

Appendix. Threshold Routing to Trade-off Waiting and Call Resolution in Call Centers

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A. Further Details on the Simulation Study

We simulate the RPT and QIR controls using C++, and in accordance with the common random numbers variance reduction technique. For each control, we have 10 fixed random seeds, and for each seed, we simulate 200,000 units of time (hours). The first 40,000 time units are the warm-up periods and we calculate the time-average queue-length and callback rate over the remaining 160,000 time units. Then, we report the average over the 10 runs. With the average queue-length and callback rate, we can calculate the average waiting time and call resolution. To have the common random numbers, for each fixed random seed, we apriori generate a sequence of customer arrival times and a sequence of service times and indicator random variables on whether the call is resolved. Then, every time a call is routed to an agent, we use that agent's sequence of service times and indicators to determine how long that caller spends with that agent and whether the call is resolved. (Note that due to the callbacks, we cannot generate one sequence of customer arrival and service times, as would be possible for an inverted-V model without callbacks.)

The parameters for our simulation study are chosen in accordance with the empiric data in Mehrotra et al. (2012), which is summarized in Table 3. Note that our simulation parameters are consistent in the sense that within any one customer type, the agent service speeds and resolution probabilities used in our simulations are approximately within the range of service speeds and resolution probabilities shown in Table 3 below.

Table EC-1 Summary of Tables 1 and 2 in Appendix B in Mehrotra et al. (2012)

customer type	1	2	3	4
min. service speed	2.76	2.65	3.14	7.03
max. service speed	13.82	12.80	14.78	27.60
min. resolution prob.	0.62	0.24	0.50	0.73
max. resolution prob.	1.00	0.96	0.92	0.97

Finally, we provide additional simulation results to supplement the main body.

Simulations for a 2-pool System with Different System Loads:

We also perform a simulation study that investigates the impact of system load on performance. We keep using the system parameters in Figure 8(a) in the main body, except we change the system load. Figure A shows the results of the comparison of the RPT control and the QIR control for 3 different system loads: 0.85, 0.90, and 0.95. We observe that when the system load is low ($\rho = 0.85$), the waiting times are smaller. On the other hand, when the system load is high ($\rho = 0.95$), the change in call resolution is smaller because the system is crowded, so there is not as much opportunity to choose between idle agents across pools when routing.

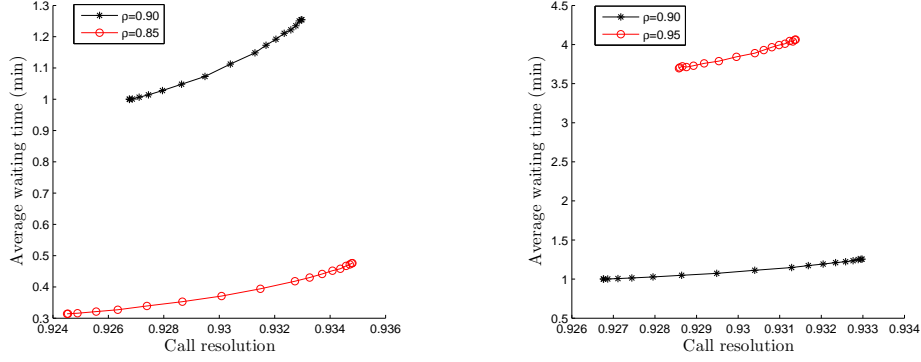
Simulation of RPT and QIR with Gamma Service Times:

To complement the simulation study we perform that assumes lognormal distributed service times (Figure 8 in the main body), we also perform a simulation study that assumes Gamma distributed service times. We simulate two systems with different coefficients of variation, but keep the mean service rates fixed at 3 for pool 1 and 6 for pool 2. In Figure EC-2(a) we let the service times of pool 1 agents follow $Gamma(\frac{1}{4}, \frac{4}{3})$, where $\frac{1}{4}$ is the shape parameter and $\frac{4}{3}$ is the scale parameter. The service times of pool 2 agents follow $Gamma(\frac{1}{4}, \frac{2}{3})$. In Figure EC-2(b) we change the service time distributions of pool 1 and 2 to $Gamma(4, \frac{1}{12})$ and $Gamma(4, \frac{1}{24})$ respectively. From the plots of the RPT and QIR controls, we again see that the RPT controls are on the efficient frontier, and that the variation in the average waiting time increases when the coefficient of variation increases.

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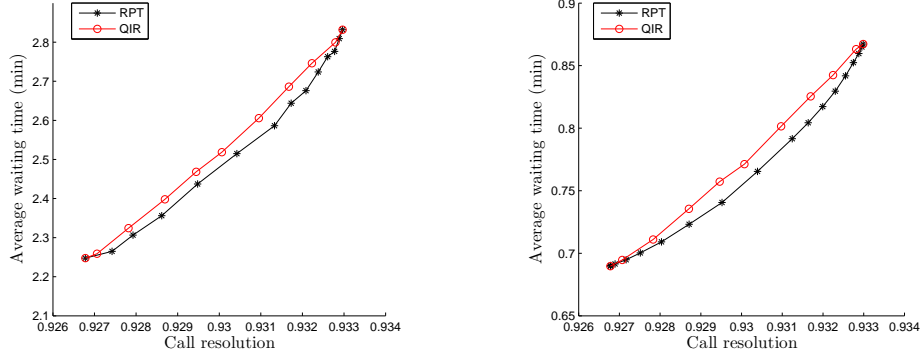
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Figure EC-1 The trade-off between waiting time and call resolution in a 2-pool system with different ρ .



$$\vec{\mu} = (3, 6), \vec{p} = (0.99, 0.90), \text{ and } \vec{N} = (25, 25)$$

Figure EC-2 Simulated comparison between RPT and QIR in a 2-pool system with Gamma service times



(a) Gamma service times with CV=2

(b) Gamma service times with CV=0.5

$$\vec{p} = (0.99, 0.90), \vec{N} = (25, 25), \rho = 0.9.$$

B. Proofs

Proof of Proposition 1:

Under any non-idling control, the state-space for the continuous time Markov chain (CTMC) $\mathcal{M} := \{Q(t), I_1(t), \dots, I_J(t); t \geq 0\}$ is: $\mathcal{S} = \mathbb{Z}_+ \times \{0, 1, \dots, N_1\} \times \dots \times \{0, 1, \dots, N_J\}$ where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The attainable states are:

$$\mathcal{S} = \begin{cases} (q, 0, \dots, 0) & \text{for } q \in \{1, 2, \dots\} \\ (0, i_1, \dots, i_J) & \text{for } (i_1, \dots, i_J) \in \{0, 1, \dots, N_1\} \times \dots \times \{0, 1, \dots, N_J\} \end{cases},$$

and for ease of notation we represent the states as q when $q > 0$ and (i_1, \dots, i_J) when $q = 0$. The transitions for the CTMC \mathcal{M} occur when there is: 1) an arrival; 2) a service completion, with no callback; 3) a service completion, followed by a callback. The control decisions are made when case 1 or 3 occurs, *and* there are idle agents in more than one pool. We begin by "splitting" the CTMC \mathcal{M} into two "sub" CTMC's:

1. $\mathcal{Q} := \{Q(t), t \geq 0\}$ having state space $\{1, 2, \dots\}$ and
2. $\mathcal{I} := \{I_1(t), \dots, I_J(t), t \geq 0\}$ having state space $\{0, 1, \dots, N_1\} \times \dots \times \{0, 1, \dots, N_J\}$.

