Online Supplement: "Price Commitments with Strategic Consumers: Why it can be Optimal to Discount More Frequently...Than Optimal"

A Proofs

Proof of Lemma 1. Under the "no commitment" policy, the indifferent consumer solves

$$\frac{\int_0^{\frac{v_l}{v_h}\frac{k}{\gamma}} xf(x)dx}{\mu}(v_h - v_l) = c. \tag{1}$$

As the left-hand-side (LHS) strictly decreases with γ and the right-hand-side (RHS) is constant, there either exists a unique $\gamma \in [0,1]$ which solves (1), or, if $(v_h - v_l)\int_0^{\frac{v_l}{v_h}k} xf(x)dx > \mu c$, there does not exist a γ which solves (1), in which case $\gamma_0 = 1$. $\square$

Proof of Lemma 2. First, note that the expected sales function is given by

$$S\left(\frac{k}{\gamma}\right) = \int_0^{k/\gamma} xf(x)dx + \frac{k}{\gamma}\bar{F}\left(\frac{k}{\gamma}\right)$$

and that

$$S'\left(\frac{k}{\gamma}\right) = \frac{dS(k/\gamma)}{d\gamma} = -\frac{k}{\gamma^2}\bar{F}\left(\frac{k}{\gamma}\right).$$

Differentiating $R_s^h(\gamma)$ with respect to γ, we get:

$$\begin{aligned}\zeta_s(\gamma) &= \frac{dR_s^h(\gamma)}{d\gamma} = v_h\left(S(k/\gamma) + \gamma S'(k/\gamma)\right) - \mu c \\ &= v_h \int_0^{k/\gamma} xf(x)dx - \mu c.\end{aligned}$$

$R_s^h(\gamma)$ is concave because $\zeta_s(\gamma)$ is decreasing in γ. The optimal γ_s may be 1 (a corner solution) if $\zeta_s(1) \geq 0$ or interior, in which case solving the first-order condition $\zeta_s(\gamma) = 0$ gets the desired result. Note that $\gamma_s \neq 0$, because we assume that $v_h > c$. $\square$

Proof of Lemma 3. Under the discount-frequently policy, the indifferent consumer solves

$$(v_h - v_l)\int_0^{\delta k/\gamma_f} xf(x)dx = \mu c. \tag{2}$$

As the left-hand-side (LHS) strictly decreases with γ and the right-hand-side (RHS) is constant, there either exists a unique $\gamma \in [0,1]$ which solves (2), or, if

$$(v_h - v_l) \int_0^{\delta k/\gamma_f} x f(x) dx > \mu c,$$

there does not exist a γ which solves (2), in which case $\gamma_f = 1$. □

Proof of Theorem 1. It is useful to think of γ_f as a function of δ. γ_f can be rewritten in terms of $\hat{k}$ by comparing (2) with the definition of $\hat{k}$. There are three regimes that specify the relationship:

$$\gamma_f = \begin{cases} \delta k/\hat{k} & k \leq \hat{k} \\ \begin{cases} \delta k/\hat{k} & \frac{v_l}{v_h} \leq \delta \leq \frac{\hat{k}}{k} \\ 1 & \frac{\hat{k}}{k} \leq \delta \leq 1 \end{cases} & \hat{k} < k \leq \hat{k} v_h/v_l \\ 1 & k > \hat{k} v_h/v_l \end{cases}$$

Whenever $\gamma_f < 1$, $(v_h - v_l) \int_0^{\delta k/\gamma_f} x f(x) dx = \mu c$ holds and whenever $\gamma_f = 1$, $(v_h - v_l) \int_0^{\delta k} x f(x) dx > \mu c$ holds. Define $\hat{\delta}$ as follows:

$$\hat{\delta} = \begin{cases} 1 & k \leq \hat{k} \\ \frac{\hat{k}}{k} & \hat{k} < k \leq \hat{k} v_h/v_l \\ \frac{v_l}{v_h} & \hat{k} v_h/v_l < k. \end{cases}$$

