Online Supplement for "Understanding How Generation Flexibility and Renewable Energy Affect Power Market Competition"

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Proof of Lemma 1. The inverse supply function is defined as $S_k^{-1}(q) = \inf\{p : S_k(p) > q\}$. Under Assumption 2(i) and (ii), $S_k^{-1}(0) = p_k^{\min}$ and $S_k^{-1}(q)$ is continuous and increasing in q for $q \geq 0$. Assumption 2(iii) implies that $S_k^{-1}(q) > 0$ for $q > 0$. Hence, $C_k(q) = \int_0^q S_k^{-1}(x)dx$ is continuously differentiable, convex, and strictly increasing in q for $q \geq 0$.

Consider the optimal allocation problem in (2), rewritten below

$$C^I(q) = \min\Big\{ \sum_{i \in G^I} C_i(q_i) : q_i \geq 0, \sum_{i \in G^I} q_i = q \Big\}. \tag{A.1}$$

For any $\overline{q} > 0$, the objective in (A.1) is convex on a closed convex set $\big\{(q, q_i, i \in G^I) : q \in [0, \overline{q}], q_i \in [0, \overline{q}], \sum_{i \in G^I} q_i = q\big\}$. Hence, the theorem on convexity preservation (Heyman and Sobel 1984, p. 525) implies that $C^I(q)$ is convex in q.

For a given $q > 0$, let the minimizer for (A.1) be $\{q_i^*\}$, which has two properties: 1) If $q_i^* > 0$, then $C_i'(q_i^*) = p_0$, for some constant $p_0 > 0$ that depends on the given q; 2) If $q_i^* = 0$, then $C_i'(0) \geq p_0$.[1]

Define $G_+^I \stackrel{\text{def}}{=} \big\{i \in G^I : C_i'(q_i^*) = p_0\big\}$. For $k \notin G_+^I$, we have $q_k^* = 0$ and $C_k'(0) > p_0$. Then, for sufficiently small $\varepsilon > 0$, we have $C^I(q + \varepsilon) = \sum_{i \in G_+^I} C_i(q_i^* + \varepsilon_i)$, for some $\varepsilon_i \geq 0$ and $\sum_{i \in G_+^I} \varepsilon_i = \varepsilon$. Using Taylor's expansion, we can write

$$C^I(q + \varepsilon) = \sum_{i \in G_+^I} [C_i(q_i^*) + \varepsilon_i C_i'(q_i^*) + o(\varepsilon_i)] = C^I(q) + \varepsilon p_0 + o(\varepsilon).$$

Similarly, we can show that $C^I(q) - C^I(q - \varepsilon) = \varepsilon p_0 + o(\varepsilon)$. Hence, $C^I(q)$ is differentiable with derivative $(C^I)'(q) = p_0 > 0$.

For $i \in G_+^I$, we have $p_0 = C_i'(q_i^*) = S_i^{-1}(q_i^*)$, implying $S_i(p_0) = q_i^*$. For $k \notin G_+^I$, we have $C_k'(0) > p_0$, implying $S_k^{-1}(0) = p_k^{\min} > p_0$, which in turn implies $S_k(p_0) = 0 = q_k^*$ due to Assumption 2(i). Hence, $S^I(p_0) = \sum_{i \in G^I} S_i(p_0) = \sum_{i \in G^I} q_i^* = q$, which leads to $p_0 = (S^I)^{-1}(q) = (C^I)'(q)$. Note that $(S^I)^{-1}(q)$ is continuous in q because $S^I(p)$ satisfies Assumption 2. Hence, $(C^I)'(q)$ is continuous in q.

Similar results can be shown for the problem in (3), which completes the proof. ∎

Proof of Theorem 1. We first prove that (12) is optimal in the scenario of $L_s - q^I - W_s \geq 0$. In this scenario, constraints (10)-(11) imply that $q_s^F \geq L_s - q^I - W_s \geq 0$. If we set q_s^F at the lower bound $L_s - q^I - W_s$, then $q_s^V = W_s$ and $e_s = 0$, which clearly minimize the objective function in (9).

[1] To see property 1), note that if $q_i^*, q_k^* > 0$ and $C_i'(q_i^*) < C_k'(q_k^*)$, we can strictly reduce the objective by increasing q_i^* by ε and reducing q_k^* by ε, where $\varepsilon > 0$ is small. Thus, for $q_i^*, q_k^* > 0$, $C_i'(q_i^*) = C_k'(q_k^*) = p_0$. Note that $p_0 > 0$ because $C_i(q_i)$ is convex and strictly increasing in q_i. To see property 2), note that if $C_i'(0) < p_0 = C_k'(q_k^*)$, we can strictly reduce the objective by setting $q_i^* = \varepsilon$ and reducing q_k^* by ε, where $\varepsilon > 0$ is small.

When $L_s - q^I - W_s < 0$, we have $q_s^{F*} = 0$ because: (i) if $q_s^F > 0$ and $e_s > 0$, then a lower q_s^F reduces the objective value in (9); (ii) if $q_s^F > 0$ and $e_s = 0$, then $q_s^V = L_s - q^I - q_s^F < W_s$, and we can reduce q_s^F and increase q_s^V to lower the objective in (9). Hence, $q_s^{F*} = 0$, and (9)-(11) becomes

$$\min_{q_s^V \in [0, W_s]} \big\{ - r\, q_s^V + h(q^I + q_s^V - L_s) \big\},$$

where we set $h(e) = 0$ for $e < 0$ without affecting the optimal solution. An interior optimal solution satisfies $h'(q^I + q_s^{V*} - L_s) = r$, or $q_s^{V*} = L_s - q^I + \mu(r)$, which is in the interior if $0 < L_s - q^I + \mu(r) < W_s$. If $L_s - q^I + \mu(r) \geq W_s$, then $q_s^{V*} = W_s$. If $L_s - q^I + \mu(r) < 0$, then $q_s^{V*} = 0$. This proves (12).