We next observe that \mathcal{Q} evolves as a one-dimensional birth-and-death process with constant birth rate λ and constant death rate $\sum_{j \in \mathcal{J}} p_j \mu_j N_j$, that are not affected by the control. The state 1 is a reflecting boundary state. When (2) in the main body holds, \mathcal{Q} has a unique stationary distribution that we denote by $\nu^{\mathcal{Q}}$. Next, we assume that the (possibly state-dependent) transition rates of the sub-CTMC \mathcal{I} between any two states (i_1, \dots, i_J) when $\sum_{j \in \mathcal{J}} i_j > 0$ are exactly

the same as for the CTMC \mathcal{M} (and so, in contrast to \mathcal{Q} , these rates *are* affected by the control). The state $(0, \dots, 0)$ is a reflecting boundary state. Then, \mathcal{I} is a finite-state, irreducible, aperiodic, J -dimensional birth-and-death process under any non-idling stationary Markovian control. Hence there exists a unique stationary distribution for \mathcal{I} , that we denote by ν^I .

Define the candidate stationary distribution ν for \mathcal{M} by

$$\nu(s) := \begin{cases} \theta^Q \nu^Q(s) & \text{if } s = q > 0 \\ \theta^I \nu^I(s) & \text{if } s = (i_1, \dots, i_J) \text{ for } i_j \in \{0, 1, \dots, N_j\} \end{cases},$$

where $\theta^Q, \theta^I \in (0, 1)$ satisfy both the flow balance equation from \mathcal{M}

$$\theta^Q \nu^Q(1) \sum_{j \in \mathcal{J}} p_j \mu_j N_j = \theta^I \nu^I((0, \dots, 0)) \lambda \quad (1)$$

and the normalization of the probabilities

$$\sum_{s \in \mathcal{S}} \nu(s) = \theta^Q + \theta^I = 1. \quad (2)$$

Together, (1) and (2) uniquely define θ^Q and θ^I , which uniquely defines $\nu(s)$. Furthermore, ν is a probability distribution and satisfies all the balance equations for the CTMC \mathcal{M} (since ν^Q and ν^I satisfy the balance equations in their respective parts of the state space). We conclude that ν is the unique stationary distribution for \mathcal{M} .

In the case that (2) in the main body is not satisfied, \mathcal{Q} is not positive recurrent. Since the transition rates of \mathcal{Q} are the same regardless of the control, it follows that there does not exist a stationary distribution under any routing control.

Proof of Theorem 1:

It follows very similarly to the proof of Lemma 2 in Armony and Ward (2010) that $\mathcal{C}(v^*) = d$. It follows very similarly to the proof of Lemma 4 in Armony and Ward (2010) that $\mathcal{C}(v) \geq \mathcal{C}(v^*)$, for any $v \in \mathcal{V}$. Hence we omit the details.

Proof of Lemma 1:

It is straightforward to verify the first two inequalities using the explicit expression for C in (16) in the main body. We show the algebra for the last three inequalities. Recall that $\hat{h}(x) = \Phi(x)/\phi(x)$, and note that $\Phi'(x) = \phi(x)$, $\phi'(x) = -x\phi(x)$. It is helpful to first observe that for $x(p_2) = \beta/\sqrt{p_2\mu_2}$,

$$\frac{\partial \left([x(p_2)\hat{h}(x(p_2))]^{-1} \right)}{\partial p_2} = \frac{\partial \left([x\hat{h}(x)]^{-1} \right)}{\partial x} \frac{\partial x}{\partial p_2} = \frac{1}{2p_2} \left(\frac{\phi(x)^2}{\Phi(x)^2} + \frac{\phi(x)}{x\Phi(x)} + \frac{x\phi(x)}{\Phi(x)} \right).$$

Furthermore,

$$\frac{\phi(x)^2}{\Phi(x)^2} + \frac{x\phi(x)}{\Phi(x)} < 1,$$

because

- $\hat{h}(x)$ increasing in x implies $\frac{1}{\hat{h}(x)^2} = \frac{\phi(x)^2}{\Phi(x)^2}$ is decreasing in x , so that $\frac{\phi(x)^2}{\Phi(x)^2} < \frac{\phi(0)^2}{\Phi(0)^2} < 0.64$ for all $x > 0$;
 - the first order condition $\frac{\partial(x\phi(x)/\Phi(x))}{\partial x} = 0$ and use of Mathematica for numeric calculation show that the function $\frac{x\phi(x)}{\Phi(x)}$ has the unique maximum on $[0, \infty)$ located at $x = 0.8399$, so that $\frac{x\phi(x)}{\Phi(x)} \leq \frac{0.8399\phi(0.8399)}{\Phi(0.8399)} < 0.3$.
- Then,

$$\begin{aligned} \frac{\partial C}{\partial p_2} &= \beta^2 \mu_1 \left(\frac{p_1^2 \mu_1 + p_2^2 \mu_2 - 2p_1 p_2 \mu_2}{p_2^2 (p_2 \mu_2 - p_1 \mu_1)^2} \left(1 + \frac{\phi(x)}{x\Phi(x)} \right) + \frac{(p_1 - p_2)}{2p_2^2 (p_2 \mu_2 - p_1 \mu_1)} \left(\frac{\phi(x)^2}{\Phi(x)^2} + \frac{\phi(x)}{x\Phi(x)} + \frac{x\phi(x)}{\Phi(x)} \right) \right) \\ &< \beta^2 \mu_1 \left(\frac{p_1^2 \mu_1 + p_2^2 \mu_2 - 2p_1 p_2 \mu_2}{p_2^2 (p_2 \mu_2 - p_1 \mu_1)^2} \left(1 + \frac{\phi(x)}{x\Phi(x)} \right) + \frac{(p_1 - p_2)}{2p_2^2 (p_2 \mu_2 - p_1 \mu_1)} \left(1 + \frac{\phi(x)}{x\Phi(x)} \right) \right) \\ &= \beta^2 \mu_1 \left(\frac{-2p_2 \mu_2 (p_1 - p_2) - (p_1 + p_2)(p_2 \mu_2 - p_1 \mu_1)}{2p_2^2 (p_2 \mu_2 - p_1 \mu_1)^2} \left(1 + \frac{\phi(x)}{x\Phi(x)} \right) \right) < 0 \end{aligned}$$

