Online Appendix

A. Acronyms

Table 6: Summary of the acronyms used in this paper.

Acronym	Abbreviated Words
RI	Rolling Intrinsic
BH	Bucket Hedging
FH	Factor Hedging
NFH	Naïve Factor Hedging
FTFH	Fine-tuned Factor Hedging
MVH	Minimum Variance Hedging
CMVH	Constrained Minimum Variance Hedging
PCA	Principal Component Analysis
HEM	Hedging Effectiveness Metric
BHEM	Backtesting Hedging Effectiveness Metric

Table 6 summarizes the acronyms used throughout this paper.

B. Deltas with Exact Storage Valuation

Proposition 9 in this appendix extends the pathwise delta characterization established in Proposition 2 to the case of exact storage valuation. We let $\boldsymbol{F}_n' := (F_{n,m}, m \in \mathcal{N}_{n+1})$, $\forall n \in \mathcal{N} \setminus \{N-1\}$ denote the futures curve at time T_n exclusive of the spot price $s_n \equiv F_{n,n}$. We also define $\boldsymbol{F}_{N-1}' := 0$. The stochastic dynamic program (SDP) for exact storage valuation (Lai et al. 2010, §2) is

$$V_n(x_n, \boldsymbol{F}_n) = \max_{a \in \mathcal{A}(x_n)} p(a, s_n) + \delta \mathbb{E}\left[V_{n+1}(x_n - a, \boldsymbol{F}_{n+1}) | \boldsymbol{F}_n'\right], \tag{17}$$

for all $n \in \mathcal{N}$ and $(x_n, \boldsymbol{F}_n) \in \mathcal{X} \times \Re_+^{N-n}$, where $V_n(x_n, \boldsymbol{F}_n)$ denotes the optimal value function in stage n and state $(x_n, \boldsymbol{F}_n)$, and $V_N(x_N, \boldsymbol{F}_N) := 0$, for all $x_N \in \mathcal{X}$ (recall that $\boldsymbol{F}_N \equiv 0$).

Proposition 7 characterizes the optimal value function and an optimal operating policy of this SDP. The proof of this result is a simple adaptation of the proofs of Lai et al. (2010, Theorem 1) and Secomandi (2010, Theorem 1).

Proposition 7 (Concavity and basestock optimality)**.** *In every stage n, the function $V_n(x_n, \boldsymbol{F}_n)$ is concave in x_n for each given $\boldsymbol{F}_n$, and the optimal policy for the SDP (17) features two basestock targets, $\underline{b}_n(\boldsymbol{F}_n)$, $\bar{b}_n(\boldsymbol{F}_n) \in \mathcal{X}$, such that $\underline{b}_n(\boldsymbol{F}_n) \leq \bar{b}_n(\boldsymbol{F}_n)$ and an optimal decision rule $A_n^*(x_n, \boldsymbol{F}_n)$ satisfies*

$$A_n^*(x_n, \boldsymbol{F}_n) = \begin{cases} C^I \vee [x_n - \underline{b}_n(\boldsymbol{F}_n)], & x_n \in [0, \underline{b}_n(\boldsymbol{F}_n)), \\ 0, & x_n \in [\underline{b}_n(\boldsymbol{F}_n), \bar{b}_n(\boldsymbol{F}_n)], \\ C^W \wedge [x_n - \bar{b}_n(\boldsymbol{F}_n)], & x_n \in (\bar{b}_n(\boldsymbol{F}_n), \overline{x}]. \end{cases} \tag{18}$$

Proposition 8, based on Lemma 1, establishes basic properties of the optimal value function and basestock targets, which are used to establish Proposition 9. All of these results are based on Assumption 2.

Assumption 2 (Capacities and maximum space)**.** The capacity limits C^{I} and C^{W} as well as the maximal inventory level $\overline{x}$ are integer multiples of some positive real number Q.

Lemma 1 is related to Secomandi (2010, Propositions 2 and 3). We define the set $\mathcal{X}^Q$ as $\{0, Q, 2Q, \ldots, \overline{x}\}$.

Lemma 1 (Characterization)**.** *Under Assumption 2, in every stage $n \in \mathcal{N}$: (a) The function $V_n(x, \boldsymbol{F}_n)$ is piecewise linear continuous in inventory $x \in \mathcal{X}$ with break points in set $\mathcal{X}^Q$ for each given futures curve $\boldsymbol{F}_n \in \Re_+^{N-n}$; (b) There exist optimal basestock levels $\underline{b}_n(\boldsymbol{F}_n)$ and $\overline{b}_n(\boldsymbol{F}_n)$ in set $\mathcal{X}^Q$, $\forall \boldsymbol{F}_n \in \Re_+^{N-n}$.*

Following Lemma 1 we define the finite set of feasible actions at inventory level $x \in \mathcal{X}$ as

$$\begin{aligned}\mathcal{A}'(x) \quad &:= \quad \{(x-\overline{x}) \vee C^{\mathrm{I}}, \ldots, x - (\lfloor x/Q \rfloor + 2)Q, x - (\lfloor x/Q \rfloor + 1)Q\} \cup \{0\} \\ &\qquad \cup \{x - \lfloor x/Q \rfloor Q, x - (\lfloor x/Q \rfloor - 1)Q, \ldots, x \wedge C^{\mathrm{W}}\},\end{aligned}$$

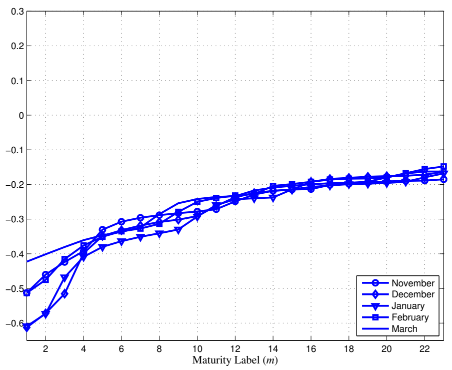
with $\{0\}$ removed if it is a duplicate.

Proposition 8 (Representation and Lipschitz continuity)**.** *Under Assumption 2, in every stage $n \in \mathcal{N}$: (a) For all inventory levels $x \in \mathcal{X}$ and futures curves $\boldsymbol{F}_n \in \Re_+^{N-n}$ it holds that $V_n(x, \boldsymbol{F}_n) = \max_{a \in \mathcal{A}'(x)} p(a, s_n) + \delta \mathbb{E}\left[V_{n+1}(x_n - a, \boldsymbol{F}_{n+1}) | \boldsymbol{F}'_n\right]$; (b) For each given inventory $x \in \mathcal{X}$ it holds that the function $V_n(x, \boldsymbol{F}_n)$ is Lipschitz continuous in the futures curve $\boldsymbol{F}_n \in \Re_+^{N-n}$; i.e., there exists $\mathsf{L}_n(x) \in \Re_+$ such that $\left|V_n(x, \boldsymbol{F}_n^2) - V_n(x, \boldsymbol{F}_n^1)\right| \leq \mathsf{L}_n(x) \sum_{m=n}^{N} |F_{n,m}^2 - F_{n,m}^1|$, $\forall \boldsymbol{F}_n^1, \boldsymbol{F}_n^2 \in \Re_+^{N-n}$.*

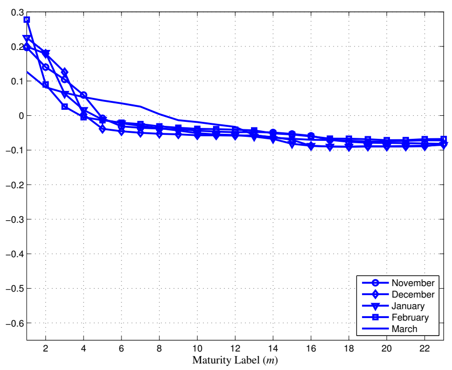
In Proposition 9 we indicate by x_m^* the optimal inventory level in stage $m \in \mathcal{N} \setminus \{0\}$ and by $\Delta_m(t, x_n, \check{\boldsymbol{F}}_n(t))$ the delta under an optimal policy.

