Online Supplement to
Determining Optimal Parameters of Expediting Policies

A. Proofs

Proof of Proposition 1. If one or more of the previous l orders have been expedited in period t, then $\sum_{i=1}^{l} o^+_{t,i} = K$. Therefore, if $\sum_{i=1}^{l} o^+_{t,i} < K$, then none of the previous l orders has been expedited in period t. The newest outstanding order, $o^+_{t,1}$, corresponds to the demand of the previous period, d_{t-1}, and all other outstanding orders, $o^+_{t,2}, \ldots, o^+_{t,l}$, correspond to their respective outstanding order of the previous period, $o^+_{t-1,1}, \ldots, o^+_{t-1,l-1}$:

$$o^+_{t,i} = \begin{cases} d_{t-1}, & \text{for } i = 1, \\ o^+_{t-1,i-1}, & \text{for } i = 2 \ldots l. \end{cases}$$

It remains to be proven that $o^+_{t,i} = d_{t-i}$ for $i = 2 \ldots l$. Note that

$$\sum_{i=1}^{l-1} o^+_{t-1,i} = \sum_{i=2}^{l} o^+_{t-1,i-1} = \sum_{i=2}^{l} o^+_{t,i} \le \sum_{i=1}^{l} o^+_{t,i} < K,$$

i.e., the total number of units outstanding from the previous $l-1$ orders after the expediting decision in period $t-1$ is below K. Therefore, along the same line of arguments as above, we obtain

$$o^+_{t,i} = o^+_{t-1,i-1} = \begin{cases} d_{t-2}, & \text{for } i = 2, \\ o^+_{t-2,i-2}, & \text{for } i = 3 \ldots l, \end{cases}$$

and $\sum_{i=1}^{l-2} o^+_{t-2,i} < K$. Iteratively, we can show that we obtain $o^+_{t,i} = d_{t-i}$ for $i = 1 \ldots l$. □

Proof of Proposition 2. To prove that $S^*(K) = (F_O^{L_e,L_n})^{-1}(b/(b+h))$ is increasing in K, we prove that $F_O^{L_e,L_n}(k)$ is decreasing in K. $F_O^{L_e,L_n}(\cdot)$ can be written as (see Equation (11))

$$F_O^{L_e,L_n}(k) = \sum_{j=0}^{k} f_O^{L_e,L_n}(j) = \sum_{j=0}^{k} \sum_{i=0}^{j} f_D^{L_n+1}(i) f_{O^+}^{L_e}(j-i) = \cdots = \sum_{i=0}^{k} f_D^{L_n+1}(i) F_{O^+}^{L_e}(k-i), \tag{18}$$

where $F_{O^+}^{L_e}(\cdot)$ can be determined from Equation (3),

$$F_{O^+}^{L_e}(k) = \begin{cases} F_D^{L_e}(k), & \text{for } k < K, \\ 1, & \text{for } k \ge K. \end{cases} \tag{19}$$

From Equation (18), we obtain the first-order finite difference of $F_O^{L_e,L_n}(\cdot)$ with respect to K,

$$\Delta_K F_O^{L_e,L_n}(k) = \cdots = \sum_{i=0}^{k} f_D^{L_n+1}(i) \Delta_K F_{O^+}^{L_e}(k-i)$$

and from Equation (19), we obtain the first-order finite difference of $F_{O^+}^{L_e}(\cdot)$ with respect to K,

$$\Delta_K F_{O^+}^{L_e}(k) = \begin{cases} F_D^{L_e}(K) - 1, & \text{for } k = K, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\Delta_K F_{O^+}^{L_e}(k) \le 0$, which implies that $\Delta_K F_O^{L_e,L_n}(k) \le 0$. Thus, $F_O^{L_e,L_n}(k)$ is decreasing in K and $S^*(K)$ is increasing in K. □

Proof of Proposition 3. To prove quasi-convexity of $Z^s(S,K)$ in K for a given S, we show that $Z^s(S,K)$ is decreasing in K for $K < K^*(S)$ and increasing in K for $K \geq K^*(S)$ by showing that the first-order finite difference $\Delta_K Z^s(S,K)$ (Equation (16)) is negative for $K < K^*(S)$ and positive for $K \geq K^*(S)$.

Note that the first term of Equation (16), $1 - F_D^{L_e}(K)$, is always positive and it suffices to show that the second term of Equation (16), $b - c_v - (b+h)F_D^{L_n+1}(S-K-1)$, is negative for $K < K^*(S)$ and positive for $K \geq K^*(S)$.

