Managing Production-Inventory Systems with Scarce Resources

Online Supplement

Proof of Lemma 1: Consider the following dynamic program:

$$\bar{g}_t(x,z) = \max_{x \le y \le \min(x+u,z)} \{-cy + E\bar{f}_t(y,z,D)\}, \tag{7}$$

where

$$\bar{f}_t(y,z,d) = \max_{0 \le l \le \min(d,z)} \{\bar{L}(y-l) + py + \alpha\bar{g}_{t-1}(y-l,z-l)\}, \tag{8}$$

and $\bar{g}_0(x,z) = 0$, for $z \ge x \ge 0$, where $\bar{L}(x) = (-p+\alpha c)x - h(x)^+ - b(-x)^+$.

Let $\bar{G}_t(y,z) = -cy + E\bar{f}_t(y,z,D)$ and $\bar{F}_t(y,z,l) = \bar{L}(y-l) + py + \alpha\bar{g}_{t-1}(y-l,z-l)$. It can be shown by induction that $0 \le \frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) \le p-c$. Note that

$$\frac{\partial}{\partial l}\bar{F}_t(y,z,l) = \begin{cases} p-\alpha c+h-\alpha(\frac{\partial}{\partial x}\bar{v}_{t-1}(y-l,z-l) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-l,z-l)), & \text{if } l \le y, \\ p-\alpha c-b-\alpha(\frac{\partial}{\partial x}\bar{v}_{t-1}(y-l,z-l) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-l,z-l)), & \text{if } l > y. \end{cases}$$

Clearly, when $b \ge p-\alpha c$, $\frac{\partial}{\partial l}\bar{F}_t(y,z,l) \ge 0$ if $l \le y$; and $\frac{\partial}{\partial l}\bar{F}_t(y,z,l) \le 0$ if $l > y$. This implies that $\bar{f}_t(y,z,d) = \bar{F}_t(\min(y,d),z,d)$ for $y \le z$ and

$$\begin{aligned} \bar{g}_t(x,z) &= \max_{x \le y \le \min(x+u,z)} \{-cy + E\bar{F}_t(\min(y,D),z,D)\}, \\ &= \max_{x \le y \le \min(x+u,z)} \{(p-c)y + (p-\alpha c+h)E(y-D)^+ + \alpha E\bar{g}_{t-1}((y-D)^+, z-\min(y,d))\}. \end{aligned}$$

Compare the above equation with Equation (2), we have $\bar{v}_t(x,z) = \bar{g}_t(x,z)$ for all x,z and $t = 0,\ldots,T$, and $\bar{V}_t(x,z) = \bar{G}_t(y,z)$ for all y,z and $t = 0,\ldots,T$, when $b \ge p-\alpha c$. Therefore, we have $0 \le \frac{\partial}{\partial x}\bar{v}_t(x,z) + \frac{\partial}{\partial z}\bar{v}_t(x,z) \le p-c$.

To show that $\bar{V}_t(y,z)$ satisfies properties (a), (b) and (c), we only need to show that $-\bar{V}(y,z)$ is $L^\natural$-convex, or equivalently, $-\bar{G}(y,z)$ is $L^\natural$-convex. In the following, we prove by induction that $-\bar{g}_t(x,z)$ and $-\bar{G}_t(y,z)$ are both $L^\natural$-convex for $t = 1,\ldots,T$.

We have $\bar{g}_0(x,z) = 0$ for $z \ge x \ge 0$. Clearly, $-\bar{g}_0(x,z)$ is $L^\natural$-convex. Suppose $-\bar{g}_{t-1}(x,z)$ is $L^\natural$-convex. It can be easily verified that $\bar{L}(x)$ is concave. By Lemma 1 of Zipkin (2008),

$-\bar{g}_{t-1}(y-l,z-l)$ is $L^\natural$-convex in (y,z,l) and $-L(y-l)$ is $L^\natural$-convex in (y,l). Then, $\bar{F}_t(y,z,l) = L(y-l) + py + \alpha\bar{g}_{t-1}(y-l,z-l)$ is $L^\natural$-convex. By property (b) of Lemma 2 of Huh and Janakiraman (2010), we see that $-\bar{f}_t(y,z,d) = \min_{0\le l\le \min(d,z)}\{-\bar{F}_t(y,z,l)\}$ is $L^\natural$-convex in (y,z) for fixed d. Thus, $-\bar{G}_t(y,z) = cy - E\bar{f}_t(y,z,D)$ is $L^\natural$-convex. Again, by property (b) of Lemma 2 of Huh and Janakiraman (2010), $-\bar{g}_t(x,z) = \min_{x\le y\le \min(x+u,z)}\{-\bar{G}_t(y,z)\}$ is $L^\natural$-convex. This completes the induction and the proof. □

Proof of Theorem 2: Let $\bar{y}_t(z) = \operatorname*{argmax}_{0\le y\le z} \bar{V}_t(y,z)$ and $y_t^*(x_t,q_t) = \bar{y}_t(x_t+q_t)$. Since $\bar{V}_t(y,z)$ is concave in y, the optimal production threshold is given by

$$y_t(x_t,q_t) = \operatorname*{argmax}_{x_t\le y\le \min(x_t+u,x_t+q_t)} \bar{V}_t(y,x_t+q_t) = \begin{cases} x_t, & \text{if } x_t \ge y_t^*(x_t,q_t), \\ y_t^*(x_t,q_t), & \text{if } x_t \le y_t^*(x_t,q_t) \le x_t+u, \\ x_t+u, & \text{if } x_t+u \le y_t^*(x_t,q_t). \end{cases}$$

Clearly, $y_t^*(x_t,q_t)$ depends only on the sum x_t+q_t.

