## Online Appendix to Inventory Management in Humanitarian Operations: Impact of Amount, Schedule, and Uncertainty in Funding

LEMMA 1. *$G_t^\lambda(x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}, R_t, OF_t)$ is jointly convex in $x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}$ for fixed $R_t$ and $OF_t$.*

**Proof**

The proof proceeds through induction on the number of periods to go. We know that $G_0^\lambda(.) = 0 \ \forall \ (x_0, w_0^0, ..., w_o^{\lambda-1}, R_0, 0)$. For $t \leq \lambda$, no new orders will be placed in period $t$ since the order will not be received before the end of the horizon. Therefore, for $t \leq \lambda$, it follows that for every $(R_t, OF_t)$,

$$\begin{aligned} &G_t^\lambda(x_t, w_t^0, ..., w_t^{\lambda-1}, R_t, OF_t) \\ &= \tilde{f}_t(x_t + w_t^0) + \mathsf{E}_{\zeta_t}\tilde{f}_{t-1}(x_t + w_t^0 + w_t^1 - \zeta_t) + ... + \mathsf{E}_{\zeta_t,...,\zeta_2}\tilde{f}_1(x_t + w_t^0 + ... + w_t^{\lambda-1} - \zeta_t - \zeta_{t-1} - ... - \zeta_2) \end{aligned} \quad (10)$$

where $\tilde{f}_i(x) = h\mathsf{E}_{\zeta_i}[x - \zeta_i]^+ + b\mathsf{E}_{\zeta_i}[\zeta_i - x]^+$. From equation (10), it follows directly that $G_t^\lambda$ is jointly convex in $x_t, w_t^0, ..., w_t^{\lambda-1}$ for every $t \leq \lambda$. Now,

$$\begin{aligned} &G_{\lambda+1}^\lambda(x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}, R_{\lambda+1}, OF_{\lambda+1}) \\ &= \min_{0 \leq z_{\lambda+1} \leq \frac{R_{\lambda+1}}{c} - IP_{\lambda+1}} \left\{ \begin{array}{l} cz_{\lambda+1} + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w_{\lambda+1}^0]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w_{\lambda+1}^0 - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{OF_\lambda|OF_{\lambda+1}}\mathsf{E}_{\zeta_{\lambda+1}}G_\lambda^\lambda(x_{\lambda+1} + w_{\lambda+1}^0 - \zeta_{\lambda+1}, ..., w_{\lambda+1}^{\lambda-1}, z_{\lambda+1}, R_{\lambda+1} - c\zeta_{\lambda+1} + OF_{\lambda+1} - OF_\lambda, OF_\lambda) \end{array} \right\} \end{aligned} \quad (11)$$

From the convexity of $G_\lambda^\lambda$ in $x_\lambda, w_\lambda^0, ..., w_\lambda^{\lambda-1}$, it follows that for fixed $(R_{\lambda+1}, OF_{\lambda+1})$, the function to be minimized in expression (11) is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}$ and $z_{\lambda+1}$. Now notice that for fixed $R_{\lambda+1}$, C=$\{(x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}, z_{\lambda+1}) : 0 \leq z_{\lambda+1} \leq \frac{R_{\lambda+1}}{c} - IP_{\lambda+1}\}$ is a convex set. Then using proposition B-4 from Heyman and Sobel (1984), we see that $G_{\lambda+1}^\lambda$ is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}$ for fixed $(R_{\lambda+1}, OF_{\lambda+1})$.

If we let the induction hypothesis be that $G_t^\lambda$ is jointly convex in $x_t, w_t^0, ..., w_t^{\lambda-1}$ for any given $t$ and fixed $(R_t, OF_t)$, then using an argument identical to the one following expression (11), we can show that $G_{t+1}^\lambda$ is jointly convex in $x_{t+1}, w_{t+1}^0, ..., w_{t+1}^{\lambda-1}$ for fixed $(R_{t+1}, OF_{t+1})$. This completes the induction.

**Proof of Theorem 1**

The proof proceeds through induction on the number of periods to go. We know that $NV_0^\lambda(.) = G_0^\lambda(.) = 0$. For $t \leq \lambda$, orders placed in period $t$ will not be received before the end of the horizon and hence, no orders will be placed during those periods. Therefore, on–hand or future funding inflows become irrelevant and hence, for every $(R_t, OF_t)$, $G_t^\lambda(x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}, R_t, OF_t) = NV_t^\lambda(x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}) \ \forall \ t \leq \lambda$. Then, from expression (10), it follows that $NV_t^\lambda$ is jointly convex in $x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1} \ \forall \ t \leq \lambda$. Now,

$$NV_{\lambda+1}^\lambda(x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}) = \min_{z_{\lambda+1} \geq 0} \left\{ \begin{array}{l} cz_{\lambda+1} + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w_{\lambda+1}^0]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w_{\lambda+1}^0 - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{\zeta_{\lambda+1}}NV_\lambda^\lambda(x_{\lambda+1} + w_{\lambda+1}^0 - \zeta_{\lambda+1}, w_{\lambda+1}^1, ..., w_{\lambda+1}^{\lambda-1}, z_{\lambda+1}) \end{array} \right\} \quad (12)$$

From the convexity of $NV_\lambda^\lambda$, we have that the expression to be minimized in equation (12) is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}$ and $z_{\lambda+1}$. Two important results follow: (i) $NV_{\lambda+1}^\lambda$ is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^0, ..., w_{\lambda+1}^{\lambda-1}$. (ii) The specific form of equation (10) implies that there exists a base stock level $y_{\lambda+1}^{\lambda*}$, independent of $x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}$, such that $z^*_{\lambda+1}$=max $(y^{\lambda*}_{\lambda+1} - IP_{\lambda+1}$,0) minimizes (12). In general, we can show through induction that $NV^\lambda_t$ is jointly convex in $x_t, w^0_t, ..., w^{\lambda-1}_t$ and that there exists a base stock level $y^*_t$, independent of $x_t, w^0_t, ..., w^{\lambda-1}_t$, such that $z^*_t$=max $(y^{\lambda*}_t - IP_t$,0) is the optimal order quantity in period $t$.