The objective function in (9) is convex on a closed convex set $\{(q^I, L_s, W_s, q_s^F, q_s^V) : q^I \in [0, \overline{q}], L_s \in [0, \overline{L}], W_s \in [0, \overline{W}], q_s^F \in [0, \overline{q}], (10) \text{ and } (11)\}$ for any $\overline{q} > 0$, $\overline{L} > 0$, and $\overline{W} > 0$. By the theorem on convexity preservation under minimization (Heyman and Sobel 1984, p. 525) we conclude that $\widehat{C}(q^I, L_s, W_s)$ is jointly convex in (q^I, L_s, W_s). ∎

Proof of Lemma 2. We verify the consistency between (13) and (14) by considering each of the four regions. The inequality in (14) is

$$S^F(p) + W \mathbf{1}_{\{p \geq -r\}} - \mu(-p) \;\geq\; L - q^I. \tag{A.2}$$

In region A_1, $L - q^I \leq -\mu(r)$, and (A.2) clearly holds if $p = p^o \stackrel{\text{def}}{=} -h'(q^I - L)$. Note that $p^o \leq -r$. Thus, for any other price $p_1 < p^o$, the left side of (A.2) becomes $-\mu(-p_1)$, which is strictly less than $L - q^I$. Hence, p^o is the minimum price for (A.2) to hold.

In region A_2, $L - q^I \in (-\mu(r),\, W - \mu(r))$. If $p = -r$, then (A.2) holds because $W - \mu(r) > L - q^I$. For any other $p_1 < -r$, (A.2) does not hold because $-\mu(-p_1) < -\mu(r) < L - q^I$.

In region A_3, $L - q^I \in [W - \mu(r),\, W]$. If $p = -h'(q^I + W - L) \in [-r, 0]$, then (A.2) holds with equality: $W - (q^I + W - L) = L - q^I$.

Lastly, in region A_4, $L - q^I > W$. If $p = (C^F)'(L - W - q^I) > (C^F)'(0)$, then (A.2) also holds with equality: $(L - W - q^I) + W = L - q^I$.

Hence, the minimum price p for (A.2) to hold is exactly $P(q^I, L, W, S^F)$ defined in (13). ∎

Proof of Theorem 2. Using Theorem 1, we can write

$$\widehat{C}(q^I, L, W) = C^F(q^{F*}) - r\, q^{V*} + h(q^I + q^{F*} + q^{V*} - L),$$

where q^{F*} and q^{V*} are given in Figure 1 under the four events with the following indicator functions:

$$\begin{aligned} \mathbf{1}_{A_1} &= \mathbf{1}_{L \leq q^I - \mu(r)}, \\ \mathbf{1}_{A_2} &= -\mathbf{1}_{L \leq q^I - \mu(r)} + \mathbf{1}_{L < q^I + W - \mu(r)}, \\ \mathbf{1}_{A_3} &= -\mathbf{1}_{L < q^I + W - \mu(r)} + \mathbf{1}_{L \leq q^I + W}, \\ \mathbf{1}_{A_4} &= \mathbf{1}_{L > q^I + W}. \end{aligned} \tag{A.3}$$

We denote $\widehat{C}_{A_i}(\cdot,\cdot,\cdot) = \widehat{C}(\cdot,\cdot,\cdot)$ when A_i occurs. Then, using the optimal policy in Figure 1, we have

$$\begin{aligned}
\widehat{C}_{A_1}(q^I, L, W) &= h(q^I - L), \\
\widehat{C}_{A_2}(q^I, L, W) &= -r\,(L - q^I + \mu(r)) + h(\mu(r)), \\
\widehat{C}_{A_3}(q^I, L, W) &= -r\,W + h(q^I + W - L), \\
\widehat{C}_{A_4}(q^I, L, W) &= C^F(L - q^I - W) - r\,W.
\end{aligned} \tag{A.4}$$

Using (A.3), we can write the expected second-stage cost as

$$\begin{aligned}
\mathsf{E}\big[\widehat{C}(q^I, L, W)\big] &= \sum_{i=1}^{4} \mathsf{E}\big[\widehat{C}_{A_i}(q^I, L, W)\mathbf{1}_{A_i}\big] \\
&= \mathsf{E}\Big[\big(\widehat{C}_{A_1}(q^I, L, W) - \widehat{C}_{A_2}(q^I, L, W)\big)\mathbf{1}_{L \le q^I - \mu(r)}\Big] \\
&\quad + \mathsf{E}\Big[\big(\widehat{C}_{A_2}(q^I, L, W) - \widehat{C}_{A_3}(q^I, L, W)\big)\mathbf{1}_{L < q^I + W - \mu(r)}\Big] \\
&\quad + \mathsf{E}\Big[\widehat{C}_{A_3}(q^I, L, W)\mathbf{1}_{L \le q^I + W}\Big] - \mathsf{E}\Big[\widehat{C}_{A_4}(q^I, L, W)\mathbf{1}_{L > q^I + W}\Big].
\end{aligned} \tag{A.5}$$

It can be verified that $\widehat{C}(q^I, L_s, W_s)$ is differentiable in q^I except at $q^I = L_s - W_s$, where the left derivative is $-(C^F)'(0)$ and the right derivative is $h'(0)$. Because L and W have continuous distributions, $\mathsf{E}\big[\widehat{C}(q^I, L, W)\big]$ is differentiable in q^I everywhere. Next, we compute its derivative.