Since $\bar{V}_t(y,z)$ is $L^\natural$-convex and $\bar{y}_t(z) = \operatorname*{argmin}_{0\le y\le z}\{-\bar{V}_t(y,z)\}$, by property (c) of Lemma 2 of Huh and Janakiraman (2010), we have $0 \le \frac{\partial}{\partial z}\bar{y}_t(z) \le 1$. This proves properties (a) and (b). □

Proof of Proposition 3: It can be shown by induction that $0 \le \frac{\partial}{\partial z}\bar{v}_t(x,z,u) \le p-c$. Note that $v_t(x,q,u) = \bar{v}_t(x,x+q,u) + cx$. Thus, $\frac{\partial}{\partial q}v_t(x,q,u) = \frac{\partial}{\partial z}\bar{v}_t(x,x+q,u) \in [0,p-c]$ and $\frac{\partial^2}{\partial q^2}v_t(x,q,u) = \frac{\partial^2}{\partial z^2}\bar{v}_t(x,x+q,u) \le 0$. This proves properties (a) and (b).

(c) If we can show $\bar{v}_t(x,z,u) = v_t(x,z-x,u) - cx$ is supermodular in (z,u), then we have $\frac{\partial^2}{\partial q\partial u}v_t(x,q,u) = \frac{\partial^2}{\partial z\partial u}v_t(x,z,u)|_{z=x+q} \ge 0$. In the following, we prove by induction that $\bar{v}_t(x,z,u)$ is supermodular in (z,u), $\pi_t(z,u) = \bar{v}_t(z,z,u)$ is supermodular in (z,u), and $\frac{\partial}{\partial x}\bar{v}_t(x,z,u) + \frac{\partial}{\partial z}\bar{v}_t(x,z,u)$ is nondecreasing in u.

Clearly, this is true for $t=0$ since $\bar{v}_0(x,z,u) = 0$. Now, suppose $\bar{v}_{t-1}(x,z,u)$ is supermodular in (z,u), $\pi_{t-1}(z,u)$ is supermodular in (z,u), and $\frac{\partial}{\partial x}\bar{v}_{t-1}(x,z,u) + \frac{\partial}{\partial z}\bar{v}_{t-1}(x,z,u)$ is nondecreasing in u. Then, it is easy to verify that $\bar{V}_t(y,z,u)$ is supermodular in (z,u) and $\Pi_t(z,u) = V_t(z,z,u)$ is supermodular in (z,u). Moreover,

$$\frac{\partial}{\partial y}\bar{V}_t(y,z,u) + \frac{\partial}{\partial z}\bar{V}_t(y,z,u) = L'(y) + \alpha\int_0^y [\frac{\partial}{\partial x}\bar{v}_{t-1}(y-\xi,z-\xi,u) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-\xi,z-\xi,u)]\phi(\xi)d\xi,$$

which is nonincreasing in u. Equivalently, $\frac{\partial}{\partial u}\bar{V}_t(\bar{y}+\delta,z,u+\delta) \le \frac{\partial}{\partial u}\bar{V}_t(\bar{y},z,u)$ for $\delta \ge 0$. Similar to the proofs of Lemma 1 and Theorem 2, we can show that the optimal production threshold $y_t(x,q,u)$ is nonincreasing in u and $|\frac{\partial}{\partial u}y_t(x,z,u)| \le 1$. Thus, for fixed x and z, there exist u_1

and u_2, with $u_1 \le u_2$, such that

$$y_t(x,z,u) = \begin{cases} x+u, & \text{if } u \in [0,u_1], \\ \bar{y}_t(z,u), & \text{if } u \in [u_1,u_2], \\ x, & \text{if } u \in [u_2,\infty). \end{cases}$$

When $u \in [0,u_1]$, $\bar{v}_t(x,z,u) = \bar{V}_t(x,z,u)$, which is supermodular in (z,u). When $u \in [u_2,\infty)$, $\bar{v}_t(x,z,u) = \bar{V}_t(x+u,z,u)$, which is supermodular in (z,u) since $\bar{V}_t(y,z,u)$ is supermodular in (z,u) and (y,z). When $u \in [u_1,u_2]$, if $\bar{y}_t(z,u) = z$, then $\bar{v}_t(x,z,u) = \bar{V}_t(z,z,u)$, which is supermodular in (z,u). This also implies that $\pi_t(z,u) = \bar{v}_t(z,z,u) = \bar{V}_t(z,z,u)$ is supermodular in (z,u). If $\bar{y}_t(z,u) < z$, we have $\bar{v}_t(x,z,u) = \bar{V}_t(y_t(z,u),z,u)$ and $\frac{\partial}{\partial u}\bar{v}_t(x,z,u) = \frac{\partial}{\partial u}\bar{V}_t(\bar{y}_t(z,u),z,u)$. Thus, for $\delta \ge 0$,

$$\begin{aligned} \frac{\partial}{\partial u}\bar{v}_t(x,z+\delta,u) &= \frac{\partial}{\partial u}\bar{V}_t(\bar{y}_t(z+\delta,u),z+\delta,u) \\ &\ge \frac{\partial}{\partial u}\bar{V}_t(\bar{y}_t(z,u)+\delta,z+\delta,u) \ge \frac{\partial}{\partial u}\bar{V}_t(\bar{y},z,u) = \frac{\partial}{\partial u}\bar{v}_t(x,z,u). \end{aligned}$$

