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Online Appendix to "Joint Selling of Complementary Components under Brand and Retail Competition"

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Table A.1 Parameters and Decision Variables

Symbol	Definition
a, b	Complementary components a, b, respectively.
n	Number of competing brands of component b.
m	Number of packages of complementary components sold jointly to retailer, $m \in \{0, ..., n\}$.
i	A brand or supplier of component b in alliance with supplier a.
j	A brand or supplier of component b not in alliance with supplier a.
α	Degree of substitution among final products, where $\alpha \in [0, \infty)$.
θ	Relative (or normalized) degree of substitution from a single competitor, or individual competition intensity, where $\theta \in [0, \frac{1}{n-1})$.
$(n-1)\theta$	Relative (or normalized) degree of substitution from all competitors, or aggregate competition intensity, where $(n-1)\theta \in [0, 1)$.
w_a	Wholesale price of component a sold individually to retailer.
w_j	Wholesale price of brand j of component b sold individually to retailer.
w_{ak}	Total wholesale price of component a and brand k of component b.
p_{ak}	Retailer's selling price of final product ak.
q_{ak}	Retailer's demand or order quantity of final product ak.
Π_{ai}	Total profit of coalition ai, where $\Pi_{ai} = \Pi_{ai}^a + \Pi_{ai}^i$.
Π_{ai}^a	Share of coalitional profit Π_{ai} to supplier a.
$\Pi_i = \Pi_{ai}^i$	Supplier i's profit, which is also the share of coalitional profit Π_{ai} to supplier i of component b.
Π_j	Supplier j's profit.
Π_a^T	Supplier a's total profit.

Proof of Proposition 1. By following the definition of the pair-wise stability concept, a structure with m coalitions is stable if all single-link deviations are deterred. Recall that we use index i to stand for suppliers of component b that are in alliance with supplier a, and index j to represent suppliers of component b that are independent of supplier a. In order to verify the stability of a given structure, we need to examine two types of deviations – Type I is with regards to the termination of an existing alliance and Type II is about the formation of a new alliance.

For Type I deviations, it only applies to structures with $m \geq 1$. We need to verify that neither supplier a nor i has an incentive to terminate their alliance unilaterally. In a structure with $m \geq 1$, if supplier a's equilibrium profit satisfies a condition such that $\Pi_a^T(m) \geq \Pi_a^T(m-1)$, then it implies that supplier a has no incentive to unilaterally break an existing alliance. If this condition also implies that $\Pi_i(m) \geq \Pi_i'(m-1)$, for any $i \in \{1, ..., m\}$, then we can further conclude that supplier i would not unilaterally break the alliance either as long as supplier a chooses not to do so.

For Type II deviations, it only applies to structures with $m \leq n-1$. Under such a structure, if supplier a's equilibrium profit satisfies a condition such that $\Pi_a^T(m) \leq \Pi_a^T(m+1)$, then it implies that supplier a would benefit from forming a new alliance with some supplier j. If this condition also leads to an inequality such that $\Pi_j(m) \leq \Pi_j'(m+1)$, for any $j \in \{m+1, ..., n\}$, then we obverse that supplier j would always want to form a new alliance with supplier a if supplier a prefers to form such a new alliance.

From the discussions about the two types of possible single-link deviations above, we can conclude that the properties expressed in Proposition 1(1) indicate that supplier a's performance/preferences play a dominant role in shaping the stability conditions for a structure with m coalitions. That is, we only need to ensure that supplier a has no incentive to deviate from this structure to a structure with either $m-1$ or $m+1$ coalitions. This leads to a sufficient and necessary condition where $\Pi_a^T(m) \geq \max\{\Pi_a^T([m-1]^+), \Pi_a^T(n-[n-(m+1)]^+)\}$. □

Equilibrium Analysis of the 1-by-n Model with a Single Retailer and Linear Demand in §4.1. In this section, we solve for the equilibrium decisions and profits in the three-stage 1-by-n base model presented in Figure 1(a) with a monopolist retailer under a given coalition structure with m alliances. Backward induction is used to solve for equilibrium decisions. We first solve for the equilibrium retail prices under a given coalition structure. Suppose that supplier a has formed joint selling coalitions with m suppliers of component b, i.e., there are m packages of complementary components, and the rest are sold individually to the retailer, where $m \in \{0, 1, ..., n\}$. Due to symmetry of suppliers of component b in the base model, without loss of generality, we assume that the first m suppliers of component b are in alliance with supplier a. We will use index "i" to denote suppliers of component b in alliance with supplier a, and index "j" to denote suppliers of component b independent of supplier a. Without loss of generality, we assume that the retailer does not incur any extra cost in addition to the purchasing costs. Facing $\{w_{a1},...,w_{am},w_a,w_{m+1},...,w_n\}$, the retailer determines retail prices $\{p_{a1},p_{a2},...,p_{an}\}$ to:

$$\text{Maximize } \Pi_R = \sum_{k=1}^{n}(p_{ak}-w_{ak})q_{ak}, \tag{1}$$

where demands q_{ak} is given in equation (1) in the main body of the paper and $w_{ak}=w_a+w_k$ for components a and k sold independently. That is, $k\in\{m+1,...,n\}$. It is implied in the retailer's profit function above that $w_a \le \min(w_{a1},..,w_{am})$. This condition is preferred by supplier a to prevent the retailer from an arbitrary opportunity which allows him to use package ai to replace component a needed to meet demand of product aj. Solving, simultaneously, the first-order conditions of the retailer's objective function above with respect to retail prices, we obtain the best response functions:

