

Online Supplement

Appendix A

In Proposition A1, we show that the expected client's profit defined by (6) is concave for each value of g_i (holding all values of g_j constant for $j \neq i$) such that the problem of finding equilibrium \hat{g}_i values can be decomposed into n subproblems if solved in a backward sequence $n, n-1, n-2, \dots, 1$. The latter result leads to the development of an efficient backward search procedure for finding the equilibrium values \hat{g}_i .

Proposition A1. Given values of $\beta_i \geq 0$, the expected client's profit $E[\Pi_C^{\mathbb{I}}(\beta_1, \dots, \beta_n)]$ defined by (6) is concave for $g_i = \frac{\alpha}{\alpha + r_i}$ when g_j (for all $j \neq i$) are fixed. Furthermore, the problem of finding unique equilibrium \hat{g}_i values that maximize (6) can be decomposed into n subproblems.

Proof: See Appendix C.

Using the results from Proposition A1, we can define a search procedure that we denote as Algorithm *SIP* (“Search for Incentive Payments”) to find the unique equilibrium values of g_i . The correctness of Algorithm *SIP* is implied by the proof of Proposition A1. While the algorithm is presented for finding equilibrium subcontractor payments when contract \mathbb{I} is used, it is immediately applicable for contract \mathbb{F} by setting $\beta_i = 0$ for all i , and it can be easily modified to find equilibrium subcontractor payments for other related contracts as well.

Algorithm SIP.

- 1) Set $j = n$.
- 2) Use binary search on g_j to find the value \hat{g}_j that satisfies

$$\frac{\alpha k_j}{\alpha + \beta_j} \left[\beta_j \left(1 + \frac{1}{(g_j)^2} - 2g_j \right) + \alpha \left(\frac{2}{(g_j)^3} - \frac{1}{(g_j)^2} - 1 \right) \right] = \text{Constant}(j)$$

where

$$\text{Constant}(j) = \begin{cases} Q \prod_{h=j+1}^n (1 - \hat{g}_h) - \sum_{m=j+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{h=j+1}^{m-1} (1 - \hat{g}_h), & \text{for } j < n \\ Q, & \text{for } j = n. \end{cases}$$

- 3) Calculate $\hat{r}_j = \frac{\alpha(1 - \hat{g}_j)}{\hat{g}_j}$ and \hat{p}_j using (4).

- 4) Set $j = j-1$ and go to step 2, letting $g_k = \hat{g}_k$ and $p_k = \hat{p}_k$ for all $k = j+1, \dots, n$.

The equation used in step 2 to find \hat{g}_j is defined by setting

$$\frac{\partial E[\Pi_C^j(\beta_1, \dots, \beta_n)]}{\partial g_j} = 0; \text{ details are given in the proof of Proposition A1 in Appendix C.}$$

When searching for \hat{g}_j in step 2 of the algorithm, it should be noted that the term $\text{Constant}(j)$ is defined by the values $(\hat{g}_{j+1}, \dots, \hat{g}_n)$ that were calculated in the previous iterations (since we use a backward search method). Given values of \hat{g}_i , the equilibrium expected client's profit $E[\Pi_C^{j*}(\beta_1, \dots, \beta_n)]$ can be found using (6); conversely, we can find values $\hat{r}_i = \frac{\alpha(1 - \hat{g}_i)}{\hat{g}_i}$ and \hat{p}_i from (4) and the maximum expected client's profit using (5).

Appendix B: Modified SIP Algorithm When Opportunity Costs $O_i > 0$

B.1 Modifications for Contract \mathbb{F}

When opportunity costs are positive, the SIP Algorithm (described in Appendix A) can be modified for contract \mathbb{F} as follows:

- 1) Calculate the prices $p_n^{SIP}, p_{n-1}^{SIP}, \dots, p_1^{SIP}$ from Algorithm SIP.
- 2) For $j=1, \dots, n$, $p_i^{\mathbb{F}} = \max\{p_i^{SIP}, \underline{p}_i(O_i)\}$.