Now, $G^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, R_{\lambda+1}, OF_{\lambda+1})$

$$= \min_{0 \le z_{\lambda+1} \le \frac{R_{\lambda+1}}{c} - IP_{\lambda+1}} \left\{ \begin{array}{l} cz_{\lambda+1} + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{OF_\lambda|OF_{\lambda+1}}\mathsf{E}_{\zeta_{\lambda+1}} G^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, z_{\lambda+1}, R_{\lambda+1} - c\zeta_{\lambda+1} + OF_{\lambda+1} - OF_\lambda, OF_\lambda) \end{array} \right\}$$

$$= \min_{0 \le z_{\lambda+1} \le \frac{R_{\lambda+1}}{c} - IP_{\lambda+1}} \left\{ \begin{array}{l} cz_{\lambda+1} + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{\zeta_{\lambda+1}} NV^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, w^1_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, z_{\lambda+1}) \end{array} \right\} \quad (13)$$

where the second equality follows from the fact that $NV^\lambda_\lambda(.) = G^\lambda_\lambda(., R_\lambda, OF_\lambda)\ \forall\ (R_\lambda, OF_\lambda)$. The expressions to be minimized in (12) and (13) are the same, implying that $z^*_{\lambda+1}$=max(min$(y^{\lambda*}_{\lambda+1} - IP_{\lambda+1}, \frac{R_{\lambda+1}}{c} - IP_{\lambda+1})$,0) minimizes equation (13).

Now consider the following cases.

Case 1.1: $IP_{\lambda+1} \ge y^{\lambda*}_{\lambda+1}$

$$NV^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}) = G^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, R_{\lambda+1}, OF_{\lambda+1})$$
$$= h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+ + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + \mathsf{E}_{\zeta_{\lambda+1}} NV^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, w^1_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, 0)$$

where we have again used the fact that $NV^\lambda_\lambda(.) = G^\lambda_\lambda(., R_\lambda, OF_\lambda)$.

Case 1.2: $IP_{\lambda+1} < y^{\lambda*}_{\lambda+1}$, $\frac{R_{\lambda+1}}{c} \ge y^{\lambda*}_{\lambda+1}$

$$NV^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}) = G^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, R_{\lambda+1}, OF_{\lambda+1}) = h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+$$
$$+ b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + c(y^{\lambda*}_{\lambda+1} - IP_{\lambda+1}) + \mathsf{E}_{\zeta_{\lambda+1}} NV^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, w^1_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, y^{\lambda*}_{\lambda+1} - IP_{\lambda+1})$$

In cases 1.1 and 1.2, $NV^\lambda_{\lambda+1}(.) = G^\lambda_{\lambda+1}(., R_{\lambda+1}, OF_{\lambda+1})$. Therefore, $\frac{\partial NV^\lambda_{\lambda+1}}{\partial x_{\lambda+1}} = \frac{\partial G^\lambda_{\lambda+1}}{\partial x_{\lambda+1}}$ and $\frac{\partial NV^\lambda_{\lambda+1}}{\partial w^i_{\lambda+1}} = \frac{\partial G^\lambda_{\lambda+1}}{\partial w^i_{\lambda+1}}, i = 0, 1, ..., \lambda - 1$.

Case 1.3: $\frac{R_{\lambda+1}}{c} < y^{\lambda*}_{\lambda+1}$

$$NV^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}) = h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+ + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + c(y^{\lambda*}_{\lambda+1} - IP_{\lambda+1})$$
$$+ \mathsf{E}_{\zeta_{\lambda+1}} NV^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, w^1_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, y^{\lambda*}_{\lambda+1} - IP_{\lambda+1})$$

$$G^\lambda_{\lambda+1}(x_{\lambda+1}, w^0_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, R_{\lambda+1}, OF_{\lambda+1}) = h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}]^+ + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w^0_{\lambda+1}]^+ + R_{\lambda+1} - cIP_{\lambda+1}$$
$$+ \mathsf{E}_{\zeta_{\lambda+1}} NV^\lambda_\lambda(x_{\lambda+1} + w^0_{\lambda+1} - \zeta_{\lambda+1}, w^1_{\lambda+1}, ..., w^{\lambda-1}_{\lambda+1}, \frac{R_{\lambda+1}}{c} - IP_{\lambda+1})$$

Using expression (10), we see that

$$\frac{\partial NV^\lambda_{\lambda+1}}{\partial x_{\lambda+1}} = \frac{\partial NV^\lambda_{\lambda+1}}{\partial w^0_{\lambda+1}} = -c + hF_{\lambda+1}(x_{\lambda+1} + w^0_{\lambda+1}) - b\bar{F}_{\lambda+1}(x_{\lambda+1} + w^0_{\lambda+1}) + \mathsf{E}_{\zeta_{\lambda+1}} \frac{\partial \tilde{f}_\lambda(x_{\lambda+1} + w^0_{\lambda+1} + w^1_{\lambda+1} - \zeta_{\lambda+1})}{\partial x_{\lambda+1}}$$
$$+ ... + \mathsf{E}_{\zeta_{\lambda+1},...,\zeta_2} \frac{\partial \tilde{f}_1(y^{\lambda*}_{\lambda+1} - \zeta_{\lambda+1} - ... - \zeta_2)}{\partial x_{\lambda+1}}$$