$K < K^*(S)$: From Equation (17), we know that $K^*(S)$ is the largest integer satisfying $F_D^{L_n+1}(S-K^*(S)) \geq (b-c_v)/(b+h)$. Thus, $F_D^{L_n+1}(S-K) \geq (b-c_v)/(b+h)$ for $K \leq K^*(S)$ and we obtain $b-c_v-(b+h)F_D^{L_n+1}(S-K-1) \leq 0$ for $K < K^*(S)$.

$K \geq K^*(S)$: Because $K^*(S)$ is the largest integer satisfying $F_D^{L_n+1}(S-K^*(S)) \geq (b-c_v)/(b+h)$, we know that $F_D^{L_n+1}(S-K) < (b-c_v)/(b+h)$ for $K \geq K^*(S)+1$. Thus, we obtain $b-c_v-(b+h)F_D^{L_n+1}(S-K-1) > 0$ for $K \geq K^*(S)$. ☐

Proof of Proposition 4. Because $K^*(S) = S - \left(F_D^{L_n+1}\right)^{-1}\left(\frac{b-c}{b+h}\right)$, we obtain $K^*(S+k) = K^*(S)+k$, i.e., if S increases, then $K^*(S)$ increases by the same quantity. Therefore, we can show that $Z^s(S,K^*(S))$ is convex in S by showing that $Z^s(S,K)$ is convex in S and K, given $K = K^*(S)$.

The second-order finite difference function of $Z^s(S,K)$ with respect to S and K is

$$\Delta^2_{S,K} Z^s(S,K) = (b+h)\left[\sum_{j=0}^{K-1} f_D^{L_e}(j) f_D^{L_n+1}(S-j) + f_D^{L_e}(K)F_D^{L_n+1}(S-K)\right] - (b-c_v) f_D^{L_e}(K).$$

Setting $K = K^*(S)$, we obtain

$$\begin{aligned}\Delta^2_{S,K} Z^s(S,K^*(S)) = (b+h) &\sum_{j=0}^{K^*(S)-1} f_D^{L_e}(j) f_D^{L_n+1}(S-j)\\ &+ (b+h) f_D^{L_e}(K^*(S))F_D^{L_n+1}(S-K^*(S)) - (b-c_v) f_D^{L_e}(K^*(S)).\end{aligned}$$

Because $F_D^{L_n+1}(S-K^*(S)) = F_D^{L_n+1}((F_D^{L_n+1})^{-1}(\frac{b-c}{b+h})) \geq \frac{b-c}{b+h}$, we can replace $F_D^{L_n+1}(S-K^*(S))$ by $\frac{b-c}{b+h}$ to obtain the lower bound

$$\begin{aligned}\Delta^2_{S,K} Z^s(S,K^*(S)) \geq (b+h) &\sum_{j=0}^{K^*(S)-1} f_D^{L_e}(j) f_D^{L_n+1}(S-j)\\ &+ (b+h) f_D^{L_e}(K^*(S))\frac{b-c}{b+h} - (b-c_v) f_D^{L_e}(K^*(S))\\ = (b+h) &\sum_{j=0}^{K^*(S)-1} f_D^{L_e}(j) f_D^{L_n+1}(S-j)\\ \geq 0&,\end{aligned}$$

which proves that $Z^s(S,K^*(S))$ is convex is S. ☐

Proof of Proposition 5. The proof of Proposition 5 uses the following Lemma:

LEMMA 1. *The expected number of orders from which units are expedited, $\mathbb{E}C^o(K)$, has the following property:*

$$\Delta_K \mathbb{E}C^o(K) \geq F_D^{L_e}(K) - 1.$$

Proof of Lemma 1. Before we derive the lower bound on $\Delta_K \mathbb{E}C^o(K) = -\sum_{l=1}^{L_e} \Delta_K P(e_l = 0)$, we calculate an upper bound on $\Delta_K P(e_l = 0)$, the probability that no units from the order placed l periods before are expedited with respect to the expediting level K. For $l \geq 3$, we have

$$\begin{aligned}\Delta_K P(e_l = 0) = {} & f_D^l(K+1) - f_D(0) f_D^{l-1}(K+1) - f_D(0)\left[f_D^{l-1}(K+1) - f_D(0) f_D^{l-2}(K+1)\right] \\ & - [1 - f_D(0)][1 - f_D(0)] f_D^{l-2}(K).\end{aligned}$$