Recall that Algorithm SIP necessarily calculates the prices in reverse order: $p_n^{SIP}, p_{n-1}^{SIP}, \dots, p_1^{SIP}$. However, the second step in our procedure must be processed in the natural order $p_1^{\mathbb{F}}, p_2^{\mathbb{F}}, \dots, p_n^{\mathbb{F}}$ since increasing subcontractor i 's price will also increase her work rate, which will result in a *decrease* in the minimum price $\underline{p}_j(O_j)$ for all successors $j > i$ (due to the structure of the profit function). Therefore, there is a “backwards pass” to determine the SIP prices, and then a “forward pass” to determine the adjusted prices.

B.2. Modifications for Contract \mathbb{I}

In this section, for simplicity, we assume that $K_i = 0$ for all i . Under contract \mathbb{I} , the price adjustments require a more involved procedure. Subcontractor i 's expected profit, given in Equation (1), evaluated at the equilibrium work rate r_i^* of Proposition 1, can be written as

$$E[\pi_i^{\mathbb{I}}(r_i^*, \beta_i)] = \left(\frac{k_i r_i^* (r_i^{*2} + 2r_i^* \alpha)}{(\alpha + \beta_i)(\alpha + r_i^*)} - \frac{k_i r_i^{*2}}{\alpha + r_i^*} \right) \prod_{j=1}^{i-1} \frac{r_j^*}{\alpha + r_j^*}.$$

Next, we use the substitution $g_i = \frac{\alpha}{\alpha + r_i^*}$, that results in the profit expression

$$E[\pi_i^{\mathbb{I}}(g_i, \beta_i)] = \alpha k_i \left(\frac{\alpha}{(\alpha + \beta_i)} \left(\frac{1}{g_i^2} - 1 - \frac{1}{g_i} + g_i \right) - \left(\frac{1}{g_i} - 2 + g_i \right) \right) \prod_{j=1}^{i-1} (1 - g_j).$$

For a given β_i , Algorithm SIP gives $\hat{g}_i(\beta_i)$, which maximizes the client's profit; suppressing the dependence on β_i for clarity (i.e., $\hat{g}_i = \hat{g}_i(\beta_i)$), the subcontractor's (maximized) profit is now only a function of β_i : $E[\pi_i^{\mathbb{I}}(\beta_i)]$. The derivative of this profit with respect to β_i can be written as

$$\frac{\partial E[\pi_i^{\mathbb{I}}(\beta_i)]}{\partial \beta_i} = \frac{\alpha k_i}{(\alpha + \beta_i) \hat{g}_i^2} \left[(1 - \hat{g}_i) \left(\frac{\beta_i \hat{g}_i (1 + \hat{g}_i) - 2\alpha}{\hat{g}_i} \right) \frac{\partial \hat{g}_i}{\partial \beta_i} - \frac{\alpha}{(\alpha + \beta_i)} (1 + \hat{g}_i) (1 - \hat{g}_i)^2 \right] \prod_{j=1}^{i-1} (1 - g_j)$$

The one unknown quantity in this expression is the partial derivative $\frac{\partial \hat{g}_i}{\partial \beta_i}$. However,

recalling step 2 of SIP Algorithm, we see that \hat{g}_i is determined by setting

$$\frac{\alpha k_j}{\alpha + \beta_j} \left[\beta_j \left(1 + \frac{1}{(g_j)^2} - 2g_j \right) + \alpha \left(\frac{2}{(g_j)^3} - \frac{1}{(g_j)^2} - 1 \right) \right]$$

equal to a constant; the derivative of this expression with respect to β_i can be shown to be negative, which implies that $\frac{\partial \hat{g}_i}{\partial \beta_i} < 0$. In addition, since $\hat{g}_i \in (0,1)$ for all $\beta_i \geq 0$, the derivative $\frac{\partial \hat{g}_i}{\partial \beta_i} > -1$ for all $\beta_i \geq 0$, except possibly on a set of β_i with measure strictly less than 1; in other words, \hat{g}_i decreases more slowly than β_i increases. Therefore, for a large enough β_i , the derivative $\frac{\partial E[\pi_i^{\mathbb{I}}(\beta_i)]}{\partial \beta_i}$ is negative.

When opportunity costs are positive, the SIP Algorithm (described in Appendix A) can be modified for contract \mathbb{I} as follows:

In the natural order $i = 1, \dots, n$, we evaluate the subcontractor's expected profit under the SIP prices when $\beta_i = 0$.

