\Lambda_{N-1})$.

In period $(N-1)$, the first order condition with respect to μ_{N-1}, given starting state Λ_{N-1}, is

$$\begin{aligned}
\mathcal{R}'^{\mu}_{N-1}(\Lambda_{N-1}) &= \frac{\gamma\lambda_{N-1}}{(\mu_{N-1}-\lambda_{N-1})^2} + \left(p_N - \frac{\gamma\mu_N}{(\mu_N-\lambda_N)^2}\right)\frac{\partial\Lambda_N}{\partial\mu_{N-1}} + \theta\frac{\partial\Lambda_{N+1}}{\partial\mu_{N-1}} = 0 \\
&= \frac{\gamma\lambda_{N-1}}{(\mu_{N-1}-\lambda_{N-1})^2} - \delta\lambda_{N-1}\left(p_N - \frac{\gamma\mu_N}{(\mu_N-\lambda_N)^2}\right) - \theta\delta\lambda_{N-1}(1-\delta\mu_N) = 0 \\
&= \lambda_{N-1}\left[\frac{\gamma}{(\mu_{N-1}-\lambda_{N-1})^2} - \delta\left(p_N - \frac{\gamma\mu_N}{(\mu_N-\lambda_N)^2}\right) - \theta\delta(1-\delta\mu_N)\right] = 0 \quad (39)
\end{aligned}$$

where

$$\mathcal{R}'^{\mu}_{N-1}(\Lambda_{N-1}) := \frac{\partial\mathcal{R}_{N-1}}{\partial\mu_{N-1}}$$

and $\mu^*_{N-1}(\Lambda_{N-1})$ is the solution to (39) given starting state Λ_{N-1}.

Now since $\mu^*_N - \lambda^*_N \geq \mu^*_{N-1} - \lambda^*_{N-1}$,

$$\frac{\gamma}{(\mu^*_{N-1}-\lambda^*_{N-1})^2} \geq \frac{\gamma}{(\mu^*_N-\lambda^*_N)^2}.$$

Therefore, from (38) and (39),

$$\delta\left(p^*_N - \frac{\gamma\mu^*_N}{(\mu^*_N-\lambda^*_N)^2}\right) + \theta\delta(1-\delta\mu^*_N) > \theta\delta. \quad (40)$$

Let $\tilde{\mu}_{N-1}(\Lambda_{N-1})$ be the solution to the following N-th period first order condition with starting state Λ_{N-1},

$$\mathcal{R}'^{\mu}_{N}(\Lambda_{N-1}) := \frac{\gamma\lambda_{N-1}}{(\mu_{N-1}-\lambda_{N-1})^2} - \theta\delta\lambda_{N-1} = 0.$$

We know that $\tilde{\mu}_{N-1}(\Lambda_{N-1}) = \mu^*_N(\Lambda_{N-1}) < \mu^*_N(\Lambda_N)$. From (39) and (40), given starting state Λ_{N-1},

$$\mathcal{R}'^{\mu}_{N-1}(\Lambda_{N-1}) = \frac{\partial\mathcal{R}_{N-1}}{\partial\mu_{N-1}} < 0 \text{ at } \mu_{N-1} = \tilde{\mu}_{N-1}(\Lambda_{N-1}).$$

Hence, $\tilde{\mu}_{N-1} > \mu^*_{N-1}$ since $\mathcal{R}_i$ is concave in μ_i and so $\mu^*_N > \mu^*_{N-1}$. Using the same reasoning inductively, one can show that $\mu^*_{i+1} > \mu^*_i$ if $\mu^*_i < \hat{\mu}$. We note that since the analysis above only involves the scenario $\mu^*_i < \hat{\mu}$ and $\hat{\mu} < \frac{2}{\delta}$, the condition $\mu^*_i < \frac{2}{\delta}$ is satisfied.

Next, we show that p^*_i is increasing over time. In period N, the first order condition with respect to p_N, given Λ_N, is,

$$\mathcal{R}'^{p}_{N}(\Lambda_N) := \frac{\partial\mathcal{R}_N}{\partial p_N} = (\Lambda_N - 2\alpha p_N) + \frac{\gamma\alpha\mu_N}{(\mu_N-\lambda_N)^2} + \delta\alpha\theta(\mu_N - \hat{\mu}) = 0. \quad (41)$$

Let $p^*_N(\Lambda_N)$ be the solution to (41) given state Λ_N. From single period analysis, we know that $p^*_N(\Lambda_N) \geq p^*_N(\Lambda_{N-1})$ since $\Lambda_N \geq \Lambda_{N-1}$ because $\mu^*_N \leq \hat{\mu}$.