Thus, if the firm chooses $\delta \in [v_l/v_h, \hat{\delta}]$, then $\gamma_f = \delta k/\hat{k}$. But if the firm chooses $\delta \in [\hat{\delta}, 1]$, then $\gamma_f = 1$. The revenue function can be written in terms of δ and $\hat{k}$. As $\hat{k}$ is fixed, the revenue function can be expressed exclusively in terms of δ (without an implicit function defining another term):

$$R_f(\delta; \hat{k}) = \begin{cases} v_l k F\left(\hat{k}\right) + v_h \left(\frac{\delta k}{\hat{k}}\right) \int_{\hat{k}}^{\hat{k}/\delta} x f(x) dx + v_h k \bar{F}\left(\frac{\hat{k}}{\delta}\right) & v_l/v_h \leq \delta \leq \hat{\delta} \\ v_l k F(\delta k) + v_h \int_{\delta k}^{k} x f(x) dx + v_h k \bar{F}(k) & \hat{\delta} < \delta \leq 1. \end{cases}$$

Differentiate the revenue function:

$$\frac{dR_f}{d\delta} = \begin{cases} v_h \left(\frac{k}{\hat{k}}\right) \int_{\hat{k}}^{\hat{k}/\delta} x f(x) dx & v_l/v_h \leq \delta \leq \hat{\delta} \\ -v_h k^2 f(\delta k) \left(\delta - \frac{v_l}{v_h}\right) & \hat{\delta} < \delta \leq 1 \end{cases}$$

It is immediately clear that $\delta > \hat{\delta}$ is not optimal - in this case the firm is marking down more frequently than optimal, but there is no benefit in terms of increased high type demand (they are already all visiting). It is also apparent that it is optimal to choose $\delta = \hat{\delta}$. □

Proof of Theorem 2. To search for the optimal pricing policy, start by fixing γ, the high-type's search strategy. For a given γ, find a set of prices, one for each possible demand realization, such that the firm's revenue is maximized and γ is the optimal strategy for high-type consumers. The strategy γ is optimal if the high-types' expected value of search equals their cost of search conditional that γ fraction of high-type consumers visit. Next, inspect the set of chosen prices to confirm that the set can be implemented with discount-frequently. Finally, if discount-frequently is optimal for any given γ, then it must be the optimal policy overall.

Begin with some definitions. There exists a threshold demand realization, $\hat{x} = k/\gamma$, such that for each $x < \hat{x}$, high-type demand, γx, is strictly less than capacity. For $x < \hat{x}$, let $s(r,x)$ be the sales function for demand realization x and price r:

$$s(r,x) = \begin{cases} k & r \leq v_l \\ \gamma x & v_l < r \leq v_h \end{cases} \tag{3}$$

For $x \geq \hat{x}$, the firm sells k units for every $r \leq v_h$.

Let $\mathcal{P} = \{p : p : \mathbb{R}^+ \to \mathbb{R}^+\}$ be the class of price functions that maps each demand realization x to its price $p(x)$. The firm's objective is to choose a function $p \in \mathcal{P}$ that maximizes revenue conditional on a search constraint that stipulates that the expected value a high-type receives from search is at least as great as the cost of search:

$$\begin{aligned} \max_{p \in \mathcal{P}} R(p) &= \max_{p \in \mathcal{P}} \int_0^{\hat{x}} p(x) s(p(x), x) f(x) dx + k \int_{\hat{x}}^{\infty} p(x) f(x) dx \\ s.t. \quad & \int_0^{\hat{x}} (v_h - p(x)) \frac{x f(x)}{\mu} dx + \frac{k}{\gamma} \int_{\hat{x}}^{\infty} (v_h - p(x)) \frac{f(x)}{\mu} dx \geq c \end{aligned} \tag{4}$$