This implies that $\bar{v}_t(x,z,u)$ is supermodular when $u \in [u_1,u_2]$. Therefore, for fixed x and z, $\frac{\partial}{\partial z}\bar{v}(x,z,u)$ in nondecreasing in u on $[0,u_1)$, (u_1,u_2), and (u_2,∞). Moreover, it can be verified that $\lim_{u\uparrow u_1} \frac{\partial}{\partial z}\bar{v}(x,z,u) \le \lim_{u\downarrow u_1} \frac{\partial}{\partial z}\bar{v}(x,z,u)$ and $\lim_{u\uparrow u_2} \frac{\partial}{\partial z}\bar{v}(x,z,u) \le \lim_{u\downarrow u_2} \frac{\partial}{\partial z}\bar{v}(x,z,u)$. Thus, $\frac{\partial}{\partial z}\bar{v}(x,z,u)$ in nondecreasing in u for all u and $\bar{v}_t(x,u,z)$ in supermodular in (u,z). Similarly, we can show $\frac{\partial}{\partial x}\bar{v}_t(x,z,u) + \frac{\partial}{\partial z}\bar{v}_t(x,z,u)$ is nondecreasing in u. This completes the induction and the proof. □

Proof of Theorem 4: Let $\hat{V}_t(y,z) = L(y) + E\hat{v}_{t-1}(z-y+(y-D)^+)$. Similar to the proofs of Lemma 1 and Theorem 2, we can show that $\hat{v}_t(z)$ is concave, $\hat{V}_t(y,z)$ is jointly concave and supermodular, and $0 \le \hat{y}_t'(z) \le 1$. We can also show by induction that $v_t(z)$ is continuously differentiable, $0 \le \hat{v}_t'(z) \le p-c$, and $\hat{v}'(0) = p-c$.

Next, we show by induction that $\hat{y}_t(z)$ satisfies the following properties:

(a) if $z \ge ty^b$, then $\hat{y}_t(z) = y^b$,

(b) if $z < ty^b$, then $\frac{z}{t} \le \hat{y}_t(z) \le y^b$,

(c) if $z \le \check{y}$, then $\hat{y}_t(z) = z$, and

(d) $\hat{y}_t(z)$ is nonincreasing in t.

For (a), we show by induction that when $z \ge ty^b$, $v_t'(z) = 0$. For $t=0$, $v_t'(z) = 0$. Now, suppose that when $z \ge (t-1)y^b$, $v_{t-1}'(z) = 0$. Then, when $z \ge ty^b$, we have $z - \xi \ge (t-1)y^b$ for $\xi \le y^b$. Thus, $v_{t-1}'(z-\xi) = 0$ for $\xi \le y^b$ and

$$\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=y^b} = L'(y^b) - \alpha \int_{y^b}^{\infty} \hat{v}_{t-1}'(z-y^b)\phi(\xi)d\xi = 0.$$

This implies that $\hat{y}_t(z) = y^b$. Therefore, when $z \geq ty^b$,

$$v_t'(z) = \alpha \int_0^{y^b} \hat{v}_{t-1}'(z-\xi)\phi(\xi)d\xi - \alpha \int_{y^b}^{\infty} \hat{v}_{t-1}'(z-y^b)\phi(\xi)d\xi = 0.$$

This completes the induction. Moreover, from the above analysis, we can see that the fact $v_t'(z) = 0$ when $z \geq ty^b$ implies that $\hat{y}_{t+1}(z) = y^b$. Therefore, (a) holds for $t = 1, \ldots, T$.

Next, we prove (c).

Note that for $z \leq \check{y}$, we have

$$\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=z} = L'(z) - \alpha \int_z^{\infty} \hat{v}_{t-1}'(0)\phi(\xi)d\xi = (1-\alpha)p - c - ((1-\alpha)p + h)\Phi(z) \geq 0.$$

This implies that $\hat{y}_t(z) = z$ when $z \leq \check{y}$ and $\frac{\partial V_t}{\partial y}(y,z)\Big|_{y=\hat{y}_t(z)} = 0$ when $z \geq \check{y}$.

For (b), first note that for $t = 1, \ldots, T$, when $z \geq \check{y}$, we have $\frac{\partial V_t}{\partial y}(y,z)\Big|_{y=\hat{y}_t(z)} = 0$, i.e., $L'(\hat{y}_t(z)) - \alpha(1-\Phi(\hat{y}_t(z)))\hat{v}_{t-1}'(z-\hat{y}_t(z)) = 0$. Therefore, for $t = 1, \ldots, T$, when $z \geq \check{y}$,

$$\begin{aligned}
\hat{v}_t'(z) &= \alpha(\int_0^{\hat{y}_t(z)} \hat{v}_{t-1}'(z-\xi)\phi(\xi)d\xi + \int_{\hat{y}_t(z)}^{\infty} \hat{v}_{t-1}'(z-\hat{y}_t(z))\phi(\xi)d\xi) \\
&= L'(\hat{y}_t(z)) + \alpha \int_0^{\hat{y}_t(z)} \hat{v}_{t-1}'(z-\xi)\phi(\xi)d\xi \\
&\leq L'(\hat{y}_t(z)) + \alpha\Phi(\hat{y}_t(z))\hat{v}_{t-1}'(z-\hat{y}_t(z)) \\
&= L'(\hat{y}_t(z)) + \frac{\Phi(\hat{y}_t(z))}{1-\Phi(\hat{y}_t(z))}L'(\hat{y}_t(z)) \\
&= \frac{1}{1-\Phi(\hat{y}_t(z))}L'(\hat{y}_t(z)).
\end{aligned} \tag{9}$$