$$p^*_{ak} = \frac{A+w_{ak}}{2} \text{ for } k\in\{1,...,n\}. \tag{2}$$

Knowing the retailer's best response functions of the retail prices, the suppliers determine their wholesale prices. Recall that complementary suppliers who decide to sell their components in a package will jointly determine a single wholesale price for the whole package, and suppliers whose components are sold independently to the retailer will set their individual wholesale prices. This results in three categories of wholesale prices, i.e, wholesale price of a package, w_{ai}; wholesale price of a brand of component b sold independently to the retailer, w_j; and wholesale price of component a sold independently to the retailer, w_a. Note that not all the three categories of wholesale prices need to be present simultaneously, depending on the coalition structure. For price w_{ai}, we assume that it is determined to maximize the total profit generated from selling package ai to the retailer. This assumption is appropriate since such a wholesale price decision involves two participating suppliers' joint profit. For price w_j, we assume that it is determined to maximize supplier j's profit, which is the only relevant objective function in this case. Finally, for the individual wholesale price of component a, w_a (where $w_a \le \min(w_{a1},..,w_{am})$), we assume that it is determined to maximize supplier a's total profit generated from individual selling of this component to the retailer, i.e., $\sum_{j=m+1}^{n}[w_a q_{aj}]$. We further assume that all suppliers have a zero production cost. Accordingly:

- For packaged components of suppliers a and i (or simply, coalition ai), wholesale price w_{ai} is to maximize $\Pi_{ai}=w_{ai}q_{ai}=w_{ai}[A-(1+\alpha)p_{ai}+\frac{\alpha}{n}\sum_{k=1}^{n}(p_{ak})]$, where prices p_{ak} are given in (2).

• For components sold independently, for supplier j of component b, wholesale price w_j is to maximize $\Pi_j = w_j q_{aj} = w_j[A-(1+\alpha)p_{aj} + \frac{\alpha}{n}\sum_{k=1}^{n}(p_{ak})]$. For supplier a, wholesale price w_a, where $w_a \le \min(w_{a1},..,w_{am})$, is to maximize $\Pi_a = \sum_{j=m+1}^{n}[w_a q_{aj}] = \sum_{j=m+1}^{n}(w_a[A-(1+\alpha)p_{aj} + \frac{\alpha}{n}\sum_{k=1}^{n}(p_{ak})])$.

Since all prices are simultaneously set by suppliers, solving three first-order conditions $\frac{\partial \Pi_{ai}}{\partial w_{ai}} = 0$, $\frac{\partial \Pi_a}{\partial w_a} = 0$ and $\frac{\partial \Pi_j}{\partial w_j} = 0$ yields equilibrium prices w_{ai}^*, w_a^* and w_j^*, with the condition $w_a^* \le w_{ai}^*$ automatically satisfied for this problem setting. The equilibrium retail prices and profits can be derived accordingly and are summarized in the table below.

Table A.2 Equilibrium Prices, Demands and Profits under a Given Coalition Structure with m Coalitions

	Product ai for $i \in \{1,...,m\}$	Product aj for $j \in \{m+1,..,n\}$
Wholesale Price	$w_{ai}^* = \frac{An(b_1+b_2n+b_3n^2)}{d_0+d_1n+d_2n^2+d_3n^3}$	$w_a^* = \frac{An(\alpha(n-1)+n)(-\alpha+2(1+\alpha)n)}{d_0+d_1n+d_2n^2+d_3n^3}$ $w_j^* = \frac{An(\alpha m+n)(-\alpha+2(1+\alpha)n)}{d_0+d_1n+d_2n^2+d_3n^3}$
Retail Price	$p_{ai}^* = \frac{A(d_0+(d_1+b_1)n+(d_2+b_2)n^2+(d_3+b_3)n^3)}{2(d_0+d_1n+d_2n^2+d_3n^3)}$	$p_{aj}^* = \frac{A(g_0+g_1n+g_2n^2+g_3n^3)}{2(d_0+d_1n+d_2n^2+d_3n^3)}$
Demand	$q_{ai}^* = \frac{A(\alpha(n-1)+n)(b_1+b_2n+b_3n^2)}{2(d_0+d_1n+d_2n^2+d_3n^3)}$	$q_{aj}^* = \frac{A(\alpha(n-1)+n)(\alpha m+n)(-\alpha+2(1+\alpha)n)}{2(d_0+d_1n+d_2n^2+d_3n^3)}$
Profit	$\Pi_{ai}^* = \frac{A^2n(\alpha(n-1)+n)(b_1+b_2n+b_3n^2)^2}{2(d_0+d_1n+d_2n^2+d_3n^3)^2}$	$\Pi_a^* = \frac{A^2n(n-m)(\alpha m+n)(-\alpha+2(1+\alpha)n)^2(\alpha(n-1)+n)^2}{2(d_0+d_1n+d_2n^2+d_3n^3)^2}$ $\Pi_j^* = \frac{A^2n(\alpha(n-1)+n)(\alpha m+n)^2(-\alpha+2(1+\alpha)n)^2}{2(d_0+d_1n+d_2n^2+d_3n^3)^2}$