The search constraint (4) can be rewritten as:

$$\int_0^{\hat{x}} p(x) x f(x) dx + \frac{k}{\gamma} \int_{\hat{x}}^{\infty} p(x) f(x) dx \leq g(\gamma),$$

where

$$g(\gamma) = v_h S\left(\frac{k}{\gamma}\right) - \mu c.$$

Note that $g(\gamma)$ is independent of the chosen prices. Define the slack in the search constraint as:

$$g(\gamma) - \int_0^{\hat{x}} p(x) x f(x) dx - \frac{k}{\gamma} \int_{\hat{x}}^{\infty} p(x) f(x) dx.$$

An increase in any price $p(x)$ has two effects: (i) it increases revenue and (ii) it decreases the slack in the constraint. Thus, an optimal $p(x)$ can be found by continuously increasing the set of prices so as to maximize the ratio of the marginal increase in revenue to the marginal decrease in the slack.

Define $p_l \in \mathcal{P}$ as the constant function $p(x) = v_l \; \forall x$. With this policy the firm generates $v_l k$ in revenue, which is a lower bound on the revenue that can be achieved with the optimal policy. Given k, if the search constraint is not satisfied with this pricing policy, then γ cannot be the equilibrium search strategy in the optimal policy (because it does not generate at least $v_l k$ in revenue). Thus, it is sufficient to consider values of γ such that the search constraint is satisfied with p_l: i.e., it must be that γ is sufficiently small so that

$$(v_h - v_l) S\left(\frac{k}{\gamma}\right) \geq \mu c.$$

In other words, with the pricing strategy p_l, there must be some slack in the search constraint.

Starting with p_l, we next increase prices for some values of x so as to increase revenue while not violating the search constraint. For the most part, revenue increases and the slack decreases smoothly in price for each x, except for the very first increase in the price above v_l. The first incremental price increase above v_l yields a discontinuous decrease in revenue (because all low-type shoppers abandon their purchases). Thus, for all $x \leq \hat{x}$, the first incremental price increase is particularly costly - it decreases the slack without increasing revenue. In fact, the firm generates the same revenue with price v_l as it does with price $\hat{p}(x) = v_l k/(\gamma x)$.

Thus, for all $x \leq \hat{x}$, the optimal policy either charges v_l, or some price $\hat{p}(x) \leq p(x) \leq v_h$. In that range, additional increases in price generate smooth increases in revenue and smooth decreases in the slack. In particular, there are two cases to consider:

Case 1: $x \leq \hat{x}$, $\hat{p}(x) \leq p(x) \leq v_h$. The marginal increase in revenue is $\partial R(p)/\partial p(x) = \gamma x f(x)$ and the marginal decrease in the slack with respect to price is $xf(x)$. Thus, the increase in revenue per unit of decrease in slack is:

$$\gamma x f(x)/xf(x) = \gamma.$$

Case 2: $\hat{x} < x$. The marginal increase in revenue is $\partial R(p)/\partial p(x) = kf(x)$ and the marginal decrease in the slack with respect to price is $(k/\gamma)f(x)$. Thus, the increase in revenue per unit of decrease is slack is:

$$kf(x)/(k/\gamma)f(x) = \gamma.$$

To repeat, for all $x \leq \hat{x}$, the first incremental price increase above v_l actually decreases revenue and slack. Hence, any price increase above v_l should first be done in the $\hat{x} < x$ demand states. In these states, all price increases generate the same constant increase in the ratio of revenue to slack (case 2), the optimal price is the maximum price so long as the search constraint is satisfied. Therefore, starting with the highest demand states in the range $\hat{x} < x$, increase the price from v_l to the maximum price, v_h, until either the search constraint binds (i.e., all slack is consumed) or the price is increased in all of these demand states. This yields the following price function:

$$p(x) = \begin{cases} v_l & x \leq \tilde{x} \\ v_h & otherwise \end{cases}$$

for some $\tilde{x} \geq \hat{x}$. If $\tilde{x} > \hat{x}$, then no slack remains and the above pricing strategy is the optimal solution. If $\tilde{x} = \hat{x}$, then some slack remains in the search constraint and price increases in the $x \leq \hat{x}$ can be considered, which is done next.