Proposition 9 (Pathwise deltas with an optimal policy)**.** *Under Assumption 2, for every $n \in \mathcal{N} \setminus \{0\}$ it holds for all $m \in \mathcal{N}_n$, $x_n \in \mathcal{X}$, $t \in [T_{n-1}, T_n)$, and $\check{\boldsymbol{F}}^n(t) \in \Re_+^{N-n}$ that*

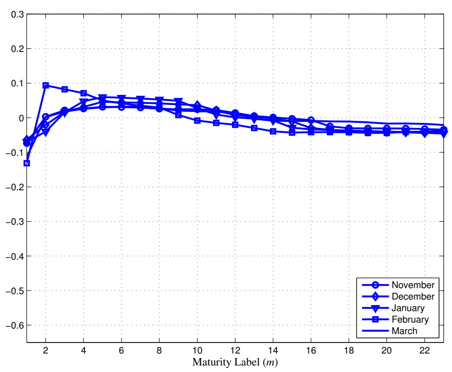
$$\begin{aligned}\Delta_m(t, x_n, \check{\boldsymbol{F}}^n(t)) \quad &= \quad \frac{\overline{\delta}(t, T_m)}{F(t, T_m)} \mathbb{E}[(\phi^I 1\{x_m^* \in [0, \underline{b}_m(\boldsymbol{F}_m))\} + \phi^W 1\{x_m^* \in (\overline{b}_m(\boldsymbol{F}_m), \overline{x}]\}) \\ &\qquad s_m A_m^*(x_m^*, \boldsymbol{F}_m) | x_n, \check{\boldsymbol{F}}^n(t)].\end{aligned} \tag{19}$$



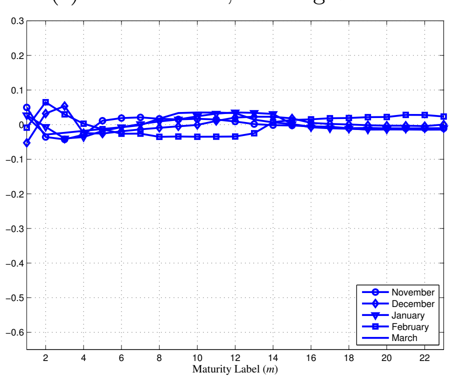
(a) First Factor, Heating Season



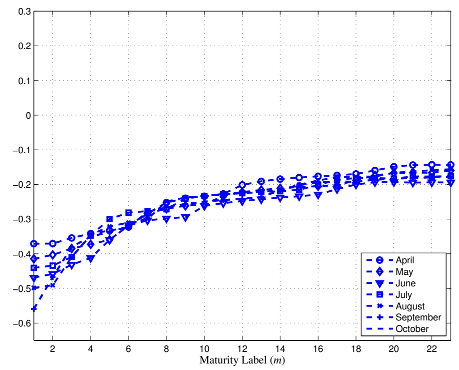
(b) Second Factor, Heating Season



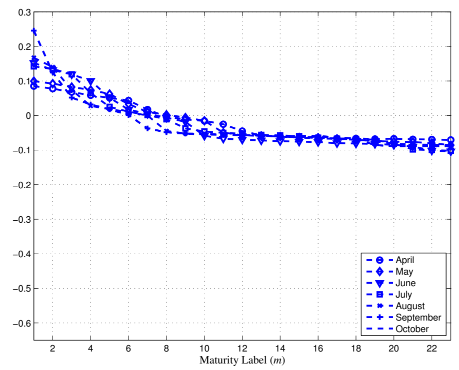
(c) Third Factor, Heating Season



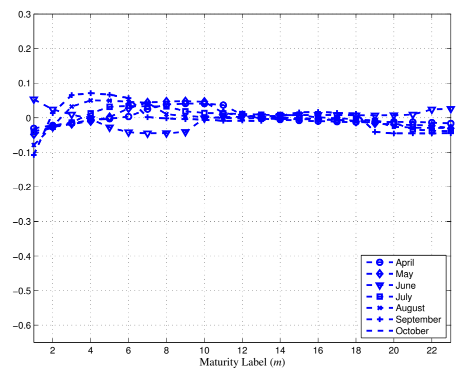
(d) Fourth Factor, Heating Season



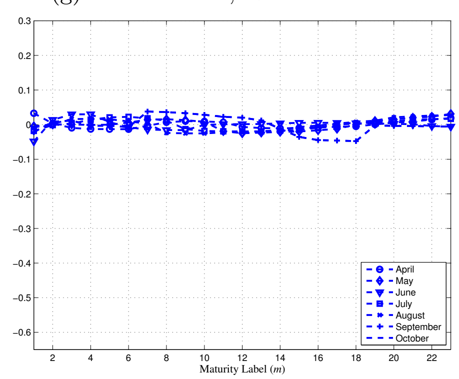
(e) First Factor, Rest of the Year



(f) Second Factor, Rest of the Year

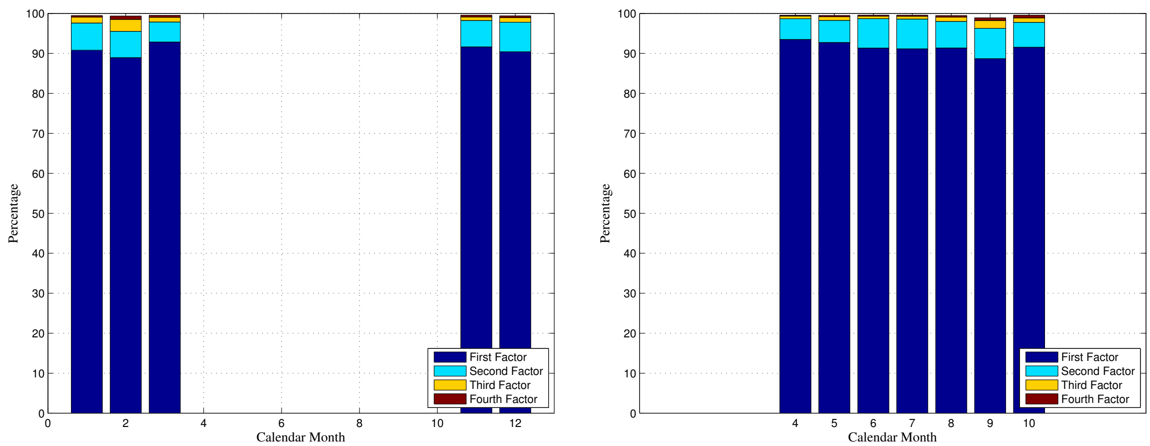


(g) Third Factor, Rest of the Year



(h) Fourth Factor, Rest of the Year

Figure 6: The estimated loading coefficients of the first four factors in the heating season (panels (a)-(d)) and rest of the year (panels (e)-(h)) monthly PCAs over the period from January 1997 to December 2012.