1) If $E[\pi_i^{\mathbb{I}}(0)] < O_i$, then apply the adjustment analysis from Contract \mathbb{F} to raise subcontractor i 's profit to O_i by determining the adjusted price $p_i^{\mathbb{F}} = \max\{p_i^{\text{SIP}}, \underline{p}_i(O_i)\}$.

2) Alternatively, if $E[\pi_i^{\mathbb{I}}(0)] > O_i$, β_i is increased to lower subcontractor i 's profit to O_i .

Appendix C

Proof of Proposition 1.

Given that $E[\pi_i^{\mathbb{I}}] = \left(\frac{p_i r_i}{\alpha + \beta_i + r_i} - \frac{K_i + k_i r_i^2}{\alpha + r_i} \right) \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j}$,

setting the first order condition (FOC) to zero and holding all values of r_j ($l \neq i$) constant, we get

$$\frac{\partial E[\pi_i^1]}{\partial r_i} = \left[\frac{p_i(\alpha + \beta_i)}{(\alpha + \beta_i + r_i)^2} - \frac{k_i(r_i^2 + 2\alpha r_i) - K_i}{(\alpha + r_i)^2} \right] \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j} = 0, \text{ implying that the equilibrium}$$

work rate for subcontractor i is equal to

$$r_i^* = \sqrt{\alpha^2 + \frac{p_i(\alpha + \beta_i)(\alpha + r_i^*)^2}{k_i} + \frac{K_i}{k_i}} - \alpha. \quad (C0)$$

The second order condition (SOC) confirms that this maximizes $E[\pi_i^1]$; that is,

$$\frac{\partial^2 E[\pi_i^1]}{\partial r_i^2} = \left[-\frac{2(\alpha + \beta_i)p_i}{(\alpha + \beta_i + r_i)^3} - \frac{2\alpha^2 k_i + K_i}{(\alpha + r_i)^3} \right] \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j} < 0.$$

To prove uniqueness, rewrite (C0) as

$$\frac{p_i(\alpha + \beta_i)(\alpha + r_i)^2}{k_i(\alpha + \beta_i + r_i)^2} + \frac{K_i}{k_i} = (r_i)^2 + 2\alpha r_i. \quad (C1)$$

Denote the left-hand-side of (C1) as LHS, and similarly define the RHS. From (C1) the $LHS_{(r_i=0)} > RHS_{(r_i=0)}$ and $\lim_{r_i \rightarrow \infty} LHS < \lim_{r_i \rightarrow \infty} RHS$. Moreover, the derivative of $LHS_{(r_i)}$ is positive and strictly decreasing in r_i while the derivative of $RHS_{(r_i)}$ is positive and strictly increasing in r_i . Consequently, for a given β_i and p_i the positive solution r_i^* that satisfies (C0) is unique.

Q.E.D.

Proof of Proposition 2.

When subcontractor i selects its optimal work rate as defined in (2), we can rewrite p_i as a function of r_i^* :

$$p_i = \frac{\left[(r_i^*)^2 + 2\alpha r_i^* \right] (\alpha + \beta_i + r_i^*)^2 k_i}{(\alpha + \beta_i) (\alpha + r_i^*)^2}. \quad (\text{C2})$$

Substituting this expression for p_i into (1) we have

$$E[\pi_i^{\text{II}}] = \left[\frac{k_i (r_i^*)^2 \left((r_i^*)^2 + 2\alpha r_i^* + \alpha^2 + \alpha\beta \right)}{(\alpha + \beta_i) (\alpha + r_i^*)^2} \right] \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j} > 0. \quad (\text{C3})$$

Q.E.D.

Proof of Corollary 1

Replace all r_i^* with $\alpha + \beta_i$ in the expression for $\frac{\partial r_i^*}{\partial \beta_i}$, and we have

$$\frac{\alpha^2}{(2\alpha + \beta_i)^2} + \frac{p_i}{4k_i} \frac{1}{(\alpha + \beta_i)} = 1.$$