Because $f_D^{l-1}(K+1) - f_D(0) f_D^{l-2}(K+1) = \sum_{k=0}^{K} f_D^{l-2}(k) f_D(K+1-k) \geq 0$, we can simplify this expression to

$$\Delta_K P(e_l = 0) \leq f_D^l(K+1) - f_D(0) f_D^{l-1}(K+1).$$

Similarly, we obtain $\Delta_K P(e_2 = 0) \leq f_D^2(K+1) - f_D(0) f_D^1(K+1)$ for $l = 2$ and $\Delta_K P(e_1 = 0) \leq f_D^1(K+1)$ for $l = 1$. Therefore,

$$\Delta_K \mathbb{E}C^o(K) \geq -f_D^l(K+1) + \sum_{l=2}^{L_e} \left[f_D(0) f_D^{l-1}(K+1) - f_D^l(K+1)\right].$$

After decomposing the sum, we shift the index of the first sum by 1 and re-compose the sums. This results in

$$\Delta_K \mathbb{E}C^o(K) \geq -f_D^{L_e}(K+1) + \sum_{l=1}^{L_e - 1} [f_D(0) - 1] f_D^l(K+1).$$

Using

$$\begin{aligned}[f_D(0) - 1] f_D^l(K+1) &= F_D^{l+1}(K+1) - F_D^{l+1}(K+1) + f_D(0) f_D^l(K+1) - f_D^l(K+1) \\ &= F_D^{l+1}(K+1) - \sum_{k=0}^{K} f_D^l(k) F_D(K+1-k) - f_D^l(K+1) \\ &\geq F_D^{l+1}(K+1) - \sum_{k=0}^{K} f_D^l(k) - f_D^l(K+1) \\ &= F_D^{l+1}(K+1) - F_D^l(K+1),\end{aligned}$$

we obtain

$$\Delta_K \mathbb{E}C^o(K) \geq -f_D^{L_e}(K+1) + \sum_{l=1}^{L_e - 1} \left[F_D^{l+1}(K+1) - F_D^l(K+1)\right].$$

After decomposing the sum, shifting the index of the first sum by 1, and simplifying results, we obtain

$$\Delta_K \mathbb{E}C^o(K) \geq F_D^{L_e}(K) - F_D^1(K+1) \geq F_D^{L_e}(K) - 1.$$

□

The inequality $0 \leq K^*$ follows from the definition of the expediting level. To prove that $K^* \leq \widetilde{K}^*$ holds, it remains to be shown that $\widetilde{Z}(S^*(K), K) - Z(S^*(K), K)$ is decreasing in K.

The cost difference between $Z(S^*(K), K)$ and $\widetilde{Z}(S^*(K), K)$ for a given expediting level K can be fully attributed to the difference in the expected expediting cost, because $S^*(K)$ is independent of the expediting cost components. To prove that $\widetilde{Z}(S^*(K), K) - Z(S^*(K), K)$ is decreasing in K, we show that

$\Delta_K \left[\widetilde{Z}(S^*(K),K) - Z(S^*(K),K)\right] = c_f[\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^f] + c_b[\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^b] + c_o[\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^o]$ is negative.

The first term can be written as $\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^f = \left[F_D^{L_e}(K) - 1\right] + \sum_{k=0}^{K} f_D^{L_e-1}(k) f_D(K+1-k) = \left[F_D^{L_e}(K) - 1\right] + f_D^{L_e}(K+1) - f_D^{L_e-1}(K+1) f_D(0) \leq F_D^{L_e}(K+1) - 1$ and is negative. The second term can be written as $\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^b = \left[F_D^{L_e}(K) - 1\right] + \sum_{i=0}^{\infty} \sum_{k=0}^{K} f_D^{L_e-1}(k) f_D(K+1+i\cdot q-k) = \left[F_D^{L_e}(K) - 1\right] + \sum_{i=0}^{\infty} \left[f_D^{L_e}(K+1+i\cdot q) - \sum_{k=K+1}^{K+1+i\cdot q} f_D^{L_e-1}(k) f_D(K+1+i\cdot q-k)\right] \leq F_D^{L_e}(K) + \sum_{i=0}^{\infty} f_D^{L_e}(K+1+i\cdot q) - 1$ and is also negative. The third term can be written as $\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^o = \left[F_D^{L_e}(K) - 1\right] + \sum_{l=1}^{L_e} \Delta_K P(e_l = 0)$. Using Lemma 1, we obtain $\Delta_K \mathbb{E}C^v - \Delta_K \mathbb{E}C^o \leq \left[F_D^{L_e}(K) - 1\right] - \left[F_D^{L_e}(K) - 1\right] = 0$, which completes the proof. □