Note that $b_1 = -\alpha^2 m$, $b_2 = \alpha(-2-\alpha+2(1+\alpha)m)$, $b_3 = (1+\alpha)(3+\alpha)$, $d_0 = 2m\alpha^3$, $d_1 = \alpha^2[2-(6+5\alpha)m]$, $d_2 = \alpha[-7-6\alpha+(1+\alpha)(5+3\alpha)m]$, $d_3 = 2(1+\alpha)(3+2\alpha)$, $g_0 = d_0 = 2m\alpha^3$, $g_1 = \alpha^2[3-m(7+5\alpha)]$, $g_2 = \alpha[-11-9\alpha+m(1+\alpha)(7+3\alpha)]$ and $g_3 = 2(1+\alpha)(5+3\alpha)$.

Supplier a's total profit can be written as $\Pi_a^T = \sum_{i=1}^{m}[\Pi_{ai}^a] + \Pi_a$, where Π_{ai}^a is the share of the total profit of coalition ai (i.e., Π_{ai}) to supplier a. Correspondingly, we denote supplier i's share of coalitional profit Π_{ai} as Π_{ai}^i, or simply Π_i. Clearly, both Π_{ai}^a and Π_{ai}^i depend on the proportional profit allocation rule. □

Proof of Proposition 2. We prove this result in two steps. In step I, we follow Proposition 1 of the paper to derive the stability conditions for a coalition structure with m alliances (as described in Proposition 1). That is, we first show that, for any given parameters n and α (or equivalently, θ), there exits an integer value of m, denoted by m^*, which satisfies the stability conditions characterized in step I. And then, we show that m^* decreases in α.

Step I: For a given coalition structure with m alliances, to ensure its stability, we need to ensure no single-link profitable deviations. Specifically, supplier a would neither terminate an existing alliance nor form a new one; supplier i would not terminate the alliance with supplier a, for any $i \in \{1, ..., m\}$; and supplier j would not form an alliance with supplier a, for any $j \in \{m+1, ..., n\}$. Essentially, we follow Proposition 1 to establish these conditions. Recall that we denote by $\Pi_a^T(m)$, $\Pi_i(m)$ and $\Pi_j(m)$ the equilibrium profits for suppliers a, i and j, respectively, where $i \in \{1, ..., m\}$ and $j \in \{m+1, ..., n\}$. For $m \geq 1$, if an alliance is terminated between suppliers a and i, we denote by $\Pi_a^T(m-1)$ and $\Pi_i'(m-1)$ their corresponding equilibrium profits in the new structure with $m-1$ alliances. Similarly, for $m \leq n-1$, if a new alliance is formed between suppliers a and j, we denote by $\Pi_a^T(m+1)$ and $\Pi_j'(m+1)$ their corresponding equilibrium profits in the new structure with $m+1$ alliances. We also denote by $\Pi_{aj}(m+1)$ the total profit of selling the new package aj (i.e., the coalitional profit of alliance aj) in the new structure. Due to the symmetry between different packages, we have $\Pi_{aj}(m+1) = \Pi_{ai}(m+1)$ for all $i \in \{1, ..., m\}$. Note that these profit functions are presented in Table A.2 above.

Following from Proposition 1(1), we need to show two separate conditions, one related to suppliers a and i and the other related to suppliers a and j. (i) For any $i \in \{1, ..., m\}$, we need to show that $\Pi_i(m) \geq \Pi_i'(m-1)$ when $\Pi_a^T(m) \geq \Pi_a^T(m-1)$; and (ii) for any $j \in \{m+1, ..., n\}$, we need to show that $\Pi_j(m) \leq \Pi_j'(m+1)$ when $\Pi_a^T(m) \leq \Pi_a^T(m+1)$. Let us work on condition (ii) in the sequel and condition (i) can be worked out similarly. Note that condition (ii) is mathematically equivalent to show that if $\Pi_j(m) \geq \Pi_j'(m+1)$ then $\Pi_a^T(m) \geq \Pi_a^T(m+1)$. In the structure with m coalitions, if a new alliance is formed between supplier a and j, recall that $\Pi_j'(m+1) = (1-\beta_{m+1})\Pi_{aj}(m+1)$. If $\Pi_j(m) \geq \Pi_j'(m+1)$, then we have $\Pi_a(m) \geq \frac{(n-m)(n+\alpha(n-1))}{\alpha m+n}(1-\beta_{m+1})\Pi_{aj}(m+1) = \frac{(n-m)(n+\alpha(n-1))}{\alpha m+n}(1-\beta_{m+1})\Pi_{ai}(m+1)$, since, recall from Table A.2 that $\frac{\Pi_j(m)}{\Pi_a(m)} = \frac{\alpha m+n}{(n-m)(n+\alpha(n-1))}$. Recall also that $\Pi_a^T(m) = \sum_{i=1}^m [\Pi_{ai}^a(m)] + \Pi_a(m) = [m\beta_m R_m + 1]\Pi_a(m)$, where $R_m = \frac{\Pi_{ai}(m)}{\Pi_a(m)} = \frac{[-a^2m+a(-2-a+2(1+a)m)n+(1+a)(3+a)n^2]^2}{(n-m)(am+n)(a(-1+n)+n)(a-2(1+a)n)^2}$. Accordingly, $\Pi_a^T(m) \geq S_1\Pi_{ai}(m+1)$, where $S_1 = [m\beta_m R_m + 1]\frac{(n-m)(n+\alpha(n-1))}{\alpha m+n}(1-\beta_{m+1})$. Similarly, we have $\Pi_a^T(m+1) = S_2\Pi_{ai}(m+1)$, where $S_2 = ((m+1)\beta_{m+1} + \frac{1}{R_{m+1}})$ and $R_{m+1} = \frac{\Pi_{ai}(m+1)}{\Pi_a(m+1)}$. With some algebra, we can show that $S_1 - S_2 \geq 0$, which implies that $\Pi_a^T(m) \geq \Pi_a^T(m+1)$. This verifies that condition (ii) is satisfied with the special case with a linear deterministic demand function and zero costs for production and assembly. Combination of these two conditions indicates that if supplier a would like to form a new alliance with a type-j supplier, this type-j supplier would also like to do the same. Also, if supplier a would maintain the alliance with a type-i supplier, then this type-i supplier would do that as well. Essentially, the conditions for a structure with m coalitions to be stable only rely on supplier a's performance/preference. That is, we only need to ensure that supplier a has no incentive to deviate from this structure to a structure with either $m-1$ or $m+1$ coalitions, or equivalently, $\Pi_a^T(m) \geq max\{\Pi_a^T([m-1]^+), \Pi_a^T(n-[n-(m+1)]^+)\}$. This is indeed Proposition 1(2) of the paper.