Define $f(\beta_i) = \frac{\alpha^2}{(2\alpha + \beta_i)^2} + \frac{p_i}{4k_i} \frac{1}{(\alpha + \beta_i)}$, which is strictly decreasing in β_i . Note that β_i is defined for $\beta_i \geq 0$. If $f(0) < 1$, then $p_i < 3\alpha k_i$. Hence, there does not exist a value of β_i where $r_i^* = \alpha + \beta_i$. Alternatively, if $p_i \geq 3\alpha k_i$, then $f(0) \geq 1$. If $f(0) = 1$, since $f(\beta_i)$ is strictly decreasing, $\hat{\beta}_i = 0$ is the only value of β_i that allows $r_i^* = \alpha + \beta_i$. If $f(0) > 1$, we need to make sure that there exists a value of $\beta_i > 0$ where $f(\beta_i) \leq 1$; this is easily seen by taking the limit

$$\lim_{\beta_i \rightarrow \infty} f(\beta_i) = 0.$$

Therefore, there exists a unique $\hat{\beta}_i > 0$ where $r_i^* = \alpha + \hat{\beta}_i$.

Q.E.D.

Proof of Corollary 2.

We provide this proof by contradiction. Taking the derivative of (C3) with respect to β_i , we have

$$\frac{\partial E[\pi_i^{I^*}]}{\partial \beta_i} = 2k_i r_i^* \left[\frac{\left[(r_i^*)^3 + 3(r_i^*)^2 + 3\alpha^2 r_i^* + \alpha^2(\alpha + \beta_i) \right] (\alpha + \beta_i) \frac{\partial r_i^*}{\partial \beta_i} - \left(\alpha + \frac{r_i^*}{2} \right) (\alpha + r_i^*) (r_i^*)^2}{(\alpha + \beta_i)^2 (\alpha + r_i^*)^3} \right] \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j},$$

which implies

$$\frac{\partial E[\pi_i^{I^*}]}{\partial \beta_i} > 0 \quad \text{iff} \quad \frac{\partial r_i^*}{\partial \beta_i} > \frac{\left(\alpha + \frac{r_i^*}{2} \right) (\alpha + r_i^*) (r_i^*)^2}{\left[(r_i^*)^3 + 3(r_i^*)^2 + 3\alpha^2 r_i^* + \alpha^2(\alpha + \beta_i) \right] (\alpha + \beta_i)}. \quad (C4)$$

Given that $\frac{\partial r_i^*}{\partial \beta_i} = \frac{(\alpha + r_i^*) \left[\frac{r_i^* - (\alpha + \beta_i)}{\alpha + \beta_i + r_i^*} \right]}{2(\alpha + \beta_i) \left[\frac{\alpha^2}{r_i^* (2\alpha + r_i^*)} + \frac{\alpha + r_i^*}{\alpha + \beta_i + r_i^*} \right]}$,

we can substitute $\frac{\partial r_i^*}{\partial \beta_i}$ from this expression into (C4) and after a couple of iterations of

rearranging terms we can observe the following equivalence:

$$\frac{\partial E[\pi_i^{I^*}]}{\partial \beta_i} > 0 \quad \text{iff} \quad -\frac{1}{2} \frac{(\alpha + \beta_i)(2\alpha + r_i^*) r_i^*}{(r_i^*)^3 + 3(r_i^*)^2 + 3\alpha^2 r_i^* + \alpha^2(\alpha + \beta_i)} > 0.$$

However, the later inequality is impossible, since $\alpha, \beta_i, r_i^* \geq 0$. Hence

$$\frac{\partial E[\pi_i^{I^*}]}{\partial \beta_i} < 0 \quad \forall \beta_i \geq 0. \quad Q.E.D.$$

Proof of Proposition A1.

Taking the partial derivative of (6) with respect to g_n and setting it to zero, we have

$$\frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_n} = \left\{ \frac{\alpha k_n}{\alpha + \beta_n} \left[\beta_n \left(1 + \frac{1}{(g_n)^2} - 2g_n \right) + \alpha \left(\frac{2}{(g_n)^3} - \frac{1}{(g_n)^2} - 1 \right) \right] - Q \right\} \prod_{j=1}^{n-1} (1 - g_j) = 0,$$

implying that

$$\beta_n \left(1 + \frac{1}{(g_n)^2} - 2g_n \right) + \alpha \left(\frac{2}{(g_n)^3} - \frac{1}{(g_n)^2} - 1 \right) - \frac{Q(\alpha + \beta_n)}{\alpha k_n} = 0. \quad (C5)$$

We denote the left-hand-side of (C5) as LHS. From (C5) $\lim_{g_n \rightarrow 0} \text{LHS} = \infty > 0$
 $\text{LHS}_{(g_n=1)} = -\frac{Q(\alpha + \beta_n)}{\alpha k_n} < 0$, and LHS is strictly decreasing in g_n . We can conclude that

$E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave in g_n and that there exists a unique optimal solution \hat{g}_n that maximizes the Client's expect profit. Note that \hat{g}_n does not depend on g_i for any $i < n$.