and

$$\frac{\partial G^\lambda_{\lambda+1}}{\partial x_{\lambda+1}} = \frac{\partial G^\lambda_{\lambda+1}}{\partial w^0_{\lambda+1}} = -c + hF_{\lambda+1}(x_{\lambda+1} + w^0_{\lambda+1}) - b\bar{F}_{\lambda+1}(x_{\lambda+1} + w^0_{\lambda+1}) + \mathsf{E}_{\zeta_{\lambda+1}} \frac{\partial \tilde{f}_\lambda(x_{\lambda+1} + w^0_{\lambda+1} + w^1_{\lambda+1} - \zeta_{\lambda+1})}{\partial x_{\lambda+1}}$$

$$+ ... + \mathsf{E}_{\zeta_{\lambda+1},...,\zeta_2} \frac{\partial \tilde{f}_1(\frac{R_{\lambda+1}}{c} - \zeta_{\lambda+1} - ... - \zeta_2)}{\partial x_{\lambda+1}}$$

For fixed $R_{\lambda+1}$, the last term is 0 in both the above equations. Therefore $\frac{\partial NV^{\lambda}_{\lambda+1}(.)}{\partial x_{\lambda+1}} = \frac{\partial NV^{\lambda}_{\lambda+1}(.)}{\partial w^0_{\lambda+1}} = \frac{\partial G^{\lambda}_{\lambda+1}(.,R_{\lambda+1},OF_{\lambda+1})}{\partial x_{\lambda+1}} = \frac{\partial G^{\lambda}_{\lambda+1}(.,R_{\lambda+1},OF_{\lambda+1})}{\partial w^0_{\lambda+1}}$. Similarly,

$$\frac{\partial NV^{\lambda}_{\lambda+1}}{\partial w^1_{\lambda+1}} = -c + \mathsf{E}_{\zeta_{\lambda+1}} \frac{\partial \tilde{f}_\lambda(x_{\lambda+1} + w^0_{\lambda+1} + w^1_{\lambda+1} - \zeta_{\lambda+1})}{\partial w^1_{\lambda+1}} + ... + \mathsf{E}_{\zeta_{\lambda+1},...,\zeta_2} \frac{\partial \tilde{f}_1(y^{\lambda*}_{\lambda+1} - \zeta_{\lambda+1} - ... - \zeta_2)}{\partial w^1_{\lambda+1}}$$

and

$$\frac{\partial G^{\lambda}_{\lambda+1}}{\partial w^1_{\lambda+1}} = -c + \mathsf{E}_{\zeta_{\lambda+1}} \frac{\partial \tilde{f}_\lambda(x_{\lambda+1} + w^0_{\lambda+1} + w^1_{\lambda+1} - \zeta_{\lambda+1})}{\partial w^1_{\lambda+1}} + ... + \mathsf{E}_{\zeta_{\lambda+1},...,\zeta_2} \frac{\partial \tilde{f}_1(\frac{R_{\lambda+1}}{c} - \zeta_{\lambda+1} - ... - \zeta_2)}{\partial w^1_{\lambda+1}}$$

Again, for fixed $R_{\lambda+1}$, the last term is 0 in both the above equations. Therefore, $\frac{\partial NV^{\lambda}_{\lambda+1}(.)}{\partial w^1_{\lambda+1}} = \frac{\partial G^{\lambda}_{\lambda+1}(.,R_{\lambda+1},OF_{\lambda+1})}{\partial w^1_{\lambda+1}}$. For any given $(R_{\lambda+1}, OF_{\lambda+1})$, following similar steps, it is easy to show that $\frac{\partial NV^{\lambda}_{\lambda+1}(.)}{\partial w^i_{\lambda+1}} = \frac{\partial G^{\lambda}_{\lambda+1}(.,R_{\lambda+1},OF_{\lambda+1})}{\partial w^i_{\lambda+1}}$ $\forall\ i = 1, ..., \lambda - 1$.

Let the induction hypothesis be that $\frac{\partial NV^{\lambda}_t(.)}{\partial x_t} = \frac{\partial G^{\lambda}_t(.,R_t,OF_t)}{\partial x_t}$ and $\frac{\partial NV^{\lambda}_t(.)}{\partial w^i_t} = \frac{\partial G^{\lambda}_t(.,R_t,OF_t)}{\partial w^i_t}$ for any given $(R_t, OF_t)$ and $i = 0, 1, ..., \lambda - 1$. We will show that they hold true for $t + 1$ as well. We know that

$$G^{\lambda}_{t+1}(x_{t+1}, w^0_{t+1}, ..., w^{\lambda-1}_{t+1}, R_{t+1}, OF_{t+1})$$
$$= \min_{0 \le z_{t+1} \le \frac{R_{t+1}}{c} - IP_{t+1}} \left\{ \begin{array}{l} cz_{t+1} + b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w^0_{t+1}]^+ + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w^0_{t+1} - \zeta_{t+1}]^+ \\ +\mathsf{E}_{OF_t|OF_{t+1}}\mathsf{E}_{\zeta_{t+1}} G^{\lambda}_t(x_{t+1} + w^0_{t+1} - \zeta_{t+1}, ..., w^{\lambda-1}_{t+1}, z_{t+1}, R_{t+1} - c\zeta_{t+1} + OF_{t+1} - OF_t, OF_t) \end{array} \right\} \quad (14)$$

From the convexity of $G^{\lambda}_t$ and the induction assumptions $\frac{\partial NV^{\lambda}_t}{\partial x_t} = \frac{\partial G^{\lambda}_t}{\partial x_t}$ and $\frac{\partial NV^{\lambda}_t}{\partial w^i_t} = \frac{\partial G^{\lambda}_t}{\partial w^i_t}$, it follows that $z^*_{t+1}$=max(min( $y^{\lambda*}_{t+1} - IP_{t+1}$, $\frac{R_{t+1}}{c} - IP_{t+1}$),0) minimizes equation (14).