Step II: In this step, we first show that there exists such an integer m so that the condition in Proposition 1(2) is satisfied for given n and α. And then, we show that this value of m decreases in α. First, we check the sign of $\Pi_a^T(m) - \Pi_a^T([m-1]^+)$.

For $m = 0$, $\Pi_a^T(m) = \Pi_a^T([m-1]^+)$. For $m \geq 1$, from the profit expressions in Table 2 of the paper, we can rewrite $\Pi_a^T(m) - \Pi_a^T([m-1]^+) = \Pi_a^T(m) - \Pi_a^T(m-1) = B(m)C(m)$, where $B(m) = \sum_{i=0}^{4} u_i m^i$ with u_i's being polynomial functions of α and n only[1] and $C(m) = \frac{A^2 n((\alpha(-1+n)+n)^2)}{(2n+\alpha(-3+m+n))(2n+\alpha(-2+m+n))D(m)} > 0$ and $D(m) = [2\alpha^3 m + \alpha^2(2-(6+5\alpha)m)n + \alpha(-7-6\alpha+(1+\alpha)(5+3\alpha)m)n^2 + 2(1+\alpha)(3+2\alpha)n^3]^2[(6n^3 + \alpha^3(-1+m)(-1+n)(-2+3n) + \alpha n^2(-12+5m+10n) + 2\alpha^2 n(4+n(-7+2n)+m(-3+4n))]^2$. Clearly, the sign of $\Pi_a^T(m) - \Pi_a^T(m-1)$ depends on the sign of $B(m)$ alone. The expressions of u_i's in $B(m)$ are quite involved, but we observe that $B(m)$ is a polynomial function of m with the highest exponent being 4. The sign of $B(m)$ can be explored by taking derivatives of $B(m)$ w.r.t. m. Denote by $B_1 = \frac{\partial B(m)}{\partial m}$, $B_2 = \frac{\partial^2 B}{\partial^2 m} = \frac{\partial B_1}{\partial m}$, $B_3 = \frac{\partial^3 B}{\partial^3 m} = \frac{\partial B_2}{\partial m}$ and $B_4 = \frac{\partial^4 B}{\partial^4 m} = \frac{\partial^2 B_2}{\partial^2 m}$. Since $B_4 = -24\alpha^5(-\alpha+2(1+\alpha)n)\left(2\alpha^2 - \alpha(6+5\alpha)n + (1+\alpha)(5+3\alpha)n^2\right)^2\left(n(3+2n)+2\alpha\left(-1+n+n^2\right)\right) < 0$, B_2 is concave in m. By solving $B_3 = 0$, B_2 reaches its maximum at a value of m smaller than one. So, B_2 is always decreasing in m. Since $B_2(m=1) \leq 0$, we have $B_2 \leq 0$, and hence, B_1 is always decreasing in m. Again, since $B_1(m=1) \leq 0$, we have $B_1 \leq 0$ always and $B(m)$ decreases in m, an important property. Since $B(m=1)$ and $B(m=n)$ can be negative or positive depending on n and α, we consider three possible scenarios in terms of different parameter ranges w.r.t. n and α.