Similarly,

$$\frac{\partial C}{\partial \mu_2} = \beta^2 \mu_1 \frac{p_1 - p_2}{p_2} \left(\frac{-p_2}{(p_2 \mu_2 - p_1 \mu_1)^2} \left(1 + \frac{\phi(x)}{x\Phi(x)} \right) + \frac{1}{3^2 \mu_2 (p_2 \mu_2 - p_1 \mu_1)} \left(\frac{\phi(x)^2}{\Phi(x)^2} + \frac{\phi(x)}{x\Phi(x)} + \frac{x\phi(x)}{\Phi(x)} \right) \right) < 0.$$

Finally, to see $\frac{\partial C}{\partial \beta} > 0$, first note that for $x(p_2) = \beta/\sqrt{p_2\mu_2}$

$$C = \frac{\mu_1(p_1 - p_2)}{p_2(p_2\mu_2 - p_1\mu_1)} p_2\mu_2 x \left(\frac{\phi(x)}{\Phi(x)} + x \right).$$

It is sufficient to show

$$\frac{\partial \left(\frac{\phi(x)}{\Phi(x)} + x \right)}{\partial x} = \frac{\Phi(x)^2 - x\phi(x)\Phi'(x) - \phi(x)^2}{\Phi(x)^2} > 0.$$

This follows because

1. $\frac{\partial(\Phi(x) - x\phi(x))}{\partial x} = x^2\phi(x) \geq 0$ so that $\Phi(x) - x\phi(x)$ is nondecreasing in x ;
2. $\Phi(x) - x\phi(x)$ is increasing in x and $\phi(x)$ is decreasing in x for $x > 0$, so that $\Phi(x)^2 - x\phi(x)\Phi'(x) - \phi(x)^2 > \Phi(0)^2 - \phi(0)^2 = \frac{1}{4} - \frac{1}{2\pi} > 0$.

Proof of Theorem 2:

For simplicity of notation, define $T(0, 1) := \infty$, $T(K, K + 1) := 0$, so that $T(0, 1) > T(1, 2) > T(2, 3) > \dots > T(K - 1, K) > T(K, K + 1)$. We begin under the assumption that c is arbitrarily large. Suppose we can find functions V_1', \dots, V_{K-1}' , a constant d , and threshold values $l_0 := 0 < l_1 < l_2 < \dots < l_{K-1} < l_K := \infty$, that solve

$$\begin{aligned} V_0''(x) - \beta V_0'(x) + cx &= d, & x \geq 0 \\ V_1''(x) - (\beta + p_1\mu_1 x)V_1'(x) + (1 - p_1)\mu_1 x &= d, & -l_1 \leq x < 0 \\ V_2''(x) - (\beta + p_2\mu_2 x)V_2'(x) + (1 - p_2)\mu_2 x &= d, & -l_2 \leq x < -l_1 \\ & \vdots & \vdots \\ V_K''(x) - (\beta + p_K\mu_K x)V_K'(x) + (1 - p_K)\mu_K x &= d, & x < -l_{K-1} \end{aligned} \quad (3)$$

having

$$V_0'(0) = V_1'(-l_1), \text{ and } V_j'(-l_j) = V_{j+1}'(-l_j) = T(j, j + 1), \quad j \in \{1, 2, \dots, K - 1\}, \quad (4)$$

and for which $V(x) = \begin{cases} V_0(x) & x \geq 0 \\ V_1(x) & -l_1 \leq x < 0 \\ \vdots & \vdots \\ V_K(x) & x < -l_{K-1} \end{cases}$ satisfies

$$|V(x)| \leq b_1 x^2 + b_2 \text{ for all } x \in \mathcal{R}, \text{ and some } b_1, b_2 \in \mathcal{R}. \quad (5)$$

We first argue that such a (V, d) satisfies the conditions of Theorem 1 and has an associated optimal control that is of threshold structure, i.e., $v^*(x) = v_{\vec{L}}(x)$ with $\vec{L} = (l_1, \dots, l_K)$. This requires the following claim, which we verify at the end of the proof.

Claim 1. If V satisfies (3) and (4), then V' is increasing.

Also, recall that j^* defined in (19) in the main body as $j^*(x) := \min_{j \in \mathcal{J}} \left\{ \arg \min \{V'(x)p_j\mu_j - (1 - p_j)\mu_j\} \right\}$ gives the argmin in (12) in Theorem 1 that defines the optimal control v^* . The conditions of Theorem 1 are satisfied because:

- For any $x > -L_1$, $V'(x) > T(1, 2) = \frac{(1-p_2)\mu_2 - (1-p_1)\mu_1}{p_2\mu_2 - p_1\mu_1}$ (because V' is increasing by Claim 1) implies $V'(x)p_2\mu_2 - (1 - p_2)\mu_2 > V'(x)p_1\mu_1 - (1 - p_1)\mu_1$. Since $T(1, 2) > T(i, i + 1)$ for $i \in \{2, 3, \dots, K - 1\}$, also $V'(x) > T(i, i + 1)$, so that $V'(x)p_{i+1}\mu_{i+1} - (1 - p_{i+1})\mu_{i+1} > V'(x)p_i\mu_i - (1 - p_i)\mu_i$. Hence $j^*(x) = 1$.
- For any $x \in (-l_{i-1}, -l_i)$, $i \in \{2, 3, \dots, K\}$, similar reasoning shows that $T(i - 1, i) > V'(x) > T(i, i + 1)$ implies $V'(x)p_j\mu_j - (1 - p_j)\mu_j > V'(x)p_i\mu_i - (1 - p_i)\mu_i$ for all $j \neq i$. Hence $j^*(x) = i$.
- The above two bullet points imply that (13) in Theorem 1 is equivalent to (3).
- The condition (4) implies $V_0''(0) = V_1''(-l_1)$ and $V_{j+1}''(-l_j) = V_j''(-l_j)$, for all $j \in \{1, 2, \dots, K - 1\}$, so that V is twice-continuously differentiable.

In summary, to conclude from Theorem 1 that $v_{\vec{L}}$ with $K - 1$ non-zero thresholds is optimal, it is sufficient to show that there exists $V_0', V_1', \dots, V_{K-1}', d$ and l_1, \dots, l_{K-1} that solve (3) and satisfy (4)-(5).