Since $\bar{v}_{t-1}'(z) \geq 0$, we have

$$\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=y^b} = L'(y^b) - \alpha \int_{y^b}^{\infty} \hat{v}_{t-1}'(z-y^b)\phi(\xi)d\xi \leq 0.$$

This implies that $\hat{y}_t(z) \leq y^b$. Next we show that $\hat{y}_t(z) \geq \frac{z}{t}$ when $z < ty^b$. It can be easily verified that $\hat{y}_1(z) = \min(z, y^b)$. Thus, $\hat{y}_1(z) \geq z$ when $z < y^b$. Now, suppose that $\hat{y}_{t-1}(z) \geq \frac{z}{t-1}$ when $z < (t-1)y^b$. Then, in period t, we have $\frac{\partial V_t}{\partial y}(y,z)\Big|_{y=\frac{z}{t}} = L'(\frac{z}{t}) - \alpha(1-\Phi(\frac{z}{t}))\hat{v}_{t-1}'(\frac{t-1}{t}z)$. We consider two cases: $\frac{t-1}{t}z \leq \check{y}$ and $\frac{t-1}{t}z > \check{y}$. For the first case, we have $\frac{z}{t} \leq \frac{t-1}{t}z \leq \check{y}$. Thus,

$$\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=\frac{z}{t}} \geq L'(\frac{z}{t}) - \alpha(1-\Phi(\frac{z}{t}))(p-l) = (1-\alpha)p - c - ((1-\alpha)p + h)\Phi(\frac{z}{t}) \geq 0.$$

This implies that $\hat{y}_t(z) \geq \frac{z}{t}$. For the second case, by (9), we have

$$\hat{v}_{t-1}'(\frac{t-1}{t}z) \leq \frac{1}{1-\Phi(\hat{y}_{t-1}(\frac{t-1}{t}z))}L'(\hat{y}_{t-1}(\frac{t-1}{t}z)) = \theta(\hat{y}_{t-1}(\frac{t-1}{t}z)),$$

where $\theta(y) = \frac{L'(y)}{1-\Phi(y)} = \frac{p-c-(p-\alpha c+h)\Phi(y)}{1-\Phi(y)}$. It can be verified that $\theta(y)$ is nonnegative and nonincreasing on $[0, y^b]$. Therefore,

$$\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=\frac{z}{t}} = L'(\frac{z}{t}) - \alpha(1-\Phi(\frac{z}{t}))\hat{v}_{t-1}'(\frac{t-1}{t}z)$$

$$\geq \frac{1}{1-\Phi(\frac{z}{t})}(\theta(\frac{z}{t}) - \alpha\theta(\hat{y}_{t-1}(\frac{t-1}{t}z))) \geq \frac{1}{1-\Phi(\frac{z}{t})}(\theta(\frac{z}{t}) - \theta(\hat{y}_{t-1}(\frac{t-1}{t}z))) \geq 0,$$

where the last inequality is due to the inductive assumption that $\hat{y}_{t-1}(\frac{t-1}{t}z) \geq \frac{1}{t-1}\frac{t-1}{t}z = \frac{z}{t}$. This implies that $\hat{y}_t(z) \geq \frac{z}{t}$.

For (d), we show by induction that $\hat{y}_t(z) \geq \hat{y}_{t+1}(z)$ and $\hat{v}'_t(z) \leq \hat{v}'_{t+1}(z)$. We have $\hat{v}'_0(z) = 0$ and $\hat{v}'_t(z) \geq 0$ for all t. Thus, $\hat{v}'_0(z) \leq \hat{v}'_1(z)$. Now, suppose $\hat{v}'_{t-1}(z) \leq \hat{v}'_t(z)$. When $z \leq \check{y}$, we have $\hat{y}_t(z) = \hat{y}_{t+1}(z) = z$ and

$$\hat{v}'_t(z) - \hat{v}'_{t+1}(z) = \alpha \int_0^z (\hat{v}'_{t-1}(z-\xi) - \hat{v}'_t(z-\xi))\phi(\xi)d\xi \leq 0.$$

When $z > \check{y}$, we have

$$\begin{aligned}\frac{\partial V_t}{\partial y}(y,z)\bigg|_{y=\hat{y}_{t+1}(z)} &= L'(\hat{y}_{t+1}(z)) - \alpha(1-\Phi(\frac{z}{t}))\hat{v}'_{t-1}(z-\hat{y}_{t+1}(z)) \\ &\geq L'(\hat{y}_{t+1}(z)) - \alpha(1-\Phi(\frac{z}{t}))\hat{v}'_t(z-\hat{y}_{t+1}(z)) = 0.\end{aligned}$$

This implies that $\hat{y}_t(z) \geq \hat{y}_{t+1}(z)$. We also have

$$\begin{aligned}\hat{v}'_t(z) - \hat{v}'_{t+1}(z) &= L'(\hat{y}_t) - L'(\hat{y}_{t+1}) + \alpha(\int_0^{\hat{y}_t} \bar{v}'_{t-1}(z-\xi)\phi(\xi)d\xi - \int_0^{\hat{y}_{t+1}} \bar{v}'_t(z-\xi)\phi(\xi)d\xi) \\ &\leq -(p-\alpha c+h)(\Phi(\hat{y}_t) - \Phi(\hat{y}_{t+1})) + \alpha\int_{\hat{y}_{t+1}}^{\hat{y}_t} v'_{t-1}(z-\xi)\phi(\xi)d\xi \\ &\leq -(p-\alpha c+h)(\Phi(\hat{y}_t) - \Phi(\hat{y}_{t+1})) + \alpha\int_{\hat{y}_{t+1}}^{\hat{y}_t} (p-c)\phi(\xi)d\xi \\ &= -((1-\alpha)c+h)(\Phi(\hat{y}_t) - \Phi(\hat{y}_{t+1})) \\ &\leq 0,\end{aligned}$$

where $\hat{y}_t = \hat{y}_t(z)$, $\hat{y}_{t+1} = \hat{y}_{t+1}(z)$, and the last inequality is due to the fact that $\hat{y}_t(z) \geq \hat{y}_{t+1}(z)$.