The first three expectations in (A.5) all assume the form of $\mathsf{E}\big[g(q^I, L, W)\mathbf{1}_{L \le b(q^I, W)}\big]$, for some functions $g(q^I, L, W)$ and $b(q^I, W)$. Let $L \in (\underline{L}, \overline{L})$, $W \in (\underline{W}, \overline{W})$, and let the joint probability density function of (L, W) be $f(l, w)$. We use "$\vee$" and "$\wedge$" for max and min operations. We have

$$\begin{aligned}
\frac{d}{dq^I}\mathsf{E}\Big[g(q^I, L, W)\mathbf{1}_{L \le b(q^I, W)}\Big] &= \frac{d}{dq^I}\Big[\int_{\underline{W}}^{\overline{W}} \int_{\underline{L}}^{\underline{L} \vee b(q^I, w) \wedge \overline{L}} g(q^I, l, w) f(l, w)\, dl\, dw\Big]
\end{aligned} \tag{A.6}$$

$$= \mathsf{E}\Big[\frac{\partial g(q^I, L, W)}{\partial q^I}\mathbf{1}_{L \le b(q^I, W)}\Big] + \int_{\underline{W}}^{\overline{W}} g\big(q^I, b(q^I, w), w\big)\, f\big(b(q^I, w), w\big) \frac{\partial b(q^I, w)}{\partial q^I}\mathbf{1}_{b(q^I, w) \in [\underline{L}, \overline{L}]}\, dw.$$

The last expectation in (A.5) is in the form of $\mathsf{E}\big[g(q^I, L, W)\mathbf{1}_{L > b(q^I, W)}\big]$ and its derivative is

$$\begin{aligned}
\frac{d}{dq^I}\mathsf{E}\Big[g(q^I, L, W)\mathbf{1}_{L > b(q^I, W)}\Big] &= \frac{d}{dq^I}\Big[\int_{\underline{W}}^{\overline{W}} \int_{\underline{L} \vee b(q^I, w) \wedge \overline{L}}^{\overline{L}} g(q^I, l, w) f(l, w)\, dl\, dw\Big]
\end{aligned} \tag{A.7}$$

$$= \mathsf{E}\Big[\frac{\partial g(q^I, L, W)}{\partial q^I}\mathbf{1}_{L > b(q^I, W)}\Big] - \int_{\underline{W}}^{\overline{W}} g\big(q^I, b(q^I, w), w\big)\, f\big(b(q^I, w), w\big) \frac{\partial b(q^I, w)}{\partial q^I}\mathbf{1}_{b(q^I, w) \in [\underline{L}, \overline{L}]}\, dw.$$

Now, applying (A.6)-(A.7) to the derivatives of the expectations in (A.5), we find that the integral term $g\big(q^I, b(q^I, w), w\big)$ in (A.6)-(A.7) becomes

$$\begin{aligned}
\big(\widehat{C}_{A_1}(q^I, l, w) - \widehat{C}_{A_2}(q^I, l, w)\big)\big|_{l = q^I - \mu(r)} &= 0, \\
\big(\widehat{C}_{A_2}(q^I, l, w) - \widehat{C}_{A_3}(q^I, l, w)\big)\big|_{l = q^I + w - \mu(r)} &= 0, \\
\widehat{C}_{A_3}(q^I, l, w)\big|_{l = q^I + w} &= -r\,w, \\
\widehat{C}_{A_4}(q^I, l, w)\big|_{l = q^I + w} &= -r\,w,
\end{aligned}$$

where we have used the value of $\widehat{C}_{A_i}(\cdot,\cdot,\cdot)$ given by (A.4). This leads to

$$\begin{aligned}\frac{d}{dq^I}\mathsf{E}\big[\widehat{C}(q^I,L,W)\big] &= \sum_{i=1}^{4}\mathsf{E}\left[\frac{\partial \widehat{C}_{A_i}(q^I,L,W)}{\partial q^I}\mathbf{1}_{A_i}\right]\\ &= \mathsf{E}\Big[-(C^F)'(L-W-q^I)\mathbf{1}_{A_4}+h'(q^I+W-L)\mathbf{1}_{A_3}+r\mathbf{1}_{A_2}+h'(q^I-L)\mathbf{1}_{A_1}\Big]\\ &= \mathsf{E}\big[-P(q^I,L,W,S^F)\big],\end{aligned}$$

where the last equality follows from Lemma 1 and the definition in (13).

Now, because the objective in (15) is convex in q^I, $q^{I*}>0$ is optimal if and only if

$$(C^I)'(q^{I*})+\frac{d}{dq^I}\mathsf{E}\big[\widehat{C}(q^{I*},L,W)\big] \;=\; (C^I)'(q^{I*})-\mathsf{E}\big[P(q^{I*},L,W,S^F)\big] \;=\; 0,$$

which is equivalent to $(S^I)^{-1}(q^{I*})=\overline{P}(q^{I*},S^F)$ or $q^{I*}=S^I\big(\overline{P}(q^{I*},S^F)\big)$.