It is convenient to have the piecewise general solution to (3), which is straightforward to find because each ODE is linear. Define $H_j(x) := \frac{\beta \Phi(\sqrt{p_j \mu_j} x + \frac{\beta}{\sqrt{p_j \mu_j}})}{\sqrt{p_j \mu_j} \phi(\sqrt{p_j \mu_j} x + \frac{\beta}{\sqrt{p_j \mu_j}})}$, $j \in \mathcal{K}^*$. Then, the general solution to (3) is

$$\begin{aligned} V_0'(x) &= \frac{c}{\beta^2} + \frac{cx-d}{\beta}, & x \geq 0 \\ V_1'(x) &= \left(\frac{d}{\beta} + \frac{1-p_1}{p_1}\right)H_1(x) + \frac{1-p_1}{p_1} + \frac{\alpha_1}{\phi(\sqrt{p_1 \mu_1} x + \frac{\beta}{\sqrt{p_1 \mu_1}})}, & -l_1 \leq x < 0 \\ V_2'(x) &= \left(\frac{d}{\beta} + \frac{1-p_2}{p_2}\right)H_2(x) + \frac{1-p_2}{p_2} + \frac{\alpha_2}{\phi(\sqrt{p_2 \mu_2} x + \frac{\beta}{\sqrt{p_2 \mu_2}})}, & -l_2 \leq x < -l_1. \\ & \vdots & \vdots \\ V_K'(x) &= \left(\frac{d}{\beta} + \frac{1-p_K}{p_K}\right)H_K(x) + \frac{1-p_K}{p_K} + \frac{\alpha_K}{\phi(\sqrt{p_K \mu_K} x + \frac{\beta}{\sqrt{p_K \mu_K}})}, & x < -l_{K-1} \end{aligned}$$

In order that condition (5) holds, we must have $\alpha_K = 0$. Then, $\lim_{x \rightarrow -\infty} V_K'(x) = \frac{1-p_K}{p_K} < T(K-1, K)$, which is consistent with V_K' increasing up to some $-l_{K-1} < 0$.

The next step is to derive the $2K-1$ equations, that d, l_1, \dots, l_{K-1} , and $\alpha_1, \dots, \alpha_{K-1}$ must satisfy. In particular, it follows from the general solution to (3) and the condition (4) that we must solve the following equations:

$$\left(\frac{d}{\beta} + \frac{1-p_1}{p_1}\right)H_1(0) + \frac{1-p_1}{p_1} + \frac{\alpha_1}{\phi(\frac{\beta}{\sqrt{p_1 \mu_1}})} = \frac{c}{\beta^2} - \frac{d}{\beta} \quad (6)$$

$$\left(\frac{d}{\beta} + \frac{1-p_1}{p_1}\right)H_1(-l_1) + \frac{1-p_1}{p_1} + \frac{\alpha_1}{\phi(-\sqrt{p_1 \mu_1} l_1 + \frac{\beta}{\sqrt{p_1 \mu_1}})} = T(1, 2) \quad (7)$$

$$\left(\frac{d}{\beta} + \frac{1-p_2}{p_2}\right)H_2(-l_1) + \frac{1-p_2}{p_2} + \frac{\alpha_2}{\phi(-\sqrt{p_2 \mu_2} l_1 + \frac{\beta}{\sqrt{p_2 \mu_2}})} = T(1, 2) \quad (8)$$

\vdots

$$\left(\frac{d}{\beta} + \frac{1-p_{K-2}}{p_{K-2}}\right)H_{K-2}(-l_{K-3}) + \frac{1-p_{K-2}}{p_{K-2}} + \frac{\alpha_{K-2}}{\phi(-\sqrt{p_{K-2} \mu_{K-2}} l_{K-3} + \frac{\beta}{\sqrt{p_{K-2} \mu_{K-2}}})} = T(K-3, K-2) \quad (9)$$

$$\left(\frac{d}{\beta} + \frac{1-p_{K-2}}{p_{K-2}}\right)H_{K-2}(-l_{K-2}) + \frac{1-p_{K-2}}{p_{K-2}} + \frac{\alpha_{K-2}}{\phi(-\sqrt{p_{K-2} \mu_{K-2}} l_{K-2} + \frac{\beta}{\sqrt{p_{K-2} \mu_{K-2}}})} = T(K-2, K-1) \quad (10)$$

$$\left(\frac{d}{\beta} + \frac{1-p_{K-1}}{p_{K-1}}\right)H_{K-1}(-l_{K-2}) + \frac{1-p_{K-1}}{p_{K-1}} + \frac{\alpha_{K-1}}{\phi(-\sqrt{p_{K-1} \mu_{K-1}} l_{K-2} + \frac{\beta}{\sqrt{p_{K-1} \mu_{K-1}}})} = T(K-2, K-1) \quad (11)$$

$$\left(\frac{d}{\beta} + \frac{1-p_{K-1}}{p_{K-1}}\right)H_{K-1}(-l_{K-1}) + \frac{1-p_{K-1}}{p_{K-1}} + \frac{\alpha_{K-1}}{\phi(-\sqrt{p_{K-1} \mu_{K-1}} l_{K-1} + \frac{\beta}{\sqrt{p_{K-1} \mu_{K-1}}})} = T(K-1, K) \quad (12)$$

$$\left(\frac{d}{\beta} + \frac{1-p_K}{p_K}\right)H_K(-l_{K-1}) + \frac{1-p_K}{p_K} = T(K-1, K) \quad (13)$$

Claim 2. There exists C_{K-1} (that can be found by a one-dimensional search) such that for all $c > C_{K-1}$, there exists d, l_1, \dots, l_{K-1} , and $\alpha_1, \dots, \alpha_{K-1}$ that solve (6)-(13). The threshold values l_1, \dots, l_{K-1} can be found by a sequence of one-dimensional searches.

Once c falls below C_{K-1} , we set $l_1 = 0$. Then, to conclude from Theorem 1 that v_L with $K-2$ non-zero thresholds is optimal, it is sufficient to show that there exists $V_0', V_2', V_3', \dots, V_K', d$, and L_2, \dots, L_{K-1} , that solve (3) and satisfy (4) for $j \in \{2, \dots, K-1\}$ and also satisfy (5). Note that when $l_1 = 0$, V_1' does not appear in the equations. This follows by repeating the same argument. Continued repetition of the argument evidences $C_1 < C_2 < \dots < C_{K-1}$ that can be found by a sequence of one-dimensional searches, and that v_L with $K-1$ non-zero thresholds is optimal when $c \in (C_{K-1}, C_K]$.

Proof of Claim 2:

We first evaluate equations (11)-(13), which have the variables l_{K-2}, l_{K-1} , and d . We consider l_{K-2} as fixed, and solve for α_{K-1}, l_{K-1} , and d as functions of l_{K-2} . More precisely, we find d as a function of l_{K-1} , α_{K-1} as a function of d , and l_{K-1} as a function of l_{K-2} . From (13),

$$d(l_{K-1}) = \beta \left(\left(T(K-1, K) - \frac{1-p_K}{5p_K} \right) / H_K(-l_{K-1}) - \frac{1-p_K}{p_K} \right) \quad (14)$$

From (11),

$$\alpha_{K-1}(d(l_{K-1})) = \phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-2} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right) \times \left(T(K-2, K-1) - \frac{1-p_{K-1}}{p_{K-1}} - \left(\frac{d(l_{K-1})}{\beta} + \frac{1-p_{K-1}}{p_{K-1}} \right) H_{K-1}(-l_{K-2}) \right). \quad (15)$$