To prove the induction for $t$+1, the following recursive equation for $NV^{\lambda}_t$, $t > \lambda + 1$, would be useful.

$$\begin{aligned} NV^{\lambda}_t(x_{t+1} + w^0_{t+1} - \zeta_{t+1}, w^1_{t+1}, ..., w^{\lambda-1}_{t+1}, z_{t+1}) &= c(max(y^{\lambda*}_t - IP_t, 0)) + \tilde{f}_t(x_{t+1} + w^0_{t+1} + w^1_{t+1} - \zeta_{t+1}) \\ &+ \mathsf{E}_{\zeta_t} \tilde{f}_{t-1}(x_{t+1} + w^0_{t+1} + w^1_{t+1} + w^2_{t+1} - \zeta_{t+1} - \zeta_t) + ... + \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+2}} \tilde{f}_{t-\lambda+1}(IP_{t+1} + z_{t+1} - \zeta_{t+1} - \zeta_t - ... - \zeta_{t-\lambda+2}) \\ &+ \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+1}} NV^{\lambda}_{t-\lambda}(x_{t-\lambda}, w^0_{t-\lambda}, ..., w^{\lambda-1}_{t-\lambda}) \end{aligned} \quad (15)$$

Now consider three cases similar to the ones we considered earlier for $t = \lambda + 1$.

Case 2.1: $IP_{t+1} \ge y^{\lambda*}_{t+1}$.

$$\begin{aligned} G^{\lambda}_{t+1}(x_{t+1}, w^0_{t+1}, ..., w^{\lambda-1}_{t+1}, R_{t+1}, OF_{t+1}) &= b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w^0_{t+1}]^+ + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w^0_{t+1} - \zeta_{t+1}]^+ \\ &+ \mathsf{E}_{OF_t|OF_{t+1}}\mathsf{E}_{\zeta_{t+1}} G^{\lambda}_t(x_{t+1} + w^0_{t+1} - \zeta_{t+1}, ..., w^{\lambda-1}_{t+1}, 0, R_{t+1} - c\zeta_{t+1} + OF_{t+1} - OF_t, OF_t) \end{aligned}$$

and

$$\begin{aligned} NV^{\lambda}_{t+1}(x_{t+1}, w^0_{t+1}, ..., w^{\lambda-1}_{t+1}) &= b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w^0_{t+1}]^+ + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w^0_{t+1} - \zeta_{t+1}]^+ \\ &+ \mathsf{E}_{\zeta_{t+1}} NV^{\lambda}_t(x_{t+1} + w^0_{t+1} - \zeta_{t+1}, w^1_{t+1}, ..., w^{\lambda-1}_{t+1}, 0) \end{aligned}$$

Since $\frac{\partial NV^{\lambda}_t(.)}{\partial x_t} = \frac{\partial G^{\lambda}_t(.,R_t,OF_t)}{\partial x_t}$ and $\frac{\partial NV^{\lambda}_t(.)}{\partial w^i_t} = \frac{\partial G^{\lambda}_t(.,R_t,OF_t)}{\partial w^i_t}$ for $i = 1, 2, ..., \lambda - 1$, it follows that, for fixed $(R_{t+1}, OF_{t+1})$, $\frac{\partial NV^{\lambda}_{t+1}(.)}{\partial x_{t+1}} = \frac{\partial G^{\lambda}_{t+1}(.,R_{t+1},OF_{t+1})}{\partial x_{t+1}}$ and $\frac{\partial NV^{\lambda}_{t+1}(.)}{\partial w^i_{t+1}} = \frac{\partial G^{\lambda}_{t+1}(.,R_{t+1},OF_{t+1})}{\partial w^i_{t+1}}$ $\forall\ i = 0, 1, ..., \lambda - 1$.

Case 2.2: $IP_{t+1} < y_{t+1}^{\lambda *}$, $\frac{R_{t+1}}{c} \geq y_{t+1}^{\lambda *}$.

$$G_{t+1}^{\lambda}(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1}, R_{t+1}, OF_{t+1}) = c(y_{t+1}^{\lambda *} - IP_{t+1}) + b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w_{t+1}^{0}]^{+} + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}]^{+}$$
$$+ \mathsf{E}_{OF_t|OF_{t+1}}\mathsf{E}_{\zeta_{t+1}} G_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, ..., w_{t+1}^{\lambda-1}, y_{t+1}^{\lambda *} - IP_{t+1}, R_{t+1} - c\zeta_{t+1} + OF_{t+1} - OF_t, OF_t)$$

and

$$NV_{t+1}^{\lambda}(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1}) = c(y_{t+1}^{\lambda *} - IP_{t+1}) + b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w_{t+1}^{0}]^{+} + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}]^{+}$$
$$+ \mathsf{E}_{\zeta_{t+1}} NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, w_{t+1}^{1}, ..., w_{t+1}^{\lambda-1}, y_{t+1}^{\lambda *} - IP_{t+1})$$

Again, the induction hypothesis concerning the derivatives of $NV_t^{\lambda}$ and $G_t^{\lambda}$ directly yields $\frac{\partial NV_{t+1}^{\lambda}(.)}{\partial x_{t+1}} = \frac{\partial G_{t+1}^{\lambda}(.,R_{t+1},OF_{t+1})}{\partial x_{t+1}}$ and $\frac{\partial NV_{t+1}^{\lambda}(.)}{\partial w_{t+1}^{i}} = \frac{\partial G_{t+1}^{\lambda}(.,R_{t+1},OF_{t+1})}{\partial w_{t+1}^{i}}$ $\forall\ i = 0, 1, ..., \lambda - 1$.