(a) Heating Season

(b) Rest of the Year

Figure 7: The percentage of the total futures price log return variance explained by the first four factors in the heating season (panel (a)) and rest of the year (panel (b)) monthly PCAs over the period from January 1997 to December 2012.

Table 7: Estimated standard errors of the HEM estimates reported in Table 2.

	$K^{\diamond} = 3$			$K^{\diamond} = 23$			$K^{\diamond} = 3$ plus Noise		
K	NFH	FTFH	BH	NFH	FTFH	BH	NFH	FTFH	BH
1	1.81	1.85	0.43	1.89	1.89	0.35	1.88	1.88	0.23
2	7.56	0.67	0.27	62.05	0.52	0.22	31.22	0.46	0.14
3	0.40	0.25	0.24	167.01	0.26	0.18	183.37	0.23	0.12
5	6.47	0.24	0.24	913.91	0.21	0.18	5,045.80	0.19	0.12
10	0.30	0.24	0.24	391.49	0.17	0.17	4,572.57	0.14	0.12
15	0.33	0.24	0.24	304.62	0.18	0.17	4,021.19	0.13	0.12
20	0.26	0.24	0.24	49.82	0.18	0.18	675.35	0.12	0.12
23	0.24	0.24	0.24	0.18	0.18	0.18	0.12	0.12	0.12

C. Loading Coefficients and Futures Price Log Return Variance Explained

Figures 6 and 7 display the estimated monthly loading coefficients and the percentage of the total futures price log return variance explained by the first four factors, respectively, in the heating season and the rest of the year.

D. Standard Errors of the Hedging Effectiveness Metric and Backtesting Hedging Effectiveness Metric Estimates

Tables 7 and 8 include the standard errors (estimated by bootstrapping) of the HEM estimates in Tables 2 and 4, respectively, discussed in §8.2.2. Table 9 reports the standard errors (also estimated by bootstrapping) of the BHEM estimates in Table 5 analyzed in §8.2.3.

Table 8: Estimated standard errors of the HEM estimates reported in Table 4.

	$K^\diamond = 3$			$K^\diamond = 23$			$K^\diamond = 3$ plus Noise		
	L			L			L		
K	3	5	10	3	5	10	3	5	10
1	0.98	0.85	0.34	1.44	1.13	0.44	1.05	0.79	0.38
2	1.01	0.76	0.25	1.39	1.03	0.35	0.98	0.69	0.29
3	1.03	0.75	0.25	1.39	1.02	0.34	0.95	0.67	0.27
5	1.04	0.74	0.25	1.39	1.00	0.34	0.95	0.68	0.26
10	1.05	0.75	0.25	1.40	1.00	0.34	0.94	0.68	0.26
15	1.05	0.75	0.25	1.40	1.00	0.34	0.94	0.67	0.26
20	1.05	0.75	0.25	1.40	1.00	0.34	0.94	0.68	0.26
23	1.05	0.75	0.25	1.40	1.00	0.34	0.94	0.68	0.26

Table 9: Estimated standard errors of the BHEM estimates reported in Table 5.

				CMVH		
				L		
	NFH	FTFH	BH	3	5	10
K	In-Sample					
1	13.61	14.23	0.54	13.94	8.63	1.43
2	3,360.35	0.23	0.54	12.67	7.65	1.32
3	7,835.22	0.52	0.48	10.01	5.71	0.90
5	1,771.76	0.72	0.49	9.04	5.03	0.82
10	1,576.81	0.52	0.46	8.49	4.63	0.75
15	63.23	0.45	0.46	8.47	4.62	0.75
20	2.19	0.46	0.46	8.49	4.63	0.75
23	0.46	0.46	0.46	8.48	4.62	0.75
	Out-of-Sample					
1	11.13	13.63	0.26	15.08	9.62	0.65
2	1,753.42	3.70	0.96	12.07	6.41	0.16
3	162,487.42	2.21	1.35	11.16	5.55	0.28
5	5,661.23	1.61	1.55	10.88	5.26	0.37
10	717.60	1.65	1.67	10.65	5.06	0.42
15	656.86	1.68	1.69	10.65	5.05	0.43
20	4.37	1.68	1.68	10.62	5.04	0.43
23	1.68	1.68	1.68	10.63	5.04	0.43

E. Proofs

We denote by $\boldsymbol{Y}$ the vector $(Y_k, k \in \mathcal{K})$ of K uncorrelated standard normals. Given $n \in \mathcal{N} \setminus \{N-1\}$ and $m \in \mathcal{N}_{n+1}$, we define $\beta_m^{n,n+1}(\boldsymbol{Y})$ as the quantity

$$\exp\left[-\frac{1}{2}(T_{n+1}-T_n)\sum_{k\in\mathcal{K}}\sigma_{m,k,n}^2 + \sqrt{T_{n+1}-T_n}\sum_{k\in\mathcal{K}}\sigma_{m,k,n}Y_k\right],$$

and $\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y})$ as the column vector $(\beta_m^{n,n+1}(\boldsymbol{Y}), m \in \mathcal{N}_{n+1})$. Under price model (2), we can equivalently express $F_{n+1,m}$ given $F_{n,m}$ as $F_{n,m}\beta^{n,n+1}(\boldsymbol{Y})$ and $\boldsymbol{F}_{n+1}$ given $\boldsymbol{F}_n'$ as $\mathrm{diag}(\boldsymbol{F}_n')\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y})$ – the notation $\boldsymbol{F}_n'$ is introduced at the beginning of Online Appendix B. The equalities $F_{n,m} = \mathbb{E}\left[F_{n+1,m}|F_{n,m}\right] = F_{n,m}\mathbb{E}\left[\beta_m^{n,n+1}(\boldsymbol{Y})\right]$ imply that

$$\mathbb{E}\left[\beta_m(t,t',\boldsymbol{Y})\right] = 1. \tag{20}$$

We use this notation and (20) in the proofs of Lemma 2 and Proposition 8.