(1) If $B(m=1) < 0$, then $B(m) < 0$ and $\Pi_a^T(m) < \Pi_a^T(m-1)$ for any m, which means that Π_a^T is monotone and decreases in m. Hence, the independent structure is uniquely stable in this scenario. Further examining $B(m=1)$ indicates that $B(m=1) = -n^2(2n+\alpha n-2\alpha)(2n-\alpha+2\alpha n)^2 Bm1$, where $Bm1 = [(3n-2)^2(n-1)^3]\alpha^7 + [2(2n^4+27n^3-52n^2+28n-4)(n-1)^2]\alpha^6 + [4n(8n^3+28n^2-40n+11)(n-1)^2]\alpha^5 + [n^2(-180n^2+252n+95n^4-78-90n^3)]\alpha^4 + [n^3(29+128n^3+48n-213n^2)]\alpha^3 + 2n^4(35n^2+22-68n)$. Note that the sign of $B(m=1)$ can be determined by the sign of $Bm1$. One can verify that $Bm1$ first decreases and then increases in α. Moreover, since $Bm1(\alpha=0) < 0$ and $Bm1(\alpha=1) > 0$, we can conclude that there exists an α, namely, $\alpha_1(n)$ (≤ 1), such that $Bm1 > 0$ if and only if $\alpha > \alpha_1(n)$ for any given n. This implies that $B(m=1) < 0$ if and only if $\alpha > \alpha_1(n)$.

[1] For space considerations, expressions for u_i's are not provided here, but they are available from authors upon request. A similar comment applies to expressions of v_i's, x_i's, y_i's and z_i's which appear in the sequel.

(2) If $B(m=n)>0$, then $B(m)>0$ and $\Pi_a^T(m)>\Pi_a^T(m-1)$ for any m, which means that Π_a^T is monotone and increases in m. Hence, the full-alliance structure is uniquely stable. Similar to scenario (1) above, one can show that $B(m=n)>0$ if and only if $\alpha<\alpha_n(n)$ for any given n.

(3) If $B(m=1)\geq 0$ and $B(m=n)\leq 0$, then there exists a unique $m_1\in[1,n]$ satisfying $B(m)=0$, and $\Pi_a^T(m)-\Pi_a^T(m-1)\geq 0$ for any $m\leq m_1$ and $\Pi_a^T(m)-\Pi_a^T(m-1)\leq 0$ otherwise. This scenario is applicable if and only if $\alpha\in[\alpha_n(n),\alpha_1(n)]$. Refining $m=m'+1$, we can rewrite the above argument as follows. For $\alpha\in[\alpha_n(n),\alpha_1(n)]$, there exists a unique $m_1'=m_1-1\geq 0$ satisfying $B(m'+1)=0$, and $\Pi_a^T(m'+1)-\Pi_a^T(m'+1-1)\geq 0$ for any $m\leq m_1'$ and $\Pi_a^T(m'+1)-\Pi_a^T(m'+1-1)\leq 0$ otherwise. Equivalently, we can say that in this scenario, $\Pi_a^T(m+1)-\Pi_a^T(m)\leq 0$ for any $m\geq m_1-1$. Combining this with the first statement of this case we conclude that there exists an integer in the range of $[m_1-1,m_1]$, denoted by m^*, such that a structure with m^* coalitions satisfies the condition $\Pi_a^T(m)\geq max\{\Pi_a^T([m-1]^+),\Pi_a^T(n-[n-(m+1)]^+)\}$ and hence is stable.

These three scenarios conclude that there exists a pairwise stable structure for any given n and α. In what follows, we show that m^* (weakly) decreases in α, or equivalently, $m_1(\alpha)$ (weakly) decreases in α since $m^*\in[m_1-1,m_1]$.

Note that we defined $B(m)=\sum_{i=0}^{4}u_im^i$ previously, which has a solution $m_1(\alpha,n)$ for $\alpha\in[\alpha_n(n),\alpha_1(n)]$. For any given n, we will show next that m_1 continuously decreases in α. Taking derivatives and simplifying yields $\frac{\partial m_1(\alpha)}{\partial\alpha}=\frac{E(m_1(\alpha,n),\alpha,n)}{F(m_1(\alpha,n),\alpha,n)}$, where $E(m_1(\alpha,n),\alpha,n)=\sum_{i=0}^{4}x_im_1^i(\alpha,n)$ and $F(m_1(\alpha,n),\alpha,n)=\sum_{i=0}^{3}v_im_1^i(\alpha,n)$ with x_i's and v_i's being polynomial functions of n and α only. Next, we treat $m_1(\alpha,n)$ in $\frac{E(m_1(\alpha,n),\alpha,n)}{F(m_1(\alpha,n),\alpha,n)}$ as a free variable, m, instead of a function of α and n, and show that $\frac{E(m)}{F(m)}\leq 0$ for any m. Hence, $\frac{E(m)}{F(m)}\leq 0$ at $m=m_1(\alpha,n)$. Let us consider the denominator $F(m)$ first. Denote by $F_1=\frac{\partial F}{\partial m}$, $F_2=\frac{\partial F_1}{\partial m}$ and $F_3=\frac{\partial^2 F_1}{\partial^2 m}$, where $F_3=-24\alpha^5[2n^2(3+2n)+n(-7+8n+8n^2)\alpha+2(1-3n+n^2+2n^3)\alpha^2][2\alpha^2-n\alpha(6+5\alpha)+n^2(5+8\alpha+3\alpha^2)]^2\leq 0$. Hence, F_1 is concave in m. Further analysis indicates that $F_2=0$ gives a unique solution at a negative value of m, which implies that $F_1(m)$ decreases in m (≥ 1). Since $F_1(m=1)\leq 0$ and $F(m=1)\leq 0$, we have $F(m)$ decreases in m and $F(m)\leq 0$ for all $m\geq 1$. A similar approach can show that the numerator $E(m)$ always increases in $m\geq 1$ and $E(m=1)\geq 0$ with $n\geq 6$ if $\alpha\leq 0.149$. The last inequality is satisfied since $B(m=n)>0$ for any $\alpha<0.149$, which leads to $\alpha_n(n)\geq 0.149$ for given n. Hence, $\frac{E(m)}{F(m)}\leq 0$ which means that $m_1(\alpha)$ decreases in α, if $n\geq 6$.