Last we show that the optimal production thresholds of the system with only the allowance constraint are exactly the same as those of the relaxed system and therefore satisfy properties (a), (b), (c) and (d) in Theorem 4. We only need to show that $\bar{v}_t(x,z) = \hat{v}_t(z)$ for $x \leq \hat{y}_t(z)$, and $\frac{\partial}{\partial y}\bar{V}_t(y,z) \leq 0$ for $y > \hat{y}_t(z)$. This then implies that

$$\operatorname*{argmax}_{x\leq y\leq z} \bar{V}_t(y,z) = \begin{cases} \hat{y}_t(z), & \text{if } x \leq y_t(z), \\ x, & \text{if } x \geq y_t(z). \end{cases}$$

In other words, $y^*_t(x_t,q_t) = \bar{y}_t(x_t+q_t) = \hat{y}_t(x_t+q_t)$.

When $t=0$, we have $\bar{v}_0(x,z) = \hat{v}_0(z) = 0$. Now, suppose that $\bar{v}_{t-1}(x,z) = \hat{v}_{t-1}(z)$ for $x \leq \hat{y}_{t-1}(z)$, and $\frac{\partial}{\partial y}\bar{V}_{t-1}(y,z) \leq 0$ for $y > \hat{y}_{t-1}(z)$. Then, when $x \leq \hat{y}_t(z)$, since $0 \leq \hat{y}'_t(z) \leq 1$ and $\hat{y}'_{t-1}(z) \geq \hat{y}'_t(z)$, we have $\hat{y}_t(z) - \xi \leq \hat{y}_t(z-\xi) \leq \hat{y}_{t-1}(z-\xi)$ for $\xi \leq \hat{y}_t(z)$. This implies that $(\hat{y}_t(z)-D)^+ \leq \hat{y}_{t-1}(z-y+(y-D)^+)$. Thus, $\bar{v}_{t-1}((\hat{y}_t(z)-D)^+, z-y+(y-D)^+) = \hat{v}_{t-1}(z-y+(y-D)^+)$ and $\hat{V}_t(\hat{y}_t(z),z) = \bar{V}_t(\hat{y}_t(z),z)$. As a consequence, $\hat{v}_t(z) = \hat{V}_t(\hat{y}_t(z),z) =$

$\bar{V}_t(\hat{y}_t(z), z) \leq \bar{v}_t(x, z)$ for $x \leq \hat{y}_t(z)$. Clearly, $\bar{v}_t(x, z) \leq \hat{v}_t(z)$. Therefore, $\bar{v}_t(x, z) = \hat{v}_t(z)$ for $x \leq \hat{y}_t(z)$. Similarly, we can show that $\hat{V}_t(y, z) = \bar{V}_t(y, z)$ for $y \leq \hat{y}_t(z)$. We can also prove by induction that $\bar{V}_t(y, z)$ is continuously differentiable. Since $\bar{V}_t(y, z)$ is concave in y, then for $y > \hat{y}_{t-1}(z)$, we have

$$\frac{\partial}{\partial y}\bar{V}_t(y, z) \leq \frac{\partial}{\partial y}\bar{V}_t(\hat{y}_t(z), z) = \bar{V}_t(\bar{y}_t(z), z) = 0.$$

This completes the induction and the proof. □

Proof of Theorem 5: The proof is similar to the proofs of Lemma 8.5 and Theorem 8.4 in Porteus (2002). □

Proof of Proposition 6: Similar to the proof of Lemma 1, we can show that $\bar{v}_t(x, z, u)$ is jointly concave in (x, z, u). Therefore, $v_t(x, q, u) = \bar{v}_t(x, x + q, u)$ is jointly concave in (q, u). This implies that $F_T(c_q, c_u, x, q, u) = -c_q q - c_u u + v_T(x, q, u)$ is jointly concave in (q, u). □

Proof of Proposition 7: (a) Note that $\frac{\partial^2 F_T}{\partial q^2}\frac{\partial q_T^*}{\partial c_q} + \frac{\partial^2 F_T}{\partial q \partial u}\frac{\partial u_T^*}{\partial c_q} + \frac{\partial^2 F_T}{\partial q \partial c_q} = 0$ and $\frac{\partial^2 F_T}{\partial q \partial u}\frac{\partial q_T^*}{\partial c_q} + \frac{\partial^2 F_T}{\partial u^2}\frac{\partial u_T^*}{\partial c_q} + \frac{\partial^2 F_T}{\partial u \partial c_q} = 0$. Since $\frac{\partial^2 F_T}{\partial q \partial c_q} = -1$ and $\frac{\partial^2 F_T}{\partial u \partial c_q} = 0$, we have $\frac{\partial q_T^*}{\partial c_q} = -\frac{1}{A}\frac{\partial^2 F_T}{\partial q \partial u}$ and $\frac{\partial q_T^*}{\partial c_q} = \frac{1}{A}\frac{\partial^2 F_T}{\partial u^2}$, where $A = \frac{\partial^2 F_T}{\partial q^2}\frac{\partial^2 F_T}{\partial u^2} - (\frac{\partial^2 F_T}{\partial q \partial u})^2$. Since F_T is jointly concave in (q, u), $A \geq 0$ and $\frac{\partial^2 F_T}{\partial u^2} \leq 0$. Moreover, $\frac{\partial^2 F_T}{\partial q \partial u} = \frac{\partial^2 v_T}{\partial q \partial u} \geq 0$. Therefore, $\frac{\partial q_T^*}{\partial c_q} \leq 0$ and $\frac{\partial u_T^*}{\partial c_q} \leq 0$. Similarly, we have $\frac{\partial q_T^*}{\partial c_u} \leq 0$ and $\frac{\partial u_T^*}{\partial c_u} \leq 0$.