Next, we use strong induction to show that $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave in g_i , with a unique optimal solution \hat{g}_i for all i , where \hat{g}_i depends only on g_j for $j > i$. This characteristic implies that we are able to decompose the Client's problem into n sub-problems and solve them in reverse order $j=n, \dots, 1$. The base case, where $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave in g_n with a unique optimal solution \hat{g}_n , has already been proven. For the induction step, we assume that $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave in g_j for all $j > i$, with unique optimal solutions \hat{g}_j ; furthermore \hat{g}_j does not depend on g_k , for $k \leq i$ and we set $g_j = \hat{g}_j$ for $j \geq i+1$. We then show that $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave in g_i with a unique optimal solution \hat{g}_i . Take the derivative of $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ with respect to g_i , and letting it equal zero, we have (for $i < n$):

$$\begin{aligned} \frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_i} &= \frac{\alpha k_i}{\alpha + \beta_i} \left[\beta_i \left(1 + \frac{1}{(g_i)^2} - 2(g_i) \right) + \alpha \left(\frac{2}{(g_i)^3} - \frac{1}{(g_i)^2} - 1 \right) \right] \\ &\quad - \underbrace{Q \prod_{j=i+1}^n (1 - \hat{g}_j) + \sum_{m=i+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{j=i+1}^{m-1} (1 - \hat{g}_j)}_{f(\hat{g}_{i+1}, \dots, \hat{g}_n)} = 0. \end{aligned} \tag{C6}$$

Here we invoke the inductive hypothesis: at the optimal \hat{g}_{i+1} , we have that

$$\frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_{i+1}} = 0. \text{ Multiplying this equality (Equation (C6) with } i \text{ replaced by } i+1)$$

by $(1 - \hat{g}_{i+1})$, we have

$$\begin{aligned} \frac{\alpha k_{i+1}}{\alpha + \beta_{i+1}} (1 - \hat{g}_{i+1}) &\left[\beta_{i+1} \left(1 + \frac{1}{(\hat{g}_{i+1})^2} - 2\hat{g}_{i+1} \right) + \alpha \left(\frac{2}{(\hat{g}_{i+1})^3} - \frac{1}{(\hat{g}_{i+1})^2} - 1 \right) \right] \\ &- Q \prod_{j=i+1}^n (1 - \hat{g}_j) + \sum_{m=i+2}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{j=i+1}^{m-1} (1 - \hat{g}_j) = 0 \end{aligned} \tag{C7}$$

Denote the left-hand-side of (C7) as LHS2; note that LHS2 = 0. Let $f(\hat{g}_{i+1}, \dots, \hat{g}_n)$

denote the constant

$$\left\{ -Q \prod_{j=i+1}^n (1 - \hat{g}_j) + \sum_{m=i+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{j=i+1}^{m-1} (1 - \hat{g}_j) \right\} \text{ in Equation (C6).}$$

Subtracting $f(\hat{g}_{i+1}, \dots, \hat{g}_n)$ from LHS2, we get the result that

$$\frac{\beta_{i+1}(\hat{g}_{i+1}^2 + 1)(1 - \hat{g}_{i+1})}{\hat{g}_{i+1}^2} + \frac{2\alpha(1 - \hat{g}_{i+1})}{\hat{g}_{i+1}^3} > 0.$$

Since LHS2 = 0 and the difference of LHS2 and $f(\hat{g}_{i+1}, \dots, \hat{g}_n)$ is positive, we can infer

that

$$-Q \prod_{j=i+1}^n (1 - \hat{g}_j) + \sum_{m=i+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{j=i+1}^{m-1} (1 - \hat{g}_j) < 0.$$