Note that we have $B(m=1)\leq 0$ at $\alpha=\alpha_1(n)$ and we have $B(m=n)\geq 0$ at $\alpha=\alpha_n(n)$. Since m_1 decreases in $\alpha\in[\alpha_n(n),\alpha_1(n)]$, we conclude that when α increases within this range, $m_1(\alpha)$ decreases from a number higher than n to a number less than 1, and consequently, the number of coalitions in the stable structure decreases from n to 0. Together with the previous observation that for $\alpha < \alpha_n(n)$, the full-alliance is uniquely stable, and that for $\alpha > \alpha_1(n)$, the independent structure is uniquely stable, we can conclude that the number of coalitions in the pairwise stable structure is continuously decreasing in α from 0 to n for $\alpha \in [0, \infty)$ when $n \geq 6$.

For a given n, where $n \leq 5$, we directly derive the condition in terms of $\alpha \in [0, \infty)$ under which a structure with m coalitions is stable, i.e., satisfying the stability condition $\Pi_a^T \geq max\{\Pi_a^T([m-1]^+), \Pi_a^T(n-[n-(m+1)]^+)\}$, for each $m \in [0, n]$. The table below presents the range of α where a structure with m coalitions is stable. Combination of the results in the table for $n \leq 5$ and the analysis for $n \geq 6$ proves Proposition 2. □

Table A.3 **The Stable Coalition Structures for a Given α and n with $n \leq 5$**

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
2	$\alpha \geq 0.577$	$0.278 \leq \alpha \leq 0.577$	$\alpha \leq 0.278$	–	–	–
3	$\alpha \geq 0.426$	$0.274 \leq \alpha \leq 0.426$	$0.215 \leq \alpha \leq 0.274$	$\alpha \leq 0.274$	–	–
4	$\alpha \geq 0.377$	$0.274 \leq \alpha \leq 0.377$	$0.225 \leq \alpha \leq 0.274$	$0.193 \leq \alpha \leq 0.225$	$\alpha \leq 0.193$	–
5	$\alpha \geq 0.353$	$0.274 \leq \alpha \leq 0.353$	$0.231 \leq \alpha \leq 0.274$	$0.203 \leq \alpha \leq 0.231$	$0.183 \leq \alpha \leq 0.203$	$\alpha \leq 0.183$

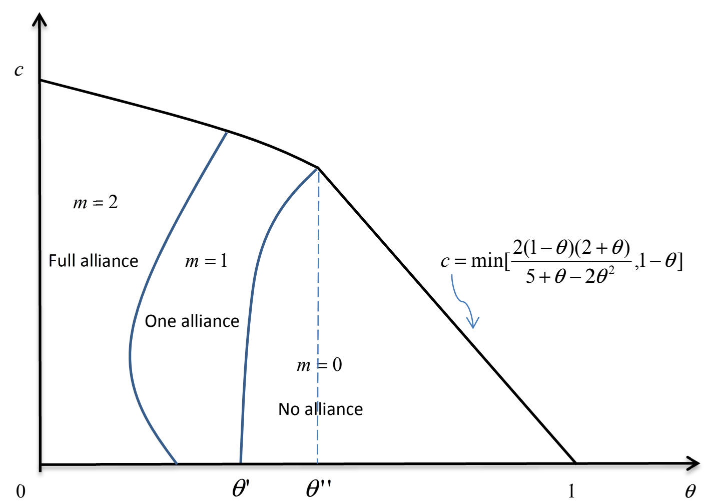
Extensions I. In this section, we briefly discuss some implications based on several extensions to the base model with a single retailer.

(i) *An alternative stability concept – strong stability.* To test whether allowing for coalitional deviations (or multi-link deviations) would alter the effect of brand competition on alliance formation, we have extended the stability analysis in Section 4 to use the strong (Nash) equilibrium concept. Strong stability refines pairwise stability and it has also been studied in network formation games, see, e.g., Dutta and Mutuswami (1997) and Jackson and van den Norweland (2005). Jackson (2005) provides a comprehensive survey on these concepts and their applications. Our analysis shows that both pairwise and strong stability concepts demonstrate a discouraging effect of brand competition on alliances, even though the actual stable outcomes may be different under the two concepts.