On the other hand, $q^{I*}=0$ is optimal if and only if $(C^I)'(0)\geq\overline{P}(0,S^F)$, which is equivalent to $S^I\big(\overline{P}(0,S^F)\big)=0$. (To see the equivalence, note that $S^I(p)=0$ if and only if $p\leq(C^I)'(0)$.) Thus, the same condition $q^{I*}=S^I\big(\overline{P}(q^{I*},S^F)\big)$ is necessary and sufficient for $q^{I*}=0$. ∎

Proof of Theorem 3. ***IG's profit function:*** Given a first-stage price $p_0>0$, IG $i\in G^I$ commits to produce $S_i(p_0)=\beta_i p_0$, incurring a cost of $\frac{1}{2}c_i(\beta_i p_0)^2$. Thus, the profit is $\beta_i p_0^2-\frac{1}{2}c_i(\beta_i p_0)^2=\beta_i\big(1-\frac{1}{2}c_i\beta_i\big)p_0^2$. Note that the first-stage price p_0 is equal to the expected second-stage price, which can be written as (using (23)):

$$\overline{P}(q^I,\beta^F) \;=\; \frac{q^I}{\beta^I} \;=\; \frac{q^{I*}(\beta^I,\beta^F)}{\beta^I}.$$

We define $\beta_{-i}\stackrel{\text{def}}{=}\beta^I-\beta_i$ and write generator i's expected profit as

$$\pi_i(\beta_i;\beta_{-i},\beta^F)=\frac{\beta_i\big(1-\frac{1}{2}c_i\beta_i\big)}{(\beta_i+\beta_{-i})^2}\left(q^{I*}(\beta_i+\beta_{-i},\beta^F)\right)^2, \tag{A.8}$$

which proves (25).

FG's profit function: Given the first-stage price p_0, FG $j\in G^F$ plans to produce $S_j(p_0)=\beta_j p_0$ and receives a predispatch payment of $\beta_j p_0^2$. When uncertainties are realized and the price is p_s, generator j's actual production is $\beta_j p_s^+$, incurring a cost of $\frac{1}{2}c_j(\beta_j p_s^+)^2$. The production deviation $\beta_j(p_s^+-p_0)$ is settled at p_s, resulting in a revenue of $\beta_j(p_s^+-p_0)p_s$ (can be negative if $p_s\in(0,p_0)$). Thus, the total expected revenue of generator j is

$$\begin{aligned}\beta_j p_0^2+\beta_j\mathsf{E}\big[(p_s^+-p_0)p_s\big] &= \beta_j p_0^2-\beta_j p_0\mathsf{E}\big[p_s\big]+\beta_j\mathsf{E}\big[p_s^+p_s\big]\\ &= \beta_j p_0^2-\beta_j p_0^2+\beta_j\mathsf{E}\big[p_s^+(p_s^++p_s^-)\big]\\ &= \beta_j\mathsf{E}\big[(p_s^+)^2\big],\end{aligned}$$

where the second equality is due to $p_0=\mathsf{E}[p_s]$, which follows from Theorem 2, and the last equality is due to the fact that $p_s^+ \cdot p_s^- \equiv 0$. Thus, the expected profit is

$$\beta_j \mathsf{E}\big[(p_s^+)^2\big] - \frac{1}{2}c_j \mathsf{E}\big[\big(\beta_j p_s^+\big)^2\big] = \beta_j\big(1 - \frac{1}{2}c_j\beta_j\big)\mathsf{E}\big[(p_s^+)^2\big].$$

Equation (21) implies that p_s^+ can be expressed as

$$p_s^+ = \frac{1}{\beta^F}(L_s - W_s - q^I)^+ \ = \ \frac{1}{\beta^F}\big(L_s - W_s - q^{I*}(\beta^I, \beta^F)\big)^+.$$

We define $\beta_{-j} \overset{\text{def}}{=} \beta^F - \beta_j$ and write generator j's expected profit as

$$\pi_j(\beta_j; \beta_{-j}, \beta^I) = \frac{\beta_j\big(1 - \frac{1}{2}c_j\beta_j\big)}{(\beta_j + \beta_{-j})^2}\,\mathsf{E}\Big[\big(\big(L - W - q^{I*}(\beta^I, \beta_j + \beta_{-j})\big)^+\big)^2\Big], \tag{A.9}$$

which proves (26).

Monotonicity of $q^{I*}(\beta^I, \beta^F)$: Recall that the average price $\overline{P}(q^I, \beta^F)$ decreases in q^I, as discussed after the definition in (17). Furthermore, because L and W have continuous distributions, $\overline{P}(q^I, \beta^F)$ is differentiable in q^I everywhere. Denote $\overline{P}_1 \equiv \partial\overline{P}/\partial q^I$. We have $\overline{P}_1 \leq 0$.

Equation (23), $q^{I*} - \beta^I\overline{P}(q^{I*}, \beta^F) = 0$, implicitly determines q^{I*} as a function of β_k, $k \in G^I \cup G^F$. Therefore,

$$\frac{\partial q^{I*}}{\partial \beta_i} = \frac{\overline{P}}{1 - \beta^I\overline{P}_1} = \frac{q^{I*}}{\beta^I(1 - \beta^I\overline{P}_1)} > 0, \qquad i \in G^I, \tag{A.10}$$

which proves that q^{I*} strictly increases in β_i.

We finally derive $\partial q^{I*}/\partial\beta_j$. To simplify notations, let $D = L - W$ denote the net demand. For a continuous random variable X, we use $f_X(x)$ and $F_X(x)$ to denote the probability density and cumulative distribution functions, respectively, and let $\overline{F}_X(x) = 1 - F_X(x)$.