In period $N-1$, the first order condition with respect to p_{N-1}, given state Λ_{N-1} is

$$\mathcal{R}'^{p}_{N-1}(\Lambda_{N-1}) := \frac{\partial\mathcal{R}_{N-1}}{\partial p_{N-1}} = (\Lambda_{N-1} - 2\alpha p_{N-1}) + \frac{\gamma\alpha\mu_{N-1}}{(\mu_{N-1}-\lambda_{N-1})^2} + \delta\alpha(\mu_{N-1}-\hat{\mu})\frac{\partial\mathcal{R}_N}{\partial\Lambda_N} = 0. \quad (42)$$

But we have that

$$\frac{\partial \mathcal{R}_N}{\partial \Lambda_N} = \frac{\partial \pi_N}{\partial \Lambda_N} + \frac{\partial \mathcal{R}_{N+1}}{\partial \Lambda_N} = \frac{\partial \imath_N}{\partial \Lambda_N} + \frac{\partial \mathcal{R}_{N+1}}{\partial \Lambda_{N+1}}\frac{\partial \Lambda_{N+1}}{\partial \Lambda_N} = \frac{\partial \pi_N}{\partial \Lambda_N} + \theta\left(1 - \delta(\mu_N - \hat{\mu})\right)$$

where π_N is the profit in period N, i.e. $\pi_N = (\Lambda_N - 2\alpha p_N) + \frac{\gamma\alpha\mu_N}{(\mu_N - \lambda_N)^2}$.

Since $\mu_N \leq \hat{\mu}$ and $\frac{\partial \pi_N}{\partial \Lambda_N} > 0$, we can infer that

$$\frac{\partial \mathcal{R}_N}{\partial \Lambda_N} > \theta. \tag{43}$$

Let p^*_{N-1} be the solution to (42) given state Λ_{N-1}. Now let $\tilde{p}_{N-1}(\Lambda_{N-1})$ be the solution to the following N-th period first order condition (41) given state Λ_{N-1},

$$\mathcal{R}'^{p}_{N}(\Lambda_{N-1}) = (\Lambda_N - 2\alpha p_N) + \frac{\gamma\alpha\mu_N}{(\mu_N - \lambda_N)^2} + \delta\alpha\theta(\mu_N - \hat{\mu}) = 0.$$

We know that $\tilde{p}_{N-1}(\Lambda_{N-1}) = p^*_N(\Lambda_{N-1}) < p^*_N(\Lambda_N)$. From (42) and (43), given starting state Λ_{N-1},

$$\mathcal{R}'^{p}_{N-1}(\Lambda_{N-1}) = \frac{\partial \mathcal{R}_{N-1}}{\partial p_{N-1}} < 0 \text{ at } p_{N-1} = \tilde{p}_{N-1}(\Lambda_{N-1})$$

because from (43), $\theta < \frac{\partial \mathcal{R}_N}{\partial \Lambda_N}$ and $\mu_N - \hat{\mu} < 0$.

Since $\mathcal{R}_i$ is concave in p_i,

$$\tilde{p}_{N-1}(\Lambda_{N-1}) > p^*_{N-1} \text{ and } p^*_N > p^*_{N-1}.$$

Using the same reasoning inductively, one can show that $p^*_{i+1} > p^*_i$, if $\mu^*_i < \hat{\mu}$. The approach for showing that $\lambda^*_{i+1} > \lambda^*_i$, if $\mu^*_i < \hat{\mu}$ is similar to the proof above for price. □

Proof of Theorem 6

(i) (a) The demand potential can also be written as

$$\Lambda_i = \Lambda_{i-1} - \delta(\mu - \hat{\mu})\lambda_{i-1} = \Lambda_1 - \delta(\mu - \hat{\mu})\sum_{j=1}^{i-1}\lambda_j$$

for $2 \leq i \leq N+1$, and its derivative with respect to λ_j will be

$$\frac{\partial \Lambda_i}{\partial \lambda_j} = \begin{cases} -\delta(\mu - \hat{\mu}) & \text{for } j < i \\ 0 & \text{otherwise} \end{cases}.$$

