Online Supplement: "Advance Demand Information in a Multi-Product System"

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Appendix A – Proofs of Results

Proof of Proposition 1. We first note that the multivariate random vector $(D_1, \cdots, D_{n-1}, X)$ is obtained through a linear transformation of the multivariate normal random vector $(D_1, \cdots, D_n)$. Therefore, $(D_1, \cdots, D_{n-1}, X)$ is itself multivariate normal. The mean of this vector is $(\mu, \cdots, \mu, n\mu)$ and the variance-covariance matrix is

$$\begin{pmatrix} \sigma^2 & 0 & \cdots & \sigma^2 \\ 0 & \sigma^2 & \cdots & \sigma^2 \\ \vdots & \ddots & & \vdots \\ \sigma^2 & \sigma^2 & \cdots & n\sigma^2 \end{pmatrix}.$$

Therefore, the marginal mean of the conditional distribution is

$$= \mu + \begin{pmatrix} \sigma^2 \\ \vdots \\ \sigma^2 \end{pmatrix} \frac{1}{n\sigma^2}(x - n\mu) = \frac{x}{n},$$

while the covariance matrix is

$$\begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{pmatrix} - \begin{pmatrix} \frac{\sigma^2}{n} & \frac{\sigma^2}{n} & \cdots & \frac{\sigma^2}{n} \\ \frac{\sigma^2}{n} & \frac{\sigma^2}{n} & \cdots & \frac{\sigma^2}{n} \\ \vdots & \ddots & & \vdots \\ \frac{\sigma^2}{n} & \frac{\sigma^2}{n} & \cdots & \frac{\sigma^2}{n} \end{pmatrix} = \begin{pmatrix} \frac{n-1}{n}\sigma^2 & -\frac{\sigma^2}{n} & \cdots & -\frac{\sigma^2}{n} \\ -\frac{\sigma^2}{n} & \frac{n-1}{n}\sigma^2 & \cdots & -\frac{\sigma^2}{n} \\ \vdots & \ddots & & \vdots \\ -\frac{\sigma^2}{n} & -\frac{\sigma^2}{n} & \cdots & \frac{n-1}{n}\sigma^2 \end{pmatrix}.$$

From this covariance matrix, corresponding to the conditional demand vector, we conclude that the conditional demands have identical marginal standard deviations $\sigma\sqrt{(n-1)/n}$ and identical pairwise correlation coefficients $-1/(n-1)$. $\square$

Proof of Proposition 2. From Proposition 1, the vector of demand random variables $(D_1, \cdots, D_{n-1})|(\widehat{X} = \widehat{x}, \Delta = \delta)$ is jointly normal with marginal mean $(\widehat{x}+\delta)/n$, marginal standard deviation $\sigma\sqrt{(n-1)/n}$, and pairwise correlation coefficients $-1/(n-1)$. Let Σ denote the covariance matrix of $(D_1,\cdots,D_{n-1})|(\widehat{X}=\widehat{x},\Delta=\delta)$. Since Δ is normally distributed with mean 0 and standard deviation $\sigma\sqrt{n(1-\beta)}$, the density for $(D_1,\cdots,D_{n-1})|\widehat{X}=\widehat{x}$ can be written as

$$f_{(D_1,\cdots,D_{n-1})|\widehat{x}}(d)=\int_{-\infty}^{\infty}\frac{1}{(\sqrt{2\pi})^{n-1}\sqrt{\det(\Sigma)}}\exp\left\{-\frac{1}{2}\left(d-\frac{\hat{x}+\delta}{n}e\right)^t\Sigma^{-1}\left(d-\frac{\hat{x}+\delta}{n}e\right)\right\}$$
$$\times\frac{1}{\sigma\sqrt{n(1-\beta)}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\delta}{\sigma\sqrt{n(1-\beta)}}\right)^2\right\}d\delta, \qquad (1)$$

where $d=(d_1,...,d_{n-1})$ is a generic vector of demand realizations and $e=(1,\cdots,1)$ is the $n-1$-dimensional unit vector. One can show that $\det(\Sigma)=\frac{(\sigma^2)^{n-1}}{n}$ and

$$\Sigma^{-1}=\frac{1}{\sigma^2}\begin{pmatrix}2&\cdots&1\\\vdots&\ddots&\vdots\\1&\cdots&2\end{pmatrix}.$$