We define by $W_n^\pi(x,\boldsymbol{F}_n') := \delta\mathbb{E}\left[V_{n+1}^\pi(x,\boldsymbol{F}_{n+1})|\boldsymbol{F}_n'\right]$ the policy π continuation function in stage $n \in \mathcal{N} \setminus \{N-1\}$ given the inventory level x and the futures curve $\boldsymbol{F}_n'$. We use this notation in the proof of Proposition 2. This notation allows us to write

$$\begin{aligned}
V_n^\pi(x,\boldsymbol{F}_n) &= \sum_{m=n}^{N-1}\delta^{m-n}\mathbb{E}[p(A_m^\pi(x_m^\pi,\boldsymbol{F}_m),s_m)|x,\boldsymbol{F}_n] \\
&= p(A_n^\pi(x,\boldsymbol{F}_n),s_n) + \sum_{m=n+1}^{N-1}\delta^{m-n}\mathbb{E}[p(A_m^\pi(x_m^\pi,\boldsymbol{F}_m),s_m)|x-A_n^\pi(x,\boldsymbol{F}_n),\boldsymbol{F}_n'] \\
&= p(A_n^\pi(x,\boldsymbol{F}_n),s_n) + \delta\mathbb{E}\left[V_{n+1}^\pi(x-A_n^\pi(x,\boldsymbol{F}_n),\boldsymbol{F}_{n+1})|\boldsymbol{F}_n'\right] \\
&= p(A_n^\pi(x,\boldsymbol{F}_n),s_n) + W_n^\pi(x-A_n^\pi(x,\boldsymbol{F}_n),\boldsymbol{F}_n').
\end{aligned}$$

We define by $W_n(x,\boldsymbol{F}_n') := \delta\mathbb{E}\left[V_{n+1}(x,\boldsymbol{F}_{n+1})|\boldsymbol{F}_n'\right]$ the optimal continuation function in stage $n \in \mathcal{N} \setminus \{N-1\}$ given the inventory level x and the futures curve $\boldsymbol{F}_n'$. We use this notation in the proofs of Lemma 1 and Propositions 8 and 9. We denote by $v_n(x,a,\boldsymbol{F}_n)$ the objective function of the maximization on the right hand side of the SDP (17). We use this notation in the proofs of Propositions 8 and 9 and Lemma 3.

Proof of Proposition 1 (FH positions). The system of linear equations (7) can be expressed as $B_{n-1}^\mathsf{T}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}(t))\boldsymbol{q}^{\mathcal{H}_n}(t) = B_{n-1}^\mathsf{T}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}(t))\boldsymbol{\Delta}^{\pi,\mathcal{H}_n}(t) + E_{n-1}^\mathsf{T}\mathrm{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{H}}_n}(t))\boldsymbol{\Delta}^{\pi,\overline{\mathcal{H}}_n}(t)$. Premultiplying both sides of this expression by $\left(B_{n-1}^\mathsf{T}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}(t))\right)^{-1} = \mathrm{diag}^{-1}\left(\check{\boldsymbol{F}}^{\mathcal{H}_n}(t)\right)\left(B_{n-1}^\mathsf{T}\right)^{-1}$ yields the claimed result. □

Lemma 2 is needed in the proof of Proposition 2.

Lemma 2 (Pathwise derivatives)**.** *Under Assumption 1(a), given $x \in \mathcal{X}$, for every $n \in \mathcal{N} \setminus \{N-1\}$ and $m \in \mathcal{N}_{n+1}$ it holds that*

$$\frac{dV^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{dF_{n,m}} = \frac{\partial V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{\partial F_{n+1,m}} \frac{F_{n+1,m}}{F_{n,m}}, \tag{21}$$

$$\frac{\partial \mathbb{E}[V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1}) | \boldsymbol{F}'_n]}{\partial F_{n,m}} = \mathbb{E}\left[\frac{dV^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{dF_{n,m}} | \boldsymbol{F}'_n\right]. \tag{22}$$

Proof. We interpret $F_{n,m}$ as a parameter and write $\boldsymbol{F}_{n+1}(F_{n,m})$ to explicitly indicate the dependence of $\boldsymbol{F}_{n+1}$ on this parameter. In particular, as $F_{n+1,m} = F_{n,m}\beta^{n,n+1}_m(\boldsymbol{Y})$, $F_{n+1,m}$ depends on $F_{n,m}$ but every other $F_{n+1,\ell}$, with $\ell > m$, does not.

We first show that conditions (A1)-(A4) in Appendix A of Broadie and Glasserman (1996) hold with respect to $\mathbb{E}[V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1}(F_{n,m})) | \boldsymbol{F}'_n]$.

(A1) Under model (2) the quantity $\partial F_{n+1,\ell}(F_{n,m})/\partial F_{n,m}$ exists with probability 1 because it is equal to $\beta^{n,n+1}_m(\boldsymbol{Y})$ if $\ell = m$ and to 0 when $\ell > m$.

(A2) Fix x and let $\mathcal{D}^{V^{\pi}}_{n+1}$ denote the set of futures curves $\boldsymbol{F}_{n+1}$ at which $V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})$ is differentiable with respect to each element of $\boldsymbol{F}_{n+1}$. Assumption 1(a) and Rademacher's theorem imply that $V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})$ is differentiable almost everywhere on $\Re^{N-n-1}_{+}$ with respect to each element of $\boldsymbol{F}_{n+1}$. Price model (2) implies that $\mathbb{P}(\boldsymbol{F}_{n+1}(F_{n,m}) \in \mathcal{D}^{V^{\pi}}_{n+1} | \boldsymbol{F}'_n) = 1$, where $\mathbb{P}$ denotes probability, for all $F_{n,m} \in \Re_{+}$.

(A3) This is Assumption 1(a).

(A4) The random variable $F_{n+1,m}(F_{n,m})$ is almost surely Lipschitz continuous with integrable modulus $\beta^{n,n+1}_m(\boldsymbol{Y})$ because $|F_{n+1,m}(F^2_{n,m}) - F_{n+1,m}(F^1_{n,m})| = \beta^{n,n+1}_m(\boldsymbol{Y})|F^2_{n,m} - F^1_{n,m}|$, for all $F^1_{n,m}$, $F^2_{n,m} \in \Re_{+}$, and $\mathbb{E}[\beta^{n,n+1}_m(\boldsymbol{Y})] = 1$. Every other random variable $F_{n+1,\ell}(F_{n,m})$, with $m < \ell$, is almost surely Lipschitz with integrable modulus 0 because each such random variable does not depend on $F_{n,m}$.

Following Broadie and Glasserman (1996, p. 280), (21) holds because under conditions (A1)-(A2) the pathwise derivative $dV_{n+1}(x, \boldsymbol{F}_{n+1})/dF_{n,m}$ satisfies

$$\frac{dV^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{dF_{n,m}} = \sum_{\ell=n+1}^{N-1} \frac{\partial V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{\partial F_{n+1,\ell}} \frac{\partial F_{n+1,\ell}}{\partial F_{n,m}} = \frac{\partial V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{\partial F_{n+1,m}} \beta^{n,n+1}_m(\boldsymbol{Y})$$

$$= \frac{\partial V^{\pi}_{n+1}(x, \boldsymbol{F}_{n+1})}{\partial F_{n+1,m}} \frac{F_{n+1,m}}{F_{n,m}}.$$

Expression (22) follows from Proposition 1 in Broadie and Glasserman (1996). □

Proof of Proposition 2 (Pathwise deltas). For simplicity of exposition, we prove the claimed result for $t = 0$ and $n = 1$. The proof for the general case follows similar steps. As $\boldsymbol{F}'_0 \equiv \check{\boldsymbol{F}}(T_0)$, in this proof we use the notation $\boldsymbol{F}_0'$ in lieu of the notation $\check{\boldsymbol{F}}(T_0)$.