For $x \leq \hat{x}$, an increase in price from v_l to $\hat{p}(x)$ increases revenue by $(\gamma xp - kv_l)f(x)$ and decreases slack by $(p - v_l)xf(x)$. The relative increase in revenue to slack consumed is the ratio:

$$\frac{(\gamma xp - kv_l)f(x)}{(p - v_l)xf(x)} = \frac{\gamma}{p - v_l}\left(p - \frac{kv_l}{\gamma x}\right)$$

which is increasing in x. This implies that, for $x \leq \hat{x}$, if the price is increased above v_l, then it should be increased for the highest demand state with the price still at v_l. Furthermore, because the marginal increase in revenue to consumed slack equals γ for all price increases above $\hat{p}(x)$ (case 1) and the initial price increase from v_l to $\hat{p}(x)$ is costly, if a price is increased above $\hat{p}(x)$, then it should be increased all the way to v_h. This leads to a pricing policy in which the firm charges either v_l or v_h, and the search constraint is binding. In particular,

$$p(x) = \begin{cases} v_l & x \leq x' \\ v_h & otherwise \end{cases}$$

where $x' \leq \tilde{x}$. The above can be implemented as a discount-frequently policy. Therefore, for a given γ, discount-frequently maximizes revenue, which implies it is an optimal policy. $\square$

Proof of Theorem 3. (1) Conditional on observing γ, the firm's revenue is maximized by discounting to v_l

when realized high-type demand is less than $(v_l k)/(v_h \gamma)$, otherwise the firm charges the high price v_h. If the firm follows this policy and all high-type consumers are myopic (i.e., they surely visit the firm), then the firm's profit is $\Pi_m = v_l k + v_h \left(S(k) - S\left(\frac{v_l k}{v_h}\right)\right) - c_k k$. Differentiate Π_m with respect to k: $\eta(k) = d\Pi_m/dk = v_l F\left(\frac{v_l k}{v_h}\right) + v_h \bar{F}(k) - c_k$. Π_m is quasi-concave because $\eta(0) = v_h - c_k > 0$, $\lim_{k\to\infty} \eta(k) = v_l - c_k < 0$ and $\eta(k)$ is decreasing when $\eta(k) = 0$ (which implies there is a unique k such that $\eta(k) = 0$). To demonstrate the latter, note that there is a unique solution to $\eta(k)/v_h \bar{F}(k) = 0$ because $\eta(k)/v_h \bar{F}(k)$ is increasing in k :

$$\eta(k)/v_h \bar{F}(k) = \frac{v_l F\left(\frac{v_l k}{v_h}\right) - c_k}{v_h \bar{F}(k)} + 1. \tag{5}$$

(2) Differentiate the profit function, $\Pi_0 = R_0 - c_k k$ with respect to k :

$$\frac{d\Pi_0}{dk} = \begin{cases} v_l + \frac{v_l}{\hat{k}}\left(S\left(\frac{v_h \hat{k}}{v_l}\right) - S(\hat{k})\right) - c_k & k < \frac{v_h}{v_l}\hat{k} \\ v_l F\left(\frac{v_l k}{v_h}\right) + v_h \bar{F}(k) - c_k & k \geq \frac{v_h}{v_l}\hat{k}. \end{cases}$$