It follows that $f_{(D_1,\cdots,D_{n-1})|\widehat{x}}(d)=$

$$\int_{-\infty}^{\infty}\frac{1}{(\sqrt{2\pi}\sigma)^n\sqrt{1-\beta}}\exp\left\{-\frac{1}{2\sigma^2}\left[\left(\sum_{i=1}^{n-1}d_i-\frac{n-1}{n}(\hat{x}+\delta)\right)^2+\sum_{i=1}^{n-1}\left(d_i-\frac{\hat{x}+\delta}{n}\right)+\frac{\delta^2}{n(1-\beta)}\right]\right\}.$$

We can write the integrand as the product of two factors, namely

$$\frac{\sqrt{n}}{(\sqrt{2\pi}\sigma)^{n-1}\sqrt{\beta+n(1-\beta)}}\exp\left\{-\frac{1}{2}\frac{\beta\left(\sum_{i=1}^{n-1}d_i-\frac{n-1}{n}\hat{x}\right)^2+(\beta+n(1-\beta))\sum_{i=1}^{n-1}\left(d_i-\frac{\hat{x}}{n}\right)}{\sigma^2(\beta+n(1-\beta))}\right\}$$

and

$$\frac{\sqrt{\beta+n(1-\beta)}}{\sqrt{2\pi}\sigma\sqrt{n(1-\beta)}}\exp\left\{-\frac{1}{2}\frac{\left(\delta-\frac{n(1-\beta)\sum_{i=1}^{n-1}d_i-(n-1)(1-\beta)\hat{x}}{\beta+n(1-\beta)}\right)^2}{\frac{n(1-\beta)\sigma^2}{\beta+n(1-\beta)}}\right\}.$$

The first factor is independent of δ. The second factor is the density of a normal random variable with mean $\frac{n(1-\beta)\sum_{i=1}^{n-1}d_i-(n-1)(1-\beta)\hat{x}}{\beta+n(1-\beta)}$ and standard deviation $\frac{\sqrt{n(1-\beta)}\sigma}{\sqrt{\beta+n(1-\beta)}}$. Therefore, the density $f_{(D_1,\cdots,D_{n-1})|\widehat{x}}(d)$ is equal to the first factor above. Consider now an $n-1$-dimensional multivariate normal distribution with mean vector $\frac{\hat{x}}{n}e$ and $(n-1)\times(n-1)$-dimensional covariance matrix

$$\bar{\Sigma}=\begin{pmatrix}\sigma^2\left(\frac{n-\beta}{n}\right)&\cdots&-\sigma^2\frac{\beta}{n}\\\vdots&\ddots&\vdots\\-\sigma^2\frac{\beta}{n}&\cdots&\sigma^2\left(\frac{n-\beta}{n}\right)\end{pmatrix}.$$

(These are the mean vector and covariance matrix of an $n-1$-dimensional multivariate normal distribution with identical marginal means $\widehat{x}/n$, identical marginal standard deviations $\sigma\sqrt{(n-\beta)/n}$, and identical pairwise correlation coefficients $-\beta/(n-\beta)$ for $i \neq j$.) One can show by induction that $\det(\bar{\Sigma}) = \frac{(\sigma^2)^{n-1}}{n}(\beta + n(1-\beta))$ and that

$$\bar{\Sigma}^{-1} = \frac{1}{\sigma^2(\beta + n(1-\beta))}\begin{pmatrix} 2\beta + n(1-\beta) & \cdots & \beta \\ \vdots & \ddots & \vdots \\ \beta & \cdots & 2\beta + n(1-\beta) \\ & & \end{pmatrix}.$$

Thus, the density function associated with this $n-1$-dimensional multivariate normal distribution is

$$\frac{\sqrt{n}}{(\sqrt{2\pi}\sigma)^{n-1}\sqrt{\beta + n(1-\beta)}} \exp\left\{-\frac{1}{2}\frac{\beta\left(\sum_{i=1}^{n-1} d_i - \frac{n-1}{n}\hat{x}\right)^2 + (\beta + n(1-\beta))\sum_{i=1}^{n-1}\left(d_i - \frac{\hat{x}}{n}\right)}{\sigma^2(\beta + n(1-\beta))}\right\},$$

which coincides with the first factor above.