Since $\lim_{g_i \rightarrow 0} \frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_i} = \infty > 0$, $\lim_{g_i \rightarrow 1} \frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_i} = f(\hat{g}_{i+1}, \dots, \hat{g}_n) < 0$ and

$\frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial g_i}$ is strictly decreasing in g_i , we can infer that $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is concave

in g_i with a unique optimal solution $\hat{g}_i \forall i$. Since g_i has a one-to-one correspondence with r_i , $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ has a set of unique optimal solutions r_1^*, \dots, r_n^* . *Q.E.D.*

Proof of Proposition 3

The derivative of (6) with respect to β_i satisfies

$$\frac{\partial E[\Pi_C^I(\beta_1, \dots, \beta_n)]}{\partial \beta_i} = \frac{\alpha^2 k_i (1 - g_i)^3}{(\alpha + \beta_i)^2 g_i} \prod_{j=1}^{i-1} (1 - g_j) > 0.$$

This result implies that $E[\Pi_C^I(\beta_1, \dots, \beta_n)]$ is increasing in $\beta_i, \forall i$ and hence the profit function can be reduced to

$$E \left[\lim_{\beta_1, \dots, \beta_n \rightarrow \infty} E[\Pi_C^I(\beta_1, \dots, \beta_n)] \right] = Q \prod_{j=1}^n (1 - g_j) - \sum_{m=1}^n \alpha k_m (1 - g_m)^2 \left(\frac{1}{g_m} + 1 \right) \prod_{j=1}^{m-1} (1 - g_j).$$

Then from (C6), the optimality condition for $i = 1$ (with $\beta_i \rightarrow \infty \forall i$), namely

$$\frac{\partial E \left[\lim_{\beta_1, \dots, \beta_n \rightarrow \infty} E[\Pi_C^I(\beta_1, \dots, \beta_n)] \right]}{\partial g_1} = 0, \text{ is given by}$$

$$Q \prod_{j=2}^n (1 - \hat{g}_j) - \sum_{m=2}^n \alpha k_m (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \prod_{j=2}^{m-1} (1 - \hat{g}_j) = \alpha k_1 \left(1 + \frac{1}{\hat{g}_1^2} - 2\hat{g}_1 \right).$$

Multiply both sides by $(1 - \hat{g}_1)$ and then subtract $\alpha k_1 (1 - \hat{g}_1)^2 \left(\frac{1}{\hat{g}_1} + 1 \right)$, we have

$$\underbrace{Q \prod_{j=1}^n (1 - \hat{g}_j) - \sum_{m=1}^n \alpha k_m (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \prod_{j=1}^{m-1} (1 - \hat{g}_j)}_{E[\Pi_C^I(\hat{g}_1, \dots, \hat{g}_n)]} = \alpha k_1 (1 - \hat{g}_1)^2 \left(1 + \frac{1}{\hat{g}_1^2} \right) > 0.$$

where the left-hand side of this equation is the optimized Client's profit. *Q.E.D.*

Proof of Proposition 4

From (C6), the optimality condition for i is given by

$$\begin{aligned} & \frac{\alpha k_i}{\alpha + \beta_i} \left[\underbrace{\beta_i \left(1 + \frac{1}{(g_i)^2} - 2(g_i) \right) + \alpha \left(\frac{2}{(g_i)^3} - \frac{1}{(g_i)^2} - 1 \right)}_{\text{strictly decreasing in } g_i} \right] \\ &= Q \prod_{j=i+1}^n (1 - \hat{g}_j) - \sum_{m=i+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{j=i+1}^{m-1} (1 - \hat{g}_j). \end{aligned}$$

Since

$$\frac{\partial \text{LHS}}{\partial \beta_i} = \frac{2\alpha^2 k_n (1 - \hat{g}_i) \left((\hat{g}_i)^3 - 1 \right)}{(\alpha + \beta_i) (\hat{g}_i)^3} < 0, \text{ the LHS is strictly decreasing in } \beta_i, \text{ and the RHS is}$$

not affected by β_i ; hence r_i^* is increasing in β_i .