Case 2.3: $\frac{R_{t+1}}{c} < y_{t+1}^{\lambda *}$

$$NV_{t+1}^{\lambda}(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1}) = c(y_{t+1}^{\lambda *} - IP_{t+1})$$
$$+ b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w_{t+1}^{0}]^{+} + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}]^{+} + \mathsf{E}_{\zeta_{t+1}} NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, ..., w_{t+1}^{\lambda-1}, y_{t+1}^{\lambda *} - IP_{t+1})$$

From equation (15), we get

$$NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, w_{t+1}^{1}, ..., w_{t+1}^{\lambda-1}, y_{t+1}^{\lambda *} - IP_{t+1}) = c(max(y_t^{\lambda *} - (y_{t+1}^{\lambda *} - \zeta_{t+1}), 0)) + \tilde{f}_t(x_{t+1} + w_{t+1}^{0} + w_{t+1}^{1} - \zeta_{t+1})$$
$$+ \mathsf{E}_{\zeta_t}\tilde{f}_{t-1}(x_{t+1} + w_{t+1}^{0} + w_{t+1}^{1} + w_{t+1}^{2} - \zeta_{t+1} - \zeta_t) + ... + \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+2}}\tilde{f}_{t-\lambda+1}(y_{t+1}^{\lambda *} - \zeta_{t+1} - \zeta_t - ... - \zeta_{t-\lambda+2})$$
$$+ \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+1}} NV_{t-\lambda}^{\lambda}(x_{t-\lambda}, w_{t-\lambda}^{0}, ..., w_{t-\lambda}^{\lambda-1}) \quad (16)$$

In equation (16), $x_{t-\lambda} = y_{t+1}^{\lambda *} - \zeta_{t+1} - \zeta_t - ... - \zeta_{t-\lambda+1}$, i.e., it is a function only of $y_{t+1}^{\lambda *}, \zeta_{t+1}, \zeta_t, ..., \zeta_{t-\lambda+1}$. $w_{t-\lambda}^{0}$, which is the order placed in $t$ equals $max(y_t^{\lambda *} - (y_{t+1}^{\lambda *} - \zeta_{t+1}), 0)$, i.e., it is a function only of $y_{t+1}^{\lambda *}, y_t^{\lambda *}$, and $\zeta_{t+1}$. Following similar logic, it is easy to see that the state variables $(x_{t-\lambda}, w_{t-\lambda}^{0}, ..., w_{t-\lambda}^{\lambda-1})$ are independent of $(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1})$.

$$G_{t+1}^{\lambda}(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1}, R_{t+1}, OF_{t+1}) = R_{t+1} - cIP_{t+1} + b\mathsf{E}_{\zeta_{t+1}}[\zeta_{t+1} - x_{t+1} - w_{t+1}^{0}]^{+} + h\mathsf{E}_{\zeta_{t+1}}[x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}]^{+}$$
$$+ \mathsf{E}_{OF_t|OF_{t+1}}\mathsf{E}_{\zeta_{t+1}} G_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, ..., w_{t+1}^{\lambda-1}, \frac{R_{t+1}}{c} - IP_{t+1}, R_{t+1} - c\zeta_{t+1} + OF_{t+1} - OF_t, OF_t)$$

Since the derivatives of $G_t^{\lambda}$ and $NV_t^{\lambda}$ are equal (for a given $R_t$ and $OF_t$) by the induction hypothesis, to analyze the derivative of $G_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, ..., w_{t+1}^{\lambda-1}, \frac{R_{t+1}}{c} - IP_{t+1}, R_{t+1} - c\zeta_{t+1} + OF_{t+1} - OF_t, OF_t)$ with respect to state variables $x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1}$, we focus on the following expression.

$$NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, w_{t+1}^{1}, ..., w_{t+1}^{\lambda-1}, \frac{R_{t+1}}{c} - IP_{t+1}) = c(max(y_t^{\lambda *} - (\frac{R_{t+1}}{c} - \zeta_{t+1}), 0)) + \tilde{f}_t(x_{t+1} + w_{t+1}^{0} + w_{t+1}^{1} - \zeta_{t+1})$$
$$+ \mathsf{E}_{\zeta_t}\tilde{f}_{t-1}(x_{t+1} + w_{t+1}^{0} + w_{t+1}^{1} + w_{t+1}^{2} - \zeta_{t+1} - \zeta_t) + ... + \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+2}}\tilde{f}_{t-\lambda+1}(\frac{R_{t+1}}{c} - \zeta_{t+1} - \zeta_t - ... - \zeta_{t-\lambda+2})$$
$$+ \mathsf{E}_{\zeta_t,...,\zeta_{t-\lambda+1}} NV_{t-\lambda}^{\lambda}(x_{t-\lambda}, w_{t-\lambda}^{1}, ..., w_{t-\lambda}^{\lambda-1}) \quad (17)$$