So far, we have identified the distribution of $(D_1, \cdots, D_{n-1})|\widehat{X} = \widehat{x}$. We next prove the result for the case of $n = 2$. Given an imperfect observation $\widehat{x}$, the conditional random variable $\Delta|D_1 = d_1$ is normal with mean

$$\left(\frac{1-\beta}{2-\beta}\right)(2d_1 - \widehat{x}) \tag{2}$$

and standard deviation

$$\sigma\sqrt{\frac{2(1-\beta)}{2-\beta}}. \tag{3}$$

We can see the preceding as follows. First, (using $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $k(\cdot)$ as generic notation for densities) note that

$$k(\Delta|D_1 = d_1) = \frac{f(d_1|\delta)g(\delta)}{h(d_1)}, \tag{4}$$

where all of those densities are also conditioned on $\widehat{X} = \widehat{x}$. Note that, given $\widehat{x}$ and δ we know $x = \widehat{x} + \delta$. Then, we use the result in Proposition 1 regarding the marginal distribution of D_1 given perfect information to obtain the density f – i.e., it is normal with mean $\frac{\widehat{x}+\delta}{2}$ and standard deviation $\sigma/\sqrt{2}$. Next, since $\widehat{X}$ and δ are independent, conditioning on $\widehat{X} = \widehat{x}$ does not change the distribution of δ. As a result, g is the density of a normal random variable with mean 0 and standard deviation $\sigma\sqrt{2(1-\beta)}$. Finally, we have shown earlier that the marginal distribution of D_1 given imperfect signal $\widehat{X} = \widehat{x}$ is normal with mean $\frac{\widehat{x}}{2}$ and standard deviation $\sigma\sqrt{\frac{2-\beta}{2}}$, so h is the corresponding density. Plugging all of these densities in (4) and simplifying yields a normal density with mean (2) and standard deviation (3). Thus, since $D_2 = \widehat{X} + \delta - D_1$, then given $\widehat{X} = \widehat{x}$

and $D_1 = d_1$, the conditional random variable $D_2|D_1 = d_1$ is normal with mean

$$\left(\frac{1-\beta}{2-\beta}\right)(2d_1 - \widehat{x}) + \widehat{x} - d_1 = \frac{\widehat{x} - \beta d_1}{2-\beta}$$

and standard deviation as in (3). We can then write $f(d_1, d_2) = f_{D_2|D_1}(d_2|d_1) f_{D_1}(d_1)$ (all of these are conditioned on observing $\widehat{x}$). We know the marginal density $f_{D_1}(d_1)$ given $\widehat{x}$. Writing this out and gathering/simplifying yields a bivariate density having marginal means equal to $\widehat{x}/2$, marginal standard deviations equal to $\sigma\sqrt{1-\frac{\beta}{2}}$, and correlation $\rho = -\frac{\beta}{2-\beta}$.

Next, following the same argument, we can compute the distribution of the conditional random variable $\Delta|(D_1, \cdots, D_{n-1}) = d$ using the result in Proposition 1 and the result demonstrated in the first part of this proof. That allows us to calculate the density of $D_n = \widehat{X} + \delta - \sum_{i=1}^{n-1} D_i$ given $\widehat{X} = \widehat{x}$. Finally, writing $f(d_1, \cdots, d_n) = f_{D_n|(D_1,\cdots,D_{n-1})}(d_n|d) f_{(D_1,\cdots,D_{n-1})}(d)$, we obtain the desired result. □

Proof of Proposition 3. It is easy to verify that $\int_0^{\pi/2} \tilde{q}(\theta)d\theta = \int_{-\sqrt{n}\mu/\sigma}^{\mu/\sigma} \phi(z)dz$, which converges to 1 as $\mu/\sigma \to \infty$. We therefore restrict attention to values of θ in $[0, \pi/2]$. We can now write $\tilde{q}(\theta)$ as

$$\frac{\sqrt{n-1}s_\theta}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(n-1)(\sin\theta - \cos\theta)^2} \left(\int_{-\infty}^0 \frac{r}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2} dr + \int_0^\infty \frac{r}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2} dr \right),$$

so that

$$\tilde{q}(\theta) - q(\theta) = \frac{\sqrt{n-1}s_\theta}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(n-1)(\sin\theta-\cos\theta)^2} \int_{-\infty}^0 \frac{r}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2} dr =$$