B. First and Second Order Finite Differences of the Objective Functions

In this section, we derive the first- and second-order finite differences of the simplified objective function

$$Z^s(S,K) = h\mathbb{E}I(S,K) + b\mathbb{E}B(S,K) + c_v\mathbb{E}C^v(K),$$

with the expected on-hand inventory level $\mathbb{E}I(S,K) = \sum_{k=0}^{S}(S-k) f_O^{L_e,L_n}(k)$, the expected backorder level, $\mathbb{E}B(S,K) = \sum_{k=S}^{\infty}(k-S) f_O^{L_e,L_n}(k)$, and the expected total number of expedited lead time periods $\mathbb{E}C^v(K) = \sum_{l=1}^{L_e} \sum_{k=K}^{\infty}(k-K) f_{O^-}^{l}(k)$. Then, we derive the first-order finite differences of the probability that units are expedited in a period, $\mathbb{E}C^f(K) = \left[1 - F_{O^-}^{L_e}(K)\right]$, the expected number of expediting batches, $\mathbb{E}C^b(K) = \sum_{i=0}^{\infty}\left[1 - F_{O^-}^{L_e}(K+i\cdot q)\right]$, and the expected number of orders from which units are expedited, $\mathbb{E}C^o(K) = L_e - \sum_{l=1}^{L_e}\left[\sum_{m=0}^{K}\sum_{k=1}^{m} f_{O^-,O_l^-}^{l}(m,k) + \sum_{m=0}^{\infty} f_{O^-,O_l^-}^{l}(m,0)\right]$.

First- and Second-Order Difference of $Z^s(S,K)$ w.r.t. S

The first-order finite difference of the expected on-hand inventory level, $\mathbb{E}I(S,K)$, with respect to the order-up-to level S is

$$\Delta_S \mathbb{E}I(S,K) = \sum_{k=0}^{S+1}(S+1-k) f_O^{L_e,L_n}(k) - \sum_{k=0}^{S}(S-k) f_O^{L_e,L_n}(k) = F_O^{L_e,L_n}(S).$$

The expected pipeline stock at the end of a period is $\mathbb{E}P(K) = \sum_{k=0}^{\infty} k f_O^{L_e,L_n}(k)$. Note that $\mathbb{E}P(K)$ is independent of S and, thus, $\Delta_S \mathbb{E}P(K) = 0$. The first-order finite difference of the expected backorder level, $\mathbb{E}B(S,K) = \mathbb{E}I(S,K) + \mathbb{E}P(K) - S$, with respect to the order-up-to level S can be written as

$$\Delta_S \mathbb{E}B(S,K) = \Delta_S \mathbb{E}I(S,K) + \Delta_S \mathbb{E}P(K) - 1 = F_O^{L_e,L_n}(S) - 1.$$

The expected total number of expedited lead time periods, $\mathbb{E}C^v(K)$, is also independent of S and we obtain $\Delta_S \mathbb{E}C^v(K) = 0$. Thus, the first-order finite difference of our simplified objective function with respect to the order-up-to level S is

$$\Delta_S Z^s(S,K) = h\Delta_S \mathbb{E}I(S,K) + b\Delta_S \mathbb{E}B(S,K) + c_v\Delta_S \mathbb{E}C^v(K) = (b+h)F_O^{L_e,L_n}(S) - b,$$

and the second-order finite difference of our simplified objective function with respect to the order-up-to level S can be written as

$$\Delta_S^2 Z^s(S,K) = [(b+h)F_O^{L_e,L_n}(S) - b] - [(b+h)F_O^{L_e,L_n}(S-1) - b] = (b+h) f_O^{L_e,L_n}(S).$$

First-Order Difference of $Z^s(S,K)$ w.r.t. K

From Equation (3), we obtain the first-order finite difference of the distribution of the number of units outstanding from the previous l orders after expediting, $f^l_{O+}(\cdot)$, with respect to the expediting level K,

$$\Delta_K f^l_{O+}(k) = f^{l,K+1}_{O+}(k) - f^{l,K}_{O+}(k) = \begin{cases} F^l_D(K) - 1, & \text{for } k = K,\\ 1 - F^l_D(K), & \text{for } k = K+1,\\ 0, & \text{else.} \end{cases} \tag{20}$$