Suppose $m = 1$. Formula (5) and expression (22) in Lemma 2 (applied with $n = 0$ and $n+1 = 1$, because n in the current proof corresponds to $n + 1$ in Lemma 2, $m = 1$, and $x_1 = x$) imply that

$$\frac{\Delta_1^\pi(0, x_1, \boldsymbol{F}_0')}{\overline{\delta}(0, T_1)} = \frac{1}{\overline{\delta}(0, T_1)}\frac{\partial U^\pi(0, x_1, \boldsymbol{F}_0')}{\partial F_{0,1}} = \frac{\partial \mathbb{E}[V_1^\pi(x_1, \boldsymbol{F}_1)|\boldsymbol{F}_0']}{\partial F_{0,1}} = \mathbb{E}\left[\frac{dV_1^\pi(x_1, \boldsymbol{F}_1)}{dF_{0,1}}|\boldsymbol{F}_0'\right]. \tag{23}$$

It follows from (21) in Lemma 2 and $s_1 \equiv F_{1,1}$ that

$$\frac{dV_1^\pi(x_1, \boldsymbol{F}_1)}{dF_{0,1}} = \frac{\partial V_1^\pi(x_1, \boldsymbol{F}_1)}{\partial s_1}\frac{s_1}{F_{0,1}}. \tag{24}$$

Pick a vector $\overline{\boldsymbol{F}}_1$ at which $V_1^\pi(x_1, \boldsymbol{F}_1)$ is differentiable. It follows from Assumption 1(b) that

$$\left.\frac{\partial V_1^\pi(x_1, \boldsymbol{F}_1)}{\partial s_1}\right|_{\boldsymbol{F}_1=\overline{\boldsymbol{F}}_1} = \left.\frac{\partial p(a_1^\pi(x_1, \overline{\boldsymbol{F}}_1), s_1)}{\partial s_1}\right|_{s_1=\overline{s}_1} + \left.\frac{\partial W_1^\pi(x_1 - a_1^\pi(x_1, \overline{\boldsymbol{F}}_1), \boldsymbol{F}_1')}{\partial s_1}\right|_{\boldsymbol{F}_1'=\overline{\boldsymbol{F}}_1'}. \tag{25}$$

The first term on the right hand side of (25) can be expressed as follows:

$$\begin{aligned}\left.\frac{\partial p(a_1^\pi(x_1, \overline{\boldsymbol{F}}_1), s_1)}{\partial s_1}\right|_{s_1=\overline{s}_1} &= \left.\frac{\partial(\phi^{\mathrm{I}} s_1 + c^{\mathrm{I}})a_1^\pi(x_1, \overline{\boldsymbol{F}}_1)}{\partial s_1}\right|_{s_1=\overline{s}_1} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1) < 0\} \\ &\quad + \left.\frac{\partial(\phi^{\mathrm{W}} s_1 - c^{\mathrm{W}})a_1^\pi(x_1, \overline{\boldsymbol{F}}_1)}{\partial s_1}\right|_{s_1=\overline{s}_1} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1)) > 0\} \\ &= \left(\phi^{\mathrm{I}} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1) < 0\} + \phi^{\mathrm{W}} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1) > 0\}\right) a_1^\pi(x_1, \overline{\boldsymbol{F}}_1). \end{aligned}\tag{26}$$

As $\boldsymbol{F}_1'$ does not depend on s_1, the second term on the right hand side of (25) is zero:

$$\left.\frac{\partial W_1^\pi(x_1 - a_1^\pi(x_1, \overline{\boldsymbol{F}}_1), \boldsymbol{F}_1')}{\partial s_1}\right|_{\boldsymbol{F}_1'=\overline{\boldsymbol{F}}_1'} = 0. \tag{27}$$

It follows from (26) and (27) that (25) can be expressed as

$$\left.\frac{\partial V_1^\pi(x_1, \boldsymbol{F}_1)}{\partial s_1}\right|_{\boldsymbol{F}_1=\overline{\boldsymbol{F}}_1} = \left(\phi^{\mathrm{I}} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1) < 0\} + \phi^{\mathrm{W}} 1\{a_1^\pi(x_1, \overline{\boldsymbol{F}}_1) > 0\}\right) a_1^\pi(x_1, \overline{\boldsymbol{F}}_1). \tag{28}$$

Expressions (23), (24), and (28) imply that

$$\Delta_1^\pi(0, x_1, \boldsymbol{F}_0') = \frac{\overline{\delta}(0, T_1)}{F_{0,1}}\mathbb{E}\left[\left(\phi^{\mathrm{I}} 1\{A_1^\pi(x_1, \boldsymbol{F}_1) < 0\} + \phi^{\mathrm{W}} 1\{A_1^\pi(x_1, \boldsymbol{F}_1) < 0\}\right) s_1 A_1^\pi(x_1, \boldsymbol{F}_1)|x_1, \boldsymbol{F}_0'\right].$$

Thus, the claimed property holds for $m = 1$ (recall that $x_1 \equiv x_1^\pi$).

The cases corresponding to $m = 2, \ldots, N - 1$ can be dealt with by recursively applying a logic similar to the case $m = 1$. □

Proof of Proposition 3 (Bounds on deltas). We derive the inequality $\Delta^{\pi}_{m}(t, x_n, \check{\boldsymbol{F}}(t)) \leq \overline{\delta}(t, T_m)\phi^{\mathrm{W}}C^{\mathrm{W}}$. The inequality $\Delta^{\pi}_{m}(t, x_n, \check{\boldsymbol{F}}(t)) \geq \overline{\delta}(t, T_m)\phi^{\mathrm{I}}C^{\mathrm{I}}$ can be established in an analogous manner. It holds that

$$\begin{aligned}
\Delta^{\pi}_{m}(t, x_n, \check{\boldsymbol{F}}(t)) &= \frac{\overline{\delta}(t, T_m)}{F(t, T_m)}\mathbb{E}[(\phi^{\mathrm{I}}A^{\pi}_{m}(x^{\pi}_{m}, \boldsymbol{F}_m)1\{A^{\pi}_{m}(x^{\pi}_{m}, \boldsymbol{F}_m) < 0\} \\
&\qquad +\phi^{\mathrm{W}}A^{\pi}_{m}(x^{\pi}_{m}, \boldsymbol{F}_m)1\{A^{\pi}_{m}(x^{\pi}_{m}, \boldsymbol{F}_m) > 0\})s_m|x_n, \check{\boldsymbol{F}}(t)] \\
&\leq \frac{\overline{\delta}(t, T_m)}{F(t, T_m)}\mathbb{E}[\phi^{\mathrm{W}}C^{\mathrm{W}}s_m|x_n, \check{\boldsymbol{F}}(t)] \\
&= \overline{\delta}(t, T_m)\phi^{\mathrm{W}}C^{\mathrm{W}}\frac{\mathbb{E}[s_m|F(t, T_m)]}{F(t, T_m)} \\
&= \overline{\delta}(t, T_m)\phi^{\mathrm{W}}C^{\mathrm{W}},
\end{aligned}$$

where the first and last equalities follow from Proposition 2 and the property $\mathbb{E}[s_m|F(t, T_m)] = F(t, T_m)$, respectively. □