$$= \frac{\sqrt{n-1}s_\theta}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(n-1)(\sin\theta-\cos\theta)^2} \; \frac{-2e^{-\frac{1}{2}\frac{m_\theta^2\mu^2}{s_\theta^2\sigma^2}} + \sqrt{2\pi}\frac{m_\theta\mu}{s_\theta\sigma}\left(1 - \text{Erf}\left(\frac{m_\theta\mu}{\sqrt{2}s_\theta\sigma}\right)\right)}{2\sqrt{2\pi}}, \tag{5}$$

where $\text{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. For $\theta \in [0, \pi/2]$, the expression in (5) converges to 0 as $\mu/\sigma \to \infty$. Therefore, $\tilde{q}(\theta) \approx q(\theta)$ for $\theta \in [0, \pi/2]$ and large values of μ/σ. □

Proof of Proposition 4. Treating $\widetilde{h}(r|\theta)$ as an approximation of the density $h(r|\theta)$, we first compute the approximate mean

$$\int_0^\infty r\widetilde{h}(r|\theta)dr = \frac{1}{m_\theta\mu}\int_0^\infty \frac{r^2}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left[\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2\right]} dr \approx \frac{1}{m_\theta\mu}\int_{-\infty}^\infty \frac{r^2}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left[\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2\right]} dr \equiv \widetilde{m}_\theta.$$

This is the second moment of a normal random variable with mean $m_\theta\mu$ and standard deviation $s_\theta\sigma$, so $\int_0^\infty r\widetilde{h}(r|\theta)dr \approx \widetilde{m}_\theta = \frac{m_\theta^2\mu^2 + s_\theta^2\sigma^2}{m_\theta\mu}$. To determine the approximate variance, we first compute

$$\int_0^\infty r^2\widetilde{h}(r|\theta)dr = \frac{1}{m_\theta\mu}\int_0^\infty \frac{r^3}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left[\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2\right]} dr \approx \frac{1}{m_\theta\mu}\int_{-\infty}^\infty \frac{r^3}{\sqrt{2\pi}s_\theta\sigma} e^{-\frac{1}{2}\left[\left(\frac{r-m_\theta\mu}{s_\theta\sigma}\right)^2\right]} dr,$$

which is the third moment of a normal random variable with mean $m_\theta\mu$ and standard deviation $s_\theta\sigma$. Since for any random variable Y, $Var[Y] = E[Y^2] - E[Y]^2$, we can approximate the variance by

$$(m_\theta^2\mu^2 + 3s_\theta^2\sigma^2) - \left(\frac{m_\theta^2\mu^2 + s_\theta^2\sigma^2}{m_\theta\mu}\right)^2 = s_\theta^2\sigma^2\left(1 - \frac{s_\theta^2\sigma^2}{m_\theta^2\mu^2}\right) \equiv \widetilde{s}_\theta^2.$$

(These approximations are accurate for μ/σ relatively large, following similar arguments as in Proposition 3.) Let $m(q|\theta)$ be the density of

$$Q_\theta = \frac{R_\theta - \widetilde{m}_\theta}{\widetilde{s}_\theta},$$

so $m(q|\theta) = \left(\frac{s_\theta\sigma}{m_\theta\mu}\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}\right) h\left(\frac{m_\theta^2\mu^2+s_\theta^2\sigma^2}{m_\theta\mu} + \left(\frac{s_\theta\sigma}{m_\theta\mu}\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}\right) q\Big|\theta\right)$. Then,

$$m(q|\theta) = \left(\frac{s_\theta\sigma}{m_\theta\mu}\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}\right) \times$$

$$\frac{\frac{m_\theta^2\mu^2+s_\theta^2\sigma^2}{m_\theta\mu} + \left(\frac{s_\theta\sigma}{m_\theta\mu}\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}\right) q}{\sqrt{2\pi}(s_\theta\sigma)^2} \frac{e^{-\frac{1}{2}\left[\left(\frac{\frac{m_\theta^2\mu^2+s_\theta^2\sigma^2}{m_\theta\mu} + \left(\frac{s_\theta\sigma}{m_\theta\mu}\sqrt{m_\theta^2\mu^2-s_\theta^2\sigma^2}\right)q - m_\theta\mu}{s_\theta\sigma}\right)^2\right]}}{\phi\left(\frac{m_\theta\mu}{s_\theta\sigma}\right) + \frac{m_\theta\mu}{s_\theta\sigma}\Phi\left(\frac{m_\theta\mu}{s_\theta\sigma}\right)}$$