Using Equation (20), the first-order finite difference of the distribution of the number of units outstanding from the previous l orders at the end of the period, $f^l_O(\cdot)$, with respect to the expediting level K can be written as

$$\Delta_K f^l_O(k) = \sum_{j=0}^{k} \Delta_K f^l_{O+}(j) f_D(k-j) = \left[F^l_D(K) - 1\right]\left[f_D(k-K) - f_D(k-(K+1))\right]. \tag{21}$$

From Equations (11) and (20), we obtain the first-order finite difference of the distribution of the number of units outstanding at the end of the period with expeditable lead time L_e and non-expeditable lead time L_n, $f^{L_e,L_n}_O(\cdot)$, with respect to the expediting level K,

$$\Delta_K f^{L_e,L_n}_O(k) = \sum_{j=0}^{k} \Delta_K f^{L_e}_{O+}(j) f^{L_n+1}_D(k-j) = \left[F^{L_e}_D(K) - 1\right]\left[f^{L_n+1}_D(k-K) - f^{L_n+1}_D(k-(K+1))\right]. \tag{22}$$

The first-order finite difference of the expected on-hand inventory level, $\mathbb{E}I(S,K)$, with respect to the expediting level K can be written as

$$\Delta_K \mathbb{E}I(S,K) = \sum_{k=0}^{S} (S-k) f^{L_e,L_n,K+1}_O(k) - \sum_{k=0}^{S} (S-k) f^{L_e,L_n,K}_O(k) = \sum_{k=0}^{S} (S-k) \Delta_K f^{L_e,L_n}_O(k).$$

Using $\Delta_K f^{L_e,L_n}_O(k) = \left[F^{L_e}_D(K) - 1\right]\left[f^{L_n+1}_D(k-K) - f^{L_n+1}_D(k-(K+1))\right]$ from Equation (22) and re-arranging terms results in

$$\Delta_K \mathbb{E}I(S,K) = \left[F^{L_e}_D(K) - 1\right]\left[\sum_{k=K}^{S} (S-k) f^{L_n+1}_D(k-K) - \sum_{k=K+1}^{S} (S-k) f^{L_n+1}_D(k-(K+1))\right].$$

We now shift the indices of the two sums by K and $K+1$, respectively, and obtain

$$\begin{aligned} \Delta_K \mathbb{E}I(S,K) &= \left[F^{L_e}_D(K) - 1\right]\left[\sum_{k=0}^{S-K} (S-K-k) f^{L_n+1}_D(k) - \sum_{k=0}^{S-K-1} (S-K-1-k) f^{L_n+1}_D(k)\right] \\ &= \left[F^{L_e}_D(K) - 1\right] F^{L_n+1}_D(S-K-1). \end{aligned}$$

The first-order finite difference of the expected pipeline stock, $\mathbb{E}P(K)$, with respect to the expediting level K can be determined similarly and is $\Delta_K \mathbb{E}P(K) = 1 - F^{L_e}_D(K)$. The first-order finite difference of the expected backorder level, $\mathbb{E}B(S,K) = \mathbb{E}I(S,K) + \mathbb{E}P(K) - S$, with respect to the expediting level K can be written as

$$\Delta_K \mathbb{E}B(S,K) = \Delta_K \mathbb{E}I(S,K) + \Delta_K \mathbb{E}P(K) = \left[1 - F^{L_e}_D(K)\right]\left[1 - F^{L_n+1}_D(S-K-1)\right].$$

Using $f^l_{O-}(k) = f^{l-1}_O(k)$ from Equation (5) and $\Delta_K f^l_O(k) = [F^l_D(K) - 1]\,[f_D(k-K) - f_D(k-(K+1))]$ from Equation (21), the first-order finite difference of the expected total number of expedited lead time periods, $\mathbb{E}C^v(K)$, with respect to the expediting level K can be re-formulated as

$$\Delta_K \mathbb{E}C^v(K) = \sum_{l=1}^{L_e}\left[\sum_{k=0}^{K} f^{l-1,K+1}_O(k) - 1\right] + \sum_{l=1}^{L_e}\left[F^{l-1}_D(K) - 1\right]\left[\sum_{k=1}^{\infty} k f_D(k) - \sum_{k=0}^{\infty} (k+1) f_D(k)\right].$$