The expected profit is defined as

$$\mathcal{R}(\mu, \lambda_i) = \sum_{i=1}^{N}\left[\frac{\Lambda_i - \lambda_i}{\alpha}\lambda_i - \beta\frac{\lambda_i}{\alpha(\mu - \lambda_i)}\right] + \theta\Lambda_{N+1}.$$

The optimal demand rate λ_i will satisfy $\frac{\partial \mathcal{R}}{\partial \lambda_i} = 0$, or equivalently,

$$\frac{\Lambda_i - 2\lambda_i}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda_i)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}\sum_{j=i+1}^{N}\lambda_j = 0. \tag{44}$$

We first consider the case where $\mu > \hat{\mu}$. A similar argument holds for $\mu < \hat{\mu}$. (44) for i and $(i-1)$ is,

$$\begin{aligned} \frac{\Lambda_i - 2\lambda_i}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda_i)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}\sum_{j=i+1}^{N}\lambda_j &= 0 \\ \frac{\Lambda_{i-1} - 2\lambda_{i-1}}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda_{i-1})^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})\lambda_i}{\alpha} - \frac{\delta(\mu-\hat{\mu})}{\alpha}\sum_{j=i}^{N}\lambda_j &= 0 \end{aligned}$$

and combining both, we have

$$\frac{\Lambda_{i-1} - \delta\lambda_{i-1}(\mu-\hat{\mu})}{\alpha} - \frac{2\lambda_i}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda_i)^2} = \frac{\Lambda_{i-1} - 2\lambda_{i-1}}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda_{i-1})^2} - \frac{\delta(\mu-\hat{\mu})\lambda_i}{\alpha}.$$

By rearranging the terms, we end up with

$$\delta\lambda_{i-1}(\mu-\hat{\mu}) + 2\lambda_i + \frac{\beta\mu}{(\mu-\lambda_i)^2} = \delta(\mu-\hat{\mu})\lambda_i + 2\lambda_{i-1} + \frac{\beta\mu}{(\mu-\lambda_{i-1})^2}.$$

In the above equation, both the right and the left hand side are positive with the same form. Therefore, there exists an optimal solution wherein $\lambda_i = \lambda_{i-1}$. The same is true for every i. Therefore, $\lambda_N = \lambda_{N-1} = \ldots = \lambda_{i+1} = \lambda$ and the result follows.

(b) Writing (44) for i and using part (a), we have that

$$\frac{\Lambda_i - 2\lambda^*}{\alpha} - \frac{\beta\mu}{\alpha(\mu-\lambda^*)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}(N-i)\lambda^* = 0.$$

Now we can use the fact that $\lambda^* = \Lambda_i - \alpha p_i - \beta\frac{1}{\mu-\lambda^*}$ and so

$$p_i - \frac{\lambda^*}{\alpha} - \frac{\beta\lambda}{\alpha(\mu-\lambda^*)^2} - \delta\theta(\mu-\hat{\mu}) - \frac{\delta(\mu-\hat{\mu})}{\alpha}(N-i)\lambda^* = 0.$$

Therefore, the optimal pricing policy will be

$$p_i^* = \frac{\beta\lambda^*}{\alpha(\mu^*-\lambda^*)^2} + \delta\theta(\mu^*-\hat{\mu}) + \frac{\lambda^*}{\alpha} + \frac{(N-i)\lambda^*\delta(\mu^*-\hat{\mu})}{\alpha}.$$

When speed is constant and equal to $\hat{\mu}$, the optimal price reduces to:

$$p^* = \frac{\beta\hat{\mu}}{\alpha(\hat{\mu}-\lambda^*)^2} + \frac{\lambda^*}{\alpha}.$$

As we proved in part (a), demand rate remains constant and thus λ^* will satisfy

$$\Lambda_1 - 2\lambda^* - \frac{\beta\mu^*}{(\mu^* - \lambda^*)^2} - \alpha\delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\delta(\mu^* - \hat{\mu}) = 0. \tag{45}$$