(b) By Envelope Theorem, $\frac{\partial}{\partial c_q}F_T^*(c_q, c_u, x) = -q_T^*(c_q, c_u, x) \leq 0$ and $\frac{\partial}{\partial c_u}F_T^*(c_q, c_u, x) = -u_T^*(c_q, c_u, x) \leq 0$. Moreover, $F_T(c_q, c_u, x, q, u)$ is convex in (c_q, c_u). Since convexity is preserved under maximization, $F_T^*(c_q, c_u, x)$ is convex in (c_q, c_u).

(c) When $c_u = 0$, the system can be viewed as one with only the allowance constraint. Thus, the second stage problem can be formulated by optimality equation (3), and the first stage problem can be formulated as $\max_{q \geq 0}\{-c_q q + \bar{v}_T(q)\}$. Let $\bar{F}(c_q, q) = -c_q q + \bar{v}_T(q)$ and let K be the smallest q such that $\bar{v}'_T(q) = 0$. Then for $q < K$, $\bar{v}'_T(q) > 0$, and for $q \geq K$, $\bar{v}'_T(q) = 0$. This implies that $q_T^*(0, 0, 0) = K$. Note that $v_T(z)$ is twice differentiable. Thus we have $\bar{v}''_T(K) = 0$. As a consequence,

$$\frac{\partial}{\partial c_q}q_T^*(c_q, 0, 0)\Big|_{c_q=0} = -\frac{\frac{\partial^2}{\partial q \partial c_q}\bar{F}(c_q, q_T^*(c_q, 0, 0))}{\frac{\partial^2}{\partial q^2}\bar{F}(c_q, q_T^*(c_q, 0, 0))}\Bigg|_{c_q=0} = -\frac{-1}{v''_T(q_T^*(c_q, 0, 0))}\Bigg|_{c_q=0} = -\infty. \quad \square$$

Proof of Proposition 8: Since $c_u = 0$, the system can be viewed as one with only the allowance constraint. Let $Q_t(x, q)$ denote the expected cumulative amount produced, under the optimal policy, from period t to the end of the planning horizon, with starting inventory level x and remaining allowance q. Then, $u_T(w, x) = \frac{Q_T(x, c_T^*(w, x))}{c_T^*(w, x)}$. Note that $Q_t(x, q)$, for $t = T, ..., 1$, can be computed recursively as

$$Q_t(x,q) = \begin{cases} EQ_{t-1}((x-D)^+, q), & \text{if } x \geq y_t^*(x,c), \\ y_t^*(x,q) - x + EQ_{t-1}((y_t^*(x,q)-D)^+, x+c-y_t^*(x,c)), & \text{otherwise,} \end{cases}$$

and $Q_0(x,q) = 0$. We can then show by induction that, for $1 \leq t \leq T$, $\bar{v}_t'(z) < 0$ when $z < ty^b$, and $\bar{v}_t'(z) = 0$ when $z \geq ty^b$. This implies that $q_T^*(0,0,x) = (Ty^b - x)^+$. We can also show by induction that $\frac{\partial}{\partial q} Q_t(x,q) = 0$ when $x + q \geq ty^b$. As a consequence,

$$\begin{aligned} \frac{\partial \eta_T}{\partial c_q}(c_q,0,0)\bigg|_{c_q=0} &= \frac{q_T^*(0,0,0)\frac{\partial}{\partial q}Q_t(0,q_T^*(0,0,0))\frac{\partial}{\partial c_q}q_T^*(0,0,0) - Q_t(0,q_T^*(0,0,0))\frac{\partial}{\partial c_q}q_T^*(0,0,0)}{(q_T^*(0,0,0))^2} \\ &= \frac{Ty^b\frac{\partial}{\partial q}Q_t(0,Ty^b) - Q_t(0,Ty^b)}{(Ty^b)^2}\frac{\partial}{\partial c_q}q_T^*(0,0,0) = \infty. \quad \square \end{aligned}$$

Proof of Theorem 9: Let $\bar{g}_t(x,z) = v(x, z-x) - cx$, $\bar{f}_t(y,z,d) = f_t(y, z-y, d)$, and $\bar{L}(x) = (-p+\alpha c)x - h(x)^+ - b(-x)^+$. Then we can rewrite the optimality equations in (5) and (6) as (7) and (8). In the proof of Lemma 1, we show that $0 \leq \frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) \leq p - c$ and $\bar{F}_t(y,z,l) = \bar{L}(y-l) + py + \alpha\bar{g}_{t-1}(y-l, z-l)$ is concave in l. Let $\bar{l}_t(y,z) = \underset{0\leq l\leq z}{\operatorname{argmax}}\{\bar{L}(y-l) + py + \alpha\bar{g}_{t-1}(y-l,z-l)\}$. When $l \leq y$, we have

$$\frac{\partial}{\partial l}\bar{F}_t(y,z,l) = p - \alpha c + h - \alpha(\frac{\partial}{\partial x}\bar{g}_{t-1}(y-l,z-l) + \frac{\partial}{\partial z}\bar{g}_{t-1}(y-l,z-l)) \geq 0.$$