For $k \leq K < K+1$, we obtain $f_O^{l-1,K+1}(k) = f_D^l(k)$ from Equation (3) and (4). This results in

$$\Delta_K \mathbb{E}C^v(K) = \sum_{l=1}^{L_e} F_D^l(K) - \sum_{l=1}^{L_e} F_D^{l-1}(K) = F_D^{L_e}(K) - 1.$$

Thus, the first-order finite difference of our simplified objective function with respect to the expediting level K can be written as

$$\begin{aligned} \Delta_K Z^s(S,K) &= h\Delta_K \mathbb{E}I(S,K) + b\Delta_K \mathbb{E}B(S,K) + c_v \Delta_K \mathbb{E}C^v(K) \\ &= \left[1 - F_D^{L_e}(K)\right]\left[b - c_v - (b+h)F_D^{L_n+1}(S-K-1)\right]. \end{aligned}$$

First- and Second-Order Difference of $Z^s(S,K)$ w.r.t. S and K

The first-order finite difference of the expected on-hand inventory level, $\mathbb{E}I(S,K)$, with respect to the order-up-to level S and the expediting level K can be re-formulated as

$$\Delta_{S,K}\mathbb{E}I(S,K) = \sum_{k=0}^{S}\sum_{j=0}^{k} f_{O^+}^{L_e,K+1}(j) f_D^{L_n+1}(k-j) + \Delta_K \mathbb{E}I(S,K). \tag{23}$$

Since we already derived a simplified expression for $\Delta_K \mathbb{E}I(S,K)$, we now focus on the first term of Equation (23). Re-arranging the sums and shifting the index of the last sum results in

$$\sum_{k=0}^{S}\sum_{j=0}^{k} f_{O^+}^{L_e,K+1}(j) f_D^{L_n+1}(k-j) = \sum_{j=0}^{K+1} f_{O^+}^{L_e,K+1}(j) F_D^{L_n+1}(S-j).$$

For $j < K+1$, we obtain $f_{O^+}^{L_e,K+1}(j) = f_D^{L_e}(j)$ from Equation (3) and for $j > K+1$, we have $f_{O^+}^{L_e,K+1}(j) = 0$. This results in

$$\sum_{k=0}^{S}\sum_{j=0}^{k} f_{O^+}^{L_e,K+1}(j) f_D^{L_n+1}(k-j) = \sum_{j=0}^{K} f_D^{L_e}(j) F_D^{L_n+1}(S-j) + [1 - F_D^{L_e}(K)]F_D^{L_n+1}(S-K-1),$$

for the first term of Equation (23). With $\Delta_K \mathbb{E}I(S,K) = [F_D^{L_e}(K) - 1]F_D^{L_n+1}(S-K-1)$ for the second term of Equation (23), we obtain

$$\Delta_{S,K}\mathbb{E}I(S,K) = \sum_{j=0}^{K} f_D^{L_e}(j) F_D^{L_n+1}(S-j).$$

The first-order finite difference of the expected pipeline stock, $\mathbb{E}P(K)$, with respect to S and K corresponds to the first-order finite difference of the expected pipeline stock, $\mathbb{E}P(K)$, with respect to K, because $\mathbb{E}P(K)$ is independent of S. The first-order finite difference of the expected backorder level, $\mathbb{E}B(S,K) = \mathbb{E}I(S,K) + \mathbb{E}P(K) - S$, with respect to the order-up-to level S and the expediting level K can be written as

$$\Delta_{S,K}\mathbb{E}B(S,K) = \Delta_{S,K}\mathbb{E}I(S,K) + \Delta_K \mathbb{E}P(K) - 1 = \sum_{j=0}^{K} f_D^{L_e}(j) F_D^{L_n+1}(S-j) + \left[1 - F_D^{L_e}(K)\right] - 1.$$

Note that the expected total number of expedited lead time periods, $\mathbb{E}C^v(K)$, is independent of S. Thus, the first-order finite difference of our simplified objective function with respect to the order-up-to level S and the expediting level K is

$$\Delta_{S,K} Z^s(S,K) = h\Delta_{S,K}\mathbb{E}I(S,K) + b\Delta_{S,K}\mathbb{E}B(S,K) + c_v \Delta_K \mathbb{E}C^v(K)$$

$$= (b+h)\sum_{j=0}^{K} f_D^{L_e}(j)F_D^{L_n+1}(S-j) + (b-c_v)\left[1-F_D^{L_e}(K)\right] - b,$$

and the second-order finite difference of our simplified objective function with respect to the order-up-to level S and the expediting level K can be written as

$$\begin{aligned}\Delta^2_{S,K} Z^s(S,K) = {} & (b+h)\sum_{j=0}^{K} f_D^{L_e}(j)F_D^{L_n+1}(S-j) + (b-c_v)\left[1-F_D^{L_e}(K)\right] - b \\ & - (b+h)\sum_{j=0}^{K-1} f_D^{L_e}(j)F_D^{L_n+1}(S-1-j) - (b-c_v)\left[1-F_D^{L_e}(K-1)\right] + b.\end{aligned}$$