$$= \frac{\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}}{m_\theta\mu\sqrt{2\pi}}\left(\frac{m_\theta^2\mu^2 + s_\theta^2\sigma^2}{m_\theta^2\mu^2} + \left(\frac{s_\theta\sigma}{m_\theta^2\mu^2}\sqrt{m_\theta^2\mu^2 - s_\theta^2\sigma^2}\right) q\right) \frac{e^{-\frac{1}{2}\left[\left(\frac{s_\theta\sigma}{m_\theta\mu} + \frac{\sqrt{m_\theta^2\mu^2-s_\theta^2\sigma^2}}{m_\theta\mu}q\right)^2\right]}}{\frac{s_\theta\sigma}{m_\theta\mu}\phi\left(\frac{m_\theta\mu}{s_\theta\sigma}\right) + \Phi\left(\frac{m_\theta\mu}{s_\theta\sigma}\right)}$$

$$= \frac{\sqrt{1 - \left(\frac{s_\theta}{m_\theta}\right)^2\left(\frac{\sigma}{\mu}\right)^2}}{\sqrt{2\pi}}\left(1 + \left(\frac{s_\theta}{m_\theta}\right)^2\left(\frac{\sigma}{\mu}\right)^2 + \left(\left(\frac{s_\theta}{m_\theta}\right)\left(\frac{\sigma}{\mu}\right)\sqrt{1 - \left(\frac{s_\theta}{m_\theta}\right)^2\left(\frac{\sigma}{\mu}\right)^2}\right) q\right)$$

$$\times \frac{e^{-\frac{1}{2}\left[\left(\left(\frac{s_\theta}{m_\theta}\right)\left(\frac{\sigma}{\mu}\right) + q\sqrt{1-\left(\frac{s_\theta}{m_\theta}\right)^2\left(\frac{\sigma}{\mu}\right)^2}\right)^2\right]}}{\left(\frac{s_\theta}{m_\theta}\right)\left(\frac{\sigma}{\mu}\right)\phi\left(\left(\frac{m_\theta}{s_\theta}\right)\left(\frac{\mu}{\sigma}\right)\right) + \Phi\left(\left(\frac{m_\theta}{s_\theta}\right)\left(\frac{\mu}{\sigma}\right)\right)}.$$

Consider now any sequences σ_k and μ_k such that $\frac{\mu_k}{\sigma_k} \to \infty$ as $k \to \infty$. Then,

$$\lim_{k\to\infty} m(q|\theta, \mu = \mu_k, \sigma = \sigma_k) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}q^2},$$

which is the density of the standard normal distribution (note that $\lim_{k\to\infty}\phi\left(\left(\frac{m_\theta}{s_\theta}\right)\left(\frac{\mu_k}{\sigma_k}\right)\right) = 0$ and $\lim_{k\to\infty}\Phi\left(\left(\frac{m_\theta}{s_\theta}\right)\left(\frac{\mu_k}{\sigma_k}\right)\right) = 1$). $\square$

Appendix B – Non-Identical and Correlated Demands

The fundamental results of the preceding sections can be extended to the case when demands are correlated and have non-identical distributions. We first discuss how to approach this extension for the case of n products. We then present the details for the case of $n = 2$ products and perfect information.

Let $(D_1, \cdots, D_n)$ be a multivariate normal vector with mean $\mu = (\mu_1, \cdots, \mu_n)$ and covariance matrix Σ. Let σ_i be the standard deviation of the marginal distribution of D_i and ρ_{ij} the correlation between the marginals D_i and D_j. The (perfect) volume signal X is normally distributed with mean $\sum_{i=1}^{n} \mu_i$ and variance $\det(\Sigma)$ (i.e., the determinant of the covariance matrix Σ). We next compute the conditional density function corresponding to the distribution of a particular product demand D_i given the volume information update, i.e., the density of $D_i|X = x$. To that end, first define Σ_{-i} to be the covariance matrix corresponding to the $n-1$ dimensional multivariate normal vector $(D_1, \cdots, D_{i-1}, D_{i+1}, \cdots, D_n)$ – in other words, Σ_{-i} is equal to the matrix Σ with the i-th column and the i-th row removed. We compute