This implies that $\bar{l}_t(y,z) = \underset{y\leq l\leq z}{\operatorname{argmax}}\{(-p+\alpha c)(y-l) - b(l-y) + py + \alpha\bar{g}_{t-1}(y-l,z-l)\}$. Let $w_t^*(q) = \underset{0\leq w\leq q}{\operatorname{argmax}}\{(p-\alpha c-b)w + \alpha\bar{g}_{t-1}(-w, q-w)\}$. Then $\bar{l}_t(y,z) = y + w_t^*(z-y)$. Given inventory level y_t, remaining allowance q_{t-1}, and realized demand d, the optimal amount of demand to fulfill equals $\min(d, y_t + w^*(q_{t-1}))$. By property (c) of Lemma 2 of Huh and Janakiraman (2010), we have $0 \leq \frac{d}{dq}w_t^*(q) \leq 1$.

For (c), note that $w_t^*(q) = \underset{0\leq w\leq q}{\operatorname{argmax}}\{(p-\alpha c-b)w + \alpha\bar{g}_{t-1}(-w,q-w)\}$. If we can show $\frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) \geq \frac{\partial}{\partial x}\bar{g}_{t-1}(x,z) + \frac{\partial}{\partial z}\bar{g}_{t-1}(x,z)$, then we have $w_t^*(q) \geq w_{t+1}^*(q)$. Next we show by induction that $\frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) \leq \frac{\partial}{\partial x}\bar{g}_{t+1}(x,z) + \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z)$ and $\frac{\partial}{\partial z}\bar{g}_t(x,z) \leq \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z)$.

Clearly, $\frac{\partial}{\partial x}\bar{g}_1(x,z)+\frac{\partial}{\partial z}\bar{g}_1(x,z) \geq 0 = \frac{\partial}{\partial x}\bar{g}_0(x,z)+\frac{\partial}{\partial z}\bar{g}_0(x,z)$ and $\frac{\partial}{\partial z}\bar{g}_1(x,z) \geq 0 = \frac{\partial}{\partial z}\bar{g}_0(x,z)$. Now, suppose $\frac{\partial}{\partial x}\bar{g}_{t-1}(x,z)+\frac{\partial}{\partial z}\bar{g}_{t-1}(x,z) \leq \frac{\partial}{\partial x}\bar{g}_t(x,z)+\frac{\partial}{\partial z}\bar{g}_t(x,z)$ and $\frac{\partial}{\partial z}\bar{g}_{t-1}(x,z) \leq \frac{\partial}{\partial z}\bar{g}_t(x,z)$. Then, we have $w_t^*(q) \geq w_{t+1}^*(q)$. By the structure of the optimal fulfillment policy,

$$\begin{aligned} \bar{G}_t(y,z) = -cy &+ \int_0^y [(-p+\alpha c)(y-\xi) - h(y-\xi) + py + \alpha\bar{g}_{t-1}(y-\xi, z-\xi)]\phi(\xi)d\xi \\ &+ \int_y^{y+w_t^*} [(-p+\alpha c)(y-\xi) - b(\xi-y) + py + \alpha\bar{g}_{t-1}(y-\xi,z-\xi)]\phi(\xi)d\xi \\ &+ \int_{y+w_t^*}^{\infty} [(-p+\alpha c)(-w_t^*) - bw_t^* + py + \alpha\bar{g}_{t-1}(-w_t^*, z-y-w_t^*)]\phi(\xi)d\xi, \end{aligned}$$

where $w_t^* = w_t^*(z-y)$. Hence,

$$\frac{\partial}{\partial y}\bar{G}_t(y,z) + \frac{\partial}{\partial z}\bar{G}_t(y,z) = p - c - (h+b)\Phi(y) - (p-\alpha c - b)\Phi(y+w_t^*) + \alpha\int_0^{y+w_t^*}[\frac{\partial}{\partial x}\bar{v}_{t-1}(y-\xi,z-\xi) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-\xi,z-\xi)]\phi(\xi)d\xi.$$

By the definition of w_t^*, $\frac{\partial}{\partial x}\bar{v}_{t-1}(y-w,z-w) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-w,z-w) \leq (p-\alpha c-b)$ for any $w \in [0, w_t^*]$ and $\frac{\partial}{\partial x}\bar{v}_{t-1}(y-w,z-w) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-w,z-w) \leq (p-\alpha c-b)$ for any $w \in [w_t^*, z]$. This implies that

$$\begin{aligned}\frac{\partial}{\partial y}\bar{G}_t(y,z) + \frac{\partial}{\partial z}\bar{G}_t(y,z) &\leq p - c - (h+b)\Phi(y) - (p-\alpha c - b)\Phi(y+w_{t+1}^*)\\ &\quad + \alpha\int_0^{y+w_{t+1}^*}[\frac{\partial}{\partial x}\bar{v}_{t-1}(y-\xi,z-\xi) + \frac{\partial}{\partial z}\bar{v}_{t-1}(y-\xi,z-\xi)]\phi(\xi)d\xi\\ &= p - c - (h+b)\Phi(y) - (p-\alpha c - b)\Phi(y+w_{t+1}^*)\\ &\quad + \alpha\int_0^{y+w_{t+1}^*}[\frac{\partial}{\partial x}\bar{v}_{t}(y-\xi,z-\xi) + \frac{\partial}{\partial z}\bar{v}_{t}(y-\xi,z-\xi)]\phi(\xi)d\xi\\ &= \frac{\partial}{\partial y}\bar{G}_{t+1}(y,z) + \frac{\partial}{\partial z}\bar{G}_{t+1}(y,z).\end{aligned}$$