The optimal speed of the system will satisfy the first order condition, $\frac{\partial \mathcal{R}(\mu,\lambda_i)}{\partial \mu} = 0$, that is

$$\sum_{i=1}^{N} \left\{ \frac{\lambda}{\alpha}\frac{\partial \Lambda_i}{\partial \mu} + \frac{\beta\lambda}{\alpha(\mu-\lambda)^2} \right\} + \theta\frac{\partial \Lambda_{N+1}}{\partial \mu} = 0$$

given that $\lambda_i = \lambda$. Furthermore, for $i > 0$, $\frac{\partial \Lambda_i}{\partial \mu} = -(i-1)\delta\lambda$ and hence,

$$-\frac{\delta(N-1)\lambda}{2\alpha} + \frac{\beta}{\alpha(\mu-\lambda)^2} - \theta\delta = 0. \tag{46}$$

Combining (45) and (46) suggests that λ^* and μ^* should be a solution of

$$\mu^* = \frac{\Lambda_1 - 2\lambda^* + \hat{\mu}\delta\left(\alpha\theta + (N-1)\lambda^*\right)}{\delta\left(2\alpha\theta + \frac{3}{2}(N-1)\lambda^*\right)}$$

and

$$-\frac{\delta\lambda^*(N-1)}{2} + \frac{\beta}{(\mu^* - \lambda^*)^2} - \alpha\theta\delta = 0.$$

Suppose that $\mu^* = \hat{\mu}$, then (45) becomes

$$\Lambda_1 - 2\lambda^* - \frac{\beta\mu}{(\mu^* - \lambda^*)^2} = 0.$$

Let $\hat{\hat{\mu}}$ be defined as the value of $\hat{\mu}$ that satisfies the above and so, if $\hat{\hat{\lambda}}$ is the corresponding demand rate when service rate is $\hat{\hat{\mu}}$, then $\hat{\hat{\lambda}}$ and $\hat{\hat{\mu}}$ will also solve (46) for $\mu^* = \hat{\mu} = \hat{\hat{\mu}}$ and $\lambda^* = \hat{\hat{\lambda}}$,

$$-\frac{\delta(N-1)\hat{\hat{\lambda}}}{2\alpha} + \frac{\beta}{\alpha(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \theta\delta = 0.$$

Therefore, after some algebra, $\hat{\hat{\lambda}}$ and $\hat{\hat{\mu}}$ will satisfy,

$$\Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\beta\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0 \quad \text{and} \quad \hat{\hat{\mu}} = \frac{2(\Lambda_1 - 2\hat{\hat{\lambda}})}{\delta\left(\hat{\hat{\lambda}}(N-1) + 2\alpha\theta\right)}.$$

Now let us discuss the case when $\hat{\mu} > \hat{\hat{\mu}}$ and observe the following equations that come from (45),

$$\frac{\Lambda_1 - 2\hat{\hat{\lambda}}}{\alpha} - \frac{\beta\hat{\hat{\mu}}}{\alpha(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0 \tag{47}$$

$$\frac{\Lambda_1 - 2\lambda^*}{\alpha} - \frac{\beta\mu^*}{\alpha(\mu^* - \lambda^*)^2} - \delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\frac{\delta(\mu^* - \hat{\mu})}{\alpha} = 0 \tag{48}$$

where the second one is the optimality condition when $\hat{\mu} > \hat{\hat{\mu}}$. Now suppose that $\hat{\mu} > \hat{\hat{\mu}}$ but $\mu^* < \hat{\mu}$ is not true in the optimality equation, i.e. (48) is not satisfied when $\mu^* < \hat{\mu}$. Consider the following solution to the scenario $\hat{\mu} > \hat{\hat{\mu}}$. Let $\mu^* > \hat{\hat{\mu}}$ be the optimal service speed and let λ^* be such that

$$\frac{\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = \frac{\mu^*}{(\mu^* - \lambda^*)^2},$$

$$\text{or equivalently,} \qquad \lambda^* = \hat{\hat{\lambda}}\sqrt{\frac{\mu^*}{\hat{\hat{\mu}}}} + \mu^* - \sqrt{\mu^*\hat{\hat{\mu}}}.$$