$$f_x(d_i) = \frac{f\left(D_i = d_i, \sum_{j \neq i} D_j = x - d_i\right)}{g(x)},$$

where $g(\cdot)$ is the normal density with mean $\sum_{i=1}^{n} \mu_i$ and standard deviation $\sqrt{\det(\Sigma)}$ and $f(\cdot, \cdot)$ is the density of the bivariate normal with mean $(\mu_i, \sum_{j \neq i} \mu_j)$ and covariance matrix

$$\begin{pmatrix} \sigma_i^2 & \rho_{i,x}\sigma_i\sqrt{\det(\Sigma_{-i})} \\ \rho_{i,x}\sigma_i\sqrt{\det(\Sigma_{-i})} & \det(\Sigma_{-i}) \\ \end{pmatrix},$$

with $\rho_{i,x} = \frac{\sum_{j \neq i} \rho_{ij}\sigma_i\sigma_j}{\sigma_i\sqrt{\det(\Sigma_{-i})}}$. In particular, consider the case of $n = 2$. Let ρ be the correlation between D_1 and D_2. The signal $X = D_1 + D_2$ is normally distributed with mean $\mu_1 + \mu_2$ and standard deviation $\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$. Following the arguments for the general case, the distribution of $D_1|D_1 + D_2 = x$ is given by:

$$f_x(d_1) = \frac{1}{\sqrt{2\pi}s} \exp\left[-\frac{1}{2}\frac{(d_1 - m)^2}{s^2}\right],$$

with

$$m = \frac{\mu_1\sigma_2(\rho\sigma_1 + \sigma_2) - \mu_2\sigma_1(\sigma_1 + \rho\sigma_2) + \sigma_1(\sigma_1 + \rho\sigma_2)x}{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \quad \text{and} \quad s = \frac{\sqrt{1-\rho^2}\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}.$$

This is the density of a normal distribution with mean m and standard deviation s. Note that the standard deviation of the conditional demand distribution is decreasing in the correlation ρ. This makes sense as volume information is "orthogonal" to positive correlation.

We now turn to the case of mix information. The analysis in Section 5 is based on the derivation of the function $q(\theta)$, which is the density of the mix signal information, and the function $h(r|\theta)$. Both functions are built from the bivariate normal density of $\left(D_i, \sum_{j\neq i} D_j/(n-1)\right)$ expressed in polar coordinates, that is, $f(r\cos\theta, r\sin\theta)$. In the case of n non-identical and correlated demand distributions, $f(\cdot,\cdot)$ is the density of the bivariate normal with mean $\left(\mu_i, \sum_{j\neq i}\mu_j/(n-1)\right)$ and covariance matrix

$$\begin{pmatrix} \sigma_i^2 & \rho_{i,x}\sigma_i\frac{\sqrt{\det(\Sigma_{-i})}}{n-1} \\ \rho_{i,x}\sigma_i\frac{\sqrt{\det(\Sigma_{-i})}}{n-1} & \frac{\det(\Sigma_{-i})}{(n-1)^2} \end{pmatrix},$$

where, as before, $\rho_{i,x} = \frac{\sum_{j\neq i}\rho_{ij}\sigma_i\sigma_j}{\sigma_i\sqrt{\det(\Sigma_{-i})}}$. All the steps of the analysis in Section 5 can be extended to the case of a general multivariate normal demand distribution by using the bivariate normal density $f(\cdot,\cdot)$ above. We illustrate this extension with an analysis of the case of two products. We let ρ denote the correlation between D_1 and D_2. We further define

$$m_\theta = \mu_2\sigma_1^2\sin\theta + \mu_1\sigma_2^2\cos\theta - \rho\sigma_1\sigma_2(\mu_1\sin\theta + \mu_2\cos\theta)$$

and

$$s_\theta = \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta - \rho\sigma_1\sigma_2\sin(2\theta).$$