Then, we can write the average price function in (22) as

$$\begin{aligned}\overline{P}(q^I, \beta^F) = \frac{1}{\beta^F}\mathsf{E}\left[(D - q^I)^+\right] &- c_h\int_{q^I-\mu(r)}^{q^I}(q^I - x)f_D(x)\,dx - c_h\int_{-\infty}^{q^I-\mu(r)}(q^I - x)f_L(x)\,dx \\ &- a_hF_D(q^I) + (a_h - r)F_D(q^I - \mu(r)) + (r - a_h)F_L(q^I - \mu(r)).\end{aligned} \tag{A.11}$$

Thus,

$$\begin{aligned}\overline{P}_2 \equiv \frac{\partial\overline{P}}{\partial\beta^F} &= -\frac{1}{(\beta^F)^2}\mathsf{E}\left[(D - q^I)^+\right] \ < \ 0, \\ \frac{\partial q^{I*}}{\partial\beta_j} = \frac{\beta^I\overline{P}_2}{1 - \beta^I\overline{P}_1} &= -\frac{\beta^I\mathsf{E}\left[(D - q^{I*})^+\right]}{(\beta^F)^2(1 - \beta^I\overline{P}_1)} \ < \ 0, \qquad j \in G^F.\end{aligned} \tag{A.12}$$

This completes the proof. ∎

Proof of Corollary 1. When $D = L - W > q^{I*}$ (i.e., event A_4) occurs with probability one, equations (21)-(22) imply that $\overline{P}(q^{I*}, \beta^F) = (\mathsf{E}[D] - q^{I*})/\beta^F$. Solving equation (23) for q^{I*} gives

$$q^{I*}(\beta^I, \beta^F) = \frac{\beta^I}{\beta^I + \beta^F}\mathsf{E}[D].$$

Consequently, (25) becomes

$$\pi_i = \frac{\beta_i(1-\frac{1}{2}c_i\beta_i)}{(\beta^I)^2}\Big(\frac{\beta^I}{\beta^I+\beta^F}\mathsf{E}[D]\Big)^2 = \frac{\beta_i(1-\frac{1}{2}c_i\beta_i)}{(\beta^I+\beta^F)^2}\big(\mathsf{E}[D]\big)^2 = \lambda(i, G^I \cup G^F)\big(\mathsf{E}[D]\big)^2,$$

and (26) simplifies to

$$\begin{aligned}\pi_j &= \frac{\beta_j(1-\frac{1}{2}c_j\beta_j)}{(\beta^F)^2}\mathsf{E}\Big[\Big(D - \frac{\beta^I}{\beta^I+\beta^F}\mathsf{E}[D]\Big)^2\Big] \\ &= \frac{\beta_j(1-\frac{1}{2}c_j\beta_j)}{(\beta^F)^2}\mathsf{E}\Big[\Big(D - \mathsf{E}[D] + \frac{\beta^F}{\beta^I+\beta^F}\mathsf{E}[D]\Big)^2\Big] \\ &= \frac{\beta_j(1-\frac{1}{2}c_j\beta_j)}{(\beta^F)^2}\Big(\mathsf{Var}[D] + \Big(\frac{\beta^F}{\beta^I+\beta^F}\mathsf{E}[D]\Big)^2\Big) \\ &= \lambda(j, G^F)\mathsf{Var}[D] + \lambda(j, G^I \cup G^F)\big(\mathsf{E}[D]\big)^2.\end{aligned}$$

This completes the proof of Corollary 1. ∎

Proof of Lemma 3. We first bound the average price in (A.11). Note that

$$\begin{aligned}&\int_{q^I-\mu(r)}^{q^I}(q^I-x)f_D(x)\,dx \ge 0, \qquad \text{and} \\ &\int_{-\infty}^{q^I-\mu(r)}(q^I-x)f_L(x)\,dx > \mu(r)F_L(q^I-\mu(r)) \\ &\qquad\qquad = \frac{(r-a_h)^+}{c_h}F_L(q^I-\mu(r)) \ \ge\ \frac{(r-a_h)}{c_h}F_L(q^I-\mu(r)).\end{aligned}$$

Using these inequalities, the average price in (A.11) is bounded above by

$$\overline{P}(q^I,\beta^F) < \frac{1}{\beta^F}\mathsf{E}\left[(D-q^I)^+\right] - a_hF_D(q^I) + (a_h - r)F_D(q^I-\mu(r)). \tag{A.13}$$

If $a_h \ge r$, then $\mu(r)=0$ and (A.13) becomes $\overline{P}(q^I,\beta^F) < \frac{1}{\beta^F}\mathsf{E}\left[(D-q^I)^+\right] - rF_D(q^I)$. If $a_h < r$, then (A.13) implies $\overline{P}(q^I,\beta^F) < \frac{1}{\beta^F}\mathsf{E}\left[(D-q^I)^+\right] - a_hF_D(q^I)$. Combining these two cases, we obtain

$$\overline{P}(q^I,\beta^F) \ < \ \frac{1}{\beta^F}\mathsf{E}\left[(D-q^I)^+\right] - \min\{r, a_h\}F_D(q^I). \tag{A.14}$$