Since $\mu^* > \hat{\hat{\mu}}$, the above implies that $\lambda^* > \hat{\hat{\lambda}}$. Subtracting (47) from (48) gives

$$\frac{2\hat{\hat{\lambda}}}{\alpha} - \frac{2\lambda^*}{\alpha} + \frac{\beta\hat{\hat{\mu}}}{\alpha(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \frac{\beta\mu^*}{\alpha(\mu^* - \lambda^*)^2} - \delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\frac{\delta(\mu^* - \hat{\mu})}{\alpha} = 0. \tag{49}$$

But,

$$\frac{2\hat{\hat{\lambda}}}{\alpha} - \frac{2\lambda^*}{\alpha} < 0, \text{ because } \lambda^* > \hat{\hat{\lambda}}$$

$$\frac{\beta\hat{\hat{\mu}}}{\alpha(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} - \frac{\beta\mu^*}{\alpha(\mu^* - \lambda^*)^2} = 0, \text{ due to the definition of } \lambda^*.$$

(49) can be satisfied only if $\mu^* \leq \hat{\mu}$ and so we have a contradiction. A similar argument holds for the opposite case and the result follows.

Now we show that $\mu^* > \hat{\hat{\mu}}$ when $\mu^* < \hat{\mu}$. From (48), we know that μ^* is such that,

$$\Lambda_1 - 2\lambda^* - \frac{\beta\mu^*}{(\mu^* - \lambda^*)^2} - \alpha\delta\theta(\mu^* - \hat{\mu}) - (N-1)\lambda^*\delta(\mu^* - \hat{\mu}) = 0$$

which implies

$$\Lambda_1 - 2\lambda^* - \frac{\beta\mu^*}{(\mu^* - \lambda^*)^2} = \alpha\delta\theta(\mu^* - \hat{\mu}) + (N-1)\lambda^*\delta(\mu^* - \hat{\mu}) < 0.$$

But $\hat{\hat{\mu}}$ satisfies

$$\Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\beta\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = 0.$$

Combining the last two relationships, we have

$$\Lambda_1 - 2\lambda^* - \frac{\beta\mu^*}{(\mu^* - \lambda^*)^2} < 0 = \Lambda_1 - 2\hat{\hat{\lambda}} - \frac{\beta\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2}$$

which is equivalent to

$$2\lambda^* + \frac{\beta\mu^*}{(\mu^* - \lambda^*)^2} > 2\hat{\hat{\lambda}} + \frac{\beta\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2}$$

and implies that $\mu^* > \hat{\hat{\mu}}$ taking into consideration that

$$\frac{\hat{\hat{\mu}}}{(\hat{\hat{\mu}} - \hat{\hat{\lambda}})^2} = \frac{\mu^*}{(\mu^* - \lambda^*)^2},$$

and also, if $\hat{\hat{\lambda}} > (<)\lambda^*$, then $\hat{\hat{\mu}} > (<)\mu^*$, as shown earlier.

We can use a similar argument to show that if $\mu^* > \hat{\mu}$, then $\hat{\mu} < \mu^* < \hat{\hat{\mu}}$. The result on prices increasing if $\hat{\hat{\mu}} < \mu^* < \hat{\mu}$ follows (and similarly for prices decreasing) because λ is constant and when $\mu^* < \hat{\mu}$, Λ_i will increase and so p_i^* has to increase to keep the demand rate constant.

(ii) (a)

$$\begin{aligned} Max \quad &: \quad R_t(\Lambda_t) = p\lambda_t + R_{t+1}(\Lambda_{t+1}) \ \forall t = 1, \ldots, N \\ \text{where } \Lambda_{t+1} \quad &= \quad \Lambda_t - \delta\lambda_t(\mu_t - \hat{\mu}) \text{ and } R_{N+1} = \theta\Lambda_{N+1} \text{ where } \theta > 0. \end{aligned}$$

Let $\pi_t(\Lambda_t) = p\lambda_t,\ t = 1, \ldots, N$. $\pi_t(\Lambda_t)$ is effectively the single-period profit in period t.