Next, we use (12) to show that there exists (finite) $l_{K-1} > l_{K-2}$ so that (12) is satisfied when d and α_{K-1} are determined by (14) and (15). For that, we view the left-hand side of (12) as a function of l_{K-1}

$$f(l_{K-1}) := \left(\frac{d(l_{K-1})}{\beta} + \frac{1-p_{K-1}}{p_{K-1}} \right) H_{K-1}(-l_{K-1}) + \frac{1-p_{K-1}}{p_{K-1}} + \frac{\alpha_{K-1}(d(l_{K-1}))}{\phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-1} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right)}$$

Substituting in for $\alpha_{K-1}(d(l_{K-1}))$ in the above shows

$$f(l_{K-1}) = \frac{1-p_{K-1}}{p_{K-1}} (1 + H_{K-1}(-l_{K-1})) + \frac{g(l_{K-1})}{\phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-1} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right)} \quad (16)$$

for

$$g(l_{K-1}) = \left(\frac{d(l_{K-1})}{\sqrt{p_{K-1}\mu_{K-1}}} \left(\Phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-1} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right) - \Phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-2} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right) \right) + \phi \left(-\sqrt{p_{K-1}\mu_{K-1}}l_{K-2} + \frac{\beta}{\sqrt{p_{K-1}\mu_{K-1}}} \right) \left(T(K-2, K-1) - \frac{1-p_{K-1}}{p_{K-1}} (1 + H_{K-1}(-l_{K-2})) \right) \right).$$

Note that $\lim_{l_{K-1} \rightarrow \infty} H_j(-l_{K-1}) = 0, j \in \mathcal{K}^*$ and, from (14), $d(l_{K-1}) \rightarrow \infty$ as $l_{K-1} \rightarrow \infty$. Then, recalling that l_{K-2} is fixed, it follows from (16) that

$$f(l_{K-1}) \rightarrow -\infty \text{ as } l_{K-1} \rightarrow \infty.$$

Since by construction

$$f(l_{K-1}) \rightarrow T(K-2, K-1) > T(K-1, K) \text{ as } l_{K-1} \downarrow l_{K-2},$$

it follows that there exists $l_{K-1} > l_{K-2}$ that satisfies (12).

Now we find l_{K-1} as a function of $l_{K-2} \geq 0$, denoted by $L_{K-1}(l_{K-2})$. The next step is to show $L_{K-1}(l_{K-2})$ is increasing in l_{K-2} . The argument is by contradiction. Consider $l_{K-2}^A < l_{K-2}^B$, and suppose we have used (11)-(13) to solve for the corresponding $l_{K-1}^A = L_{K-1}(l_{K-2}^A)$ and $l_{K-1}^B = L_{K-1}(l_{K-2}^B)$. Then, the functions $V_{K-1}^{A'}$ and $V_{K-1}^{B'}$ in the general solution to (3) are uniquely determined, and, by construction

$$V_{K-1}^{A'}(l_{K-1}^A) = V_{K-1}^{B'}(l_{K-1}^B) = T(K-1, K) \text{ and } V_{K-1}^{A'}(l_{K-2}^A) = V_{K-1}^{B'}(l_{K-2}^B) = T(K-2, K-1) > T(K-1, K).$$

Suppose $l_{K-1}^A \geq l_{K-1}^B$. Then, since $V_{K-1}^{A'}$ and $V_{K-1}^{B'}$ are increasing by Claim 1, it follows that there must be an intersection point. The ODE

$$V_{K-1}''(x) = (\beta + p_{K-1}\mu_{K-1}x)V_{K-1}'(x) - (1-p_{K-1})\mu_{K-1}x + d := g(x, V_{K-1}')$$

has g being Lipschitz continuous in V_{K-1}' and continuous in x , so that the Picard-Lindelof theorem guarantees a unique V_{K-1}' in an interval around the intersection point. This is a contradiction, therefore, $l_{K-1}^A < l_{K-1}^B$, and we conclude $L_{K-1}(l_{K-2})$ is increasing in l_{K-2} .

We next evaluate the equations (9) and (10), in addition to (11)-(13). We consider l_{K-3} as fixed, and solve for α_{K-2} and l_{K-2} that satisfy (9) and (10). Similar to our previous argument, we can conclude that there exists $l_{K-2} > l_{K-3}$ so that (10) is satisfied. Furthermore, similar argument also shows that $l_{K-2} = L_{K-2}(l_{K-3})$ is increasing in l_{K-3} .

Continued iteration of the above argument shows that given $l_1 \geq 0$, we can find $L_2(l_1), L_3(L_2(l_1)), \dots, L_{K-1}(L_{K-2}(\dots(L_2(l_1))\dots))$ and $d_1(l_1) := d(L_{K-1}(L_{K-2}(\dots(L_2(l_1))\dots)))$. We need to show that there exists l_1 and α_1 that solve (6) and (7). From (6), we can solve for α_1 as a function of $d(l_1)$. Next, we view the left-hand side of (7) as a function of l_1

$$f(l_1) := \left(\frac{d_1(l_1)}{\beta} + \frac{1-p_1}{p_1} \right) + \frac{1-p_1}{p_1} + \frac{\alpha_1(d_1(l_1))}{\phi \left(-\sqrt{p_1\mu_1}l_1 + \frac{\beta}{\sqrt{p_1\mu_1}} \right)}.$$

As $l_1 \rightarrow \infty, l_{K-1} \rightarrow \infty$, it follows from (14) that $d(l_1) \rightarrow \infty$. Then, it follows from (6) that $\alpha_1(d(l_1)) \rightarrow -\infty$. Similar argument to that in the second paragraph of the proof of this claim shows that $\lim_{l_1 \rightarrow \infty} \frac{d(l_1)}{\beta} H_1(-l_1)$ is finite.

Hence, $f(l_1) \rightarrow -\infty$ as $l_1 \rightarrow \infty$. Furthermore, by construction, $f(l_1) \rightarrow \frac{c}{\beta} - \frac{d_1(0)}{\beta}$ as $l_1 \downarrow 0$. Then, provided $c \geq C_1 := T(1, 2)\beta^2 + d_1(0)\beta$, it follows that $\frac{c}{\beta^2} - \frac{d_1(0)}{\beta} > T(1, 2) = V'_1(-l_1)$, so that there exists l_1 and $\alpha(l_1)$ satisfying (6) and (7).

Proof of Claim 1:

We know that $V'_K(x)$ is monotone in x on $(-\infty, -l_{K-1}]$ and $\lim_{x \rightarrow -\infty} V'_K(x) \rightarrow \frac{1-p_K}{p_K}$. If $l_{K-1} > 0$, then $V'_K(-l_{K-1}) = T(K-1, K) > \frac{1-p_K}{p_K}$, so $V'_K(x)$ is increasing in x ; if $L_{K-1} = 0$, then we need to connect $V'_0(0)$ and $V'_K(0)$. Then, the condition (6) with $\alpha_1 = 0$ implies

$$\frac{d}{\beta} + \frac{1-p_K}{p_K} = \frac{c}{\beta^2(1+H_K(0))} > 0.$$

Therefore, $V'_K(x)$ is increasing in x .