We can express and bound $\mathsf{E}\left[(D-q^I)^+\right]$ as follows:

$$\begin{aligned}\mathsf{E}\left[(D-q^I)^+\right] &= \int_{q^I}^{\infty}(x-\mu_D+\mu_D-q^I)f_D(x)\,dx \\ &= \int_{q^I}^{\infty}\frac{x-\mu_D}{\sqrt{2\pi}\sigma_D}\exp\left(-\frac{(x-\mu_D)^2}{2\sigma_D^2}\right)dx + \int_{q^I}^{\infty}(\mu_D-q^I)f_D(x)\,dx \\ &= \frac{\sigma_D}{\sqrt{2\pi}}\int_{\frac{q^I-\mu_D}{\sigma_D}}^{\infty} y\exp\left(-y^2/2\right)dy + (\mu_D-q^I)\overline{F}_D(q^I) \\ &\le \frac{\sigma_D}{\sqrt{2\pi}} + (\mu_D-q^I)\overline{F}_D(q^I).\end{aligned} \tag{A.15}$$

The inequalities (A.14) and (A.15) lead to

$$\overline{P}(q^I,\beta^F) \ < \ \frac{\sigma_D}{\sqrt{2\pi}\beta^F} + \frac{(\mu_D-q^I)\overline{F}_D(q^I)}{\beta^F} - \min\{r, a_h\}F_D(q^I). \tag{A.16}$$

Using (23) and (A.16), we have

$$q^{I\max} = \beta^{I\max}\overline{P}(q^{I\max}, \beta^{F\min}) \tag{A.17}$$
$$< \beta^{I\max}\left[\frac{\sigma_D}{\sqrt{2\pi}\beta^{F\min}} + \frac{(\mu_D - q^{I\max})\overline{F}_D(q^{I\max})}{\beta^{F\min}} - \min\{r, a_h\}F_D(q^{I\max})\right].$$

We now prove $q^{I\max} < \mu_D$. If the opposite is true, $q^{I\max} \geq \mu_D$, then $F_D(q^{I\max}) \geq \frac{1}{2}$ and (A.17) implies

$$q^{I\max} < \beta^{I\max}\left[\frac{\sigma_D}{\sqrt{2\pi}\beta^{F\min}} - \frac{\min\{r, a_h\}}{2}\right] \leq \beta^{I\max}\left[\frac{\sigma_D^*}{\sqrt{2\pi}\beta^{F\min}} - \frac{\min\{r, a_h\}}{2}\right] = \mu_D,$$

where $\sigma_D^* \equiv \sqrt{2\pi}\,\beta^{F\min}\left[\frac{\mu_D}{\beta^{I\max}} + \frac{\min\{r, a_h\}}{2}\right]$. This contradicts $q^{I\max} \geq \mu_D$. Therefore, we conclude that $q^{I\max} < \mu_D$ when $\sigma_D \leq \sigma_D^*$. ∎

Proof of Theorem 4. Because generator k's pure strategy space is a finite interval $[\beta_k^{\min}, c_k^{-1}]$, it suffices to show that, $\forall\, k \in G^I \cup G^F$, generator k's profit function is quasi-concave with respect to β_k to prove the existence of a pure strategy Nash equilibrium (Debreu 1952).

The proof of the quasi-concavity will use the derivatives of $\overline{P}(q^I, \beta^F)$. Differentiating $\overline{P}(q^I, \beta^F)$ in (A.11) with respect to q^I and using $\mu(r) = (r - a_h)^+/c_h$, we obtain

$$\begin{aligned}\overline{P}_1(q^I, \beta^F) \equiv \frac{\partial \overline{P}}{\partial q^I} &= -\frac{1}{\beta^F}\overline{F}_D(q^I) - c_h\left[F_D(q^I) - F_D(q^I - \mu(r)) + F_L(q^I - \mu(r))\right] \\ &\quad - a_h f_D(q^I) + \left[c_h\mu(r) + (a_h - r)\right]\left[f_D(q^I - \mu(r)) - f_L(q^I - \mu(r))\right] \\ &= -\frac{1}{\beta^F}\overline{F}_D(q^I) - c_h\left[F_D(q^I) - F_D(q^I - \mu(r)) + F_L(q^I - \mu(r))\right] \\ &\quad - a_h f_D(q^I) + (a_h - r)^+\left[f_D(q^I - \mu(r)) - f_L(q^I - \mu(r))\right],\end{aligned} \tag{A.18}$$

$$\begin{aligned}\overline{P}_{11}(q^I, \beta^F) \equiv \frac{\partial^2 \overline{P}}{\partial {q^I}^2} &= \frac{1}{\beta^F} f_D(q^I) - c_h\left[f_D(q^I) - f_D(q^I - \mu(r)) + f_L(q^I - \mu(r))\right] \\ &\quad - a_h f_D'(q^I) + (a_h - r)^+\left[f_D'(q^I - \mu(r)) - f_L'(q^I - \mu(r))\right].\end{aligned} \tag{A.19}$$

By Lemma 3, if $\sigma_D \leq \sigma_D^*$, then we have $q^{I\max} < \mu_D$. When $q^I < \mu_D$, and $\sigma_D \to 0$, all the distribution functions in (A.18)-(A.19) approach zero, except for $\overline{F}_D(q^I)$, which approaches one. Therefore, when σ_D is small, $\overline{P}_1$ is close to $-1/\beta^F$ and $\overline{P}_{11}$ is close to zero.

Quasi-concavity of IG's profit function. The profit function of IG $i \in G^I$ is expressed as $\pi_i(\beta_i; \beta_{-i}, \beta^F)$ in (25). To prove its quasi-concavity in β_i, we will show that its derivative $\partial\pi_i/\partial\beta_i$ can cross zero value from above at most once as β_i increases, while holding β_{-i} and β^F constant.