We have,

$$\frac{\partial\mathcal{R}_t}{\partial\Lambda_t} = \frac{\partial\pi_t}{\partial\Lambda_t} + \frac{\partial\mathcal{R}_{t+1}}{\partial\Lambda_t} = \frac{\partial\pi_t}{\partial\Lambda_t} + \frac{\partial\mathcal{R}_{t+1}}{\partial\Lambda_{t+1}}\frac{\partial\Lambda_{t+1}}{\partial\Lambda_t}$$

and

$$\frac{\partial\Lambda_{t+1}}{\partial\Lambda_t} = 1 - \delta(\mu_t - \hat{\mu})\frac{(\mu_t - \lambda_t)^2}{(\mu_t - \lambda_t)^2 + \beta}$$

due to the fact that $\lambda_t = \Lambda_t - \alpha p - \beta\frac{1}{\mu_t - \lambda_t}$ which implies that $\frac{\partial\lambda_t}{\partial\Lambda_t} = \frac{(\mu_t - \lambda_t)^2}{(\mu_t - \lambda_t)^2 + \beta}$. Then, $\frac{\partial\Lambda_{t+1}}{\partial\Lambda_t} > 1$ when $\mu_t < \hat{\mu}$. From the single-period analysis, we know that $\frac{\partial\pi_t}{\partial\Lambda_t} > 0$. Note that $\frac{\partial\mathcal{R}_{N+1}}{\partial\Lambda_{N+1}} = \theta > 0$ and it follows recursively that $\frac{\partial\mathcal{R}_t}{\partial\Lambda_t} > \frac{\partial\mathcal{R}_{t+1}}{\partial\Lambda_{t+1}}$.

We can derive the following

$$\begin{aligned}
\frac{\partial \lambda_t}{\partial \mu_t} &= \frac{\beta}{(\mu_t - \lambda_t)^2 + \beta} > 0 \\
\frac{\partial \Lambda_{t+1}}{\partial \mu_t} &= -\delta\lambda_t - \delta(\mu_t - \hat{\mu})\frac{\partial \lambda_t}{\partial \mu_t} = -\delta\lambda_t - \delta(\mu_t - \hat{\mu})\frac{\beta}{(\mu_t - \lambda_t)^2 + \beta}.
\end{aligned}$$

Also observe that $\mu_t = \frac{\beta}{\Lambda_t - \alpha p - \lambda_t} + \lambda_t$ and Λ_t increases when $\mu_t < \hat{\mu}$. Then, if λ_t decreases, μ_t and $\mu_t - \lambda_t$ has to decrease.

Now,

$$\frac{\partial \mathcal{R}_t}{\partial \mu_t} = p\frac{\partial \lambda_t}{\partial \mu_t} + \frac{\partial \mathcal{R}_{t+1}}{\partial \mu_t} = p\frac{\beta}{(\mu_t - \lambda_t)^2 + \beta} + \frac{\partial \Lambda_{t+1}}{\partial \mu_t}\frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}} = 0. \tag{50}$$

So,

$$p - \delta\left\{\lambda_t + \frac{\lambda_t}{\beta}(\mu_t - \lambda_t)^2 + (\mu_t - \hat{\mu})\right\}\frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}} = 0 \tag{51}$$

and since $\frac{\partial \mathcal{R}_t}{\partial \Lambda_t} > \frac{\partial \mathcal{R}_{t+1}}{\partial \Lambda_{t+1}}$, we need $\lambda_t + \frac{\lambda_t}{\beta}(\mu_t - \lambda_t)^2 + (\mu_t - \hat{\mu})$ to increase. If we assume that λ_t decreases, we are led to a contradiction and so λ_t has to increase and in turn μ_t will increase. Moreover, since $\frac{\partial \lambda_t}{\partial \mu_t} < 1$, $\mu_t - \lambda_t$ will increase as well.

(b) The optimal price should satisfy the first order condition $\frac{\partial \mathcal{R}}{\partial p} = 0$. The derivative of the demand potential with respect to price is

$$\frac{\partial \Lambda_{t+1}}{\partial p} = -\alpha\left\{\prod_{j=1}^{t}\left(1 + \frac{\delta}{\beta}\lambda_t(\mu_t - \lambda_t)^2\right) - 1\right\}$$

when $t \geq 1$. The first order condition will then be

$$\sum_{t=1}^{N}\lambda_t + \theta\frac{\partial \Lambda_{N+1}}{\partial p} = 0$$

that leads to

$$\sum_{t=1}^{N}\lambda_t - \alpha\theta\left\{\prod_{j=1}^{N}\left(1 + \frac{\delta}{\beta}\lambda_t(\mu_t - \lambda_t)^2\right) - 1\right\} = 0.$$

□