This can be re-formulated as

$$\Delta^2_{S,K} Z^s(S,K) = (b+h)\left[\sum_{j=0}^{K-1} f_D^{L_e}(j)f_D^{L_n+1}(S-j) + f_D^{L_e}(K)F_D^{L_n+1}(S-K)\right] - (b-c_v)f_D^{L_e}(K).$$

First-Order Differences of $\mathbb{E}C^f(K)$, $\mathbb{E}C^b(K)$ and $\mathbb{E}C^o(K)$ w.r.t. K

Using $f^l_{O^-}(k) = f_O^{l-1}(k)$ from Equation (5) and $\Delta_K f_O^l(k) = [F_D^l(K)-1]\,[f_D(k-K) - f_D(k-(K+1))]$ from Equation (21), the first-order finite difference of the probability that units are expedited in a period, $\mathbb{E}C^f(K)$, with respect to the expediting level K can be re-formulated as

$$\Delta_K \mathbb{E}C^f(K) = \sum_{k=0}^{K} -\left[F_D^{L_e-1}(K)-1\right]\left[f_D(k-K) - f_D(k-(K+1))\right] - f_O^{L_e-1,K+1}(K+1).$$

Now, re-arranging the sums in the first term and using Equation (4) on the second term results in

$$\Delta_K \mathbb{E}C^f(K) = \left[1-F_D^{L_e-1}(K)\right] f_D(0) - \sum_{k=0}^{K+1} f_{O^+}^{L_e-1,K+1}(k) f_D(K+1-k).$$

From Equation (3), we obtain $f_{O^+}^{l,K+1}(K+1) = 1 - F_D^l(K)$ and, for $k < K+1$, we obtain $f_{O^+}^{l,K+1}(k) = f_D^l(k)$. This results in

$$\Delta_K \mathbb{E}C^f(K) = -\sum_{k=0}^{K} f_D^{L_e-1}(k) f_D(K+1-k).$$

Using $f^l_{O^-}(k) = f_O^{l-1}(k)$ from Equation (5) and $\Delta_K f_O^l(k) = [F_D^l(K)-1]\,[f_D(k-K) - f_D(k-(K+1))]$ from Equation (21), the first-order finite difference of the expected number of expediting batches, $\mathbb{E}C^b(K)$, with respect to the expediting level K can be written as

$$\Delta_K \mathbb{E}C^b(K) = \sum_{i=0}^{\infty}\left[\sum_{k=0}^{K+i\cdot q} -\left[F_D^{L_e-1}(K)-1\right]\left[f_D(k-K) - f_D(k-(K+1))\right] - f_O^{L_e-1,K+1}(K+1+i\cdot q)\right].$$

Now, re-arranging the sums in the first term and using Equation (4) on the second term results in

$$\Delta_K \mathbb{E}C^b(K) = \sum_{i=0}^{\infty}\left[\left[1-F_D^{L_e-1}(K)\right] f_D(i\cdot q) - \sum_{k=0}^{K+1} f_{O^+}^{L_e-1,K+1}(k) f_D(K+1+i\cdot q-k)\right].$$

From Equation (3), we obtain $f_{O^+}^{l,K+1}(K+1) = 1 - F_D^l(K)$ and, for $k < K+1$, we obtain $f_{O^+}^{l,K+1}(k) = f_D^l(k)$. This results in

$$\Delta_K \mathbb{E}C^b(K) = -\sum_{i=0}^{\infty}\sum_{k=0}^{K} f_D^{L_e-1}(k) f_D(K+1+i\cdot q-k).$$