(ii) *An alternative profit allocation rule – Nash bargaining solution.* Under the allocation rule based on the Nash bargaining solution, members of a coalition negotiate on how to split the coalitional profit. Negotiation between business partners is prevalent in many industries, e.g., manufacturing, retailing, and service, and they could negotiate on various terms of their trade, e.g., prices, transaction quantities, payment formats, and delivery times, etc. (see Nagarajan and Bassok 2008 for a detailed discussion on these examples). To check the robustness of stability results derived under proportional allocation, we have extended the stability analysis assuming that coordinating component suppliers will do a bilateral negotiation to split the coalitional profit. We have used the two-person bargaining problem proposed by Nash (1950) to model the negotiation between suppliers of an alliance. For simplicity, we assume that suppliers of component b are symmetric in terms of their bargaining power relative to that of supplier a. Our analytical study for the case when $n=2$ and extensive numerical study for cases when $n \geq 3$ indicate a robust effect of brand competition on alliance formation under both proportional and Nash bargaining allocation rules.

(iii) *Asymmetric suppliers.* We discuss two kinds of asymmetry between competing suppliers of component b: production cost asymmetry and market potential asymmetry. Due to the difficulty in analytically characterizing the stability conditions, we have conducted extensive numerical studies in a 1-by-2 model. For simplicity, we also assume that supplier 1 of component b has a zero production cost, while supplier 2 of component b has a cost $c \geq 0$. All other assumptions remain the same as that in the base model. The stable outcomes are presented on a θ-c plane in Figure A.1 below. The top curve in the figure implies an upper bound on c, i.e., $c \leq \min(\frac{2(1-\theta)(2+\theta)}{5+\theta-2\theta^2}, 1-\theta)$, which is imposed to ensure the existence of both competing brands of component b.

Figure A.1 Stable Coalition Structures in a 1-by-2 Model with Suppliers' Cost Asymmetry



Following this figure, we observe: (1) At any given value of c, stronger brand competition leads to a lower number of alliances in equilibrium, which is consistent with Proposition 2 where $c=0$. (2) The effect of cost asymmetry is as follows. Let us focus on the issue whether or not alliances are formed. If the level of brand competition is on the extreme, i.e., θ being either low as $\theta \leq \theta'$ or high as $\theta \geq \theta''$, whether an alliance is formed or not is irrespective of the degree of cost asymmetry.

In these cases, following the previous study on the effect of brand competition, it is known that either the profit loss for products not involved in joint selling dominates the gain for products under joint selling or vice versa. So, alliances will always be formed when θ is low while no alliances will be formed otherwise. However, for a medium level of brand competition (i.e., $\theta' \leq \theta \leq \theta''$), a high degree of cost asymmetry encourages joint selling since an alliance will be formed between supplier a and the less cost-efficient supplier of component b only when c is relatively high. This is probably due to the fact that such an alliance could guarantee a lion's share of the joint profit allocated to the monopolist supplier a, so she is incentivized to enter a partnership with the less cost-efficient supplier of component b. Recall that joint profit is allocated to participating suppliers in a way such that individual players' shares are positively correlated (e.g., proportionally or Nash bargaining solution-based) to their corresponding profits before coordination.

Under market size asymmetry, we adopt the following demand function to represent asymmetric market potentials for the two final products in the 1-by-2 model:

$$q_{a1} = 1 - (1+\alpha)p_{a1} + \frac{\alpha}{2}(p_{a1} + p_{a2}) \text{ and } q_{a2} = \bar{A} - (1+\alpha)p_{a2} + \frac{\alpha}{2}(p_{a1} + p_{a2}), \tag{3}$$

where $\bar{A} \leq 1$. Again, all other assumptions remain the same as those in the base model. Similar to cost asymmetry, we were not able to analytically characterize the stability conditions for a given coalition structure due to the number of model parameters involved in the model, so we resort to a numerical study in a 1-by-2 model to characterize the stable outcomes. In general, we may conclude that brand competition again makes alliances less likely. In terms of the effect of market potential asymmetry, if brand competition is too strong, alliances will not be formed, irrespective of the degree of market potential asymmetry; if brand competition is relatively low, alliances will occur only when $\bar{A}$ is sufficiently high which implies a low degree of market potential asymmetry.

Finally, we have also analyzed a 2-by-2 model where both components have two competing brands and a 1-by-2 model with exclusive selling only (i.e., components are sold by their suppliers in packages only). Our analysis in both extensions demonstrates a consistent negative effect of brand competition on alliance formation. Also, relative to the base model, exclusive selling makes alliances more likely. □

Proof of Proposition 3. (1) The approach to derive the equilibrium decisions and profits for a given structure is very similar to the base model with a single monopolist retailer. The only difference is in the stage when retail prices are determined. In this scenario, retailers determine their retail prices p_{ak} simultaneously to maximize their individual profits $\Pi_R^k = (p_{ak} - w_{ak})q_{ak}$, where $q_{ak} = A - (1+\alpha)p_{ak} + \frac{\alpha}{n}\Sigma_{t=1}^{n}(p_{at})$. Recall that in the base model, retail prices are chosen to maximize the total profit of all final products. Using backward induction, we can derive the equilibrium decisions and profits of all players. Again, like what we have proved in Proposition 2, only supplier a's profit performance matters in determining whether a coalition structure is stable. Hence, we report supplier a's equilibrium profit under a structure with m coalitions and write it as follows: $(\Pi_a^T)^{DR}(m) = \frac{A^2\left(n(n+(-1+n)\alpha)(-\alpha+2n(1+\alpha))\left(\alpha^2-2n\alpha(1+\alpha)+n^2(1+\alpha)(2+\alpha)\right)^2 \sum_{i=0}^{9} y_i n^i\right.}{\left(4n^2+n(-4+m+5n)\alpha+(-1+n)(-2+m+n)\alpha^2\right)(-\alpha+n(2+\alpha)) \sum_{i=0}^{6} z_i m^i}$, where y_i's and z_i's are polynomial functions of n and α only. Based on this profit function, we can follow a similar proof process as that in the proof of Proposition 2 (through defining an analog $B^{DR}(m)$ in the stability condition) to show that the number of coalitions in the stable structure continuously decreases from n to 0 when α increases from 0 to ∞. We omit the details here due to limited space.