When $k < \frac{v_h}{v_l}\hat{k}$, $\Pi_0(k)$ is linear and increasing if $c_k < v_l + \frac{v_l}{\hat{k}}\left(S\left(\frac{v_h \hat{k}}{v_l}\right) - S(\hat{k})\right)$. If $k \geq \frac{v_h}{v_l}\hat{k}$, $\Pi_0(k)$ is quasi-concave with a unique solution given by k_m (see part (1)). Therefore, $k_0 = \max\left\{v_h \hat{k}/v_l, k_m\right\}$, if $c_k < c_k^0$ and 0 otherwise. (3) If $v_h \bar{F}(k/\gamma_s) \leq v_l$, the firm charges v_l and will choose $k_s = 0$ because $c_k > 0$. Assume that $v_h \bar{F}(k/\gamma_s) > v_l$, so the firm charges $p_s > v_l$. Differentiate the profit function, $\Pi_s = R_s - c_k k$ with respect to k :

$$\frac{d\Pi_s}{dk} = \begin{cases} v_h \bar{F}\left(\tilde{k}\right) - c_k & k < \tilde{k} \\ v_h \bar{F}(k) - c_k & k \geq \tilde{k}. \end{cases}$$

Observe that for $k < \tilde{k}$ the function is linear and increasing if $c_k < v_h \bar{F}\left(\frac{k}{\gamma_s}\right)$. If $k \geq \tilde{k}$, $d\Pi_s(k)/dk$ is decreasing and hence $\Pi_s(k)$ is concave, which guarantees a solution exists, is unique and is given by $\bar{F}(k') = c_k/v_h$ if $k' > \tilde{k}$ and by $\tilde{k}$ otherwise. (4) Differentiate the profit function $\Pi_f = R_f - c_k$ with respect to k:

$$\frac{d\Pi_f}{dk} = \begin{cases} v_l + (v_h - v_l)\,\bar{F}\left(\hat{k}\right) - c_k & k \leq \hat{k} \\ v_l F\left(\hat{k}\right) + v_h \bar{F}(k) - c_k & \hat{k} < k \leq \hat{k} v_h/v_l \\ v_l F\left(\frac{v_l}{v_h} k\right) + v_h \bar{F}(k) - c_k & \hat{k} v_h/v_l < k \end{cases} \tag{6}$$

From the $k < \hat{k}$ part, it follows that $k_f > 0$ if and only if $c_k < c_k^f$. $d\Pi_f/dk$ is continuous and decreasing (strictly decreasing for $k \geq \hat{k}$. This implies that there exists a unique capacity level k that maximizes profits. Solving $v_l F\left(\hat{k}\right) + v_h \bar{F}(k) - c_k = 0$, we get the first candidate k''. $k'' \geq \hat{k}$ if $c_k < c_k^f$ and comparing it with k_m, we get the desired result. □

Proof of Theorem 4 . (1) Proof of $k_s < k_m$: Let $\tilde{\tau}_s = v_h \bar{F}(\tilde{k})$, $\tau_s(k) = v_h \bar{F}(k)$, and $\tau_m(k) = v_l F\left(\frac{v_l k}{v_h}\right) + v_h \bar{F}(k)$. If $\tilde{\tau}_s \leq c_k$, then $k_m > k_s = 0$. Suppose now that $\tilde{\tau}_s > c_k$. Since $\tau_s(k) < \tau_m(k)\ \forall k$ and $\tau_s'(k) < 0$ and $\tau_m'(k) < 0$, it must be that $k_s < k_m$. Note that $k_s = F^{-1}(c_k/v_h) < \tilde{k}$ since $\tilde{\tau}_s > c_k$. Proof of $k_m \leq k_f$: This immediately follows from Theorem 3. If $k_m \geq k''$, then $k_f = k_m$. Otherwise, $k_f > k_m$. (2) If $c_k = \underline{c}_k$, then $k_m = v_h \hat{k}/v_l = k_0 = k'' = k_f$. Thus, for $c_k \leq \underline{c}_k$ we have $k_m = k_0 = k_f$. For $c_k > \underline{c}_k$, $k'' > k_m$ and k'' is decreasing in c_k. Thus, $k_f = k'' < v_h \hat{k}/v_l = k_0$. □