Proof of Proposition 4 (BH optimality). The objective function of model (13) is nonnegative for any value of $\boldsymbol{q}(t)$. Thus, setting $\boldsymbol{q}(t) = \boldsymbol{\Delta}^{\pi}(t)$ optimally solves the version of this model obtained by replacing $\boldsymbol{\Delta}^{\pi,\diamond}(t)$ with $\boldsymbol{\Delta}^{\pi}(t)$. □

Proof of Proposition 5 (CMVH positions). We omit the suffix (t) for ease of exposition. Using the constraints in set $\mathcal{Q}^{L}_{n}$ to set to zero in the objective function the vector $\boldsymbol{q}^{\overline{\mathcal{L}}_n}$, rearranging, and ignoring constant terms yields the unconstrained model

$$\begin{aligned}
\min_{\boldsymbol{q}^{\mathcal{L}_n}} \quad & \left(\boldsymbol{q}^{\mathcal{L}_n}\right)^{\mathsf{T}}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\mathcal{L}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\boldsymbol{q}^{\mathcal{L}_n} - 2\left(\boldsymbol{q}^{\mathcal{L}_n}\right)^{\mathsf{T}}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\mathcal{L}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\boldsymbol{\Delta}^{\pi,\mathcal{L}_n} \\
& -2\left(\boldsymbol{q}^{\mathcal{L}_n}\right)^{\mathsf{T}}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\overline{\mathcal{L}}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{L}}_n})\boldsymbol{\Delta}^{\pi,\overline{\mathcal{L}}_n}.
\end{aligned}$$

Because $\xi^{N-1,\mathcal{L}_n}_{n-1}$ is assumed positive definite, the necessary and sufficient optimality conditions for this model are

$$\begin{aligned}
\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\mathcal{L}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\boldsymbol{q}^{\pi,\mathcal{L}_n} - \mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\mathcal{L}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\boldsymbol{\Delta}^{\pi,\mathcal{L}_n}& \\
- \mathrm{diag}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\xi^{N-1,\overline{\mathcal{L}}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{L}}_n})\boldsymbol{\Delta}^{\pi,\overline{\mathcal{L}}_n} &= 0.
\end{aligned}$$

The optimal solution is thus

$$\boldsymbol{q}^{\pi,\mathcal{L}_n} = \boldsymbol{\Delta}^{\pi,\mathcal{L}_n} + \mathrm{diag}^{-1}(\check{\boldsymbol{F}}^{\mathcal{L}_n})\left(\xi^{N-1,\mathcal{L}_n}_{n-1}\right)^{-1}\xi^{N-1,\overline{\mathcal{L}}_n}_{n-1}\mathrm{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{L}}_n})\boldsymbol{\Delta}^{\pi,\overline{\mathcal{L}}_n}. \square$$

Proof of Proposition 6 (Approximate equivalence). We suppress the suffix (t) to simplify the exposition. For a given K factor model let Σ_{n-1} be its corresponding matrix of factor loading coefficients and ξ^K_{n-1} the analogue for this model of the matrix ξ^{N-1}_{n-1}. Interpret $\xi^K_{n-1}(\Delta t)$ as a function of Δt. For sufficiently small Δt, taking a first-order Taylor series approximation of each element of $\xi^K_{n-1}(\Delta t)$ around 0 allows us to write $\xi^K_{n-1}(\Delta t) \approx \Delta t \Sigma_{n-1} (\Sigma_{n-1})^\mathsf{T}$. The resulting version of model (15) is

$$\min_{\boldsymbol{q} \in \mathcal{Q}^L} \Delta t \left(\boldsymbol{q} - \boldsymbol{\Delta}^\pi\right)^\mathsf{T} \operatorname{diag}\left(\check{\boldsymbol{F}}\right) \Sigma_{n-1} \left(\Sigma_{n-1}\right)^\mathsf{T} \operatorname{diag}\left(\check{\boldsymbol{F}}\right) \left(\boldsymbol{q} - \boldsymbol{\Delta}^\pi\right). \tag{29}$$

Recall the meaning of the matrices B_{n-1} and E_{n-1} introduced just before Proposition 1. We have $\mathcal{H}_n \equiv \mathcal{L}_n$ and $\overline{\mathcal{H}}_n \equiv \overline{\mathcal{L}}_n$, so that

$$\Sigma_{n-1} \equiv \begin{bmatrix} B_{n-1} \\ E_{n-1} \end{bmatrix}$$

and

$$\Sigma_{n-1} \left(\Sigma_{n-1}\right)^\mathsf{T} \equiv \begin{bmatrix} B_{n-1}B_{n-1}^\mathsf{T} & B_{n-1}E_{n-1}^\mathsf{T} \\ E_{n-1}B_{n-1}^\mathsf{T} & E_{n-1}E_{n-1}^\mathsf{T} \end{bmatrix}.$$

Hence, solving model (29) is equivalent to solving the following model:

$$\begin{aligned} \min_{\boldsymbol{q}^{\mathcal{H}_n}} \quad & \left(\boldsymbol{q}^{\mathcal{H}_n}\right)^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} B_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) \boldsymbol{q}^{\mathcal{H}_n} \\ & -2 \left(\boldsymbol{q}^{\mathcal{H}_n}\right)^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} B_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) \boldsymbol{\Delta}^{\pi,\mathcal{H}_n} \\ & -2 \left(\boldsymbol{q}^{\mathcal{H}_n}\right)^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} E_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{H}}_n}) \boldsymbol{\Delta}^{\pi,\overline{\mathcal{H}}_n}. \end{aligned}$$

Because the matrix $B_{n-1}B_{n-1}^\mathsf{T}$ is positive definite, by the assumption on the matrix B_{n-1}, proceeding as in the proof of Proposition 5 yields the following optimality conditions for this model:

$$\begin{aligned} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} B_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) \boldsymbol{q}^{\pi,\mathcal{H}_n} - \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} B_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) \boldsymbol{\Delta}^{\pi,\mathcal{H}_n} & \\ - \operatorname{diag}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) B_{n-1} E_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{H}}_n}) \boldsymbol{\Delta}^{\pi,\overline{\mathcal{H}}_n} & = 0. \end{aligned}$$

The solution to this system of linear inequalities is

$$\boldsymbol{q}^{\pi,\mathcal{H}_n} = \boldsymbol{\Delta}^{\pi,\mathcal{H}_n} + \operatorname{diag}^{-1}(\check{\boldsymbol{F}}^{\mathcal{H}_n}) \left(B_{n-1}^\mathsf{T}\right)^{-1} E_{n-1}^\mathsf{T} \operatorname{diag}(\check{\boldsymbol{F}}^{\overline{\mathcal{H}}_n}) \boldsymbol{\Delta}^{\pi,\overline{\mathcal{H}}_n}. \square$$