We next show that $V'_{K-1}(x)$ is increasing in x on $[-l_{K-1}, 0]$. To do this, we first show (step 1) that $V''_{K-1}(-l_{K-1}) > 0$. We then show (step 2) that if there exists a stationary point it must be a local maximum. We finally show (step 3) that having a local maximum leads to a contradiction.

Step 1: From the ODE's, at the point $-l_{K-1}$,

$$\begin{aligned} V''_{K-1}(-l_{K-1}) - (\beta - p_{K-1}\mu_{K-1}L_{K-1})V'_{K-1}(-l_{K-1}) - (1-p_{K-1})\mu_{K-1}l_{K-1} &= d \\ V''_K(-l_{K-1}) - (\beta - p_K\mu_K l_{K-1})V'_K(-l_{K-1}) - (1-p_K)\mu_K l_{K-1} &= d \end{aligned}$$

It follows from the above equations and the fact that $V'_{K-1}(-l_{K-1}) = V'_K(-l_{K-1}) = T(K-1, K)$ that

$$V''_{K-1}(-l_{K-1}) = V''_K(-l_{K-1})$$

Since $V'_K(x)$ is increasing in x , $V''_K(-l_{K-1}) > 0$.

Step 2: Suppose there exists a stationary point $x_0 > -l_{K-1}$ (We let x_0 to be the smallest stationary point if there is more than one), where $V''(x_0) = 0$. Then, taking the derivative of the ODE shows that

$$V'''_{K-1}(x) - (\beta + p_{K-1}\mu_{K-1}x)V''_{K-1}(x) - (1-p_{K-1})\mu_{K-1} = 0$$

At the point x_0 , we have

$$V'''_{K-1}(x_0) + p_{K-1}\mu_{K-1}V'(x_0) - (1-p_{K-1})\mu_{K-1} = 0$$

Since $V'_{K-1}(x_0) = V'_{K-1}(-l_{K-1}) = T(K-1, K)$, it follows that $V'''_{K-1}(x_0) = (1-p_{K-1})\mu_{K-1} - p_{K-1}\mu_{K-1}V'_{K-1}(x_0) < (1-p_{K-1})\mu_{K-1} - p_{K-1}\mu_{K-1}T(K-1, K) < 0$. Hence x_0 is a local maximum.

Step 3: If there exists a local maximum, then there exists a horizontal line at $H_0 \in (T(K-1, K)V'(x_0))$ and $-l_{K-1} < l_1^c < x_0 < l_2^c$ having $V'_{K-1}(l_1^c) = V'_{K-1}(l_2^c) = H_0$ and $V''_{K-1}(l_1^c) > 0$, $V''_{K-1}(l_2^c) < 0$. From the ODE,

$$V''_{K-1}(l_1^c) - V''_{K-1}(l_2^c) = (l_2^c - l_1^c)((1-p_{K-1})\mu_{K-1} - p_{K-1}\mu_{K-1}H_0)$$

This is a contradiction because $V''_{K-1}(l_1^c) - V''_{K-1}(l_2^c) > 0$ but $(l_2^c - l_1^c)((1-p_{K-1})\mu_{K-1} - p_{K-1}\mu_{K-1}H_0) < (l_2^c - l_1^c)((1-p_{K-1})\mu_{K-1} - p_{K-1}\mu_{K-1}T(K-1, K)) < 0$. We conclude that such a local maximum cannot exist and so $V'_{K-1}(x)$ is increasing in x on $[-l_{K-1}, 0]$. In the same way, we can show that $V'_j(x)$ is increasing in x on $[-l_j, 0]$, $j \in \mathcal{K}^* - \{K\}$. Note that $V'_0(x)$ is increasing in x on $[0, \infty)$. Therefore, $V'(x)$ is increasing on $(-\infty, \infty)$.

Proof of Lemma 2:

The proof requires use of the functions V'_1, \dots, V'_{K-1} , the constant d , and the threshold values L_1, L_2, \dots, L_{K-1} that were shown in the proof of Theorem 2 to solve (3) and satisfy the conditions of Theorem 1. Recall that the threshold values can all be expressed as functions of the smallest non-zero threshold value l_k . Without loss of generality, we assume $k = 1$. We have already shown in the proof of Theorem 2 that L_j is increasing in L_{j-1} for $j \in \{2, 3, \dots, K-1\}$. Therefore, it is sufficient to show that the threshold L_1 increases as c increases. For this, we regard $V'_1(0)$ that satisfies the left-hand side of (6) and connects the ODE solutions V'_0 and V'_1 in (3) in a twice-continuously differentiable manner as a function of l_1 .

Suppose we can show that $V'_1(0)$ is increasing in l_1 . From (6), we have that $c = d\beta + \beta^2 V'_1(0)$. Furthermore, from the last paragraph of the proof of Claim 2 in the proof of Theorem 2, $d = d(L_{K-1}(L_{K-2}(\dots(L_2(l_1))\dots)))$, and so is increasing in l_1 . Therefore, c is increasing in l_1 , which also implies l_1 is increasing in c .

The argument to show $V'_1(0)$ is increasing in l_1 is by contradiction. It is helpful to first note that since V'_1 satisfies (3)

$$V'_1(x) = V'_1(-l_1) + \int_{-l_1}^x d + (\beta + p_1\mu_1 y)V'_1(y) - (1-p_1)\mu_1 y dy \text{ for } x \in [-l_1, 0]. \quad (17)$$

Let $l_1^A < l_1^B$, $V'_{1,A}(0) := V'_1(0; l_1^A)$, $V'_{1,B}(0) := V'_1(0; l_1^B)$, and $d_1^A := d(L_{K-1}(L_{K-2}(\dots(L_2(l_1^A))\dots)))$, $d_1^B := d(L_{K-1}(L_{K-2}(\dots(L_2(l_1^B))\dots)))$. Suppose $V'_{1,A}(0) \geq V'_{1,B}(0)$. Then, since $V'_{1,A}(-l_1) = V'_{1,B}(-l_1) = T(1, 2)$ by construction, there must exist at least one intersection point z on $(-l_1, 0]$ where $V'_{1,A}(z) = V'_{1,B}(z)$. We let z be the smallest intersection point if there is more than one. Since from (17),

$$\begin{aligned} V'_{1,A}(z) &= V'_{1,A}(-l_1) + \int_{-l_1}^z d + (\beta + p_1\mu_1y)V'_{1,A}(y) - (1-p_1)\mu_1ydy \\ V'_{1,B}(z) &= V'_{1,B}(-l_1) + \int_{-l_1}^z d + (\beta + p_1\mu_1y)V'_{1,B}(y) - (1-p_1)\mu_1ydy, \end{aligned}$$

and $V'_{1,A}(y) < V'_{1,B}(y)$ on $[-l_1, z)$, it follows that $V'_{1,A}(z) < V'_{1,B}(z)$. This is a contradiction.