In equation (17), $x_{t-\lambda} = \frac{R_{t+1}}{c} - \zeta_{t+1} - \zeta_t - ... - \zeta_{t-\lambda+1}$, i.e., it is a function only of $\frac{R_{t+1}}{c}, \zeta_{t+1}, \zeta_t, ..., \zeta_{t-\lambda+1}$. $w_{t-\lambda}^{0} = max(y_t^{\lambda *} - (\frac{R_{t+1}}{c} - \zeta_{t+1}), 0)$ is a function only of $\frac{R_{t+1}}{c}, y_t^{\lambda *}$ and $\zeta_{t+1}$. Using similar logic, we see that for a given $R_{t+1}$, the state variables $(x_{t-\lambda}, w_{t-\lambda}^{0}, ..., w_{t-\lambda}^{\lambda-1})$ are independent of $(x_{t+1}, w_{t+1}^{0}, ..., w_{t+1}^{\lambda-1})$ in this case as well. Therefore it follows that the derivatives of $NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, w_{t+1}^{1}, ..., w_{t+1}^{\lambda-1}, y_{t+1}^{\lambda *} - IP_{t+1})$ and $NV_t^{\lambda}(x_{t+1} + w_{t+1}^{0} - \zeta_{t+1}, w_{t+1}^{1}, ..., w_{t+1}^{\lambda-1}, \frac{R_{t+1}}{c} - IP_{t+1})$ with respect to $x_{t+1}$ and $w_{t+1}^{i}, i = 0, 1, ..., \lambda - 1$ are equal. This in turn implies that $\frac{\partial NV_{t+1}^{\lambda}(.)}{\partial x_{t+1}} = \frac{\partial G_{t+1}^{\lambda}(.,R_{t+1},OF_{t+1})}{\partial x_{t+1}}$ and $\frac{\partial NV_{t+1}^{\lambda}(.)}{\partial w_{t+1}^{i}} = \frac{\partial G_{t+1}^{\lambda}(.,R_{t+1},OF_{t+1})}{\partial w_{t+1}^{i}}$ for any given $(R_{t+1}, OF_{t+1})$ and $i = 0, 1, ..., \lambda - 1$. Hence the induction is complete.

**Proof of Theorem 2**

We use a sample path approach to prove this result. For expositional clarity, let us define a new variable $\bar{Z}_t^n=(\bar{z}_1^n,\bar{z}_2^n,...,\bar{z}_{t-1}^n)$, $n$=1,2, where $\bar{z}_i^n$ is the amount received in period $i$ under funding scenario $n$, given that $OF_t^n$=$j$. Of course, $\bar{Z}_t^1$ and $\bar{Z}_t^2$ are different for different sample paths. Let $\zeta=(\zeta_1,\zeta_2,...,\zeta_t)$ be the vector of realized demands in periods $1,2,...,t$ along a particular sample path. Given $\zeta$ and $\bar{Z}_t^n$, let $G^{\lambda,n}_{t,\zeta,\bar{Z}_t^n}\left(x_t,w_t^0,...,w_t^{\lambda-1},R_t,j\right)$ be the cost incurred in periods $1,2,...,t$ along a particular sample path under funding scenario $n$, following the optimal replenishment policy specified in (1). If random variables $\zeta$, $\bar{Z}_t^1$, $\bar{Z}_t^2$, $\{OF_{t'}^1,t'=1,2,...,t-1\}$ and $\{OF_{t'}^2,t'=1,2,...,t-1\}$ are defined on the same probability space, then given that $OF_t^1=OF_t^2=j$, Probability($OF_{t'}^2\geq OF_{t'}^1$)=1 $\forall\ t'=1,2,...,t-1$. This follows from condition (2). Also, from condition (3), we have that Probability($OF_{t'-1}^n\mid OF_{t'}^n=i'\geq OF_{t'-1}^n\mid OF_{t'}^n=i$)=1 for $i'>i,\ t'=2,3,...,t-1$ and $n=1,2$. Combined, they imply that $\sum\limits_{j=t-1-i}^{t-1}\bar{z}_j^1\geq\sum\limits_{j=t-1-i}^{t-1}\bar{z}_j^2$ w.p. 1 $\forall\ i=0,1,...,t-2$. Therefore, every replenishment decision feasible under scenario 2 is also feasible under scenario 1 along every sample path. Hence, $G^{\lambda,2}_{t,\zeta,\bar{Z}_t^2}\left(x_t,w_t^0,...,w_t^{\lambda-1},R_t,j\right)\geq G^{\lambda,1}_{t,\zeta,\bar{Z}_t^1}\left(x_t,w_t^0,...,w_t^{\lambda-1},R_t,j\right)$. w.p.1. Since, this result holds for every sample path, the result also holds in expectation, i.e, $G_t^{\lambda,2}\left(x_t,w_t^0,...,w_t^{\lambda-1},R_t,j\right)\geq G_t^{\lambda,1}\left(x_t,w_t^0,...,w_t^{\lambda-1},R_t,j\right)$.

**Proof of Theorem 3**

The proof proceeds through induction. We use an equivalent value function $\hat{G}_t^\lambda$ instead of $G_t^\lambda$ in the proof. Define a new state variable $TF_t$=$R_t+OF_t$ and let

$$\begin{aligned}
&\hat{G}_t^\lambda(x_t,w_t^0,w_t^2,...,w_t^{\lambda-1},R_t,TF_t)\\
&\quad=\min_{0\leq z\leq\frac{r_t}{c}}\left\{\begin{array}{l}cz+b\mathsf{E}_{\zeta_t}[\zeta_t-x_t-w_t^0]^++h\mathsf{E}_{\zeta_t}[x_t+w_t^0-\zeta_t]^+\\+\mathsf{E}_{OF_{t-1}|OF_t}\mathsf{E}_{\zeta_t}\hat{G}_{t-1}^\lambda(x_t+w_t^0-\zeta_t,w_t^1,...,w_t^{\lambda-1},z,R_t-c\zeta_t+(OF_t-OF_{t-1}),TF_t-c\zeta_t)\end{array}\right\}\\
&\quad=\min_{0\leq z\leq\frac{r_t}{c}}\left\{\begin{array}{l}cz+b\mathsf{E}_{\zeta_t}[\zeta_t-x_t-w_t^0]^++h\mathsf{E}_{\zeta_t}[x_t+w_t^0-\zeta_t]^+\\+\mathsf{E}_{(OF_t-OF_{t-1})|OF_t}\mathsf{E}_{\zeta_t}\hat{G}_{t-1}^\lambda(x_t+w_t^0-\zeta_t,w_t^1,...,w_t^{\lambda-1},z,R_t-c\zeta_t+(OF_t-OF_{t-1}),TF_t-c\zeta_t)\end{array}\right\}
\end{aligned}$$