(2) It is equivalent to show that for any given α, the stable structure under dedicated retailers always has a higher or equal number of coalitions relative to that under a single retailer.

First, let us show that for the model with a single retailer, for any given n, the independent structure is always uniquely stable if $\alpha > 0.577$ or $\alpha_1(n) \leq 0.577$. Recall from Proposition 2 that $\alpha_1(n)$ is the unique solution to $\Pi_a^T(m=1) - \Pi_a^T(m=0) = 0$, which can be converted to the solution to $G_1(\alpha, n) = 0$, where $G_1(\alpha, n)$ is a polynomial function of n with the highest exponent being 7. Define $G_{i+1}(\alpha, n) = G_i(\alpha, n+1) - G_i(\alpha, n)$, where $i \in [1, 7]$. Then $G_7(\alpha, n) = 5040(1+\alpha)^2(-18 + 27\alpha + 104\alpha^2 + 91\alpha^3 + 32\alpha^4 + 4\alpha^5)$, which increases in α and reaches zero at $\alpha = 0.281$. Since $G_7(\alpha, n) = G_6(n+1) - G_6(n)$, we can conclude that at $\alpha = 0.281$, $G_6(\alpha, n)$ is a negative constant for any n. Or, the lines $G_6(\alpha, n)$ for all n come across the same point at $\alpha = 0.281$. Also, we have that $G_6(\alpha = 0, n)$ decreases in n and $G_6(\alpha = 0, n = 2) \leq 0$, and that $\frac{\partial G_6(\alpha, n)}{\partial \alpha} \geq 0$ implying that $G_6(\alpha, n)$ increases in α. These properties of $G_6(\alpha, n)$ indicate that the solution to $G_6(\alpha, n) = 0$ in terms of α decreases in n. A similar approach can show that the solution to $G_i(\alpha, n) = 0$ in terms of α is always decreasing in n, for any $i \in [1, 5]$. Hence, $\alpha_1(n)$ decreases in n and $\alpha_1(n = 2) = 0.577$.

Secondly, we show that for the model with dedicated retailers, for any given n, the full-alliance structure is always uniquely stable if $\alpha < 0.298$ or $\alpha_n^{DR}(n) \geq 0.298$. A similar analysis as that in the case of $\alpha_1(n)$ can show that $\alpha_n^{DR}(n)$ also decreases in n and $\alpha_n^{DR}(n \to \infty) = 0.298$.

The discussion in the above two paragraphs indicates that we only need to show that for $\alpha \in [0.298, 0.577]$, in equilibrium, the number of coalitions in the case with dedicated retailers is more than or equal to that of the case with a single retailer. Here, we use the relation between m_1 and m_1^{DR}, i.e., the solution to $B(m) = 0$ and $B^{DR}(m) = 0$, respectively. Note that $\frac{\partial B(m)}{\partial m} < \frac{\partial B^{DR}(m)}{\partial m} < 0$ and that $B(m = 0) < B^{DR}(m = 0)$ when $\alpha \in [0.298, 0.577]$. Thus, we can conclude that the $m_1^{DR} >$

m_1, which implies that we observe more alliances in the model with dedicated retailers than that with a single retailer, for any given n, when $0.298 \leq \alpha \leq 0.577$. This completes the proof. □

Extensions II. So far, we have considered two extreme scenarios in terms of retail decentralization – a centralized structure with one retailer controlling all final products and a fully decentralized structure with a dedicated retailer for each final product. It is natural to consider how a continuous level of retail decentralization would alter the effect of brand and retail competition. Indeed, this extension may introduce another type of system asymmetry – retailer asymmetry. Moreover, this asymmetry leads to a much higher number of possible coalition structures, which further complicates the stability analysis. For simplicity, we limit our analysis to small values of n, i.e., $n \leq 4$. When $n = 2$, stable outcomes directly follow from Proposition 2 in the paper. When $n = 3$ or $n = 4$, our analytical results show a generally consistent effect of brand and retail competition as established previously in Propositions 2 and 3 in the paper. However, a few exceptions may occur for a very small range of brand competition in models with asymmetric retailers. For example, in the 1-by-3 model with two retailers one selling a single final product and the other selling two final products, in general, an increase in the level of product substitutability decreases the number of alliances formed in equilibrium. However, our numerical experiments indicate that there is a very small range of α, i.e., $\alpha \in [0.35, 0.45]$, within which more alliances could be formed when α increases. For example, when α increases from 0.35 to 0.4, the number of alliances formed can increase from one to two. A similar observation can be made for the 1-by-4 model with two asymmetric retailers where one controls one final product and the other controls three final products. Finally, we can also analytically show that the effect of brand competition on stable outcomes remains the same when suppliers sell directly to consumers. □