Before we analyze the finite differences of the expected number of orders from which units are expedited, $\mathbb{E}C^o(K) = L_e - \sum_{l=1}^{L_e} P\left(e_l = 0\right)$, we simplify the term $P\left(e_l = 0\right)$, the probability that no units from the order placed l periods before are expedited. For $l \geq 3$, we obtain

$$P\left(e_l = 0\right) = \sum_{m=0}^{K} \sum_{k=1}^{m} f^l_{O^-, O^-_l}(m,k) + \sum_{m=0}^{\infty} f^l_{O^-, O^-_l}(m,0).$$

Using $f^l_{O^-, O^-_l}(m,k) = \sum_{n=0}^{m-k} f^{l-1}_{O^+, O^+_{l-1}}(m-n,k) f_D(n)$ from Equation (9) and (10), and with $f^{l-1}_{O^+, O^+_{l-1}}(m,k) = f_{O^+_{l-1}}\left(k \middle| \sum_{i=1}^{l-2} O^+_{t,i} = m-k\right) f^{l-2}_{O^+}(m-k)$ from Equation (8), we obtain

$$\begin{aligned} P\left(e_l = 0\right) = &\sum_{n=0}^{\infty} \sum_{m=0}^{K-n} \sum_{k=1}^{m} f_{O^+_{l-1}}\left(k \middle| \sum_{i=1}^{l-2} O^+_{t,i} = m-k\right) f^{l-2}_{O^+}(m-k) f_D(n) \\ &+ \sum_{m=0}^{K} f_{O^+_{l-1}}\left(0 \middle| \sum_{i=1}^{l-2} O^+_{t,i} = m\right) f^{l-2}_{O^+}(m). \end{aligned}$$

With Equations (3) and (7) and, in particular, with $f^{l-2}_{O^+}(m-k) = f^{l-2}_D(m-k)$ for $m-k < m \leq K$ for $k \geq 1$, we obtain

$$\begin{aligned} P\left(e_l = 0\right) = &\sum_{n=0}^{\infty} \sum_{m=0}^{K-n} \sum_{k=1}^{m} f_D(k) f^{l-2}_D(m-k) f_D(n) + \sum_{k=1}^{K} \left[1 - F_D(k)\right] f^{l-2}_D(K-k) f_D(0) \\ &+ \sum_{m=0}^{K-1} f_D(0) f^{l-2}_D(m) + 1 - F^{l-2}_D(K-1), \end{aligned}$$

which can be simplified to

$$P\left(e_l = 0\right) = F^l_D(K) - 2 f_D(0) F^{l-1}_D(K) - [1 - 2 f_D(0)] F^{l-2}_D(K-1) + f_D(0) f_D(0) f^{l-2}_D(K) + 1. \tag{24}$$

Along the same line of arguments as above, we obtain for $l = 2$

$$P\left(e_2 = 0\right) = \begin{cases} F^2_D(K) + 2[1 - F^1_D(K)] f_D(0), & \text{for } K > 0, \\ 1, & \text{for } K = 0, \end{cases}$$

and for $l = 1$, we obtain $P\left(e_1 = 0\right) = F^1_D(K)$. The expected number of orders from which units are expedited, $\mathbb{E}C^o(K)$, with respect to the expediting level K can be re-formulated as

$$\Delta_K \mathbb{E}C^o(K) = L_e - L_e - \sum_{l=1}^{L_e} P\left(e^{K+1}_l = 0\right) + \sum_{l=1}^{L_e} P\left(e^K_l = 0\right) = -\sum_{l=1}^{L_e} \Delta_K P\left(e_l = 0\right),$$

which depends on $\Delta_K P\left(e_l = 0\right)$, the probability that no units from the order placed l periods before are expedited, $P\left(e_l = 0\right)$, with respect to the expediting level K. Using Equation (24), we obtain for $l \geq 3$

$$\begin{aligned} \Delta_K P\left(e_l = 0\right) = &f^l_D(K+1) - f_D(0) f^{l-1}_D(K+1) - f_D(0) \left[f^{l-1}_D(K+1) - f_D(0) f^{l-2}_D(K+1)\right] \\ &- [1 - f_D(0)][1 - f_D(0)] f^{l-2}_D(K), \end{aligned}$$

and for $l = 2$, we obtain

$$\Delta_K P\left(e_2 = 0\right) = \begin{cases} f^2_D(K+1) - 2 f_D(0) f^1_D(K+1), & \text{for } K > 0, \\ -[1 - F^1_D(K)][1 - F^1_D(K)], & \text{for } K = 0, \end{cases}$$

and for $l = 1$, we obtain $\Delta_K P\left(e_1 = 0\right) = f^1_D(K+1)$.