The terminal condition is $\hat{G}_0^\lambda(x_0,w_0^0,w_0^2,...,w_0^{\lambda-1},R_0,TF_0)=0$. Clearly, $\hat{G}_0^\lambda$ is jointly convex in $x_0,w_0^0,...,w_0^{\lambda-1},R_0$ and $TF_0$. For $t\leq\lambda$, it is not optimal to order in period $t$ and hence, from expression (10), we see that $\hat{G}_t^\lambda$ is jointly convex in $x_t,w_t^0,...,w_t^{\lambda-1},R_t$ and $TF_t$. Now,

$$\begin{aligned}
&\hat{G}_{\lambda+1}^\lambda(x_{\lambda+1},w_{\lambda+1}^0,...,w_{\lambda+1}^{\lambda-1},R_{\lambda+1},TF_{\lambda+1})\\
&=\min_{0\leq z\leq\frac{r_t}{c}}\left\{\begin{array}{l}cz+b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1}-x_{\lambda+1}-w_{\lambda+1}]^0+\hbar\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1}+w_{\lambda+1}\overset{0}{-}\zeta_{\lambda+1}]\qquad^+\\+\mathsf{E}_{(OF_{\lambda+1}-OF_\lambda)|OF_{\lambda+1}}\mathsf{E}_{\zeta_{\lambda+1}}\hat{G}_\lambda^\lambda(x_{\lambda+1}+w_{\lambda+1}^0-\zeta_{\lambda+1},...,w_{\lambda+1}^{\lambda-1},z,R_{\lambda+1}-c\zeta_{\lambda+1}+(OF_{\lambda+1}-OF_\lambda),TF_{\lambda+1}-c\zeta_{\lambda+1})\end{array}\right\}
\end{aligned}$$

Pick any $(\bar{x}_{\lambda+1},\bar{w}_{\lambda+1}^0,...,\bar{w}_{\lambda+1}^{\lambda-1},\bar{R}_{\lambda+1},T\bar{F}_{\lambda+1})$, $(\bar{\bar{x}}_{\lambda+1},\bar{\bar{w}}_{\lambda+1}^0,...,\bar{\bar{w}}_{\lambda+1}^{\lambda-1},\bar{\bar{R}}_{\lambda+1},T\bar{\bar{F}}_{\lambda+1})$, $\bar{z}$, $\bar{\bar{z}}$ and $0\leq\bar{\lambda}\leq1$. Define $\hat{x}_{\lambda+1}=\bar{\lambda}\bar{x}_{\lambda+1}+(1-\bar{\lambda})\bar{\bar{x}}_{\lambda+1}$ and define $\hat{w}_{\lambda+1}^0$, $\hat{w}_{\lambda+1}^1$,... etc. similarly. Now, for fixed $\zeta_{\lambda+1}$,

$$\begin{aligned}
&\hat{G}_\lambda^\lambda(\hat{x}_{\lambda+1}+\hat{w}_{\lambda+1}^0-\zeta_{\lambda+1},...,\hat{w}_{\lambda+1}^{\lambda-1},\hat{z},\hat{R}_{\lambda+1}-c\zeta_{\lambda+1}+\hat{OF}_{\lambda+1}-(OF_\lambda\mid\hat{OF}_{\lambda+1}),\hat{TF}_{\lambda+1}-c\zeta_{\lambda+1})\\
&\leq\hat{G}_\lambda^\lambda(\hat{x}_{\lambda+1}+\hat{w}_{\lambda+1}^0-\zeta_{\lambda+1},...,\hat{w}_{\lambda+1}^{\lambda-1},\hat{z},\hat{R}_{\lambda+1}-c\zeta_{\lambda+1}+\hat{OF}_{\lambda+1}-\bar{\lambda}(OF_\lambda\mid\bar{OF}_{\lambda+1})-(1-\bar{\lambda})(OF_\lambda\mid\bar{\bar{OF}}_{\lambda+1}),\hat{TF}_{\lambda+1}-c\zeta_{\lambda+1})\\
&\leq\bar{\lambda}\hat{G}_\lambda^\lambda(\bar{x}_{\lambda+1}+\bar{w}_{\lambda+1}^0-\zeta_{\lambda+1},...,\bar{w}_{\lambda+1}^{\lambda-1},\bar{z},\bar{R}_{\lambda+1}-c\zeta_{\lambda+1}+\bar{OF}_{\lambda+1}-(OF_\lambda\mid\bar{OF}_{\lambda+1}),\bar{TF}_{\lambda+1}-c\zeta_{\lambda+1})\\
&\qquad+(1-\bar{\lambda})\hat{G}_\lambda^\lambda(\bar{\bar{x}}_{\lambda+1}+\bar{\bar{w}}_{\lambda+1}^0-\zeta_{\lambda+1},...,\bar{\bar{w}}_{\lambda+1}^{\lambda-1},\bar{\bar{z}},\bar{\bar{R}}_{\lambda+1}-c\zeta_{\lambda+1}+\bar{\bar{OF}}_{\lambda+1}-(OF_\lambda\mid\bar{\bar{OF}}_{\lambda+1}),\bar{\bar{TF}}_{\lambda+1}-c\zeta_{\lambda+1})
\end{aligned}$$