Noting $\partial q^{I*}/\partial\beta_i$ in (A.10), we differentiate (25) with respect to β_i to obtain

$$\begin{aligned}\frac{\partial \pi_i}{\partial \beta_i} &= \frac{\beta_i(2 - c_i\beta_i)}{(\beta_i + \beta_{-i})^2}\, q^{I*}\frac{\partial q^{I*}}{\partial \beta_i} + \frac{\beta_{-i}(1 - c_i\beta_i) - \beta_i}{(\beta_i + \beta_{-i})^3}\left(q^{I*}\right)^2 \\ &= \frac{\beta_i(2 - c_i\beta_i)}{(\beta^I)^2}\frac{\left(q^{I*}\right)^2}{\beta^I(1 - \beta^I\overline{P}_1)} + \frac{\beta_{-i}(1 - c_i\beta_i) - \beta_i}{(\beta^I)^3}\left(q^{I*}\right)^2\end{aligned}$$

$$= \frac{(q^{I*})^2}{(\beta^I)^3(1-\beta^I\overline{P}_1)}\Big[\beta_i(2-c_i\beta_i) + \big(\beta_{-i}(1-c_i\beta_i)-\beta_i\big)(1-\beta^I\overline{P}_1)\Big]$$
$$= \frac{(q^{I*})^2}{(\beta^I)^3(1-\beta^I\overline{P}_1)}\Big[\beta^I(1-c_i\beta_i) - \big(\beta_{-i}(1-c_i\beta_i)-\beta_i\big)\beta^I\overline{P}_1\Big]$$
$$= \frac{(q^{I*})^2}{(\beta^I)^2(1-\beta^I\overline{P}_1)}X(\beta_i;\beta_{-i},\beta^F),$$

where $X(\beta_i;\beta_{-i},\beta^F) \stackrel{\text{def}}{=} 1-c_i\beta_i+\big(\beta_i(1+c_i\beta_{-i})-\beta_{-i}\big)\overline{P}_1$. To show $\partial\pi_i/\partial\beta_i$ can cross zero value from above at most once, it suffices to show X decreases in β_i. Differentiating X with respect to β_i, we have

$$\frac{\partial X}{\partial \beta_i} = -c_i + (1+c_i\beta_{-i})\overline{P}_1 + \big(\beta_i(1+c_i\beta_{-i})-\beta_{-i}\big)\overline{P}_{11}\,\frac{q^{I*}}{\beta^I(1-\beta^I\overline{P}_1)},$$

where $\overline{P}_{11}$ is derived in (A.19). Note that $-c_i+(1+c_i\beta_{-i})\overline{P}_1 < 0$. Thus, if $\overline{P}_{11}(q^{I*},\beta^F)$ is sufficiently small, we can establish $\partial X/\partial\beta_i \le 0$. Based on the discussion after (A.18) and (A.19), there exists $\widehat{\sigma}_D$, such that when $\sigma_D < \widehat{\sigma}_D$, we have $\partial X/\partial\beta_i \le 0$ and, therefore, π_i is quasi-concave in β_i.

Quasi-concavity of FG's profit function. Denote $D = L - W$ and $q^{I*} = q^{I*}(\beta^I,\beta^F)$, we can write FG's profit function in (26) as

$$\pi_j(\beta_j;\beta_{-j},\beta^I) = \frac{\beta_j\big(1-\frac{1}{2}c_j\beta_j\big)}{(\beta_j+\beta_{-j})^2}\mathsf{E}\left[\big((D-q^{I*})^+\big)^2\right].$$

We will show that $\partial\pi_j/\partial\beta_j$ can cross zero value at most once from above when β_j increases.

Differentiating π_j with respect to β_j and using $\partial q^{I*}/\partial\beta_j$ from (A.12) and the following fact

$$\frac{\partial}{\partial q^{I*}}\mathsf{E}\left[\big((D-q^{I*})^+\big)^2\right] = \frac{\partial}{\partial q^{I*}}\int_{q^{I*}}^{\infty}(x-q^{I*})^2 f_D(x)dx$$
$$= \int_{q^{I*}}^{\infty} -2(x-q^{I*})f_D(x)dx = -2\mathsf{E}\big[(D-q^{I*})^+\big],$$

we obtain

$$\frac{\partial \pi_j}{\partial \beta_j} = \frac{\beta_{-j}(1-c_j\beta_j)-\beta_j}{(\beta^F)^3}\mathsf{E}\left[\big((D-q^{I*})^+\big)^2\right] + \frac{\beta_j(2-c_j\beta_j)}{(\beta^F)^2}\mathsf{E}\left[(D-q^{I*})^+\right]\frac{\beta^I\mathsf{E}\left[(D-q^{I*})^+\right]}{(\beta^F)^2(1-\beta^I\overline{P}_1)}$$
$$= \frac{\beta_j\mathsf{E}\big[((D-q^{I*})^+)^2\big]}{(\beta^F)^3(1-\beta^I\overline{P}_1)}\Big[Y(\beta_j,\beta_{-j},\beta^I) + \beta^I Z(\beta_j,\beta_{-j},\beta^I)\Big],$$

where

$$Y(\beta_j,\beta_{-j},\beta^I) \stackrel{\text{def}}{=} \Big(\frac{\beta_{-j}}{\beta_j} - (1+c_j\beta_{-j})\Big)(1-\beta^I\overline{P}_1),$$
$$Z(\beta_j,\beta_{-j},\beta^I) \stackrel{\text{def}}{=} \frac{2-c_j\beta_j}{\beta_j+\beta_{-j}}\psi(q^{I*}),$$
$$\psi(q^{I*}) \stackrel{\text{def}}{=} \frac{\big(\mathsf{E}\left[(D-q^{I*})^+\right]\big)^2}{\mathsf{E}\left[((D-q^{I*})^+)^2\right]}.$$