Similarly, we can show that $\frac{\partial}{\partial z}\bar{G}_t(y,z) \leq \frac{\partial}{\partial z}\bar{G}_{t+1}(y,z)$. It can be verified that, for $n = 1, \ldots, T$,

$$\frac{\partial}{\partial x}\bar{g}_n(x,z) + \frac{\partial}{\partial z}\bar{g}_n(x,z) = (\frac{\partial}{\partial y}\bar{G}_n(y,z) + \frac{\partial}{\partial z}\bar{G}_n(y,z))\Big|_{y=y_n(x,z)},$$

where $y_n(x,z)$ is the optimal order-up-to level in period n. Let $\bar{y}_t(z) = \underset{0\leq y\leq z}{\text{argmax}}\{-cy + E\bar{f}_t(y,z,D)\}$. If $\bar{y}_t(z) \geq \bar{y}_{t+1}(z)$, then $y_t(x,z) \geq y_{t+1}(x,z)$ and

$$\begin{aligned}\frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) &\leq (\frac{\partial}{\partial y}\bar{G}_t(y,z) + \frac{\partial}{\partial z}\bar{G}_t(y,z))\Big|_{y=y_{t+1}(x,z)}\\ &\leq (\frac{\partial}{\partial y}\bar{G}_{t+1}(y,z) + \frac{\partial}{\partial z}\bar{G}_{t+1}(y,z))\Big|_{y=y_{t+1}(x,z)}\\ &= \frac{\partial}{\partial x}\bar{g}_{t+1}(x,z) + \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z),\end{aligned}$$

where the first inequality is due to the $L^\natural$-convexity of $-\bar{G}_t(y,z)$. If $\bar{y}_t(z) < \bar{y}_{t+1}(z)$, then $y_t(x,z) \leq y_{t+1}(x,z)$. It can be verified that $\frac{\partial}{\partial x}\bar{g}_t(x,z) \leq \frac{\partial}{\partial x}\bar{g}_{t+1}(x,z)$. Moreover,

$$\begin{aligned}\frac{\partial}{\partial z}\bar{g}_t(x,z) = \frac{\partial}{\partial z}\bar{G}_t(y,z)\Big|_{y=y_t(x,z)} &\leq \frac{\partial}{\partial z}\bar{G}_t(y,z)\Big|_{y=y_{t+1}(x,z)}\\ &\leq \frac{\partial}{\partial z}\bar{G}_{t+1}(y,z))\Big|_{y=y_{t+1}(x,z)} = \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z),\end{aligned}$$

where the first inequality is due to the supermodularity of $\bar{G}_t(y,z)$. Therefore, $\frac{\partial}{\partial x}\bar{g}_t(x,z) + \frac{\partial}{\partial z}\bar{g}_t(x,z) \leq \frac{\partial}{\partial x}\bar{g}_{t+1}(x,z) + \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z)$. Similarly, we can show $\frac{\partial}{\partial z}\bar{g}_t(x,z) \leq \frac{\partial}{\partial z}\bar{g}_{t+1}(x,z)$. This completes the induction and the proof. □

Proof of Proposition 10: Note that $w_t^*(q) = \underset{0\leq w\leq q}{\text{argmax}}\{(-p+\alpha c-b)w + \alpha\bar{g}_{t-1}(-w,q-w)$. When $p-\alpha c \leq b$, we have $p-\alpha c-b-\alpha(\frac{\partial}{\partial x}\bar{g}_{t-1}(-w,q-w)+\frac{\partial}{\partial z}\bar{g}_{t-1}(-w,q-w)) \leq p-\alpha c-b \leq 0$. This implies that $w_t^*(q) = 0$. When $(1-\alpha c)p \leq b$, we have $p-\alpha c-b-\alpha(\frac{\partial}{\partial x}\bar{g}_{t-1}(-w,q-w)+$

$\frac{\partial}{\partial z}\bar{g}_{t-1}(-w, q-w)) \geq p - \alpha c - b - \alpha(p-c) \geq 0$. This implies that $w_t^*(q) = q$. (a) and (b) are proved.

For (c), note that given x_t and q_t, it is optimal to back order at most $w_t^*(x_t + q_t - y_t(x_t, q_t))$ units of demand, where $y_t(x_t, q_t)$ is defined in the proof of Theorem 2. Since $w_t^*(q)$ is nondecreasing in q and $x + q - y_t(x, q)$ is nondecreasing in q, $w_t^*(x_t + q_t - y_t(x_t, q_t))$ is nondecreasing in q_t. Clearly, $w_t^*(x_t + q_t - y_t(x_t, q_t)) > 0$ when q_t is sufficiently large. Let $q_t^*(x_t)$ be the largest q_t such that $w_t^*(x_t + q_t - y_t(x_t, q_t)) = 0$. Then $w_t^*(x_t + q_t - y_t(x_t, q_t)) = 0$ when $q_t \leq q_t^*(x_t)$ and $w_t^*(x_t + q_t - y_t(x_t, q_t)) > 0$ when $q_t > q_t^*(x_t)$. □

References

Huh WT, Janakiraman G (2010). On the Optimal Policy Structure in Serial Inventory Systems with Lost Sales. *Operations Research.* 58(2):486-491.

Porteus E (2002). *Foundations of Stochastic Inventory Theory.* (Stanford University Press, Redwood City, CA)

Zipkin P (2008). On the Structure of Lost-Sales Inventory Models. *Operations Research.* 56(4):937-944.