with probability 1. The first inequality is obtained by combining conditions (3) and (5) and the second inequality follows from the convexity of $\hat{G}_{\lambda}^{\lambda}$ in $x_{\lambda}, w_{\lambda}^{0}, ..., w_{\lambda}^{\lambda-1}, R_{\lambda}$ and $TF_{\lambda}$. Therefore, it follows that the function to be minimized in the expression for $\hat{G}_{\lambda+1}^{\lambda}$ is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^{0}, ..., w_{\lambda+1}^{\lambda-1}, R_{\lambda+1}, TF_{\lambda+1}$ and $z$. Then, using proposition B-4 from Heyman and Sobel (1984), we see that $\hat{G}_{\lambda+1}^{\lambda}$ is jointly convex in $x_{\lambda+1}, w_{\lambda+1}^{0}, ..., w_{\lambda+1}^{\lambda-1}, R_{\lambda+1}$ and $TF_{\lambda+1}$. A similar induction argument can be used to demonstrate that $\hat{G}_{t}^{\lambda}$ is jointly convex in $x_t, w_t^0, ..., w_t^{\lambda-1}, R_t$ and $TF_t$ for a general $t$.

Now, we make use of the convex ordering of the funding received in any given period under the two funding scenarios. To begin with, since no orders are placed in periods $t \leq \lambda$, $\hat{G}_t^{\lambda,1}(x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}, R_t, TF_t) = \hat{G}_t^{\lambda,2}(x_t, w_t^0, w_t^1, ..., w_t^{\lambda-1}, R_t, TF_t)$ for $t \leq \lambda$. Now,

$$\begin{aligned}
&\hat{G}_{\lambda+1}^{\lambda,1}(x_{\lambda+1}, w_{\lambda+1}^{0}, ..., w_{\lambda+1}^{\lambda-1}, R_{\lambda+1}, TF_{\lambda+1}) \\
&= \min_{0 \leq z \leq \frac{r_t}{c}} \left\{ \begin{array}{l} cz + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w_{\lambda+1}^{0}]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{(OF_{\lambda+1} - OF_{\lambda}^{1})|OF_{\lambda+1}} \mathsf{E}_{\zeta_{\lambda+1}} \hat{G}_{\lambda}^{\lambda,1}(x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}, ..., w_{\lambda+1}^{\lambda-1}, z, R_{\lambda+1} - c\zeta_{\lambda+1} + (OF_{\lambda+1} - OF_{\lambda}^{1}), TF_{\lambda+1} - c\zeta_{\lambda+1}) \end{array} \right\} \\
&= \min_{0 \leq z \leq \frac{r_t}{c}} \left\{ \begin{array}{l} cz + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w_{\lambda+1}^{0}]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{(OF_{\lambda+1} - OF_{\lambda}^{1})|OF_{\lambda+1}} \mathsf{E}_{\zeta_{\lambda+1}} \hat{G}_{\lambda}^{\lambda,2}(x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}, ..., w_{\lambda+1}^{\lambda-1}, z, R_{\lambda+1} - c\zeta_{\lambda+1} + (OF_{\lambda+1} - OF_{\lambda}^{1}), TF_{\lambda+1} - c\zeta_{\lambda+1}) \end{array} \right\} \\
&\leq \min_{0 \leq z \leq \frac{r_t}{c}} \left\{ \begin{array}{l} cz + b\mathsf{E}_{\zeta_{\lambda+1}}[\zeta_{\lambda+1} - x_{\lambda+1} - w_{\lambda+1}^{0}]^+ + h\mathsf{E}_{\zeta_{\lambda+1}}[x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}]^+ \\ +\mathsf{E}_{(OF_{\lambda+1} - OF_{\lambda}^{2})|OF_{\lambda+1}} \mathsf{E}_{\zeta_{\lambda+1}} \hat{G}_{\lambda}^{\lambda,2}(x_{\lambda+1} + w_{\lambda+1}^{0} - \zeta_{\lambda+1}, ..., w_{\lambda+1}^{\lambda-1}, z, R_{\lambda+1} - c\zeta_{\lambda+1} + (OF_{\lambda+1} - OF_{\lambda}^{2}), TF_{\lambda+1} - c\zeta_{\lambda+1}) \end{array} \right\} \\
&= \hat{G}_{\lambda+1}^{\lambda,2}(x_{\lambda+1}, w_{\lambda+1}^{0}, ..., w_{\lambda+1}^{\lambda-1}, R_{\lambda+1}, TF_{\lambda+1})
\end{aligned}$$

In the above, the first equality is by definition, the second equality follows from the fact that $\hat{G}_{\lambda}^{\lambda,1} = \hat{G}_{\lambda}^{\lambda,2}$, and the inequality follows from the convex ordering $(OF_t - OF_{t-1}^2) \mid OF_t \geq_{cvx} (OF_t - OF_{t-1}^1) \mid OF_t$ and the convexity of $\hat{G}_{\lambda}^{\lambda}$ in $R_{\lambda}$. Thus, $\hat{G}_{\lambda+1}^{\lambda,2} \geq \hat{G}_{\lambda+1}^{\lambda,1}$. Using an induction argument identical to the one used above, it is easy to show that $\hat{G}_t^{\lambda,2} \geq \hat{G}_t^{\lambda,1}$ for any general $t$.

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