Recall that the density of the angle θ is given by $q(\theta) = \int_0^\infty rf(r\cos\theta, r\sin\theta)dr$, with the appropriate bivariate density $f(\cdot,\cdot)$. For the case of two products,

$$q(\theta) = \frac{1}{\sqrt{2\pi s_\theta}}e^{-\frac{1}{2}\frac{(\mu_1\sin\theta-\mu_2\cos\theta)^2}{s_\theta}}\int_0^\infty \frac{r}{\sqrt{2\pi}\sqrt{1-\rho^2}\frac{\sigma_1\sigma_2}{\sqrt{s_\theta}}}e^{-\frac{1}{2}\left(\frac{r-m_\theta/s_\theta}{\sqrt{1-\rho^2}\frac{\sigma_1\sigma_2}{\sqrt{s_\theta}}}\right)^2}dr.$$

Following the steps in Section 5, we next determine an approximation $\tilde{q}(\theta)$ of the density of the offset normal distribution. This approximation is presented in the next result.

Proposition 5 *Let*

$$\tilde{q}(\theta) = \frac{m_\theta}{\sqrt{2\pi}(s_\theta)^{3/2}}e^{-\frac{1}{2}\frac{(\mu_1\sin\theta-\mu_2\cos\theta)^2}{s_\theta}}.$$

Then, $\lim_{\mu/\sigma\to\infty}\ \tilde{q}(\theta) - q(\theta) = 0$ *for* $\theta\in[0,\pi/2]$.

The proof of Proposition 5 is similar to that of Proposition 1, and it is therefore omitted. We next use $\tilde{q}(\theta)$ to determine the approximate density function of R_θ, which denotes the value of the radius R conditional on a realization of the angle Θ.

Proposition 6 *The conditional radius R_θ is approximately normal with mean* $\frac{m_\theta}{s_\theta} + (1-\rho^2)\ \frac{\sigma_1^2\sigma_2^2}{m_\theta}$ *and variance* $\sigma_1^2\sigma_2^2(1-\rho^2)\left[\frac{1}{s_\theta} - (1-\rho^2)\ \frac{\sigma_1^2\sigma_2^2}{m_\theta^2}\right]$.

This result follows by computing the function $\tilde{h}(r|\theta) = \frac{rf(r\cos\theta, r\sin\theta)}{\tilde{q}(\theta)}$ (associated with the conditional density of the radius) for the appropriate bivariate density $f(\cdot,\cdot)$ and the approximate density of the offset normal $\tilde{q}(\theta)$ derived in Proposition 5. Based on the function $\tilde{h}(r|\theta)$, we calculate $\int_{-\infty}^{\infty} r\tilde{h}(r|\theta)dr$ and $\int_{-\infty}^{\infty} r^2\tilde{h}(r|\theta)dr$ to derive the approximate mean and variance of R_θ, respectively. We conclude from Proposition 6 that $D_1|\Theta = \theta$ is approximately normal (for sufficiently large μ/σ), with mean

$$\mu_1(\theta) = \cos\theta\left[\frac{m_\theta}{s_\theta} + (1-\rho^2)\ \frac{\sigma_1^2\sigma_2^2}{m_\theta}\right]$$

and standard deviation

$$\sigma_1(\theta) = \cos\theta\ \sigma_1\sigma_2\sqrt{(1-\rho^2)\left[\frac{1}{s_\theta} - (1-\rho^2)\ \frac{\sigma_1^2\sigma_2^2}{m_\theta^2}\right]}.$$

Similarly, $D_2|\Theta = \theta$ is approximately normal with mean $\tan\theta\mu_1(\theta)$ and standard deviation $\tan\theta\sigma_1(\theta)$. One can verify that these distributions coincide with those derived in Section 5 when $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\rho = 0$.

Appendix C

In this appendix we provide evidence in support of using the random variable $D_i|A_i$ for the random variable $D_i|\{A_j, j = 1, ..., n\}$. This discussion is relevant for $n \geq 3$ products. The idea behind this restricted information structure is that a product's own mix signal (A_i) contains the most relevant information about that product's demand distribution, and that a limited amount of information is lost by ignoring the mix signals of other products. For tractability, we present analytical arguments based on a simple three-point demand distribution. (We also experimented with various numerical approaches for exploring this assumption using the actual demand distributions, and the results were consistent with the arguments below – with high probability, mix signals of other products had significantly less impact on a given product's demand distribution than that product's own mix signal.)