It suffices to show that Y and Z decrease in β_j. Differentiating Y with respect to β_j, we have

$$\frac{\partial Y}{\partial \beta_j} = -\frac{\beta_{-j}}{\beta_j^2}(1-\beta^I\overline{P}_1) + \Big(\frac{\beta_{-j}}{\beta_j} - (1+c_j\beta_{-j})\Big)\beta^I\overline{P}_{11}\frac{\beta^I\mathsf{E}\left[(D-q^{I*})^+\right]}{(\beta^F)^2(1-\beta^I\overline{P}_1)}.$$

By the same argument used for the quasi-concavity of π_i, we see that when σ_D is sufficiently small, $\overline{P}_1$ is close to $-1/\beta^F$ and $\overline{P}_{11}$ is close to zero. Thus, there exists $\widetilde{\sigma}_D$, such that when $\sigma_D < \widetilde{\sigma}_D$, we have $\partial Y/\partial\beta_j \leq 0$.

Next, we show that Z decreases in β_j. Note that $\frac{\partial}{\partial q^{I*}}\mathsf{E}\left[(D-q^{I*})^+\right] = -\overline{F}_D(q^{I*})$ and

$$\psi'(q^{I*}) = -\frac{2\mathsf{E}\left[(D-q^{I*})^+\right]\overline{F}_D(q^{I*})}{\mathsf{E}\left[\left((D-q^{I*})^+\right)^2\right]} + \frac{2\left(\mathsf{E}\left[(D-q^{I*})^+\right]\right)^3}{\left(\mathsf{E}\left[\left((D-q^{I*})^+\right)^2\right]\right)^2} = -\frac{2\psi(q^{I*})(\overline{F}_D(q^{I*})-\psi(q^{I*}))}{\mathsf{E}[(D-q^{I*})^+]}.$$

Using this derivative and $\partial q^{I*}/\partial\beta_j$ in (A.12), we have

$$\begin{aligned}\frac{\partial Z}{\partial\beta_j} &= \frac{-c_j\beta^F-(2-c_j\beta_j)}{(\beta^F)^2}\,\psi(q^{I*}) + \frac{2-c_j\beta_j}{\beta^F}\,\psi'(q^{I*})\frac{\partial q^{I*}}{\partial\beta_j}\\ &= \frac{-c_j\beta^F-(2-c_j\beta_j)}{(\beta^F)^2}\,\psi(q^{I*}) + \frac{2-c_j\beta_j}{\beta^F}\,\frac{2\psi(q^{I*})(\overline{F}_D(q^{I*})-\psi(q^{I*}))\beta^I}{(\beta^F)^2\,(1-\beta^I\overline{P}_1)}\\ &= \frac{\psi(q^{I*})}{(\beta^F)^2}\left[-c_j\beta^F-(2-c_j\beta_j)\left(1-\frac{2(\overline{F}_D(q^{I*})-\psi(q^{I*}))\beta^I}{\beta^F(1-\beta^I\overline{P}_1)}\right)\right].\end{aligned}$$

We will show that $\psi(q^{I*})$ is close to $\overline{F}_D(q^{I*})$ when σ_D is sufficiently small and $q^{I*} < \mu_D$ to complete the proof.

For a normal random variable $X \sim \mathcal{N}(\mu,\sigma)$, we can show that $\mathsf{E}[X^+] = \mu\overline{F}_X(0) + \sigma^2 f_X(0)$ and $\mathsf{E}\left[(X^+)^2\right] = (\mu^2+\sigma^2)\overline{F}_X(0) + \mu\sigma^2 f_X(0)$. Then,

$$\begin{aligned}\mathsf{E}\left[(D-q^{I*})^+\right] &= (\mu_D-q^{I*})\overline{F}_D(q^{I*}) + \sigma_D^2 f_D(q^{I*}),\\ \mathsf{E}\left[((D-q^{I*})^+)^2\right] &= ((\mu_D-q^{I*})^2+\sigma_D^2)\overline{F}_D(q^{I*}) + \sigma_D^2(\mu_D-q^{I*})f_D(q^{I*}),\\ \psi(q^{I*}) &= \frac{\left[(\mu_D-q^{I*})\overline{F}_D(q^{I*}) + \sigma_D^2 f_D(q^{I*})\right]^2}{\left[(\mu_D-q^{I*})^2+\sigma_D^2\right]\overline{F}_D(q^{I*}) + \sigma_D^2(\mu_D-q^{I*})f_D(q^{I*})}.\end{aligned}$$

If $\sigma_D \leq \sigma_D^*$, we have $q^{I\max} < \mu_D$ (Lemma 3). The above expression for $\psi(q^{I*})$ implies that as $\sigma_D \to 0$, we have $\overline{F}_D(q^{I*}) \to 1$, $f_D(q^{I*}) \to 0$, and $\psi(q^{I*}) \to 1$. Hence, there exists $\sigma_D^\dagger$, such that when $\sigma_D < \sigma_D^\dagger$, we have $\partial Z/\partial\beta_j \leq 0$.

To summarize, when $\sigma_D < \min\{\sigma_D^*, \widehat{\sigma}_D, \widetilde{\sigma}_D, \sigma_D^\dagger\}$, the profit function π_j is quasi-concave in β_j. This establishes the existence of a pure strategy equilibrium, i.e., the linear supply function equilibrium. ∎

Reference

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