Proof of Lemma 1 (Characterization). The claimed properties are true in stage $N-1$, because $V_{N-1}(x_{N-1}, s_{N-1}) = \left(\phi^\mathrm{W} s_{N-1} - c^\mathrm{W}\right)^+ (x_{N-1} \wedge C^\mathrm{W})$, and $\underline{b}_{N-1}(s_{N-1}) = 0$ and $\bar{b}_{N-1}(s_{N-1}) = \overline{x} 1\{\phi^\mathrm{W} s_{N-1} - c^\mathrm{W} \leq 0\}$. Make the induction hypothesis that these properties also hold in stages $n+1, \ldots, N-2$. Consider stage n. By definition, $W_n(\cdot, \boldsymbol{F}_n')$ is the discounted expected value of piecewise linear continuous functions, each with possible break points only in set $\mathcal{X}^Q$, which implies that this function satisfies the same property. The quantities $\underline{b}_n(\boldsymbol{F}_n)$ and $\bar{b}_n(\boldsymbol{F}_n)$ are optimal solutions to the following maximizations, respectively (see the proof of Theorem 1 in Secomandi 2010):

$$\max_{x_{n+1}\in\mathcal{X}} W_n(x_{n+1}, \boldsymbol{F}'_n) - \left(\phi^{\mathrm{I}} s_n + c^{\mathrm{I}}\right) x_{n+1},$$
$$\max_{x_{n+1}\in\mathcal{X}} W_n(x_{n+1}, \boldsymbol{F}'_n) - \left(\phi^{\mathrm{W}} s_n - c^{\mathrm{W}}\right) x_{n+1}.$$

Thus, $\underline{b}_n(\boldsymbol{F}_n)$ and $\bar{b}_n(\boldsymbol{F}_n)$ can be taken to belong to set $\mathcal{X}^Q$. It follows that (recalling that $C^{\mathrm{I}} < 0$)

$$V_n(x_n, \boldsymbol{F}_n) = \begin{cases} \left(\phi^{\mathrm{I}} s_n + c^{\mathrm{I}}\right) C^{\mathrm{I}} + W_n(x_n - C^{\mathrm{I}}, \boldsymbol{F}'_n), & x_n \in [0, \underline{b}_n(\boldsymbol{F}_n) + C^{\mathrm{I}}), \\ \left(\phi^{\mathrm{I}} s_{n-1} + c^{\mathrm{I}}\right) [x_n - \underline{b}_n(\boldsymbol{F}_n)] + W_n(\underline{b}_n(\boldsymbol{F}_n), \boldsymbol{F}'_n), & x_n \in [\underline{b}_n(\boldsymbol{F}_n) + C^{\mathrm{I}}, \underline{b}_n(\boldsymbol{F}_n)), \\ W_n(x_n, \boldsymbol{F}'_n), & x_n \in [\underline{b}_n(\boldsymbol{F}_n), \bar{b}_n(\boldsymbol{F}_n)], \\ \left(\phi^{\mathrm{W}} s_{N-1} - c^{\mathrm{W}}\right) [x_n - \bar{b}_n(\boldsymbol{F}_n)] + W_n(\bar{b}_n(\boldsymbol{F}_n), \boldsymbol{F}'_n), & x_n \in (\underline{b}_n(\boldsymbol{F}_n), \bar{b}_n(\boldsymbol{F}_n) + C^{\mathrm{W}}], \\ \left(\phi^{\mathrm{W}} s_{N-1} - c^{\mathrm{W}}\right) C^{\mathrm{W}} + W_n(x_n - C^{\mathrm{W}}, \boldsymbol{F}'_n), & x_n \in (\bar{b}_n(\boldsymbol{F}_n) + C^{\mathrm{W}}, \overline{x}]. \end{cases}$$

It is easy to check that this function is piecewise linear and continuous in x_n with break points in set $\mathcal{X}^Q$ for each given $\boldsymbol{F}_n$. Therefore, the claimed properties are true in stage n. By the principle of mathematical induction, they hold in every stage. □

Proof of Proposition 8 (Representation and Lipschitz continuity). (a) This part follows directly from part (b) of Lemma 1.

(b) The claimed property is trivially true in stage N with $\mathsf{L}_N(x) = 0$ for all $x \in \mathcal{X}$. Make the induction hypothesis that this property holds in stages $n+1, \ldots, N-1$. Consider stage n and pick $\boldsymbol{F}^1_n$, $\boldsymbol{F}^2_n \in \times\Re^{N-n}_+$.

Define $C := |C^{\mathrm{I}}| \vee C^{\mathrm{W}}$ and $C' := (\phi^{\mathrm{I}} \vee \phi^{\mathrm{W}})C$. For each $a \in [-C, 0)$, we have

$$\begin{aligned} |p(a, s^2_n) - p(a, s^1_n)| &= |(\phi^{\mathrm{I}} s^2_n + c^{\mathrm{I}})a - (\phi^{\mathrm{I}} s^1_n + c^{\mathrm{I}})a| \\ &\leq \phi^{\mathrm{I}}|a||s^2_n - s^1_n| \leq \phi^{\mathrm{I}} C|s^2_n - s^1_n| \leq C'|s^2_n - s^1_n|. \end{aligned}$$

For $a = 0$, it holds that $|p(a, s^2_n) - p(a, s^1_n)| = 0 \leq \phi^{\mathrm{W}} C|s^2_n - s^1_n| \leq C'|s^2_n - s^1_n|$. For each $a \in (0, C]$, we have

$$\begin{aligned} |p(a, s^2_n) - p(a, s^1_n)| &= |(\phi^{\mathrm{W}} s^2_n - c^{\mathrm{W}})a - (\phi^{\mathrm{W}} s^1_n - c^{\mathrm{W}})a| \\ &\leq \phi^{\mathrm{W}}|a||s^2_n - s^1_n| \leq \phi^{\mathrm{W}} C|s^2_n - s^1_n| \leq C'|s^2_n - s^1_n|. \end{aligned}$$