Consider n i.i.d. random variables with distribution:

$$D_i = \begin{cases} \mu - a\sigma & \text{with probability } 1/3 \\ \mu & \text{with probability } 1/3 \\ \mu + a\sigma & \text{with probability } 1/3 \end{cases}$$

The calculations below do not depend on the value of a.

For convenience, we work with the market share random variable $D_1/\sum_{j=1}^n D_j$, which is equivalent to A_1. This random variable can take one of the following values:

$$\begin{cases} \frac{1}{n} & \\ \frac{\mu}{n\mu+ka\sigma} & \text{for } \; -n+1 \leq k \leq n-1 \\ \frac{\mu-a\sigma}{n\mu+ia\sigma} & \text{for } \; -n+1 \leq i \leq n-2 \\ \frac{\mu+a\sigma}{n\mu+ia\sigma} & \text{for } \; -n+2 \leq j \leq n-1 \end{cases}$$

Each outcome in the last three sets can only occur for a given value of D_1. Therefore, having information that the outcome of $D_1/\sum_{j=1}^n D_j$ is equal to any of the values in one of the last three sets above uniquely characterizes the outcome of D_1. For example, if $D_1/\sum_{j=1}^n D_j = \frac{\mu}{n\mu+ka\sigma}$ then it must be that $D_1 = \mu$. This unique identification of the outcomes occurs as long as $\mu/\sigma > a$, which is the case in the settings of interest. This means that having information about the market share of any other product will not be of help in those cases, as D_1 is already identified.

However, if $D_1/\sum_{j=1}^n D_j = 1/n$, then various scenarios are possible. For example, we could have that $D_j = \mu$ for all j, or that $D_j = \mu \pm a\sigma$ for all j, or, if n is odd, we could have that $D_1 = \mu$, $D_j = \mu - a\sigma$ for $j = 2, \cdots, (n-1)/2$, and $D_j = \mu + a\sigma$ for $j = (n+1)/2, \cdots, n$. Other similar examples exist. Now, if $D_1/\sum_{j=1}^n D_j = 1/n$ and the outcome of all other market shares is also $1/n$, then nothing new is learned – D_1 can take any of the three original values with the same probability 1/3. However, if at least one of the other market share outcomes is not $1/n$, then it must mean that not all the demand outcomes are equal, so that additional market share information will help to provide more accuracy on the outcome of D_1.

We want to compute the probability that the outcome of the market shares will be such that having information about $D_i/\sum_{j=1}^n D_j$ for $i \neq 1$ will improve the accuracy of the distribution of D_1. We then count the outcomes in which this will be the case. If n is even, then there are

$$E_n \stackrel{def}{=} \sum_{k=1}^{n/2-1} \begin{pmatrix} n-2k \\ n/2-k \end{pmatrix}$$

possible combinations of $D_i's$ that will lead to $D_1/\sum_{j=1}^n D_j = 1/n$, while having that not all $D_i's$ are equal. Each term k in the sum corresponds to the case in which $2k$ demand outcomes are equal to μ, including that of D_1, while half of the remaining $n-2k$ demands are equal to $\mu - a\sigma$ and the other half are equal to $\mu + a\sigma$. If n is odd, then there are

$$O_n \stackrel{def}{=} \sum_{k=0}^{(n-1)/2-1} \begin{pmatrix} n-1-2k \\ (n-1)/2-k \end{pmatrix}$$

such combinations. In either case, the number of outcomes needs to be multiplied by the corresponding probability of occurrence, which is $(1/3)^n$. It is easy to see that $O_n = E_{n+1}$ for n odd, and $(1/3)^n > (1/3)^{n+1}$, so we report the probabilities for n odd (and those for even n will be 1/3 smaller than the corresponding probability for $n-1$). Therefore, using the formula

$$(1/3)^n \sum_{k=0}^{(n-1)/2-1} \begin{pmatrix} n-1-2k \\ (n-1)/2-k \end{pmatrix},$$

we have that the probability that information about other market shares will help is 0.074 for $n = 3$, 0.033 for $n = 5$, 0.013 for $n = 7$, 0.005 for $n = 9$, 0.002 for $n = 11$, and less than 0.001 for larger n. We conclude that: (1) In this setting, the maximum loss (across all parameters) for not using the other market shares is 0.074; (2) The loss decreases with n.