We next examine the influence of successors. Again, from (C6), the optimality condition for g_i is

$$\begin{aligned} & \frac{\alpha k_i}{\alpha + \beta_i} \left[\underbrace{\beta_i \left(1 + \frac{1}{(g_i)^2} - 2(g_i) \right) + \alpha \left(\frac{2}{(g_i)^3} - \frac{1}{(g_i)^2} - 1 \right)}_{\text{strictly decreasing in } g_i} \right] \\ &= Q \prod_{j=i+1}^n (1 - \hat{g}_j) - \sum_{m=i+1}^n \frac{\alpha k_m}{\alpha + \beta_m} (1 - \hat{g}_m)^2 \left(\frac{1}{\hat{g}_m} + 1 \right) \left(\frac{\alpha}{\hat{g}_m} + \beta_m \right) \prod_{k=i+1}^{m-1} (1 - \hat{g}_k). \end{aligned}$$

From (6), we can interpret the RHS as the expected profit of a reduced problem consisting of only subcontractors $i+1$ to n . Hence we can denote the RHS as

$E[\Pi_C^I(\beta_{i+1}, \dots, \beta_n)]$. From Proposition 3, we can conclude that the RHS is increasing in β_j , $\forall j = i+1, \dots, n$, that is

$$\frac{\partial E[\Pi_C^I(\beta_{i+1}, \dots, \beta_n)]}{\partial \beta_j} = \frac{\alpha^2 k_j (1 - g_j)^3}{(\alpha + \beta_j)^2 g_j} \prod_{k=1}^{j-1} (1 - g_k) > 0, \quad \forall j = i+1, \dots, n.$$

Since the LHS is not affected by $\beta_j, j = i+1, \dots, n$ and is strictly decreasing in g_i ,

decreasing \hat{g}_i increases r_i^* .

Q.E.D.

Proof of Proposition 5.

Given that $E[\pi_i^F] = p_i \prod_{k=1}^i \frac{r_k}{\alpha + r_k} - \frac{K_i + k_i r_i^2}{\alpha + r_i} \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j}$,

setting the first order condition (FOC) to zero and holding all values of r_l ($l \neq i$) constant, we get

$$\frac{\partial E[\pi_i^F]}{\partial r_i} = \left(\frac{\alpha p_i - 2\alpha k_i r_i - k_i r_i^2 + K_i}{(\alpha + r_i)^2} \right) \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j} = 0,$$

implying that the optimal work rate for subcontractor i is equal to

$$r_i^* = \sqrt{\alpha^2 + \frac{p_i \alpha + K_i}{k_i}} - \alpha.$$

The second order condition (SOC) confirms that this maximizes $E[\pi_i^F]$; that is,

$$\frac{\partial^2 E[\pi_i^F]}{\partial r_i^2} = -\frac{2\alpha(p_i + \alpha k_i + K_i)}{(\alpha + r_i)^3} \prod_{j=1}^{i-1} \frac{r_j}{\alpha + r_j} < 0.$$

Q.E.D.

Proof of Proposition 6.

From (1) and (8), we have,

$$E[\pi_i^I] = E \left[e^{-\alpha \sum_{j=1}^{i-1} t_j} \right] \times E \left[\left(p_i e^{-\beta t_i} \right) e^{-\alpha t_i} - \int_0^{t_i} (K_i + k_i r_i^2) e^{-\alpha t} dt \right].$$

and

$$E[\pi_i^D] = E \left[e^{-\alpha \sum_{j=1}^{i-1} t_j} \right] \times E \left[\left(p_i e^{-\beta t_i} \right) e^{-\alpha t_i} - \int_0^{t_i} (K_i + k_i r_i^2) e^{-\alpha t} dt \right].$$

Letting $p_i' = p_i E \left[e^{-\alpha \sum_{j=i+1}^n t_j} \right]$, then $E[\Pi_C^I] = E \left[Q e^{-\alpha T} - \sum_{i=1}^n (p_i' e^{-\beta t_i}) e^{-\alpha \sum_{j=1}^i t_j} \right]$, that is

equivalent to the expected client's profit under contract I as defined by (3). Since, $E[\Pi_C^D] = E[\Pi_C^I]$ and $E[\pi_i^D] = E[\pi_i^I] \quad \forall i$, we can conclude that the equilibrium solutions of p_i and p_i' $\forall i$ are also equivalent. Therefore, the equilibrium work rates for all subcontractors are equal under the two contracts.

Q.E.D.