Thus, given $x \in \mathcal{X}$ and $a \in \mathcal{A}(x) \subseteq [-C, C]$, it holds that

$$|p(a, s^2_n) - p(a, s^1_n)| \leq C'|s^2_n - s^1_n|. \tag{30}$$

Replacing $W_n(x_{n+1}, \boldsymbol{F}_n^{2\prime}) - W_n(x_{n+1}, \boldsymbol{F}_n^{1\prime})$ with $\Delta W_n(x_{n+1}, \boldsymbol{F}_n^{2\prime}, \boldsymbol{F}_n^{1\prime})$ for expositional convenience, we have

$$\begin{aligned}
\left|\Delta W_n(x_{n+1}, \boldsymbol{F}_n^{2\prime}, \boldsymbol{F}_n^{1\prime})\right| &= \delta \left|\mathbb{E}\left[V_{n+1}(x_{n+1}, \mathrm{diag}(\boldsymbol{F}_n^{2\prime})\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y})) - V_{n+1}(x_{n+1}, \mathrm{diag}(\boldsymbol{F}_n^{1\prime})\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y}))\right]\right| \\
&\leq \delta \mathbb{E}\left[\left|V_{n+1}(x_{n+1}, \mathrm{diag}(\boldsymbol{F}_n^{2\prime})\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y})) - V_{n+1}(x_{n+1}, \mathrm{diag}(\boldsymbol{F}_n^{1\prime})\boldsymbol{\beta}^{n,n+1}(\boldsymbol{Y}))\right|\right] \\
&\leq \delta \mathbb{E}\left[\mathsf{L}_{n+1}(x_{n+1}) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1| \beta_m^{n,n+1}(\boldsymbol{Y})\right] \\
&= \delta \mathsf{L}_{n+1}(x_{n+1}) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1| \mathbb{E}\left[\beta_m^{n,n+1}(\boldsymbol{Y})\right] \\
&= \delta \mathsf{L}_{n+1}(x_{n+1}) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1|, \qquad (31)
\end{aligned}$$

where the first inequality holds by the modulus inequality Resnick (1999, p. 128), the second inequality follows from the induction hypothesis, and the last equality follows from (20).

Given $x \in \mathcal{X}$ and $a \in \mathcal{A}(x)$, inequalities (30) and (31) imply that for all $\boldsymbol{F}_n^1$, $\boldsymbol{F}_n^2 \in \Re_+^{N-n}$ it holds that

$$\begin{aligned}
\left|v_n(x, a, \boldsymbol{F}_n^2) - v_n(x, a, \boldsymbol{F}_n^1)\right| &= \left|p(a, s_n^2) - p(a, s_n^1) + W_n(x_n - a, \boldsymbol{F}_n^{2\prime}) - W_n(x_n - a, \boldsymbol{F}_n^{1\prime})\right| \\
&\leq \left|p(a, s_n^2) - p(a, s_n^1)\right| + \left|W_n(x_n - a, \boldsymbol{F}_n^{2\prime}) - W_n(x_n - a, \boldsymbol{F}_n^{1\prime})\right| \\
&\leq C'|s_n^2 - s_n^1| + \delta \mathsf{L}_{n+1}(x_n - a) \sum_{m=n+1}^{N} |F_{n,m}^2 - F_{n,m}^1| \\
&\leq \{C' \vee [\delta \mathsf{L}_{n+1}(x_n - a)]\} \sum_{m=n}^{N} |F_{n,m}^2 - F_{n,m}^1|.
\end{aligned}$$

Thus, the function $v_n(x, a, \boldsymbol{F}_n)$ is Lipschitz continuous in $\boldsymbol{F}_n \in \Re_+^{N-n}$ for each given $x \in \mathcal{X}$ and $a \in \mathcal{A}(x)$.

Part (a) of this proposition and Dudley (2002, Proposition 11.2.2(a), p. 391) imply that the claimed property holds in stage n. It follows from the principle of mathematical induction that the claimed property holds in every stage. □

Lemma 3 is needed in the proof of Proposition 9.

Lemma 3 (Differentiability and unique optimal action)**.** *Under Assumption 2, for every $n \in \mathcal{N}$, if $V_n(x_n, \boldsymbol{F}_n)$ is differentiable with respect to each element of $\boldsymbol{F}_n$ at a given futures curve $\overline{\boldsymbol{F}}_n$ for given $x_n \in \mathcal{X}$, then at $(x_n, \overline{\boldsymbol{F}}_n)$ there is a unique optimal action, denoted as $a_n^*(x_n, \overline{\boldsymbol{F}}_n)$, and*

$$\left.\frac{\partial V_n(x_n, \boldsymbol{F}_n)}{\partial F_{n,m}}\right|_{\boldsymbol{F}_n = \overline{\boldsymbol{F}}_n} = \left.\frac{\partial v_n(x_n, a_n^*(x_n, \overline{\boldsymbol{F}}_n), \boldsymbol{F}_n)}{\partial F_{n,m}}\right|_{\boldsymbol{F}_n = \overline{\boldsymbol{F}}_n}, \ \forall m \in \mathcal{N}_n.$$

Proof. Consider stage n. Part (b) of Proposition 8 and Rademacher's theorem imply that the function $V_n(x_n, \boldsymbol{F}_n)$ is differentiable almost everywhere in each element of $\boldsymbol{F}_n$ for each given $x_n \in \mathcal{X}$. Fix x_n and pick $\overline{\boldsymbol{F}}_n$ such that $V_n(x_n, \boldsymbol{F}_n)$ is differentiable in each element of $\boldsymbol{F}_n$ at $\overline{\boldsymbol{F}}_n$. In particular, this means that $\partial V_n(x_n, \boldsymbol{F}_n)/\partial s_n$ exists at $\overline{s}_n$. Each function $v_n(x_n, a, \boldsymbol{F}_n)$ with $a \in \mathcal{A}'(x_n)$ is linear in s_n. This property and part (a) of Proposition 8 imply that there is a unique optimal action at $\overline{\boldsymbol{F}}_n$, because otherwise $\partial V_n(x_n, \boldsymbol{F}_n)/\partial s_n$ would not exist at $\overline{s}_n$. This implies that $V_n(x_n, \overline{\boldsymbol{F}}_n) \equiv v_n(x_n, a_n^*(x_n, \overline{\boldsymbol{F}}_n), \overline{\boldsymbol{F}}_n)$ in a neighborhood of $\boldsymbol{F}_n$. The assumed differentiability of $V_n(x_n, \boldsymbol{F}_n)$ in each element of $\boldsymbol{F}_n$ at $\overline{\boldsymbol{F}}_n$ implies that $\partial V_n(x_n, \boldsymbol{F}_n)/\partial F_{n,m} = \partial v_n(x_n, a_n^*(x_n, \overline{\boldsymbol{F}}_n), \boldsymbol{F}_n)/\partial F_{n,m}$ at $\overline{F}_{n,m}$ for all $m \in \mathcal{N}_n$. $\square$

Proof of Proposition 9 (Pathwise deltas with an optimal policy). Under Assumption 2, an optimal policy satisfies the conditions (a) and (b) stated in Assumption 1 by Proposition 8(b) and Lemma 3, respectively. Expression (19) then follows from Proposition 2. $\square$

References

Broadie, M., P. Glasserman. 1996. Estimating security price derivatives using simulation. *Management Science* **42**(2) 269–285.

Dudley, R. M. 2002. *Real Analysis and Probability*. 2nd ed. Cambridge University Press, Cambridge, UK.

Lai, G., F. Margot, N. Secomandi. 2010. An approximate dynamic programming approach to benchmark practice-based heuristics for natural gas storage valuation. *Operations Research* **58**(3) 564–582.

Resnick, S. I. 1999. *A Probability Path*. Birkhäuser, Boston, MA, USA.

Secomandi, N. 2010. Optimal commodity trading with a capacitated storage asset. *Management Science* **56**(3) 449–467.