Next, we show that when $c \rightarrow \infty$, then $l_j \rightarrow \infty$, $j \in \{1, 2, \dots, K-1\}$. For this, we extend $V'_K(x)$ to $(-\infty, 0)$ so that

$$V'_K(x) = \left(\frac{d}{\beta} + \frac{1-p_K}{p_K}\right)H_K(x) + \frac{1-p_K}{p_K}, \text{ for } x \leq 0. \quad (18)$$

Here d is the same d that appears in the second paragraph of this proof, and ensures that V satisfies the conditions of Theorem 1. It is helpful to note that when $c \rightarrow \infty$, then $d \rightarrow \infty$. This follows from Theorem 1, since

$$d = \mathcal{C}(v^*) = cE \left[\hat{X}(\infty; v^*)^+ \right] - \sum_{j \in \mathcal{J}} (1-p_j)\mu_j E \left[v_j^* \left(\hat{X}(\infty; v^*) \right) \hat{X}(\infty; v^*)^- \right].$$

and it is straightforward to see that the second term on the right-hand side of the above expression is bounded and $E \left[\hat{X}(\infty; v^*)^+ \right] > 0$. Then, we may assume in the remaining of this proof that c and d are arbitrarily large. Finally, recall that V' given in Theorem 2 has $V'(x) = V'_K(x)$ for $x \leq -L_{K-1}$.

Suppose we can establish that $V'(x) \geq V'_K(x)$ for all $x \leq 0$. Since V' and V'_K are both increasing by Theorem 2, they have unique inverse functions V'^{-1} and V'^{-1}_K . Define $-L_j^K := V'^{-1}_K(T(j, j+1))$, and recall $-L_j := V'^{-1}(T(j, j+1))$, $j \in \{1, 2, \dots, K-1\}$. (Note that $-L_j^K$ is well-defined from (18) for large enough d and $-L_j$ is well-defined from the proof of Theorem 2 for large enough c .) Then, since $V'(x) \geq V'_K(x)$, it follows that

$$L_j = -V'^{-1}(T(j, j+1)) \geq -V'^{-1}_K(T(j, j+1)) = L_j^K. \quad (19)$$

From (18),

$$L_j^K = -H_K^{-1} \left((T(j, j+1) - \frac{1-p_K}{p_K}) / (\frac{d}{\beta} + \frac{1-p_K}{p_K}) \right). \quad (20)$$

Then, $L_j^K \rightarrow \infty$ as $d \rightarrow \infty$ since $H_K(x) \downarrow 0$ as $x \rightarrow -\infty$ and $H_K(x)$ is monotone increasing in x . The fact that $L_j \rightarrow \infty$ as $c \rightarrow \infty$ follows from (19) and the fact that $d \rightarrow \infty$ as $c \rightarrow \infty$.

To complete the proof, we show $V'(x) \geq V'_K(x)$ for all $x \leq 0$. It is equivalent to show $V'^{-1}(y) \leq V'^{-1}_K(y)$ whenever both inverse functions are defined. We know $V'^{-1}(y) = V'^{-1}_K(y)$ on $(\frac{1-p_K}{p_K}, T(K-1, K)]$. Next consider the interval $(T(K-1, K), T(K-2, K-1)]$. First we show that there cannot exist $z > T(K-1, K)$ such that $V'^{-1}(z) = V'^{-1}_K(z)$, and second we use this fact to show $V'^{-1}(y) < V'^{-1}_K(y)$ for $y \in (T(K-1, K), T(K-2, K-1)]$. Suppose such a z exists. Note that there can be at most one such z because at any such point V' has a larger slope than V'_K . To see that, note from (3) and $K-1$ is the index that minimize $p_j\mu_j V'(x) - (1-p_j)\mu_j x$, $j \in \mathcal{K}$ when $V'(x) \in [T(K-1, K), T(K-2, K-1)]$ from the proof of Theorem 2, then

$$V''(z) - V''_K(z) = ((p_{K-1}\mu_{K-1} - (1-p_{K-1})\mu_{K-1}) - (p_{K-1}\mu_{K-1} - (1-p_{K-1})\mu_{K-1})) V'^{-1}(z) > 0$$

Furthermore, it follows that $V'_K(x) \geq V'(x)$ for $x \in (-L_{K-1}, -L_{K-2})$, so that

$$V'^{-1}_K(y) \leq V'^{-1}(y) \text{ for } y \in (T(K-1, K), T(K-2, K-1)). \quad (21)$$

We will derive a contradiction in order to conclude that no such z exists. The first step is to take the derivative of the inverse function to find

$$V'^{-1}(z) = -l_{K-1} + \int_{T(K-1, K)}^z \left(d + (\beta + p_{K-1}\mu_{K-1}V'^{-1}(y))y - (1-p_{K-1})\mu_{K-1}V'^{-1}(y) \right)^{-1} dy,$$

$$V'^{-1}_K(z) = -l_{K-1} + \int_{T(K-1, K)}^z \left(d + (\beta + p_K\mu_KV'^{-1}(y))y - (1-p_K)\mu_KV'^{-1}(y) \right)^{-1} dy.$$

For each $y \in (T(K-1, K), T(K-2, K-1))$,

$$\begin{aligned} \left(d + \beta y + (p_{K-1}\mu_{K-1}y - (1-p_{K-1})\mu_{K-1})V'^{-1}(y)\right)^{-1} &\geq \left(d + \beta y + (p_K\mu_Ky - (1-p_K)\mu_K)V'^{-1}(y)\right)^{-1} \\ &> \left(d + \beta y + (p_K\mu_Ky - (1-p_K)\mu_K)V_K'^{-1}(y)\right)^{-1} \end{aligned} \quad (22)$$

The first inequality follows because $K-1$ is the index that minimize $p_j\mu_jV'(x) - (1-p_j)\mu_jx$, $j \in \mathcal{K}$ when $V(x) \in [T(K-1, K), T(K-2, K-1)]$ and $V'^{-1}(y) < 0$. The second inequality follows by (21) and $p_K\mu_Ky - (1-p_K)\mu_K > 0$. We conclude from (22) that $V'^{-1}(z) < V_K'^{-1}(z)$, which is a contradiction. Since no intersection point z can exist, either $V'^{-1}(y) > V_K'^{-1}(y)$ or $V'^{-1}(y) < V_K'^{-1}(y)$ on $(T(K-1, K), T(K-2, K-1))$. Similar argument shows that $V'^{-1}(y) > V_K'^{-1}(y)$ leads to a contradiction. Finally, iterate this argument we can show $V'^{-1}(y) < V_K'^{-1}(y)$ on each $[T(j-1, j), T(j, j+1)]$, $j \in \{1, 2, \dots, K-1\}$. That is,

$$V'^{-1}(y) \leq V_K'^{-1}(y), y \in \left(-\frac{1-p_K}{p_K}, V_K'^{-1}(0)\right]. \quad (23)$$

Proof of Corollary 1:

First we show that $T(i, j) \leq T(i, k) \leq T(j, k)$. It is straightforward to verify that for any a, b, c, d satisfying $\frac{a}{b} \geq \frac{c}{d}$, $b > 0$ and $d > 0$, also $\frac{a}{b} \geq \frac{a+c}{b+d} \geq \frac{c}{d}$. From the definition of $T(i, j)$ in (20) in the main body, $T(i, j) + 1 = \frac{\mu_j - \mu_i}{p_j\mu_j - p_i\mu_i}$ and $T(j, k) + 1 = \frac{\mu_k - \mu_j}{p_k\mu_k - p_j\mu_j}$. Hence $T(i, j) \leq T(j, k)$ is equivalent to $\frac{\mu_j - \mu_i}{p_j\mu_j - p_i\mu_i} \leq \frac{\mu_k - \mu_j}{p_k\mu_k - p_j\mu_j}$. Both denominators in the inequality are positive, so we can apply $\frac{a}{b} \geq \frac{a+c}{b+d} \geq \frac{c}{d}$ and get $\frac{\mu_j - \mu_i}{p_j\mu_j - p_i\mu_i} \leq \frac{\mu_k - \mu_i}{p_k\mu_k - p_i\mu_i} \leq \frac{\mu_k - \mu_j}{p_k\mu_k - p_j\mu_j}$. The middle term is exactly $T(i, k) + 1$. Therefore, $T(i, j) \leq T(i, k) \leq T(j, k)$.

It follows by assumption that V and d satisfy (12) in the main body so that

$$V''(x) + cx^+ - \beta V'(x) + \min_{v \in \mathcal{V}(\mathcal{J}')} \left\{ \sum_{n \in \mathcal{J}'} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} x^- = d \text{ for all } x \in \mathcal{R}.$$

We next show that this same V and d satisfy

$$V''(x) + cx^+ - \beta V'(x) + \min_{v \in \mathcal{V}(\mathcal{J})} \left\{ \sum_{n \in \mathcal{J}} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} x^- = d \text{ for all } x \in \mathcal{R}.$$

Since $T(i, j) \leq T(i, k) \leq T(j, k)$, from the text following (20) in the main body, j cannot be the index to minimize the expression $V'(x)p_j\mu_j - (1-p_j)\mu_j$, and so

$$\min_{v \in \mathcal{V}(\mathcal{J}')} \left\{ \sum_{n \in \mathcal{J}'} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} = \min_{v \in \mathcal{V}(\mathcal{J})} \left\{ \sum_{n \in \mathcal{J}} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\}.$$

We conclude that the same V and d that satisfy the conditions of Theorem 1 with $\mathcal{V} = \mathcal{V}(\mathcal{J}')$ also satisfy the conditions of Theorem 1 with $\mathcal{V} = \mathcal{V}(\mathcal{J})$. Therefore, $(v_1^{\mathcal{J}'}, \dots, v_{i-1}^{\mathcal{J}'}, 0, v_{i+1}^{\mathcal{J}'}, \dots, v_{j-1}^{\mathcal{J}'})$ is an optimal control when $\mathcal{V} = \mathcal{V}(\mathcal{J})$.

Proof of Corollary 2:

First, reduce all pools j for which $j > i$ (equivalently, $p_j\mu_j > p_i\mu_i$) and $p_j \geq p_i$ for some $i \in \mathcal{J}$, and group the remaining pools in the set \mathcal{S} . Then, the $S = |\mathcal{S}|$ pools in the set \mathcal{S} can be re-labeled so that

$$p_1\mu_1 < p_2\mu_2 < \dots < p_S\mu_S \text{ and } p_1 > p_2 > \dots > p_S$$

Furthermore, $p_S = \min_{j \in \mathcal{J}} \{p_j\}$. Also, the pool re-labeled as S must be an element in $\mathcal{K}^* \subset \mathcal{S}$. Hence $p_S = p_K$, for p_K in the re-labeling (21) in the main body.

It follows from Theorem 2 that there exists V and d that satisfy the conditions of Theorem 1; specifically, that solve (12) in the main body with \mathcal{K}^* replacing \mathcal{J} . Then, to complete the proof, it is sufficient to show that this same V and d satisfy

$$V''(x) + cx^+ - \beta V'(x) + \min_{v \in \mathcal{V}(\mathcal{J})} \left\{ \sum_{n \in \mathcal{J}} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} x^- = d \text{ for all } x \in \mathcal{R} \quad (24)$$

and

$$V''(x) + cx^+ - \beta V'(x) + \min_{v \in \mathcal{V}(\mathcal{J}')} \left\{ \sum_{n \in \mathcal{J}'} \left(V'(x)p_n\mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} x^- = d \text{ for all } x \in \mathcal{R}. \quad (25)$$

Consider any pool $k \in \mathcal{J}$ but $k \notin \mathcal{S}$. Then, there exists $n \in \mathcal{S}$ for which $p_n \mu_n < p_k \mu_k$ and $p_n \leq p_k$. Since $n \in \mathcal{S}$, $p_n \geq p_K$. Recall that V' is increasing from Theorem 2 and $V'(x) \rightarrow \frac{1-p_K}{p_K}$ as $x \rightarrow -\infty$ from the proof of Theorem 2. It follows that

$$V'(x) \geq \frac{1-p_K}{p_K} \geq \frac{1-p_n}{p_n} \geq \frac{1-p_k}{p_k}.$$

Furthermore, also noting the definition (20) in the main body,

$$\frac{1-p_k}{p_k} - T(n, k) = \frac{(p_k - p_n)\mu_n}{p_k(p_k \mu_k - p_n \mu_n)} \geq 0.$$

Hence

$$V'(x) \geq T(n, k) = \frac{(1-p_k)\mu_k - (1-p_n)\mu_n}{p_k \mu_k - p_n \mu_n},$$

which is equivalent to $p_k \mu_k V'(x) - (1-p_k)\mu_k \geq p_n \mu_n V'(x) - (1-p_n)\mu_n$. In particular, letting $\Delta = \mathcal{K}^* \cup \{k\}$,

$$\min_{v \in \mathcal{V}(\Delta)} \left\{ \sum_{n \in \mathcal{K}^* \cup \{k\}} \left(V'(x) p_n \mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} = \min_{v \in \mathcal{V}(\mathcal{K}^*)} \left\{ \sum_{n \in \mathcal{K}^* \cup \{k\}} \left(V'(x) p_n \mu_n - (1-p_n)\mu_n \right) v_n(x) \right\} \quad (26)$$

Repeating the above argument for each pool $k \in \mathcal{J}$ but $k \notin \mathcal{S}$ in turn implies (26) also holds with $\Delta = \mathcal{K}^* \cup \mathcal{J} - \mathcal{S}$. Finally, note that any pool $k \in \mathcal{S}$ but $k \notin \mathcal{K}^*$ must satisfy the conditions of Corollary 1. Then, iteratively applying Corollary 1 shows that (26) also holds with any Δ satisfying $\mathcal{K}^* \subset \Delta \subset \mathcal{V}$, from which we conclude (24) and (25) is valid.

Proof of Theorem 3:

This theorem follows from Theorem 2, Corollary 1 and 2.

References

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